C.P. No. 418
(20,570)
A.R.C. Technical Report

C.P. No. 418
(20,570)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL

CURRENT PAPERS

An Improvement of the Velocity Distribution Predicted by Linear Theory for Wings with Straight Subsonic Leading Edges

by

D. G. Randall, B.Sc.

LONDON: HER MAJESTY'S STATIONERY OFFICE

1959

THREE SHILLINGS NET

LIST OF CONTENTS

LIST OF ILLUSTRATIONS

C.P. No.418

U.D.C. No. 533.6.011.5: 533.693.1

Technical Note No. Aero.2577
September, 1958

ROYAL AIRCRAFT ESTABLISHMENT

AN IMPROVEMENT OF THE VELOCITY DISTRIBUTION PREDICTED BY LINEAR THEORY FOR WINGS WITH STRAIGHT SUBSONIC LEADING EDGES

by

D. G. Randall, B.Sc.

SUMMARY

When the supersonic flow over a non-lifting wing with straight subsonic leading edges is calculated by thin wing theory, the results are unacceptable in a small region near the leading edge. By satisfying the boundary condition on the wing more exactly near the leading edge, a solution of the linearised equation is obtained which gives plausible results there.

Elsewhere on the wing it gives results which are in agreement with thin wing theory.

C.P. No.418

<u>R.A.E. Technical Note No. Aero 2577</u>

<u>Errata and Addenda</u>

Page 7, second equation from bottom for $x/4$ read $\pi/4$; for $\tan(\frac{1}{\lambda})$ read $\tan^{-1}(\frac{1}{\lambda})$.

Page 10, line 10. Delete the sentence beginning "It seems ..."
Line 16. Delete the sentence beginning "It is ......"
Replace by "It is seen that k is of order unity for most wings with subsonic leading edges. Hence, it is likely that the last four integrals in Equation (11) can be neglected for most wings with subsonic leading edges".

Page 12, equations (17) and (19). For ϕ_n read ϕ_x .

Throughout the paper. For $\sin h^{-1}$ read $\sinh^{-1}$.

The critical nature of the approximate solution of the integral equation should be emphasized. Only in the limiting case of slender wings $(\lambda \rightarrow 1)$ can the approximations made be justified.

1 INTRODUCTION

The assumptions and approximations made in deriving the linearised equation of supersonic flow from the full equations of motion are so well known that it is unnecessary to state them again. The only assumption of interest here is that the deviations of the flow quantities from those of the free stream are everywhere small compared with the free stream speed. This requirement is too stringent, and the linearised equation can sometimes be used to determine a flow field containing regions in which the deviations are not small, provided that these regions are of sufficiently limited extent. The quantities predicted for such regions may then be considerably in error, but the quantities elsewhere will be given to an acceptable accuracy.

Perhaps the most familiar example of the application of linear theory to a flow field containing small regions where the theory is invalid is the determination of the supersonic flow over a non-lifting sweptback wing (whose profiles are not cuspidal) with straight leading edges. At a sonic or subsonic leading edge the component of velocity normal to the edge is zero and, in general, the deviation of this component from its value in the free stream is not small compared with the free stream speed. Near the edge it is not permissible either to linearise the equations of motion (except for slender wings) or to replace the boundary condition on the wing by the usual simplified condition, so that the predictions of linear theory in a small region containing the leading edge are likely to be unacceptable. In fact, the component of velocity (in the plane of the wing) normal to the edge is infinite according to linear theory.

Large local errors in the flow quantities should be of little consequence when only the forces acting on the wing are required, although the predictions of even these, using linear theory, are of doubtful value in certain circumstances[1]. When, as in boundary layer calculations, it is necessary to know the values of the flow quantities all over the wing, linear theory cannot be applied directly. Either non-linear differential equations must be solved, or a semi-empirical method must be developed which gives acceptable results in the vicinity of the leading edge, but does not significantly alter the linear theory results elsewhere.

This Note describes such a method. A solution of the linearised equation is found which, at the leading edge, satisfies the full boundary condition on the surface of a certain non-lifting wing with straight subsonic leading edges. This treatment is inconsistent (except for slender wings) because it retains second order terms in the boundary condition, but drops them in the differential equations; nevertheless, the flow quantities obtained by using it behave correctly near the leading edge while they do not significantly differ from the linear-theory values over most of the wing. The problem therefore, is solved by one of those methods the sole justification for which is that they give plausible results which can be checked by experiment.

2 DERIVATION OF AN INTEGRAL EQUATION FOR THE SOURCE DISTRIBUTION

X, Y and Z are rectangular Cartesian coordinates; the free stream flows in the X direction. The origin of coordinates is the apex of the wing, which is symmetrical about the planes $Y = 0$, $Z = 0$. The equation of the upper surface of the wing is

$$Y > 0, \quad Z = \delta(X - Y \tan \Lambda), \tag{1a}$$

$$Y < 0, \quad Z = \delta(X + Y \tan \Lambda). \tag{1b}$$

Λ is the sweepback angle of the leading edge; δ is a quantity small compared with unity. The wing has a straight leading edge and a wedge profile.

A disturbance velocity potential, ϕ, is introduced; ϕ is such that

$$\text{velocity in X direction} = U(1 + \phi_X),$$

$$\text{velocity in Y direction} = U\phi_Y,$$

$$\text{velocity in Z direction} = U\phi_Z,$$

where U is the free stream speed. ϕ is assumed to satisfy the linearised equation of supersonic flow,

$$B^2 \phi_{XX} = \phi_{YY} + \phi_{ZZ} .$$

B^2 is written for $(M^2 - 1)$, M being the free stream Mach number. Only subsonic leading edges are considered, so that $B \cot \Lambda < 1$.

ϕ_X, ϕ_Y and ϕ_Z must vanish everywhere outside the Mach cone from the apex of the wing, and the flow must be tangential to the wing surface. The latter requirement yields the following boundary condition to be satisfied on the special wing surface considered.

$$\delta(1 + \phi_X) - \delta \phi_Y \tan \Lambda - \phi_Z = 0 . \tag{2}$$

This differs from the usual form in that ϕ_x is not ignored altogether. No significant difference is found when this is satisfied on the actual wing surface rather than on the plane $Z = 0$, and the latter procedure will be followed (as in thin-wing theory). Eqn.(2) is the condition on that part only of the wing for which $Y > 0$.

