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The Incompressible Potential Flow past Two-dimensional Aerofoils with Arbitrary Surface Suction

By

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The Incompressible Potential Flow past Two-dimensional Aerofoils with Arbitrary Surface Suction

- By -
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Presented by Prof. S. Goldstein, F.R.S.

20th June, 1950

SUMMARY

The problem of the incompressible potential flow past a conventional aerofoil which has an arbitrary distribution of normal velocity across its surface is solved by the application of conformal transformation to the solution derived for the corresponding problem for the unit circle. Several simple worked examples are given. Although it is theoretically possible to obtain any desired tangential velocity distribution on an aerofoil by the correct choice of suction velocity distribution, and this use of "sink effect" gives promise of considerable lift increments, it is considered that the quantities of suction required to produce such desired effects are prohibitively large for immediate practical use. For suction quantities of the order of those at which boundary-layer porous suction is designed to act, the effect of the suction on the main stream potential flow is negligible.

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CONTENTS

Introduction/

1. Introduction

Suction as a boundary-layer control device may be used in two ways: by sucking through one or more slots or by sucking continuously through a porous portion of the surface. These methods are based on rather different principles. Suction at a slot either removes part of the boundary layer at that point and forms a new, thinner, and if the original layer is laminar, more stable boundary layer downstream of the slot, or else removes much more fluid than that contained in the boundary layer, thereby altering the pressure field along the surface for some distance from the slot. Separation of flow is thus delayed, and several suitably spaced slots may be used to suck away parts of the upstream boundary layer as it tends to thicken and separate from the surface. On the other hand, distributed suction, instead of allowing the boundary layer to approach a condition near to separation before removing all or part of it, aims to maintain a favourable velocity profile through the boundary layer, which is kept thin all the time. Distributed, or porous, suction also has a stabilising effect on the whole boundary layer. A considerable amount of theoretical work has been done on the solution of the boundary-layer equations for flow past a porous surface through which there is a continuous normal velocity, and it is assumed throughout this work that the velocity distribution outside the boundary layer is that calculated on potential theory for an impermeable surface of similar shape. That is, it is assumed that surface suction, of the scale envisaged in boundary-layer distribution suction work, does not affect the potential flow past that surface. The question of the validity of this assumption led to the investigation below.

We are concerned with investigating the potential flow past an aerofoil of conventional shape which has an arbitrary distribution of suction on its surface. By "conventional" we mean not specially designed for use with suction. A very thorough discussion of aerofoils designed to incorporate surface suction has been given by Professor S. Goldstein[1]. The words "arbitrary distribution" cover the cases of "overall" distributed suction, where the entire aerofoil is constructed of porous material, distributed suction localised over a given porous region of the aerofoil surface, single and multi-slot suction. In particular we want to know the tangential velocity distribution on the aerofoil when the suction, or inward normal, velocity distribution is known, and we also enquire what suction distribution is necessary to produce a prescribed tangential velocity distribution on the aerofoil at a given incidence. We are interested, too, in the effect of surface suction on the theoretical lift and drag on the aerofoil. The problem is tackled by considering the corresponding problem for the unit circle, the solution for the flow past the aerofoil being obtained from the conformal transformation of the circle into the aerofoil.

Section 2 deals exclusively with the case of a porous circle under certain conditions of surface suction. The most general potential flow round the circle is solved in the cases when either the normal or the tangential velocity on the circle is specified, formulae being given for the complex potential, complex velocity, surface velocity distributions and the lift and drag on the circle.

In section 3, the transformation to the aerofoil is discussed

Several simple examples of distributed suction and one of slot suction applied to a symmetrical Joukowski aerofoil are given in section 4. The results are illustrated in the diagrams.

In the Appendix, the Blasius theorem for the force on a closed curve is extended to the case when the curve is not necessarily a streamline.

2. The Potential Flow Past the Unit Circle with Arbitrary Surface Suction

2.1 Statement of the problem

Consider the unit circle in a uniform stream of "perfect" fluid of velocity U in a direction parallel to the negative real axis of $z (= r e^{i\theta})$, with either the normal or tangential velocity distribution on the circle prescribed. It is required to find the complex potential $W(z) = \phi + i\psi$ of the flow past the circle, such that

(1) $W(z)$ is analytic for $|z| \geqslant 1$,

either (2a) $\left(\dfrac{\partial\phi}{\partial r}\right)_{r=1} = f(\theta)$, is given,

or (2b) $\left(\dfrac{\partial\phi}{\partial\theta}\right)_{r=1} = g(\theta)$, is given,

(3) $\dfrac{dW}{dz} \longrightarrow -U + O\left(\dfrac{1}{z}\right)$ as $z \longrightarrow \infty$.

In particular we want the unknown velocity distribution on the circle in terms of the given one. The problem is stated above for the main stream at zero incidence, but the solution to the case of non-zero incidence can easily be deduced from the solution to this problem.

Put

$$W(z) = -Uz + W_1(z) , \quad W_1 = \phi_1 + i\psi_1 . \tag{1}$$

Then $W_1(z)$ must satisfy

(1) $W_1(z)$ is analytic for $|z| \geqslant 1$,

either (2a) $\left(\dfrac{\partial\phi_1}{\partial r}\right)_{r=1} = f(\theta) + U\cos\theta$,

or (2b) $\left(\dfrac{\partial\phi_1}{\partial\theta}\right)_{r=1} = g(\theta) - U\sin\theta$,

(3)/

(3) $$\frac{dW_1}{dz} \to 0\left(\frac{1}{z}\right) \text{ as } z \to \infty .$$

Having enunciated the problem for the flow past the circle, we shall consider separately the cases when the suction i.e., inward normal, and tangential velocity distributions are specified, since although the methods of solution at zero incidence are precisely similar, the extensions to general incidence are different.

2.2 Solution for prescribed suction velocity distribution

2.2.1 Zero incidence

Consider the function

$$P(z) = z\frac{dW_1}{dz} = r\frac{\partial\phi_1}{\partial r} - i\frac{\partial\phi_1}{\partial\theta} .$$

Then $P(z)$ is analytic in $|z| > 1$, is finite at infinity and has its real part known on the unit circle. In fact

$$[\mathrm{Re}(P(z))]_{r=1} = \left(\frac{\partial\phi_1}{\partial r}\right)_{r=1} = f(\theta) + U\cos\theta = f_1(\theta) \text{, say.} \quad ..(2)$$

Hence, provided $f(\theta)$ is absolutely integrable, we have, by Poisson's integral[2]

$$P(z) - P(\infty) = \frac{1}{\pi i}\oint_C \frac{f_1(t)}{z - t}\,dt , \quad |z| > 1 ,$$

where $t = e^{i\tau}$, C denotes the unit circle and the integral is taken round C in the positive sense. $f_1(t)$ or $f_1(\tau)$ will be used to denote the value of f_1 on C as the context requires. Moreover,

$$P(\infty) = \frac{1}{2\pi i}\oint_C \frac{P(t)}{t}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(\tau)\,d\tau + iK , \quad (3)$$

where K is an arbitrary constant, since $[\mathrm{Im}(P(z))]_{r=1}$ is unknown. Thus

$$P(z) = iK + \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(\tau)\,d\tau + \frac{1}{\pi i}\oint_C \frac{f_1(t)}{z - t}\,dt , \quad |z| > 1 .$$

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$$\frac{dW_1}{dz} = \frac{iK}{z} + \frac{1}{2\pi z}\int_{-\pi}^{\pi} f_1(\tau)\,d\tau + \frac{1}{\pi i}\oint_C \frac{f_1(t)}{z(z-t)}\,dt\,, \quad |z| > 1\,,$$

and, neglecting an arbitrary constant,

$$W_1(z) = iK\log z + \frac{1}{2\pi}\log z \int_{-\pi}^{\pi} f_1(\tau)\,d\tau$$

$$+ \frac{1}{\pi i}\oint_C f_1(t)\log\left(\frac{z-t}{z}\right)\frac{dt}{t}\,, \quad |z| > 1\,,$$

or

$$W_1(z) = iK\log z - \frac{1}{2\pi}\log z \int_{-\pi}^{\pi} f_1(\tau)\,dt$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(\tau)\log(z - e^{i\tau})d\tau\,, \quad |z| > 1\,.$$

Using (1) and (3) and the relation

$$\int_{-\pi}^{\pi} \cos\tau\log(z - e^{i\tau})d\tau = -\frac{\pi}{z}\,,$$

we have

$$W(z) = -U\left(z + \frac{1}{z}\right) + iK\log z - \frac{1}{2\pi}\log z\int_{-\pi}^{\pi} f(\tau)\,d\tau$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)\log(z - e^{i\tau})d\tau, \qquad |z| > 1\,, \qquad \ldots\ldots(4)$$

$$\frac{dW}{dz} = -U\left(1 - \frac{1}{z^2}\right) + \frac{iK}{z} - \frac{1}{2\pi z}\int_{-\pi}^{\pi} f(\tau)d\tau + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(\tau)}{z-e^{i\tau}}\,d\tau\,, \qquad |z| > 1\,.$$

$$\ldots\ldots(5)$$

The/

The equation for $W(z)$ holds also for $|z| = 1$, but in equation (5) $\int_{-\pi}^{\pi} \frac{f(\tau)}{z-e^{i\tau}} d\tau$ becomes divergent when $z = e^{i\theta}$. To find the tangential velocity distribution on the circle, consider

$$\mathrm{Im}\,(P(z)) = -\frac{\partial \phi_1}{\partial \theta} = K - \mathrm{Re}\left\{\frac{1}{\pi}\oint_C \frac{f_1(t)}{z-t}\,dt\right\} .$$

As $z \to e^{i\theta}$,

$$\mathrm{Re}\left\{\frac{1}{\pi}\int_C \frac{f_1(t)}{z-t}\,dt\right\} \to \frac{1}{2\pi}\,P\int_{-\pi}^{\pi} f_1(\tau)\cot\tfrac{1}{2}(\theta-\tau)\,d\tau ,$$

where P denotes that the Cauchy principal value of the integral is to be taken. Hence

$$\left(\frac{\partial \phi_1}{\partial \theta}\right)_{r=1} = -K + \frac{1}{2\pi}\,P\int_{-\pi}^{\pi} f_1(\tau)\cot\tfrac{1}{2}(\theta-\tau)\,d\tau ,$$

and since

$$P\int_{-\pi}^{\pi} \cos\tau\cot\tfrac{1}{2}(\theta-\tau)\,d\tau = 2\pi\sin\theta$$

we have by (1) and (3)

$$g(\theta) = 2U\sin\theta - K + \frac{1}{2\pi}\,P\int_{-\pi}^{\pi} f(\tau)\cot\tfrac{1}{2}(\theta-\tau)\,d\tau , \quad -\pi < \theta \leqslant \pi ,$$

where $g(\theta)$ denotes the tangential velocity distribution on the circle induced at zero incidence by the outward normal velocity distribution $f(\theta)$.

2.2.2 General incidence

(i) The suction distribution, when once chosen, must remain the same with respect to the circle whatever the direction of the incident stream. We may now generalise the above results to the case when the direction of the uniform stream is at an angle α to the negative real axis.

where

$$Z = r\,e^{i\Theta} = ze^{i\alpha}\,, \quad \Phi(\Theta) = f(\Theta - \alpha) = f(\theta)\,.$$

Since $f(\theta)$ has period 2π, then

$$\int_{-\pi}^{\pi} \Phi(\tau)d\tau = \int_{-\pi}^{\pi} f(\tau)d\tau\,,$$

$$\int_{-\pi}^{\pi} \Phi(\tau)\log(Z - e^{i\tau})d\tau = \int_{-\pi}^{\pi} f(\tau)\log(z - e^{i\tau})d\tau + i\alpha\int_{-\pi}^{\pi} f(\tau)d\tau.$$

Thus, neglecting the constant term we have

$$W_\alpha(z) = -U\left(ze^{i\alpha} + \frac{e^{-i\alpha}}{z}\right) + iK\log z - \frac{1}{2\pi}\log z\int_{-\pi}^{\pi} f(\tau)d\tau$$

$$+\frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)\log(z - e^{i\tau})d\tau\,, \quad |z| \geqslant 1\,, \qquad \ldots\ldots (6)$$

and it follows that

$$\frac{dW_\alpha}{dz} = -U\left(e^{i\alpha} - \frac{e^{-i\alpha}}{z^2}\right) + \frac{iK}{z} - \frac{1}{2\pi z}\int_{-\pi}^{\pi} f(\tau)d\tau + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(\tau)}{z-e^{i\tau}}d\tau\,, \quad |z| > 1\,,$$

$$\ldots\ldots (7)$$

$$g_\alpha(\theta) = 2U\sin(\theta + \alpha) - K + \frac{1}{2\pi}P\int_{-\pi}^{\pi} f(\tau)\cot\tfrac{1}{2}(\theta - \tau)d\tau\,, \; -\pi < \theta < \pi\,.$$

$$\ldots\ldots (8)$$

Since K is the coefficient of $i\log z$ in the expression for $W_\alpha(z)$, we see that $-2\pi K$ is the positive circulation in the flow.

The above theory holds provided $f(\theta)$ is absolutely integrable. Hence the suction distribution may be chosen non-zero over just a portion of the circle - the case of "localised" suction - and the results still apply. Thus suppose we define

$$\left(\frac{\partial\phi}{\partial r}\right)_{r=1} = \begin{cases} f(\theta) & a \leqslant \theta \leqslant b\,, \\ 0 & \text{elsewhere}\,. \end{cases}$$

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$$g_a(\theta) = 2U \sin(\theta + a) - K + \frac{1}{2\pi} P \int_a^b f(\tau) \cot \tfrac{1}{2}(\theta - \tau) d\tau \, , \; -\pi < \theta \leqslant \pi \, .$$

But

$$P \int_a^b f(\tau) \cot \tfrac{1}{2}(\theta - \tau) d\tau = -2 \{ f(b) \log \left| \sin \tfrac{1}{2}(\theta - b) \right|$$

$$- f(a) \log \left| \sin \tfrac{1}{2}(\theta - a) \right| \} + 2 \int_a^b \log \left| \sin \tfrac{1}{2}(\theta - \tau) \right| \frac{df}{d\tau} \cdot d\tau \, .$$

Hence unless $f(a) = 0$, $f(b) = 0$, the tangential velocity becomes logarithmically infinite at $\theta = a$, $\theta = b$, the end points of the suction region. So, when the suction distribution is non-zero only in the range $a \leqslant \theta \leqslant b$, it must satisfy the conditions

$$f(a) = 0 \, , \quad f(b) = 0 \, ,$$

which make $f(\theta)$ continuous, in order that the tangential velocity remains finite everywhere.

(ii) The extent of the suction region is quite arbitrary. Suppose then, that we define

$$\left(\frac{\partial \phi}{\partial r}\right)_{r=1} = \begin{cases} f(\theta) & \beta - \varepsilon \leqslant \theta \leqslant \beta + \varepsilon \, , \\ 0 & \text{elsewhere} \, , \end{cases}$$

and that as $\varepsilon \longrightarrow 0$, $\int_{\beta-\varepsilon}^{\beta+\varepsilon} [-f(\tau)] d\tau$ tends to a finite limit m, say. Consider

$$I = \int_{\beta-\varepsilon}^{\beta+\varepsilon} f(\tau) \left[\log (z - e^{i\tau}) - \log (z - e^{i\beta}) \right] d\tau \, , \quad |z| > 1 \, .$$

Since $\log (z - e^{i\tau})$ is continuous at $\tau = \beta$, there exists a positive $\varepsilon' = \varepsilon'(\varepsilon)$, such that

$$|I| = \varepsilon' \left| \int_{\beta-\varepsilon}^{\beta+\varepsilon} f(\tau) \, d\tau \right| \, .$$

Hence/

Hence $I \to 0$ as $\varepsilon \to 0$, since $\left| \int_{\beta-\varepsilon}^{\beta+\varepsilon} f(\tau) d\tau \right|$ is bounded. Thus

$$\lim_{\varepsilon \to 0} \int_{\beta-\varepsilon}^{\beta+\varepsilon} f(\tau) \log (z - e^{i\tau}) d\tau = - m \log (z - e^{i\beta}) ,$$

and therefore from (6), as $\varepsilon \to 0$

$$W_\alpha(z) \to - U \left(z e^{i\alpha} + \frac{e^{-i\alpha}}{z} \right) + iK \log z + \frac{m}{2\pi} \log z - \frac{m}{\pi} \log (z - e^{i\beta}) , \quad |z| > 1 . \qquad \text{.....(9)}$$

As $z \to e^{i\theta}$, this equation still holds except at $\theta = \beta$, and it follows that, as $\varepsilon \to 0$,

$$\frac{dW_\alpha}{dz} \to - U \left(e^{i\alpha} - \frac{e^{-i\alpha}}{z^2} \right) + \frac{iK}{z} + \frac{m}{2\pi} \frac{1}{z} - \frac{m}{\pi} \frac{1}{(z - e^{i\beta})} , \quad |z| > 1 , \qquad \text{.....(10)}$$

$$g_\alpha(\theta) \to 2U \sin (\theta + \alpha) - K - \frac{m}{2\pi} \cot \tfrac{1}{2} (\theta - \beta) , \quad - \pi < \theta < \pi . \qquad \text{.....(11)}$$

Equations (9), (10), (11) are the expressions for the complex potential, complex velocity and tangential velocity on the circle respectively for the flow of a uniform stream U at incidence α past the unit circle with a sink of strength m at the point $z = e^{i\beta}$, the positive circulation in the flow being $- 2\pi K$. We may therefore conclude that as the size of the porous suction region about a given point tends to zero, the conditions approach those due to a single slot of appropriate sink strength at that point. However, the conditions represented by these equations may be a good approximation to the existing state of affairs when porous suction is applied over a sufficiently small region.

