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A One-Dimensional Theory of Liquid-Fuel Rocket Combustion III: The Effect of Non-Uniform Droplet Radii, Injection Velocities and Physical Properties

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SUMMARY

The one-dimensional liquid-fuel rocket combustion theory of Spalding is extended to apply to models consisting of two different droplet groups. In one model, droplet radii and injection velocities are varied, and in the other the particle densities and mass transfer numbers are non-uniform. Solutions are obtained for the variation of droplet radius, velocity and gas velocity with distance from the injection end.

1. Introduction

A one-dimensional theory of liquid-fuel rocket combustion has been considered in a previous paper by Spalding[1]. In it the assumptions have been made that the injected droplets all have the same initial radii and velocities and that the fuel and oxidizer have the same physical properties. The additional assumption of taking a chemical loading parameter* E, defined by Spalding, to be zero, makes it possible to obtain analytical solutions to the equation describing the model. This assumption corresponds to taking the temperature inside the combustion chamber to be everywhere at its maximum attainable value. In a subsequent paper[2] we have considered the effect of chemical reaction, and have shown that under conditions for which combustion is possible the solutions of Spalding do not differ much from those which take account of reaction. Since this procedure produces considerable mathematical simplification and only small errors in the solutions, it is worth while adopting it for the consideration of other one-dimensional combustion models.

In the following work we have investigated the effect of varying the initial droplet sizes and injection velocities. Two droplet groups have been considered, which might be taken to be the fuel and oxidant of a bi-propellant system. The assumptions of Model I of Spalding in which the chemical loading parameter E is zero, have here been made.

We/

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*Denoted by L in Refs.1 and 2.

We have also considered a combustion model, involving two groups of droplets, with different densities and transfer numbers. As previously, the assumption of a uniform temperature throughout the combustion chamber is also made; this is somewhat less realistic than formerly, because now the fuel-oxidant ratio in the gas phase varies.

In Model I of Spalding, the assumption E = 0 made it possible to obtain explicit analytical solutions for the differential equations describing the system. Although by adopting this procedure, we have obtained considerable simplification it has been possible to obtain explicit analytical solutions only for extreme conditions. This is due to the fact that each equation involves both groups of droplets injected into the combustion chamber. Numerical solutions to the equations have been obtained by use of a high-speed electronic digital computer.

2. Differential Equations

2.1 The equation of droplet vaporization

If one considers a number of droplet groups, denoted by the suffix i, then in Ref.1 it has been shown that the law of droplet vaporization for each droplet group may be written in the form

$$\frac{Dr_i}{Dt} = v_i \frac{dr_i}{dx} = -\sigma_i(\tau)\, R_i(r_i)\, f_{i1}(Re) \qquad \ldots (1)$$

where r_i = droplet radius of i^{th} droplet

$\frac{D}{Dt}$ = total derivative

v_i = droplet velocity of i^{th} droplet

x = axial distance

σ_i = function of reactedness τ, for i^{th} droplet ($\sigma_i = 1$ for gas in equilibrium)

R_i = function of droplet radius r_i

f_{i1} = function of i^{th} droplets Reynolds number.

It has been shown by Spalding and Jain[3,4], that in simple cases

$$R_i = \frac{1}{r_i} \frac{k}{c\rho_{\ell_i}} \log (1+B_i) \qquad \ldots (2)$$

where k = average gas conductivity

c = average gas specific heat

ρ_{ℓ_i} = density of i^{th} droplet

B_i = transfer number of i^{th} droplet.

2.2/

2.2 The equation of droplet drag

The rate of change of the velocity of the i^{th} droplet may be written as

$$v_i \frac{dv_i}{dx} = \frac{9}{2} \frac{\mu_g}{S_{1_i} r_i^2} (u-v)\, m_i(\tau)\, f_{i2}(Re) \qquad \ldots (3)$$

where μ_g = viscosity of gas in equilibrium state

m_i = function of τ for i^{th} droplet expressing variation of gas viscosity with temperature ($m_i = 1$ if $\tau = 1$)

f_{i2} = function of i^{th} droplets Reynolds number.

For small Re, $f_{i2} = 1$ and equation (3) reduces to Stokes's law. As Re increases f_{i2} rises above unity, cf., Ref.1.

2.3 Mass conservation in droplet vaporization

The mass conservation equation may be written in the form

$$G = \rho_g u\, \delta(\tau) + G \frac{\Sigma_i \rho_{\ell_i} r_i^3 N_i}{\Sigma_i \rho_{\ell_i} r_{io}^3 N_i} \qquad \ldots (4)$$

where the summation extends over all droplet groups.

G = mass flow rate of injected material per unit duct area

ρ_g = density of gas in equilibrium

u = gas velocity

δ = function of τ, expressing the dependence of the gas density on reactedness

r_i = radius of i^{th} droplet

r_{io} = initial radius of i^{th} droplet

N_i = fraction of droplets of initial radius r_{io}.

2.4 Assumption of zero chemical loading

Spalding[1] has shown, that by making a defined chemical loading parameter E equal to zero, the reactedness τ inside the gas, becomes everywhere unity. The combustion system considered consists of uniform droplets with equal properties but it is seen that, by suitable adjustment of his equations so that they apply to a system of non-uniform droplets, the result will also be applicable.

