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The Calculation of Transient Temperatures in Turbine Blades and Tapered Discs using Biot's Variational Method

By

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SUMMARY

Transient temperatures in aerofoil sections and tapered discs are calculated taking advantage of simplifications in heat flow analysis achieved in Biot's variational method. Cross-sections are represented by a line of adjacent squares of various sizes suitable for the local dimensions, e.g. small squares near the leading and trailing edges. The potential, dissipation and surface dissipation functions of Biot's method are set up, and the Lagrange equations lead, by automatic procedures, to an eigenvalue formulation in matrix form for the temperatures and their first time derivatives. Solutions are sums of exponentials in time, and are evaluated by digital computer, requiring about five minutes for each cross-section and heat transfer coefficient. Transient temperatures in a particular aerofoil section for variation of heat transfer coefficient and for external temperature depending exponentially on time agree with results obtained on an analogue computer. Maximum transient temperature differences are evaluated for tapered discs (which are not amenable to analysis by a simple electrical analogue) with variation of edge radius and heat transfer coefficient. Peculiarities in the solution for cyclic temperature external to an aerofoil over a range of frequencies indicate limitations in the mathematical formulation. A successful solution for cyclic external temperature might enable eigenvalues to be separated out in experimental measurements using electronic equipment, and this might be extended to exponential external temperature if a relationship between cyclic and exponential external temperature could be established. Eigenvalues and eigenvectors as discrete values arise fictitiously from the sub-division into squares and the possibility of an integral formulation is mentioned. There is a possible, but not immediate, extension to cooled blades, whose cross-sections are multiply-connected regions. Transient stresses due to creep, and viscoelasticity might be included.

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1.0 Introduction

Remarkable simplifications in heat flow analysis have been achieved by Biot[1], whose variational method permits considerable flexibility in formulating and solving heat flow problems. Several examples are given in Biot's Paper and they apply mainly to situations with one space dimension. The present problem of transient heat flow in an aerofoil section (or a tapered disc, which is its axisymmetric equivalent) has heat flow in both the chordwise and thickness-wise directions, but it is still possible to treat it one-dimensionally by regarding the latter heat flow as occurring essentially at the surface. A simple formulation of the heat flow equations is possible, leading to an eigenvalue* problem, which although formidable for hand calculation, can be readily solved by digital computer.

Calculated transient temperatures have been required at the National Gas Turbine Establishment in connection with an investigation of thermal fatigue[2,3]. Transient temperatures in an aerofoil section have been determined by an analogue computer[4], but this cannot readily be adapted to determine transient temperatures in tapered discs, which have been used for the bulk of the experimental work. Techniques of calculation equivalent to the analogue method, using finite differences and squares of constant size, have been used to a small extent[5,6], and these permit extension to tapered discs, but since they probably require more time on a digital computer, they are less attractive, although potentially more accurate, than an application of Biot's method.

A correlation between thermal fatigue endurances of tapered discs and mechanical properties has been obtained[7], using a few experimentally-determined transient temperature differences in tapered discs together with trends adapted from results for aerofoils, for variation of heat transfer coefficient and edge radius. The results presented below are the first calculated values for tapered discs and their use, instead of the previous curves, improves the correlation between thermal fatigue endurances and mechanical properties, especially with regard to the effect of variation of edge radius.

A previous method for the rapid calcul·tion of transient temperatures in aerofoil sections[8] was indebted to Biot's ideas, but depended on an assumption of equal rates of heat flow in a chordwise direction and across the surface. The present method is based much more rigorously on the heat flow equations, and with a digital computer available, is equally rapid.

2.0 Basic theory

Biot[1] defines a vector field B such that the heat flow at any point is $\frac{\partial B}{\partial t}$ per unit area, normal to B. Energy conservation leads to the equation

$$\text{div } B = -cT \qquad \ldots. (1)$$

---

*Eigenvalues are particular values of a parameter for which an equation (or set of equations) has non-zero solutions.

The following relationships (see Appendix I for List of Symbols) indicate that the equation of heat conduction is equivalent to an equation between the first variations of the thermal potential V

$$V = \frac{1}{2}\int cT^2 \, d\tau \qquad \text{.... (2)}$$

and the dissipation function D

$$D = \frac{1}{2}\int \frac{1}{k}\left(\frac{\partial B}{\partial t}\right)^2 d\tau \qquad \text{.... (3)}$$

The variation of V is

$$\begin{aligned} \delta V &= \int cT\delta T \, d\tau \qquad \text{.... (4)} \\ &= -\int T\,\delta\,\mathrm{div}\,B\,d\tau \\ &= -\int T\,\mathrm{div}\,\delta B\,d\tau \\ &= -\int (\mathrm{div}\,T\delta B - \mathrm{grad}\,T \cdot \delta B)\,d\tau \\ &= \int T\mathbf{n} \cdot \delta B\,dS - \int \frac{1}{k}\frac{\partial B}{\partial t} \cdot \delta B\,d\tau, \end{aligned}$$

using the equation of heat conduction, and where n is the inward drawn normal. If B, V and D are expressed in terms of coordinates q_i,

$$\delta V = \Sigma \frac{\partial V}{\partial q_i}\delta q_i, \qquad \delta B = \Sigma \frac{\partial B}{\partial q_i}\delta q_i \qquad \text{.... (5)}$$

$$\frac{\partial B}{\partial t} = \Sigma \frac{\partial B}{\partial q_i}\dot{q}_i \text{ so that } \frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial B}{\partial t}\right) = \frac{\partial B}{\partial q_i} \qquad \text{.... (6)}$$

Hence

$$\int \frac{1}{k}\frac{\partial B}{\partial t} \cdot \delta B \, d\tau = \Sigma \frac{\partial D}{\partial \mathring{q}_i} \delta q \qquad \text{.... (7)}$$

by replacing δB using Equation (5) and then applying Equation (6).

If at the surface,

$$\frac{\partial B_n}{\partial t} = -hT, \qquad \text{.... (8)}$$

then

$$\int Tn \cdot \delta B \, dS = -\int \frac{1}{h}\frac{\partial B_n}{\partial t} \cdot \delta B \, dS \qquad \text{.... (9)}$$

which is the same form as before, so that defining

$$D_s = \tfrac{1}{2}\int \frac{1}{h}\left(\frac{\partial B_n}{\partial t}\right)^2 dS, \qquad \text{.... (10)}$$

the variational principle, which applies for arbitrary variations δq leads to the equations

$$\frac{\partial V}{\partial q_i} + \frac{\partial (D + D_s)}{\partial \mathring{q}_i} = 0 \qquad \text{.... (11)}$$

If at the surface

$$\frac{\partial B_n}{\partial t} = h(T_a - T) \qquad \text{.... (12)}$$

then

$$\frac{\partial V}{\partial q_i} + \frac{\partial (D + D_s)}{\partial \mathring{q}_i} = \int T_a \frac{\partial B_n}{\partial q_i} dS = Q_i \qquad \text{.... (13)}$$

Q_i is referred to as a thermal force by Biot.

The purpose of this section has been to indicate concisely the mathematical relationships applying between the functions V, D and D_s and the coordinates q_i. These relationships are expounded more fully by Biot. The meaning behind this formulation can also be indicated in descriptive terms. Thus the function V represents the potential of the actual temperature distribution to cause heat flow back to the equilibrium state, and the function D sums a measure of the heat flow taking place throughout the body and therefore represents a dissipation function for the process by which the difference from equilibrium is reduced. The function D_s represents a similar process concentrated in the boundary. As pointed out by Biot, the formulation has close analogies with dynamical systems. Thus Equations (11) or (13) correspond to the Lagrangian equations for a mechanical dissipative system with a potential energy V and a dissipation function D. The entity Q_i of Equation (13) is referred to as a thermal force since it can be defined in exactly the same way as a mechanical force, i.e. as the virtual work done by a temperature T on a virtual displacement δB. If heat flow is regarded as belonging to the subject of irreversible thermodynamics, the function D as defined by Equation (3) has a physical significance since it is related to the rate of entropy production. The whole process of heat flow as discussed here has a complete analogy with the seepage of a compressible viscous fluid through a porous solid. The mass flow rate corresponds to the rate of heat flow, the pressure to the temperature, and the increase of fluid mass per unit volume to the heat content. The fluid compressibility represents the heat capacity and the permeability is the equivalent of the thermal conductivity.