Making the transformation

$$x = \frac{X - BY}{\sqrt{2} B}, \quad y = \frac{X + BY}{\sqrt{2} B}, \quad z = Z , \tag{3}$$

the linearised equation becomes

$$2 \phi_{xy} = \phi_{zz} , \tag{4}$$

and the boundary condition Eqn.(2) becomes

$$\delta\left\{1 + \frac{1}{\sqrt{2}} \frac{B + \cot \Lambda}{B \cot \Lambda} \phi_x - \frac{1}{\sqrt{2}} \frac{B - \cot \Lambda}{B \cot \Lambda} \phi_y\right\} = \phi_z ,$$

on $$z = 0 \quad \text{for} \quad Y \geq 0 \tag{5}$$

In these coordinates the equation in the plane $Z = 0$ of the trace of the Mach cone from the apex is

$$x = 0, \text{ for } Y \geq 0; \quad y = 0, \text{ for } Y \leq 0.$$

Eqn.(4) is solved by distributing sources over the wing;

$$\phi = -\frac{1}{\pi}\iint_S \frac{f(x_1,y_1)\ dy_1\ dx_1}{[2(x - x_1)\ (y - y_1) - z^2]^{\frac{1}{2}}} \ .$$

f gives the source distribution, and S is that part of the wing surface cut out by the Mach forecone from (x,y,z). On the wing surface, which is to be replaced by z = 0, it is known that

$$\phi_z = f(x,y), \tag{7a}$$

$$\phi_x = -\frac{1}{\sqrt{2}\pi}\frac{\partial}{\partial x}\iint_S \frac{f(x_1,y_1)\ dy_1\ dx_1}{[(x - x_1)\ (y - y_1)]^{\frac{1}{2}}} \ , \tag{7b}$$

$$\phi_y = -\frac{1}{\sqrt{2}\pi}\frac{\partial}{\partial y}\iint_S \frac{f(x_1,y_1)\ dy_1\ dx_1}{[(x - x_1)\ (y - y_1)]^{\frac{1}{2}}} \ . \tag{7c}$$

From the symmetry of the problem, $f(x,y) = f(y,x)$, and it suffices to consider points for which $x \leqslant y$. The boundary condition is then Eqn. (5).

In Fig.1 the area ABCD is S. A, B, C, D, E, F, G and H are respectively the points (0,0), $(x,\lambda x)$, (x,y), $(\lambda y,y)$, (x,x), $(\lambda y,\lambda y)$, $(\lambda x,x)$ and $(\lambda x,\lambda x)$. It is seen that

$$\iint_S \frac{f(x_1,y_1)\ dy_1\ dx_1}{[(x - x_1)\ (y - y_1)]^{\frac{1}{2}}} = \int_0^{\lambda y}\int_{x_1}^{x_1/\lambda} \frac{f(x_1,y_1)\ dy_1\ dx_1}{[(x - x_1)\ (y - y_1)]^{\frac{1}{2}}}$$

$$+ \int_{\lambda y}^{x}\int_{x_1}^{y} \frac{f(x_1,y_1)\ dy_1\ dx_1}{[(x - x_1)\ (y - y_1)]^{\frac{1}{2}}} + \int_0^{x}\int_{\lambda x_1}^{x_1} \frac{f(y_1,x_1)\ dy_1\ dx_1}{[(x - x_1)\ (y - y_1)]^{\frac{1}{2}}} \ ,$$

if the integration with respect to y_1 is performed first, and that

$$\iint_S \frac{f(x_1,y_1)\, dy_1\, dx_1}{[(x - x_1)\,(y - y_1)]^{\frac{1}{2}}} = \int_0^x \int_{\lambda y_1}^{y_1} \frac{f(x_1,y_1)\, dx_1\, dy_1}{[(x - x_1)\,(y - y_1)]^{\frac{1}{2}}}$$

$$+ \int_x^y \int_{\lambda y_1}^{x} \frac{f(x_1,y_1)\, dx_1\, dy_1}{[(x - x_1)\,(y - y_1)]^{\frac{1}{2}}} + \int_0^{\lambda x} \int_{y_1}^{y_1/\lambda} \frac{f(y_1,x_1)\, dx_1\, dy_1}{[(x - x_1)\,(y - y_1)]^{\frac{1}{2}}}$$

$$+ \int_{\lambda x}^{x} \int_{y_1}^{x} \frac{f(y_1,x_1)\, dx_1\, dy_1}{[(x - x_1)\,(y - y_1)]^{\frac{1}{2}}} \;,$$

if the integration with respect to x_1 is performed first. From the first of these formulae, after integration by parts, followed by differentiation with respect to x, it can be shown that

$$\frac{\partial}{\partial x} \iint_S \frac{f(x_1,y_1)\, dy_1\, dx_1}{[(x - x_1)\,(y - y_1)]^{\frac{1}{2}}} = \frac{f(\lambda)}{\lambda} \int_0^{\lambda y} \frac{dx_1}{[(x - x_1)\,(y - x_1/\lambda)]^{\frac{1}{2}}}$$

$$- \lambda\, f(\lambda) \int_0^x \frac{dx_1}{[(x - x_1)\,(y - \lambda x_1)]^{\frac{1}{2}}} + \iint_{S_1} \frac{f_{x_1}(x_1,y_1)}{[(x - x_1)\,(y - y_1)]^{\frac{1}{2}}}$$

$$+ \iint_{S_2} \frac{f_{x_1}(y_1,x_1)\, dx_1\, dy_1}{[(x - x_1)\,(y - y_1)]^{\frac{1}{2}}} \;.$$

S_1 and S_2 are respectively the areas AECD and ABE in Fig.1. Since the flow field is conical with A as apex (there is no determining length in the problem), $f(\lambda t,t)$ is constant; it has been written as $f(\lambda)$. Hence,

$$\frac{\partial}{\partial x} \iint_S \frac{f(x_1,y_1)\, dy_1\, dx_1}{[(x - x_1)\,(y - y_1)]^{\frac{1}{2}}} = \frac{2\, f(\lambda)}{\lambda^{\frac{1}{2}}} \sinh^{-1} \left(\frac{\lambda y}{x - \lambda y}\right)^{\frac{1}{2}}$$

$$-2\lambda^{\frac{1}{2}}\, f(\lambda)\, \sinh^{-1}\left(\frac{\lambda x}{y - \lambda x}\right)^{\frac{1}{2}} + \iint_{S_1} \frac{f_{x_1}(x_1,y_1)\, dx_1\, dy_1}{[(x - x_1)\,(y - y_1)]^{\frac{1}{2}}}$$

$$+ \iint_{S_2} \frac{f_{x_1}(y_1,x_1)\, dx_1\, dy_1}{[(x - x_1)\,(y - y_1)]^{\frac{1}{2}}} \;.$$

It follows, from Eqn. (7b), that ϕ_x will become infinite along the leading edge $(x = \lambda y)$ unless $f(\lambda) = 0$. Assuming that this is so,

$$\phi_x = -\frac{1}{\sqrt{2\pi}} \iint\limits_{S_1} \frac{f_{x_1}(x_1,y_1)\, dx_1\, dy_1}{[(x-x_1)(y-y_1)]^{\frac{1}{2}}} - \frac{1}{\sqrt{2\pi}} \iint\limits_{S_2} \frac{f_{x_1}(y_1,x_1)\, dx_1\, dy_1}{[(x-x_1)(y-y_1)]^{\frac{1}{2}}} \, .$$

