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Tensile Instability of Hollow Rotating Discs of Uniform Thickness

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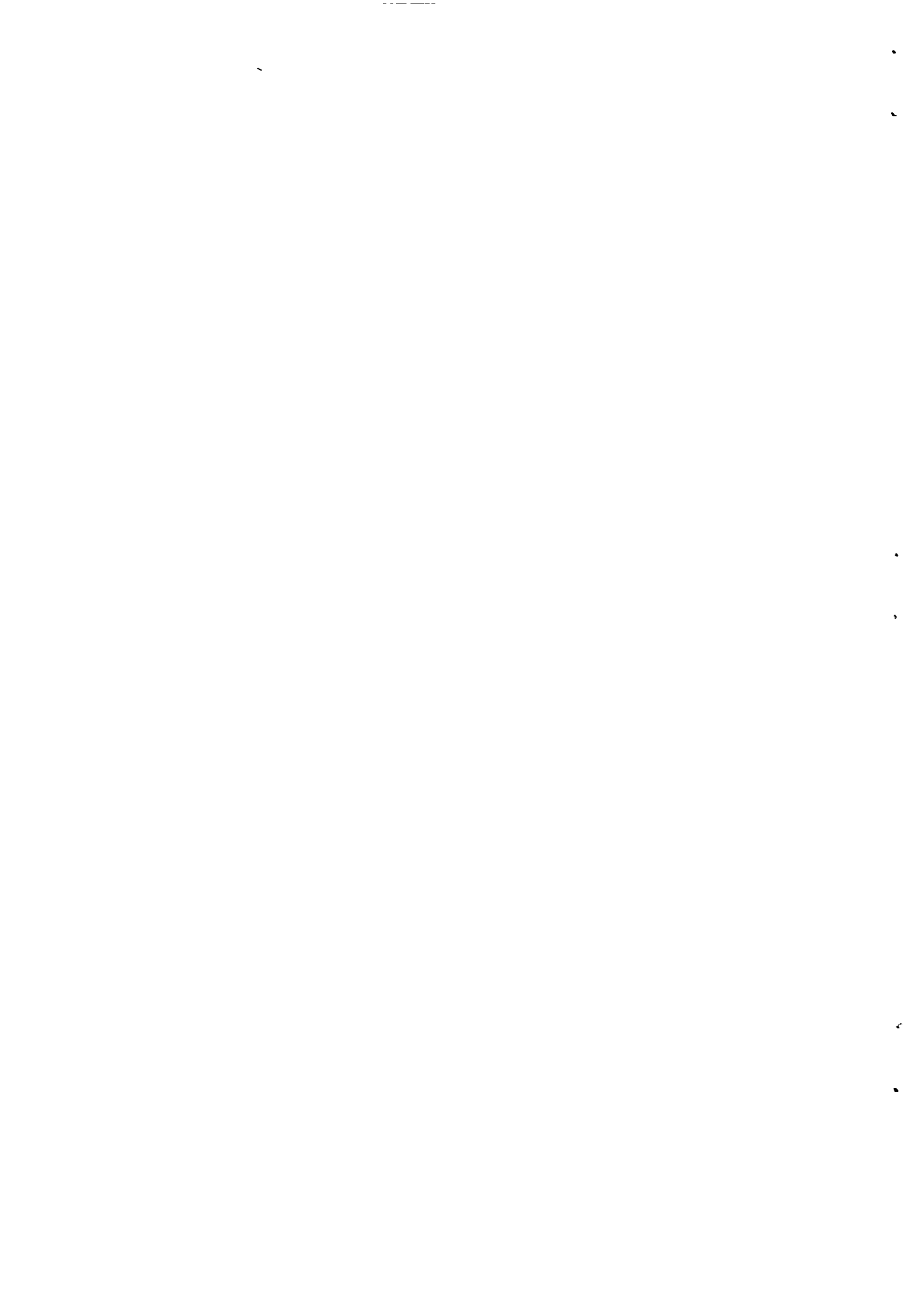
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Tensile Instability of Hollow Rotating Discs of Uniform Thickness

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SUMMARY

The paper presents an analysis of conditions at instability of hollow rotating discs of uniform initial thickness. The analysis is based on Tresca's yield criterion and associated flow rule. The analysis is given more general application by assuming a strain-hardening law of the form $\sigma = Ae^n$. It is shown that when the ratio of outer to inner radius of a disc is greater than approximately 3.3 then, at instability the circumferential strain at the bore is always greater than the axial strain at instability in a tensile test on the same material. Results indicate that it would be worthwhile to attempt correlation of the theoretical results with experimental results on vacuum melted material.

NOTATION

A	=	constant in strain-hardening relation
a, b	=	current inner and outer radii of disc
a_0, b_0	=	initial inner and outer radii of disc
e_r, e_θ e_z	=	engineering strains in radial, circumferential and axial directions
n	=	constant in strain-hardening relation
r	=	current radius to particle
r_0	=	initial radius to particle
s_r, s_θ s_z	=	nominal stress in radial, circumferential and axial directions
t	=	current thickness of disc of radius r
t_0	=	initial uniform thickness of disc
u	=	radial displacement of particle
Y	=	tensile yield stress
α	=	radius ratio, b_0/a_0

- $\epsilon_r, \epsilon_\theta$ = natural strains in radial, circumferential and axial directions
- η = u/a_0
- ξ = r_0/a_0
- ρ = density of material
- σ_r, σ_θ = true stress in radial, circumferential and axial directions
- ω = angular velocity of rotating disc.