3.0 Transient temperatures in symmetrical aerofoil

Figure 1 represents a quarter of a symmetrical aerofoil by n adjacent squares. Eight squares are shown on Figure 1. Sides of squares lying along the x and y axes are insulated. Other external complete sides of squares have heat transfer coefficient h, while incomplete segments of sides are insulated. The squares are numbered 0 .... n-1, with sides ℓ_{i-1}, temperatures are T_{i-1}, the first time derivatives of temperature are v_{i-1}, heat conduction $\frac{\partial B}{\partial t}$ inward across the external side parallel to the x-axis of square i-1 is R_{i-1}, and heat conduction $\frac{\partial B}{\partial t}$ in the direction of the x-axis across the common side of squares i-1 and i, that is, over a length ℓ_i, is L_i.

With the definitions $\ell_n = \ell_{n-1}$ and $T_n = 0$

$$V = \frac{c}{2} \sum_{i=1}^{n} \ell_{i-1}^2 \, T_{i-1}^2 \qquad \text{.... (14)}$$

$$D = \frac{1}{2k} \sum_{i=1}^{n} \varepsilon_i L_i^2 \quad \text{where} \quad \varepsilon_i = \tfrac{1}{2}(\ell_{i-1}^2 + \ell_i^2) \quad i = 1 \ldots n-1$$

$$\varepsilon_n = \tfrac{1}{2} \ell_{n-1}^2 \qquad \text{.... (15)}$$

$$D_s = \frac{1}{2h}\left(\sum_{i=1}^{n} \ell_{i-1} R_{i-1}^2 + \ell_{n-1} L_n^2\right) \qquad \text{.... (16)}$$

The equations of conservation are obtained from Equation (1), by using the theorem $\int \operatorname{div} B \, d\tau = \int B \cdot dS$, as

$$\ell_i L_i = \sum_{j=0}^{i-1} (-c\ell_j^2 v_j + \ell_j R_j) \qquad \text{.... (17)}$$

and the coordinates are taken as

$$q_i = T_{i-1}$$

$$\frac{\partial q_{n+i}}{\partial t} = R_{i-1} \qquad \text{.... (18)}$$

Equations (11) are

$$hT_{i-1} - H \sum_{j=i}^{j=n} \frac{\varepsilon_j L_j}{\ell_j} - L_n = 0 \qquad \text{.... (19)}$$

$$H \sum_{j=i}^{j=n} \frac{\varepsilon_j L_j}{\ell_j} + L_n + R_{i-1} = 0 \qquad \text{.... (20)}$$

Equations (17), (19) and (20) can be expressed in matrix form as

$$\begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix} \begin{pmatrix} c\ell_{i-1}^2 v_{i-1} \\ L_i \\ R_{i-1} \end{pmatrix} = h \begin{pmatrix} 0 \\ T_{i-1} \\ 0 \end{pmatrix} \qquad \text{.... (21)}$$

where D_{ij} are n x n matrices and T_{i-1} etc., are representative terms of n x 1 matrices. By subtracting the second row of matrices from the third, then in the first row of matrices subtracting successively row $i-1$ of elements from row i starting with row $n-1$, and similarly in the second row of matrices subtracting successively row i from row $i-1$ starting with row n, and finally subtracting rows of elements multiplied by ℓ_{i-1} in the third row of matrices from rows of elements in the first row of matrices, the following matrix equation is obtained

$$\begin{pmatrix} -1 & M & 0 \\ 0 & N & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\ell_{i-1}^2 \; v_{i-1} \\ L_i \\ R_{i-1} \end{pmatrix} = h \begin{pmatrix} \ell_{i-1} \; T_{i-1} \\ T_{i-1} - T_i \\ -T_{i-1} \end{pmatrix} \quad \ldots (22)$$

where matrices marked 0 have all elements zero, matrices marked ± 1 have ± 1 along the main diagonal and other elements zero, and non-zero elements of M are

$$M_{ii} = -\ell_i \qquad i = 1 \ldots n$$

$$M_{i+1,i} = \ell_i \qquad i = 1 \ldots n-1$$

and non-zero elements of N are

$$N_{ii} = \frac{h\varepsilon_i}{\ell_i} \qquad i = 1 \ldots n-1$$

$$N_{nn} = \frac{H\varepsilon_n}{\ell_n} + 1 \qquad \ldots . (23)$$

On multiplying out, Equation (22) leads to the matrix equations

$$-c\ell_{i-1}^2 \; v_{i-1} + ML_i = h\ell_{i-1} \; T_{i-1} \qquad \ldots . (24)$$

$$NL_i = h(T_{i-1} - T_i) \qquad \ldots . (25)$$

so that

$$-c\ell_{i-1}^2 \; v_{i-1} + MN^{-1} \; h(T_{i-1} - T_i) = h\ell_{i-1} \; T_{i-1} \qquad \ldots . (26)$$

Since N is diagonal, N^{-1} is easily obtained and the term $MN^{-1} \; T_i$ is converted to the form PT_{i-1} by putting the first column of P zero and column i of P the same as column i-1 of MN^{-1}. Hence Equation (26) can be expressed in the eigenvalue formulation

$$(G - \mu pI)\, T = 0 \qquad \ldots.\ (27)$$

where $\mu = \frac{c\ell_o}{h}$, $p = \frac{d}{dt}$, I has 1 in diagonal elements and zero elsewhere, and non-zero elements of G are

$$G_{11} = \frac{-\ell_1^2}{H\varepsilon_1 \ell_o} - 1$$

$$G_{ii} = -\frac{\ell_o\, \ell_i^2}{H\varepsilon_i\, \ell_{i-1}^2} - \frac{\ell_o}{H\varepsilon_{i-1}} - \frac{\ell_o}{\ell_{i-1}} \quad i = 2 \ldots n-1$$

$$G_{nn} = -\frac{\ell_o}{H\varepsilon_n + \ell_n} - \frac{\ell_o}{H\varepsilon_{n-1}} - \frac{\ell_o}{\ell_{n-1}}$$

$$G_{i,i+1} = \frac{\ell_i^2\, \ell_o}{H\varepsilon_i\, \ell_{i-1}^2} \quad i = 1 \ldots n-1$$

$$G_{i+1,i} = \frac{\ell_o}{H\varepsilon_i} \quad i = 1 \ldots n-1 \qquad \ldots.\ (28)$$

Equations (27) have non-zero solutions T if

$$\det\,(G - \mu pI) = 0 \qquad \ldots.\ (29)$$

This equation has n roots $\mu p = \lambda_j$, $j = 0 \ldots n-1$ and for each root λ_j

$$(G - \lambda_j I)x_{ij} = 0 \qquad \ldots.\ (30)$$

where x_{ij} has n rows $i = 0 \ldots n-1$ and one column specified by j, and is defined apart from an arbitrary multiplying constant. Equations (27) have n solutions

$$T_{ij} = A_j\, x_{ij} \exp\left(\frac{\lambda_j t}{\mu}\right) \qquad \ldots.\ (31)$$

so that the general solution is

$$T_i = \sum x_{ij} A_j \exp\left(\frac{\lambda_j t}{\mu}\right)$$

or

$$T = X\left(A_j \exp\left(\frac{\lambda_j t}{\mu}\right)\right) \quad \ldots. (32)$$

where $X = (x_{ij})$ is an n x n matrix whose columns are the eigenvectors x_{ij} with $i = 0 \ldots n-1$ for each j. If the initial values of T_i are all unity, then

$$XA = 1$$

so that

$$A = X^{-1} 1 \quad \ldots. (33)$$

where 1 is an n x 1 matrix with all elements 1. Determination of A in terms of X removes the effect of the arbitrary multiplying factor in the columns of X.

