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The Development of
the Boundary Layer in
Supersonic Shear Flow

by

Ruth H. Rogers

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SUPERSONIC SHEAR FLOW

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Ruth H. Rogers*

SUMMARY

The development of the boundary layer in a velocity shear layer is discussed for two-dimensional flow and for axisymmetric flow of both compressible and incompressible fluids. It is shown that the solutions obtained by Li^{1,3} and Glauert² for the two-dimensional flow of an incompressible fluid are applicable in the more general case after suitable transformations of coordinates have been made. New definitions are shown to be necessary, and are given, for the displacement and momentum thicknesses of such a boundary layer. Reynolds numbers based on these thicknesses are given, and it is shown that any phenomenon (such as transition to turbulence) which occurs at a constant value of such a Reynolds number will occur at a point which, as the length scale of the flow increases, first moves downstream and then moves slightly upstream. This is shown to be in qualitative agreement with experimental results on a blunt cone in a supersonic flow. A quantitative comparison of the theoretical and experimental values of displacement and momentum thicknesses is attempted, and no disagreement is obvious; unfortunately the accuracy of the experiments so far available is insufficient to give positive confirmation of the theory of this note.

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1 INTRODUCTION

In recent years, interest has grown in flows which contain vorticity: this interest is due mainly to the highly curved shock waves which occur upstream of bodies travelling at very high velocities. The fluid which has passed through the curved portion of the shock possesses vorticity, so that there exists a velocity (and density, etc.) shear in the flow even when it is far from the surface. A new problem immediately arises: the effect of the boundary itself on such a flow.

The concept of a thin boundary layer, as originally suggested by Prandtl, is so useful that we are reluctant to discard it; and all the workers in the field so far have assumed that such a layer exists, even when there is vorticity further away from the body. This assumption is not easily justified at present, except on grounds of expediency; it implies that although vorticity exists in each of two neighbouring regions, the two regions may be treated separately. The distinction between the regions is made by supposing that viscous effects are negligible in the region further from the wall (this region will be referred to as "the external flow", or "the external shear flow", throughout this note). In the boundary layer (that is, in the region nearer to the wall) the vorticity is produced mainly by viscous effects due to the presence of the boundary; in the external flow it is produced by a mechanism which is upstream of the region, and whose nature is unimportant. At present there is little evidence to show that the division between the two regions is as distinct as the corresponding one for a boundary layer developing in a uniform flow. Unfortunately no exact solutions and insufficient experimental results are at present available for comparison, and the concept can be used only in so far as it predicts correctly such phenomena as transition to turbulence.

Most of the workers in the field have considered the development of the boundary layer in an incompressible flow over a flat plate, in the case where the flow has a uniform velocity shear far from the plate. Li¹ has shown that the Howarth transformation (see equation (3)) is sufficient to convert the equations of a compressible shear flow into those of an incompressible shear flow. There has been some discussion (between Glauert² and Li³) of the appropriate boundary conditions to take at the outer "edge" of the boundary layer; the arguments involved can be found in Ref.2 and will not be repeated here. In the present paper results will be presented using the solutions obtained from both methods, the difference in the boundary conditions being specified precisely in section 2.3.

It is shown first that either solution can be extended to cover the case of a shear flow over an axisymmetric surface, both for the incompressible and for the compressible case. As with all solutions so far produced it is assumed that all the variables (velocity, density, etc.) in the boundary layer can be expressed as a power series in a parameter which is proportional to the velocity shear far from the surface. This series is not shown to be convergent, and the only justification given for its use is that of qualitative agreement with experimental results.

The solutions of the boundary layer equations assume, of course, that there is no turbulence in the flow, and it is of importance to know if, and where, turbulence occurs. Instability of a laminar flow can be due to many causes, only one of which is discussed here. This is referred to as a boundary-layer instability and it occurs for a constant value of the Reynolds number, R_{δ_2} , based on local conditions and the momentum thickness of the boundary layer. Experimental evidence⁴ suggests that this is the kind of instability which causes transition to turbulence in the boundary layer on a blunt cone in supersonic flow.

Reference has been made above to the displacement and momentum thicknesses of the boundary layer, but the meaning of these terms is less clear than for boundary layers developing in a uniform flow. A fuller discussion of this point is given in section 3, where it is shown that there can be some ambiguity in their definition. Such quantities can be defined, however, and are related respectively to the mass deficit and to the momentum deficit which occur due to the presence of the boundary layer. The definitions chosen here reduce to the usual ones when the external flow is uniform, but make the evaluation of the momentum thickness in the experimental case less accurate.

It is shown that the Reynolds number, R_x , based on conditions at the edge of the boundary layer and on the distance, x , downstream is not the most suitable parameter for predicting transition to turbulence (as was suggested by Moeckel⁵). Instead, the Reynolds number, R_{δ_2} , based on certain reference conditions (which are defined in section 4) and on the momentum thickness, δ_2 , is used. It is found that, as the length scale, r , increases, there is a region in which the position of transition moves upstream, although this position is always downstream of that obtained when r is very small. This implies (see Fig.12) that there is also a region in which the position of transition moves downstream as r increases.

Comparisons with experimental results obtained by the author⁴ are made wherever possible; these results were for boundary layer development on a blunt cone in supersonic flow. Due to the difficulty of calculating δ_2 from the experimental results, the Reynolds number R_{δ_1} is considered as well as R_{δ_2} ; there is no obvious reason why this quantity should be inferior as a transition parameter to the quantity, R_{δ_2} , which has previously been considered. The accuracy of the experiments was insufficient to give more than qualitative agreement with the theory of this paper, and a further series of experiments designed to test the theory would be desirable. As far as it goes, the agreement is satisfactory.

2 THE FUNDAMENTAL SOLUTION

2.1 The equations of the motion

Glauert² assumes that the boundary layer approximations hold throughout both the regions considered: that is, using the notation of Fig.1, the component of velocity, v , normal to the surface is of a smaller order of magnitude than that, u , along the surface in the direction of the flow. On the other hand, the rate of change of any quantity in the direction of u , is of a smaller order than that in a direction normal to the surface. He supposes further that there is no pressure gradient in the direction of u in the external flow: that is, that $\partial p/\partial x$ is zero in this region. Under the boundary layer assumptions, we have $\partial p/\partial y$ is zero everywhere; it follows that $\partial p/\partial x$ is zero in the boundary layer as well as in the external flow - that is, the pressure is constant throughout both the regions considered. Li³, on the other hand, finds an induced pressure gradient due to the presence of the boundary layer, so that the (constant) term $-1/\rho \partial p/\partial x$ must be retained in the equations of motion.

The momentum equation for the flow in either region is

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \frac{\partial}{\partial y} \left(\mu \frac{\partial u}{\partial y} \right) , \quad (1)$$

where ρ is the density and μ is the viscosity of the fluid at the point (x,y) . The equation of continuity is, to a good approximation

$$\frac{\partial}{\partial x} \left(\rho r_1^k u \right) + \frac{\partial}{\partial y} \left(\rho r_1^k v \right) = 0 , \quad (2)$$

where $k = 0$ in the two-dimensional case and $k = 1$ in the axisymmetric case. In this equation the distance of the point (x,y) from the axis is replaced (as is usual in an axisymmetric boundary layer) by the distance, r_1 , of the point $(x,0)$ from the axis of symmetry. This form is also used in the external flow, since for the important region near the edge of the boundary layer it is still small compared with r_1 and the approximation remains acceptable.

It is convenient to use a transformation of the independent variables which is a combination of the well-known ones originally used by Howarth and by Mangler. These new coordinates (X,Y) are given by the integrals

$$X = \int_0^x \left(\frac{r_1}{r} \right)^{2k} dx , \quad Y = \left(\frac{r_1}{r} \right)^k \int_0^y \frac{\rho}{\rho_0} dy , \quad (3)$$

where r is the length scale of the motion (in the case of a blunt cone, for example, r is the radius of the tip), and ρ_0 is a reference density. In the two-dimensional incompressible case, of course, $X = x$ and $Y = y$.

A stream function, ψ , can be introduced, by virtue of the equation of continuity (2). We define this by the equations

$$\rho r_1^k u = \rho_0 r^k \frac{\partial \psi}{\partial y} , \quad \rho r_1^k v = - \rho_0 r^k \frac{\partial \psi}{\partial x} .$$

In terms of the new coordinates (X,Y) defined in equation (3), the stream function is given by

$$u = \frac{\partial \psi}{\partial Y} , \quad (4)$$

and

$$v = - \frac{\rho_0}{\rho} \left(\frac{r_1}{r} \right)^k \frac{\partial \psi}{\partial X} - \frac{\rho_0}{\rho} \left(\frac{r}{r_1} \right)^k Y' \frac{\partial \psi}{\partial Y} ,$$

where $Y' = dY/dx$. Substituting these expressions in the momentum equation (1), and writing $\mu\rho = \mu_0\rho_0$, a constant*, we have after some manipulation

$$\frac{\partial\psi}{\partial Y} \frac{\partial^2\psi}{\partial X\partial Y} - \frac{\partial\psi}{\partial X} \frac{\partial^2\psi}{\partial Y^2} = -\frac{1}{\rho} \frac{\partial p}{\partial X} + \frac{\mu_0}{\rho_0} \frac{\partial^3\psi}{\partial Y^3}, \quad (5)$$

and this is identical with the form obtained for the two-dimensional incompressible case. In this equation, Glauert² takes $\partial p/\partial X = 0$, and Li³ finds that $1/\rho \partial p/\partial X$ is a function of x only, and becomes a constant ($A_1/2$, say) to the required order of the problem in the later analysis with the coordinates (ξ, η) that are introduced in the next section; these values will be assumed here without proof. The expression for u , given in equation (4), is also the same as that in the two-dimensional incompressible case.

2.2 The external flow

This is the region in which viscous effects are negligible although there may be vorticity in the flow. The vorticity is introduced into the fluid upstream of the region considered by a mechanism such as a curved shock wave in the case of supersonic flow, or a nonuniform grid in the case of subsonic flow. The nature of the mechanism is unimportant, since there is no vorticity producing mechanism in the region itself.

It is assumed that the boundary layer is contained entirely within a region of velocity shear in the external flow; in practice, however, such an external velocity shear cannot extend to an infinite distance from the wall, but is itself only a layer. The model chosen here is shown schematically for $x = 0$ in Fig.2: the shear layer, as defined below, exists for all Y less than Y_∞ ; for Y greater than Y_∞ , the velocity component in the direction parallel to the surface remains constant, and has the value u_∞ . In the cases of axisymmetric and compressible flow, the shear layer exists for all y less than y_∞ (which may be a function of x), where y_∞ and Y_∞ are related by the second integral of equation (3). This model permits a direct comparison with the blunt cone experiments: in this case the velocity u_∞ is the velocity at the edge of a boundary layer on a sharp cone (that is, far downstream from the tip).

The shear flow to be considered in the two-dimensional incompressible case is that in which the velocity shear is uniform: that is

$$\frac{du_\infty}{dY} = \frac{u_\infty \Omega}{r}$$

where u_∞ is a reference velocity (at the point $x = 0, y = 0$, of Fig.2) and Ω is a non-dimensional velocity shear. This corresponds, in the more general case, to an assumption that the shearing stress is independent of y : that is

* This is only true if the pressure is constant as Glauert assumes. Li also assumes constant pressure in Ref.1 and in his reappraisal of the problem with induced pressure gradient³ he considers only incompressible flow. It is, however, perhaps reasonable to regard the pressure gradient as being small and to neglect variations in $\mu\rho$ as a first approximation, especially since the assumption of proportionality between μ and T is itself only approximately true.

$$\mu \frac{\partial u_e}{\partial y} = \mu_o \left(\frac{r_1}{r}\right)^k \frac{\partial u_c}{\partial Y} = \frac{\mu_o u_o \Omega}{r} \left(\frac{r_1}{r}\right)^k .$$

It is of interest to note that for this form of the external flow, the right hand side of equation (5) is identically zero if we use Glauert's model ($dp/dx = 0$) - that is, that viscous forces have no effect on the flow (not merely a negligible effect, as stated in the introduction).

In the absence of a boundary layer, the shear flow is defined by

$$u_{co} = u_o \left(1 + \frac{\Omega}{r} Y\right) , \quad (6a)$$

but in the presence of a boundary layer, there may be a displacement effect which must be included in this formulation. If the displacement thickness of the boundary layer is given by $y = \delta_1$, or $Y = \Delta_1$, Glauert² has shown that the appropriate form for u_e is

$$u_e = u_o \left[1 + \frac{\Omega}{r} (Y - \Delta_1)\right] \quad (6b)$$

in the two-dimensional incompressible case. The value of Δ_1 can be determined only when the boundary layer solution has been obtained; since, however, Δ_1 is zero when $x = 0$, and the boundary condition which Glauert applies at the edge of the boundary layer is at the point $x = 0$, this does not cause any practical difficulty in the solution of the problem. It should be noticed that the displacement effect acts on the position of the edge of the shear layer as well, and this will now be in the position given by $Y = Y_\infty + \Delta_1$ instead of $Y = Y_\infty$. Li³ takes $u = u_{co}$ given by (6(a)) as the value outside the boundary layer but takes the zero for Y in the external shear layer to be at the edge of the boundary layer, i.e. Y could be replaced by $Y - \Delta$ in (6(a)) where Δ is the boundary layer thickness (not the displacement thickness) in the coordinate of Y .

The usual boundary layer transformation

$$\eta = Y \sqrt{\frac{\rho_o u_o}{\mu_o X}} \quad (7)$$

will be made, and in terms of this variable, equation (6) becomes

$$u_e = u_o \left[1 + \xi(\eta - \eta_1)\right] , \quad (8)$$

where $\eta = \eta_1$ when $Y = \Delta_1$, and where

$$\xi = \frac{\Omega}{r} \sqrt{\frac{\mu_o X}{\rho_o u_o}} \quad (9)$$

is a parameter depending on the velocity shear in the flow.

In terms of the physical variables x and y , the equation (6) for the velocity in the external shear layer $0 < y < y_\infty + \delta_1$ becomes

$$u_e = u_o \left[1 + \frac{\Omega}{r} \left(\frac{r_1}{r} \right)^k \left(\int_0^y \frac{\rho}{\rho_o} dy - \delta_1 \right) \right].$$

As shown in Fig.3 for an incompressible fluid, if $r_1(x)$ is an increasing function of x (on a blunt cone, for example, $r_1(x) \propto x$, if a suitable origin is chosen for x), the lines of constant velocity approach the surface as x increases, and the boundary layer grows into a region of higher velocity.

2.3 The flow in the boundary layer

In this region the stream function ψ must satisfy the full equation (5). At the surface $Y = 0$, we must have $u = 0$ and $v = 0$: that is $\partial\psi/\partial X = 0$, $\partial\psi/\partial Y = 0$ for all X . At the outer edge of the boundary layer, the boundary condition is either:-

(i) that u is given by u_{eo} (see equation 6(a)) when $x = 0$.

This together with the absence of a pressure gradient is a sufficient condition and automatically satisfies the relation $u \rightarrow u_e$ (see equation 6(b)) as $Y \rightarrow \infty$ for all X . Thus the value of Δ_1 does not enter into the specification of the problem. (This is the boundary condition used by Glauert².)

(ii) that $u \rightarrow u_{eo}$ as $Y \rightarrow \infty$ for all x . (This is the boundary condition used by Li³ with, however, the zero for Y taken at the edge of the boundary layer.)

The solution is expanded in powers of the parameter ξ , defined by equation (9); it is assumed that such an expansion exists and is convergent. We look for a solution of equation (5), therefore, of the form

$$\psi = \sqrt{\frac{\mu_o u_o X}{\rho_o}} \left(f_o(\eta) + \xi f_1(\eta) + \dots \right), \quad (10)$$

where η is given by equation (7). This expression satisfies equation (5) for all values of ξ if

$$\left. \begin{aligned} 2 f_o''' + f_o f_o'' &= 0 \\ 2 f_1''' + f_o f_1'' - f_o' f_1' + 2 f_o'' f_1 &= A_1 \end{aligned} \right\} \quad (11)$$

where dashes denote differentiation with respect to η . The boundary conditions may be written either as

$$\left. \begin{aligned} f_o(0) = f_o'(0) = 0, \quad f_o'(\eta) \rightarrow 1 \quad \text{as } \eta \rightarrow \infty \\ f_1(0) = f_1'(0) = 0, \quad f_1''(\eta) \rightarrow 1 \quad \text{as } \eta \rightarrow \infty \end{aligned} \right\} \quad (12a)$$

which are the same equations as those given by Li⁶ and Glauert² when $A_1 = 0$; or as

$$\left. \begin{aligned} f_0(0) = f'_0(0) = 0, \quad f'_0(\eta) \rightarrow 1 \text{ as } \eta \rightarrow \infty \\ f_1(0) = f'_1(0) = 0, \quad f'_1(\eta) \rightarrow \eta \text{ as } \eta \rightarrow \infty \end{aligned} \right\} \quad (12b)$$

which are the equations given by Li³ with $A_1 \neq 0$. The author is grateful to Mr. Glauert for making available to her the numerical solution of the second of equations (11) with $A_1 = 0$ subject to the boundary conditions (12a). The solution with $A_1 \neq 0$, subject to the boundary conditions (12b), has been obtained from the solution of similar equations given by Murray⁷. In this case A_1 is found to be 1.7207. The numerical solution of the first of equations (11), which is the familiar Blasius equation, is well known. The values of $f'_0(\eta)$ and $f'_1(\eta)$ in each case are given in Table 1, together with the asymptotic forms of the functions $f_0(\eta)$ and $f_1(\eta)$.