Similarly,

$$\phi_y = -\frac{1}{\sqrt{2\pi}} \iint\limits_{S_1} \frac{f_{y_1}(x_1,y_1)\, dx_1\, dy_1}{[(x-x_1)(y-y_1)]^{\frac{1}{2}}} - \frac{1}{\sqrt{2\pi}} \iint\limits_{S_2} \frac{f_{y_1}(y_1,x_1)\, dx_1\, dy_1}{[(x-x_1)(y-y_1)]^{\frac{1}{2}}} \, .$$

Polar coordinates are now introduced;

$$x = r \sin\theta,\ y = r\cos\theta,\ x_1 = r_1 \sin\theta_1,\ y_1 = r_1\cos\theta_1 \, . \qquad (8)$$

f is a function of θ only; $f(r\sin\theta, r\cos\theta)$ will be written as $f(\theta)$, so that $f(r\cos\theta, r\sin\theta)$ is $f(\pi/2 - \theta)$. Then,

$$f_{x_1}(x_1,y_1) = \frac{\cos\theta}{r} f'(\theta),$$

$$f_{y_1}(x_1,y_1) = \frac{\sin\theta}{r} f'(\theta),$$

$$f_{x_1}(y_1,x_1) = \frac{\cos\theta}{r} f'\left(\frac{\pi}{2} - \theta\right),$$

$$f_{y_1}(y_1,x_1) = \frac{\sin\theta}{r} f'\left(\frac{\pi}{2} - \theta\right).$$

Dashes denote differentiation with respect to θ. After an integration with respect to r_1,

$$\phi_x = -\frac{1}{\sqrt{2\pi}} \int_{\tan^{-1}\lambda}^{\pi/4} d\theta_1\, f'(\theta_1)\, (\cot\theta_1)^{\frac{1}{2}} \log \frac{(\cot\theta_1)^{\frac{1}{2}} + (\cot\theta)^{\frac{1}{2}}}{\left|(\cot\theta_1)^{\frac{1}{2}} - (\cot\theta)^{\frac{1}{2}}\right|}$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{\pi/4}^{\tan(1/\lambda)} d\theta_1\, f'\left(\frac{\pi}{2} - \theta_1\right) (\cot\theta_1)^{\frac{1}{2}} \log \frac{(\cot\theta)^{\frac{1}{2}} + (\cot\theta_1)^{\frac{1}{2}}}{(\cot\theta)^{\frac{1}{2}} - (\cot\theta_1)^{\frac{1}{2}}} \, .$$

Hence,

$$\phi_x = \frac{1}{\sqrt{2\pi}} \int_{\tan^{-1}\lambda}^{\pi/4} d\theta_1\, f(\theta_1) \frac{d}{d\theta_1} \left\{ (\cot\theta_1)^{\frac{1}{2}} \log \frac{(\cot\theta_1)^{\frac{1}{2}} + (\cot\theta)^{\frac{1}{2}}}{\left|(\cot\theta_1)^{\frac{1}{2}} - (\cot\theta)^{\frac{1}{2}}\right|} \right\}$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{\pi/4}^{\tan^{-1}(1/\lambda)} d\theta_1\, f\left(\frac{\pi}{2} - \theta_1\right) \frac{d}{d\theta_1} \left\{ (\cot\theta_1)^{\frac{1}{2}} \log \frac{(\cot\theta)^{\frac{1}{2}} + (\cot\theta_1)^{\frac{1}{2}}}{(\cot\theta)^{\frac{1}{2}} - (\cot\theta_1)^{\frac{1}{2}}} \right. \, .$$

Putting $u = (\tan\theta)^{\frac{1}{2}}$, $u_1 = (\tan\theta_1)^{\frac{1}{2}}$ in the first integral, $u_1 = (\cot\theta_1)^{\frac{1}{2}}$ in the second integral, and writing $f(u)$ for $f(\tan^{-1} u^2)$,

$$\phi_x = \frac{1}{\sqrt{2\pi}} \int_{\lambda^{\frac{1}{2}}}^{1} du_1 \; f(u_1) \frac{d}{du_1}\left\{\frac{1}{u_1} \log \frac{u_1+u}{|u_1-u|}\right\} - \frac{1}{\sqrt{2\pi}} \int_{\lambda^{\frac{1}{2}}}^{1} du_1 \; f(u_1) \frac{d}{du_1}\left\{u_1 \log \frac{1+uu_1}{1-uu_1}\right\}. \tag{9a}$$

Similarly,

$$\phi_y = \frac{1}{\sqrt{2\pi}} \int_{\lambda^{\frac{1}{2}}}^{1} du_1 \; f(u_1) \frac{d}{du_1}\left\{u_1 \log \frac{u_1+u}{|u_1-u|}\right\} + \frac{1}{\sqrt{2\pi}} \int_{\lambda^{\frac{1}{2}}}^{1} du_1 \; f(u_1) \frac{d}{du_1}\left\{\frac{1}{u_1} \log \frac{1+uu_1}{1-uu_1}\right\}. \tag{9b}$$

Using Eqn.(9a), (9b) and (7a), the boundary condition (5) becomes

$$\begin{aligned}
\delta\Bigg[1 &+ \frac{(B+\cot\Lambda)}{2\pi B\cot\Lambda} \int_{\lambda^{\frac{1}{2}}}^{1} du_1 \; f(u_1) \frac{d}{du_1}\left\{\frac{1}{u_1} \log \frac{u_1+u}{|u_1-u|}\right\} \\
&- \frac{(B+\cot\Lambda)}{2\pi B\cot\Lambda} \int_{\lambda^{\frac{1}{2}}}^{1} du_1 \; f(u_1) \frac{d}{du_1}\left\{u_1 \log \frac{1+uu_1}{1-uu_1}\right\} \\
&+ \frac{(B-\cot\Lambda)}{2\pi B\cot\Lambda} \int_{\lambda^{\frac{1}{2}}}^{1} du_1 \; f(u_1) \frac{d}{du_1}\left\{u_1 \log \frac{u_1+u}{|u_1-u|}\right\} \\
&- \frac{(B-\cot\Lambda)}{2\pi B\cot\Lambda} \int_{\lambda^{\frac{1}{2}}}^{1} du_1 \; f(u_1) \frac{d}{du_1}\left\{\frac{1}{u_1} \log \frac{1+uu_1}{1-uu_1}\right\}\Bigg] \\
&= f(u). \\
&(\lambda^{\frac{1}{2}} \leqslant u \leqslant 1).
\end{aligned} \tag{10}$$

This is an integral equation for the source distribution $f(u)$. λ is given by Eqn.(6).

sides differ by terms of order no higher than δ^2. The solution (in this sense) for $\lambda = 0$ (sonic leading edge) is

$$f = \frac{2\delta}{\pi}\sin^{-1}\frac{u}{\left[\frac{(B^2+1)\delta}{2B} + u^2\right]^{\frac{1}{2}}} ;$$

that this is a solution is shown in the Appendix to Ref. 1.