Solutions of Equation (27) with the elements of G given by Equations (28) have been obtained by Mercury digital computer, using Mercury Autocode and an R.A.E. Library programme[9]. However, starting from Equations (17), (19) and (20) all the matrix operations could have been performed by digital computer, so that Equations (28) need not have been obtained explicitly.

For a particular aerofoil section a quarter of which is represented by eight squares with sides 1, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, as shown in Figure 1, Table I shows the eigenvalues for $H\ell_0$ = 0.25, 0.5 and 1.0, and Table II gives the eigenvectors for $H\ell_0$ = 0 5. (Typical values of $H\ell_0$ encountered in gas turbine blading applications lie in the range 0 to 0.4.) Table III shows the values of A_j for unit initial temperature, and Table IV presents values of the temperatures $T_0 \ldots T_7$ at various intervals of time. Tables I to IV therefore illustrate the calculation procedure, but since the whole process is carried out by digital computer the intermediate results can be obtained just by printing out, or can be ignored altogether.

From results such as those of Table IV the familiar type of cooling curve can be plotted[2,3]. Maximum transient temperature differences are plotted in Figure 2, together with analogue results[10] for the same shape. The analogue measurements are made for eight adjacent squares, exactly as for the calculations. The question of the relative merits of fitting an aerofoil shape by adjacent squares or by an analogue mesh is not considered here. The calculated and analogue results agree fairly well although there is a slight but significant discrepancy, which is probably due to the inadequate representation of thickness-wise conduction in the mathematical formulation.

3.1 Time-dependent external temperature

Equations (13) apply if the external temperature is time-dependent instead of the step change of temperature of Section 3.0.

For coordinates, $i = 1 \ldots n$

$$q_i = T_{i-1}$$

$$\frac{\partial q_{n+i}}{\partial t} = R_{i-1}$$

and equations of conservation

$$\ell_i L_i = \sum_{j=0}^{i-1} (-c\ell_j^2 v_j + \ell_j R_j)$$

$$\frac{\partial L_n}{\partial v_{i-1}} = -\frac{c\ell_{i-1}^2}{\ell_n} \quad \text{so that} \quad \frac{\partial B_n}{\partial T_{i-1}} = \frac{c\ell_{i-1}^2}{\ell_n}$$

$$\frac{\partial L_n}{\partial R_{i-1}} = \frac{\ell_{i-1}}{\ell_n} \quad \text{so that} \quad \frac{\partial B_n}{\partial q_{n+i}} = -\frac{\ell_{i-1}}{\ell_n} \qquad \ldots . (34)$$

Hence $\quad Q_i = T_a c\ell_{i-1}^2$

$$Q_{n+i} = T_a \left\{ \ell_{i-1} - \frac{\ell_{i-1}}{\ell_n} \times \ell_n \right\} = 0 \qquad \ldots . (35)$$

Equations (13) lead to equations similar to Equations (19) and (20) but with extra terms on the right hand side. The extra term on the right hand side of Equation (22) is the $3n \times 1$ matrix

$$T_a \, h \begin{pmatrix} -\ell_{i-1} \\ \cdot \quad \cdot \\ 0 \\ 1 \\ \ldots \quad . \\ 1 \end{pmatrix} \qquad \ldots . (36)$$

whose first n elements are $-\ell_{i-1}$, whose second n elements are zero except for the last which is 1 and whose third n elements are all 1.

The equation corresponding to Equation (27) is

$$(G - \mu p I)T = T_a J \qquad \ldots\ (37)$$

where

$$J_i = -\frac{\ell_o}{\ell_{i-1}} \qquad i = 1 \ldots n-1$$

$$J_n = -\frac{\ell_o}{\ell_{n-1}} - \frac{\ell_o}{H\varepsilon_n + \ell_n} \qquad \ldots\ (38)$$

If $T = X\xi$ so that $\xi = X^{-1}T$, where X has the eigenvectors as columns,

$$(G - \mu p I)\, X\xi = T_a J$$

so that

$$(\lambda_j - \mu p)x_{ij}\, \xi_j = T_a J \qquad \ldots\ (39)$$

or

$$X \begin{bmatrix} \lambda_j - \mu p \end{bmatrix} \xi = T_a J$$

where the matrix has $\lambda_j - \mu p$ as diagonal elements.

Hence

$$\xi = \begin{bmatrix} \dfrac{1}{\lambda_j - \mu p} \end{bmatrix} X^{-1} J\, T_a$$

$$= \left(\frac{F_j}{\lambda_j - \mu p} \right) T_a \qquad \text{where } X^{-1} J = F \qquad \ldots\ (40)$$

If

$$T_a = \exp(-at) \qquad \ldots\ (41)$$

$$\xi_j = \frac{F_j}{\lambda_j - \mu p} \exp(-at)$$

$$= \frac{F_j \exp(-at)}{\lambda_j + a\mu} + A_j \exp\left(\lambda_j \frac{t}{\mu}\right) \quad \ldots (42)$$

so that

$$T = X \left\{ \left(\frac{F_j}{\lambda_j + a\mu} \right) \exp(-at) + \left(A_j \exp\left(\lambda_j \frac{t}{\mu} \right) \right) \right\} \quad \ldots (43)$$

If $T_a = b \exp(i\omega t)$, the cyclic solution without transient terms is

$$\xi_j = \frac{F_j b \exp(i\omega t)}{\lambda_j - \mu i\omega} \quad \ldots (44)$$

and if $T_a = b \cos \omega t = \frac{b}{2} (\exp(i\omega t) + \exp(-i\omega t))$, the cyclic solution for the temperatures is

$$T = X \left[\frac{F_j b}{\lambda_j^2 + \mu^2 \omega^2} (\lambda_j \cos \omega t - \mu\omega \sin \omega t) \right] \quad \ldots (45)$$

or

$$T = bX \left[\frac{F_j}{\alpha_j} \left(\frac{\lambda_j}{\alpha_j} \cos \omega t - \frac{\mu\omega}{\alpha_j} \sin \omega t \right) \right]$$

where $\alpha_j^2 = \lambda_j^2 + \mu^2 \omega^2$

If $T_a = ft$, the solution without transient terms is

$$\xi_j = \frac{f\mu F_j}{\lambda_j^2} + \frac{fF_j}{\lambda_j} t$$

so that

$$T = f\,X\left[\frac{\mu F_j}{\lambda_j^2} + \frac{F_j}{\lambda_j}\,t\right] \qquad \ldots. (46)$$

For exponential external temperature given by Equation (41), transient temperatures have been calculated for eight adjacent squares of sides 1, 0.9 ... 0.3, for $H\ell_o = 0.5$ and for $a\mu = 0.8$, 3 and 6. The eigenvalues are the same as on Table I for $H\ell_o = 0.5$ and the variation of maximum transient temperature difference with $a\mu$ is plotted on Figure 3, which shows that the reduction of $(T_o - T_7)_{max}$ from the step change case, i.e. $a\mu = \infty$ only becomes appreciable when $a\mu$ is comparable in magnitude with the smaller eigenvalues. The time to the maximum temperature difference increases as $a\mu$ decreases; for instance for $a\mu = 0.8$ this time is three times greater than for the step change case. The same results are plotted on Figure 4 on the basis of time-constants i.e. the time to e^{-1} of the original temperature, together with analogue results. As on Figure 2 the calculated and analogue results agree fairly closely but there is a significant discrepancy at high $H\ell_o$.