The significance of $E = 0$ is that the chemical reactivity of the gases is very large. For bi-propellant systems, the burning rate is then entirely determined by physical factors.

If/

If $\tau = 1$ then one can put $\sigma_i = 1$, $m_i = 1$ and $\delta = 1$ with corresponding simplification of equations (1), (3) and (4).

2.5 Equations in dimensionless form

The equations describing the behaviour of the fuel droplets and that of the surrounding gas are conveniently reduced to dimensionless form by introducing the following set of dimensionless variables:

Droplet radius : $$\zeta_i = \frac{r_i}{r_o} \quad \ldots (5)$$

Droplet velocity : $$\chi_i = \frac{\rho_g v_i}{G} \quad \ldots (6)$$

Distance : $$\xi = \frac{R_o \rho_g x}{G r_o} \quad \ldots (7)$$

Vaporization rate : $$\beta_i = \frac{R_i}{R_o} \quad \ldots (8)$$

Droplet drag : $$S_i = \frac{9}{2} \frac{\mu_g}{\rho_{\ell i} r_o R_o} \quad \ldots (9)$$

Gas velocity : $$\omega = \frac{u \rho_g}{G} \quad \ldots (10)$$

where r_o = maximum droplet radius

R_o = value of R_i at $r = r_o$ for the droplet group with initial radius r_o.

With the dimensionless quantities as defined above the differential equations become

Droplet vaporization : $$\chi_i \frac{d\zeta_i}{d\xi} = -\beta_i f_{i1}. \quad \ldots (11)$$

Droplet drag : $$\chi_i \frac{d\chi_i}{d\xi} = \frac{S_i f_{i2}}{\zeta_i^2} (\omega - \chi_i). \quad \ldots (12)$$

Gas velocity : $$\omega = 1 - \frac{\Sigma_i \zeta_i^3 \gamma_i N_i}{\Sigma_i \zeta_{io}^3 \gamma_i N_i} \quad \ldots (13)$$

where $$\gamma_i = \frac{\rho_{\ell i}}{\rho_{\ell o}} \quad \ldots (14)$$

$\rho_{\ell o}$ = density of the droplet group with initial radius r_o.

We/

We shall take R_i to be of the form given by equation (2). Then the dimensionless vaporization rate may be written as

$$\beta_i = \frac{D_i}{\gamma_i \zeta_i} \qquad \ldots (15)$$

in which we have put

$$D_i = \frac{\log(1+B_i)}{\log(1+B_o)} \qquad \ldots (16)$$

B_o = transfer number of droplet group with initial radius r_o.

Spalding has shown that for values of $Re < 30$ the ratio f_{i2}/f_{i1} does not differ by more than about 6% from unity. If in addition it is assumed that Stokes's law is operative, one will have f_{i2} equal to unity. We shall therefore assume that for all groups of particles

$$f_{i1} = f_{i2} = 1. \qquad \ldots (17)$$

Equations (11), (12) and (13) refer to an as yet unspecified number of droplet groups, the particles of each group having the same initial radius, injection velocity, density and transfer number. We shall consider particular combustion models in which these quantities are varied and determine what their effect is on the combustion system as a whole.

3. Combustion Model III(a)

3.1 The effect of varying droplet sizes and injection velocities

A combustion model is considered, consisting of two groups of droplets, having the same physical properties but different initial radii and injection velocities. For such a system one will have

$$\left.\begin{aligned} \gamma_a &= \gamma_b = 1 \\ D_a &= D_b = 1 \end{aligned}\right\} \qquad \ldots (18)$$

and

$$S_i = S = \frac{9}{2} \frac{\mu_g}{\rho_{\ell_o} r_o R_o} \qquad \ldots (19)$$

ρ_{ℓ_o} = density of both droplet groups.

The differential equation for droplet vaporization and drag and the equation for the gas velocity becomes

$$\chi_i \frac{d\zeta_i}{d\xi} = -\frac{1}{\zeta_i}. \qquad \ldots (20)$$

$$\chi_i \frac{d\chi_i}{d\xi} = \frac{S}{\zeta_i^2}(\omega - \chi_i) \qquad \ldots (21) \quad i = a,b$$

$$\omega = 1 - \frac{\zeta_a N_a + \zeta_b^3 N_b}{\zeta_{ao}^3 N_a + \zeta_{bo}^3 N_b} \qquad \ldots (22)$$

in which

$$N_a + N_b = 1.$$

The/

The boundary conditions are:

$$\left.\begin{aligned} \xi = 0: \quad \zeta_a &= \zeta_{ao} \quad \zeta_b = \zeta_{bo} \\ \chi_a &= \chi_{ao} \quad \chi_b = \chi_{bo} \\ \omega &= 0. \end{aligned}\right\} \qquad \ldots (24a)$$

$$\left.\begin{aligned} \xi = \xi^*: \quad \zeta_a &= \zeta_b = 0 \\ \chi_a &= \chi_b = 1, \quad \omega = 1. \end{aligned}\right\} \qquad \ldots (24b)$$

3.2 Parameters and results

We shall assume that equal masses of the two droplet groups are injected into the combustion chamber. By use of (23) it follows then, that

$$N_a = \frac{\zeta_{bo}^3}{\zeta_{ao}^3 + \zeta_{bo}^3}; \quad N_b = \frac{\zeta_{ao}^3}{\zeta_{ao}^3 \zeta_{bo}^3}. \qquad \ldots (25)$$

As has been done for combustion Models I(a) and II we choose the droplet drag parameter to be $S = 1$.