INTRODUCTION

When the angular velocity of a thin circular disc at room temperature is gradually increased the disc first of all behaves elastically then, at a certain velocity, plastic yielding begins and proceeds, with work-hardening, until all the material of the disc has reached the initial yield point. Further increase in speed causes unrestricted plastic flow to take place. The system is stable as long as an increase in speed is necessary to produce further plastic flow. The following analysis is concerned with conditions at the inception of instability, that is with conditions at maximum speed. It will be assumed that the material is homogeneous and isotropic and the elastic strains will be assumed to be negligible compared with the plastic strains.

Several analyses have been made of this problem, the only difference between them being in the assumed plastic stress-strain relationships. The Levy-Mises plastic stress-strain relationships are usually agreed to be nearest to physical reality for this type of work but they are difficult to handle in this problem since they predict that the strain increment ratios vary throughout the process. Lee Wu (1950), Manson (1951) and Zaid (1953) have introduced a simplification by assuming the von Mises yield criterion and a deformation theory of plasticity rather than a flow theory. Lee Wu and Zaid have sought to justify this simplification for particular cases by showing 'a posteriori' that the strain ratios are nearly constant as the speed is increased. Weiss and Prager (1954) made the simplifying assumption that the disc material obeys Tresca's yield criterion and the associated flow rule. Marie Madden Smith (1958) adopted the same assumption in an analysis of rotating plastic discs of variable thickness. In the following work the analysis of Weiss and Prager has been adopted and developed to cover much greater strains than contemplated in their paper.

BASIC THEORY

For a thin circular disc rotating about the central axis the principal stress components are the circumferential stress σ_θ , the radial stress σ_r and the axial stress σ_z . The axial stress, σ_z , will be assumed to be zero for the thin discs to be considered here. For a disc of uniform thickness it can be proved (Weiss and Prager, 1954) that σ_θ and σ_r are both tensile and that σ_θ is everywhere greater than σ_r . Tresca's yield criterion and flow rule can then be written,

$$\sigma_{\theta} = Y > \sigma_r > 0, \quad \sigma_z = 0 \quad \dots (1)$$

$$d\epsilon_r = 0, \quad d\epsilon_{\theta} = -d\epsilon_z \geq 0 \quad \dots (2)$$

where Y is the tensile yield stress and $d\epsilon_{\theta}$, $d\epsilon_r$ and $d\epsilon_z$ are respectively the circumferential, radial and axial strain increments.

Let u be the radial displacement of a particle. Then, since from the first equation (2) the radial strain rate is zero, this displacement u does not depend on the initial radius of the particle r_0 but only on the angular velocity. Thus, an annulus bounded initially by radii r_0 and $r_0 + dr_0$ is at a particular speed (after undergoing a radial displacement, u) bounded by radii $r = r_0 + u$ and $r + dr = r_0 + u + dr_0$, from which it follows that $dr = dr_0$. Also, suppose that the initial thickness of the disc t_0 has now, at this particular radius, a thickness t . Since the material is assumed to be incompressible

$$t_0 r_0 dr_0 = t r dr$$

or

$$t_0 r_0 = tr \quad \dots (3)$$

The true stress σ_r is transmitted across a section that is proportional to ' tr ' and since $tr = t_0 r_0$, σ_r is equal to the nominal radial stress s_r . The true circumferential stress σ_{θ} is transmitted across a section proportional to ' $t dr$ ' and is therefore related to the nominal circumferential stress s_{θ} by the relation,

$$s_{\theta}(t_0 dr_0) = \sigma_{\theta}(t dr)$$

and since $dr_0 = dr$,

$$s_{\theta} = \sigma_{\theta}(t/t_0) \quad \dots (4)$$

This nominal stress is related to the circumferential engineering strain,

$$e_{\theta} = \frac{u}{r_0} \quad \dots (5)$$

by the strain-hardening law

$$s = f(e) \quad \dots (6)$$

The plastic flow described by equations (2) is independent of the value of the intermediate principal stress component σ_r and it will therefore be assumed that s can be interpreted as the nominal circumferential stress s_{θ} and e as the circumferential engineering strain e_{θ} . That is,

$$s_{\theta} = f(e_{\theta}) \quad \dots (7)$$

Finally, the equation of radial equilibrium in the deformed state is

$$\delta(r\tau_r)/\delta r = \tau_\theta - \rho\omega^2 r^2 \quad \dots (8)$$

where ω is the angular velocity and ρ is the density of the material.

With reference to the undeformed state, equation (8) can be written as

$$\begin{aligned} \delta(r_0 s_r)/\delta r_0 &= s_\theta - \rho\omega^2 r_0 (r_0 + u) \\ &= f(u/r_0) - \rho\omega^2 r_0 (r_0 + u) \end{aligned} \quad \dots (9)$$

If the disc is not loaded at the bore, $r_0 = a_0$, or at the periphery, $r_0 = b_0$, then s_r is zero at these radii and the integral of the right-hand side of equation (9) between the limits a_0 and b_0 must vanish. Putting $\alpha = b_0/a_0$, $\eta = u/a_0$ and $\xi = r_0/a_0$, this condition yields

$$\rho a_0^2 \omega^2 / 6 = \int_1^\alpha f(\eta/\xi) d\xi / [2(\alpha^3 - 1) + 3\eta(\alpha^2 - 1)] \quad \dots (10)$$

This is the equation derived by Weiss and Prager. For a known strain-hardening characteristic and given initial dimensions the right-hand side of equation (10) must be evaluated, analytically or numerically, for various values of $\eta = u/a_0$. Each assumed value of η gives the corresponding value of ω^2 and a plot of η against ω^2 can be carried out to determine the maximum value of angular velocity and hence the point of instability.