The effect of time-dependent external temperature has previously been considered[11] using an operational method which while implicitly including the effect of chordwise conduction does not allow for its variation. Although conditions are not comparable it appears that the reductions in maximum transient temperature difference predicted by the present methods are somewhat smaller than those obtained by the operational method.

3.1.1 Cyclic external temperature

Equations (45) have been evaluated over a range of frequencies ω for eight squares of sides 1, 0.9 ... 0.3 and for $H\ell_o = 0.5$ and the amplitude and phase of T_o and T_7 are shown on Figures 5 and 6. The oscillation of amplitude for $\mu\omega$ in the region of the smallest eigenvalue, and the occurrence of amplitudes greater than that of the external temperature indicate the inadequacy of the mathematical formulation to describe this situation, although for large $\mu\omega$ the results show correct trends. The slight discrepancy with analogue results noted on Figures 2 and 4 is reinforced by the breakdown of the method for cyclic external temperature and suggests that this is due to an inadequate representation of thickness-wise conduction.

Thickness-wise conduction can be taken into account to some extent by writing

$$D = \frac{1}{2k}\sum_{i=1}^{n} \varepsilon_i\, L_i^2 + \frac{1}{2k}\gamma \sum_{i=1}^{n} \ell_{i-1}^2\, R_{i-1}^2 \qquad \ldots. (47)$$

where γ is a constant whose value can be estimated from the heat flow in single slabs and may be about 0.25. With a dissipation function D given by Equation (47), Equations (28) are replaced by

$$G_{11} = -\frac{\ell_1^2}{H\varepsilon_1 \ell_0} - \frac{1}{1 + \gamma H \ell_0}$$

$$G_{ii} = \frac{-\ell_0 \ell_i^2}{H\varepsilon_i \ell_{i-1}^2} - \frac{\ell_0}{H\varepsilon_{i-1}} - \frac{\ell_0}{\ell_{i-1}(1 + \gamma H \ell_{i-1})}$$

$$G_{nn} = -\frac{\ell_0}{H\varepsilon_n + \ell_n} - \frac{\ell_0}{H\varepsilon_{n-1}} - \frac{\ell_0}{\ell_{n-1}(1 + \gamma H \ell_{n-1})}$$

$$G_{i,i+1} = \frac{\ell_i^2 \ell_0}{H\varepsilon_i \ell_{i-1}^2} \qquad i = 1 \ldots n-1$$

$$G_{i+1,i} = \frac{\ell_0}{H\varepsilon_i} \qquad i = 1 \ldots n-1 \qquad \ldots. (48)$$

It may be anticipated that Equations (48) would improve the results for step change and exponential temperatures, but would not yield correct results for cyclic external temperature.

4.0 Transient temperatures in tapered discs

The analogue computer[4] cannot be readily adapted to determine transient temperatures in tapered discs, although an approximate method of using the analogue for tapered discs is described in Appendix II. In this method the important region of the tapered disc near the thin edge is least well represented, in contrast to Biot's method in which the size of squares can be chosen to obtain equally good representation everywhere. Transient temperatures in tapered discs have been determined experimentally[12] using thermocouples, but not over the ranges required in the correlation[7] between thermal fatigue endurances and mechanical properties. The present calculations have been made partly to validate this correlation.

A quarter of the cross-section of the tapered disc is represented by n squares, with square 0 starting at the radius of the bore. Defining $\ell_n = \ell_{n-1}$ and $T_n = 0$, then for i = 1 ... n and with the notation

$$r_{i-1} = r + \ell_{i-1} \qquad i = 1$$

$$r_{i-1} = r_{i-2} + \ell_{i-1} \qquad i = 2 \ldots n$$

$$s_{i-1} = \ell_{i-1}(r_{i-1}^2 - r^2) \qquad i = 1$$

$$s_{i-1} = \ell_{i-1}(r_{i-1}^2 - r_{i-2}^2) \quad i = 2 \ldots n$$

$$\varepsilon_i = \tfrac{1}{2}(s_{i-1} + s_i) \qquad i = 1 \ldots (n-1)$$

$$\varepsilon_n = \tfrac{1}{2}s_{n-1} \qquad \ldots. (49)$$

the functions V, D and D_s are

$$V = \frac{\pi c}{2} \sum_{i=1}^{n} s_{i-1} T_{i-1}^2$$

$$D = \frac{\pi}{2k} \sum_{i=1}^{n} \varepsilon_i L_i^2$$

$$D_s = \frac{\pi}{2h} \left(\sum_{i=1}^{n} \frac{s_{i-1}}{\ell_{i-1}} R_{i-1}^2 + 2\ell_{n-1} r_{n-1} L_n^2 \right) \qquad \ldots. (50)$$

The equations of conservation are

$$2r_{i-1} \ell_i L_i = \sum_{j=0}^{i-1} \left(-cs_j v_j + \frac{s_j}{\ell_j} R_j \right) \qquad \ldots. (51)$$

For the coordinates

$$q_i = T_{i-1}$$

$$\frac{\partial q_{n+i}}{\partial t} = R_{i-1}$$

Equations (11) are

$$hT_{i-1} - \frac{H}{2} \sum_{j=1}^{j=n} \frac{\varepsilon_j}{\ell_j r_{j-1}} L_j - \frac{\ell_{n-1}}{\ell_n} L_n = 0 \qquad \ldots. (52)$$

$$\frac{H}{2} \sum_{j=1}^{j=n} \frac{\varepsilon_j}{\ell_j r_{j-1}} L_j + \frac{\ell_{n-1}}{\ell_n} L_n + R_{i-1} = 0 \qquad \text{.... (53)}$$

The analysis is similar to that of Section 3.0 and leads to Equation (27) where the non-zero elements of G are

$$G_{11} = \frac{-4r_0^2 \ell_1^2 \ell_0}{H\varepsilon_1 s_0} - 1$$

$$G_{ii} = \frac{-4r_{i-2}^2 \ell_{i-1}^2 \ell_0}{H\varepsilon_{i-1} s_{i-1}} - \frac{4r_{i-1}^2 \ell_i^2 \ell_0}{H\varepsilon_i s_{i-1}} - \frac{\ell_0}{\ell_{i-1}} \quad i = 2 \ldots n-1$$

$$G_{nn} = \frac{-4r_{n-2}^2 \ell_{n-1}^2 \ell_0}{H\varepsilon_{n-1} s_{n-1}} - \frac{4r_{n-1}^2 \ell_n^2 \ell_0}{H\varepsilon_n s_{n-1} + 2r_{n-1} \ell_{n-1} s_{n-1}} - \frac{\ell_0}{\ell_{n-1}}$$

$$G_{i,i+1} = \frac{4r_{i-1}^2 \ell_i^2 \ell_0}{H\varepsilon_i s_{i-1}} \quad i = 1 \ldots n-1$$

$$G_{i+1,i} = \frac{4r_{i-1}^2 \ell_i^2 \ell_0}{H\varepsilon_i s_i} \quad i = 1 \ldots n-1 \qquad \text{.... (54)}$$

The analysis of Section 3.0 between Equations (29) and (33) again applies.

Transient temperature differences have been calculated by Mercury digital computer for tapered disc specimens used in N.G.T.E. thermal fatigue tests,[2] with edge radii of 0.010, 0.020, 0.030 and 0.040 in., which can be fitted by adjacent squares as shown in Table V. Maximum transient temperature differences for $H\ell_0$ = 0.05, 0.1 and 0.2 are shown in Table VI and plotted on Figure 7. The variations with heat transfer coefficient of the present results for tapered discs and analogue results for an aerofoil with 0.020 in. edge radius are very similar in shape. The variations with edge radius of tapered discs and aerofoils differ to some extent. As edge radius increases the maximum transient temperature difference for the particular aerofoil decreases to a greater extent than for the tapered discs.