If there are several separate sinks, of strengths m_n, at points on the circle where $z = e^{i\beta_n}$ - the idealisation of "multi-slot" suction - then it may easily be shown that:-

$$W_\alpha(z) = - U \left(z e^{i\alpha} + \frac{e^{-i\alpha}}{z} \right) + iK \log z + \frac{1}{\pi} \sum_n m_n \left[\tfrac{1}{2} \log z - \log (z - e^{i\beta_n}) \right] , \quad |z| > 1$$

$$\frac{dW_\alpha}{dz} = - U \left(e^{i\alpha} - \frac{e^{-i\alpha}}{z^2} \right) + \frac{iK}{z} + \frac{1}{\pi} \sum_n m_n \left[\frac{1}{2z} - \frac{1}{z - e^{i\beta_n}} \right] , \quad |z| > 1 ,$$

$$g_\alpha(\theta) = 2U \sin (\theta + \alpha) - K - \frac{1}{2\pi} \sum_n m_n \cot \tfrac{1}{2} (\theta - \beta_n) , \quad - \pi < \theta < \pi .$$

2.3 Solution for Prescribed Tangential Velocity Distribution

2.3.1 Zero incidence

Consider the function

$$P_o(z) \;=\; iz\,\frac{dW_1}{dz} \;=\; \frac{\partial\phi_1}{\partial\theta} + ir\,\frac{\partial\phi_1}{\partial r} \;.$$

Then $P_o(z)$ is analytic in $|z| > 1$, is finite at infinity and has its real part known on the unit circle. In fact

$$\mathrm{Re}[P_o(z)]_{r=1} \;=\; \left(\frac{\partial\phi_1}{\partial\theta}\right)_{r=1} \;=\; g(\theta) - U\sin\theta \;=\; g_1(\theta)\;,\ \text{say,}$$

where $g(\theta)$ is assumed to be absolutely integrable. Then by a precisely similar argument to that in 2.2.1, and using the relations

$$P_o(\infty) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} g_1(\tau)d\tau + iK_o\;, \qquad \ldots\ldots\ (12)$$

$$\int_{-\pi}^{\pi} \sin\tau\,\log\,(z - e^{i\tau})d\tau \;=\; -\,\frac{\pi i}{z}\;,$$

$$P\int_{-\pi}^{\pi} \sin\tau\,\cot\tfrac{1}{2}(\theta - \tau)d\tau \;=\; -\,2\pi\cos\theta\;,$$

where K_o is an arbitrary constant, we find that for the main stream U in the direction of the negative real axis.

$$W(z) \;=\; -U\left(z - \frac{1}{z}\right) + K_o\log z + \frac{i}{2\pi}\log z\int_{-\pi}^{\pi} g(\tau)d\tau$$

$$-\,\frac{i}{\pi}\int_{-\pi}^{\pi} g(\tau)\,\log\,(z - e^{i\tau})d\tau\;, \qquad |z| \geqslant 1\;, \qquad \ldots\ldots(13)$$

$$\frac{dW}{\ \ } \;=\; -U\left(1 + \frac{1}{\ \ }\right) + \frac{K_o}{\ \ } + \frac{i}{\ \ }\int^{\pi} g(\tau)d\tau - \frac{i}{\ \ }\int^{\pi} \frac{g(\tau)}{\ \ }\,d\tau\;, \qquad |z| > 1\;.$$

If $f(\theta)$ and $g(\theta)$ are the normal and tangential velocity distributions on the circle in a particular case, the complex potentials of the corresponding flows past the circle when either is prescribed should, of course, be the same. Thus, equation (13) may be derived from (4) as follows. Consider

$$|z| > 1 \text{ , from (4)}$$

$$W(z) = -U\left(z + \frac{1}{z}\right) + iK \log z - \frac{1}{2\pi} \log z \int_{-\pi}^{\pi} f(\tau)d\tau + \frac{1}{\pi} \int_{-\pi}^{\pi} f(\tau) \log (z - e^{i\tau})d\tau.$$

Since $P_o(\infty) = iP(\infty)$, we have from (3) and (12)

$$-K = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_1(\tau)d\tau = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\tau)d\tau = \frac{\kappa}{2\pi}\text{ , say,}$$

$$-K_c = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(\tau)d\tau = -\frac{1}{2\pi} \int_{-\pi}^{\tau} f(\tau)d\tau = \frac{Q}{2\pi}\text{ , say,}$$

where $\kappa = \int_{-\pi}^{\pi} g(\tau)d\tau = -2\pi K$ is the positive circulation in the flow, as we previously noted, and $Q = -\int_{-\pi}^{\pi} f(\tau)d\tau = -2\pi K_o$ is the total flux of liquid into the circle per unit time. Thus, using (14)

$$W(z) = \begin{cases} -U\left(z + \frac{1}{z}\right) - K_o \log z - \frac{1}{2\pi} \log z \int_{-\pi}^{\pi} g(\tau)d\tau \\ + \frac{1}{\pi} \int_{-\tau}^{\pi} \left[-2U \cos \phi + K_o - \frac{1}{2\pi} P \int_{-\pi}^{\pi} g(\tau) \cot \tfrac{1}{2} (\phi - \tau)d\tau \right] \\ \qquad \times \log (z - e^{i\phi})d\phi \text{ .} \end{cases}$$

Now

$$\int_{-\pi}^{\pi} \cos \phi \log (z - e^{i\phi})d\phi = -\frac{\pi}{z} \text{ .}$$

$$\int_{-\pi}^{\pi} \log (z - e^{i\phi})d\phi = 2\pi \log z \text{ ,}$$

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CONTENTS

1. Introduction

Suction as a boundary-layer control device may be used in two ways: by sucking through one or more slots or by sucking continuously through a porous portion of the surface. These methods are based on rather different principles. Suction at a slot either removes part of the boundary layer at that point and forms a new, thinner, and if the original layer is laminar, more stable boundary layer downstream of the slot, or else removes much more fluid than that contained in the boundary layer, thereby altering the pressure field along the surface for some distance from the slot. Separation of flow is thus delayed, and several suitably spaced slots may be used to suck away parts of the upstream boundary layer as it tends to thicken and separate from the surface. On the other hand, distributed suction, instead of allowing the boundary layer to approach a condition near to separation before removing all or part of it, aims to maintain a favourable velocity profile through the boundary layer, which is kept thin all the time. Distributed, or porous, suction also has a stabilising effect on the whole boundary layer. A considerable amount of theoretical work has been done on the solution of the boundary-layer equations for flow past a porous surface through which there is a continuous normal velocity, and it is assumed throughout this work that the velocity distribution outside the boundary layer is that calculated on potential theory for an impermeable surface of similar shape. That is, it is assumed that surface suction, of the scale envisaged in boundary-layer distribution suction work, does not affect the potential flow past that surface. The question of the validity of this assumption led to the investigation below.

We are concerned with investigating the potential flow past an aerofoil of conventional shape which has an arbitrary distribution of suction on its surface. By "conventional" we mean not specially designed for use with suction. A very thorough discussion of aerofoils designed to incorporate surface suction has been given by Professor S. Goldstein[1]. The words "arbitrary distribution" cover the cases of "overall" distributed suction, where the entire aerofoil is constructed of porous material, distributed suction localised over a given porous region of the aerofoil surface, single and multi-slot suction. In particular we want to know the tangential velocity distribution on the aerofoil when the suction, or inward normal, velocity distribution is known, and we also enquire what suction distribution is necessary to produce a prescribed tangential velocity distribution on the aerofoil at a given incidence. We are interested, too, in the effect of surface suction on the theoretical lift and drag on the aerofoil. The problem is tackled by considering the corresponding problem for the unit circle, the solution for the flow past the aerofoil being obtained from the conformal transformation of the circle into the aerofoil.

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In section 3, the transformation to the aerofoil is discussed and the flow round the aerofoil deduced. The Kutta-Joukowski condition of smooth flow off the trailing edge is satisfied, with a discussion of its effects on the surface velocities and the lift and drag on the aerofoil.

Several/

Several simple examples of distributed suction and one of slot suction applied to a symmetrical Joukowski aerofoil are given in section 4. The results are illustrated in the diagrams.

In the Appendix, the Blasius theorem for the force on a closed curve is extended to the case when the curve is not necessarily a streamline.

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2.1 Statement of the problem

Consider the unit circle in a uniform stream of "perfect" fluid of velocity U in a direction parallel to the negative real axis of $z (= r e^{i\theta})$, with either the normal or tangential velocity distribution on the circle prescribed. It is required to find the complex potential $W(z) = \phi + i\psi$ of the flow past the circle, such that

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or (2b) $\left(\dfrac{\partial\phi}{\partial\theta}\right)_{r=1} = g(\theta)$, is given,

(3) $\dfrac{dW}{dz} \longrightarrow -U + O\left(\dfrac{1}{z}\right)$ as $z \longrightarrow \infty$.

In particular we want the unknown velocity distribution on the circle in terms of the given one. The problem is stated above for the main stream at zero incidence, but the solution to the case of non-zero incidence can easily be deduced from the solution to this problem.

Put

$$W(z) = -Uz + W_1(z), \quad W_1 = \phi_1 + i\psi_1 . \tag{1}$$

Then $W_1(z)$ must satisfy

(1) $W_1(z)$ is analytic for $|z| \geqslant 1$,

either (2a) $\left(\dfrac{\partial\phi_1}{\partial r}\right)_{r=1} = f(\theta) + U\cos\theta$,

or (2b) $\left(\dfrac{\partial\phi_1}{\partial\theta}\right)_{r=1} = g(\theta) - U\sin\theta$,

(3) $\frac{dW_1}{dz} \to 0 \left(\frac{1}{z}\right)$ as $z \to \infty$.

Having enunciated the problem for the flow past the circle, we shall consider separately the cases when the suction i.e., inward normal, and tangential velocity distributions are specified, since although the methods of solution at zero incidence are precisely similar, the extensions to general incidence are different.

2.2 Solution for prescribed suction velocity distribution

2.2.1 Zero incidence

Consider the function

$$P(z) = z\frac{dW_1}{dz} = r\frac{\partial\phi_1}{\partial r} - i\frac{\partial\phi_1}{\partial\theta} .$$

Then $P(z)$ is analytic in $|z| > 1$, is finite at infinity and has its real part known on the unit circle. In fact

$$[\mathrm{Re}(P(z))]_{r=1} = \left(\frac{\partial\phi_1}{\partial r}\right)_{r=1} = f(\theta) + U\cos\theta = f_1(\theta) \text{ , say.} \qquad ..(2)$$

Hence, provided $f(\theta)$ is absolutely integrable, we have, by Poisson's integral[2]

$$P(z) - P(\infty) = \frac{1}{\pi i}\oint_C \frac{f_1(t)}{z - t}\,dt \ , \quad |z| > 1 \ ,$$

where $t = e^{i\tau}$, C denotes the unit circle and the integral is taken round C in the positive sense. $f_1(t)$ or $f_1(\tau)$ will be used to denote the value of f_1 on C as the context requires. Moreover,

$$P(\infty) = \frac{1}{2\pi i}\oint_C \frac{P(t)}{t}\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(\tau)\,d\tau + iK \ , \qquad (3)$$

where K is an arbitrary constant, since $[\mathrm{Im}(P(z))]_{r=1}$ is unknown. Thus

. . 1 $\int^{\pi}$ . . 1 $\int$ $f_1(t)$ . . , . .

Hence

$$\frac{dW_1}{dz} = \frac{iK}{z} + \frac{1}{2\pi z}\int_{-\pi}^{\pi} f_1(\tau)\,d\tau + \frac{1}{\pi i}\oint_C \frac{f_1(t)}{z(z-t)}\,dt\ , \quad |z|>1\ ,$$

and, neglecting an arbitrary constant,

$$W_1(z) = iK\log z + \frac{1}{2\pi}\log z\int_{-\pi}^{\pi} f_1(\tau)\,d\tau$$

$$+ \frac{1}{\pi i}\oint_C f_1(t)\log\left(\frac{z-t}{z}\right)\frac{dt}{t}\ , \quad |z|>1\ ,$$

or

$$W_1(z) = iK\log z - \frac{1}{2\pi}\log z\int_{-\pi}^{\pi} f_1(\tau)\,dt$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} f_1(\tau)\log(z - e^{i\tau})d\tau\ , \quad |z|>1\ .$$

Using (1) and (3) and the relation

$$\int_{-\pi}^{\pi} \cos\tau\log(z - e^{i\tau})d\tau = -\frac{\pi}{z}\ ,$$

we have

$$W(z) = -U\left(z + \frac{1}{z}\right) + iK\log z - \frac{1}{2\pi}\log z\int_{-\pi}^{\pi} f(\tau)\,d\tau$$

$$+ \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)\log(z - e^{i\tau})d\tau, \qquad |z|>1\ , \qquad \ldots\ldots(4)$$

$$\frac{dW}{dz} = -U\left(1 - \frac{1}{z^2}\right) + \frac{iK}{z} - \frac{1}{2\pi z}\int_{-\pi}^{\pi} f(\tau)d\tau + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(\tau)}{z-e^{i\tau}}\,d\tau\ , \qquad |z|>1\ .$$

$$\ldots\ldots(5)$$

The/

The equation for $W(z)$ holds also for $|z| = 1$, but in equation (5) $\int_{-\pi}^{\pi} \frac{f(\tau)}{z-e^{i\tau}} d\tau$ becomes divergent when $z = e^{i\theta}$. To find the tangential velocity distribution on the circle, consider

$$\text{Im}\,(P(z)) = -\frac{\partial\phi_1}{\partial\theta} = K - \text{Re}\left\{\frac{1}{\pi}\oint_C \frac{f_1(t)}{z-t}\,dt\right\}.$$

As $z \rightarrow e^{i\theta}$,

$$\text{Re}\left\{\frac{1}{\pi}\int_C \frac{f_1(t)}{z-t}\,dt\right\} \rightarrow \frac{1}{2\pi}\, P \int_{-\pi}^{\pi} f_1(\tau)\cot\tfrac{1}{2}(\theta-\tau)\,d\tau\,,$$

where P denotes that the Cauchy principal value of the integral is to be taken. Hence

$$\left(\frac{\partial\phi_1}{\partial\theta}\right)_{r=1} = -K + \frac{1}{2\pi}\, P\int_{-\pi}^{\pi} f_1(\tau)\cot\tfrac{1}{2}(\theta-\tau)\,d\tau\,,$$

and since

$$P\int_{-\pi}^{\pi} \cos\tau\cot\tfrac{1}{2}(\theta-\tau)d\tau = 2\pi\sin\theta$$

we have by (1) and (3)

$$g(\theta) = 2U\sin\theta - K + \frac{1}{2\pi}\, P\int_{-\pi}^{\pi} f(\tau)\cot\tfrac{1}{2}(\theta-\tau)d\tau\,, \quad -\pi < \theta \leqslant \pi\,,$$

where $g(\theta)$ denotes the tangential velocity distribution on the circle induced at zero incidence by the outward normal velocity distribution $f(\theta)$.