The following choice for the initial radii and injection velocities has also been made

$$\left.\begin{aligned} \zeta_{ao} &= 1 \quad \zeta_{bo} = 0.6, \quad 1 \\ \chi_{ao} &= 0.5 \quad \chi_{bo} = 0.1, \; 0.5. \end{aligned}\right\} \qquad \ldots (26)$$

In Figs.1 - 3 are shown the results obtained on integrating equations (20) - (22) with the above parameters. The corresponding solution for Model I(a) is shown for comparison by dotted lines.

4. Combustion Model III(b)

4.1 The effect of varying droplet densities and mass transfer numbers

We consider a combustion model consisting of two groups of droplets with different densities and mass transfer numbers. It is assumed that both groups have the same initial droplet radius and injection velocity.

The dimensionless equations for droplet vaporization and drag and that for the gas velocity become

$$\chi_i \frac{d\zeta_i}{d\xi} = -\frac{D_i}{\gamma_i} \frac{1}{\zeta_i}. \qquad \ldots (27)$$

$$\chi_i \frac{d\chi_i}{d\xi} = \frac{S}{\gamma_i} \frac{1}{\zeta_i^2} (\omega - \chi_i) \qquad \ldots (28) \quad i = a,b$$

$$\omega = 1 - \frac{(\zeta_a^3 \gamma_a N_a + \zeta_b^3 \gamma_b N_b)}{\zeta_{ao}^3 \gamma_a N_a + \zeta_{bo}^3 \gamma_b N_b}. \qquad \ldots (29)$$

The/

The drag parameter S is given by equation (19), the quantities ρ_{ℓ_o} and R_o referring now to any one particular droplet group, e.g., the particle group with smallest B_i value.

The boundary conditions are:

$$\left.\begin{aligned} \xi = 0 : \quad \zeta_a &= \zeta_b = \zeta_o \\ \chi_a &= \chi_b = \chi_o, \quad \omega = 0 \end{aligned}\right\} \qquad \ldots (30a)$$

$$\left.\begin{aligned} \xi = \xi^* : \quad \zeta_a &= \zeta_b = 0 \\ \chi_a &= \chi_b = 1, \quad \omega = 1. \end{aligned}\right\} \qquad \ldots (30b)$$

4.2 Parameters and results

It is assumed that equal masses of the two droplet groups are injected into the combustion chamber. Use of (23) then gives

$$N_a = \frac{\gamma_b}{\gamma_a + \gamma_b}; \quad N_b = \frac{\gamma_a}{\gamma_a + \gamma_b}. \qquad \ldots (31)$$

We have taken the drag parameter to be $S = 1$. To make comparison with Model I(a) possible, we have chosen the following boundary condition

$$\left.\begin{aligned} \zeta_{ao} &= \zeta_{bo} = 1 \\ \chi_{ao} &= \chi_{bo} = 0.5. \end{aligned}\right\} \qquad \ldots (32)$$

Reference to the work of Spalding[4] shows that for practical cases the values of γ_i and B_i are such that

$$0.6 < \gamma_i < 1.5$$

$$1 < B_i < 9.$$

We have therefore chosen the following extreme values

$$\left.\begin{aligned} \gamma_a &= 1 \qquad D_a = 1 \\ \gamma_b &= 0.6, 1 \qquad D_b = 1, 3. \end{aligned}\right\} \qquad \ldots (33)$$

Equations (27) - (29) have been integrated for values of the above parameters and the solutions obtained are displayed in Figs.4 - 6. The corresponding solution of Model I(a) is also shown for comparison.

4.3 Extreme solutions at large transfer numbers ($D_b \to \infty$)

We consider the case in which the transfer number D_b of the second group of droplets is very large (see also Ref.5). Under these conditions the droplets will almost instantaneously vaporize on entering the combustion chamber so that one may take $\zeta_b = 0$ for $\xi > 0$.

Assuming that both droplet groups have initially the same radii and that equal masses are injected into the combustion chamber the equations for the first group of droplets become

χ_a/

$$\chi_a \frac{d\zeta_a}{d\xi} = -\frac{1}{\zeta_a} . \qquad \ldots (34)$$

$$\chi_a \frac{d\chi_a}{d\xi} = S \frac{1}{\zeta_a^2} (\omega - \chi_a) \qquad \ldots (35)$$

$$\omega = 1 - \tfrac{1}{2} \zeta_a^3 \qquad \ldots (36)$$

where we have taken $\zeta_{ao} = \gamma_a = D_a = 1$.

Equations (34) - (36) may be combined to give the single equation

$$\frac{d\phi_a}{d\zeta_a} = -\frac{S}{\zeta_a} (1 - \zeta_a^3 - \phi_a) \qquad \ldots (37)$$

where $\phi_a = 2\chi_a - 1$.