Equation (10) is based on the nominal stress-engineering strain curve determined in simple tension and the equation used in this form gives valid and easily derived results provided the longitudinal strain at instability in simple tension is greater than the maximum circumferential strain at instability of the rotating disc. This is so for the particular example given by Weiss and Prager for an aluminium alloy with $\alpha = 2$. However, it will be shown that in general when α is greater than approximately 3 then the maximum strain at instability of the disc is greater than the instability strain in simple tension. It is then more convenient to base the solution on the true stress-natural strain diagram of the material. Such a work-hardening characteristic would be best determined by the torsion of a solid round bar.

SOLUTION BASED ON THE TRUE STRESS-STRAIN CURVE

It is intended to examine the effect of work-hardening and of radius ratio, α , on the instability of hollow rotating discs. A strain-hardening law

$$\sigma_\theta = A \epsilon_\theta^n \quad \dots (11)$$

will be adopted, where A and n are constants for a particular disc. This relation does not give full representation of a rigid-plastic material, since there is no initial yield point, but it is a simple and useful expression for discussion of conditions at instability.

The relation between true circumferential stress and nominal circumferential stress is

$$\sigma_{\theta} = s_{\theta}(1 + e_{\theta}) = s_{\theta}(1 + \eta/\xi) \quad \dots (12)$$

and the natural strain is related to the engineering strain by

$$\varepsilon_{\theta} = \ln(1 + e_{\theta}) = \ln(1 + \eta/\xi) \quad \dots (13)$$

Therefore, making use of equations (11), (12) and (13), the function in equation (10) to be integrated can be written,

$$\int_1^{\alpha} f(\eta/\xi) d\xi = A \int_1^{\alpha} \frac{[\ln(1 + \eta/\xi)]^n}{(1 + \eta/\xi)} \cdot d\xi$$

Now change the variable from ξ to $\ln(1 + \eta/\xi) = \varepsilon_{\theta}$. Denoting the circumferential strain at the bore by ε_{θ} and at the periphery by ε_{θ_b} and remembering that

$$\varepsilon_{\theta_a} = \ln(1 + \eta)$$

$$\varepsilon_{\theta_b} = \ln(1 + \eta/\alpha)$$

the expression becomes

$$\int_1^{\alpha} f(\eta/\xi) d\xi = A \int_{\varepsilon_{\theta_b}}^{\varepsilon_{\theta_a}} \eta \frac{\varepsilon_{\theta}^n}{(\exp. \varepsilon_{\theta} - 1)^2} \cdot d\varepsilon_{\theta}$$

For a given value of ω^2 , $\eta = u/r_0$ is constant and therefore equation (10) can be written

$$\frac{\rho a^2 \omega^2}{6A} = \frac{\eta \int_{\varepsilon}^{\varepsilon_{\theta_a}} \frac{\varepsilon_{\theta}^n}{(\exp. \varepsilon_{\theta} - 1)^2} \cdot d\varepsilon_{\theta}}{[2(\alpha^3 - 1) + 3\eta(\alpha^2 - 1)]} \quad \dots (14)$$

To determine the integral of the above equation the term

$\frac{\varepsilon_{\theta}}{(\exp. \varepsilon_{\theta} - 1)}$ will be expressed as a series.

$$\frac{\varepsilon_{\theta}}{(\exp. \varepsilon_{\theta} - 1)} = 1 - \frac{\varepsilon_{\theta}}{2} + \frac{B_1 \varepsilon_{\theta}^2}{2!} - \frac{B_3 \varepsilon_{\theta}^4}{4!} + \frac{B_5 \varepsilon_{\theta}^6}{6!} \dots \dots \quad \dots (15)$$

where $\varepsilon_{\theta} < 2$ and the Bernoulli numbers have the values $B_1 = \frac{1}{6}$, $B_3 = \frac{1}{30}$, $B_5 = 5$.

Using this series it can be shown that the right-hand side of the equation

$$\left\{ \frac{\epsilon_{\theta}}{(\text{exp. } \epsilon_{\theta} - 1)} \right\}^2 \simeq 1 - \epsilon_{\theta} + \frac{5}{12} \epsilon_{\theta}^2 - \frac{\epsilon_{\theta}^3}{12} \dots (16)$$

is correct to better than 0.5% for $0 < \epsilon_{\theta} < 0.5$. (Provided $\epsilon_{\theta} < 0.1$ taking the first two terms of the right-hand side of equation (16) gives a result correct to better than 0.5%. The error of taking the first two terms when $\epsilon_{\theta} = 0.3$ is 5% increasing to a 20% error when $\epsilon_{\theta} = 0.5$.)