The velocity distribution in the boundary layer is given by the expression

$$u = u_0 \left(f'_0(\eta) + \xi f'_1(\eta) + \dots \right) . \quad (13)$$

Graphs of u/u_0 as a function of η are shown in Fig.4 for three values of ξ , using Glauert's solution; there is no qualitative difference from Li's solution. Careful inspection shows a point of inflexion in the curve for the largest value of ξ ; that such a point exists in every case is shown by a study of the asymptotic form of equation (13) as shown in Appendix 1. It is interesting to compare this with similar "kinks" in the experimental curves obtained for the velocity profiles on a blunt cone: a typical curve (drawn on an arbitrary scale) is also shown in Fig.4. It must be pointed out, however, that this theory is only a first approximation, and the existence of this point of inflexion may not persist in higher order approximations; so the agreement may be entirely fortuitous.

Fig.5 shows a sketch of the velocity profiles at different stations of x in both the two-dimensional and the axisymmetric cases. The figures have been drawn to illustrate Glauert's solution (which is slightly more complicated): if Li's solution is used, the lines $y = y_\infty + \delta_1$ are not relevant. It should be emphasized that y_∞ is, in the axisymmetrical case, a function of x as shown in the figure.

2.4 Conditions at the edge of the boundary layer (Glauert)

Inspection of Fig.4 suggests that the velocity u may be taken to have reached its asymptotic value (that is, u_e) when $\eta \doteq 4.7$ which corresponds to a value

$$u_\delta \doteq u_0 (1 + 3\xi) \quad (14)$$

for u_δ , the value of u at the edge of the boundary layer. This is confirmed by inspection of the numerical values given in Table 1: it is found that

$u = u_e$ within 1 per cent when η is greater than about 4.7. It is now possible to calculate M_δ , the Mach number at the edge of the boundary layer. To do this we require a relation between the density and the velocity in the boundary layer. If we assume that the stagnation temperature is constant throughout the boundary layer (this is not exactly true, but is a good approximation), and that the ratio of specific heats, γ , is 1.4, it is easy to show that

$$\frac{\rho_o}{\rho} = 1 + 0.2 M_o^2 \left(1 - \frac{u^2}{u_o^2} \right), \quad (15)$$

where M_o is the Mach number corresponding to the density ρ_o and the velocity u_o . We also have the relation

$$M_\delta = M_o \left(\frac{\rho_\delta u_\delta^2}{\rho_o u_o^2} \right)^{\frac{1}{2}},$$

where ρ_δ is the value of the density at the edge of the boundary layer. Using the equations (14) and (15) in this relation, we have (to the first order in the small quantity ξ)

$$M_\delta = M_o + M_o \left(3 + 0.6 M_o^2 \right) \xi; \quad (16a)$$

or, in terms of the coordinate X defined by the first of equations (3),

$$M_\delta = M_o + M_o \left(3 + 0.6 M_o^2 \right) \frac{\Omega}{r} \sqrt{\frac{\mu_o X}{\rho_o u_o}}. \quad (16b)$$

The experimental variation of M_δ with X is shown in Fig.6, and this enables the experimental value of Ω to be determined. The value of M_o in this case is taken to be the value of the Mach number on the surface of the cone in the absence of the boundary layer when the static pressure on the surface is equal to that on a sharp cone*, and the stagnation pressure is that at the stagnation point at the tip of a blunt cone. Thus, for a blunt cone whose included angle is 15° placed in a supersonic stream whose undisturbed Mach number is 3.12, the value of M_o is 2.14; the value of M_∞ , the Mach number at the edge of the boundary layer very far from the tip (that is, the value of M at the edge of the external shear layer) is 2.92. For the same cone in a stream whose undisturbed Mach number is 3.81, the value of M_o is 2.34 and that of M_∞ is 3.35. Fitting a straight line through the experimental points of Fig.6, we find that the value of Ω is 0.20 when the free-stream Mach number is 3.12, and 0.14 when the free-stream Mach number is 3.81. Further, it can be seen that the values of ξ for which $M_\delta = M_\infty$ are 0.06 and 0.07 in the two cases respectively - the theory is not expected to be valid for values of ξ as large as this. The values obtained from Li's solution are not significantly different from these.

* As was the case in the experiments⁴.

3 DISPLACEMENT THICKNESS AND MOMENTUM THICKNESS

When a boundary layer develops in a uniform stream whose velocity u_e is a constant, the displacement thickness is defined by the integral

$$\delta_1 = \int_0^{\delta} \left(1 - \frac{\rho u}{\rho_e u_e} \right) dy \quad , \quad (17)$$

and the momentum thickness by the integral

$$\delta_2 = \int_0^{\delta} \frac{\rho u}{\rho_e u_e} \left(1 - \frac{u}{u_e} \right) dy \quad . \quad (18)$$

When u_e is not constant the form of these equations is not necessarily the same. It is important, therefore, to investigate the physical interpretation of these quantities.

Before doing so, however, it will be valuable to point out how confusion can arise due to the usual practice of using the velocity at the edge of the boundary layer as a reference velocity. In the development of a boundary layer in a uniform flow, this velocity, u_δ , is precisely defined, although the position of the edge of the boundary layer is imprecise. When the flow outside the boundary layer is a shear flow, however, the velocity u_δ is known only at the point (0,0), and it is this value (u_0 in the notation of the present paper) that is taken as the reference velocity. In the present problem, of course, there is also the given velocity, u_∞ ; but this depends on the extent of the external shear layer, and so is irrelevant to the flow in the boundary layer, at least for small x : it is not, therefore, suitable as a reference velocity for boundary layer problems.

It is not, however, sufficient to replace the reference quantities ρ_e and u_e in equations (17) and (18) by the reference quantities ρ_0 and u_0 , as the integrals would not then be convergent; and, as pointed out above, the quantities ρ_∞ and u_∞ are not suitable reference values in the boundary layer. In order to preserve as much as possible of the usual physical significance of the two thicknesses, we proceed as follows.

3.1 Displacement thickness and mass deficit

The physical significance of the displacement thickness of a boundary layer developing in a uniform flow has been discussed in some detail by Lighthill⁸. Similar arguments may be applied in the present case, and these are given in Appendix 2 for the two-dimensional incompressible case. It is sufficient here to regard the displacement thickness as a measure of the mass deficit due to the presence of the boundary layer: in this case, the surface $y = \delta_1$, where δ_1 is the displacement thickness, represents the position of a displaced boundary which, in the absence of viscous forces, would give the correct flow outside the boundary layer. It follows that the magnitude of the displacement thickness at any station satisfies the equation

$$\int_0^{\infty} \rho u \, dy = \int_{\delta_1}^{\infty} \rho_e u_e \, dy \quad , \quad (19)$$

and this is in agreement with (17) when ρ_e and u_e are constants. An alternative form of this equation is

$$\int_0^{\delta_1} \rho_e u_e dy = \int_0^{\infty} (\rho_e u_e - \rho u) dy, \quad (20)$$

where the right hand side is the mass deficit between the real flow along $y = 0$ and the fictitious inviscid flow along $y = 0$ (as distinct from $y = \delta_1$).

It may be of interest to note that δ_1 can be regarded as the difference between a total displacement thickness

$$\delta_{1t} = \int_0^{\infty} \left(1 - \frac{\rho u}{\rho_{\infty} u_{\infty}}\right) dy$$

which is appropriate to the whole shear flow (including both the boundary layer and the external shear layer), and a partial displacement thickness

$$\delta_{1e} = \int_0^{\infty} \left(1 - \frac{\rho_e u_e}{\rho_{\infty} u_{\infty}}\right) dy$$

appropriate to the external shear flow in the absence of a boundary layer. (It can, incidentally, be shown that δ_{1e} is negative in the axisymmetric case.) This definition, however, appears to be less fundamental than that given by equations (19) and (20), and will not be considered further.

It is shown in Appendix 3 that for the solution of section 2, the displacement thickness is given by

$$\delta_1 = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_o \bar{X}}{\rho_o u_o}} \left\{ 1.7207 + 0.47694 M_o^2 - \left(3.4687 + 0.73474 M_o^2 \right) \xi + \dots \right\} \quad \dots (21a)$$

using Glauert's solution, and by

$$\delta_1 = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_o \bar{X}}{\rho_o u_o}} \left\{ 1.7207 + 0.47694 M_o^2 - \left(8.6683 + 1.6392 M_o^2 \right) \xi + \dots \right\} \quad \dots (21b)$$

using Li's solution. The experimental values of δ_1 for the boundary layers on blunt cones have been recalculated (the incorrect formula was used in Ref.4) and the results are given in Table 2, and plotted as a function of ξ in Fig.7. The values of ξ have been calculated using the values of Ω found in section 2. To investigate the variation with ξ it has been found most convenient in Fig.7 to plot, not δ_1 , but the ratio

$\delta_1 \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_o \bar{X}}{\rho_o u_o}}$; the theoretical curves of equation (21) are shown on the graph, and so is the value of the ratio far downstream (that is, the value on a sharp cone). It can be seen that, although the experimental results

are rather scattered, they are not inconsistent with the theoretical curves: the accuracy of the experimental results is unfortunately insufficient to go further than this, and there is not enough evidence to say that the equation (21) is verified. It should be noted that for small values of ξ , and therefore of x , the effect of the boundary layer on the curved part of the blunt cone may be considerable, and this has not been allowed for here.

3.2 Momentum thickness and momentum deficit

There are two ways of interpreting the concept of momentum thickness physically. First, we can consider it as a measure of the deficit of momentum due to the presence of the boundary layer: this momentum deficit we write as

$$D = \int_{\delta_1}^{\infty} \rho_e u_e^2 dy - \int_0^{\infty} \rho u^2 dy \quad , \quad (22)$$

which is the difference between the flow of momentum in the inviscid flow round the fictitious surface $y = \delta_1$, and the flow of momentum in the real flow. It is not difficult to show that, if ρ_e and u_e are constants, then the definition of equation (18) corresponds to

$$\delta_2 = \frac{D}{\rho_e u_e^2} \quad .$$

Alternatively, we can integrate equation (1); then in the case of zero pressure gradient, using (2), we have

$$\tau_w - (\tau_w)_0 = \left(\frac{r}{r_1}\right)^k \frac{d}{dx} \left[\left(\frac{r_1}{r}\right)^k D \right] \quad , \quad (23)$$

where τ_w is the true shearing stress at the wall, and $(\tau_w)_0$ is that everywhere in the external shear flow (that is, the value at the wall in the absence of a boundary layer). It follows that, in a uniform flow (with no pressure gradient),

$$\frac{\tau_w}{\rho_e u_e^2} = \left(\frac{r}{r_1}\right)^k \frac{d}{dx} \left[\left(\frac{r_1}{r}\right)^k \delta_2 \right]$$

and this is sometimes used as an equation defining δ_2 .

The obvious definition uses an analogy with equation (20), in which the right hand side is the mass deficit: that is, we write

$$\int_0^{\infty} \rho u^2 dy = \int_{\delta_1 + \delta_2}^{\infty} \rho_e u_e^2 dy \quad ,$$

giving

$$\int_{\delta_1}^{\delta_1 + \delta_2} \rho_e u_e^2 dy = D \quad . \quad (24)$$

This means that the flow of momentum in the actual flow is equal to the flow of momentum, outside the surface $y = \delta_1 + \delta_2$, in the inviscid flow outside the surface $y = \delta_1$; this idea must not be taken too far, however, as the surface $y = \delta_1 + \delta_2$ is not a stream surface. A convenient way of writing this definition is

$$\int_0^{\delta_1 + \delta_2} \rho_e u_e^2 dy = \int_0^{\infty} (\rho_e u_e^2 - \rho u^2) dy, \quad (25)$$

and this is the expression used for calculating experimental values of δ_2 . (It should be noted that, since the calculation of δ_2 involves the subtraction of the two comparatively large quantities δ_1 and $\delta_1 + \delta_2$, the experimental determination of δ_2 is less accurate than that of δ_1).

Since the shearing stress, τ_w , is equal to the quantity $(\mu du/dy)_{y=0}$, the easiest way of calculating the theoretical value of D is to use the equation (23) and write the left hand side in the form $(\mu \frac{\partial u}{\partial y} - \mu_0 \frac{\partial u_{e0}}{\partial y})_{y=0}$;

this may be integrated with respect to x, using the fact that D = 0 when x = 0. The result of this integration is that

$$D = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_0 X}{\rho_0 u_0}} (0.664 - (1-c) \xi + \dots) \rho_0 u_0^2 \quad (26)$$

where $c = f_1''(0)$; Glauert gives $c = 0.795$, and Li gives $c = 2.955$. Using this result, and the definition given in equation (24) it is shown in Appendix 3 that the momentum thickness is given by

$$\delta_2 = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_0 X}{\rho_0 u_0}} \left\{ 0.664 - \left(0.646 + 0.0882 M_0^2 \right) \xi + \dots \right\} \quad \dots (27a)$$

for Glauert's solution and

$$\delta_2 = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_0 X}{\rho_0 u_0}} \left\{ 0.664 - \left(2.254 + 0.5452 M_0^2 \right) \xi + \dots \right\} \quad (27b)$$

for Li's solution. The experimental values for the boundary layers on blunt cones have been recalculated, and are compared with the theoretical curves of equations (27a) and (27b) in Fig.8. The agreement between theory and experiment is perhaps less good than that for the displacement thickness, but as pointed out above the accuracy of the experimental determination of the momentum thickness is rather low. At least there is no great discrepancy observable. It should be remarked that the accuracy of the experiments is not great enough to distinguish between the different curves of Glauert and Li.

4 REYNOLDS NUMBER AND TRANSITION

The Reynolds number of the flow at any given point depends on two quantities - the unit Reynolds number (Reynolds number per unit length) and a length which is associated with the point concerned. Various combinations of these two quantities can occur, and some may be more appropriate than others in any given problem.

In discussing the problem of transition to turbulence in a boundary layer on a blunt cone, Moeckel⁴ considers the Reynolds number, R_x , based on local conditions at the edge of the boundary layer and the distance downstream from the tip: that is, $R_x = \rho_\delta u_\delta x / \mu_\delta$. He assumes that at the position of transition R_x always has the same value, and points out that since the unit Reynolds number $\rho_\delta u_\delta / \mu_\delta$ increases with x on a blunt cone (so that, for a given value of x , $\rho_\delta u_\delta / \mu_\delta$ - and therefore R_x - is smaller on a blunt cone than on a sharp cone) transition will occur further downstream on a blunt cone than on a sharp cone. It was found experimentally, however, in Ref. 4 that this result does not apply for large values of the tip radius (for values of x/r less than 100).

A more appropriate Reynolds number is likely to be one which is based on the thickness of the boundary layer rather than on x , which takes no account of the way in which the boundary layer develops. It was found experimentally³ that transition occurred when the Reynolds number based on momentum thickness, R_{δ_2} , was between 650 and 700, and when that based on displacement thickness, R_{δ_1} , was about 5000; these values seemed to be appropriate, within the experimental accuracy, for all the cones tested (even when the various quantities have been recalculated using the definitions of this note).

For reasons similar to those discussed in section 3, the most suitable value of the unit Reynolds number is not that at the outer "edge" of the boundary layer, but the value $\rho_o u_o / \mu_o$ - the only given value of the problem. Using with this value, the values of δ_1 and δ_2 given in equations (21a) and (27a) respectively, we now have

$$R_{\delta_1} = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\rho_o u_o X}{\mu_o}} \left\{ 1.7207 + 0.47694 M_o^2 - \left(3.4687 + 0.73472 M_o^2 \right) \xi \right\} \quad (28a)$$

and

$$R_{\delta_2} = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\rho_o u_o X}{\mu_o}} \left\{ 0.664 - \left(0.646 + 0.0882 M_o^2 \right) \xi \right\} \quad (29a)$$

for Glauert's solution. Similarly, using Li's results we have

$$R_{\delta_1} = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\rho_o u_o X}{\mu_o}} \left[1.7207 + 0.47694 M_o^2 - \left(8.6683 + 1.6392 M_o^2 \right) \xi + \dots \right], \quad \dots (28b)$$

and

$$R_{\delta_2} = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\rho_o u_o X}{\mu_o}} \left[0.664 - \left(2.254 + 0.5452 M_o^2 \right) \xi + \dots \right]. \quad (29b)$$

The values for the experimental results can also be calculated using the relations

$$R_{\delta_1} = \frac{\rho_o u_o \delta_1}{\mu_o}, \quad R_{\delta_2} = \frac{\rho_o u_o \delta_2}{\mu_o} \quad (30)$$

where, as in section 3, ρ_o , u_o and μ_o are the values of ρ , u , μ respectively at the surface of a blunt cone in the absence of a boundary layer (that is, the values appropriate to the static pressure on the surface of a sharp cone, and to the stagnation temperature and pressure behind the normal bow shock in front of the cone). This value is not, however, that taken for the largest value of x in each case: here the value of δ/y_∞ is nearly unity, and it is not expected that the theory of this paper would hold as there is a gradient of vorticity in this region. For the sake of comparison the value of the Reynolds number at this point has been calculated using a value of $\rho u/\mu$ intermediate between $\rho_o u_o/\mu_o$ and $\rho_\infty u_\infty/\mu_\infty$: this intermediate value was taken to be that appropriate to the velocity when $y = 0$ on the tangent to the velocity profile $u(y)$ at the estimated outer "edge" of the boundary layer.