The boundary conditions are:

$$\left.\begin{aligned} \zeta_a = 0 : \quad \phi_a &= 1 \\ \zeta_a = 0 : \quad \phi_a &= 2\chi_o - 1 = \phi_{ao} \text{ (say).} \end{aligned}\right\} \qquad \ldots (38)$$

Equation (37) is identical with the corresponding equation of Spalding[1] for combustion Model I(a). It may be integrated to give

$$\phi_a = \left(\phi_{ao} + \frac{3}{S-3}\right) \frac{S}{\zeta_a} + 1 - \frac{S}{S-3} \zeta_a^3 . \qquad \ldots (39)$$

Integration of (34) gives the distance from the injection end at which the droplets disappear. One obtains

$$\xi^* = \tfrac{1}{2} \int_o^1 (1+\phi_a) \zeta_a \, d\zeta_a \qquad \ldots (40)$$

which on substitution gives

$$\xi^* = \frac{1}{S+2} \left(\chi_o + \frac{2}{5} S\right) . \qquad \ldots (41)$$

Comparison with equation (39) of Spalding shows that for complete combustion the minimum length of the rocket must be increased by an amount $\frac{1}{10} \frac{S}{S+2}$.

5. Conclusions

Examination of Figs.1 - 6, and Sections 3 and 4, shows that the combustion Models considered have the following properties:

(i)/

(i) Both droplet groups decrease continuously in diameter during their travel from the plane of injection. The distance of disappearance of the droplets decreases with diminishing droplet radius or injection velocity. A similar effect is obtained if the droplet density is decreased or the mass transfer number increased. The length of rocket required for complete combustion must be increased in both cases.

(ii) An upper limit for the necessary combustion chamber length is given by equation (41) where ξ^* and χ_o are evaluated for the most slowly vaporizing droplets. Consideration of Model III(a), with one of the propellants injected in the gas phase, leads to an equation similar to (41) where ξ^* and χ_o now refer to the droplet group with finite initial radius.

(iii) In practical cases the required increase in combustion chamber length is no greater than about 12%, when compared with that necessary for propellant-systems with uniform initial conditions and physical properties.

(iv) The droplet velocities initially fall, reaching their minimum values when they correspond to that of the enclosing gas. The droplets once more reach the gas velocity at their point of disappearance.

(v) The gas velocity increases monotonically from zero and reaches its maximum value at the point of disappearance of all droplets.

