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An Electronic Analogue Computer Representing Twelve Coupled Linear Differential Equations

By W. D. Hicks

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An Electronic Analogue Computer Representing Twelve Coupled Linear Differential Equations

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Summary.
This paper describes the design, construction and operation of an electronic network providing an analogue of the behaviour of up to twelve coupled, second-order, linear differential equations. It was designed specifically for the solution of problems in aircraft flutter investigations but can be used for other associated problems or where a problem can be expressed in similar form.

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1. *Introduction.*

In 1952 an electronic analogue computer designed to solve flutter equations in up to six degrees of freedom was brought into operation at the R.A.E. and made available for use by the aircraft industry. The machine proved invaluable both for design and research purposes and several machines working on similar basic principles were built by individual aircraft firms.

As flutter began to assume greater importance in design, it became apparent that the capacity of the R.A.E. computer was inadequate. Accordingly, a further machine has been built capable of solving equations in up to twelve degrees of freedom, and having features enabling it to handle a wider range of problems than its predecessor.

This machine has now been in operation for over two years, and in addition to solving flutter equations has been used for calculating response to harmonic excitation at sub-critical speeds and in determining response to buffeting. In principle, the computer will deal with any problem that can be represented by a set of up to twelve coupled, second-order, linear differential equations with constant coefficients. For applications to dynamic response generally, it is thus capable of simulating systems with up to twelve degrees of freedom, in which the direct and coupling forces are linearly proportional to the various displacements, velocities and accelerations. Provision is made for applying to each degree of freedom time-dependent forcing functions, generated either internally (of step or sine-wave form) or externally; the latter facility allowing the use of experimentally-determined inputs. Non-linear coefficients representing zero force for a given range of motion can be added if required.

Over the period in which analogue computers have been developed at the R.A.E. for the solution of flutter equations, digital computers of large capacity have become available and are now used extensively in the field of flutter and dynamic response. For the specific tasks for which it was designed, however, the analogue computer described in this Report has a number of advantages over the digital computer; these include:

(1) Showing more quickly the effect of changes in parameters.

(2) Enabling measured inputs (buffet or gust pressures, landing forces, for example) to be applied directly to the system simulated, without recourse to the synthesis necessary to reduce the inputs to a form suitable for digital computation.

(3) Allowing real components (recording equipment, for example) to be used and tested in conjunction with the simulated system; and

(4) permitting the effect of non-linearities to be represented more easily.

In the main part of this Report, firstly the form of the equations that can be represented on the computer is discussed together with the methods used to determine the rate of decay of motion in the flutter stability problem. Next, the general arrangement and operation of the machine is described. A further Section covers design features of the main components. Detailed matters relating to the use of the machine in various applications are contained in the Appendices: Appendix A outlines the recommended procedure for scaling flutter equations; Appendix B comments on the use of the computer for the determination of rates of decay at subcritical speeds by means of the vector-response technique; Appendix C suggests a possible way of rearranging the elements of the computer to facilitate the representation of multi-point or distributed excitation; and Appendix D describes the simulation of a pattern of external loading with a specified time-sequence of application.

2. *The Equations to be Simulated.*

2.1. *The Basic Problem.*

The equation of motion for a mechanical system with a single degree of freedom and subjected to external loads may be written as the second order differential equation:

$$A\ddot{x}+D\dot{x}+Ex = f. \tag{1}$$

For flutter and response investigations the coefficients A, D, and E are regarded as constants representing the generalised intertia, damping, and stiffness coefficients respectively for a degree of freedom of the structure, while the external forces (f) may derive from mechanical (e.g. engine vibration) or aerodynamic sources.

The latter originate either from disturbances in the environment – such as gusts – or are a consequence of the motion of the structure itself.