2.2.2 General incidence

(i) The suction distribution, when once chosen, must remain the same with respect to the circle whatever the direction of the incident stream. We may now generalise the above results to the case when the direction of the uniform stream is at an angle α to the negative real axis.

Taking the positive real axis of Z opposite to the direction of the incident stream, we have from (4)

$$W(Z) = -U\left(Z + \frac{1}{Z}\right) + iK\log Z - \frac{1}{2\pi}\log Z\int_{-\pi}^{\pi}\Phi(\tau)d\tau$$
$$+ \frac{1}{\pi}\int_{-\pi}^{\pi}\Phi(\tau)\log(Z - e^{i\tau})d\tau\,, \quad |Z| \geqslant 1\,,$$

where/

where

$$Z = r\,e^{i\Theta} = ze^{i\alpha}, \quad \Phi(\Theta) = f(\Theta - \alpha) = f(\theta) .$$

Since $f(\theta)$ has period 2π, then

$$\int_{-\pi}^{\pi} \Phi(\tau)d\tau = \int_{-\pi}^{\pi} f(\tau)d\tau ,$$

$$\int_{-\pi}^{\pi} \Phi(\tau) \log (Z - e^{i\tau})d\tau = \int_{-\pi}^{\pi} f(\tau) \log (z - e^{i\tau})d\tau + i\alpha \int_{-\pi}^{\pi} f(\tau)d\tau.$$

Thus, neglecting the constant term we have

$$W_\alpha(z) = -U\left(ze^{i\alpha} + \frac{e^{-i\alpha}}{z}\right) + iK \log z - \frac{1}{2\pi} \log z \int_{-\pi}^{\pi} f(\tau)d\tau$$

$$+ \frac{1}{\pi} \int_{-\pi}^{\pi} f(\tau) \log (z - e^{i\tau})d\tau , \quad |z| \geqslant 1 , \qquad \ldots\ldots (6)$$

and it follows that

$$\frac{dW_\alpha}{dz} = -U\left(e^{i\alpha} - \frac{e^{-i\alpha}}{z^2}\right) + \frac{iK}{z} - \frac{1}{2\pi z} \int_{-\pi}^{\pi} f(\tau)d\tau + \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{f(\tau)}{z - e^{i\tau}} d\tau , \quad |z| > 1 ,$$

$$\ldots\ldots (7)$$

$$g_\alpha(\theta) = 2U \sin (\theta + \alpha) - K + \frac{1}{2\pi} P \int_{-\pi}^{\pi} f(\tau) \cot \tfrac{1}{2} (\theta - \tau)d\tau , \quad -\pi < \theta \leqslant \pi .$$

$$\ldots\ldots (8)$$

Since K is the coefficient of $i \log z$ in the expression for $W_\alpha(z)$, we see that $-2\pi K$ is the positive circulation in the flow.

The above theory holds provided $f(\theta)$ is absolutely integrable. Hence the suction distribution may be chosen non-zero over just a portion of the circle - the case of "localised" suction - and the results still apply. Thus suppose we define

$$\left(\frac{\partial \phi}{\partial r}\right)_{r=1} = \begin{cases} f(\theta) & a \leqslant \theta \leqslant b , \\ 0 & \text{elsewhere} . \end{cases}$$

Then/

Then

$$g_\alpha(\theta) = 2U \sin(\theta + \alpha) - K + \frac{1}{2\pi} P \int_a^b f(\tau) \cot \tfrac{1}{2}(\theta - \tau) d\tau \ , \ -\pi < \theta \leqslant \pi \ .$$

But

$$P \int_a^b f(\tau) \cot \tfrac{1}{2}(\theta - \tau) d\tau = -2 \{ f(b) \log \left| \sin \tfrac{1}{2}(\theta - b) \right|$$

$$- f(a) \log \left| \sin \tfrac{1}{2}(\theta - a) \right| \} + 2 \int_a^b \log \left| \sin \tfrac{1}{2}(\theta - \tau) \right| \frac{df}{d\tau} \cdot d\tau \ .$$

Hence unless $f(a) = 0$, $f(b) = 0$, the tangential velocity becomes logarithmically infinite at $\theta = a$, $\theta = b$, the end points of the suction region. So, when the suction distribution is non-zero only in the range $a \leqslant \theta \leqslant b$, it must satisfy the conditions

$$f(a) = 0 \ , \ f(b) = 0 \ ,$$

which make $f(\theta)$ continuous, in order that the tangential velocity remains finite everywhere.

(ii) The extent of the suction region is quite arbitrary. Suppose then, that we define

$$\left(\frac{\partial \phi}{\partial r}\right)_{r=1} = \begin{cases} f(\theta) & \beta - \varepsilon \leqslant \theta \leqslant \beta + \varepsilon \ , \\ 0 & \text{elsewhere} \ , \end{cases}$$

and that as $\varepsilon \longrightarrow 0$, $\int_{\beta-\varepsilon}^{\beta+\varepsilon} [- f(\tau)] d\tau$ tends to a finite limit m , say. Consider

$$I = \int_{\beta-\varepsilon}^{\beta+\varepsilon} f(\tau) \left[\log (z - e^{i\tau}) - \log (z - e^{i\beta}) \right] d\tau \ , \ \ |z| > 1 \ .$$

Since $\log (z - e^{i\tau})$ is continuous at $\tau = \beta$, there exists a positive $\varepsilon' = \varepsilon'(\varepsilon)$, such that

$$|I| = \varepsilon' \left| \int_{\beta-\varepsilon}^{\beta+\varepsilon} f(\tau)\, d\tau \right| \ .$$

Hence/

Hence $I \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, since $\left| \int_{\beta-\varepsilon}^{\beta+\varepsilon} f(\tau)d\tau \right|$ is bounded. Thus

$$\lim_{\varepsilon \longrightarrow 0} \int_{\beta-\varepsilon}^{\beta+\varepsilon} f(\tau) \log (z - e^{i\tau})d\tau = - m \log (z - e^{i\beta}) ,$$

and therefore from (6), as $\varepsilon \longrightarrow 0$

$$W_\alpha(z) \longrightarrow - U \left(ze^{i\alpha} + \frac{e^{-i\alpha}}{z}\right) + iK \log z + \frac{m}{2\pi} \log z - \frac{m}{\pi} \log (z - e^{i\beta}) , \quad |z| > 1 . \qquad .....(9)$$

As $z \longrightarrow e^{i\theta}$, this equation still holds except at $\theta = \beta$, and it follows that, as $\varepsilon \longrightarrow 0$,

$$\frac{dW_\alpha}{dz} \longrightarrow - U \left(e^{i\alpha} - \frac{e^{-i\alpha}}{z^2}\right) + \frac{iK}{z} + \frac{m}{2\pi}\frac{1}{z} - \frac{m}{\pi}\frac{1}{(z-e^{i\beta})} , \quad |z| \geqslant 1 , \qquad .....(10)$$

$$g_\alpha(\theta) \longrightarrow 2U \sin (\theta + \alpha) - K - \frac{m}{2\pi} \cot \tfrac{1}{2} (\theta - \beta) , \quad - \pi < \theta \leqslant \pi . \qquad .....(11)$$

Equations (9), (10), (11) are the expressions for the complex potential, complex velocity and tangential velocity on the circle respectively for the flow of a uniform stream U at incidence α past the unit circle with a sink of strength m at the point $z = e^{i\beta}$, the positive circulation in the flow being $- 2\pi K$. We may therefore conclude that as the size of the porous suction region about a given point tends to zero, the conditions approach those due to a single slot of appropriate sink strength at that point. However, the conditions represented by these equations may be a good approximation to the existing state of affairs when porous suction is applied over a sufficiently small region.

If there are several separate sinks, of strengths m_n, at points on the circle where $z = e^{i\beta n}$ - the idealisation of "multi-slot" suction - then it may easily be shown that:-

$$W_\alpha(z) = - U \left(ze^{i\alpha} + \frac{e^{-i\alpha}}{z}\right) + iK \log z + \frac{1}{\pi} \sum_n m_n\left[\tfrac{1}{2} \log z - \log (z - e^{i\beta_n})\right], |z| \geqslant 1$$

$$\frac{dW_\alpha}{dz} = - U \left(e^{i\alpha} - \frac{e^{-i\alpha}}{z^2}\right) + \frac{iK}{z} + \frac{1}{\pi} \sum_n m_n \left[\frac{1}{2z} - \frac{1}{z-e^{i\beta n}}\right], \quad |z| \geqslant 1 ,$$

$$g_\alpha(\theta) = 2U \sin (\theta + \alpha) - K - \frac{1}{2\pi} \sum_n m_n \cot \tfrac{1}{2} (\theta - \beta_n) , \quad - \pi < \theta \leqslant \pi .$$

2.3/

2.3 Solution for Prescribed Tangential Velocity Distribution

2.3.1 Zero incidence

Consider the function

$$P_o(z) = iz \frac{dW_1}{dz} = \frac{\partial \phi_1}{\partial \theta} + ir \frac{\partial \phi_1}{\partial r} .$$

Then $P_o(z)$ is analytic in $|z| > 1$, is finite at infinity and has its real part known on the unit circle. In fact

$$\mathrm{Re}[P_o(z)]_{r=1} = \left(\frac{\partial \phi_1}{\partial \theta}\right)_{r=1} = g(\theta) - U \sin\theta = g_1(\theta) \text{ , say,}$$

where $g(\theta)$ is assumed to be absolutely integrable. Then by a precisely similar argument to that in 2.2.1, and using the relations

$$P_o(\infty) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_1(\tau)d\tau + iK_o , \qquad \dots\dots (12)$$

$$\int_{-\pi}^{\pi} \sin\tau \log(z - e^{i\tau})d\tau = -\frac{\pi i}{z} ,$$

$$P\int_{-\pi}^{\pi} \sin\tau \cot\tfrac{1}{2}(\theta - \tau)d\tau = -2\pi\cos\theta ,$$

where K_o is an arbitrary constant, we find that for the main stream U in the direction of the negative real axis

$$W(z) = -U\left(z - \frac{1}{z}\right) + K_o \log z + \frac{i}{2\pi} \log z \int_{-\pi}^{\pi} g(\tau)d\tau$$

$$- \frac{i}{\pi} \int_{-\pi}^{\pi} g(\tau) \log(z - e^{i\tau})d\tau , \qquad |z| \geqslant 1 , \qquad \dots\dots(13)$$

$$\frac{dW}{dz} = -U\left(1 + \frac{1}{z^2}\right) + \frac{K_o}{z} + \frac{i}{2\pi z} \int_{-\pi}^{\pi} g(\tau)d\tau - \frac{i}{\pi} \int_{-\pi}^{\pi} \frac{g(\tau)}{z - e^{i\tau}} d\tau , \quad |z| > 1 ,$$

$$f(\theta) = -2U\cos\theta + K_o - \frac{1}{2\pi} P\int_{-\pi}^{\pi} g(\tau) \cot\tfrac{1}{2}(\theta - \tau)d\tau , \qquad -\pi < \theta \leqslant \pi \qquad \dots\dots(14)$$

where $f(\theta)$ denotes the normal velocity distribution required to produce the given tangential velocity distribution $g(\theta)$.

If/

If $f(\theta)$ and $g(\theta)$ are the normal and tangential velocity distributions on the circle in a particular case, the complex potentials of the corresponding flows past the circle when either is prescribed should, of course, be the same. Thus, equation (13) may be derived from (4) as follows. Consider

$$|z| > 1 \text{ , from (4)}$$

$$W(z) = -U\left(z + \frac{1}{z}\right) + iK \log z - \frac{1}{2\pi} \log z \int_{-\pi}^{\pi} f(\tau)d\tau + \frac{1}{\pi} \int_{-\pi}^{\pi} f(\tau) \log (z - e^{i\tau})d\tau.$$

Since $P_o(\infty) = iP(\infty)$, we have from (3) and (12)

$$-K = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_1(\tau)d\tau = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\tau)d\tau = \frac{\kappa}{2\pi}, \text{ say,}$$

$$-K_o = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f_1(\tau)d\tau = -\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\tau)d\tau = \frac{Q}{2\pi}, \text{ say,}$$

where $\kappa = \int_{-\pi}^{\pi} g(\tau)d\tau = -2\pi K$ is the positive circulation in the flow, as we previously noted, and $Q = -\int_{-\pi}^{\pi} f(\tau)d\tau = -2\pi K_o$ is the total flux of liquid into the circle per unit time. Thus, using (14)

$$W(z) = \begin{cases} -U\left(z + \frac{1}{z}\right) - K_o \log z - \frac{i}{2\pi} \log z \int_{-\pi}^{\pi} g(\tau)d\tau \\ + \frac{1}{\pi} \int_{-\pi}^{\pi} \left[-2U \cos \phi + K_o - \frac{1}{2\pi} P \int_{-\pi}^{\pi} g(\tau) \cot \tfrac{1}{2} (\phi - \tau)d\tau \right] \\ \qquad \times \log (z - e^{i\phi})d\phi . \end{cases}$$

Now

$$\int_{-\pi}^{\pi} \cos \phi \log (z - e^{i\phi})d\phi = -\frac{\pi}{z} .$$

$$\int_{-\pi}^{\pi} \log (z - e^{i\phi})d\phi = 2\pi \log z ,$$

whence/

whence

$$W(z) = -U\left(z - \frac{1}{z}\right) + K_o \log z - \frac{i}{2\pi} \log z \int_{-\pi}^{\pi} g(\tau)d\tau - \frac{1}{\pi} J(z) , \quad ..(15)$$

where

$$J(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[P \int_{-\pi}^{\pi} g(\tau) \cot \tfrac{1}{2} (\phi - \tau) d\tau \right] \log (z - e^{i\phi}) d\phi ,$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\tau) \left[P \int_{-\pi}^{\pi} \log (z - e^{i\phi}) - i \left(\frac{e^{i\phi} + e^{i\tau}}{e^{i\phi} - e^{i\tau}} \right) d\phi \right] d\tau,$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\tau) \left[P \oint_C \log (z - s) \cdot \left(\frac{s + t}{s - t} \right) \frac{ds}{s} \right] d\tau , \quad s = e^{i\phi},$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\tau) \left[2\pi i \left(- \log z + \log (z - e^{i\tau}) \right) \right] d\tau ,$$

$$= - i \log z \int_{-\pi}^{\pi} g(\tau) d\tau + i \int_{-\pi}^{\pi} g(\tau) \log (z - e^{i\tau}) d\tau .$$

Substituting in (15) we find

$$W(z) = - U\left(z - \frac{1}{z}\right) + K_o \log z + \frac{i}{2\pi} \log z \int_{-\pi}^{\pi} g(\tau)d\tau$$

$$- \frac{i}{\pi} \int_{-\pi}^{\pi} g(\tau) \log (z - e^{i\tau}) d\tau , \quad |z| > 1 .$$

This also holds for $|z| = 1$, and so we have equation (13).

2.3.2 General incidence

Unlike the suction distribution, which must be the same for all directions of the main stream, the tangential velocity distribution must be prescribed for one given incidence. This then defines the suction distribution, obviously of the "overall" type, for all incidence. Thus let the required tangential velocity distribution be $g_{\alpha_o}(\theta)$ when the angle of incidence is α_o . Substituting in (13)

$$Z_o = re^{i\Theta_o} = ze^{i\alpha_o} , \quad \Gamma_{\alpha_o} (\Theta_o) = g_{\alpha_o} (\Theta_o - \alpha_o) = g_{\alpha_o} (\theta) ,$$

we/

we have

$$W(Z_o) = -U\left(Z_o - \frac{1}{Z_o}\right) + K_o \log Z_o + \frac{i}{2\pi} \log Z_o \int_{-\pi}^{\pi} \Gamma_{\alpha_o}(\tau)d\tau$$

$$- \frac{i}{\pi} \int_{-\pi}^{\pi} \Gamma_{\alpha_o}(\tau) \log (Z_o - e^{i\tau})d\tau ,$$

so that, neglecting an additional constant term,

$$W_{\alpha_o}(z) = -U\left(ze^{i\alpha_o} - \frac{e^{i\alpha_o}}{z}\right) + K_o \log z + \frac{i}{2\pi} \log z \int_{-\pi}^{\pi} g_{\alpha_o}(\tau)\, d\tau$$

$$- \frac{i}{\pi} \int_{-\pi}^{\pi} g_{\alpha_o}(\tau) \log (z - e^{i\tau})d\tau , \quad |z| \geqslant 1 .$$

It follows that

$$f(\theta) = -2U \cos (\theta + \alpha_o)) + K_o - \frac{1}{2\pi} P \int_{-\pi}^{\pi} g_{\alpha_o}(\tau) \cot \tfrac{1}{2} (\theta - \tau)d\tau,$$

$$-\pi < \theta \leqslant \pi . \qquad .....(16)$$

This therefore gives the suction distribution, all round the circle, which must be applied if the tangential velocity distribution on the circle is to be $g_{\alpha_o}(\theta)$ at incidence α_o. This suction distribution must now remain the same for all angles of incidence, and hence the flow for general incidence, in this case, is obtained by substituting (16) into equations (6), (7), (8). It can be checked from (8) that $[g_\alpha(\theta)]_{\alpha = \alpha_o}$ does in fact equal $g_{\alpha_o}(\theta)$ as prescribed.