The experimental values are shown in Figs.9 and 10, and compared with the curves for the quantities $R_{\delta_1} \left(\frac{r_1}{r}\right)^k \sqrt{\frac{\mu_o}{\rho_o u_o X}}$ and $R_{\delta_2} \left(\frac{r_1}{r}\right)^k \sqrt{\frac{\mu_o}{\rho_o u_o X}}$ given by equations (28) and (29). The values of these quantities very far downstream (that is, the values on a sharp cone) are also given. Again it is clear that the experimental results do not conflict with the theoretical ones.

Using the equations (28) and (29), writing R_δ for either R_{δ_1} or R_{δ_2} , and assuming that r_1 is proportional to x (as it is very nearly -except near the tip-on a blunt cone), we have

$$R_\delta = Ax^{\frac{1}{2}} - Bx^2/r^2 \quad (31)$$

where A and B are positive constants depending on the conditions in the flow upstream of the cone. This equation holds only for small x and for large r ; for large x or for small r , the value of R_δ is $A'x^2$ where A' is a constant (appropriate to the sharp cone) which is greater than A. Fig.11 is a sketch showing R_δ as a function of x^2 for cones of different tip radii in the same stream: as r increases, so do both the value of x at which R_δ departs

significantly from $Ax^{\frac{1}{2}}$ and also the value of x at which R_δ sensibly reaches $A'x^{\frac{1}{2}}$; it is therefore reasonable to fair in the curves as shown in the diagram for intermediate values of x . A line has been drawn at a constant value of R_δ , and this intersects curves corresponding to successively larger values of r in the points C_1, C_2, C_3 , etc.; it is evident from the diagram that the values of x at these points first increases and then slowly decreases again as r increases. If this constant value of R_δ is that at which transition to turbulence occurs, this means that as r increases from zero the position of transition moves first downstream and then upstream again. This is in agreement with the experiments of Ref.4.

An alternative way of expressing this is to regard (31) as a relation between x and r for a constant value of R_δ (this is not valid for large values of x/r , of course). If this value of x is plotted against r , a curve like the full line in Fig.12 is obtained. For a sharp cone, the value of x for the same value of R_δ is also shown on the graph, and it seems reasonable to complete the curve as shown by the dotted line. This again shows that, as r increases from zero, x first increases and then decreases again.

It is interesting to note that the value of x at transition for very large values of r is the same as that predicted by Moeckel². He, however, finds that this is the largest possible value of x at transition, whereas the present theory predicts that larger values are possible (and that, in any given free-stream conditions, there is an optimum value of r which gives the maximum value of x at transition). In order to predict this maximum value, a higher-order theory to that given in this paper is required, and an investigation into this is being made at the present time.

5 CONCLUSIONS

- (1) The theories of Li^{1,3} and Glauert² for the development of a boundary layer in a two-dimensional incompressible shear flow have been extended to apply to axisymmetric compressible shear flow.
- (2) The significance of the displacement thickness and the momentum thickness of such a boundary layer is discussed, and unique definitions are given which reduce to the usual form for a boundary layer developing in a uniform flow. A comparison is made with some experimental results⁴ obtained on a blunt cone: although the accuracy of the experiments was insufficient to give a positive confirmation of the theory, the results were at least not inconsistent with the theory.
- (3) Reynolds numbers based on displacement thickness and on momentum thickness are defined. It is shown that if transition to turbulence in the boundary layer occurs at a critical value of one of these, then, as the length scale of the flow increases from zero, the position of this transition moves first downstream and then slightly upstream again. This is in agreement with the experimental results on blunt cones.
- (4) The main justification of the theory of the present note seems to be the conclusion (3) above. To make further progress it would seem to be necessary either to perform a more accurate series of experiments, or to find a complete solution of the equations of motion.

LIST OF SYMBOLS

A_1, A, A', B	constants
\bar{a}, a, b, c	
C_1, C_2	points in Fig.10
D	momentum deficit due to presence of boundary layer
f_0, f_1	functions of η occurring in the stream function
k	in two dimensions, $k = 0$; in axisymmetric flow, $k = 1$
K	a constant
m	strength of source distribution equivalent to boundary layer
M	Mach number
p	static pressure in flow near surface
p_t	stagnation pressure in free-stream
r	length scale (e.g. radius of tip of cone)
r_1	distance of point $(x, 0)$ from axis of symmetry
R_x	Reynolds number based on x and conditions at edge of boundary layer
R_{δ_1}	Reynolds number based on displacement thickness and on conditions at the surface $y = \delta_1$ in external flow
R_{δ_2}	Reynolds number based on momentum thickness and on conditions at the surface $y = \delta_1$ in external flow
x, y	physical coordinates (see Fig.1)
X, Y	transformed coordinates (see equation (3))
Y'	dY/dx
\bar{Y}, \bar{Y}_0	transformed coordinates (see Appendix 3)
u, v	velocity components in x, y directions, respectively
δ	thickness of boundary layer
δ_1	displacement thickness of boundary layer
δ_2	momentum thickness of boundary layer
Δ, Δ_1	values of Y when $y = \delta, \delta_1$ respectively

LIST OF SYMBOLS (Contd)

η	non-dimensional coordinate (see equation (7))
$\bar{\eta}$	non-dimensional coordinate (see Appendix 3)
$\bar{\eta}_1, \bar{\eta}_2$	particular values of $\bar{\eta}$ as defined in Appendix 3
ξ	non-dimensional parameter (see equation (9))
λ	$\eta - 1.7207$
μ	viscosity
ρ	density
τ_w	shearing stress at the wall
ψ	stream function
Ω	non-dimensional velocity shear in external flow

Suffices

0	reference values (values at $x = 0, y = 0$) - except f_0
∞	values at outer edge of external shear layer (and on surface of sharp cone in inviscid flow)
e	values in external flow
e_0	values in external flow in absence of boundary layer
δ	values at edge of boundary layer

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APPENDIX 1

THE ASYMPTOTIC FORM OF THE VELOCITY

It is well known that the asymptotic solution of the first of equations (11) is

$$f_0 \sim \lambda + Ke^{-\lambda^2/4} \left[\frac{1}{\lambda^2} + o\left(\frac{1}{\lambda^4}\right) \right],$$

where $\lambda = \eta - 1.7207$ and K is a certain constant; it is easily verified that the asymptotic solution of the second equation takes the form

$$f_1 \sim \frac{1}{2} \lambda^2 + 3.4687 - \frac{1}{4} Ke^{-\lambda^2/4} \left[\lambda + o\left(\frac{1}{\lambda}\right) \right].$$

It follows that, near the edge of the boundary layer, the velocity, u , is given by

$$\frac{u}{u_0} \sim 1 + \lambda \xi + Ke^{-\lambda^2/4} \left[-\frac{1}{2\lambda} + \frac{1}{8} \lambda^2 \xi + \dots \right].$$

The second derivative of this quantity is

$$\frac{u''}{u_0} \sim Ke^{-\lambda^2/4} \left[-\frac{\lambda}{8} + \frac{1}{32} \lambda^4 \xi + \dots \right],$$

and we see that, at least to the order of the approximation of this note, there is always some value of λ (given by $\lambda^2 \doteq 4/\xi$) at which u'' vanishes: that is, there is always a point of inflexion in the curve of u as a function of λ (or η), whatever the value of ξ .

APPENDIX 2

DEFINITIONS OF DISPLACEMENT THICKNESS

We shall follow Lighthill⁶ and consider the two-dimensional incompressible case only (except in method 1, where we include compressibility). He considers four methods of defining the displacement thickness, δ_1 , for a boundary layer developing in a uniform flow, and shows that they are equivalent; we do the same here where the boundary layer develops in a shear flow.

(1) Flow reduction: the surface $y = 0$ is displaced to the position $y = \delta_1(x)$. Then $\delta_1(x)$ is such that, if there existed a flow (which will subsequently be referred to as the fictitious flow along $y = \delta_1(x)$) along this surface which is without viscosity and which has the same tangential velocity component as the given flow far from the surface (that is, $u = u_e$ far from the surface), then the mass flow in the fictitious flow is the same as that in the given flow (including the boundary layer) along $y = 0$. Then we have

$$\int_{\delta_1}^{\infty} \rho_e u_e dy = \int_0^{\infty} \rho u dy \quad (32)$$

or

$$\int_0^{\delta_1} \rho_e u_e dy = \int_0^{\infty} (\rho_e u_e - \rho u) dy \quad (33)$$

In the incompressible case, where ρ is constant everywhere, this reduces to the form

$$\int_0^{\delta_1} u_e dy = \int_0^{\infty} (u_e - u) dy \quad (34)$$

(2) Equivalent sources: here we consider the normal component of velocity just outside the boundary layer. This is

$$\begin{aligned} v &= \int_0^y \frac{\partial v}{\partial y} dy = - \int_0^y \frac{\partial u}{\partial x} dy = \int_0^{\infty} \frac{\partial}{\partial x} (u_e - u) dy - \int_0^y \frac{\partial u_e}{\partial x} dy \\ &= \frac{d}{dx} \int_0^{\infty} (u_e - u) dy + \int_0^y \frac{\partial v_e}{\partial y} dy \quad (35) \end{aligned}$$

The second term in this expression is the normal component of velocity due to a flow along the surface $y = 0$ which has no viscosity and which has the

same tangential component of velocity as the given flow far from the surface. The first term can be regarded as the normal velocity due to a distribution of sources of strength (volume flow rate) per unit area

$$m = \frac{d}{dx} \int_0^{\infty} (u_e - u) dy$$

along the surface $y = 0$. Now the flow of the "new" fluid past any point must equal the total outflow from the part of the surface between that point and the point of attachment, and this is

$$\int_0^x m dx = \int_0^{\delta_1} u_e dy \quad ;$$

thus the new fluid just fills a region, adjacent to the body, of thickness δ_1 as defined in (34).

(3) Velocity comparison: here we look for a streamline surface $y = \delta_1(x)$ such that the fictitious flow along it (see above) has the same normal velocity component as the given flow to the first order of small quantities. The boundary conditions of such a flow are

$$(i) \text{ at } y = \delta_1(x), \quad v_e = u_e(\delta_1) \delta_1'(x)$$

$$\begin{aligned} \text{and } (ii) \text{ when } y \rightarrow \infty, \quad v_e &= v_e(\delta_1) + \left(\frac{\partial v_e}{\partial y}\right)_{\delta_1} (y - \delta_1) + \frac{1}{2} \left(\frac{\partial^2 v_e}{\partial y^2}\right)_{\delta_1} (y - \delta_1)^2 + \dots \\ &= u_e(\delta_1) \delta_1'(x) - \left(\frac{\partial u_e}{\partial y}\right)_{\delta_1} (y - \delta_1) - \frac{1}{2} \left(\frac{\partial^2 u_e}{\partial x \partial y}\right)_{\delta_1} (y - \delta_1)^2 + \dots \end{aligned}$$

But outside the boundary layer, we have equation (35), and the two values of v must be equal outside the boundary layer. Hence

$$\begin{aligned} \frac{d}{dx} \left[\int_0^{\infty} (u_e - u) dy \right] &= \int_0^y \frac{\partial u_e}{\partial x} dy + u_e(\delta_1) \delta_1'(x) - \left(\frac{\partial u_e}{\partial x}\right)_{\delta_1} (y - \delta_1) - \frac{1}{2} \left(\frac{\partial^2 u_e}{\partial x \partial y}\right)_{\delta_1} (y - \delta_1)^2 + \dots \\ &= \int_0^y \frac{\partial u_e}{\partial x} dy + u_e(\delta_1) \delta_1'(x) - \int_{\delta_1}^y \left(\frac{\partial u_e}{\partial x}\right) dy \\ &= u_e(\delta_1) \delta_1'(x) + \int_0^{\delta_1} \frac{\partial u_e}{\partial x} dy \end{aligned}$$

and finally

$$\frac{d}{dx} \left[\int_0^{\infty} (u_e - u) dy \right] = \frac{d}{dx} \left[\int_0^{\delta_1} u_e dy \right] .$$

Since $u = u_e$ and $\delta_1 = 0$ when $x = 0$, this is equivalent to equation (34).

(4) Mean vorticity: here we replace the boundary layer by a vortex sheet at the mean distance $y = \delta_1(x)$, so that for y less than δ_1 , $u = 0$, and for large values of y , $u = u_e$ (which is the given flow). Then we can define a mean vorticity of the boundary layer by means of the equation

$$\delta_1 \left[\int_0^{\infty} \frac{\partial u}{\partial y} dy - \int_{\delta_1}^{\infty} \frac{\partial u_e}{\partial y} dy \right] = \int_0^{\infty} y \frac{\partial u}{\partial y} dy - \int_{\delta_1}^{\infty} y \frac{\partial u_e}{\partial y} dy ,$$

where the excess vorticity present in the boundary layer only is considered, and where it is assumed that the term $\partial v/\partial x$ in the expression for the vorticity is much smaller than the term $\partial u/\partial y$. Manipulation of the equation leads, as before, to the relation (34).

APPENDIX 3

EVALUATION OF THE DISPLACEMENT THICKNESS, δ_1 ,
AND THE MOMENTUM THICKNESS, δ_2

We note first that the asymptotic solutions for $f_0(\eta)$ and $f_1(\eta)$ are

$$f_0(\eta) \sim \eta - 1.7207 \quad ,$$

and

$$f_1(\eta) \sim \eta^2/2 - a\eta + b \quad ,$$

where the constants a and b are different in the solutions of Glauert² and Li³, and are tabulated below. In this table are given also the constant A_1 of equation (11) in the main text, the constant $c = f_1''(0)$, and the constant $c' = (1.7207a + 2b + 2 - 2c)/4$ which occurs in one of the integrals below.

Author	A_1	a	b	c	c'
Glauert	0	1.7207	4.9490	0.7950	3.3171
Li	-1.7207	0	7.1879	2.9525	2.6177

Substituting the asymptotic values given above for $f_0(\eta)$ and $f_1(\eta)$ in the second of equations (11), we find the relation $a - 1.7207 = A_1$; this is quite general and does not depend on the nature of the boundary conditions taken at the outer edge of the boundary layer. The relation is used below, and it is only in the final substitution to obtain the four equations (44a), (44b), (46a) and (46b) that the results of this Appendix are made dependent on the form of the boundary conditions taken.

Before proceeding to the main business of this Appendix, it is convenient to list four integrals which will occur later. These can be evaluated using the differential equations (11) and the initial conditions of (12); the asymptotic forms of these integrals are also given:

$$\int_0^{\eta} f_0'(\eta) d\eta = f_0(\eta) \sim \eta - 1.7207$$

$$\int_0^{\eta} f_1'(\eta) d\eta = f_1(\eta) \sim \eta^2/2 - a\eta + b$$

$$\int_0^{\eta} [f_0'(\eta)]^2 d\eta = f_0'(\eta) f_0(\eta) + 2f_0''(\eta) - 2f_0''(0) \sim \eta - 2.3847$$

$$\int_0^{\eta} f_0'(\eta) f_1'(\eta) d\eta = \frac{1}{4} f_0(\eta) f_1'(\eta) + \frac{1}{2} f_1(\eta) f_0'(\eta) + \frac{1}{2} f_1''(\eta) - \frac{1}{2} f_1''(0) - \frac{1}{4} A_1 \eta$$

$$\sim \eta^2/2 - a\eta + c' \quad ,$$

where $c' = (1.7207a + 2b + 2 - 2c)/4$.