A solution of Eqn. (10) can also be found if λ is not too close to zero. Eqn. (10) can be written

$$\delta\left[1 - \frac{(B + \cot\Lambda)}{2\pi\lambda^{\frac{1}{2}} B\cot\Lambda}\int_{\lambda^{\frac{1}{2}}}^{1}\frac{f(u_1)du_1}{u_1 - u} - \frac{(B + \cot\lambda)u}{2\pi\lambda^{\frac{1}{2}} B\cot\Lambda}\int_{\lambda^{\frac{1}{2}}}^{1}\frac{f(u_1)du_1}{1 - uu_1}\right.$$

$$- \frac{\lambda^{\frac{1}{2}}(B-\cot\Lambda)}{2\pi B\cot\Lambda}\int_{\lambda^{\frac{1}{2}}}^{1}\frac{f(u_1)du_1}{u_1 - u} - \frac{\lambda^{\frac{1}{2}}(B-\cot\Lambda)u}{2\pi B\cot\Lambda}\int_{\lambda^{\frac{1}{2}}}^{1}\frac{f(u_1)du_1}{1 - uu_1}$$

$$- \frac{(B+\cot\Lambda)}{2\pi B\cot\Lambda}\int_{\lambda^{\frac{1}{2}}}^{1} du_1\, f(u_1)\left\{\frac{1}{u_1^2}\log\frac{u_1+u}{|u_1-u|} - \frac{1}{u_1(u+u_1)} - \left(\frac{1}{\lambda^{\frac{1}{2}}} - \frac{1}{u_1}\right)\frac{1}{(u_1-u)}\right\}$$

$$- \frac{(B+\cot\Lambda)}{2\pi B\cot\Lambda}\int_{\lambda^{\frac{1}{2}}}^{1} du_1\, f(u_1)\left\{\log\frac{1+uu_1}{1-uu_1} - \frac{1}{1+uu_1} - \frac{(u-\lambda^{\frac{1}{2}})}{\lambda^{\frac{1}{2}}(1-uu_1)}\right\}$$

$$+ \frac{(B-\cot\Lambda)}{2\pi B\cot\Lambda}\int_{\lambda^{\frac{1}{2}}}^{1} du_1\, f(u_1)\left\{\log\frac{u_1+u}{|u_1-u|} + \frac{u_1}{u_1+u} - \frac{(u_1-\lambda^{\frac{1}{2}})}{(u_1-u)}\right\}$$

$$\left. + \frac{(B-\cot\Lambda)}{2\pi B\cot\Lambda}\int_{\lambda^{\frac{1}{2}}}^{1} du_1\, f(u_1)\left\{\frac{1}{u_1^2}\log\frac{1+uu_1}{1-uu_1} - \frac{u}{u_1(1+uu_1)} - \frac{u}{u_1} - \frac{u(u-\lambda^{\frac{1}{2}})}{1-uu_1}\right\}\right]$$

$$= f(u)\,. \qquad (\lambda^{\frac{1}{2}} \leq u \leq 1). \tag{11}$$

If λ is not too close to zero the only integrals which do not remain of at most order δ throughout the range of u are the first and third. (There would be difficulties near $u = 1$ if these two integrals only were retained and the second and fourth integrals must be kept also). The remaining integrals can be neglected. Even if f is put equal to δ everywhere (the zero of $f(u)$ at $u = \sqrt{\lambda}$ being ignored) the last four integrals combine to give an expression always of order δ (for λ not too close to zero),

$$- \frac{\delta}{2\pi \text{ B} \cot \Lambda} \left[\frac{(\text{B} + \cot \Lambda)}{\lambda^{\frac{1}{2}}} + \lambda^{\frac{1}{2}} (\text{B} - \cot \Lambda) \right] \log (u + \lambda^{\frac{1}{2}})$$

$$+ \frac{\delta}{2\pi \text{ B} \cot \Lambda} \left[\lambda^{\frac{1}{2}} (\text{B} + \cot \Lambda) + \frac{1}{\lambda^{\frac{1}{2}}} (\text{B} - \cot \Lambda) \right] \log (1 + \lambda^{\frac{1}{2}} u)$$

$$- \frac{\delta}{\pi \text{B}} \left(\lambda^{\frac{1}{2}} - \frac{1}{\lambda^{\frac{1}{2}}} \right) \log (1 - \lambda^{\frac{1}{2}} u) \ . \qquad (12)$$

If λ tends to zero the only term in Eqn.(12) which does not remain of order δ is

$$- \frac{\delta(\text{B} + \cot \Lambda)}{2\pi \text{ B} \cot \Lambda} \frac{1}{\lambda^{\frac{1}{2}}} \log \frac{u + \lambda^{\frac{1}{2}}}{1 + \lambda^{\frac{1}{2}} u} \ . \qquad (13)$$

The denominator ensures that Eqn.(13) like Eqn.(12) is finite when $\lambda = 1$. The largest value of Eqn.(13) occurs when $u = \sqrt{\lambda}$, this value being

$$- \frac{\delta(\text{B} + \cot \Lambda)}{2\pi \text{ B} \cot \Lambda} \frac{1}{\lambda^{\frac{1}{2}}} \log \frac{2\lambda^{\frac{1}{2}}}{1 + \lambda} = k\delta \ .$$

It seems likely that the last four integrals in Eqn.(11) can be neglected provided that k is of order unity. In Fig.2 curves have been drawn of μ (the free stream Mach angle, i.e. $\cot^{-1}$ B) against Λ; on these curves

$$- \frac{(\text{B} + \cot \Lambda)}{2\pi \text{ B} \cot \Lambda} \frac{1}{\lambda^{\frac{1}{2}}} \log \left(\frac{2\lambda^{\frac{1}{2}}}{1 + \lambda} \right) = k,$$

k taking the values ∞, 2.5, 0.4 and 0. $k = \infty$ is given by the line $\Lambda = \mu$ (corresponding to sonic leading edges) and $k = 0$ is given by the lines $\Lambda = \pi/2$ and $\mu = \pi/2$. It is seen that the last four integrals in Eqn.(11) can be neglected for most wings with subsonic leading edges.

Eqn.(11) can then be written

$$\delta \left[1 - \frac{1}{\pi(1-\text{B}^2 \cot^2 \Lambda)^{\frac{1}{2}} \sin \Lambda \cos \Lambda} \int_{\lambda^{\frac{1}{2}}}^{1} du_1 \ f(u_1) \left\{ \frac{1}{u_1 - u} + \frac{u}{1 - uu_1} \right\} \right] = f(u).$$

$$(\lambda^{\frac{1}{2}} \leqslant u \leqslant 1). \qquad (14)$$

Putting

$$\frac{u - \lambda^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}} = v, \quad \frac{u_1 - \lambda^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}} = v_1 ,$$

$$\frac{f(u)}{\pi(1 - B^2\cot^2\Lambda)^{\frac{1}{2}} \sin\Lambda \cos\Lambda} = g(v) ,$$

$$\frac{\delta}{\pi(1 - B^2\cot^2\Lambda)^{\frac{1}{2}} \sin\Lambda \cos\Lambda} = \varepsilon , \qquad (15)$$