Substituting expression (16) into equation (14),

$$\frac{\rho a_0^2 \omega^2}{6A} = \frac{\eta \int_{\epsilon_{\theta b}}^{\epsilon_{\theta a}} \left[\epsilon_{\theta}^{n-2} - \epsilon_{\theta}^{n-1} + \frac{5}{12} \epsilon_{\theta}^n - \frac{\epsilon_{\theta}^{n+1}}{12} \right] d\epsilon_{\theta}}{[2(\alpha^2 - 1) + 3\eta(\alpha^2 - 1)]} \dots (17)$$

Integrating equation (17)

$$\frac{\rho a_0^2 \omega^2}{6A} = \frac{\eta \left[\frac{\epsilon_{\theta}^{n-1}}{n-1} - \frac{\epsilon_{\theta}^n}{n} + \frac{5}{12} \frac{\epsilon_{\theta}^{n+1}}{n+1} - \frac{\epsilon_{\theta}^{n+2}}{12(n+2)} \right]_{\epsilon_{\theta b}}^{\epsilon_{\theta a}}}{[2(\alpha^2 - 1) + 3\eta(\alpha^2 - 1)]} \dots (18)$$

If for a particular radius ratio, α , a value of η is assumed then $\epsilon_{\theta a}$ and $\epsilon_{\theta b}$ can be calculated. Hence, the corresponding value of ω^2 can be determined from equation (18). When this is repeated a plot of ω^2 against $\epsilon_{\theta a}$ can be obtained. Such curves are shown in Figs. 1 and 2 for materials with $n = 0.05$ and $n = 0.1$, and for values of $\alpha = 2$ and $\alpha = 10$. (The value of the strain-hardening index n for turbine disc materials usually lies between 0.05 and 0.1.) The values of the maximum velocity can be determined accurately from curves of ω^2 against $\epsilon_{\theta a}$ but inspection of Figs. 1 and 2 will show that it is not possible to determine accurately the value of $\epsilon_{\theta a}$ at instability. To determine the instability strain equation (18) is differentiated with respect to η and equated to zero,

$$\begin{aligned} \frac{d(\omega^2)}{d\eta} = 0 = & \frac{2(\alpha^3-1)}{2(\alpha^3-1) + 3\eta(\alpha^2-1)} \left[\frac{(\epsilon_{\theta_a}^{n-1} - \epsilon_{\theta_b}^{n-1})}{(n-1)} - \frac{(\epsilon_{\theta_a}^n - \epsilon_{\theta_b}^n)}{n} \right. \\ & \left. + \frac{5}{12} \frac{(\epsilon_{\theta_a}^{n+1} - \epsilon_{\theta_b}^{n+1})}{(n+1)} - \frac{1}{12} \frac{(\epsilon_{\theta_a}^{n+2} - \epsilon_{\theta_b}^{n+2})}{(n+2)} \right] \\ & + \eta \left[\frac{(\epsilon_{\theta_a}^{n-2} - \epsilon_{\theta_a}^{n-1})}{(1+\eta)} - \frac{(\epsilon_{\theta_b}^{n-2} - \epsilon_{\theta_b}^{n-1})}{(\alpha+\eta)} + \frac{1}{12} \frac{(5\epsilon_{\theta_a}^n - \epsilon_{\theta_a}^{n+1})}{(1+\eta)} \right. \\ & \left. - \frac{1}{12} \frac{(5\epsilon_{\theta_b}^n - \epsilon_{\theta_b}^{n+1})}{(\alpha+\eta)} \right] \dots (19) \end{aligned}$$

Instability strains have been calculated from equation (19) with the aid of an electronic digital computer. Derived instability strains for a range of materials and for values of α between 1 and 20 are shown in Fig. 3.

For a material with a strain-hardening characteristic $\sigma = A\epsilon^n$ the axial strain at instability in the tension test is 'n'. Fig. 3 shows that the circumferential hoop strain at the bore of a disc at instability can be much greater than 'n'. In fact, it is seen that for all values of radius ratio $\alpha > 5$ and $n > 0.01$ the instability strain at the bore is greater than n.

The values for $\alpha = 1$ have been calculated as a separate case. When $\alpha = 1$, that is a thin rotating ring, the circumferential stress is

$$\sigma_{\theta} = \rho\omega^2 r^2 \dots (20)$$

where r is the current mean radius of the ring. At maximum velocity

$$\frac{d\sigma_{\theta}}{\sigma_{\theta}} = 2 \frac{dr}{r} = 2 d\epsilon_{\theta}$$

or

$$\frac{d\sigma_{\theta}}{d\epsilon_{\theta}} = 2 \sigma_{\theta} \dots (21)$$

for a material with a strain-hardening characteristic $\sigma_{\theta} = A\epsilon_{\theta}^n$. This means that the circumferential strain at instability is $n/2$, that is half the amount of strain attained at instability in simple tension.

Knowing the instability strain for a given value of α , the maximum speed is determined from equation (18). Values of $\rho\alpha^2\omega^2/6A$, calculated in this manner are plotted against α in Fig. 4. The calculated strains at instability have also been indicated on Figs. 1 and 2.

DISCUSSION

The aim of the design engineer is to produce an economic structure which will not fail catastrophically. In the particular case of the design of turbine discs for aircraft engines it is important that high strength should go with minimum weight but unfortunately in trying to obtain high strength, for a chosen metal or alloy, the ductility is reduced. According to the present analysis this in itself should not lead to a catastrophic failure. Indeed, the most important point that emerges from the analysis is that for all practical discs the instability strain at the bore is greater than the axial instability strain in simple tension. In fact for $\alpha = 20$ the predicted bore instability strain in a disc is very much greater than the instability strain in simple tension, see Fig. 3.

Nevertheless, experiments on model discs reported by Waldren and Ward (1961) demonstrate that some discs fracture at strains much smaller than predicted by the present theory. It must be remembered that the theory deals with an ideal material, which means a flawless material. If a material contains flaws, cracks or other stress raisers, the final flow and fracture might not follow the pattern set down in the theory. The crucial point is whether or not any existing crack can propagate and cause a premature fracture. If a material is ductile enough it is possible that the stress can be redistributed and a particular crack may then take no further part in the mechanism of fracture. On the other hand for less ductile materials a crack of the same geometry could cause an early failure.