We may note here that although the above method is necessary to obtain the complete solution to the flow, the relations between the normal and tangential velocities on the circle could have been deduced at the outset, since f_1 and $-g_1$ are the real and imaginary parts, on the circle, of the function $z \frac{dW_1}{dz}$, which is analytic outside the circle, and hence they may be expanded in conjugate Fourier series.

2.4 Summary of the solution

Since the arbitrary constants in the solution have now been identified, and the equivalence of the solution for prescribed suction and tangential velocity distributions demonstrated, we may for convenience summarise the solution in its simplest form. The following formulae apply when the uniform stream U is in a direction at an angle α to the negative real axis, and there is a positive circulation K in the flow.

2.4.1 Distributed suction

(i) Overall suction

If the outward normal velocity distribution on the circle $f(\theta)$ is defined in the range $-\pi < \theta \leqslant \pi$, then

$$W_\alpha(z) = -U\left(ze^{i\alpha} + \frac{e^{-i\alpha}}{z}\right) - \frac{(i\kappa - Q)}{2\pi}\log z + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)\log(z - e^{i\tau})d\tau,$$

$$|z| \geqslant 1, \qquad \ldots\ldots(17)$$

$$\frac{dW_\alpha}{dz} = -U\left(e^{i\alpha} - \frac{e^{-i\alpha}}{z^2}\right) - \frac{(i\kappa - Q)}{2\pi}\cdot\frac{1}{z} + \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{f(\tau)}{z - e^{i\tau}}d\tau, \qquad |z| > 1,$$

$$\ldots\ldots(18)$$

$$g_\alpha(\theta) = 2U\sin(\theta + \alpha) + \frac{\kappa}{2\pi} + \frac{1}{2\pi}F(\theta), \qquad -\pi < \theta \leqslant \pi, \qquad \ldots\ldots(19)$$

where

$$Q = -\int_{-\pi}^{\pi} f(\tau)d\tau, \qquad \ldots\ldots(20)$$

$$F(\theta) = P\int_{-\pi}^{\pi} f(\tau)\cot\tfrac{1}{2}(\theta - \tau)d\tau. \qquad \ldots\ldots(21)$$

Moreover, if $f(\theta)$ is such that

$$f(\theta) = -2U\cos(\theta + \alpha_o) - \frac{Q}{2\pi} - \frac{1}{2\pi}G_{\alpha_o}(\theta), \qquad -\pi < \theta \leqslant \pi,$$

$$\ldots\ldots(22)$$

where

$$G_\alpha(\theta) = P\int_{-\pi}^{\pi} g_\alpha(\tau)\cot\tfrac{1}{2}(\theta - \tau)d\tau, \qquad \ldots\ldots(23)$$

then $g_{\alpha_o}(\theta)$ is the tangential velocity distribution on the circle at incidence α_o.

(ii)/

(ii) Localised suction

If $f(\theta)$ is non-zero only in the range $a \leq \theta \leq b$, then equations (17) to (21) still hold, but in order that $g_\alpha(\theta)$ is not infinite at $\theta = a$ or $\theta = b$ it is necessary that

$$f(a) = 0, \quad f(b) = 0. \qquad \text{.....} \quad (24)$$

2.4.2 Slot suction

If there are N sinks of strengths m_n placed at points $\theta = \beta_n$, $n = 1 \ldots N$, on the circle, then

$$W_\alpha(z) = -U\left(ze^{i\alpha} + \frac{e^{-i\alpha}}{z}\right) - \frac{i\kappa}{2\pi}\log z + \frac{1}{\pi}\sum_{n=1}^{N} m_n[\tfrac{1}{2}\log z - \log(z - e^{i\beta_n})],$$

$$|z| > 1, \qquad \text{.....}(25)$$

$$\frac{dW_\alpha}{dz} = -U\left(e^{i\alpha} - \frac{e^{-i\alpha}}{z^2}\right) - \frac{i\kappa}{2\pi}\frac{1}{z} + \frac{1}{\pi}\sum_{n=1}^{N} m_n\left[\frac{1}{2z} - \frac{1}{z - e^{i\beta_n}}\right], \quad |z| > 1,$$

$$\text{.....}(26)$$

$$g_\alpha(\theta) = 2U\sin(\theta + \alpha) + \frac{\kappa}{2\pi} - \frac{1}{2\pi}\sum_{n=1}^{N} m_n \cot\tfrac{1}{2}(\theta - \beta_n), \qquad -\pi < \theta < \pi.$$

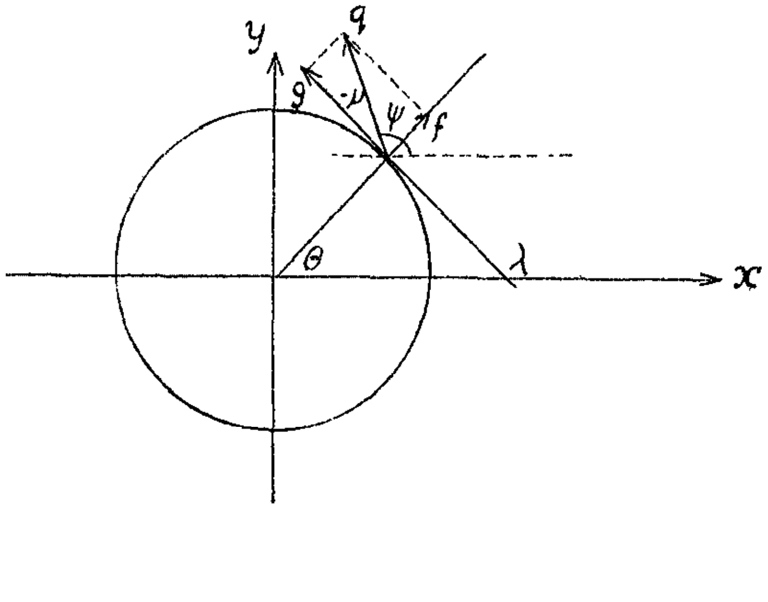
$$\text{.....}(27)$$

In this case

$$Q = \sum_{n=1}^{N} m_n. \qquad \text{.....}(28)$$

2.5 Lift and drag on the circle

In the Appendix, formulae are derived for finding the force on a closed curve C which is not a streamline. These may be applied to the case when C is the unit circle under the above conditions of surface suction.



If/

If q, χ are the magnitude and direction respectively of the velocity vector, then

$$q\,e^{-i\chi} = \frac{dW_\alpha}{dz}$$

whence

$$q^2 = \left|\frac{dW_\alpha}{dz}\right|^2 = \frac{dW_\alpha}{dz}\,\frac{d\bar{W}_\alpha}{d\bar{z}} \quad . \qquad \ldots\ldots \quad (29)$$

Since in this case $\lambda = \frac{\pi}{2} + \theta$ then when C is the unit circle equation (i) of the Appendix becomes

$$X - i\,Y = -\tfrac{1}{2}\,i\,\rho \oint_C \frac{dW_\alpha}{dz} \cdot \frac{d\bar{W}_\alpha}{d\bar{z}} \cdot e^{-2i\theta}\,dz \ ,$$

Now, on the unit circle

$$z = e^{i\theta} \ ,$$

$$\bar{z} = \frac{1}{z} \quad . \qquad \ldots\ldots \quad (30)$$

Thus we may write

$$X - i\,Y = -\tfrac{1}{2}\,i\,\rho \oint_C T(z)\,dz \ ,$$

where

$$T(z) = \frac{1}{z^2} \cdot \frac{dW_\alpha}{dz} \cdot \left(\frac{d\bar{W}_\alpha}{d\bar{z}}\right)_{\bar{z}=\frac{1}{z}} \quad . \qquad \ldots\ldots \quad (31)$$

Now, for $|z| > 1$, from (18) for distributed suction,

$$\frac{dW_\alpha}{dz} = b_o + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(\tau)}{z - e^{i\tau}}\,d\tau \ ,$$

where

$$b_o = -\,Ue^{i\alpha} \ ,$$

$$b_1 = -\frac{i}{2\pi}\,(i\kappa - Q) \ ,$$

$$b_2 = Ue^{-i\alpha} \quad .$$

Hence/

Hence

$$\left(\frac{d\bar{W}_\alpha}{d\bar{z}}\right)_{\bar{z}=\frac{1}{z}} = \bar{b}_2 z^2 + \bar{b}_1 z + \bar{b}_o - \frac{z}{\pi} \int_{-\pi}^{\pi} \frac{f(\tau)e^{i\tau}}{z - e^{i\tau}} d\tau ,$$

where

$$\bar{b}_2 = Ue^{i\alpha} = - b_o$$

$$\bar{b}_1 = - \frac{1}{2\pi}(- i\kappa - Q) = - \left(b_1 - \frac{Q}{\pi}\right),$$

$$\bar{b}_o = - Ue^{i\alpha} = - b_2 .$$

Hence, for $|z| > 1$.

$$T(z) = \left[b_o + \frac{b_1}{z} + \frac{b_2}{z^2} + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{f(\tau)}{z - e^{i\tau}} d\tau\right]\left[\bar{b}_2 + \frac{\bar{b}_1}{z} + \frac{\bar{b}_o}{z^2} - \frac{1}{\pi z}\int_{-\pi}^{\pi} \frac{f(\tau)}{z - e^{i\tau}} d\tau\right],$$

from which we see that $T(z)$ is analytic for $|z| > 1$ and is finite at infinity. Thus, since $T(z)$ is continuous on C, by Cauchy's theorem the unit circle may be deformed into a circle C_R of large radius R, so that

$$X - i Y = - \tfrac{1}{2} i \rho \oint_{C_R} T(z)\, dz .$$

For large z,

$$T(z) = B_o + \frac{B_1}{z} + ... , \qquad ..... \qquad (32)$$

since for large z

$$\frac{1}{z - e^{i\tau}} = \frac{1}{z} + \frac{e^{iz}}{z^2} + ... ,$$

where

$$B_o = b_o\bar{b}_2 = - b_o^2$$

$$B_1 = b_o\bar{b}_1 + \bar{b}_2\left(b_1 + \frac{1}{\pi}\int_{-\pi}^{\pi} f(\tau)d\tau\right) = b_o\left[\bar{b}_1 - \left(b_1 - \frac{Q}{\pi}\right)\right] = 2\, b_o\bar{b}_1 .$$

Hence/

Hence

$$X - iY = -\tfrac{1}{2} i\rho \oint_{C_R} \left[B_0 + \frac{B_1}{z} + O\left(\frac{1}{z^2}\right) \right] dz$$

$$= -\tfrac{1}{2} i\rho \int_{-\pi}^{\pi} \left[B_0 + \frac{B_1}{Re^{i\theta}} + O\left(\frac{1}{R^2}\right) \right] i\,Re^{i\theta}\, d\theta = \pi \rho B_1 + O\left(\frac{1}{R}\right).$$

Letting $R \to \infty$, we have

$$X - iY = \pi \rho B_1 = 2\pi \rho b_0 \bar{b}_1 = -\rho U e^{i\alpha} (i\kappa + Q), \qquad \ldots(33)$$

whence

$$X = \rho U\kappa \sin \alpha - \rho U Q \cos \alpha ,$$

$$Y = \rho U\kappa \cos \alpha + \rho U Q \sin \alpha .$$

Thus the lift L and the drag D on the circle, which are normal and parallel to the incident stream respectively, are given by

$$L = \rho U\kappa ,$$

$$D = \rho UQ .$$

The results for L and D are exactly the same when equation (26) for multi-slot suction is taken for $\frac{dW_\alpha}{dz}$, with Q given by (28).

We therefore deduce that for any incompressible potential flow past a circle, the lift is proportional to the circulation and the drag is proportional to the total flux per unit time into the circle.

3. The Potential Flow Past an Aerofoil with Arbitrary Surface Suction

3.1 Transformation from the unit circle

By Riemann's theorem, the space outside an aerofoil can be conformally represented on the outside of the unit circle by a unique analytic function $\zeta(z)$, where z and ζ are complex variables in the planes of the circle and aerofoil respectively, so that $\frac{d\zeta}{dz} \to 1$ as $z \to \infty$. The transformation is of the form

$$\zeta = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \ldots ,$$

$$\frac{d\zeta}{dz} = 1 - \frac{a_1}{z^2} - \frac{2a_2}{z^3} - \ldots , \qquad \ldots\ldots (34)$$

where,/

where, in general, a_1, a_2, ... are complex numbers. For an aerofoil with a sharp trailing edge, the point in the Z-plane corresponding to the trailing edge lies on the circle. We shall take this point to be $z = e^{i(-\pi + \varepsilon)}$.

If we denote by $W'(\zeta)$ the complex potential of the flow past the aerofoil corresponding to the flow $W_\alpha(z)$ past the circle, then

$$W'_\alpha(\zeta) = W_\alpha(z(\zeta)),$$

where $z(\zeta)$ is the inverse of $\zeta(z)$. The complex velocity in the aerofoil plane is

$$\frac{dW'_\alpha}{d\zeta} = \frac{dW_\alpha}{dz} \cdot \frac{dz}{d\zeta}. \qquad \ldots\ldots (35)$$

If we put

$$\frac{dz}{d\zeta} = Me^{i\nu}, \qquad \ldots\ldots (36)$$

then from (35) we have

$$q'e^{-i\psi'} = q\, e^{-i\psi}\, Me^{i\nu},$$

where q', ψ' represent the magnitude and direction respectively of the velocity vector in the flow past the aerofoil. Henceforth, primes will be used exclusively to denote quantities in the flow in the aerofoil plane thus

$$q' = M q,$$

$$\psi' = \psi - \nu. \qquad \ldots\ldots (37)$$

These relations hold at corresponding points everywhere outside and on the circle and the aerofoil, though the transformation has a singularity at $z = e^{i(-\pi + \varepsilon)}$, which corresponds to the trailing edge, where M is infinite. Now, on the circle,

$$dz = d\theta\, e^{i\left(\frac{\pi}{2}+\theta\right)},$$

and on the aerofoil

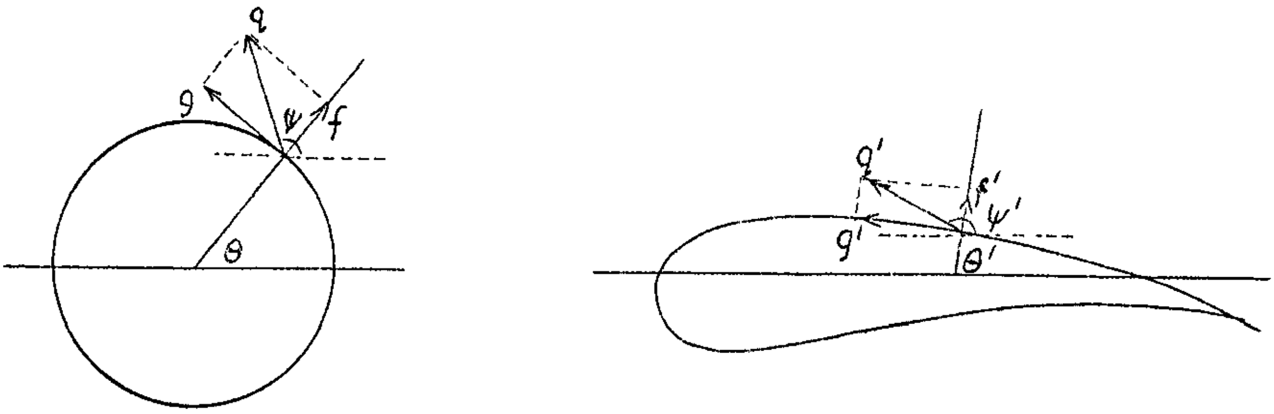
$$d\zeta = ds\, e^{i\left(\frac{\pi}{2}+\theta'\right)},$$

where ds is the element of length along the aerofoil, and θ' is the angle between the outward normal to the surface and the positive real axis. Hence, from (36), on the aerofoil we have

$$d\theta = M\, ds,$$

$$\theta = \theta' + \nu. \qquad \ldots\ldots (38)$$

Figure/



If we denote the outward normal velocities and the tangential velocities in the positive direction, at corresponding points on the circle and aerofoil by f , f' ; g , g' respectively, then on the circle

$$\arg (f - ig) = -\psi + \theta ,$$

and on the aerofoil

$$\arg (f' - ig') = -\psi' + \theta' .$$

Thus, using (37) and (38)

$$\arg (f' - ig') = \arg (f - ig) ,$$

whence

$$\frac{f}{f'} = \frac{g}{g'} . \qquad \text{..... (39)}$$

Also, on the circle

$$q = (f^2 + g^2)^{\frac{1}{2}} ,$$

and on the aerofoil

$$q' = (f'^2 + g'^2)^{\frac{1}{2}} ,$$

the positive square root being taken in each case. From (37) and (39) we then have

$$f' = Mf , \quad g' = Mg .$$

Hence, the solution for the flow past the aerofoil can be obtained from the solution in the circle plane from the relations

$$W'_\alpha(\zeta) = W_\alpha(z(\zeta)) ,$$

$$\frac{dW'_\alpha}{d\zeta} = \frac{dW_\alpha}{dz} . \frac{dz}{d\zeta} ,$$

which/

which hold outside and on the aerofoil, except at the trailing edge, and moreover the normal and tangential velocities, f' and g', on the aerofoil itself are given in terms of f and g at the corresponding points on the circle, by

$$f' = Mf \ , \quad g' = Mg \ ,$$

where

$$M = \left|\frac{dz}{d\zeta}\right|$$

on the circle.