The part of the external aerodynamic load that is associated with the motion of the structure can be transferred to the left-hand side of the above equation so that it becomes:

$$A\ddot{x}+D\dot{x}+Ex+A'\sigma\ddot{x}+BV\sqrt{\sigma}\dot{x}+CV^2x = f(t) \tag{2}$$

where A', B and C are regarded as constants and $A'\sigma$, $BV\sqrt{\sigma}$, and CV^2 represent the generalised aerodynamic inertia, damping, and stiffness coefficients, V is the equivalent airspeed and σ the ratio of the air density at altitude to the standard sea level density.

For a system with many coupled degrees of freedom the complete equations may hence be written in the matrix form:

$$[A+A'\sigma]\{\ddot{x}\}+[D+B\,V\sqrt{\sigma}]\{\dot{x}\}+[E+CV^2]\{x\} = \{f(t)\} \tag{3}$$

where [ ] signifies a square matrix and { } a column matrix.

In practice the aerodynamic inertia coefficient $A'\sigma$ is usually small compared with the structural coefficient A and, as it is also invariable with airspeed, it can for simplicity be included in A. This leads to some indefiniteness when σ is varied but this is generally sufficiently small to be neglected.

Representation of twelve coupled degrees of freedom requires the compilation of a twelve by twelve matrix of coefficients. This is, in itself, a considerable task, but the solution to the equations is the major difficulty. They are, however, amenable to simulation by analogue methods and the machine described here was designed for obtaining the particular solutions needed in flutter investigations. The equations themselves are, of course, applicable to a wide range of problems not necessarily limited to the dynamic response of aircraft although the layout of the machine is specifically associated with this field.

The displacement x and its derivatives with respect to time in the equations appear, in the electronic analogue, as voltages at the outputs of suitable amplifier circuits. These output voltages are modified, in each case, by potentiometers which can be manually adjusted to represent each coefficient. For example, the voltage $\ddot{x}$, is fed to a row of potentiometers which are set to represent $A_{11}, A_{21}, A_{31}, \ldots\ldots A_{12,1}$.

The coefficients V, V^2 and σ are also represented by potentiometer systems. In this case, however, the potentiometers are adjusted together from a central control unit to represent a change of airspeed (V) or density ratio (σ) throughout the problem.

2.2. *The Determination of the Mode Decay Rates.*

The measurement of the decay rates of the natural modes in a system of coupled degrees of freedom as described in equation (3) presents some difficulty. Two possible methods are:

(1) to excite a mode to a certain amplitude, remove the exciting force and count the cycles to half-amplitude,

(2) to obtain an amplitude-frequency curve in the vicinity of the mode frequency and calculate the decay rate from this.

Disadvantages of these methods are the extreme difficulty of exciting a particular mode without exciting other modes and thus destroying the purity of the oscillation, and the care which is necessary to maintain stable conditions when making comparative measurements of amplitude.

A superior method using additional instruments applied to the analogue machine is described by Irwin in Appendix B. Another method, which has been incorporated in the analogue network, is fully described elsewhere[2] but will be briefly outlined here.

The equation for a damped harmonic oscillation can be written as:

$$x = X_0 e^{\lambda t} \tag{4}$$

where x, X_0 are the displacements at time $t = t$, $t = 0$, and λ is complex, i.e. $(-\lambda' + j\lambda'')$.

The decay rate of an oscillation is given by the ratio, b, of the damping present to the critical damping, that is, by:

$$b = \frac{\lambda'}{\sqrt{\lambda'^2 + \lambda''^2}}$$

$$b \simeq \frac{\lambda'}{\lambda''} \tag{5}$$

provided that λ', i.e. b, is small. It should be noted that $\lambda'' = (1-b^2)^{\frac{1}{2}} \lambda_n''$ where λ_n'' is the undamped natural frequency.