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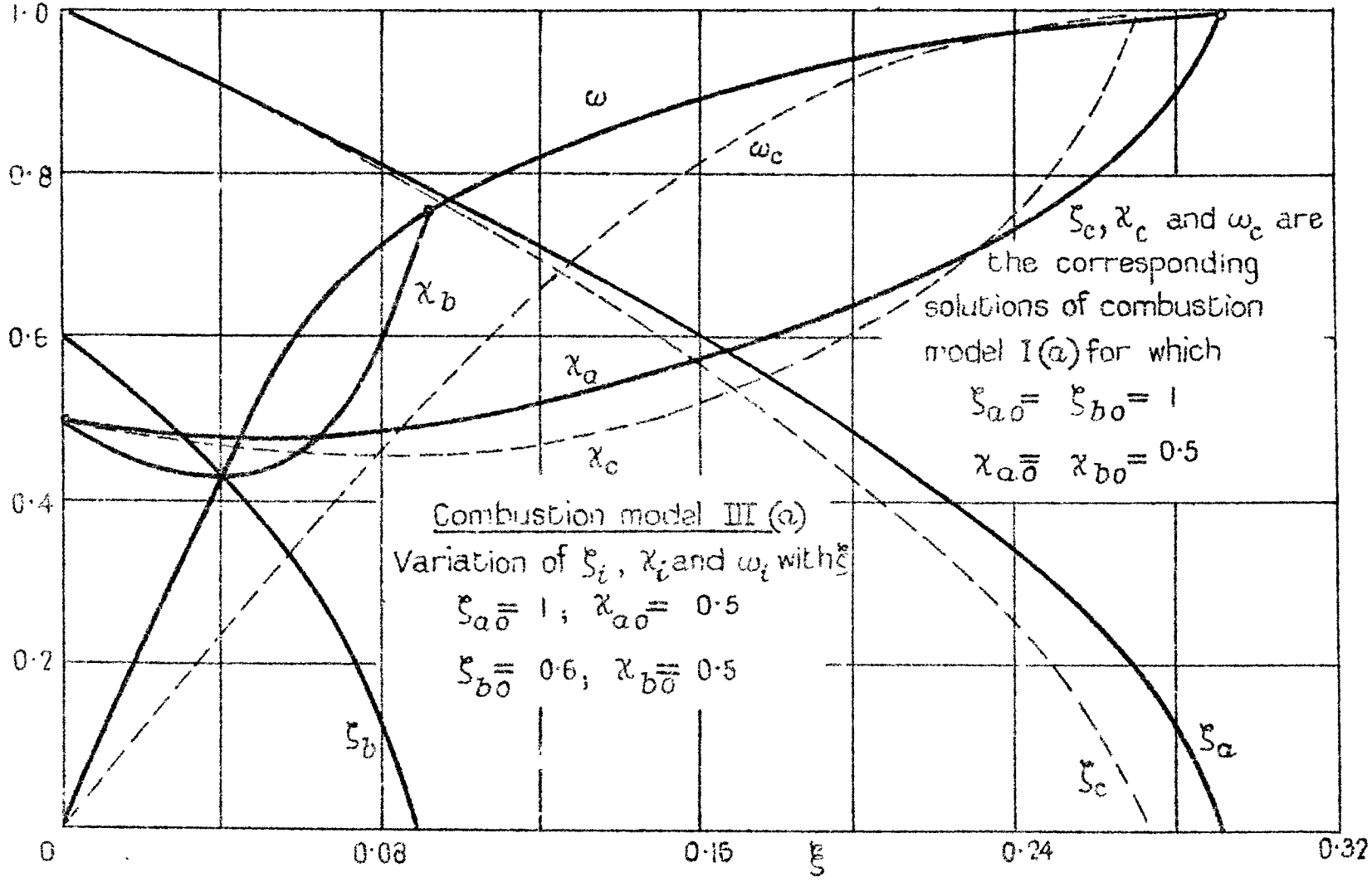
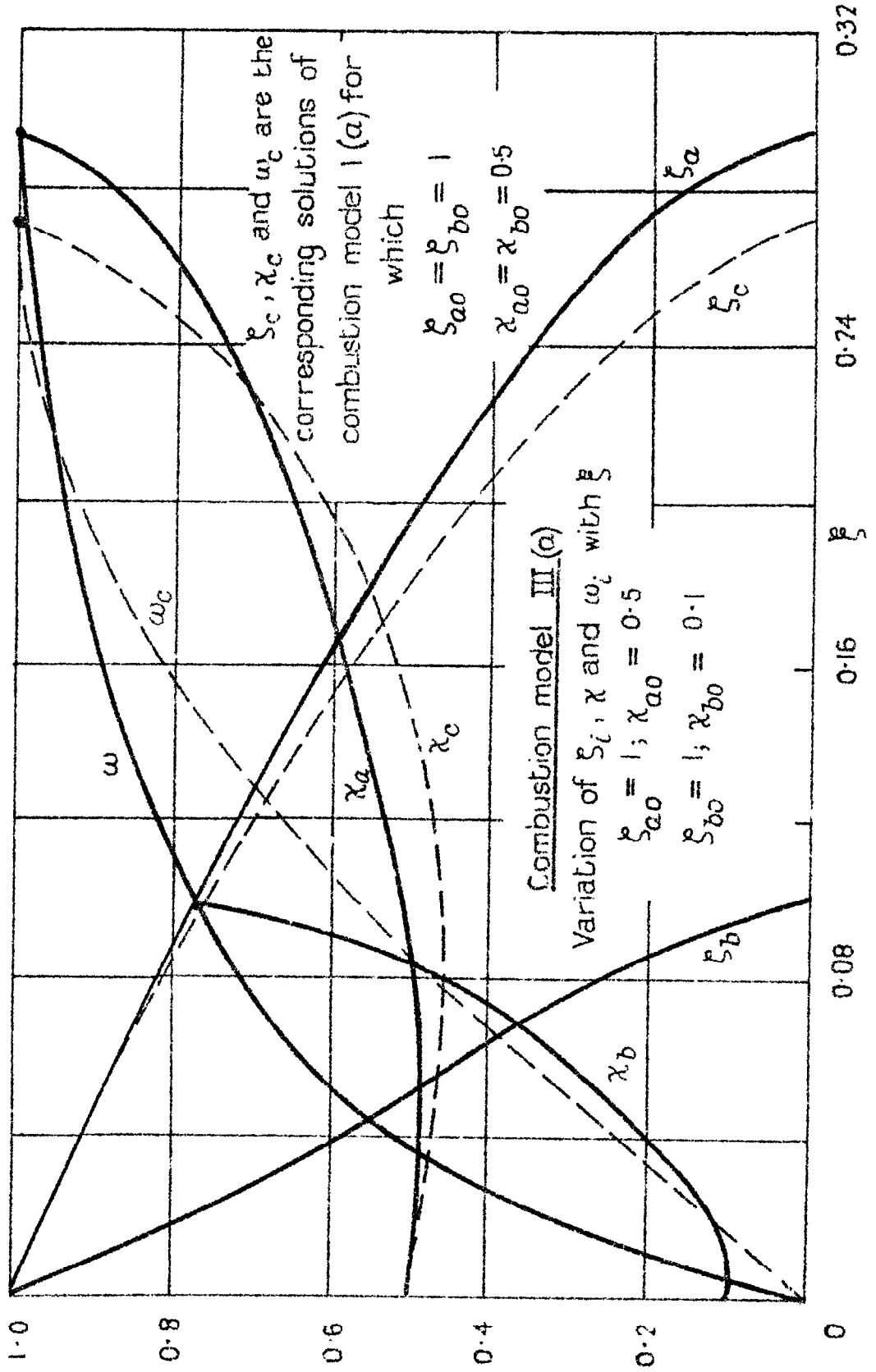