Eqn. (14) becomes

$$\varepsilon\left[1 - \int_0^1 \frac{g(v_1)dv_1}{v_1 - v} - \int_0^1 \frac{g(v_1)\{\lambda^{\frac{1}{2}} + (1-\lambda^{\frac{1}{2}})v\}\,dv_1}{(1+\lambda^{\frac{1}{2}}) - \lambda^{\frac{1}{2}}(v+v_1) - (1-\lambda^{\frac{1}{2}})vv_1}\right] = g(v) .$$

$$(0 \leqslant v \leqslant 1) .$$

The solution of this equation for small v is the same as that of

$$\varepsilon\left[1 - \int_0^1 \frac{g(v_1)\,dv_1}{v_1 - v}\right] = g(v) ,$$

since the second term is then of order ε. The latter equation is the integral equation for the source distribution which gives the incompressible flow past a wedge of angle 2ε and length unity; the solution, for small values of v, is

$$g = \varepsilon\, v^{\varepsilon} .$$

Hence, remembering that $u = (\tan\theta)^{\frac{1}{2}} = (x/y)^{\frac{1}{2}}$, the solution of Eqn. (10), when x/y is close to λ, is

$$f = \delta\left[\frac{(x/y) - \lambda^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}}\right]^{\varepsilon} .$$

From Eqn.(7a),

$$\phi_z = \delta\left[\frac{(x/y)^{\frac{1}{2}} - \lambda^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}}\right]^{\varepsilon} \quad ; \tag{16}$$

from Eqn.(5),

$$1 + \frac{1}{\sqrt{2}}\ \frac{B + \cot\Lambda}{B\cot\Lambda}\ \phi_n - \frac{1}{\sqrt{2}}\ \frac{B - \cot\Lambda}{B\cot\Lambda}\ \phi_y = \left[\frac{(x/y)^{\frac{1}{2}} - \lambda^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}}\right]^{\varepsilon} . \tag{17}$$

In the plane $z = 0$, the flow velocity in the direction normal to the leading edge is

$$U\left[\cos\Lambda + \frac{B\sin\Lambda + \cos\Lambda}{\sqrt{2}\,B}\ \phi_x - \frac{B\sin\Lambda - \cos\Lambda}{\sqrt{2}\,B}\ \phi_y\right], \tag{18}$$

and at the leading edge this vanishes because of Eqn.(17). Since Eqn.(16) also vanishes at the leading edge, the velocity there is parallel to the edge.

The above results give the velocities at and very close to the leading edge. Thin wing theory provides acceptable results for the velocities over most of the rest of the wing, and the problem is reduced to finding a plausible way of linking up the two sets of results. On thin wing theory Eqn.(17) becomes

$$1 + \frac{1}{\sqrt{2}}\ \frac{B + \cot\Lambda}{B\cot\Lambda}\ \phi_n - \frac{1}{\sqrt{2}}\ \frac{B - \cot\Lambda}{B\cot\Lambda}\ \phi_y =$$

$$1 - \frac{2\delta}{\pi(1-B^2\cot^2\Lambda)^{\frac{1}{2}}\sin\Lambda\cos\Lambda}\ \sinh^{-1}\left(\frac{\lambda y}{x-\lambda y}\right)^{\frac{1}{2}} + \frac{2(1-\cot^2\Lambda)\delta}{\pi(1-B^2\cot^2\Lambda)^{\frac{1}{2}}\cot\Lambda}\sinh^{-1}\left(\frac{\lambda x}{y-\lambda x}\right)^{\frac{1}{2}} =$$

$$1 + \varepsilon\log\frac{(x/y)^{\frac{1}{2}} - \lambda^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}} - (\sin^2\Lambda - \cos^2\Lambda)\ \varepsilon\log\frac{1 - (\lambda\, x/y)^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}}$$

$$- \varepsilon\log\left[(x/y)^{\frac{1}{2}} + \lambda^{\frac{1}{2}}\right] + (\sin^2\Lambda - \cos^2\Lambda)\,\varepsilon\log\left[1 + (\lambda x/y)^{\frac{1}{2}}\right]$$

$$+ 2\,\varepsilon\cos^2\Lambda\log\,(1 - \lambda^{\frac{1}{2}}). \qquad (\lambda \leqslant x/y \leqslant 1). \tag{19}$$

The right hand side of Eqn.(17) (which holds for $^x/y$ close to λ) may be expanded in the form

$$\left[\frac{(^x/y)^{\frac{1}{2}} - \lambda^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}}\right]^{\varepsilon} = 1 + \varepsilon \log \frac{(^x/y)^{\frac{1}{2}} - \lambda^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}} + \ldots. \tag{20}$$

Comparison of Eqn.(19) with (20) suggests writing

$$-\frac{1}{\sqrt{2}}\ \frac{B + \cot \Lambda}{B \cot \Lambda}\ \phi_x + \frac{1}{\sqrt{2}}\ \frac{B - \cot \Lambda}{B \cot \Lambda}\ \phi_y = 1 - \left[\frac{(^x/y)^{\frac{1}{2}} - \lambda^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}}\right]^{\varepsilon}$$

$$+ (\sin^2 \Lambda - \cos^2 \Lambda)\ \varepsilon \log \frac{1 - (^{\lambda x}/y)^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}} + \varepsilon \log \left[(^x/y)^{\frac{1}{2}} + \lambda^{\frac{1}{2}}\right]$$

$$- (\sin^2 \Lambda - \cos^2 \Lambda) \varepsilon \log \left[1 + (^{\lambda x}/y)^{\frac{1}{2}}\right] - 2\, \varepsilon \cos^2 \Lambda \log (1 - \lambda^{\frac{1}{2}}) \tag{21}$$

everywhere on the wing. Eqn.(21) is approximately the same as Eqn.(17) if $^x/y$ is close to λ; on the other hand, Eqn.(21) is approximately the same as Eqn.(19) if $(^x/y)^{\frac{1}{2}} - \lambda^{\frac{1}{2}}$ is not too small. Therefore Eqn.(21) will be used as a relation between ϕ_x and ϕ_y for all points on the wing. (At the leading edge the velocity component normal to the edge is then no longer zero, but it is a small quantity of order δ).