3.2 Lift and drag on the aerofoil

We may apply the relations derived in the Appendix for the force on a closed curve C which is not a streamline, to the case when C is the aerofoil C'. If X', Y' represent the forces on the aerofoil parallel to the real and imaginary axes of ζ, respectively, then from equations (ii) and (iii) of the Appendix

$$X' - iY' = \tfrac{1}{2}\, i\, \rho \oint_{C'} \left(\frac{dW'_a}{d\zeta}\right)^2 e^{2i\mu'}\, d\zeta \ ,$$

where

$$\mu' = -\tan^{-1}\frac{f'}{g'} \ .$$

We may change the integral round the aerofoil C' into one round the unit circle C. From (39),

$$\mu' = \mu = -\tan^{-1}\frac{f}{g} \ ,$$

and also

$$\psi = \mu + \lambda = \mu + \theta - \frac{\pi}{2} \ ,$$

so that

$$\frac{dW_a}{dz} = q\, e^{-i\psi} = -iq\, e^{-i\theta}\, e^{-i\mu} \ .$$

Hence/

Hence, from (29), (30), (31)

$$X' - i\,Y' = \tfrac{1}{2}\, i\, \rho \oint_{C'} \left(\frac{dW'_a}{d\zeta}\right)^2 e^{2i\mu'}\, d\zeta \,,$$

$$= \tfrac{1}{2}\, i\, \rho \oint_{C} \left(\frac{dW_a}{dz}\right)^2 e^{2i\mu}\, \frac{dz}{d\zeta}\, dz \,,$$

$$= -\tfrac{1}{2}\, i\, \rho \oint_{C} q^2\, e^{-2i\theta}\, \frac{dz}{d\zeta}\, dz \,,$$

$$= -\tfrac{1}{2}\, i\, \rho \oint_{C} T(z)\, \frac{dz}{d\zeta}\, dz \,.$$

Since $T(z)\,\frac{dz}{d\zeta}$ is an analytic function of z in $|z| > 1$, and is continuous on C except at $z = e^{i(-\pi + \varepsilon)}$, by considering the Cauchy principal value of the integral we may deform the contour into the large circle C_R so that

$$X' - i\,Y' = -\tfrac{1}{2}\, i\rho \oint_{C_R} T(z)\,.\,\frac{dz}{d\zeta}\,.\,dz\,.$$

From (34) for large z

$$\frac{dz}{d\zeta} = 1 + \frac{a_1}{z^2} + \ldots \,,$$

whence, using (32),

$$X' - iY' = \tfrac{1}{2}\, i\, \rho \oint_{C_R} \left\{B_0 + \frac{B_1}{z} + O\left(\frac{1}{z^2}\right)\right\} \left\{1 + O\left(\frac{1}{z^2}\right)\right\} dz \,,$$

$$= \pi \rho B_1 + O\left(\frac{1}{R}\right)\,.$$

Letting $R \to \infty$, we see from (33) that,

$$X' - i\,Y' = \pi \rho B_1 = X - i\,Y\,.$$

Thus/

Thus the force on the aerofoil is exactly the same as the force on the unit circle, for corresponding flows, and hence the lift L' and the drag D' on the aerofoil are given by

$$L' = \rho U k = L , \qquad \ldots\ldots (40)$$

$$D' = \rho U Q = D . \qquad \ldots\ldots (41)$$

However, if Q' represents the total flux per unit time into the aerofoil, then

$$Q' = - \oint_{C'} f' d s = - \oint_{C} M f \frac{d\theta}{M} = - \oint_{C} f d\theta = Q .$$

We therefore conclude that for any incompressible potential flow round any aerofoil, the lift is proportional to the circulation and the drag is proportional to the total flux per unit time into the aerofoil.

3.3 Satisfaction of the Kutta-Joukowski trailing-edge condition

It is well-known that in order that the velocity at the trailing edge of an aerofoil is finite, the corresponding point on the circle must be a stagnation point of the flow round the circle. This condition imposes restrictions on the permissible suction and tangential velocity distributions which may be chosen on the circle, and enables the arbitrary constants in the solution to be determined. Further, it makes it possible to calculate the change in lift and drag due to suction from the corresponding flow round an impermeable aerofoil.

3.3.1 Conditions on the surface velocities

If the stagnation point on the circle is at $z = e^{i(-\pi + \varepsilon)}$, then we must always have

$$f(-\pi + \varepsilon) = 0 , \qquad \ldots\ldots (42)$$

$$g_\alpha(-\pi + \varepsilon) = 0 , \text{ for all } \alpha . \qquad \ldots\ldots (43)$$

(i) Suppose the suction distribution is prescribed. Then it must satisfy (42) which gives a restriction on the possible porous suction distributions about the point $e^{i(-\pi + \varepsilon)}$ on the circle which may give rise to a corresponding porous suction distribution on the aerofoil, and also indicates that in the case of slot suction there must not be a slot placed at this point. If $f(-\pi + \varepsilon) \neq 0$, the corresponding flow round the aerofoil has no physical significance. Moreover, the tangential velocity $g_\alpha(\theta)$ which is produced by the prescribed $f(\theta)$ must satisfy (43) for all α, and this condition determines uniquely the circulation necessary to give the required stagnation point. Thus for distributed suction we have from (19)

$$\kappa = 4\pi U \sin(\alpha + \varepsilon) - F(-\pi + \varepsilon) , \qquad \ldots\ldots (44)$$

$$g_\alpha(\theta) = 2U[\sin(\theta + \alpha) + \sin(\alpha + \varepsilon)] + \frac{1}{2\pi}[F(\theta) - F(-\pi + \varepsilon)], \quad -\pi < \theta \leqslant \pi ,$$

whilst/

whilst for slot suction, from (27)

$$\kappa = 4\pi U \sin(\alpha + \varepsilon) + \sum_{n=1}^{N} m_n \tan \tfrac{1}{2} (\beta_n - \varepsilon) ,$$

$$g_\alpha(\theta) = 2U[\sin(\theta + \alpha) + \sin(\alpha + \varepsilon)] - \frac{1}{2\pi} \left\{ \sum_{n=1}^{N} m_n [\cot \tfrac{1}{2} (\theta - \beta_n) - \tan \tfrac{1}{2} (\beta_n - \varepsilon)] \right\} , \qquad -\pi < \theta \leqslant \pi .$$

For any type of surface suction, we may therefore write

$$\kappa = 4\pi U \sin(\alpha + \varepsilon) + k , \qquad \ldots\ldots \quad (45)$$

where for distributed suction

$$k = -F(-\pi + \varepsilon) = -\left[P \int_{-\pi}^{\pi} f(\tau) \cot \tfrac{1}{2} (\theta - \tau) d\tau \right]_{\theta = -\pi+\varepsilon} ,$$

and for slot suction

$$k = \sum_{n=1}^{N} m_n \tan \tfrac{1}{2} (\beta_n - \varepsilon) .$$

Substitution of the above value of κ in the expressions for the complex potential and complex velocity gives the flow round the circle which, under the given conditions, has a stagnation point at $z = e^{i(-\pi + \varepsilon)}$. This is the only flow which when transformed gives a physically possible flow past the aerofoil. It is apparent that the incident main stream and the suction distribution provide independent contributions to the flow.

Thus for the case of prescribed suction distribution on the circle, whether porous or slot suction, the suction velocity must be zero at the point corresponding to the trailing edge of the aerofoil, whilst for suction localised between $a \leqslant \theta \leqslant b$, we must have, also, $f(a) = 0$, $f(b) = 0$. The circulation is determined uniquely, but the total suction quantity Q may be chosen arbitrarily.

(ii) A prescribed tangential velocity distribution $g_{\alpha_0}(\theta)$ at incidence α_0 fixes the suction distribution uniquely. We must have that

$$g_{\alpha_0}(-\pi + \varepsilon) = 0 ,$$

and since also $f(-\pi + \varepsilon) = 0$, then from (22)

$$Q = 4\pi U \cos(\alpha + \varepsilon) - G_{\alpha_0}(-\pi + \varepsilon) ,$$

$$f(\theta) = -2U[\cos(\theta + \alpha_0) + \cos(\alpha_0 + \varepsilon)] - \frac{1}{2\pi} [G_{\alpha_0}(\theta) - G_{\alpha_0}(-\pi + \varepsilon)] , \qquad -\pi < \theta \leqslant \pi .$$

The/

The suction distribution is obviously of the overall type. Satisfaction of condition (43) for all α then gives the circulation κ , as in (44) above.

Hence for the case of prescribed tangential velocity distribution on the circle at a given incidence, this velocity must be zero at the point corresponding to the trailing edge. The overall suction distribution is then fixed and the total suction quantity and circulation are uniquely determined.

3.3.2 Changes in lift and drag due to suction

Since the previously arbitrary constants in the flow have now been not only identified but determined uniquely, we may calculate the changes in lift and drag on a suction aerofoil from one without suction. From (40), (41), (45) we have for the theoretical lift and drag on a suction aerofoil:-

$$L = \rho U \kappa = 4\pi \rho U^2 \sin(\alpha + \varepsilon) + \rho U k ,$$

$$D = \rho U Q .$$

Now the lift L_i and the drag D_i for a similar-shaped impermeable aerofoil in the same uniform stream are given by

$$L_i = 4\pi \rho U^2 \sin(\alpha + \varepsilon) ,$$

$$D_i = 0$$

and hence the changes ΔL and ΔD in the lift and drag due to suction are

$$\Delta L = \rho U k ,$$

$$\Delta D = \rho U Q .$$

Thus, theoretically, the use of suction gives promise of a considerable lift increase over the impermeable case, though there is a corresponding increase in drag.

It is convenient here to introduce the non-dimensional suction coefficient C_q and express the changes ΔC_L and ΔC_D in the lift and drag coefficients in terms of it. Defining

$$C_q = \frac{Q'}{Uc} = \frac{Q}{Uc} ,$$

and since

$$C_L = \frac{L}{\frac{1}{2}\rho U^2 c} , \quad C_D = \frac{D}{\frac{1}{2}\rho U^2 c} ,$$

c/

c being the chord of the aerofoil, then

$$\Delta C_L = \frac{\Delta L}{\frac{1}{2}\rho U^2 c} = 2\frac{k}{Q} \cdot C_q \,.$$

$$\Delta C_D = \frac{\Delta D}{\frac{1}{2}\rho U^2 c} = 2 \cdot C_q \,.$$

$$\frac{\Delta C_L}{\Delta C_D} = \frac{k}{Q} \,.$$

These relations enable the lift and drag increments to be analysed in any given case of surface suction. This analysis is especially simple for the case of a single slot, which has been investigated, rather inconclusively, by Smith[3].

4. Examples

4.1 Choice of surface velocity distributions

The theory for finding the potential flow round a porous aerofoil when either the tangential or normal velocity distribution on its surface is prescribed holds for aerofoils obtained from a circle by a known conformal transformation. The problem in the aerofoil plane is then reducible to a corresponding problem for the unit circle, the complete solution to which is given in 2.4. Difficulties may be encountered in evaluating analytically some of the Poisson integrals which occur in this solution, but methods exist for calculating these integrals numerically[4,5,6]. Thus for any specified velocity distribution on the porous aerofoil, the complete solution of the flow may be deduced.

We may note here that the pressure field on the aerofoil is obtained, by Bernoulli's theorem, from the distribution of total velocity on the surface and not, as in the case of an impermeable aerofoil, from the tangential velocity distribution alone. If the total velocity distribution were prescribed, we should have an integral equation for finding either of the component velocities, since if $q_{\alpha_0}(\theta)$ is the given total velocity on the circle, then

$$q^2_{\alpha_0}(\theta) = f^2(\theta) + g^2_{\alpha_0}(\theta) \,,$$

where

$$f(\theta) = -2U[\cos(\theta + \alpha_0) + \cos(\alpha_0 + \varepsilon)] - \frac{1}{2\pi}[G_{\alpha_0}(\theta) - G_{\alpha_0}(-\pi + \varepsilon)] \,, \quad -\pi < \theta \leqslant \pi \,,$$

$$g_{\alpha_0}(\theta) = 2U[\sin(\theta + \alpha_0) + \sin(\alpha_0 + \varepsilon)] + \frac{1}{2\pi}[F(\theta) - F(-\pi + \varepsilon)] \,, \quad -\pi < \theta \leqslant \pi \,,$$

whence/

whence

$$f^2(\theta) + \frac{1}{4\pi^2}\left\{S_{\alpha_o}(\theta) + P \int_{-\pi}^{\pi} f(\tau) \cot \tfrac{1}{2}(\theta - \tau)d\tau\right\}^2 = q^2_{\alpha_o}(\theta) ,$$

$$- \pi < \theta \leqslant \pi ,$$

$$g^2_{\alpha_o}(\theta) + \frac{1}{4\pi^2}\left\{C_{\alpha_o}(\theta) + P \int_{-\pi}^{\pi} g_{\alpha_o}(\tau) \cot \tfrac{1}{2}(\theta - \tau)d\tau\right\}^2 = q^2_{\alpha_o}(\theta) ,$$

$$- \pi < \theta \leqslant \pi ,$$

where

$$S_{\alpha_o}(\theta) = 4\pi U[\sin(\theta = \alpha_o) + \sin(\alpha_o + \varepsilon)] - F(-\pi + \varepsilon) ,$$

$$C_{\alpha_o}(\theta) = 4\pi U[\cos(\theta = \alpha_o) + \cos(\alpha_o + \varepsilon)] - G_{\alpha_o}(-\pi + \varepsilon) .$$

Thus, if we are interested in obtaining exactly a certain pressure field on the surface, the determination of either of the component velocities requires the solution of an integral equation of the form

$$\left[P \int_{-\pi}^{\pi} y(z) \cot \tfrac{1}{2}(x - z)dz + A(x)\right]^2 + B\, y^2(x) + C(x) = 0 .$$

However, in the cases of slot suction, suction over a small localised region, and slight suction over the whole aerofoil, the tangential velocity distribution is very nearly equal to the total velocity. In any event, knowledge of the pressure distribution is required mainly so that an estimate of the position of boundary-layer separation may be made, and if the suction velocity is sufficiently large to affect appreciably the total velocity, then it seems feasible that it will also be large enough to eliminate the possibility of the boundary layer separating at all.