The theory, then, lays down the ideal flow behaviour which is to be aimed at. The practical answer might be the better metallurgical control of disc materials. Waldren and Ward (1961) reported an experimental result on a model disc of high strength, vacuum-melted, turbine disc material. The maximum strain attained in this case was much greater than that attained in similar discs made of more usual non-vacuum-melted materials, having the same elongation at the maximum load in simple tension. More recent experimental work at the National Gas Turbine Establishment, using vacuum-melted materials, confirms that greater strains are obtained before fracture with this type of material. In view of this recent work it would seem worthwhile attempting a correlation between experimental results on vacuum-melted materials and the present theory. This would entail determining the strain-hardening characteristics of the materials to much higher values than could be obtained in simple tension.

The present analysis gives a general solution to the problem of instability in hollow rotating discs of uniform initial thickness. As far as is known the results obtained are more general than any other existing results. However, it is necessary to keep note of the simplifications that have made the solution possible. The theory is based on a flow rule of plasticity which is recognised to be more correct physically than a deformation theory, but this particular flow rule is necessarily coupled with the Tresca yield criterion. These facts have to be borne in mind when attempting any correlation with experiment. The assumed flow rule states that the radial strain is zero throughout the disc. Experiment (Waldren and Ward) shows this to be true except very near to the bore of the disc.

The behaviour of materials with the same tensile strength but with different ductility is discussed in an Appendix.

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APPENDIX

Fig. 1a shows theoretical values of bursting speed for materials having the same tensile strength but having different values of 'n', that is varying amounts of ductility. It is shown that the lower the ductility of a particular material the greater is the bursting speed. However, it is clear that such variations in speed with ductility are relatively small.

A correlation is often made between theory and experiment on the basis that at the bursting speed the nominal average tangential stress in the disc corresponds to the tensile strength of the material. This is, in effect, an approximate comparison between an instability condition in simple tension and an assumed instability condition in the rotating disc. The deformation of the disc, elastic or plastic, is not taken into account except in so far as the existence of an instability condition is assumed. This has been found to give good correlation between theory and experiment even for discs which are not of uniform thickness. The bursting speed calculated on this basis is shown in Fig. 5.

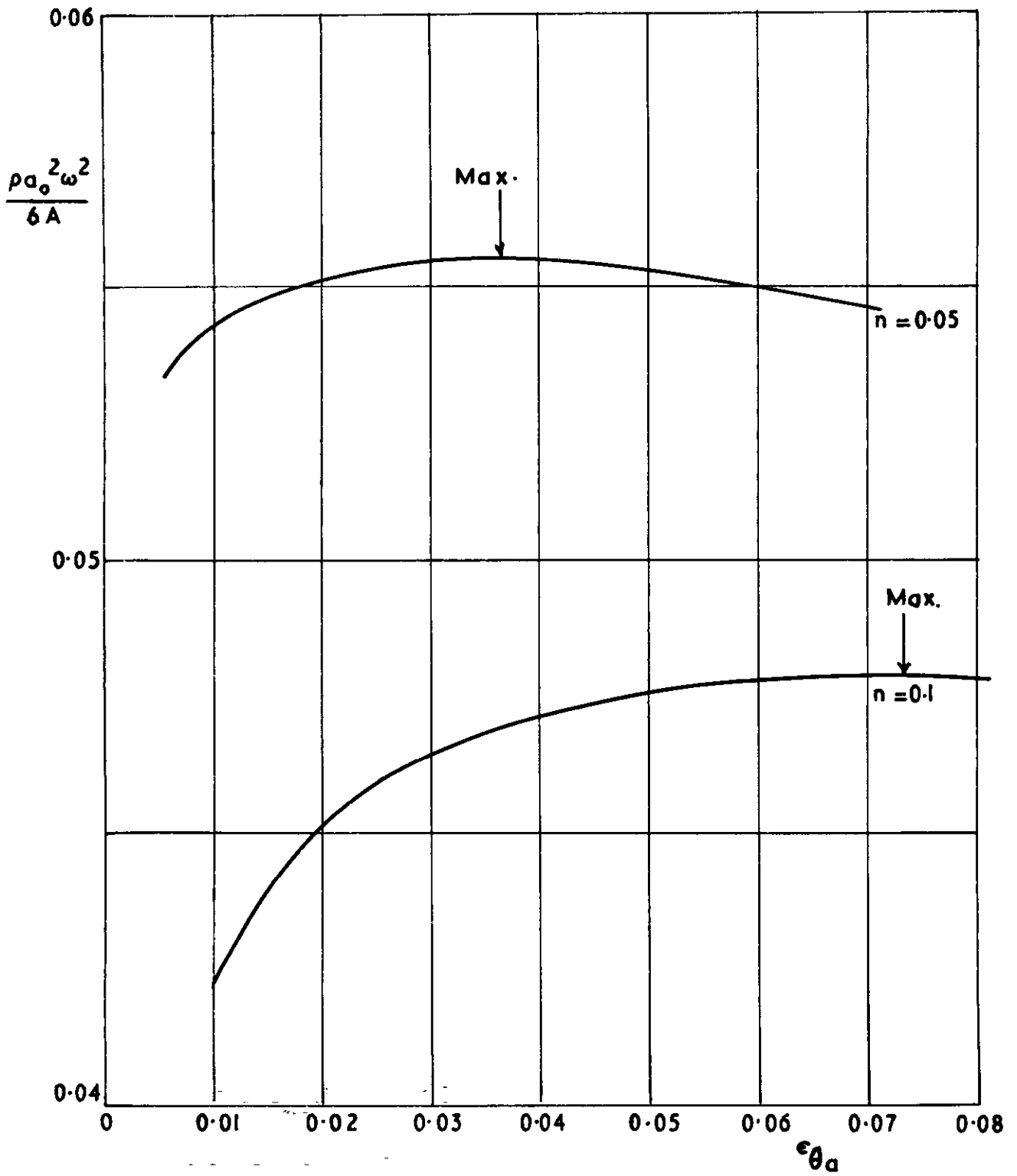


FIG.1. Variation of $\frac{\rho a_0^2 \omega^2}{6A}$ with circumferential strain at bore, $\epsilon_{\theta a}$:
 $\alpha^\circ = 2.$