A second relation between ϕ_x and ϕ_y is obtained by considering the component of velocity in the direction parallel to the leading edge. This component is

$$U \sin \Lambda \left[1 + \frac{1 - B \cot \Lambda}{\sqrt{2}\, B}\ \phi_x + \frac{1 + B \cot \Lambda}{\sqrt{2}\, B}\ \phi_y \right], \tag{22}$$

and, on thin wing theory, this is

$$U \sin \Lambda \left[1 - \frac{4\delta \cot \Lambda}{\pi (1 - B^2 \cot^2 \Lambda)^{\frac{1}{2}}} \sinh^{-1} \left(\frac{\lambda x}{y - \lambda x}\right)^{\frac{1}{2}} \right]. \tag{23}$$

Eqn.(23) gives plausible values everywhere over the wing. It will be assumed that Eqn.(22) and (23) are equal on the modified theory also, so that

$$\frac{(1-B\cot\Lambda)}{\sqrt{2}\,B\cot\Lambda}\,\phi_x + \frac{(1+B\cot\Lambda)}{\sqrt{2}\,B\cot\Lambda}\,\phi_y = \frac{-4\delta}{\pi(1-B^2\cot^2\Lambda)^{\frac{1}{2}}}\sinh^{-1}\left(\frac{\lambda x}{y-\lambda x}\right)^{\frac{1}{2}}$$

$$= -2\,\varepsilon\cos\Lambda\,\sin\Lambda\,\log\frac{1+(\lambda x/y)^{\frac{1}{2}}}{1-(\lambda x/y)^{\frac{1}{2}}}\;. \tag{24}$$

Eqn.(21) and (24) give

$$\phi_x = -\sqrt{2}\,\varepsilon\cos^2\Lambda\,\sin^2\Lambda\,(B-\cot\Lambda)\log\frac{1+(\lambda x/y)^{\frac{1}{2}}}{1-(\lambda x/y)^{\frac{1}{2}}}$$

$$+\frac{1}{\sqrt{2}}\cos\Lambda\,\sin\Lambda\,(1+B\cot\Lambda)\left\{\left[\frac{(x/y)^{\frac{1}{2}}-\lambda^{\frac{1}{2}}}{1-\lambda^{\frac{1}{2}}}\right]^{\varepsilon}\right.$$

$$-1-(\sin^2\Lambda-\cos^2\Lambda)\,\varepsilon\log\frac{1-(\lambda x/y)^{\frac{1}{2}}}{1-\lambda^{\frac{1}{2}}}-\varepsilon\log\left[(x/y)^{\frac{1}{2}}+\lambda^{\frac{1}{2}}\right]$$

$$\left.+2\,\varepsilon\cos^2\Lambda\log(1-\lambda^{\frac{1}{2}})+(\sin^2\Lambda-\cos^2\Lambda)\,\varepsilon\log\left[1+(\lambda x/y)^{\frac{1}{2}}\right]\right\}\;,$$

$$\phi_y = -\sqrt{2}\,\varepsilon\cos^2\Lambda\,\sin^2\Lambda\,(B+\cot\Lambda)\log\frac{1+(\lambda x/y)^{\frac{1}{2}}}{1-(\lambda x/y)^{\frac{1}{2}}}$$

$$-\frac{1}{\sqrt{2}}\cos\Lambda\,\sin\Lambda\,(1-B\cot\Lambda)\left\{\left[\frac{(x/y)^{\frac{1}{2}}-\lambda^{\frac{1}{2}}}{1-\lambda^{\frac{1}{2}}}\right]^{\varepsilon}\right.$$

$$-1-(\sin^2\Lambda-\cos^2\Lambda)\,\varepsilon\log\frac{1-(\lambda x/y)^{\frac{1}{2}}}{1-\lambda^{\frac{1}{2}}}-\varepsilon\log\left[(x/y)^{\frac{1}{2}}+\lambda^{\frac{1}{2}}\right]$$

$$\left.+2\,\varepsilon\cos^2\Lambda\log(1-\lambda^{\frac{1}{2}})+(\sin^2\Lambda-\cos^2\Lambda)\,\varepsilon\log\left[1+(\lambda x/y)^{\frac{1}{2}}\right]\right\}\;.$$

ϕ_z comes from Eqn.(16). λ and ε are given by equations (6) and (15) respectively; x and y are defined by Eqn.(3).

From Eqn.(18) and (21) the velocity component on the wing in the direction normal to the leading edge is, apart from a factor U cos Λ,

$$\left[\frac{(x/y)^{\frac{1}{2}} - \lambda^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}}\right]^{\varepsilon} - (\sin^2 \Lambda - \cos^2 \Lambda)\, \varepsilon \log \frac{1 - (\lambda x/y)^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}}$$

$$- \varepsilon \log \left[(x/y)^{\frac{1}{2}} + \lambda^{\frac{1}{2}}\right] + (\sin^2 \Lambda - \cos^2 \Lambda)\, \varepsilon \log \left[1 + (\lambda x/y)^{\frac{1}{2}}\right]$$

$$+ 2\, \varepsilon \cos^2 \Lambda \log (1 - \lambda^{\frac{1}{2}}) \,. \qquad (25)$$

As already stated, this does not vanish at the leading edge but has a value of order δ there. If it is necessary for this component to vanish at the edge, this can be achieved by adding to Eqn.(25)

$$\varepsilon \left\{1 - \left[\frac{(x/y)^{\frac{1}{2}} - \lambda^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}}\right]^{\varepsilon}\right\} \left\{\log (2\, \lambda^{\frac{1}{2}}) + (\sin^2 \Lambda - \cos^2 \Lambda) \log \frac{1 + \lambda^{\frac{1}{2}}}{1 + \lambda}\right.$$

$$\left. - 2 \cos^2 \Lambda \log (1 - \lambda^{\frac{1}{2}})\right\} . \qquad (26)$$

Eqn.(26) is of order δ^2 over most of the wing, but the sum of Eqn.(25) and (26) is zero at the leading edge. The velocity component on the wing in the direction parallel to the leading edge is given by Eqn.(23), and the component normal to the wing by Eqn.(16).

4 RESULTS AND DISCUSSION

Some results have been obtained for symmetrical wings whose upper surfaces have equations of the form of (1a) and (1b).* The free stream Mach number was chosen to be √2 (i.e. B = 1) and δ to be 0.1; four values of Λ were chosen, 80°, 70°, 60° and 50°.

In Fig.3 the value of the velocity component on the wing in the direction normal to the leading edge is plotted against (X cot Λ - Y)/X cot Λ + Y). (The latter quantity is zero at the leading edge and unity at the centre line). Only that part of the wing for which Y ≥ 0 is considered. The figure shows the values obtained by using thin-wing theory (broken line) and those obtained by using the modified theory of this note (full line). For all four values of Λ there is very little difference between the two curves except for a very small region near the leading edge. As X/Y tends to tan Λ the velocity tends to negative infinity on thin-wing theory and to zero on the modified theory.

*The method can be extended without difficulty to wings with profiles which are sharp at the leading edge but which have equations more complicated than (1a) and (1b). The extra terms appearing in Eqn.(5) can be treated by ordinary linear theory without any difficulties arising at the leading edge.

These results need very little discussion. It is, perhaps, advisable to repeat that the modified theory developed in this note cannot be justified rigorously, (except, perhaps, for slender wings), because it neglects second order quantities in the differential equations of motion and, at the leading edge, retains them in the boundary condition on the wing. The main justification for the method is that it predicts plausible and continuous values for the velocity on a straight-edged wing everywhere, including the region near the leading edge.