An analogue system has been devised[2] which modifies equation (4) by introduction of the factor $e^{-1/\tau_f t}$ so that it becomes:

$$x = X_0 e^{\lambda t} \mathrm{e}^{-\frac{1}{\tau_f} t} \tag{6}$$

where τ_f is a manually adjusted time constant, which can be made equal and opposite to $\frac{1}{\lambda'}$. The circuits are such that this does not affect the frequency of the damped oscillation λ''. From the definition of equation (5) the damping ratio will then be:

$$b = \frac{1}{\lambda'' \tau_f}. \tag{7}$$

This can be straightforwardly extended to apply to the equations for a number of coupled degrees of freedom. In this case the same factor $e^{-\frac{1}{\tau_f}t}$ is introduced simultaneously in all equations until zero decay rate is observed in one of the modes of a previously sufficiently damped problem. A scale is provided on the machine which is proportional to $\frac{1}{\tau_f}$. If the scale reading for zero decay rate and the frequency of the undamped oscillation are noted the damping ratio for the mode can be calculated from:

$$b = \frac{\text{scale reading}}{\text{angular frequency}}$$

any increase of the scale reading above this setting will result in the problem becoming divergent in amplitude. Hence only the mode with the smallest value of b can be obtained in this way. By changing the sign of all damping coefficients throughout a problem it is also possible to obtain the percentage of critical damping in the mode with the largest value of b in a similar way.

2.3. *Introduction of Non-linear Effects.*

Non-linear effects in the mechanical system can be represented in the electronic analogue. To do this an additional circuit representing a non-linear coefficient can be manually inserted into any one of the terms in the equation of motion. The circuit is designed to introduce the characteristics of a 'dead-space'. As examples of its action, if the non-linear unit is inserted into a direct stiffness coefficient it will represent a region of zero stiffness for a specified range of movement, as might be encountered in practice due to backlash in an aircraft control circuit; if the unit is included in a direct damping coefficient it represents a region of zero damping for a specified range of velocity.

2.4. *Matrix Analysis.*

It is necessary for the successful simulation of a problem that the inertia and stiffness coefficient matrices should be positive definite when the airspeed V is zero. This will be so if the matrices represent the properties of a physical system. This means that the determinants of all the principal minors of the matrices A and E must be positive or zero, even if 'scaling' has made the matrices asymmetric (*see* Appendix A). From physical considerations these conditions must always be true of the inertia matrix A, irrespective of airspeed, but, as the airspeed V is increased, the determinant of the total stiffness matrix $E+CV^2$ may become negative. The condition where the latter is zero corresponds to the divergence speed for the particular coefficients used, which leads to a further application of the simulator. The appropriate equations for this case are:

$$[E+CV^2]\{x\} = \{f(t)\}.$$

A switch at the control desk alters the electronic circuits to check each matrix A and $(E+CV^2)$ in turn. If they are positive definite, indicating centre-zero meters on the desk remain at zero and will return to zero if the analogue system is disturbed, but if this condition is not satisfied, they will swing off to maximum reading immediately.

3. *General Arrangement.*

3.1. *Block Diagram.*

Fig. 1 is a block diagram showing the physical layout of the major components of the simulator. The cabinets numbered 1 to 12 each contain the coefficients in the direct terms of a degree of freedom with all the coupling coefficients from all other degrees of freedom. Each cabinet therefore contains the coefficients of one row of equation (3). Also built into each of the cabinets are the amplifiers and much of the switching equipment required to operate a single degree of freedom.

The manually adjusted units which are common to all degrees of freedom – these are airspeed V, density ratio parameter $\sqrt{\sigma}$, and controlled damping parameter $\frac{1}{\tau_f}$ – are located in the control desk together with the equipment for observation of the behaviour of the degrees of freedom and for the introduction of external forces. Any or all of the degrees of freedom may be locked and thus made inoperative from the desk.

3.2. *The Coefficient Cabinets.*

Fig. 2 is a view of the row of twelve coefficient cabinets. These are built together in a common frame, the panels, each containing sixty coefficient units, being arranged in sequence from one to twelve in the top half of the racks. Below each panel is a chassis containing amplifiers and associated circuitry for each degree of freedom. On the front panel of each chassis are the controls required to adjust the time-constant of each unit separately. These controls consist of a switch with three positions multiplying the basic time-constant by one, one-tenth and one-hundredth and a knob giving continuous variation of time-constant within these ranges. This facility for varying time-constant is a considerable asset in setting up a problem as it widens the bands of modes of oscillation which can be represented and can assist in obtaining maximum accuracy in setting up the coefficients when used as described by Niblett in Appendix A.