The most important case to work out would therefore seem to be that of a specified tangential velocity distribution at a given incidence. However, we see from the theory that the suction distribution thus defined must be of the overall type, and this is a great drawback because of the immense practical difficulties involved in making a complete wing of porous material and simulating a given, presumably very complicated, suction distribution over its surface. It would be better, therefore, to approximate to the derived overall suction distribution by some form of localised or slot suction. But we already possess a general idea of the effect of the position of suction on the tangential velocity distribution - for example, we know that to increase the C_L-range on a low-drag wing the suction should be applied on the upper surface near the leading edge, whilst to reduce the profile drag on a thick high maximum C_L aerofoil we require suction somewhere towards the rear of the upper surface - so that we can in any case make good guesses as to the position of the suction to produce a certain desired effect. Hence it appears even more convenient to work out the cases of suitably-chosen suction distributions in preference to the more complicated ones derived from given tangential velocity distributions.

The/

The required surface velocities on the aerofoil are usually given as functions of ξ_A, the fraction of the chord from the leading edge, and may of course be expressed as functions of θ so that the corresponding surface velocities on the circle may be found exactly in terms of θ. These expressions for $f(\theta)$ and $g(\theta)$ will undoubtedly be complicated, and here again it is convenient to approximate to the exact required velocities. One simplification is to use the fact that in general ξ_A varies like $\cos\theta$. For the simplest possible computation, $f(\theta)$ or $g(\theta)$ may be chosen so that the Poisson integrals may be evaluated analytically.

4.2 List of worked examples

We consider the effect of several prescribed suction distributions on a 13% thick symmetrical Joukowski aerofoil, derived from the unit circle by the transformation

$$\zeta(z) = (z + \delta) + \frac{(1 - \delta)^2}{(z + \delta)}, \quad \delta = 0.1 .$$

Computations are performed only for specified suction distributions for the above mentioned reasons, in any case, the examples worked out are sufficient to enable several general conclusions to be drawn, (see the postscript to the Conclusion).

The suction distributions considered are the simplest possible which satisfy the required conditions on the circle, being obtained by putting

$$f(\theta) = A_0 + A_1 \cos\theta + B_1 \sin\theta ,$$

and finding the appropriate values of the constants. Since for a symmetrical aerofoil $\varepsilon = 0$, the point on the circle corresponding to the trailing edge is $\theta = -\pi$, and the conditions to be satisfied by $f(\theta)$ are therefore $f(-\pi) = 0$ and $f(a) = 0$, $f(b) = 0$ for localised suction. We note that since $f(\theta)$ is the outward normal velocity and we are dealing with sucking, not blowing, then we must have $f(\theta) \leqslant 0$ everywhere. An arbitrary constant is left in the expression for $f(\theta)$ to enable the total suction quantity Q and the suction coefficient C_Q to be varied.

The cases computed are as follows:-

(1) Overall suction

$$f(\theta) = -Q_0 \left[1 + \cos\theta\right], \quad -\pi < \theta \leqslant \pi .$$

$$f(-\pi) = 0 \; ; \; f(\theta) < 0 \text{ for } Q_0 > 0 .$$

(2) Localised suction

$$f(\theta) = \begin{cases} -Q_0 \dfrac{\left[\sin(\theta - a) - \sin(\theta - b) - \sin(b - a)\right]}{\sin(b - a)}, & -\pi < a \leqslant \theta \leqslant b < \pi . \\ 0 , & \text{elsewhere} . \end{cases}$$

Three/

Three different cases are considered:

(a) $a = 0^\circ$, $b = 90^\circ$. Suction applied to that part of the upper surface of the aerofoil between the leading edge and a point 0.456 of the chord from it.

(b) $a = 30^\circ$, $b = 45^\circ$. Suction applied to that part of the upper surface of the aerofoil between points 0.058 and 0.127 of the chord from the leading edge. In this particular case the streamlines past the circle are also computed, since originally some doubt existed as to whether or not a stagnation point always appears downstream of the porous region in the case of localised suction. It would seem at first sight, by analogy with the case of slot suction, that this would be the case.

(c) $a = 12^\circ$, $b = 18^\circ$. Suction applied to that part of the upper surface of the aerofoil between points 0.009 and 0.021 of the chord from the leading edge.

(3) <u>Slot suction</u>

Single sink of strength m at $\theta = 15^\circ$, i.e., at a point on the upper surface of the aerofoil 0.015 of the chord from the leading edge. This single slot case is chosen for comparison with (2(c)).

The size of C_Q permissible in practice imposes on the suction distribution a limitation in size which is treated by putting

$$Q_o = C_o U ,$$

so that for suction (1)

$$C_Q = \frac{1}{Uc} \int_{-\pi}^{\pi} Q_o \left[1 + \cos \theta\right] d\theta = \frac{2\pi}{c} C_o ,$$

and for suction (2),

$$C_Q = \frac{1}{Uc} \int_{a}^{b} Q_o \frac{\left[\sin(\theta - a) - \sin(\theta - b) - \sin(b - a)\right]}{\sin(b - a)} \, d\theta$$

$$= \frac{\left[2(1 - \cos(b - a)) - (b - a)\sin(b - a)\right]}{c \sin(b - a)} C_o .$$

For suction (3),

$$C_Q = \frac{m}{Uc} ,$$

and is chosen to take the same values as in suction (2(c)). In the computation, C_o is adjusted to give suitable ranges for C_Q.

In each of the above cases the tangential velocity distribution round the aerofoil is calculated, the total velocity also being worked out where it differs substantially from the tangential one i.e., for suctions (1) and (2a). The calculations are performed for several values of C_Q and for $\alpha = 0°, 5°, 10°$, which, for flow without suction, correspond to theoretical lift coefficients of 0, 0.602, and 1.200 respectively. The results are illustrated in the figures. A table showing the values of ΔC_L, the change in lift coefficient due to the use of suction, is given. Also included are diagrams of the streamlines past the suction region on the circle for the case (2 for $\alpha = 0$ and four values of C_Q.

5. Conclusions

The incompressible potential flow past an aerofoil having a given distribution of surface suction may be deduced by conformal transformation from the corresponding flow round the unit circle. However, not all flows with specific surface velocities on the circle may form the basis for flows round the aerofoil. In order that the flow comes smoothly off the trailing edge of the aerofoil, it is necessary that the prescribed surface velocity on the circle must be zero at the point corresponding to the trailing edge, whilst in the case of a specified localised suction distribution, the suction velocity must be zero at the end points of the suction region to prevent the tangential velocity becoming infinite there.

The incident main stream and the suction distribution provide independent contributions to the flow. By applying the correct overall suction distribution, any given tangential velocity distribution at any incidence may be obtained exactly, though if the extent, form and size of a localised suction distribution are well-chosen, a good approximation to any required tangential velocity distribution may be obtained. Strictly, the tangential velocity distribution does not determine the pressure field on the surface, which controls the separation of the boundary layer, but for small amounts of suction and suction over a small localised region it is sufficiently close to the total velocity to give a good idea of the pressure field, whilst for larger suction quantities it seems feasible that there will be no tendency for the flow to separate over the suction region. If the extent of a localised suction distribution becomes infinitesimally small whilst the total influx of fluid remains constant, the solution for distributed suction reduces to the solution for suction through a single slot of appropriate sink strength at the centre of the localised region. The lift on the aerofoil is proportional to the circulation in the flow, and the drag is proportional to the total flux per unit time into the aerofoil. The satisfaction of the Kutta-Joukowski trailing-edge condition determines the increase in lift on a suction aerofoil over a similar impermeable one, indicating a promise of a considerable lift increment to be obtained by using surface suction, though there is a corresponding increase in drag.

It appears that for values of C_Q at which boundary-layer porous suction is designed to act i.e., $C_Q < 0.001$, the effect of such suction on the potential flow outside the boundary-layer is negligible, for overall or localised suction. For higher values of C_Q, overall suction still has little effect unless the form of the suction distribution is chosen especially well, which in any case is probably equivalent to some form of localised suction. Localised suction is very effective at higher values of C_Q say $C_Q > 0.05$. The change in lift coefficients brought about by the introduction of suction is minute for $C_Q < 0.001$, and is in any case quite small. When the extent of the suction region is small, say <5% chord, the state of affairs outside the suction region may be approximated by that due to a single slot of appropriate sink strength at the centre of the region.

The/

The computed streamlines for the circle show clearly that, for C_Q below a certain value, there is no stagnation point downstream of a localised suction region. This is a consequence of the condition of continuity imposed on the suction distribution in order that the tangential velocity may remain finite.

In conclusion then, we see that the assumption that the introduction of distributed suction for the purpose of boundary-layer control does not affect the potential flow outside the boundary layer, is quite justified. The results also indicate that the use of "sink effect" to alter adequately the pressure field round an aerofoil requires such large suction quantities that its practical use is at present very doubtful.

Note. It has been pointed out by Dr. R. C. Pankhurst that the chosen examples do not give a complete picture of the lift increments due to the use of suction, since to obtain an increase in lift in potential flow, with practical suction quantities, the suction should be located in the trailing edge region. In the examples chosen, it is because of the forward location of the suction that the quantities needed for appreciable lift increment are prohibitively high. An example with trailing-edge suction has not yet been computed.

References

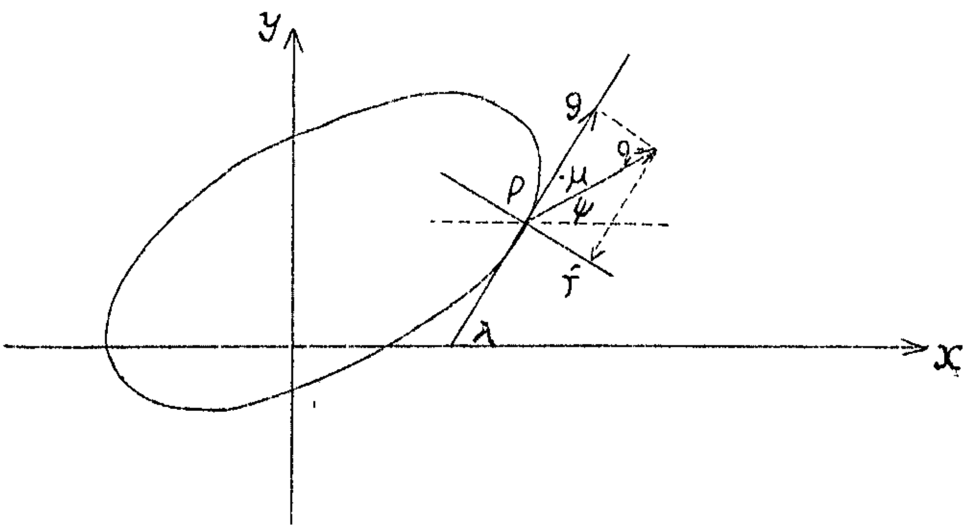
No.	Author(s)	Title, etc.
1	Goldstein, S.	Low-drag and Suction Airfoils. Journal of the Aero. Sciences. Vol.15, No.4, April, 1948.
2	Lighthill, M. J.	A Mathematical Method of Cascade Design. R. & M. No.2104. June, 1945. (Appendix).
3	Smith, C. B.	A Solution for the Lift and Drag of Airfoils with Air Inlets and Suction Slots. Journal of the Aero. Sciences, Vol.16, No.10, October, 1949.
4	Germain, P.	The Computation of Certain Functions Occurring in Profile Theory. A.R.C. Report No.8692, May, 1945.
5	Thwaites, B.	A Method by P. Germain for the Practical Evaluation of the Integral $\varepsilon(0) = -\frac{1}{2\pi}\int_0^{2\pi} \psi(\zeta) \cot \tfrac{1}{2}(\zeta - \theta) d\zeta$. A.R.C. Report No.8660. May, 1945.
6	Watson, E. J.	Formulae for the Computation of the Functions Employed for Calculating the Velocity Distribution about a Given Aerofoil. R. & M. No.2176. May, 1945.

APPENDIX

The Force on a Closed Curve which is not a Streamline

The following is an extension of the well-known Blasius theorem when the closed curve under consideration is not a streamline.

Consider the closed curve C , with radial and tangential velocity components f and g respectively at a point P , where the element of length is ds .



At P $\qquad dz = ds\,e^{i\lambda} \;,\quad d\bar{z} = ds\,e^{-i\lambda}$.

The force on the element ds is in the direction of the inward normal at P , so that

$$dF = dX + idY = ipdz \;,$$

where p is the hydrostatic pressure at P . Hence the total force on the closed curve C , having components X , Y parallel to the co-ordinate axes, is given by

$$X + i\,Y = i \oint_C p dz \;.$$

Taking conjugates

$$X - iY = -i \oint_C p\, d\bar{z} = -i \oint_C p\, e^{-2i\lambda}\, dz \;.$$

From/

From Bernoulli's equation for incompressible potential flow

$$p + \tfrac{1}{2}\rho q^2 \quad = \quad \text{constant everywhere in the fluid} ,$$

and since C is a closed curve,

$$\oint_C dz \;=\; 0 .$$

Thus

$$X - iY \;=\; \tfrac{1}{2} i \rho \oint_C q^2 \, e^{-2i\lambda} \, dz , \qquad \ldots\ldots \text{(i)}$$

and since

$$q \, e^{-i\psi} \;=\; \frac{dW}{dZ} ,$$

where q, ψ are the magnitude and direction of the velocity vector, respectively, in the flow where the complex potential is $W(z)$, then this may be written in the alternative form

$$X - iY \;=\; \tfrac{1}{2} i \rho \oint_C \left(\frac{dW}{dZ}\right)^2 e^{2i\mu} \, dz , \qquad \ldots\ldots \text{(ii)}$$

where

$$\mu \;=\; \psi - \lambda \;=\; -\tan^{-1} \frac{f}{g} . \qquad \ldots\ldots\text{(iii)}$$

We note here that since in general $e^{2i\mu}$ is not an analytic function of z, the integrand is not an analytic function and the contour of integration may not be suitably deformed without some modification of the integrand.

TABLE/

TABLE

Changes in Lift Coefficient for Various Suction Distributions

Type of Suction	C_o	C_Q	C_L
Suction (1)	0.10 0.01	0.1728 0.0173	0 0
Suction (2a)	1.0 0.1	0.1180 0.0118	0.1015 0.0101
Suction (2b)	300 200 100 20	0.1242 0.0828 0.0414 0.0083	0.0842 0.0561 0.0281 0.0056
Suction (2c)	3,000 300 30	0.0789 0.0079 0.0008	0.0237 0.0024 0.0002