LIST OF SYMBOLS

B	$(M^2 - 1)^{\frac{1}{2}}$
f	Source distribution
g	$\dfrac{f(u)}{\pi(1 - B^2 \cot^2 \Lambda)^{\frac{1}{2}} \cos \Lambda \sin \Lambda}$
k	Defined in section 3
M	Free stream Mach number
r, r_1	$(x^2 + y^2)^{\frac{1}{2}}$, $(x_1^2 + y_1^2)^{\frac{1}{2}}$
S	That part of the wing surface cut off by the Mach forecone from (x,y,z)
S_1, S_2	In Fig. 1, regions AECD and ABE respectively
U	Free stream speed
u	$(\tan \theta)^{\frac{1}{2}}$
u_1	Defined before (9a)
v	$\dfrac{u - \lambda^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}}$
v_1	$\dfrac{u_1 - \lambda^{\frac{1}{2}}}{1 - \lambda^{\frac{1}{2}}}$
X, Y, Z	Rectangular Cartesian coordinates defined in section 2
x, y, z	Defined by (3)
x_1, y_1	Variables of integration
δ	Defined by (1a) and (1b)
ε	$\dfrac{\delta}{\pi(1 - B^2 \cot^2 \Lambda)^{\frac{1}{2}} \cos \Lambda \sin \Lambda}$

LIST OF SYMBOLS (Contd.)

θ, θ_1 — $\tan^{-1} x/y$, $\tan^{-1} x_1/y_1$

Λ — Sweepback angle

λ — $(1 - B \cot \Lambda)/(1 + B \cot \Lambda)$

ϕ — Disturbance velocity potential, defined in section 2

REFERENCE

Ref.No.	Author	Title, etc.
1	Randall, D.G.	A technique for improving the predictions of linearised theory on the drag of straight-edged wings. C.P. 394 January, 1957

WT.2078.C.P.418.K3 - Printed in Great Britain

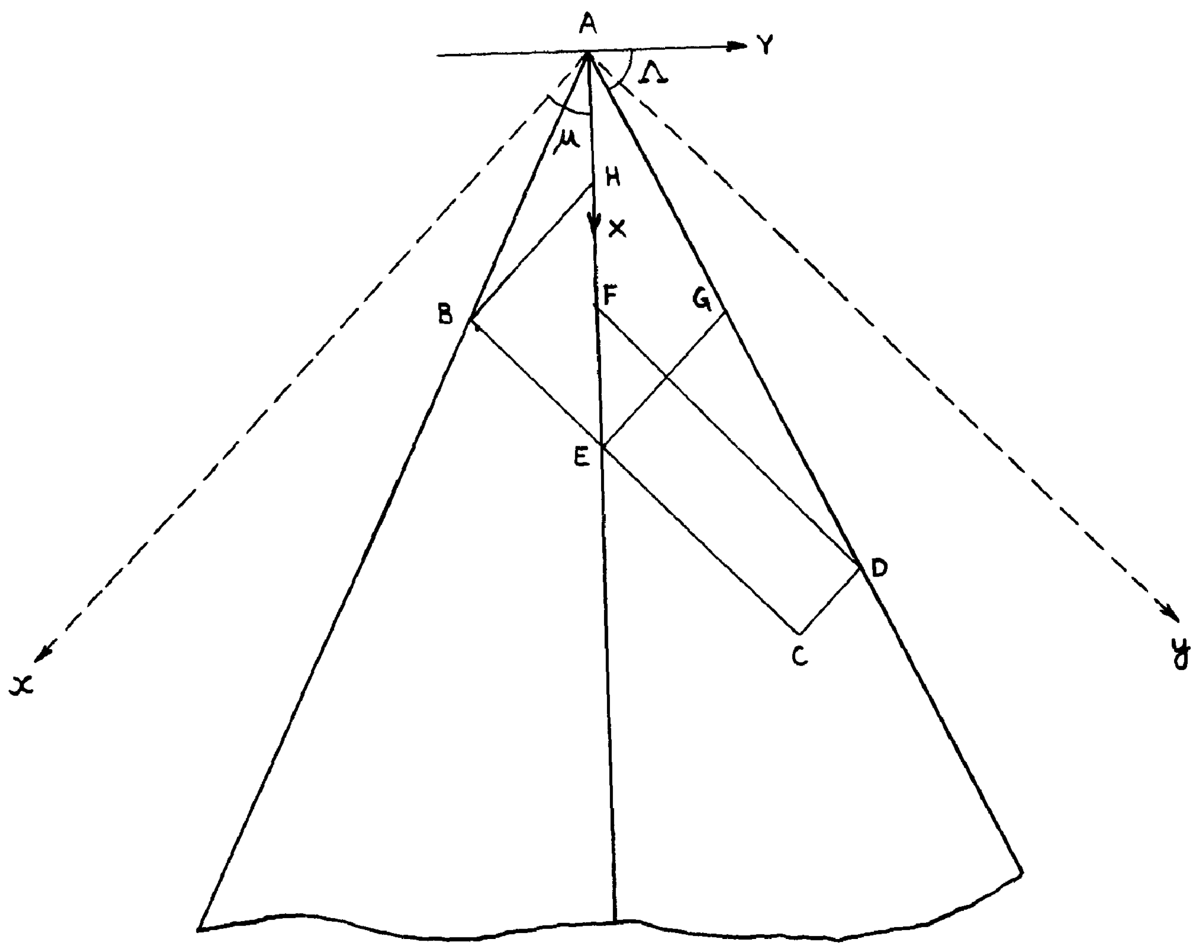

FIG. 1. GEOMETRY OF THE WING.