Mounted on top of the frame is the unit required for setting the values of the coefficients. This consists of a number of electrical components fixed on a trolley so that the whole unit can be moved along the row of coefficient panels and used to set all coefficients. As every coefficient unit is, therefore, set against a common standard included in the components on the trolley, the effects of long-term changes in them will be minimised. Because of its design, long-term changes in the standard are at a minimum in any case, but, should they occur as an overall change, are relatively unimportant because the resultant proportional settings of coefficients will remain constant. To set a coefficient, a plug attached to the standard on the trolley is inserted into a socket provided on each coefficient unit. The required value is then selected on the press-button box shown in Fig. 2 and can be a three-digit number or the maximum of 1000. The coefficient knob is then adjusted until balance is shown on the centre-zero meter on the trolley unit. This sets the coefficient as a positive number. To change it to a negative number it is necessary to insert a change-over plug unit into the coefficient socket. These plugs are held in containers on the trolley and are painted red for quick recognition of negative coefficients in the matrix. When a complete panel of coefficients has been set a transparent hinged screen can be folded up over the units to protect them from accidental rotation.

3.3. *Control Desk.*

Fig. 3 shows the control desk, and its centre panels. On the extreme right is a wheel which is used to control the density ratio parameter ($\sqrt{\sigma}$) in the problem. Above the wheel is a scale protruding through the desk top giving an indication of density ratio parameter in a hundred divisions. For most problems this scale would be initially kept at maximum reading, corresponding to sea-level. This position is indicated on the centre panel of the desk by an illuminated warning 'Altitude-Sea-level' which doubles in brightness and indicates 'Altitude' when the control is moved away from maximum.

Next to the 'Altitude' control a similar wheel with a small panel of two switches is used to control the damping parameter $\left(\frac{1}{\tau_f}\right)$ in the problem. An associated scale reading is provided through the desk top. One switch is for changing the scale range to read either ten per cent or one hundred per cent of critical damping at maximum setting, while the other switch changes the sign of the damping in each degree of freedom and also in the damping control circuits. A bright warning light appears at the centre panel of the desk unless the switch is in the positive damping position and the scale is at zero.

Resting on the desk-top in Fig. 3 is the press-button box for controlling the setting of airspeed (V) throughout the equations. The chosen speed is indicated through a window on the control desk panel and an electric motor then rotates a bank of potentiometers in the direction necessary to attain this setting. A change-over switch on the centre panel can be used to double the range of airspeed. This is shown on the control panel as an illuminated 'x1' or 'x2' against the airspeed setting.

At the top of the centre panel on the desk are two rows of six centre-zero meters. Each of these meters is connected, through a common five-position switch in the lower right-hand corner of the panel, to a coefficient cabinet.

The switch enables the operator to select which of the five direct terms in each of the twelve coupled equations shall be indicated on the meters. All meters read the similar term in their respective degrees of freedom. This avoids the confusion which would result from too many possible permutations. Beneath the meters is a row of twelve switches used to lock each rack separately.

The twelve output voltages from the five-position switch are also passed to the left-hand panel of the desk. On this panel they can be individually factored by adjustment of the controls arranged in two rows of six at the top of the panel. The switches below the central large meter can be set so that the meter will indicate either the output of any degree of freedom separately or the vector sum of all the factored outputs from all degrees of freedom. Switches are provided to rectify the output voltage before it is applied to the meter (useful at the higher frequencies where the meter response falls off) and, with balancing controls, to enable the operator to compare the amplitudes of all the degree of freedom outputs against any chosen one.