We now proceed to evaluate the displacement thickness, using the definition (19) in the main text. The rate of mass flow across the line joining the points $(x,0)$ and (x,y) , where y is large, in the flow of the viscous fluid along the surface $y = 0$ is the asymptotic value of the integral

$$\int_0^y \rho u dy = \rho_0 u_0 \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_0 X}{\rho_0 u_0}} \int_0^{\eta} (f_0' + \xi f_1' + \dots) d\eta$$

$$\sim \rho_0 u_0 \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_0 X}{\rho_0 u_0}} \left[\eta - 1.7207 + \xi(\eta^2/2 - a\eta + b) + \dots \right], \quad (36)$$

where

$$y = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_0 X}{\rho_0 u_0}} \int_0^{\eta} \frac{\rho_0}{\rho} d\eta$$

$$= \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_0 X}{\rho_0 u_0}} \int_0^{\eta} \left[1 + 0.2 M_0^2 (1 - f_0'^2 - 2\xi f_0' f_1') + \dots \right] d\eta$$

$$\sim \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_0 X}{\rho_0 u_0}} \left[\eta + 0.47694 M_0^2 - 0.4 M_0^2 \xi(\eta^2/2 - a\eta + c') + \dots \right]$$

... (37)

Now for large η , the velocity profile is of the form

$$\frac{u}{u_0} \sim 1 + \xi(\eta - a) + \dots \quad ,$$

and there is a fictitious inviscid flow along the surface $y = \delta_1(x)$ which is the same as this for large η . This flow is given by the relation

$$\frac{u_e}{u_0} = 1 + \xi(\bar{\eta} - \bar{a}) + \dots \quad ,$$

where $\bar{\eta}$ is a new variable (analogous to η) given by

$$\bar{\eta} = \left(\frac{r_1}{r}\right)^k \sqrt{\frac{\rho_0 u_0}{\mu_0 X}} \int_0^y \frac{\rho_e}{\rho_0} dy \quad , \quad (38)$$

and \bar{a} is a constant. Since for large y , $u \sim u_e$, we must have

$$\eta - a = \bar{\eta} - \bar{a} \quad . \quad (39)$$

Inverting the relation (38), we have

$$\begin{aligned} y &= \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_0 X}{\rho_0 u_0}} \int_0^{\bar{\eta}} \frac{\rho_0}{\rho_e} d\bar{\eta} \\ &= \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_0 X}{\rho_0 u_0}} \left[\bar{\eta} - 0.4 M_0^2 \xi(\bar{\eta}^2/2 - \bar{a} \bar{\eta}) + \dots \right] \quad . \quad (40) \end{aligned}$$

Comparing this relation with (37), we have that, for the same (large) values of y ,

$$\begin{aligned} \eta - \bar{\eta} = a - \bar{a} &= -0.47694 M_0^2 + 0.4 M_0^2 \xi \left\{ (\bar{\eta} - 0.47694 M_0^2)^2/2 - \bar{\eta}^2/2 + \right. \\ &\quad \left. + 0.47694 M_0^2 (\bar{\eta} + a) + c' \right\} + \dots \quad . \\ &\dots \quad (41) \end{aligned}$$

Substituting from the equation (41) in the expression (36) for the mass flow rate, we have

$$\begin{aligned}
\int_0^y \rho u \, dy &\sim \rho_0 u_0 \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_0 X}{\rho_0 u_0}} \left[\bar{\eta} - 1.7207 - 0.47694 M_0^2 + \right. \\
&+ \xi \left\{ \left(1 + 0.4 M_0^2\right) \left(\bar{\eta} - 0.47694 M_0^2\right)^2 / 2 - a \left(\bar{\eta} - 0.47694 M_0^2\right) + b + \right. \\
&\left. \left. + 0.4 M_0^2 \left(-\bar{\eta}^2 / 2 + 0.47694(\bar{\eta} + a) M_0^2 + c'\right) \right\} + \dots \right]. \quad (42)
\end{aligned}$$

But the rate of mass flow across the line joining the points (x, δ_1) and (x, y) , where y is large, in the fictitious inviscid flow along $y = \delta_1(x)$ is the integral

$$\begin{aligned}
\int_{\delta_1}^y \rho_e u_e \, dy &= \rho_0 u_0 \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_0 X}{\rho_0 u_0}} \int_{\bar{\eta}_1}^{\bar{\eta}} [1 + \xi(\bar{\eta} - \bar{a}) + \dots] \, d\bar{\eta} \\
&= \rho_0 u_0 \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_0 X}{\rho_0 u_0}} \left[\bar{\eta} - \bar{\eta}_1 + \xi \left\{ \frac{(\bar{\eta} - \bar{a})^2}{2} - \frac{(\bar{\eta}_1 - \bar{a})^2}{2} \right\} + \dots \right],
\end{aligned}$$

where $\bar{\eta} = \bar{\eta}_1$ when $y = \delta_1$. This expression has to be equated with the right hand side of equation (42). After some manipulation, the result of this is that

$$\begin{aligned}
\bar{\eta}_1 &= 1.7207 + 0.47694 M_0^2 + \xi \left[1.7207a - \left(\frac{1.7207}{2}\right)^2 - b - 0.4 M_0^2 c' - \right. \\
&\quad \left. - 0.4 M_0^2 \left(0.47694 M_0^2\right) \left(\frac{0.47694 M_0^2}{2} + a\right) \right] + \dots \\
&\dots \quad (43)
\end{aligned}$$

But we have, using (40), that

$$\delta_1 = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_0 X}{\rho_0 u_0}} \left[\bar{\eta}_1 - 0.4 M_0^2 \xi (\bar{\eta}_1^2 / 2 - \bar{a} \bar{\eta}_1) + \dots \right].$$

Substituting in this expression from equation (43), we find that

$$\begin{aligned}
\delta_1 &= \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_0 X}{\rho_0 u_0}} \left[1.7207 + 0.47694 M_0^2 - \right. \\
&\quad \left. - \xi \left\{ b + 0.4 M_0^2 c' + 1.7207(1 + 0.4 M_0^2) \left(\frac{1.7207}{2} - a\right) \right\} + \dots \right]. \\
&\dots \quad (44)
\end{aligned}$$

Using the values for the constants given in the table at the beginning of this Appendix, we have that, using Glauert's results,

$$\delta_1 = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_o X}{\rho_o u_o}} \left[1.7207 + 0.47694M_o^2 - (3.4687 + 0.73472M_o^2) \xi + \dots \right]; \quad \dots (44a)$$

and, using Li's results,

$$\delta_1 = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_o X}{\rho_o u_o}} \left[1.7207 + 0.47694M_o^2 - (8.6683 + 1.6392M_o^2) \xi + \dots \right]. \quad \dots (44b)$$

In order to calculate the momentum thickness, δ_2 , it is necessary to evaluate the integral

$$\begin{aligned} \int_{\delta_1}^{\delta_1+\delta_2} \rho_e u_e^2 dy &= \rho_o u_o^2 \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_o X}{\rho_o u_o}} \int_{\bar{\eta}_1}^{\bar{\eta}_1+\bar{\eta}_2} [1 + 2 \xi(\bar{\eta} - \bar{a}) + \dots] d\bar{\eta} \\ &= \rho_o u_o^2 \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_o X}{\rho_o u_o}} \left[\bar{\eta}_2 + \xi \left\{ \bar{\eta}_2^2 + 2\bar{\eta}_2 (\bar{\eta}_1 - \bar{a}) \right\} + \dots \right], \end{aligned}$$

where $\bar{\eta} = \bar{\eta}_1 + \bar{\eta}_2$ when $y = \delta_1 + \delta_2$.

Using equations (41) and (43) in equation (22) we find after some manipulation that

$$D = \rho_o u_o^2 \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_o X}{\rho_o u_o}} \left[0.664 - \xi \left\{ 1 - \sigma + \frac{1.7207}{2} (1.7207 - a) \right\} + \dots \right].$$

Using this and the definition (24), we have

$$0.664 - \left[1 - \sigma + \frac{1.7207}{2} (1.7207 - a) \right] \xi + \dots = \bar{\eta} + \xi \left[\bar{\eta}_2^2 + 2\bar{\eta}_2 (\bar{\eta}_1 - a) \right] + \dots$$

Thus

$$\bar{\eta}_2 = 0.664 - \xi(5.206 - \sigma - 2.188a) + \dots \quad (45)$$

Using the relation (40) to find expressions for $\bar{\eta}_1 + \bar{\eta}_2$ and $\bar{\eta}_1$, it is possible to show that

$$\delta_2 = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_o X}{\rho_o u_o}} \left[\bar{\eta}_2 - 0.4 M_o^2 \xi \left\{ \bar{\eta}_2^2/2 + \bar{\eta}_2(\bar{\eta}_1 - \bar{a}) \right\} + \dots \right],$$

and, using the relations (43) and (45) in this expression, we have

$$\delta_2 = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_o X}{\rho_o u_o}} \left[0.664 - \xi \left\{ 5.206 - \sigma - 2.188a + \right. \right. \\ \left. \left. + (0.5452 - 0.2656a) M_o^2 \right\} + \dots \right].$$

... (46)

Using Glauert's results, this is

$$\delta_2 = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_o X}{\rho_o u_o}} \left[0.664 - (0.646 + 0.0882 M_o^2) \xi + \dots \right]$$

... (46a)

and, using Li's results,

$$\delta_2 = \left(\frac{r}{r_1}\right)^k \sqrt{\frac{\mu_o X}{\rho_o u_o}} \left[0.664 - (2.254 + 0.5452 M_o^2) \xi + \dots \right]$$

... (46b)

TABLE 1

The function $f'_0(\eta)$ and $f'_1(\eta)$

η	$f'_0(\eta)$	$f'_1(\eta)$ (Glauert)	$f'_1(\eta)$ (Li and Murray)
0.0	0.0000	0.0000	0.0000
0.2	0.0664	0.1590	
0.4	0.1328	0.3179	1.1810
0.6	0.1989	0.4763	
0.8	0.2647	0.6336	2.2174
1.0	0.3298	0.7896	
1.2	0.3938	0.9429	3.0940
1.4	0.4563	1.0929	
1.6	0.5168	1.2385	3.7922
1.8	0.5748	1.3790	
2.0	0.6298	1.5137	4.3000
2.2	0.6813	1.6423	
2.4	0.7290	1.7649	4.6217
2.6	0.7725	1.8821	
2.8	0.8115	1.9947	4.7860
3.0	0.8460	2.1043	
3.2	0.8761	2.2126	4.8440
3.4	0.9018	2.3214	
3.6	0.9233	2.4328	4.8593
3.8	0.9411	2.5487	
4.0	0.9555	2.6706	4.8915
4.2	0.9670	2.7998	
4.4	0.9759	2.9371	4.9829
4.6	0.9827	3.0829	
4.8	0.9878	3.2371	5.1517
5.0	0.9915	3.3993	
5.2	0.9942	3.5688	5.3957
5.4	0.9962	3.7448	
5.6	0.9975	3.9263	5.7005
5.8	0.9984	4.1123	
6.0	0.9990	4.3021	6.0475
6.2	0.9994	4.4947	
6.4	0.9996	4.6895	6.4207
6.6	0.9998	4.8859	
6.8	0.9999	5.0835	6.4042
7.0	1.0000	5.2820	
7.2	1.0000	5.4810	7.2031
7.4	1.0000	5.6803	
7.6	1.0000	5.8799	7.6010
7.8	1.0000	6.0797	
8.0	1.0000	6.2796	8.0003
8.2	1.0000	6.4795	
8.4	1.0000	6.6794	8.4001
8.6	1.0000	6.8794	
8.8	1.0000	7.0794	8.8000
9.0	1.0000	7.2794	9.0000

For large values of η , $f'_0(\eta) \sim \eta - 1.7207$

$f'_1(\eta) \sim \frac{1}{2}(\eta - 1.7207)^2 + 3.4687$ according to Glauert
 or $\sim \frac{1}{2}\eta^2 + 7.1879$ according to Li and Murray

TABLE 2

Experimental values on blunt cones having an included angle of 15°

(a) Free-stream Mach number = 3.12

$M_0 = 2.14$, $M_\infty = 2.92$, $\Omega = 0.2$ from Fig.5

r (in.)	x (in.)	P _t (in.Hg)	M _δ	δ ₁ (in.)	δ ₂ (in.)	R _{δ₁}	R _{δ₂}	ξ (from Fig.5)
0.49	3.4	105.5	2.225	11.81×10 ⁻³	1.88×10 ⁻³	3090	492	7.78×10 ⁻⁴
0.49	5.4	105.5	2.225	13.61	2.15	3560	562	1.55×10 ⁻³
0.16	3.7	105.5	2.32	10.40	1.67	2720	437	8.26
0.16	5.5	105.5	2.40	16.29	1.57	4260	411	1.50×10 ⁻²
0.083	3.5	108	2.49	6.02	0.97	1614	260	2.80
0.083	4.2	108	2.63	8.38	1.31	2245	351	3.68
0.048	3.6	104	2.86	7.03	1.04	1813	268	8.88
0.048	4.4	104	2.90	6.34	0.99	2942	459	1.36×10 ⁻¹

(b) Free-stream Mach number = 3.81

$M_0 = 2.34$, $M_\infty = 3.35$, $\Omega = 0.14$ from Fig.5

r (in.)	x (in.)	P _t (in.Hg)	M _δ	δ ₁ (in.)	δ ₂ (in.)	R _{δ₁}	R _{δ₂}	ξ (from Fig.5)
0.49	3.4	91.5	2.32	13.36×10 ⁻³	1.56×10 ⁻³	1809	211	7.57×10 ⁻⁴
0.49	5.4	91.5	2.33	13.85	1.94	1875	263	1.51×10 ⁻³
0.16	5.5	100	2.39	7.38	1.08	1094	160	1.39×10 ⁻²
0.083	3.2	143	2.66	7.46	1.77	1578	374	1.92
0.083	4.2	143	2.90	6.00	1.41	1270	298	2.88
0.048	3.6	145	2.98	8.53	1.13	1830	242	6.80
0.048	4.4	145	3.14	8.29	1.00	3576	428	9.34

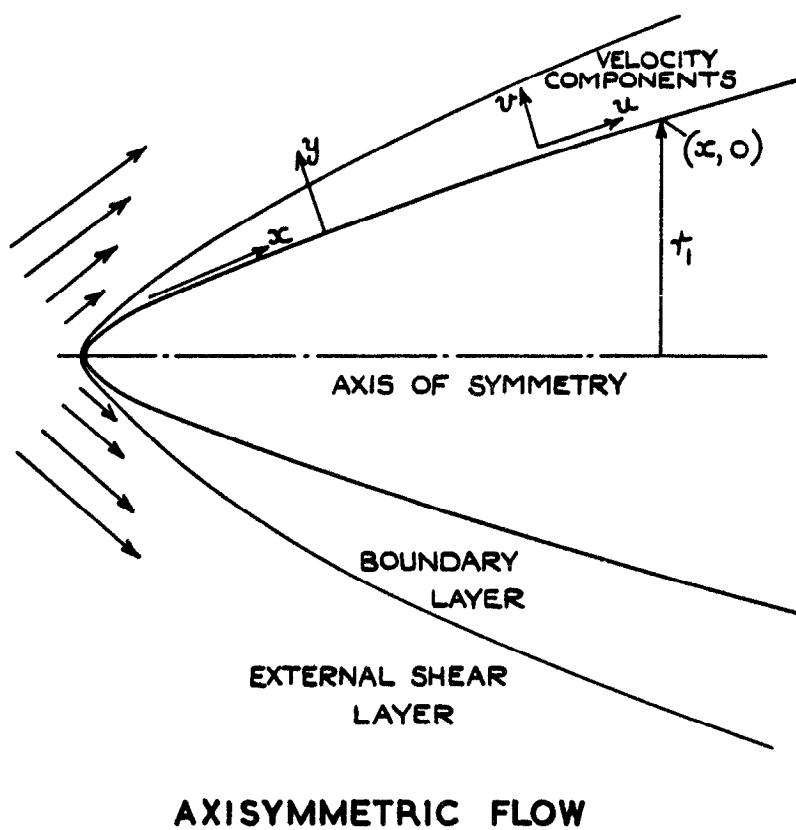
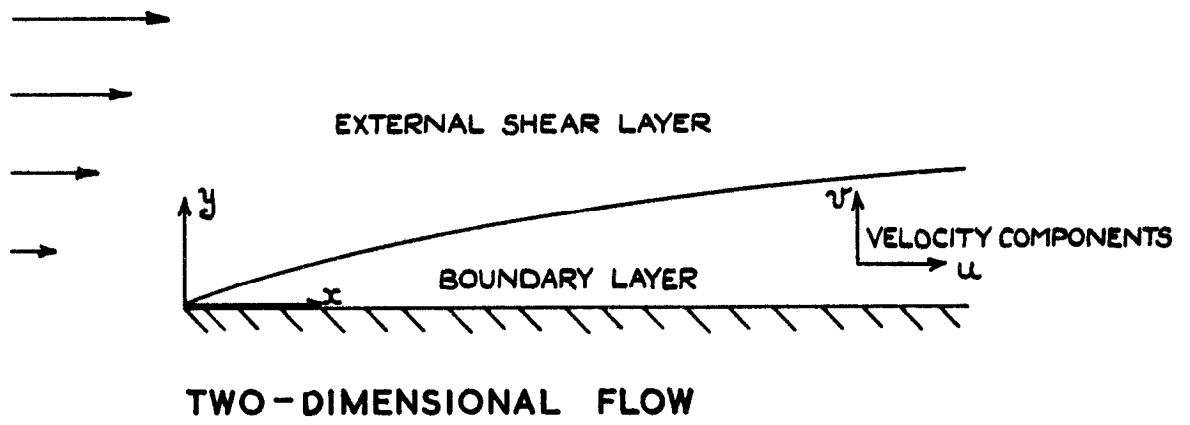


FIG.1. THE NOTATION USED.