5.0 Discussion

It can be seen from Section 3.0 that when matrix operations have been carried out to the extent that L_{i-1} and R_i can be determined individually, the formulae for these are the finite difference equations which could have been written down immediately for the segment in question. However, use of Biot's method allows these equations to be produced by automatic procedures, with various lengths taken correctly into account, and further, all the matrix operations can be processed by digital computer, so that the finite difference equations need never be produced explicitly. Use of Biot's method indicates the validity of the finite difference equations by the extent to which the original selection of coordinates q_i represents the actual heat flow. More accurate finite difference formulae, such as Simpson's rule, might be used[1]. A marked advantage of this method is that it allows shapes of varied size to be used so that heat flow in a small region is as adequately represented as in a large region. Results can be obtained quickly by digital computer; for instance, the results shown on Figure 7 (i.e. 12 cases) required about one hour on Mercury digital computer. The method can be extended to cases with variable heat transfer coefficient round the periphery, time-dependent heat transfer coefficient and temperature-dependent material properties, as indicated by Biot[1].

In attempting to determine transient temperatures in an aerofoil for cyclic external temperature, the method has been taken to one of the limits at which it breaks down. It is thought that this is due to an inadequate representation of thickness-wise conduction and this may also account for the slight discrepancy between the calculated and analogue results shown on Figures 2 and 4. An approximate method of including thickness-wise conduction has been indicated in Section 3.1 but this is probably not realistic enough to produce valid results for cyclic external temperature. Another limit of the method of Section 3.0 was reached in an attempt to apply it to a cooled blade whose cross-section, being a multiply-connected region, produces internal conduction terms L_i on more than one side of the squares. Taking the coordinates q_i as the temperatures and surface heat flows leads to an unequal number of coordinates and equations. While Biot's method, using another selection of coordinates q_i, might apply to a cooled blade, this would not be a direct extension of the method of Section 3.0. The same applies to problems treated two-dimensionally and to three-dimensional problems. An analogue computer for a particular three-dimensional problem has been designed at N.G.T.E.[13].

One of the objects in considering cyclic external temperature was to enable eigenvalues for exponential or step change external temperature to be separated out in analogue measurements. While eigenvalues could be separated out in analogue measurements for two adjacent squares, these were not obtained precisely, and for more than two exponentials the mathematical situation[14] is that even with results of high precision, exponentials cannot be separated with any certainty. The same does not apply to the separation of cyclic components, to which electronic techniques can be applied, and since there are similarities in the formulae for exponential and cyclic external temperatures, as shown in Section 3.1, it was possible that a relationship between the two might be established. However, the frequency of all temperatures T_{i-1} is the same as the external frequency so that only phase and amplitude differ, which is not the usual situation in the separation of periodicities. Thus the attempt to derive eigenvalues from measured temperatures fails because (i) the mathematical method is inadequate to produce results for cyclic external temperature, (ii) a connection between exponential and cyclic external temperature has not been fully established and (iii) electronic techniques may not be applicable.

In a sense eigenvalues and eigenvectors as discrete entities are fictitious and are due to the sub-division of the cross-section into squares. The product of the matrix of eigenvectors with the matrix containing exponentials, with eigenvalues as coefficients of time, is however meaningful and is a sum of discrete elements which could be replaced by an integral. There is often an equivalence between systems of linear equations and integral equations[15], so that for instance the eigenvalue formulations of Sections 3.0 and 3.1 could be replaced by Fredholm integral equations of the second kind, i.e. of the type

$$\phi(x) - \lambda \int_0^1 K(x,y)\, \phi(y)\, dy = f(x) \qquad 0 \leq x \leq 1$$

with the operator $p\left(= \frac{d}{dt}\right)$ being treated as a constant. Thus is may be possible to formulate the problem more in keeping, analytically, with the integral expressions used by Biot, although for numerical results it may be desirable to use systems of algebraic equations. However it is possible that an integral formulation might lend some meaning to the product of the matrix of eigenvectors with the matrix of exponentials, with eigenvalues as coefficients of time, all the terms of which are derived, although by several processes, from the shape of the cross-section.

The programme on Mercury digital computer for temperatures could be extended to determine stresses, allowing for various material properties such as secondary creep[7], or each[16] or all[17] of the three creep components[18] (primary, secondary and tertiary). This is essentially a calculation added to the temperature calculation, but since Biot originally derived his results for visocelastic behaviour[19], the present methods may be applicable at a deeper level to material behaviour, for instance to coupled thermal and stressing effects, as in thermoelastic damping.

6.0 Conclusions

Transient temperatures in symmetrical aerofoil sections and tapered discs have been calculated using Biot's variational method and a Mercury digital computer. (The computer programmes have been deposited in R.A.E. Mathematics Department Programme Library.) Cross-sections are represented by lines of adjacent squares, whose sizes are chosen to suit the local dimensions. After setting-up the potential, dissipation and surface dissipation functions, and selecting coordinates, the Lagrange equations lead, by automatic procedures, to an eigenvalue formulation in matrix form for the temperatures and their first time-derivatives. The computer time required for the solutions is about five minutes for each cross-section and heat transfer coefficient.

Transient temperatures calculated for a particular aerofoil section for (i) a step change of external temperature and variation of heat transfer coefficient and (ii) external temperature depending exponentially on time agree fairly closely with results obtained on an analogue computer.

Maximum transient temperature differences have been evaluated for tapered discs (which are not amenable to analysis by a simple electrical analogue) for variation of edge radius and heat transfer coefficient.

Peculiarities in the solution for cyclic temperature external to an aerofoil over a range of frequencies indicate limitations in the mathematical formulation. A successful solution for cyclic external temperature might enable eigenvalues to be separated out in experimental measurements using electronic equipment, and this might be extended to exponential external temperature if a relationship between cyclic and exponential external temperature could be established.

Eigenvalues and eigenvectors as discrete values arise fictitiously from the sub-division into squares and the possibility of an integral formulation has been mentioned.

There is a possible, but not immediate, extension to cooled blades, whose cross-sections are multiply-connected regions. Transient stresses due to creep, and viscoelasticity might be included.

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TABLE I

Eigenvalues for eight adjacent squares
of sides 1, 0.9, 0.8 ... 0.3 for variation of $H\ell_o$

Eigenvalues \ $H\ell_o$	0.25	0.5	1.0
1	-66.938	-35.455	-19.778
2	-37.033	-19.902	-11.363
3	-24.205	-13.184	- 7.686
4	-17.081	- 9.464	- 5.648
5	-11.682	- 6.756	- 4.245
6	- 6.876	- 4.364	- 3.030
7	- 3.325	- 2.505	- 2.004
8	- 1.288	- 1.224	- 1.166

TABLE II

Eigenvectors for eight adjacent squares of sides 1, 0.9, 0.8 ... 0.3 and $H\ell_0 = 0.5$

0	0.0006	-0.0184	0.1371	-0.3389	0.4765	-0.6327	1.0000
0	-0.0058	0.1070	-0.5113	0.7508	-0.4191	-0.1008	0.8751
-0.0006	0.0377	-0.3586	0.7907	-0.0886	-0.7017	0.5029	0.7033
0.0065	-0.1773	0.7464	-0.2807	-0.7630	-0.1830	0.8846	0.5353
-0.0499	0.5498	-0.6601	-0.7253	-0.2779	0.5035	1.0000	0.3951
0.2619	-0.9005	-0.4441	0.0572	0.5340	0.9146	0.9353	0.2885
-0.8138	0.1583	0.5781	0.8106	0.9799	1.0000	0.7917	0.2126
1.0000	1.0000	1.0000	1.0000	1.0000	0.8855	0.6356	0.1603

Note: Columns are eigenvectors

TABLE III

Coefficients A_{i-1} for eight adjacent squares of sides 1, 0.9, 0.8 ... 0.3 for $H\ell_0 = 0.5$ and step change of external temperature

0.0473
0.0504
0.0537
0.0611
0.0987
0.2013
0.4905
1.2404

TABLE IV

Transient temperatures at intervals of time for eight adjacent squares of sides 1, 0.9, 0.8 ... 0.3, $H\ell_o$ = 0.5 and step change of external temperature