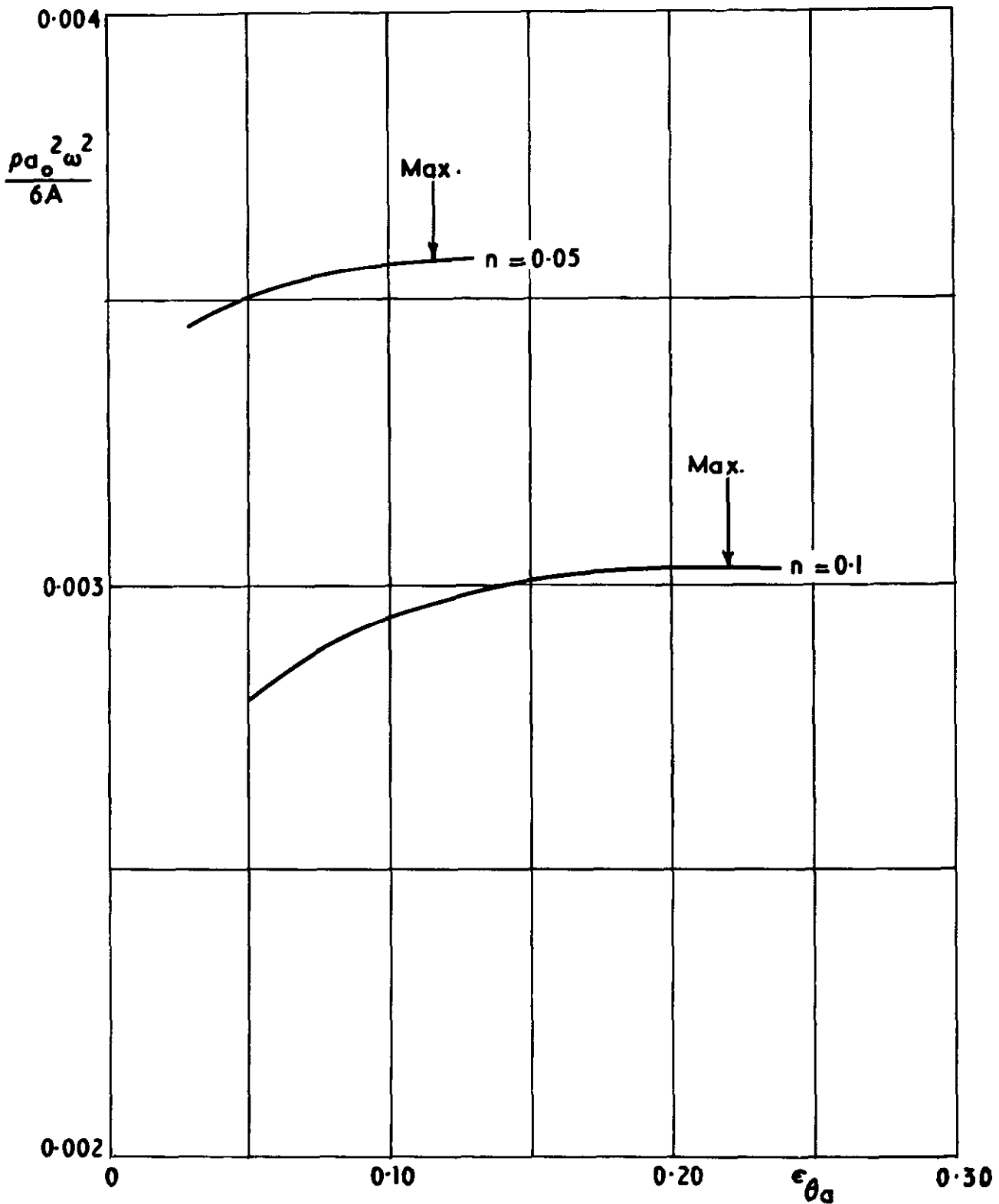


FIG.2. Variation of $\rho a_0^2 \omega^2 / 6A$ with circumferential strain at bore, $\epsilon_{\theta a}$:
 $\alpha^\circ = 10.$