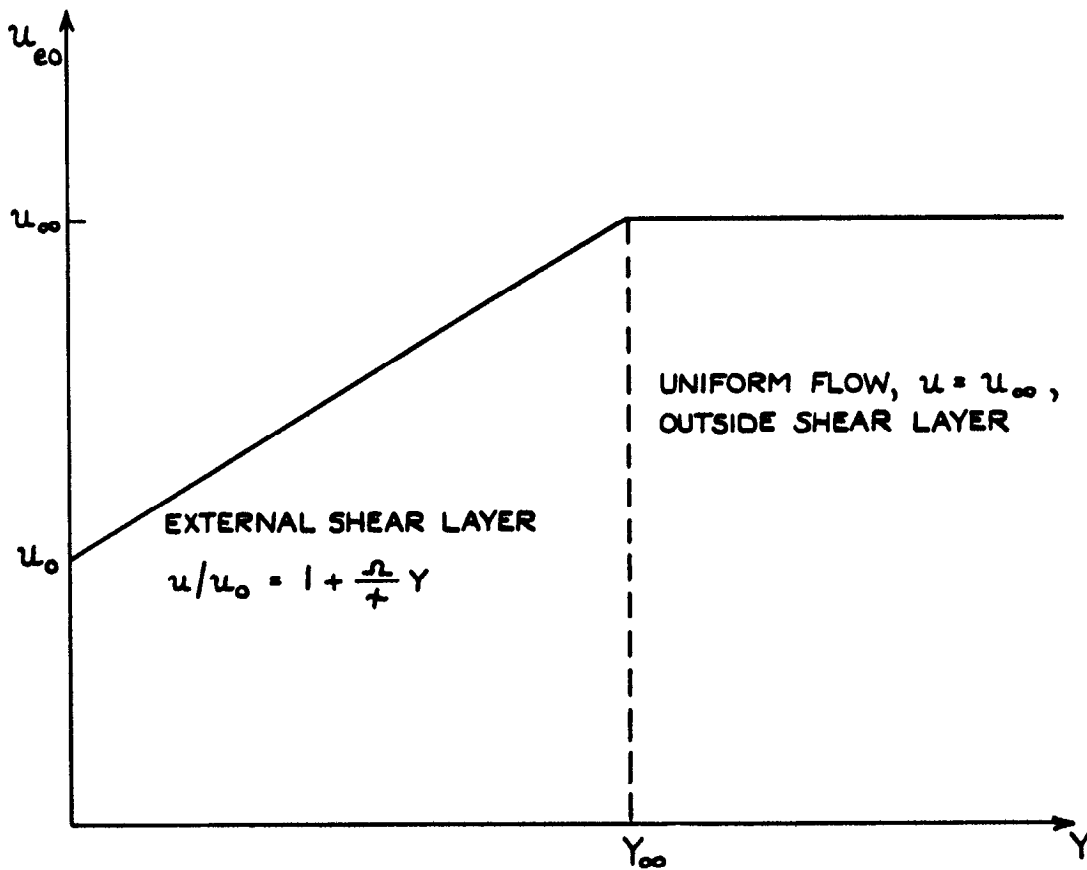


FIG. 2. THE MODEL USED FOR THE EXTERNAL FLOW AT $x = 0$.

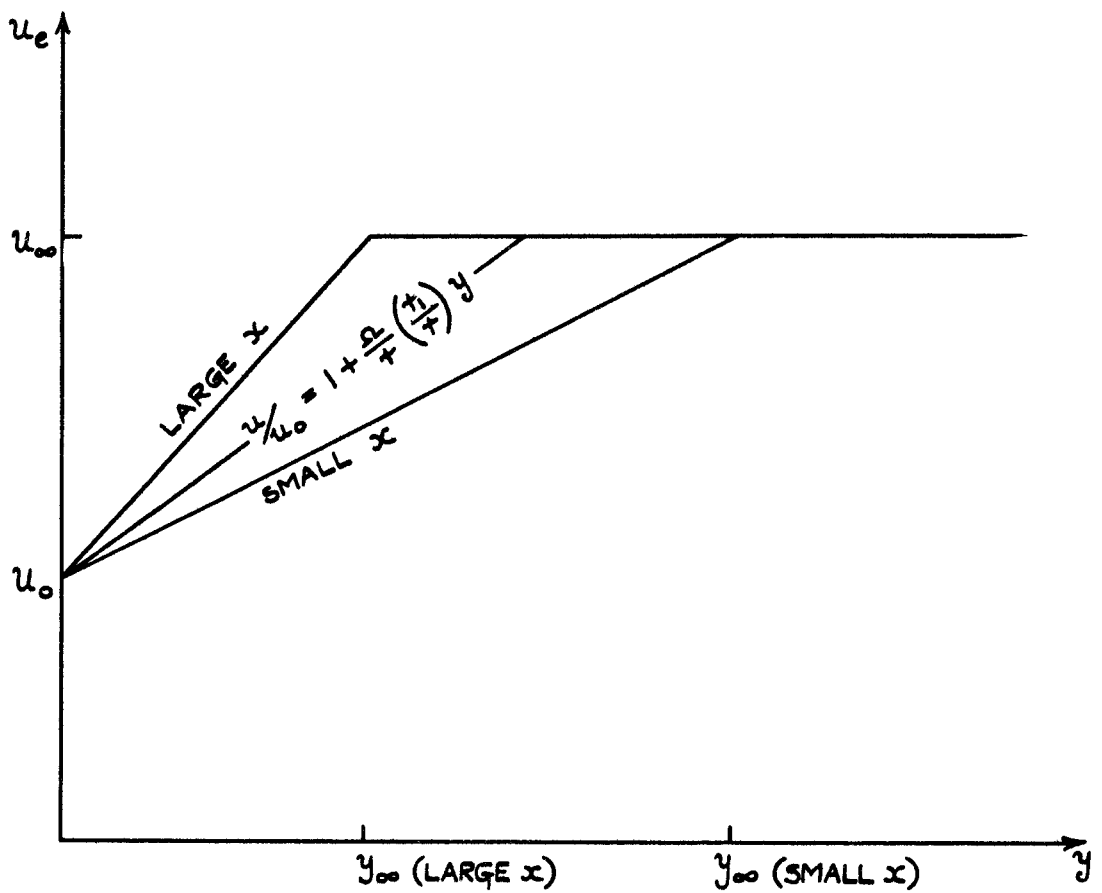


FIG.3. THE THINNING OF THE EXTERNAL SHEAR LAYER DOWNSTREAM ON AN AXISYMMETRIC BODY IN AN INCOMPRESSIBLE FLUID.

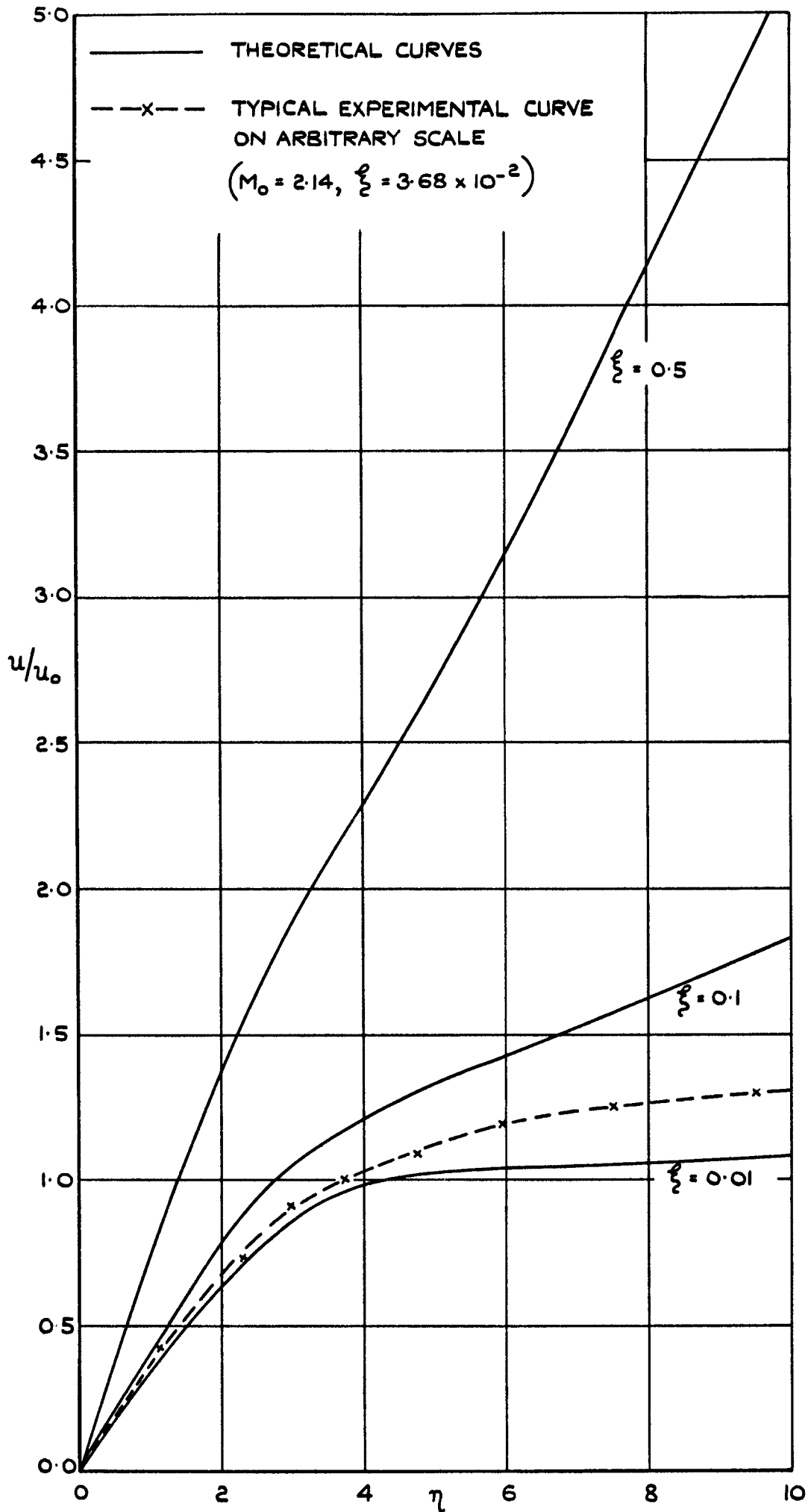


FIG. 4. THE CALCULATED VELOCITY PROFILE AT A GIVEN STATION.
 (USING GLAUERT'S SOLUTION)

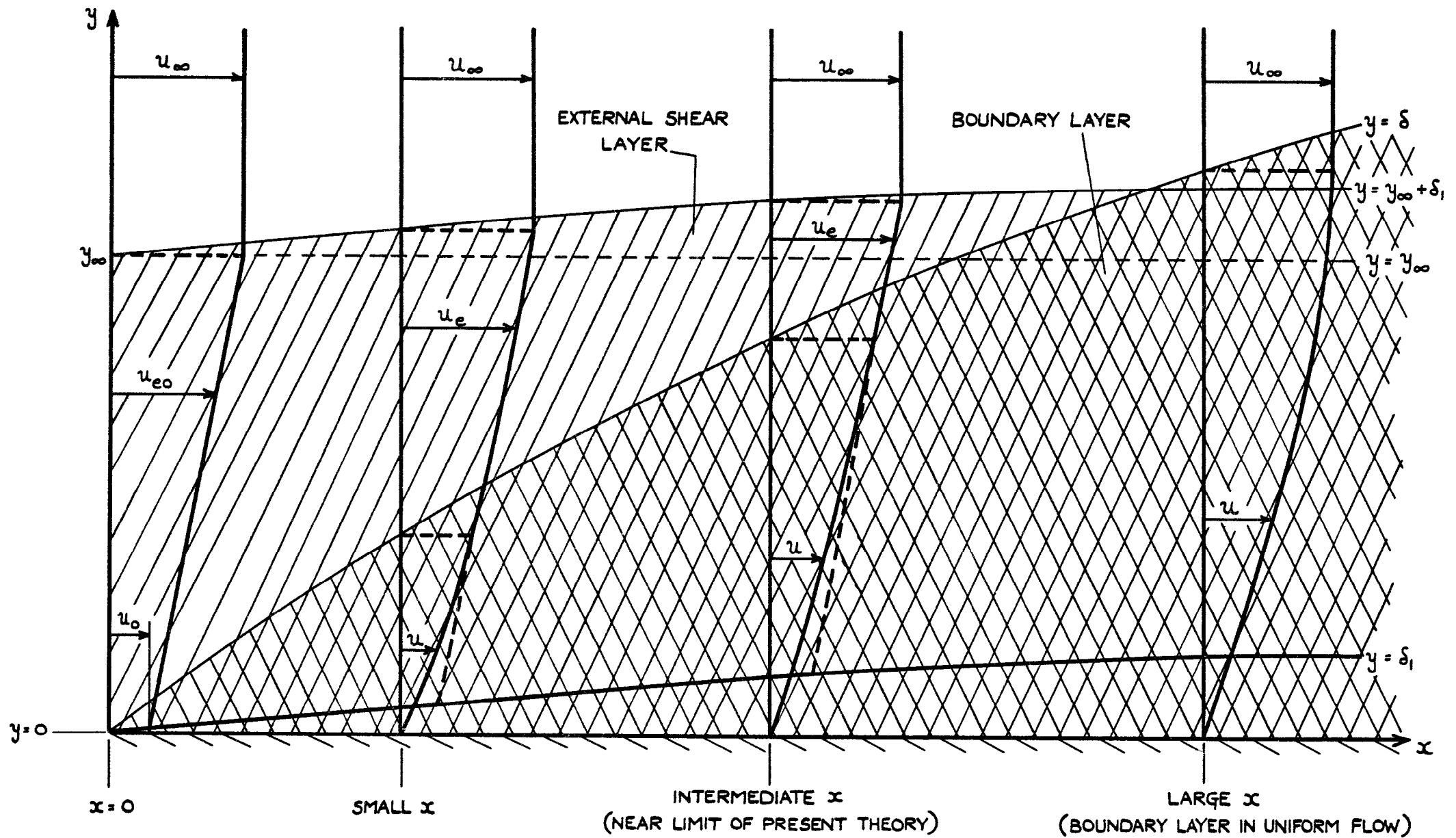
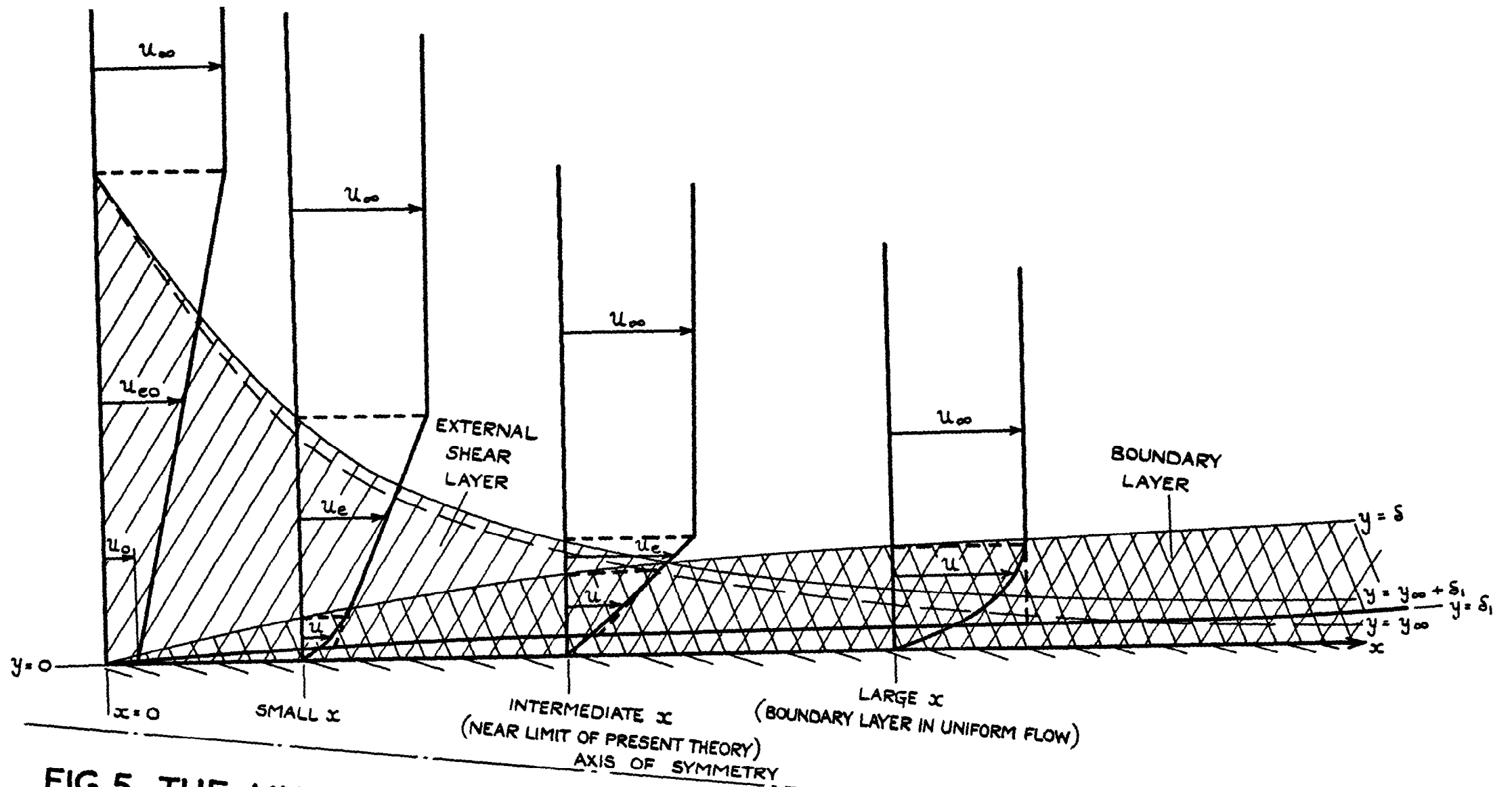
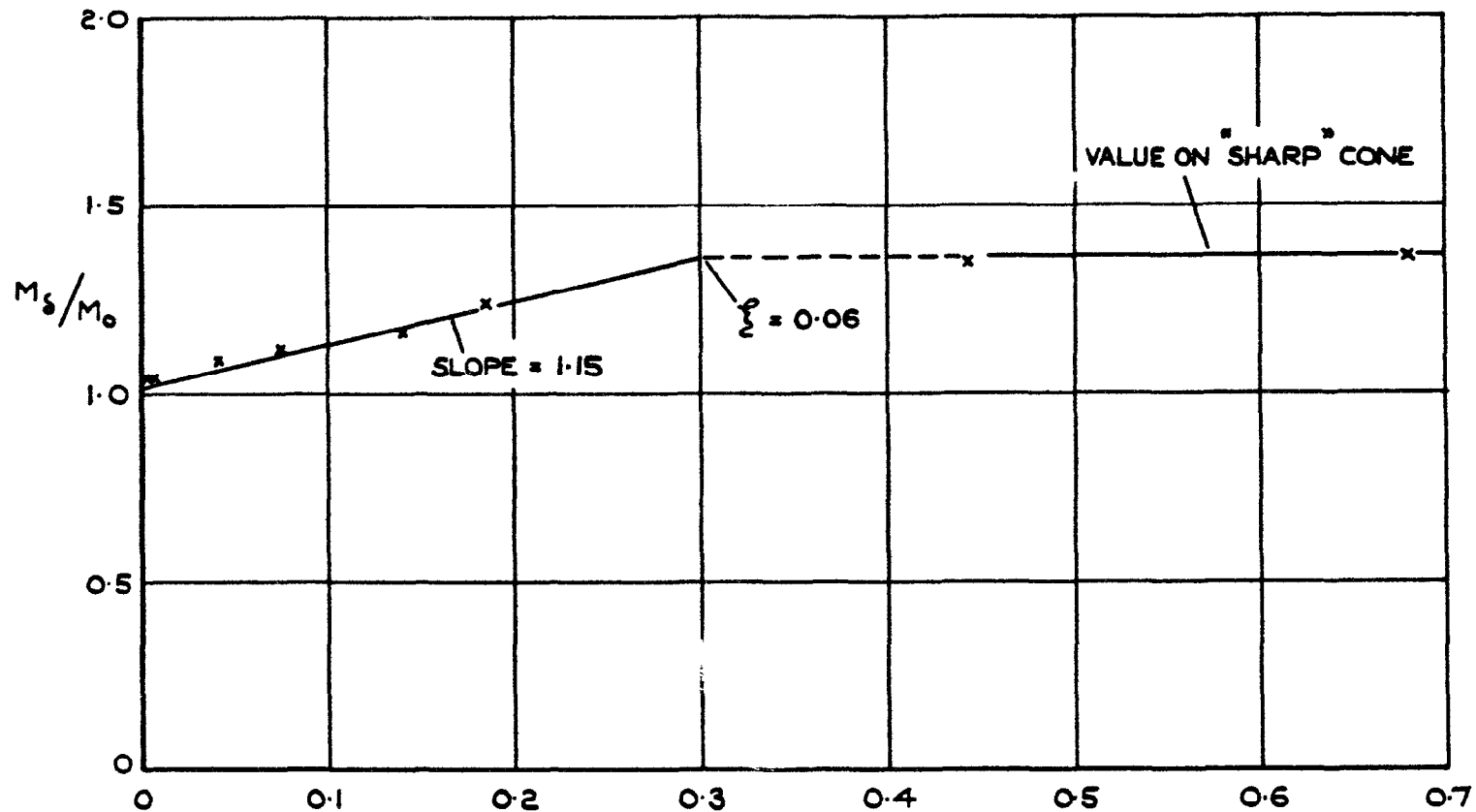