Effect of varying droplet radius ζ_{bo}

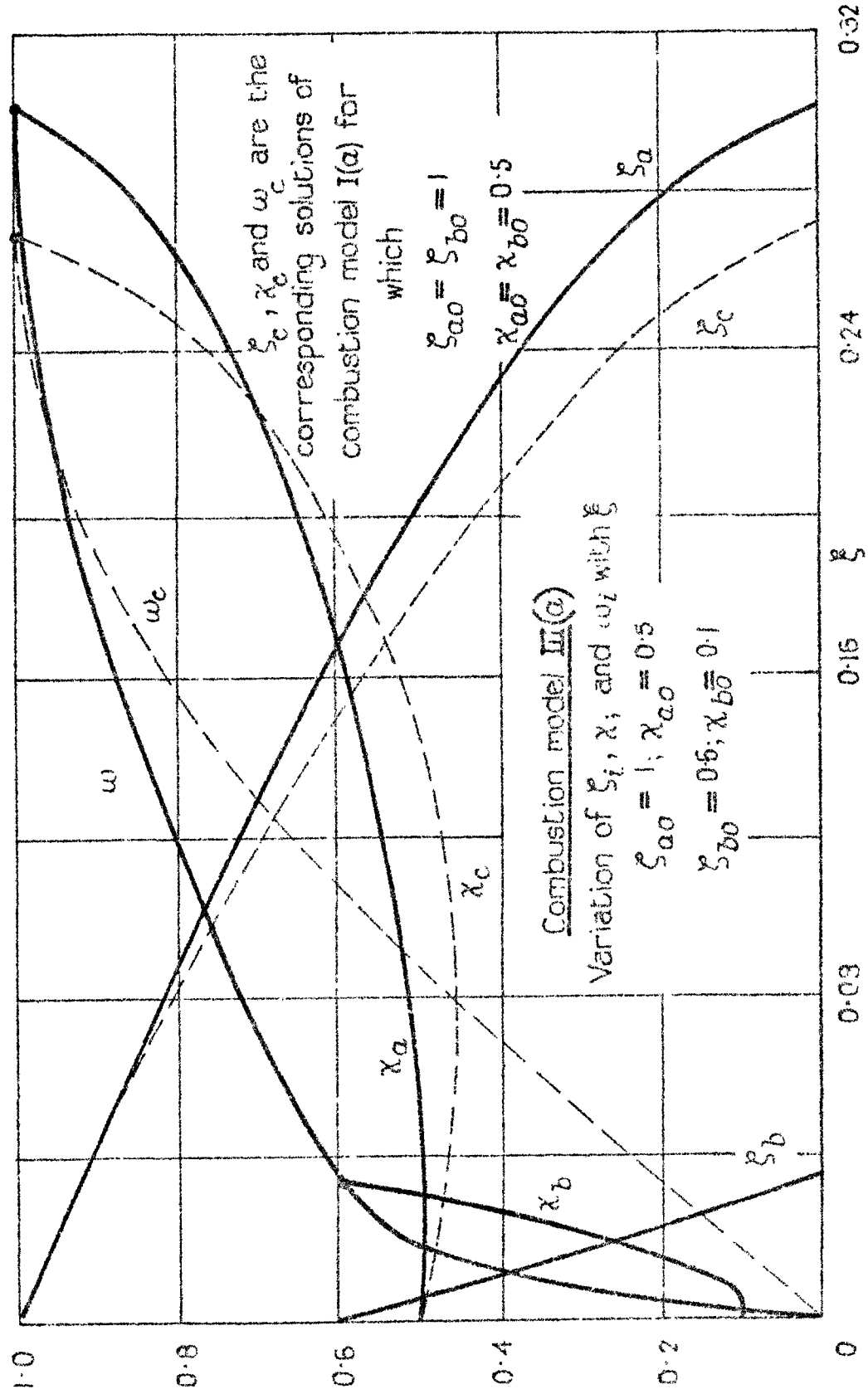
Fig. 1.

FIG. 2.