JW

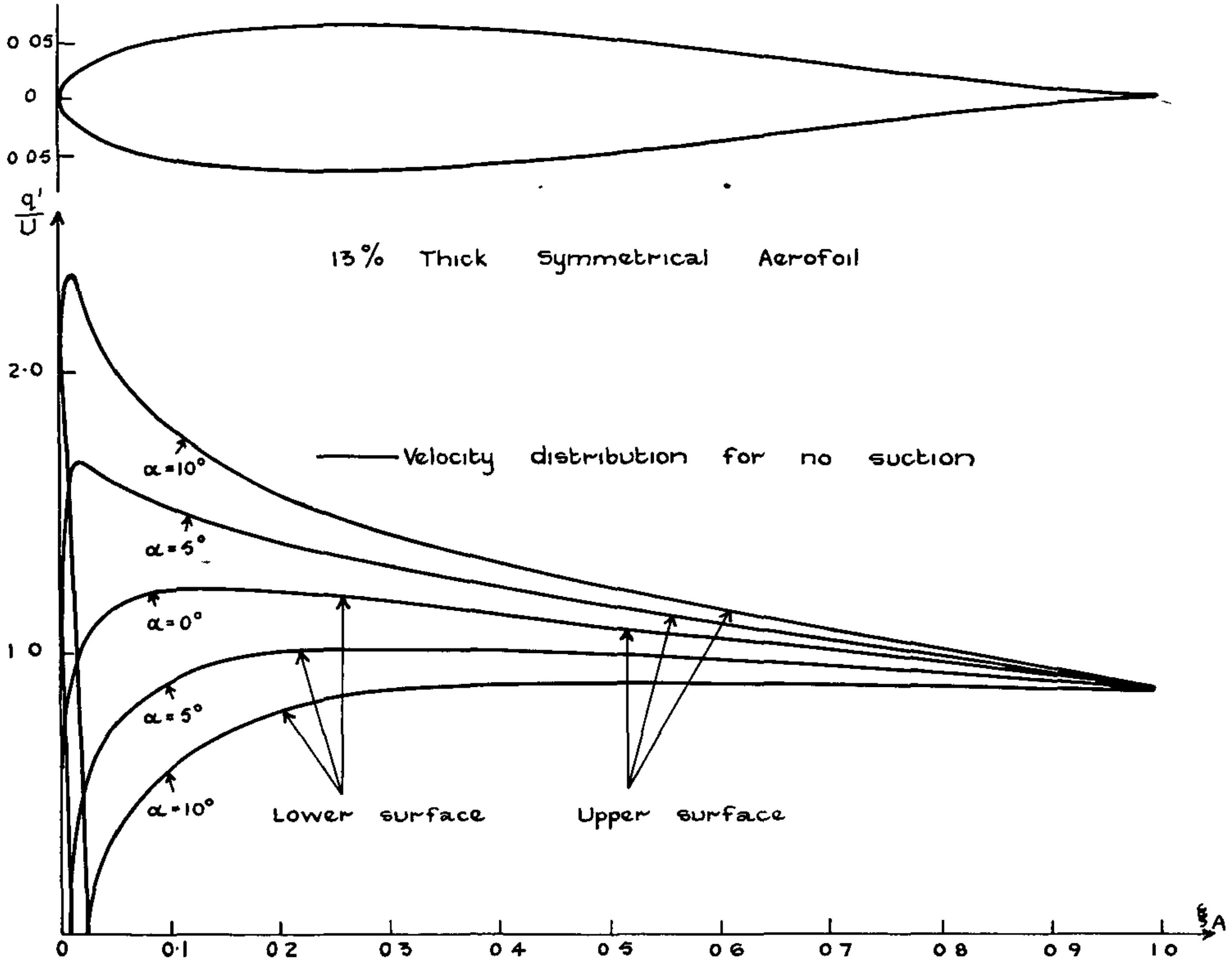


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FIG 1

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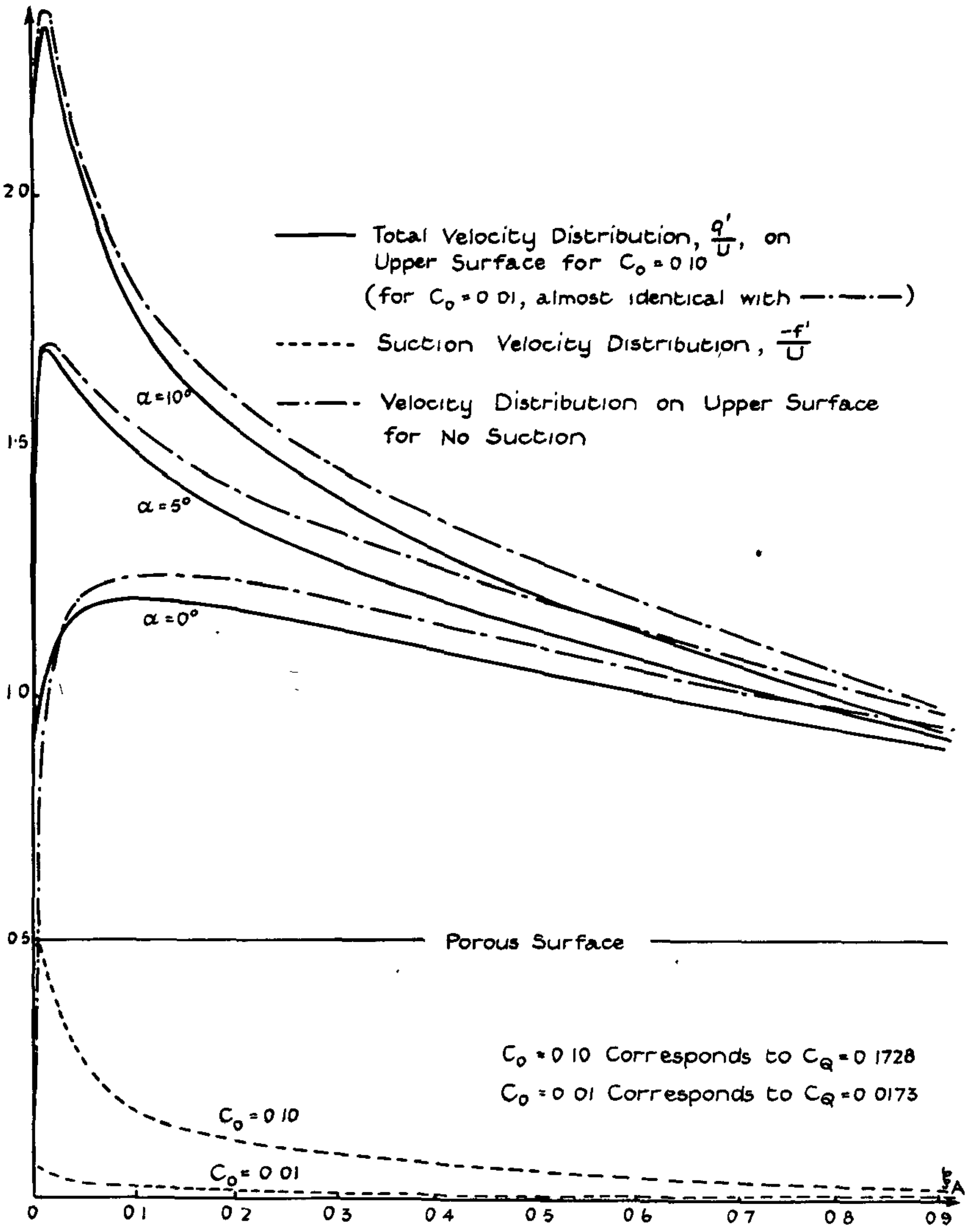
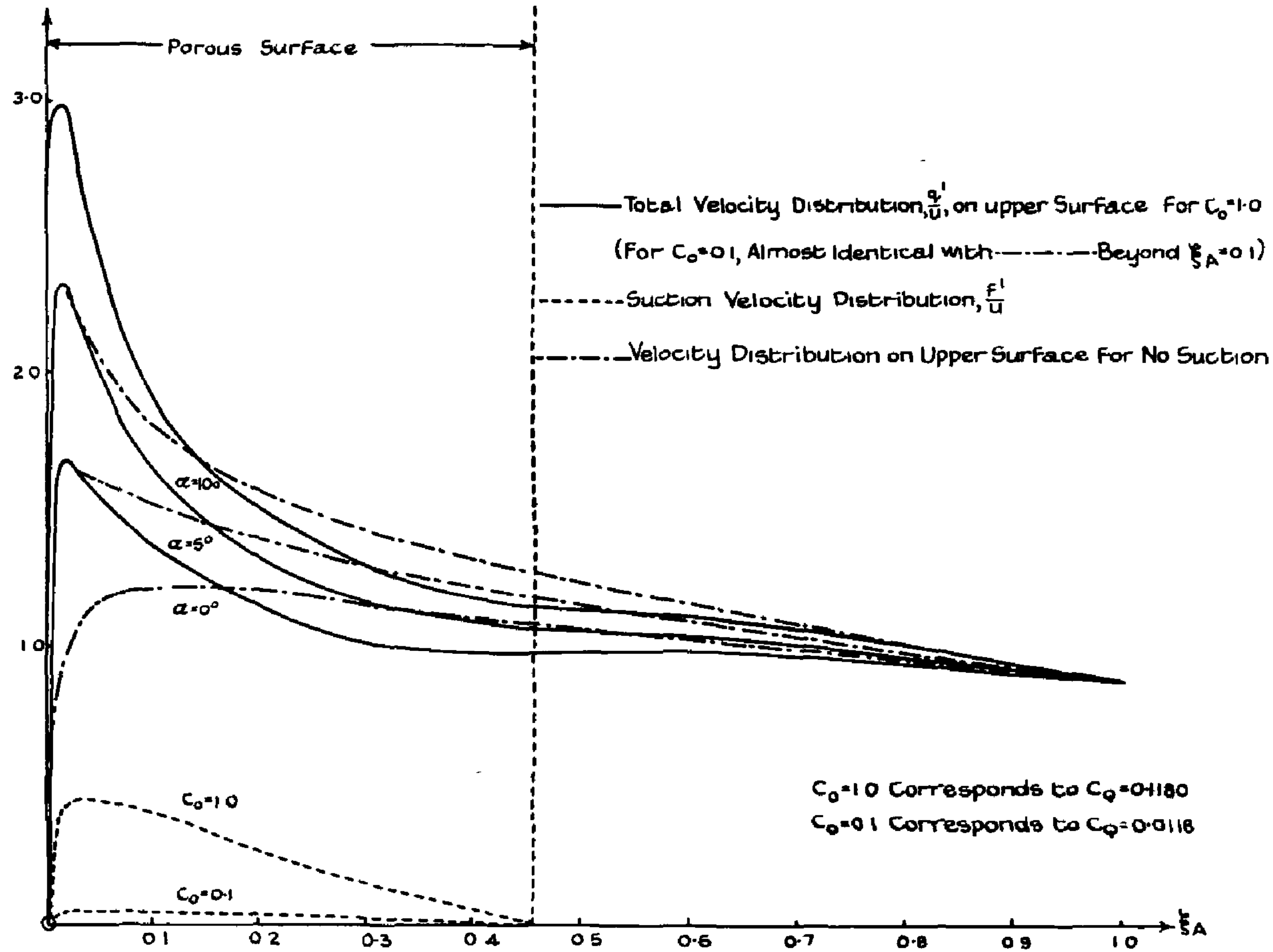
FIG 2



Suction (1)



Suction (2a)

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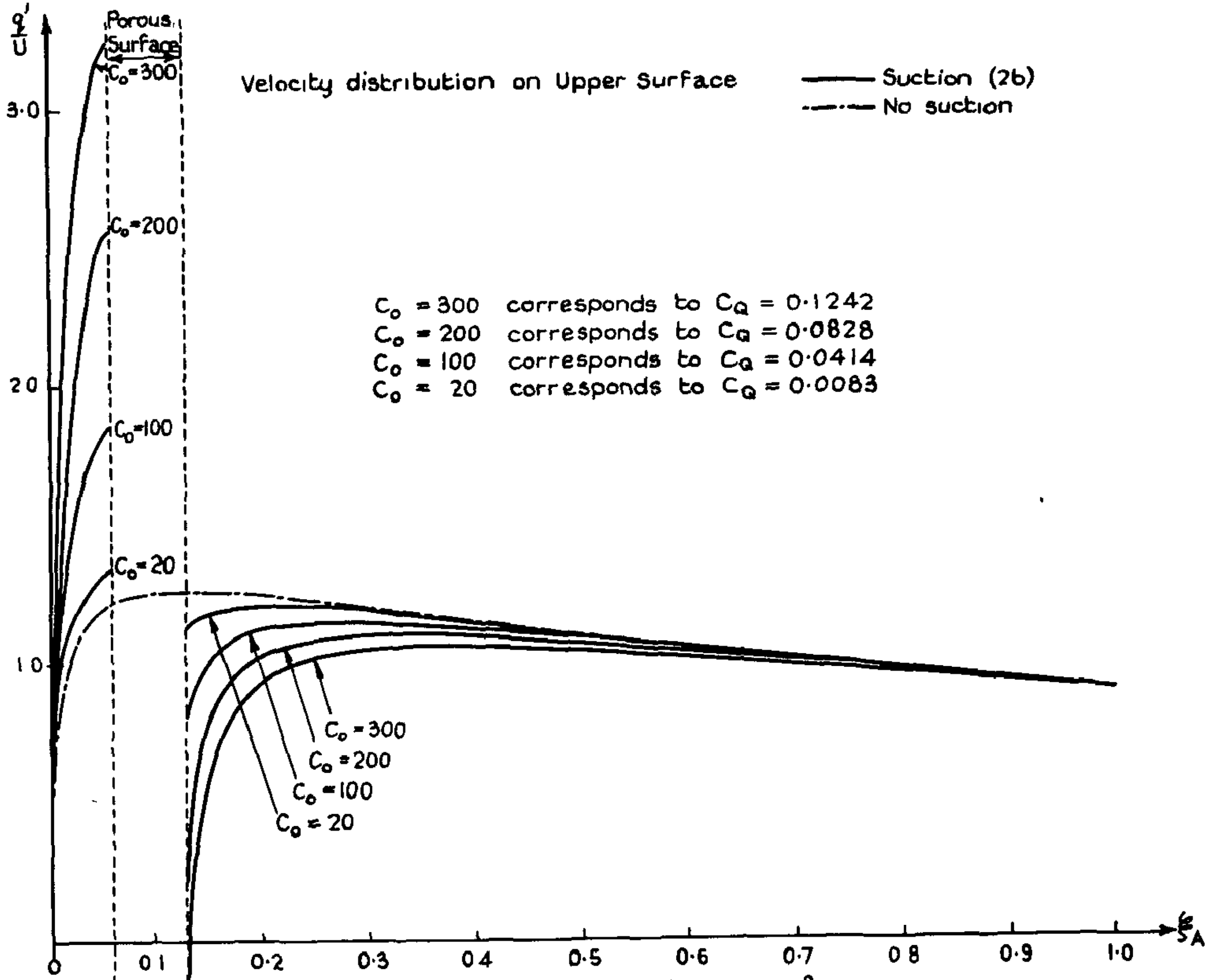
FIG 3.



Suction (2b) $\alpha = 0°$.

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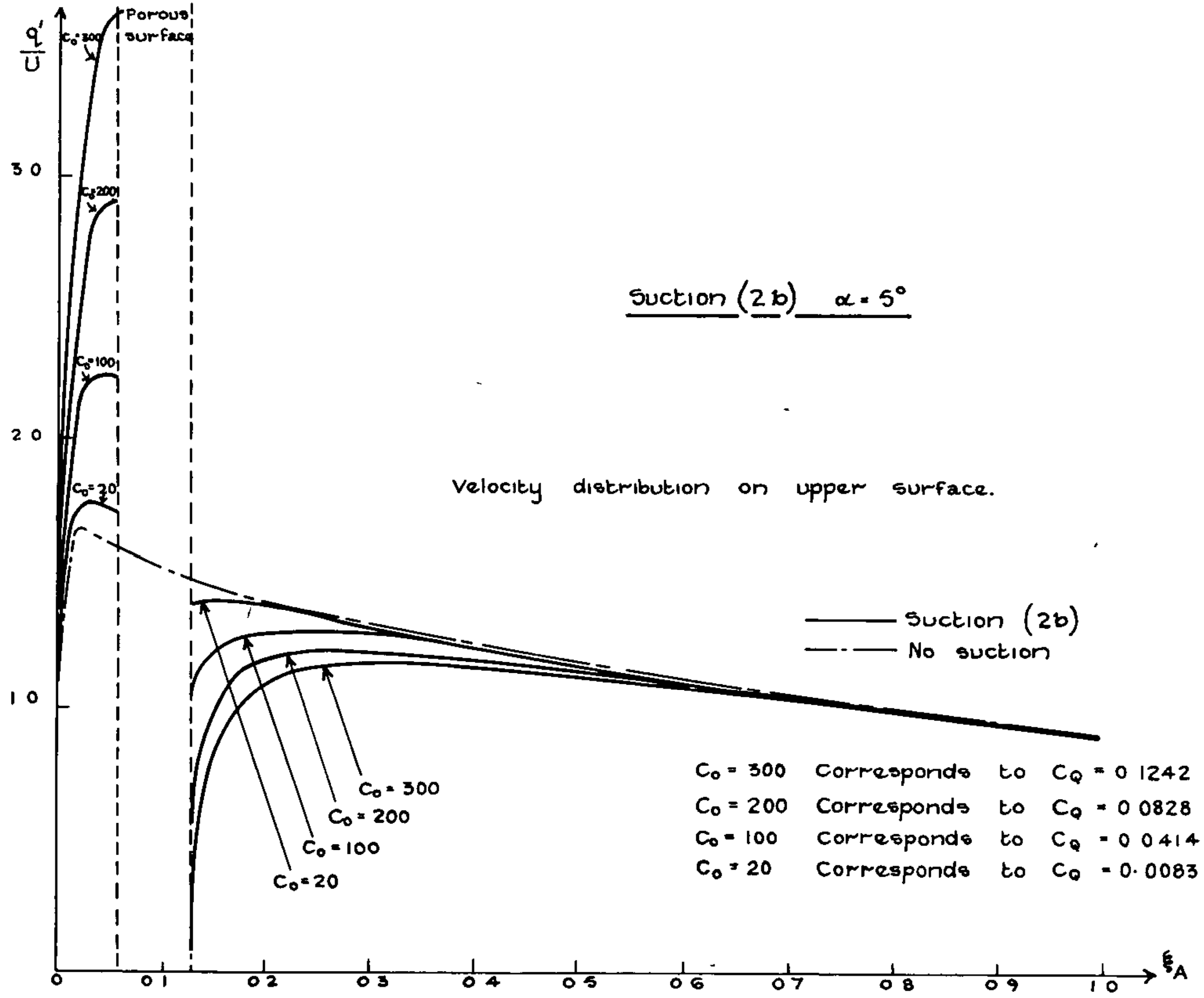
Fig. 4.