FIG. 5. THE VARIATION OF THE VELOCITY PROFILE WITH DISTANCE DOWNSTREAM.

(a) TWO-DIMENSIONAL CASE



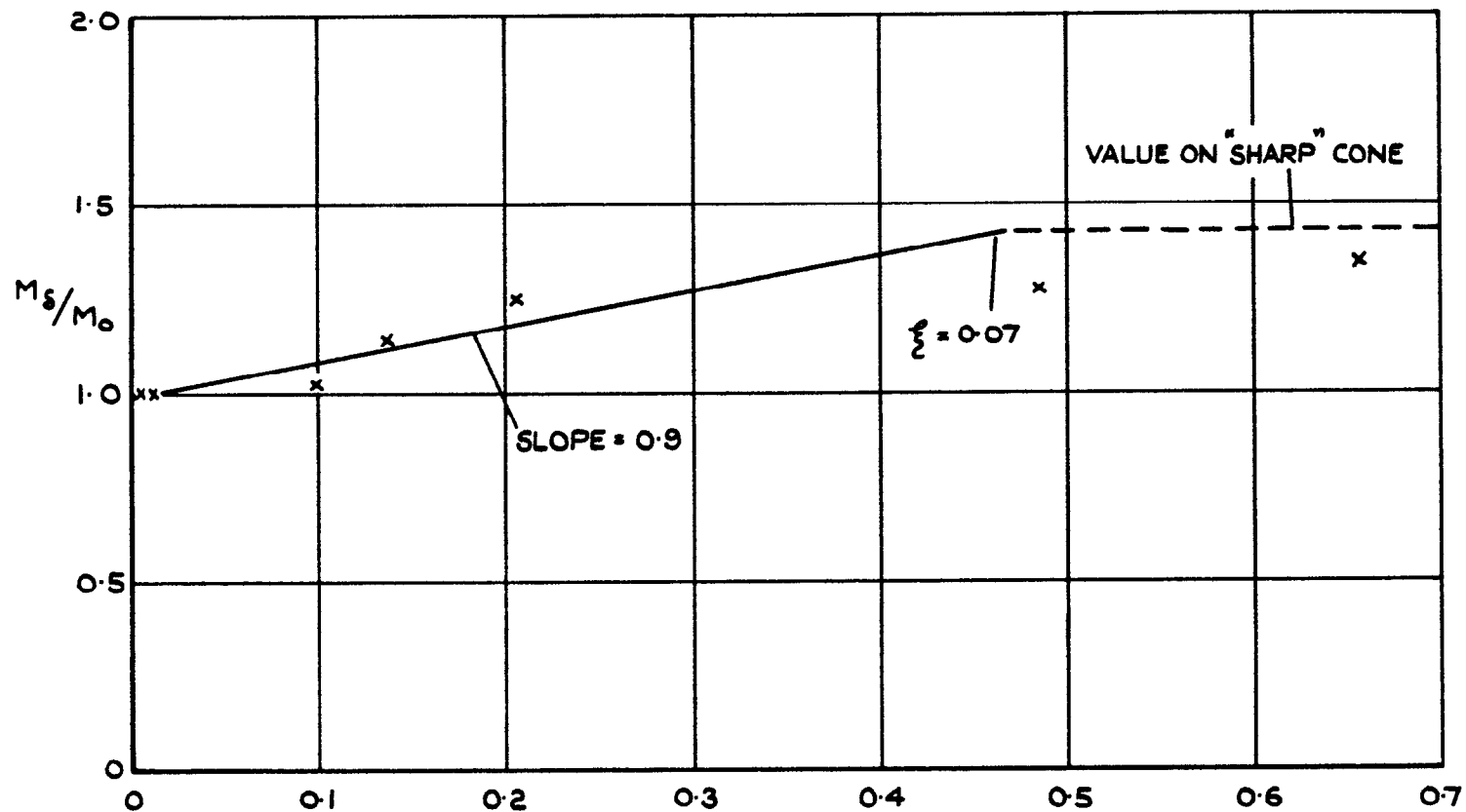
**FIG. 5. THE VARIATION OF THE VELOCITY PROFILE WITH DISTANCE
 DOWNSTREAM.**
(b) AXISYMMETRIC CASE



$$\frac{1}{\tau} \sqrt{\left(\frac{\mu_0 x}{\rho_0 u_0} \right)} = \xi / \Omega$$

FIG. 6. EXPERIMENTAL VARIATION WITH ξ OF THE MACH NUMBER, M_s , AT THE EDGE OF THE BOUNDARY LAYER ON A BLUNT CONE.

(a) FREE-STREAM MACH NUMBER = 3.12 ($M_0 = 2.14$)



$$\frac{1}{r} \sqrt{\left(\frac{\mu_0 x}{\rho_0 u_0}\right)} = \xi / \Omega$$

FIG. 6. EXPERIMENTAL VARIATION WITH ξ OF THE MACH NUMBER, M_s , AT THE EDGE OF THE BOUNDARY LAYER ON A BLUNT CONE.

(b) FREE-STREAM MACH NUMBER = 3.81 ($M_0 = 2.34$)