The right-hand panel may be regarded as the 'input' panel, representing the right-hand side of equation (3). Behind this panel is a sine-wave generator the frequency and amplitude of which are controlled from the front of the panel. The output of the generator can be switched to the two rows of controls at the top of the panel. These are used to adjust the proportion and sign of the forcing functions which can be applied to each degree of freedom. Jacks and switches are provided on this panel so that any form of external force can be plugged into all degrees of freedom at once or different forces from separate sources into any of the degrees of freedom at the operators' choice.

Also installed in the desk is an instrument for measuring frequency and phase relationship between oscillating degrees of freedom. Frequency is measured by timing the period of one cycle and phase by timing the separation between similar points on the two waveforms being compared. These times are measured to one millisecond.

The most commonly used forcing function applied to the matrix of equations is a single impulse. This is sufficient to set the circuits into oscillation. A separate push-button on the airspeed control box allows the operator to apply such a pulse at will. The pulse then passes through the controls on the right-hand panel before being applied to the input points of the twelve degrees of freedom. Also on this control box is fitted the main locking switch which controls those racks which have not been individually locked at the centre panel.

3.4. *Operating Instructions.*

The maximum number to which a coefficient may be set on the machine is 1000. Therefore the coefficients in a set of equations have to be scaled to make the greatest use of the available range with a proviso that all numbers should be made as large as possible. Scaling is fully described by Niblett in Appendix A.

Coefficients are set up against a common press-button bridge attached to the carriage shown above the coefficient cabinets in Fig. 2. A plug is inserted into each coefficient unit in turn. The required number is set into the press-button box and the coefficient potentiometer adjusted until a centre-zero reading meter on the carriage indicates a null balance. The control switches must be in the 'locked' condition while coefficients are being set up, or the setting will be inaccurate. This is, however, immediately apparent to the operator from the erratic behaviour of the needle on the balancing meter if the circuits are not locked.

In the centre of each of the bottom panels of the row of cabinets are the time-constant switches. On the left is a continuously variable control covering the range 0·000 to 1·000 and next to it is a three-position switch giving multipliers of 1·00, 10·0 and 100. If these controls are set to 1·000 × 100 the circuits are adjusted for the maximum basic operating frequency of the unit. This is $\omega = 100$ radians per second if the direct stiffness and inertia coefficients are equal. Use of these controls forms part of the complete scaling technique described in Appendix A.

At the control desk a panel of stencils are illuminated from behind to indicate the state of the various control circuits. These should be checked before commencing a run and the appropriate controls set to zero if necessary. It is advisable to switch the indicating meters on the desk to read the output of the 'E' amplifier as this is where drift, due to the amplifiers, or divergence, which may be a solution of the problem, can be most easily observed. Drift can be compensated for by adjusting the potentiometers above the central row of numbered lights and switches. The lights indicate which of the units is in operation when the main 'Locking' switch is used and the switches lock the individual degree of freedom units.

After this, the operator has only to observe the behaviour of his problem on the control desk meters and to take measurements and adjust parameters as required by the test programme. With a little experience, possible faults arising in the circuitry can be suspected by noting abnormal behaviour of the indicating meters. The five-position switch on the centre panel can be used to examine the output voltage of every amplifier and, by this means, the source of trouble can often be traced. Frequently some adjustment or balancing is sufficient to put matters right.

3.5. *Representation of the Equations.*

The five operational circuits used in representing the equations are shown in Fig. 4. These are well-known and will only be briefly described here.

Considering the summing unit of Fig. 4, which is a circuit around an amplifier designed to have a high gain and to be sign-reversing between its input and output points, it will be seen that the voltage appearing at the point A will be $\frac{X_0}{-G}$. If G (the amplifier gain) is sufficiently large this voltage can be taken to be zero. The current flowing through the resistor R_1 is therefore $\frac{X_1}{R_1}$ and the summation of the three currents shown flowing to the point A is:

$$\frac{X_1}{R_1}+\frac{X_2}{R_2}+\frac{X_0}{R_3} = 0 .$$

This assumes that the error current flowing into the amplifier at point A is negligibly small which would be so if the gain and input impedance of the amplifier were sufficiently high.