Time = multiple of 0.1μ	0	1	2	3	4	5	10	19
T_0	1	0.9040	0.8157	0.7349	0.6611	0.5938	0.3407	0.1187
T_1	1	0.8945	0.7995	0.7139	0.6368	0.5674	0.3143	0.1057
T_2	1	0.8820	0.7772	0.6839	0.6011	0.5278	0.2750	0.0874
T_3	1	0.8660	0.7480	0.6445	0.5547	0.4773	0.2302	0.0687
T_4	1	0.8440	0.7074	0.5914	0.4951	0.4160	0.1855	0.0522
T_5	1	0.8087	0.6467	0.5206	0.4234	0.3478	0.1452	0.0390
T_6	1	0.7423	0.5606	0.4364	0.3471	0.2805	0.1120	0.0292
T_7	1	0.6304	0.4586	0.3507	0.2758	0.2211	0.0864	0.0222
$T_0 - T_7$	0	0.2736	0.3571	0.3842	0.3853	0.3727	0.2543	0.0965

TABLE V

Representation of cross-section of tapered discs by adjacent squares

Edge radius	0.010 in.	0.020 in.	0.030 in.	0.040 in.
Bore	0.0935	0.0935	0.0935	0.0935
Square 1	0.138	0.138	0.138	0.138
2	0.120	0.124	0.124	0.124
3	0.097	0.099	0.100	0.103
4	0.078	0.081	0.085	0.086
5	0.064	0.065	0.067	0.073
6	0.050	0.055	0.057	0.061
7	0.042	0.045	0.047	0.052
8	0.034	0.037	0.038	0.043
9	0.025	0.028	0.033	0.040
10	0.022	0.024	0.030	
11	0.016	0.020		
12	0.013			
13	0.011			
14	0.010			
Total radius	0.813	0.810	0.812	0.813

Note: The side of the smallest square is taken equal to the edge radius since the element then has the same ratio of perimeter to area as the quardrant of the edge which it represents.

TABLE VI

Maximum transient temperature differences
in tapered discs

(ℓ_o = maximum half-thickness of disc)

$H\ell_o$ \ Edge radius of disc	0.010 in.	0.020 in.	0.030 in.	0.040 in.
0.050	0.201	0.156	0.138	0.114
0.1	0.286	0.224	0.198	0.166
0.2	0.374	0.296	0.261	0.221

APPENDIX I

List of symbols

a	coefficient for exponential external temperature
b	amplitude of cyclic external temperature
c	specific heat per unit volume
det	determinant
e	exponential
f	coefficient for linear rate of change of external temperature
h	heat transfer coefficient
i	square root of -1, or suffix
j	suffix
k	material thermal conductivity, or suffix
ℓ	length of side of square
n	vector or suffix denoting normal to surface (positive inwards)
p	operational expression for $\frac{\partial}{\partial t}$
q	generalised coordinate
r	radius of bore of tapered disc
s	proportional to the volume of a ring of square cross-section, or a variable
t	time

APPENDIX I (cont'd)

v	$\frac{\partial T}{\partial t}$
w	variable
x	component of eigenvector, or variable
y, z	variable
A	coefficients
B	heat flow vector
D	dissipation function
D_{ij}	matrices
D_s	surface dissipation function
F	coefficients, for time-dependent external temperature
G	matrix
H	h/k
I	unit matrix
J	extra matrix, over step change case, for time-dependent external temperature
K	kernel of integral equation
L	vector specifying internal (chordwise) conduction
M	matrix
N	matrix, or normal

APPENDIX I (cont'd)

Q	thermal force
R	vector specifying heat flow across surface (positive inwards)
S	surface area
T	temperature
T_a	external temperature
ΔT	difference between initial temperature and external temperature
V	thermal potential
X	matrix with eigenvectors as columns
γ	factor allowing for thickness-wise conduction
δ	small change, or first variation
ε	mean area of two adjacent squares
ϕ	function in integral equation
κ	diffusivity k/c
λ	eigenvalue
μ	$\frac{c\ell}{h}$, or $\mu^{-1} = (H\ell)\left(\frac{\kappa}{\ell^2}\right)$
τ	volume
ω	frequency of external temperature, or variable
ξ	normal coordinates of temperature

APPENDIX II

Approximate method of determining transient temperatures in tapered discs using an analogue computer

The equation of heat conduction for axially symmetrical heat flow

$$\frac{1}{\kappa}\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}$$

is transformed by the substitution $s = \log r$ into

$$\frac{r^2}{\kappa}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial s^2} + r^2\frac{\partial^2 T}{\partial z^2}$$

The coordinates (s,z) are orthogonal and this equation can be simulated by an electrical network with capacities proportional to r^2 and resistances in the z direction proportional to r^{-2}.

The further transformation $\omega = \frac{z}{r}$ leads approximately to the equation

$$\frac{r^2}{\kappa}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial s^2} + \frac{\partial^2 T}{\partial \omega^2}$$

The coordinations (s,ω) are not orthogonal, but this equation is probably valid where the transformation is nearly orthogonal, i.e. near $z = 0$, and the equation can be simulated by an electrical network with constant resistances and capacities proportional to r^2. Since $ds = \frac{dr}{r}$ and $d\omega = \frac{dz}{r}$, individual squares of the (s,ω) mesh correspond to equal-sided figures on the disc cross-section, the size of the latter being proportional to r.

The boundary condition is

$$\frac{\partial T}{\partial n} + hT = 0$$

where n is the normal in the (r,z) plane. If N is the normal in the (s,ω) plane, then $dn = r dN$ where the transformation is almost orthogonal, so that

$$\frac{\partial T}{\partial N} + (hr)T = 0$$

Hence heat transfer coefficients for the disc are multiplied by r to give heat transfer coefficients for the analogue.

Since the mesh on the tapered disc cross-section consists of lines radiating from the origin, together with lines along the z direction to produce approximate squares, the approximate squares are biggest near the thin edges, and at the surface of the thickest portion of the disc are least orthogonal. Thus this method of using a constant resistance mesh to determine transient temperatures in tapered discs has several limitations.