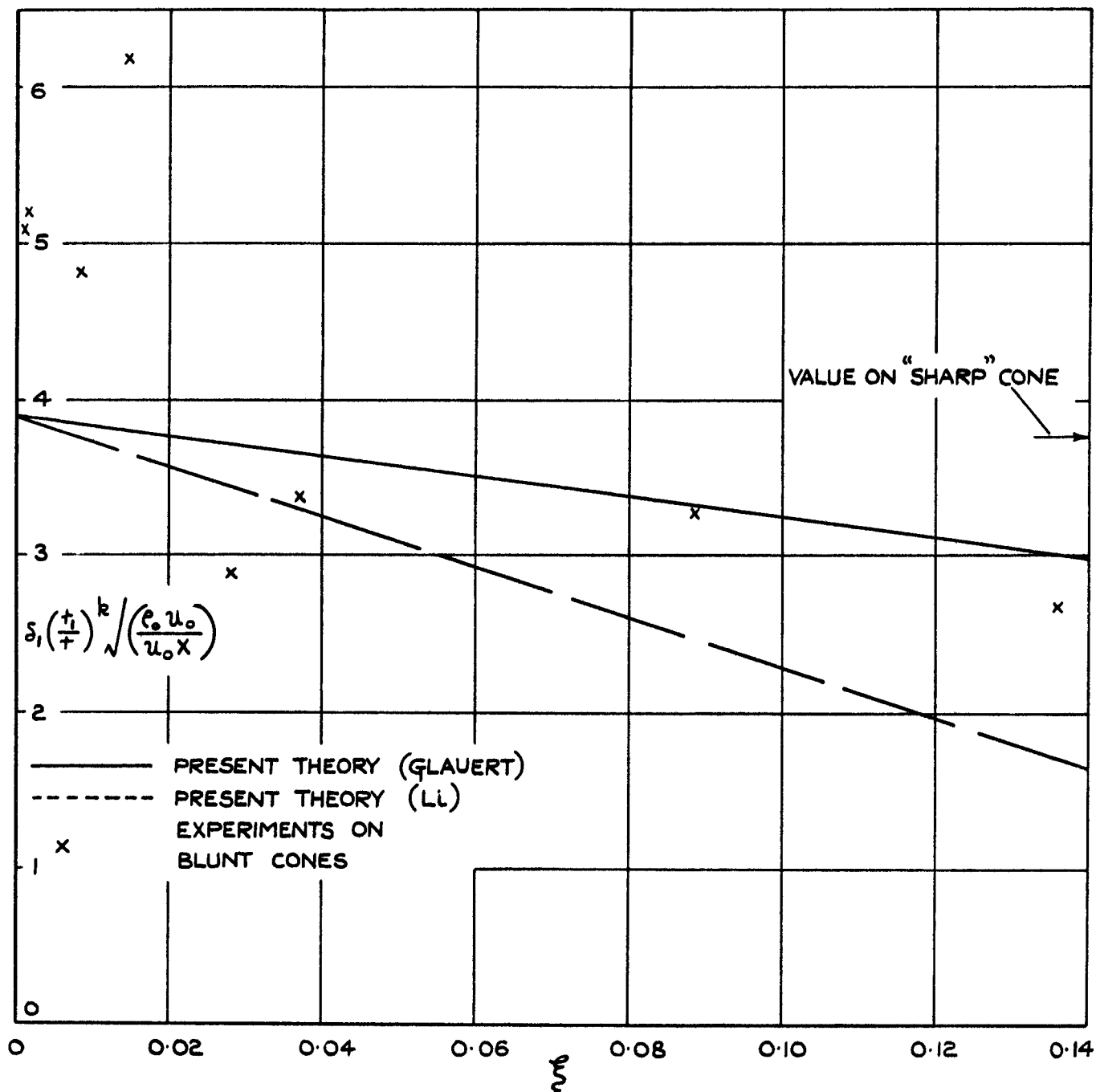


FIG. 7. VARIATION WITH ξ OF THE DISPLACEMENT THICKNESS, δ_1
 (a) $M_0 = 2.14$

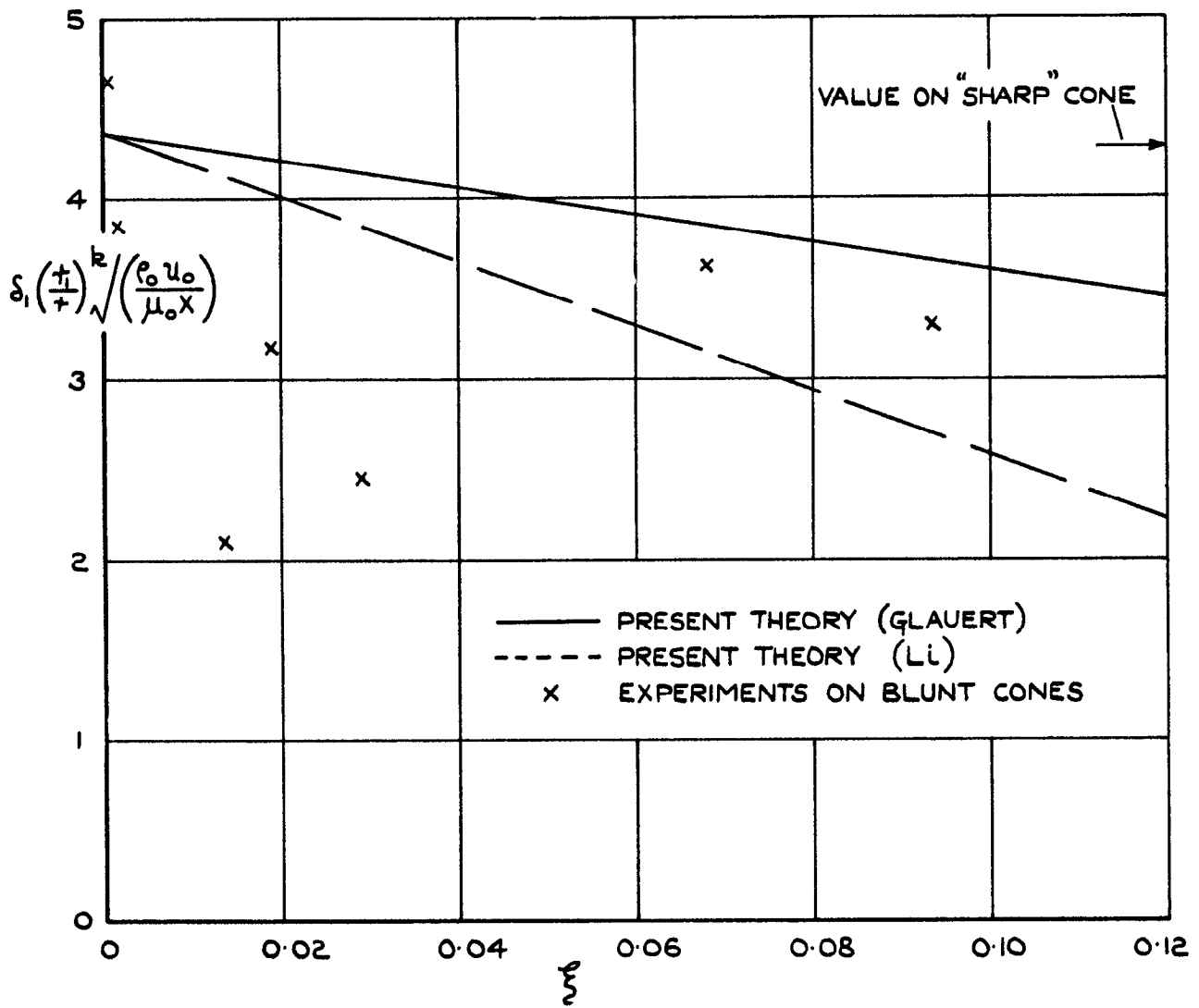


FIG. 7. VARIATION WITH ξ OF THE DISPLACEMENT THICKNESS, δ_1
 (b) $M_0 = 2.34$

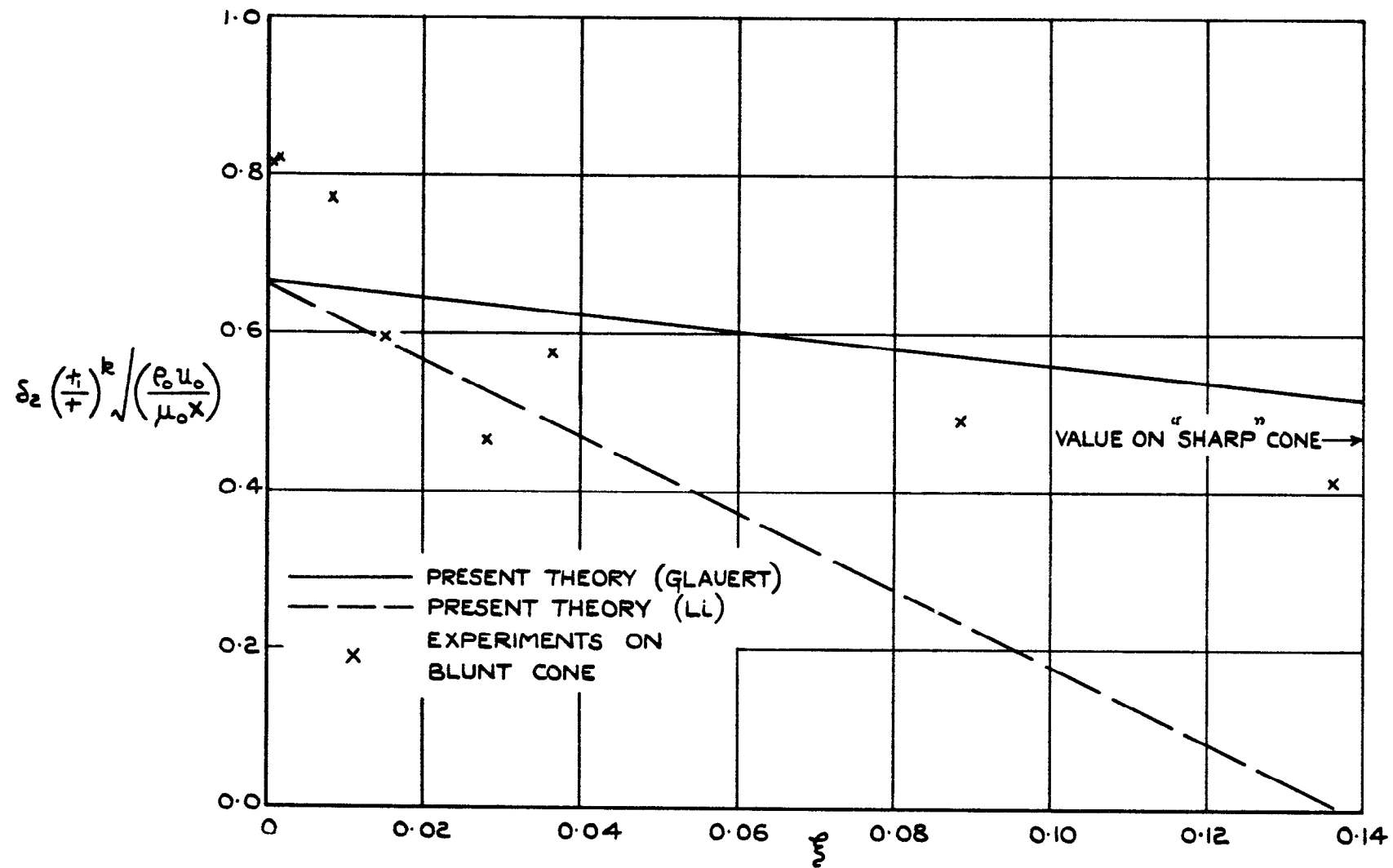


FIG. 8. VARIATION WITH ξ OF THE MOMENTUM THICKNESS, δ_2
 (a) $M_0 = 2.14$

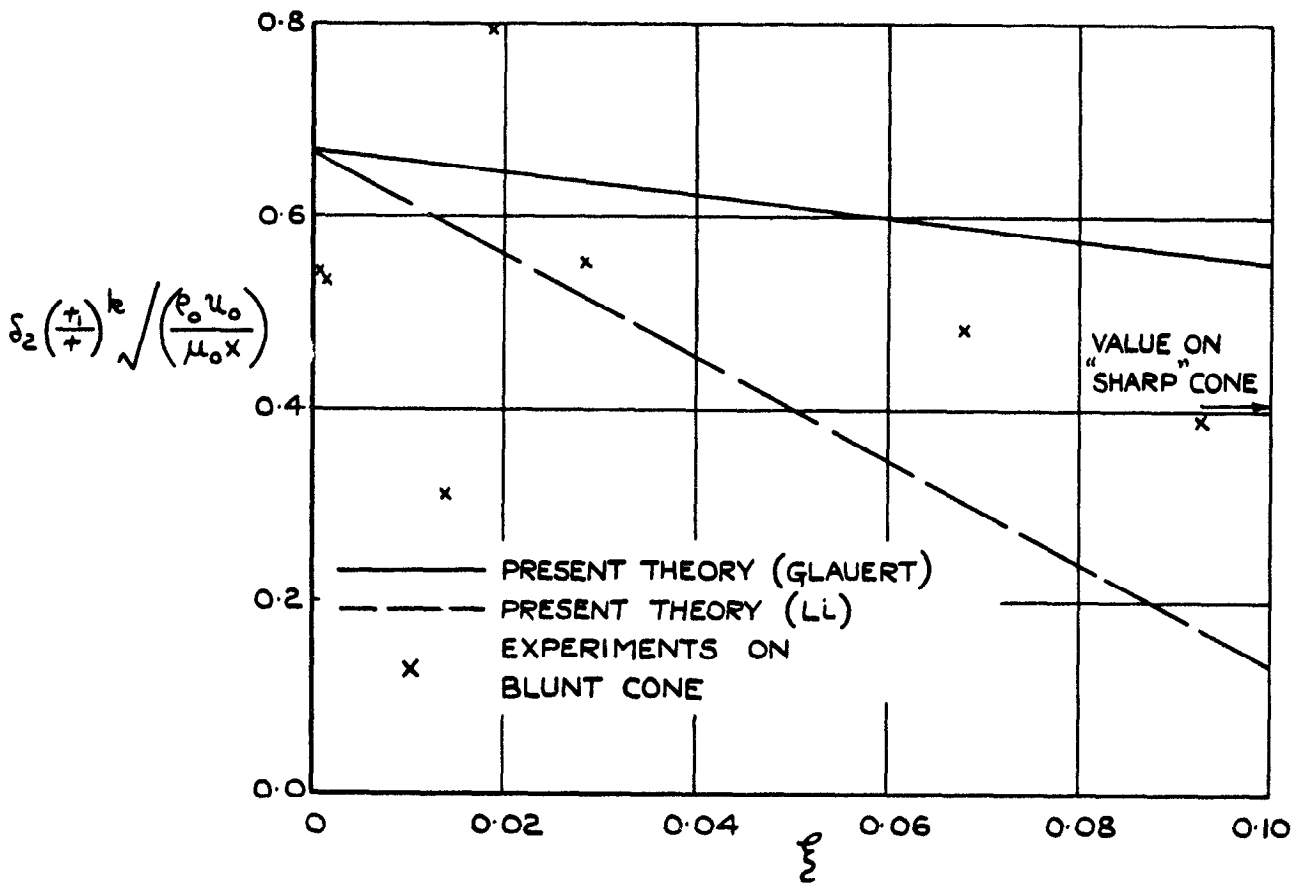


FIG. 8. VARIATION WITH ξ OF THE MOMENTUM THICKNESS, δ_2
 (b) $M_0 = 2.34$

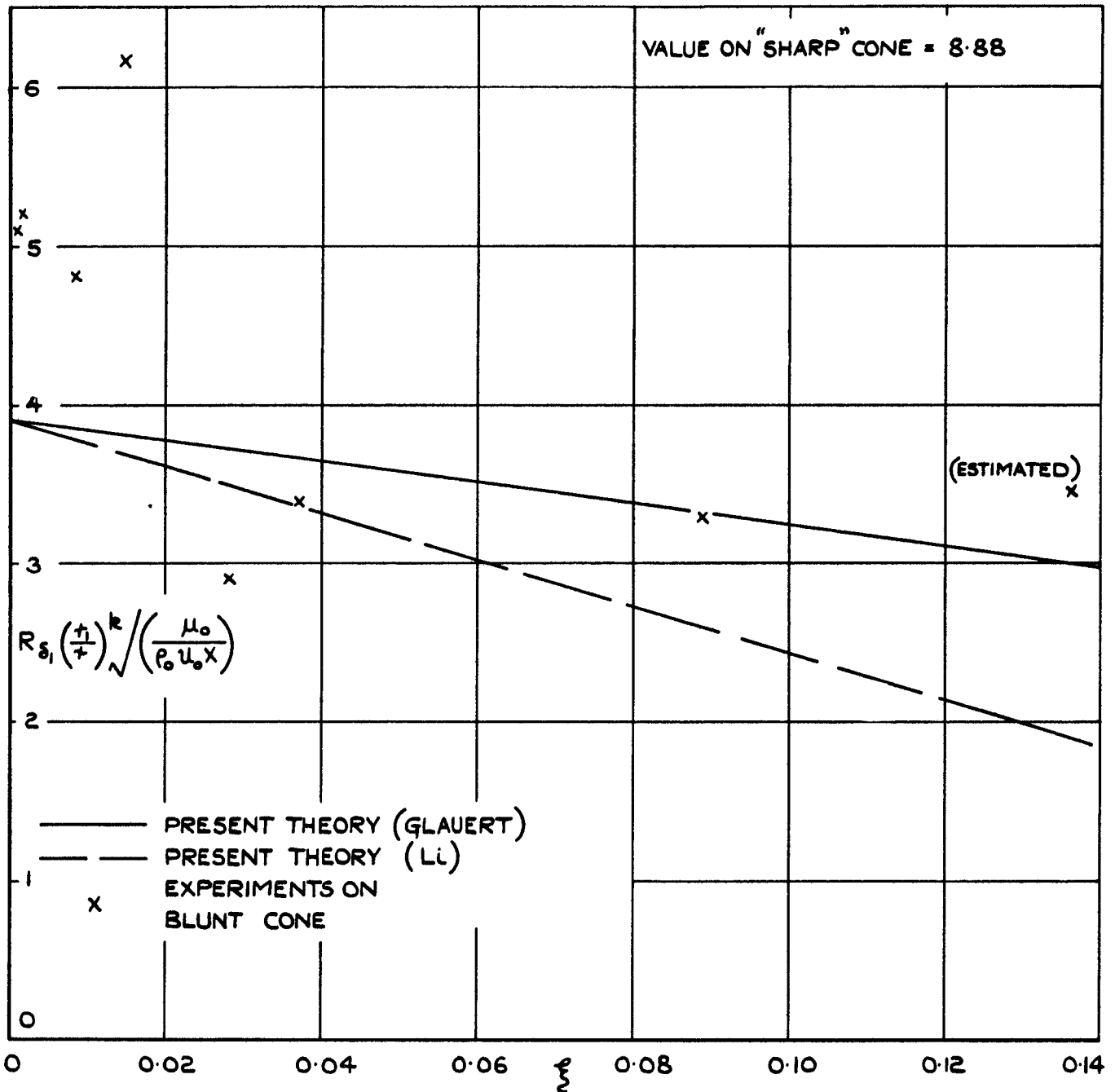


FIG. 9. VARIATION WITH ξ OF THE REYNOLDS NUMBER, R_{δ_1} , BASED ON DISPLACEMENT THICKNESS
 (a) $M_0 = 2.14$

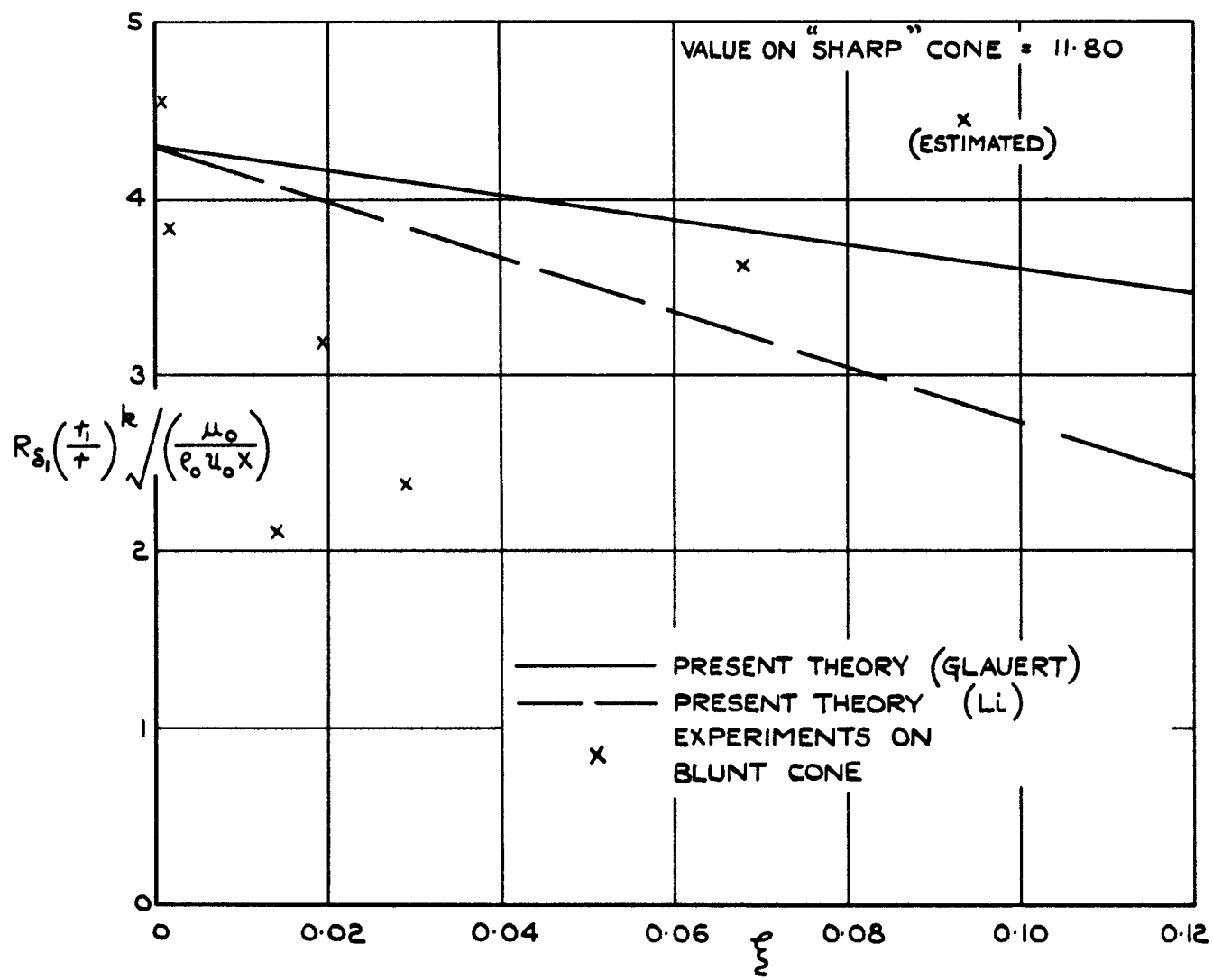


FIG.9. VARIATION WITH ξ OF THE REYNOLDS NUMBER, R_{δ_1} , BASED ON DISPLACEMENT THICKNESS
 (b) $M_0 = 2.34$

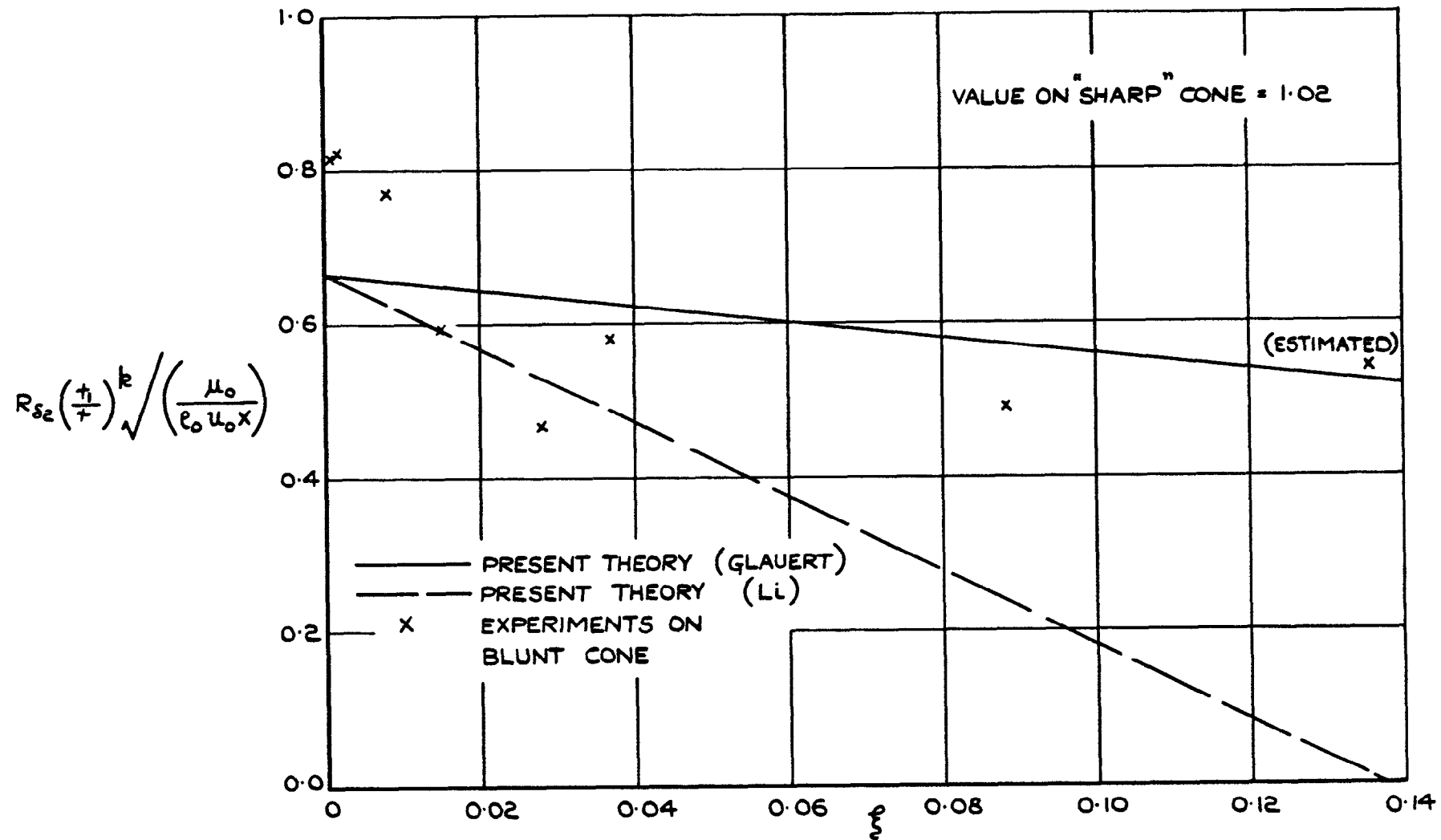


FIG.10. VARIATION WITH ξ OF THE REYNOLDS NUMBER, R_{s_2} ,
 BASED ON MOMENTUM THICKNESS
 (a) $M_0 = 2.14$

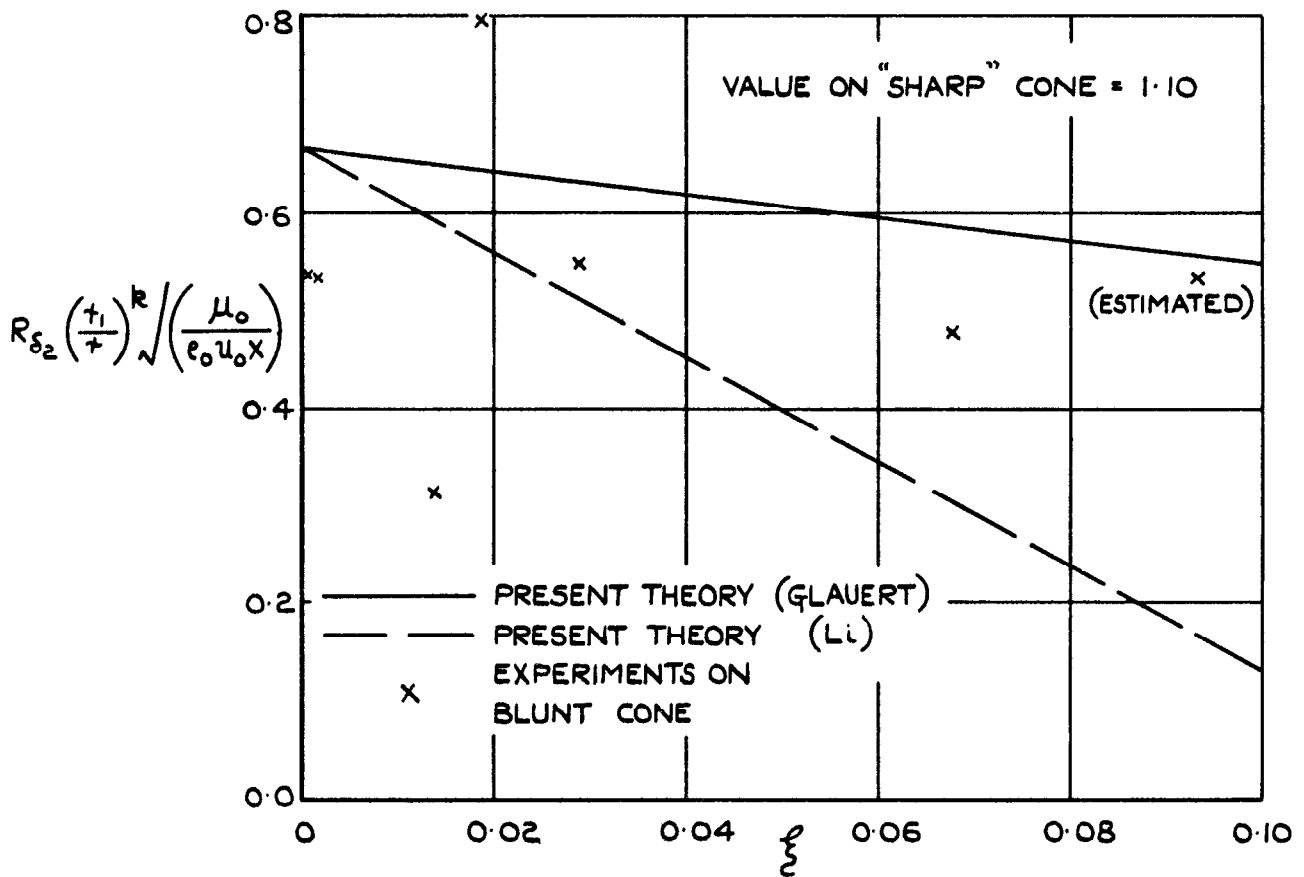


FIG.10. VARIATION WITH ξ OF THE REYNOLDS NUMBER, R_{δ_2} , BASED ON MOMENTUM THICKNESS
 (b) $M_0 = 2.34$

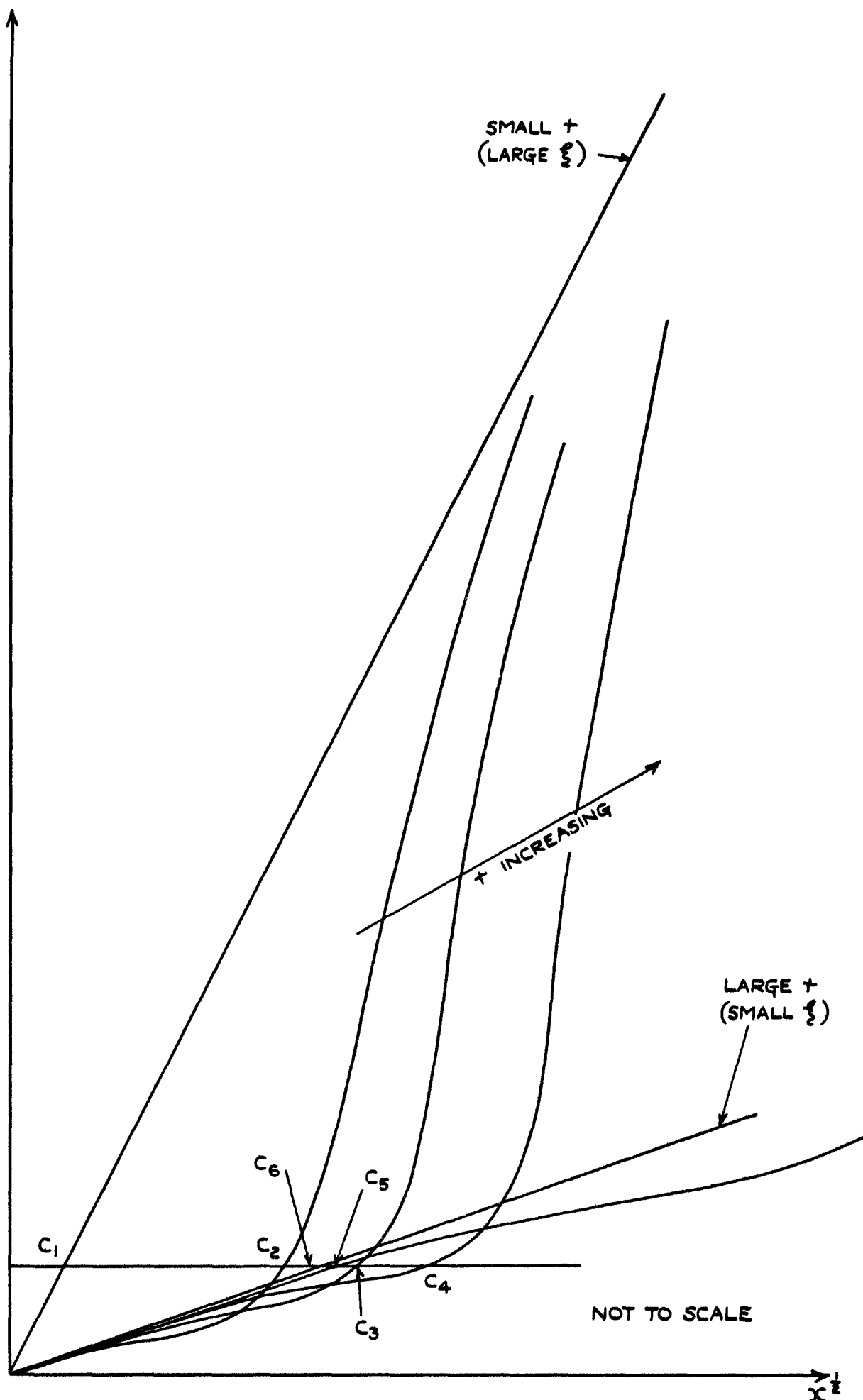


FIG.II. VARIATION WITH $x^{\frac{1}{2}}$ OF THE REYNOLDS NUMBER, R_δ , BASED ON EITHER THE DISPLACEMENT OR THE MOMENTUM THICKNESS.

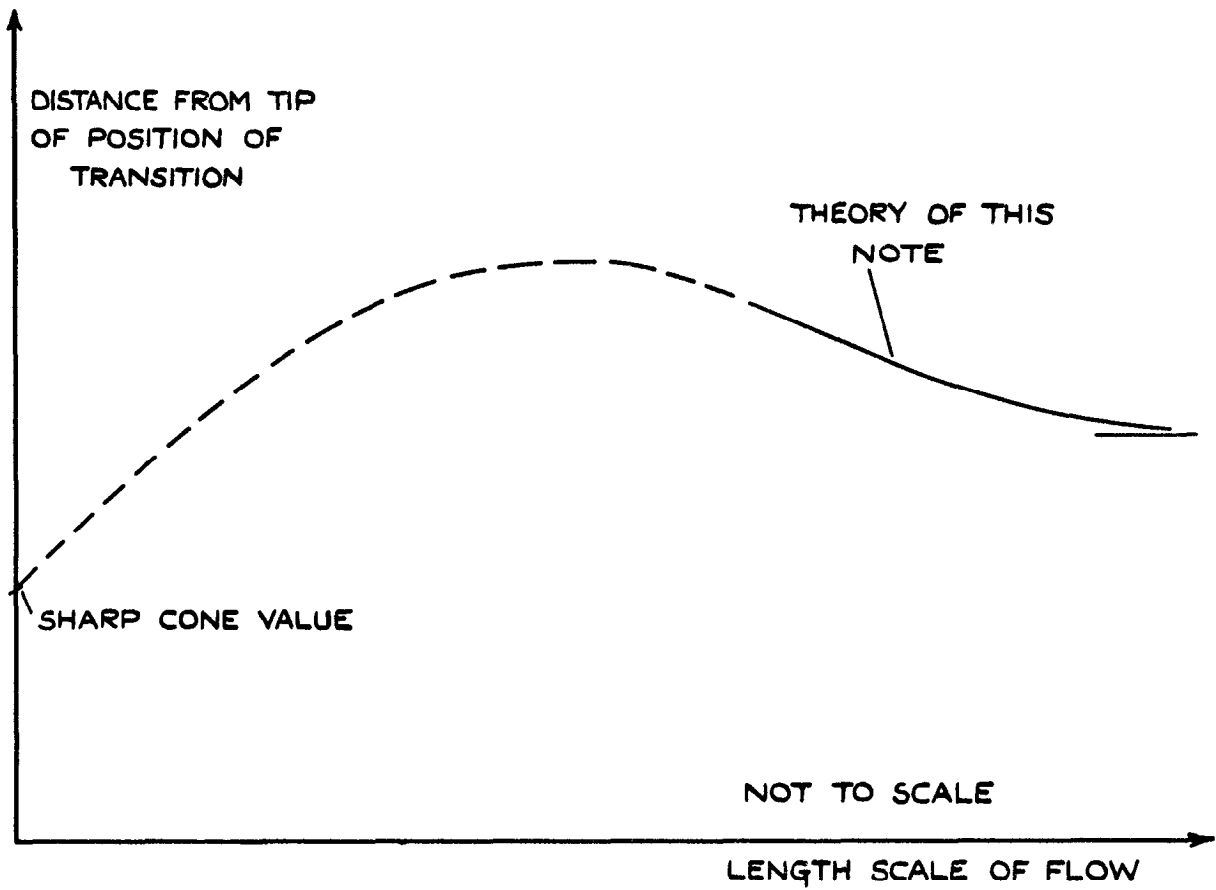


FIG. 12. VARIATION OF THE POSITION OF TRANSITION (ASSUMED TO OCCUR AT CONSTANT R_δ) WITH THE LENGTH SCALE OF THE PROBLEM.

A.R.C. C.P.No.599

532.526 :
533.6.011.5 :
532.517.43

THE DEVELOPMENT OF THE BOUNDARY LAYER IN SUPERSONIC
SHEAR FLOW. Rogers, Ruth H. January, 1961

The development of the boundary layer in a velocity shear layer is discussed for two-dimensional flow and for axisymmetric flow of both compressible and incompressible fluids. It is shown that the solutions obtained by Li^{1,3} and Glauert² for the two-dimensional flow of an