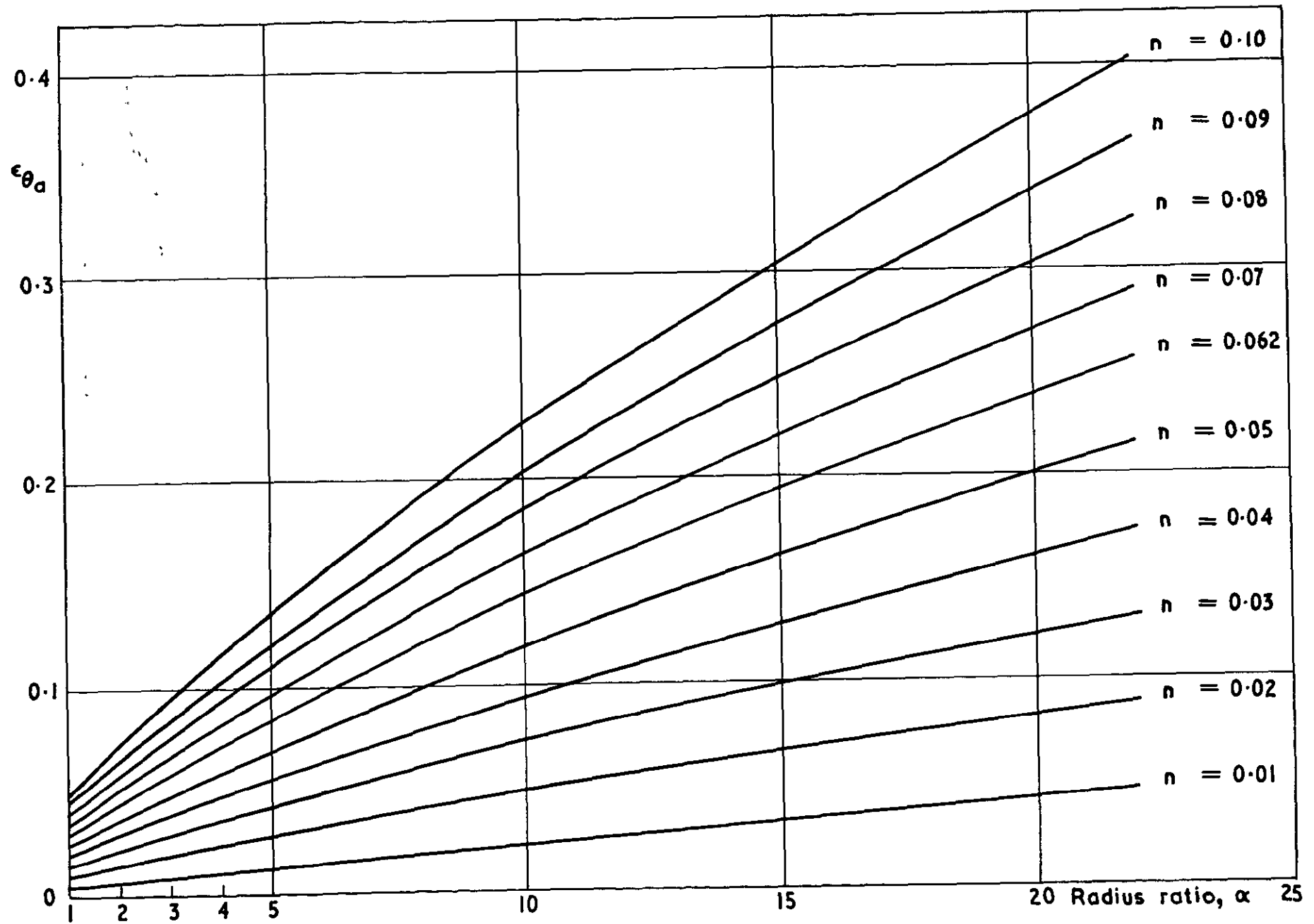


FIG. 3. Variation of maximum circumferential strain at bore with radius ratio, α

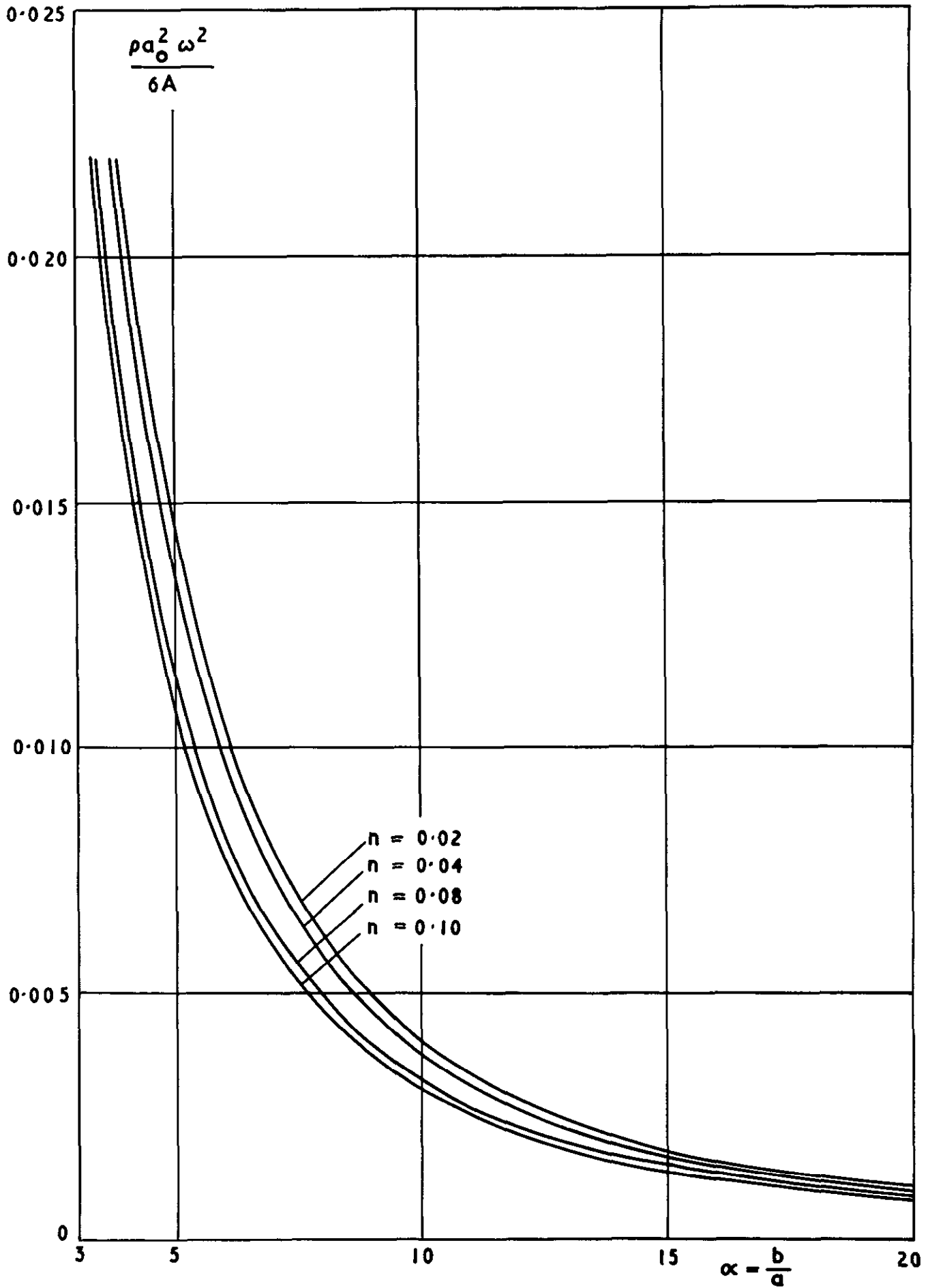


FIG. 4. Variation of $\frac{\rho