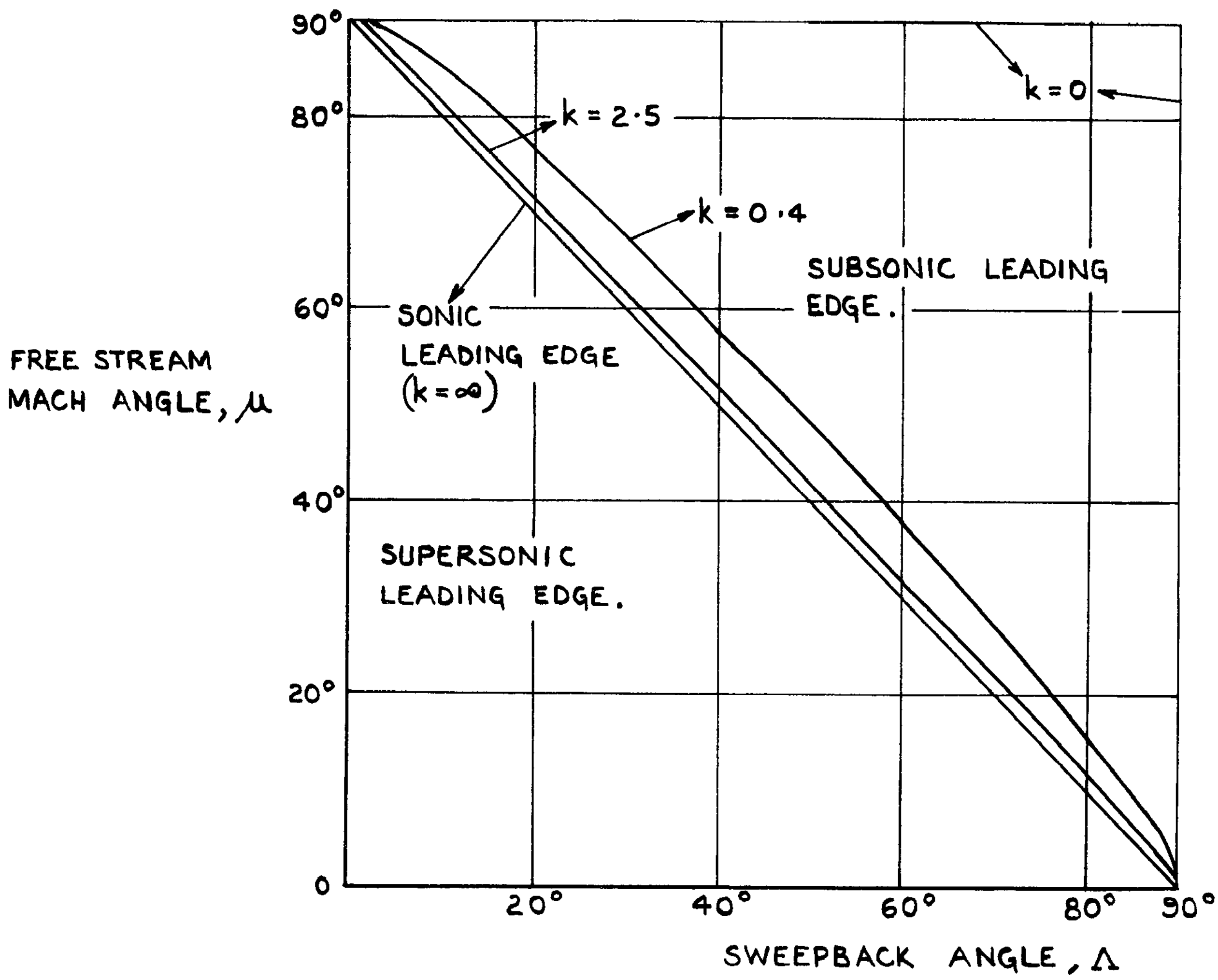

FIG. 2. PROBABLE RANGE OF VALIDITY OF THEORY.

($k\delta$ IS NEGLECTED IN COMPARISON WITH 1 IN THE THEORY.)

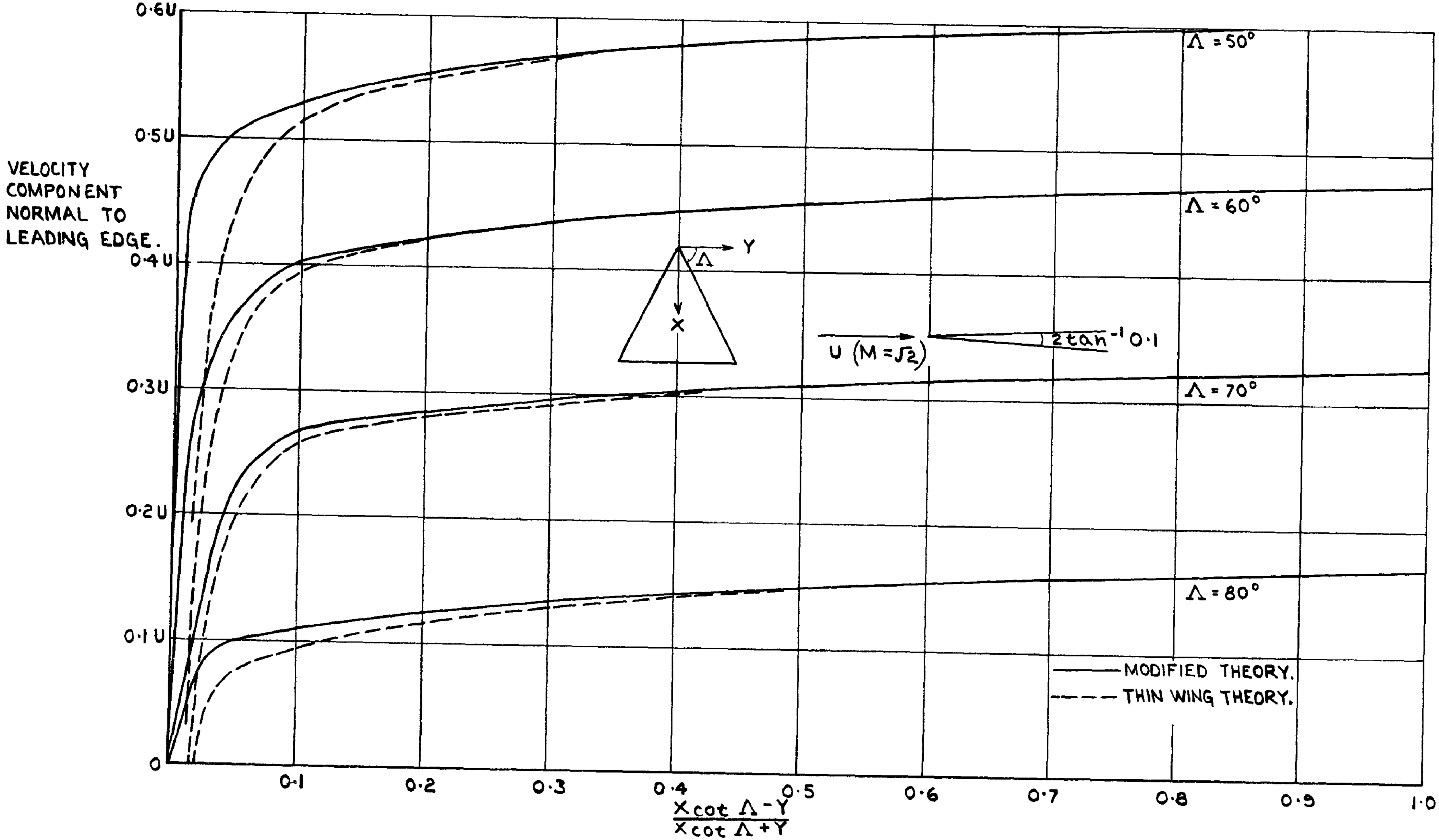

FIG. 3. VELOCITY COMPONENT ON THE WING IN DIRECTION NORMAL TO LEADING EDGE.

C.P. No. 418
(20,570)
A.R.C. Technical Report

Published by
HER MAJESTY'S STATIONERY OFFICE

To be purchased from
York House, Kingsway, London W.C.2
423 Oxford Street, London W.1
13A Castle Street, Edinburgh 2
109 St. Mary Street, Cardiff
39 King Street, Manchester 2
Tower Lane, Bristol 1
2 Edmund Street, Birmingham 3
80 Chichester Street, Belfast
or through any bookseller

Printed in Great Britain

C.P. No. 418
(20,570)
A.R.C. Technical Report