incompressible fluid are applicable in the more general case after suitable transformations of coordinates have been made. New definitions are shown to be necessary, and are given, for the displacement and momentum thicknesses of such a boundary layer. Reynolds numbers based on these thicknesses are given, and it is shown that any phenomenon (such as transition to turbulence) which occurs at a constant value of such a Reynolds number will occur at a point which, as the length scale of the

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The development of the boundary layer in a velocity shear layer is discussed for two-dimensional flow and for axisymmetric flow of both compressible and incompressible fluids. It is shown that the solutions obtained by Li^{1,3} and Glauert² for the two-dimensional flow of an incompressible fluid are applicable in the more general case after suitable transformations of coordinates have been made. New definitions are shown to be necessary, and are given, for the displacement and momentum thicknesses of such a boundary layer. Reynolds numbers based on these thicknesses are given, and it is shown that any phenomenon (such as transition to turbulence) which occurs at a constant value of such a Reynolds number will occur at a point which, as the length scale of the

(Over)

flow increases, first moves downstream and then moves slightly upstream. This is shown to be in qualitative agreement with experimental results on a blunt cone in a supersonic flow. A quantitative comparison of the theoretical and experimental values of displacement and momentum thicknesses is attempted, and no disagreement is obvious; unfortunately the accuracy of the experiments so far available is insufficient to give positive confirmation of the theory of this note.

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