a_0^2 \omega^2}{6A}$ with radius ratio, α .

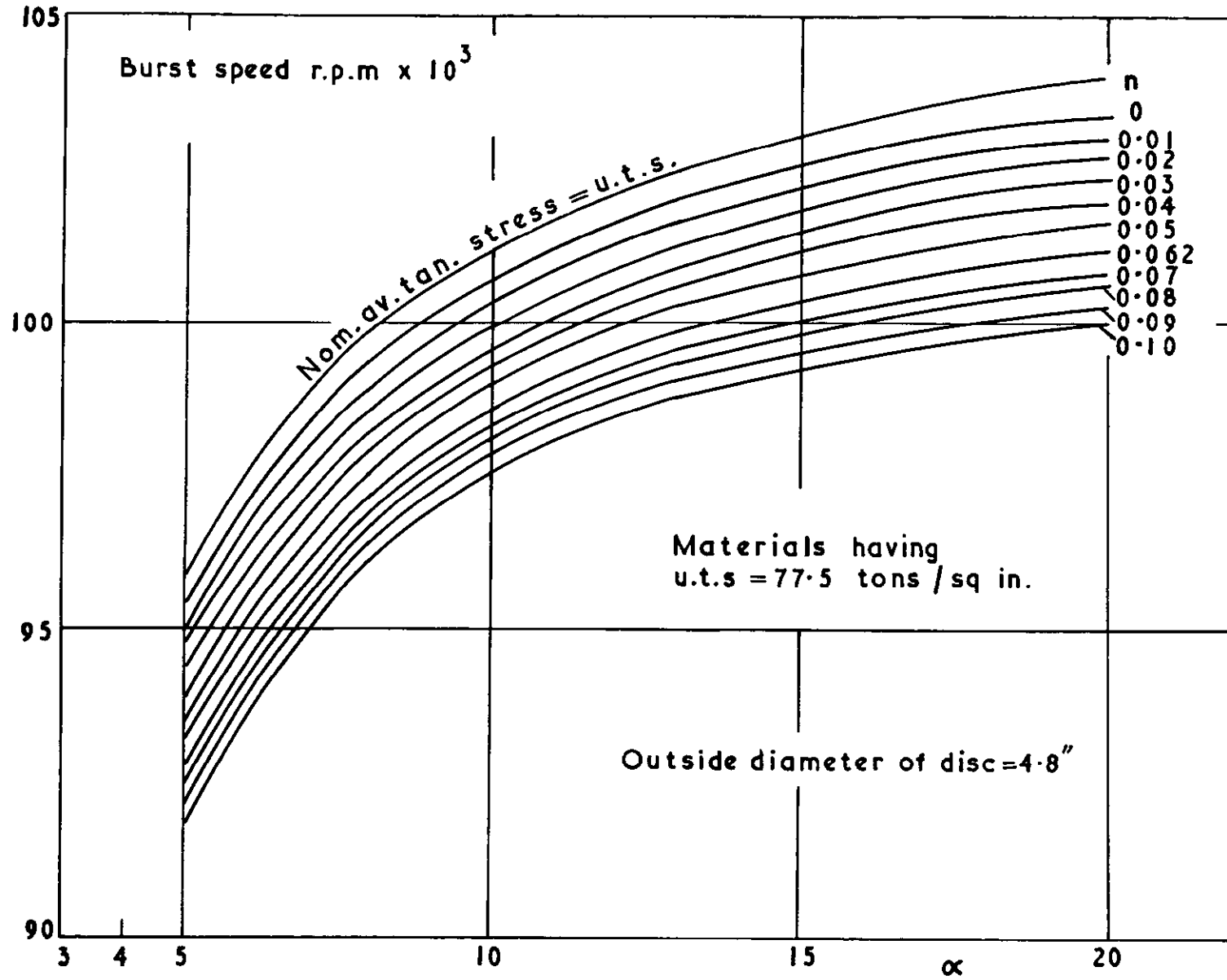


FIG. 5. Variation of burst speed with radius ratio, α .

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