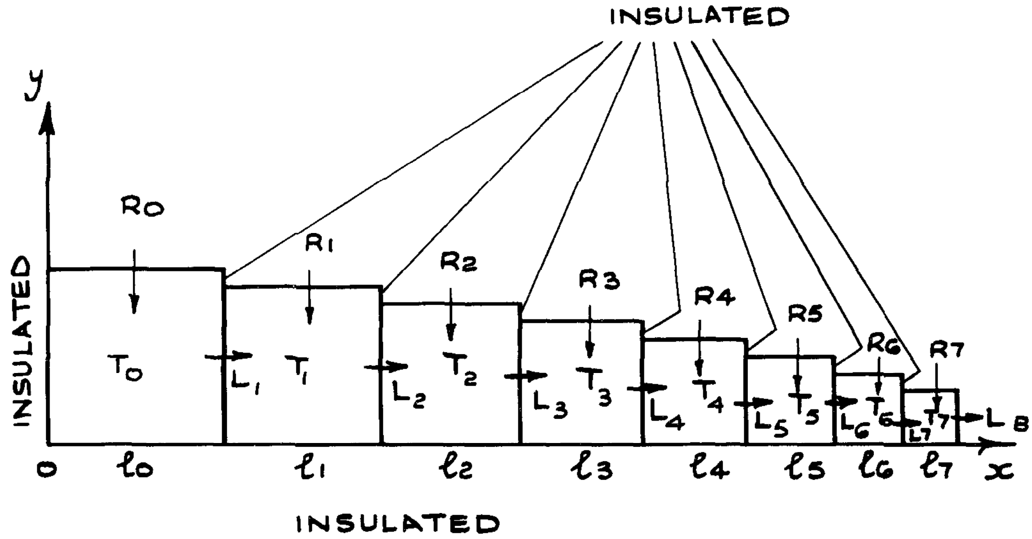

FIG 1. CO-ORDINATES FOR HEAT FLOW IN SYMMETRICAL AEROFOIL

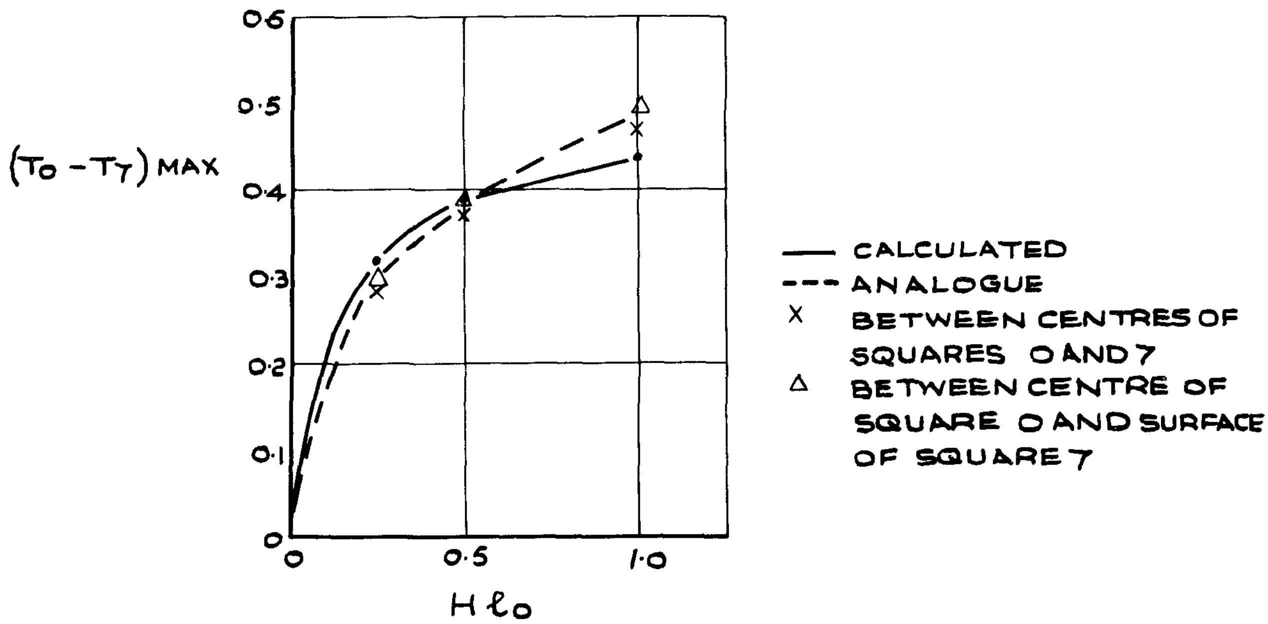

FIG. 2 VARIATION OF MAXIMUM TRANSIENT TEMPERATURE DIFFERENCE WITH HEAT TRANSFER COEFFICENT FOR SYMMETRICAL AEROFOIL

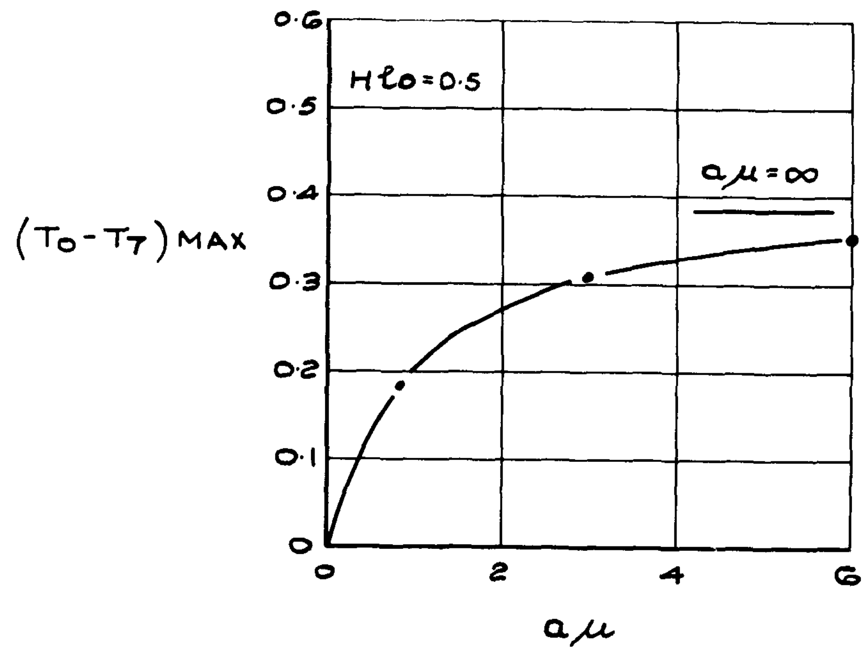

FIG 3. VARIATION OF MAXIMUM TRANSIENT TEMPERATURE DIFFERENCE WITH THE EXPONENT OF TIME-DEPENDENT EXTERNAL TEMPERATURE FOR SYMMETRICAL AEROFOIL

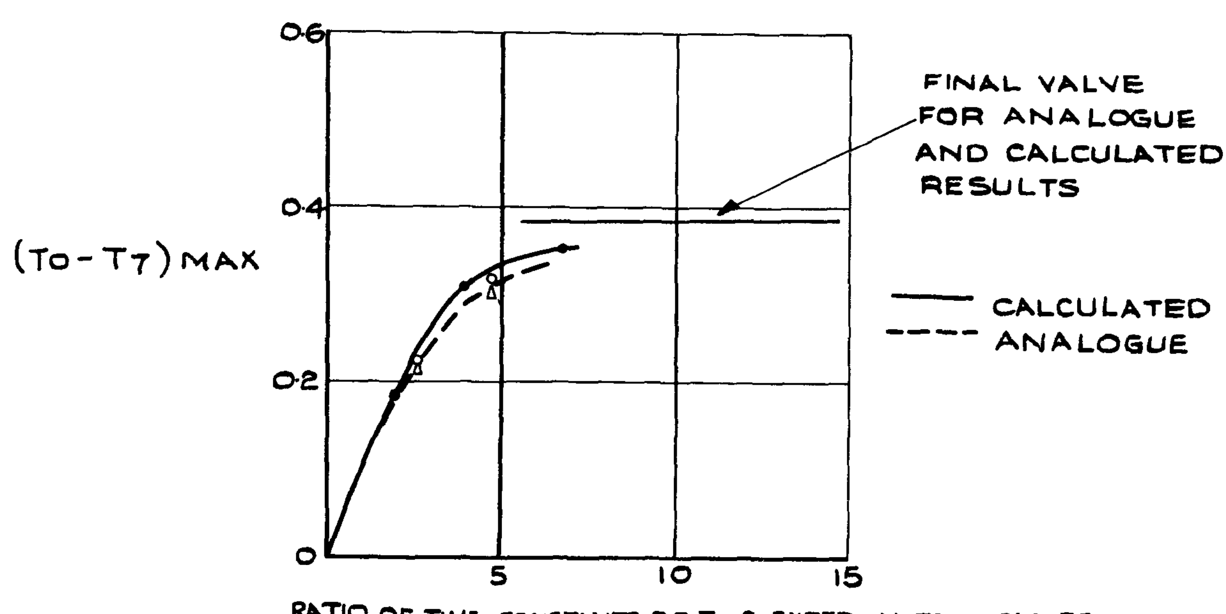

FIG 4 VARIATION OF MAXIMUM TRANSIENT TEMPERATURE DIFFERENCE WITH THE TIME CONSTANT OF EXTERNAL TEMPERATURE FOR SYMMETRICAL AEROFOIL