Suction (2b) α = 5°
Velocity distribution on upper surface.
q'/U
Porous surface
C₀ = 300
C₀ = 200
C₀ = 100
C₀ = 20
Suction (2b)
No suction
C₀ = 300 Corresponds to C_Q = 0 1242
C₀ = 200 Corresponds to C_Q = 0 0828
C₀ = 100 Corresponds to C_Q = 0 0414
C₀ = 20 Corresponds to C_Q = 0·0083
ξ_A

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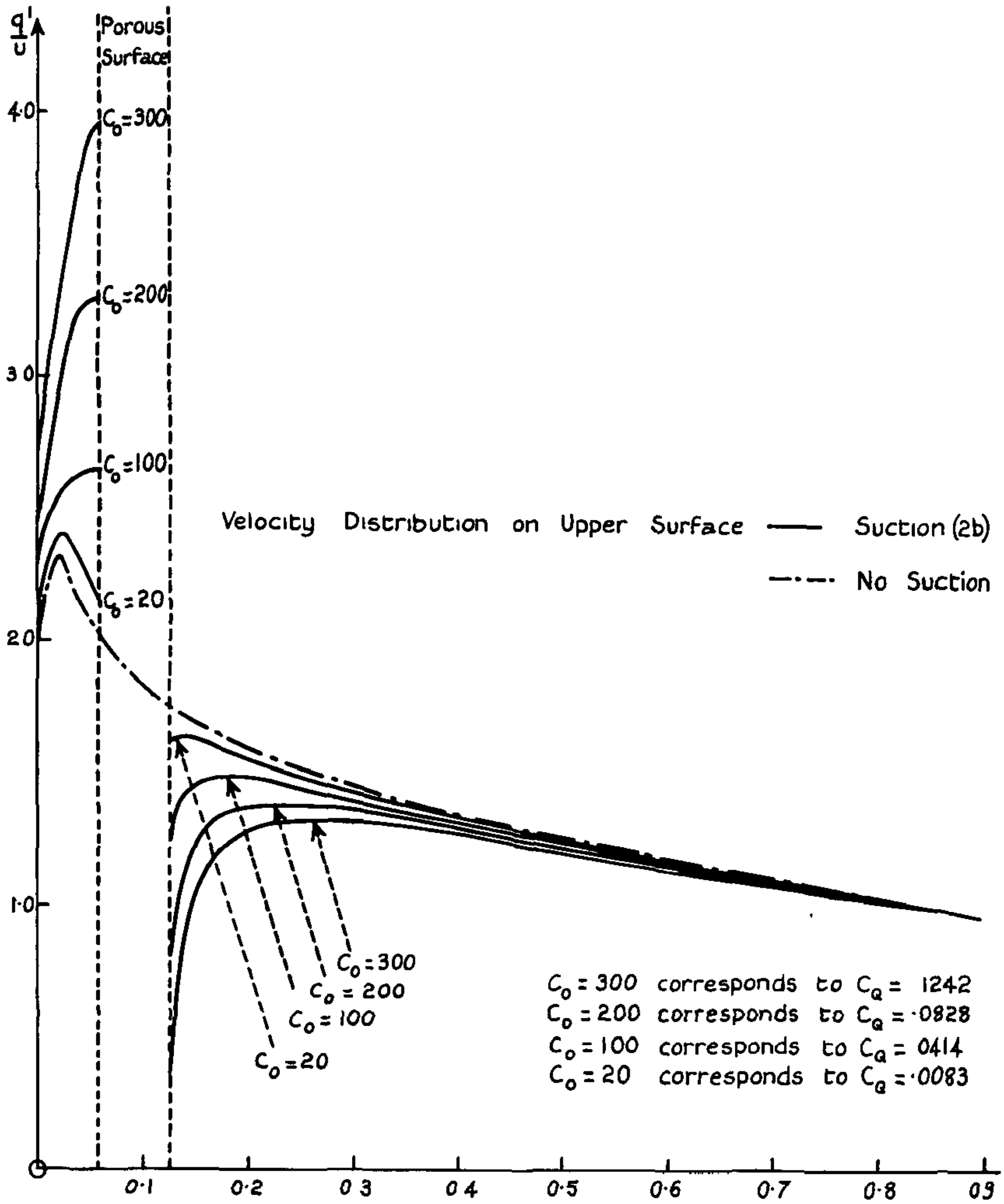
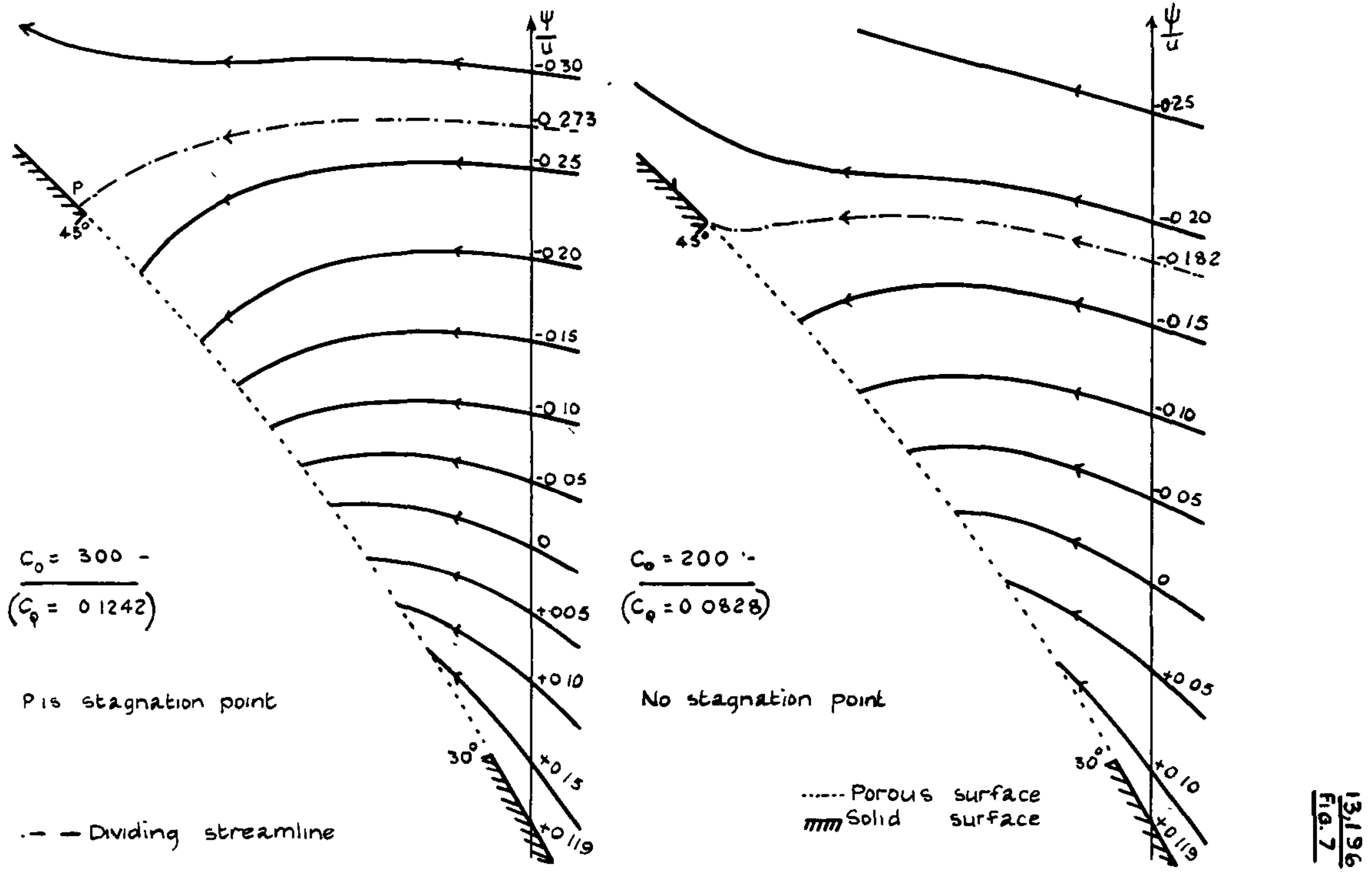
FIG 5



Suction (2b). α = 10°



Suction (2b) Streamlines past porous surface on circle

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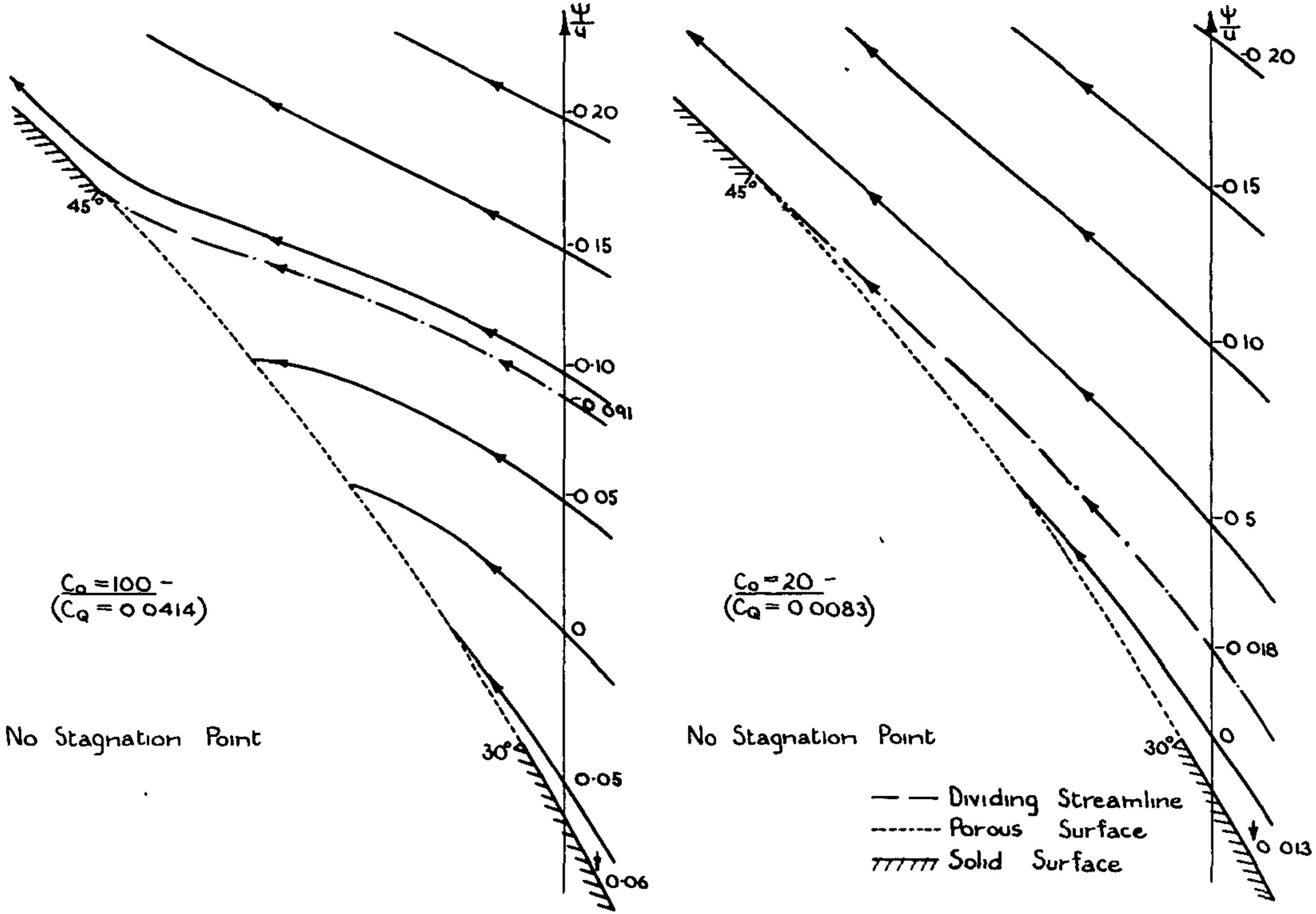
Fig. 7



Suction (2b) - Streamlines Past Porous Surface on Circle.

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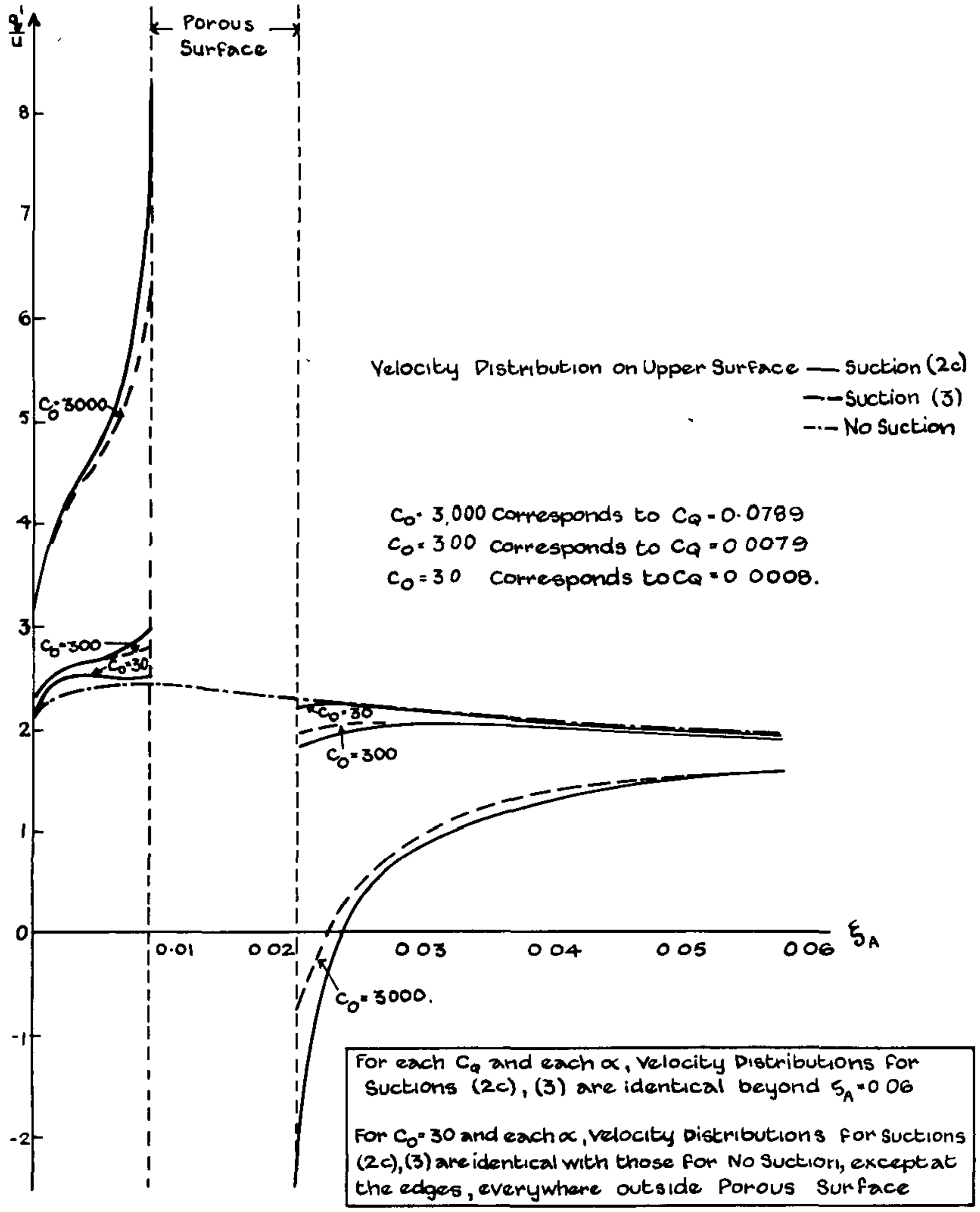
FIG 8



Suctions (2c), (3) $\alpha = 10^{\circ}$

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