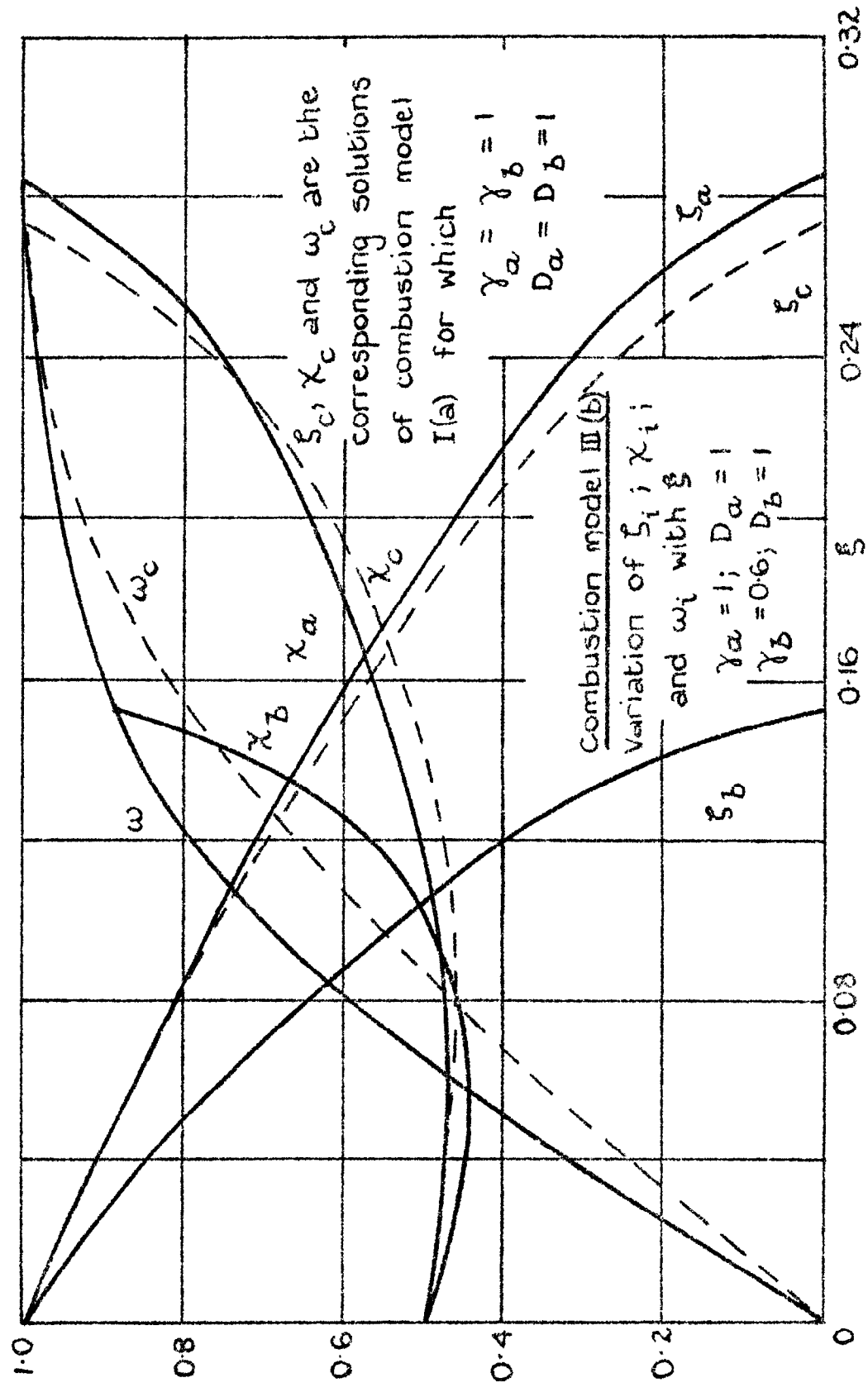
Effect of varying injection velocity χ_{bo}

FIG 3.



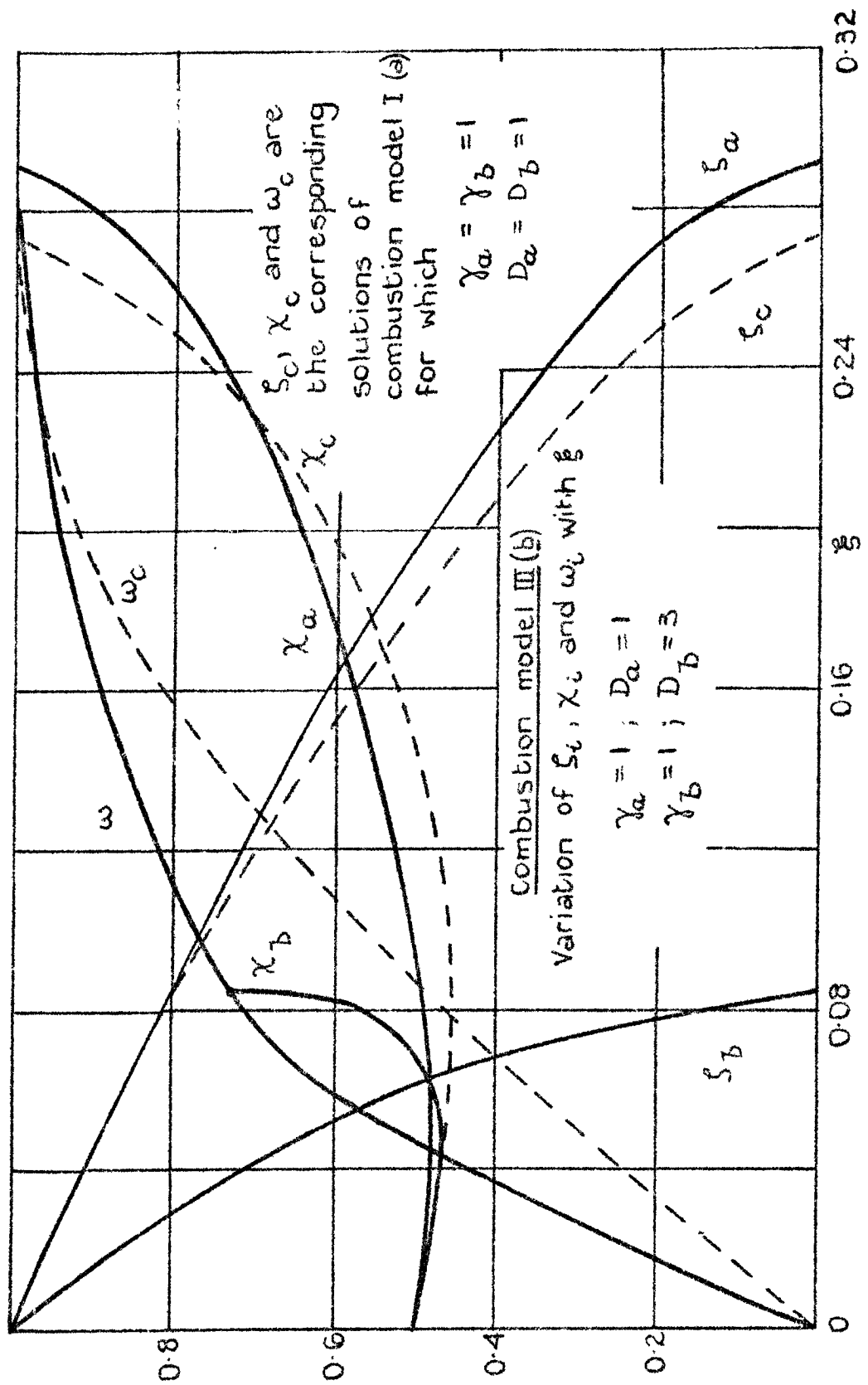
Effect of varying droplet radius ξ_{bo} and injection velocity χ_{bo}