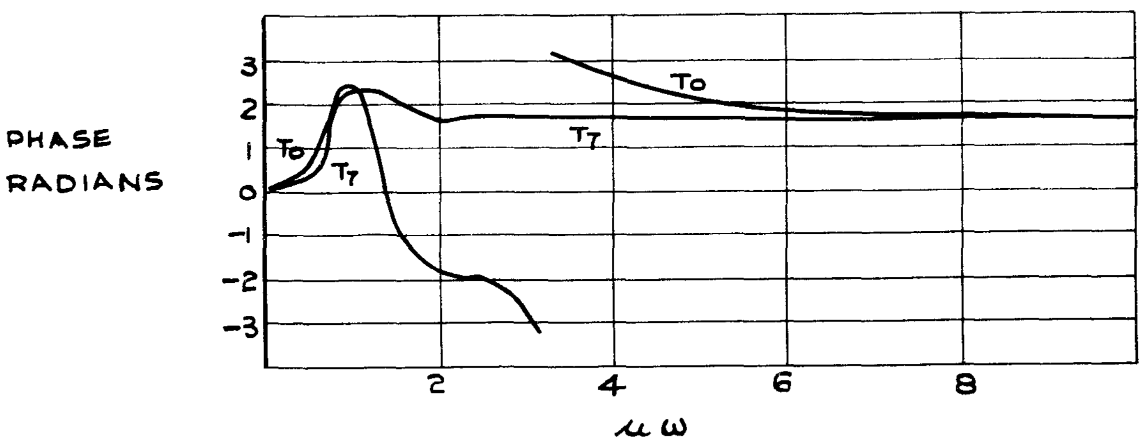

FIG. 5 CALCULATED PHASE OF TEMPERATURES IN SYMMETRICAL AEROFOIL FOR CYCLIC EXTERNAL TEMPERATURE

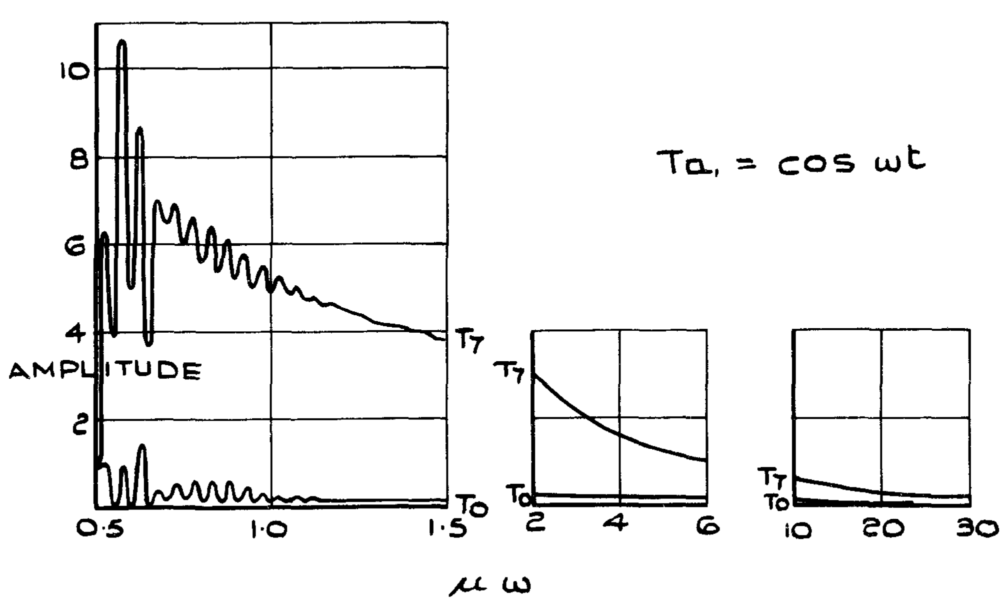

FIG. 6 LIMITATIONS OF MATHEMATICAL METHOD INDICATED BY CALCULATED AMPLITUDES FOR CYCLIC EXTERNAL TEMPERATURE

FIG. 7

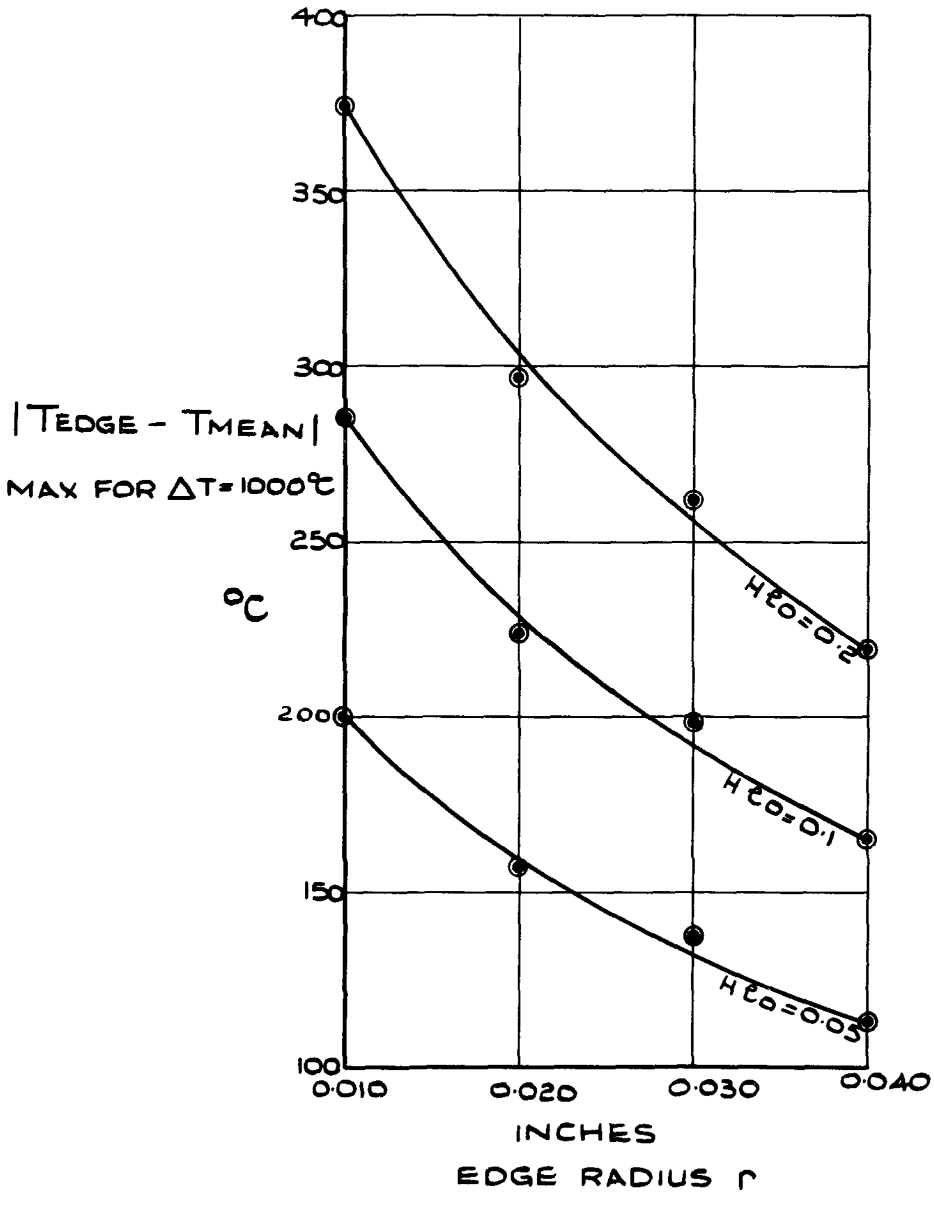

FIG 7. MAXIMUM TRANSIENT TEMPERATURE DIFFERENCES IN TAPERED DISCS

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