FIG. 4.



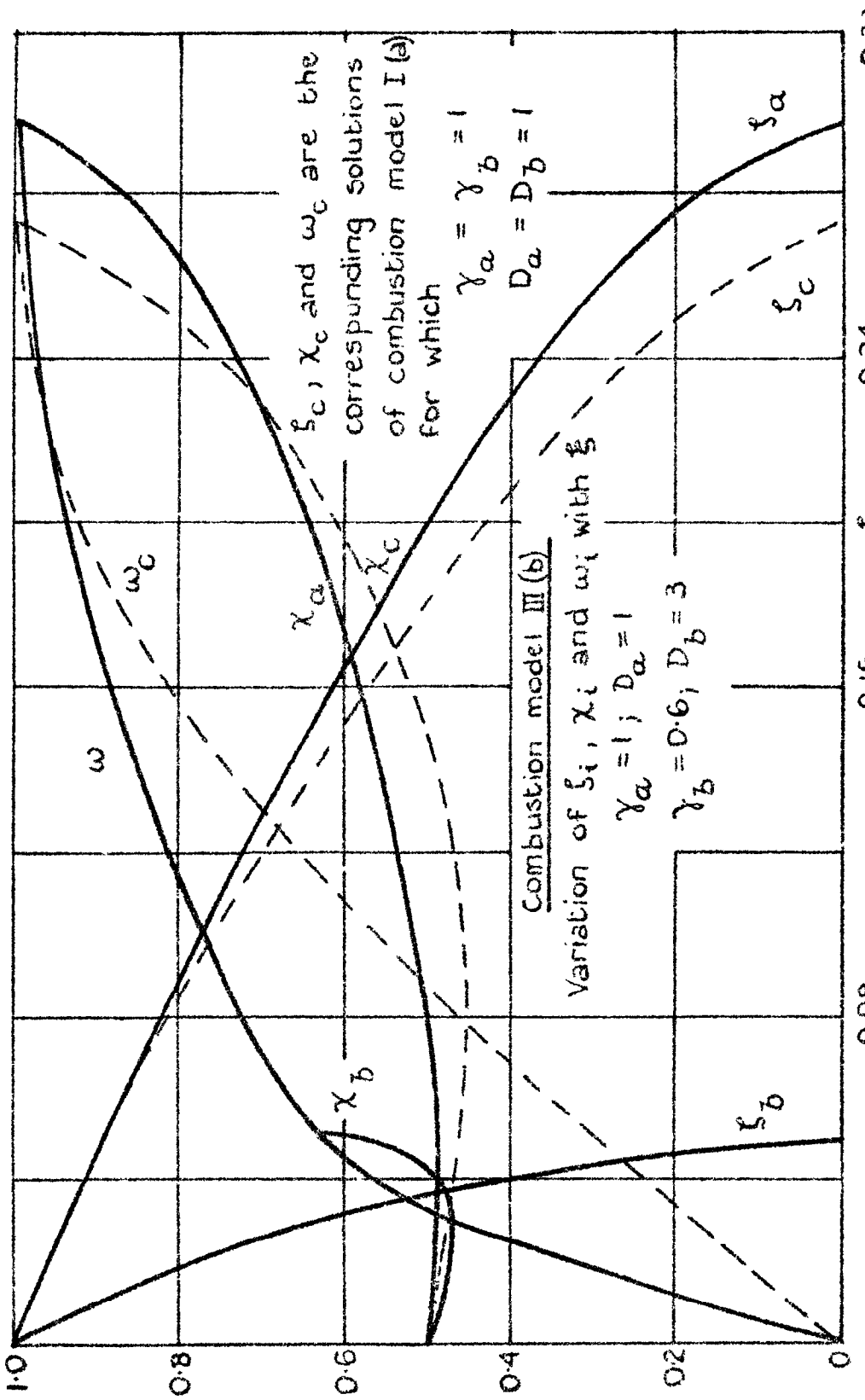
Effect of varying droplet density γ_b

FIG. 5.



Effect of varying transfer number D_b

Fig 6.



Effect of varying droplet density γ_b and transfer number D_b

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