Similarly in the integrating unit the summation of the current at the amplifier input point is:

$$\frac{X_1}{R}+\bar{C}\dot{X}_0 = 0$$

or

$$X_0 = \frac{1}{R\bar{C}}\int X_1\, dt .$$

The factor $R\bar{C}$ is known as the time constant of the circuit. This is the factor which is adjusted by the time constant controls described in Section 3.4. and Appendix A.

In the compound unit a resistor and capacitor are in parallel in the line feeding back from output to input. The summation at the input point is then:

$$\frac{X_1}{R_1}+\frac{X_0}{R_s}+\bar{C}\dot{X}_0 = 0.$$

If the voltages are sinusoidal this can be used as a phase-adjusting circuit dependent on the value given to R_s over a limited range. It is the circuit used for damping control in the twelve degrees of freedom simulator.

The airspeed and (airspeed)^2 units are special cases of the summing unit in which the three factors V are mechanically linked and controlled together in the simulator.

In the airspeed unit the summation is:

$$\frac{VX_1}{R_1}+\frac{X_0}{R_2} = 0$$

or, if R_1 is equal to R_2

$$X_0 = -VX_1.$$

In the (airspeed)^2 unit the summation is:

$$\frac{VX_1}{R_1}+\frac{X_0}{VR_2} = 0$$

or, if R_1 and R_2 are equal

$$X_0 = -V^2X_1.$$

A circuit of the type shown in Fig. 5, made up from the above units, is used to represent one row of equation (3) or one 'degree of freedom' of the simulator. In fact, in the simulator, all amplifiers are push-pull. This reduces their sensitivity to changes in the supply voltages and reduces drift, at the same time providing an easy way of changing the sign of any of the coefficients in the equations by reversing their leads at the amplifier output. In Fig. 5 the amplifiers are shown as single-sided for simplicity and sign-changes through the amplifiers are ignored as these are adjusted in the wiring. The output voltages from all five amplifiers are fed back through coefficient units to the summing point A on amplifier 1. To this point also are fed all cross-coupling coefficient units from all other degrees of freedom. Two such coefficient units are shown in Fig. 5, one being inertia coupling (A_{12}) from degree of freedom 2 and the other stiffness coupling (E_{13}) from degree of freedom 3. The complete network is indicated in Fig. 1.

An additional potentiometer across the output of amplifier 4 in Fig. 5 with a tapping resistor to the amplifier input represents the density ratio σ in the equations and is adjusted simultaneously in all degrees of freedom by use of the 'Altitude' control at the desk.

Across amplifiers 2 and 3, in parallel with the capacitor unit which forms part of the time-constant control, are shown similar potentiometers and tapping resistors (R) which are adjusted in all degrees of freedom by the 'Damping Control' wheel at the desk. Normally these are set so that no current passes through the tapping resistors back to the amplifier input. As damping control is increased so the current fed back through them increases and also the effect of this resistive component of the total current on the output voltage of the unit.

4. *Detail Design.*

4.1. *The Amplifier.*

The amplifier (Fig. 6) is a direct-coupled three stage device using paired valves in push-pull. The input and output points are designed to be at the same potential to facilitate the connection of feed-back lines. As these points are near panels and sockets they are set at earth potential for reasons of safety and the high-tension supply is not earthed. This type of amplifier is very liable to parasitic oscillation and to prevent this, a number of capacitors are included in the circuit to control the shape of the fall-off at the high frequency end of the response. The gain is about 60 000 to a frequency of approximately 200 cycles per second at an amplitude of 50 volts peak-to-peak. This is well above the frequency of the complete simulator which is limited by other considerations to about 30 cycles per second.

Each amplifier is fitted with a balancing potentiometer to give some adjustment against drift. The main adjustment for a complete degree of freedom (Fig. 5) is, however, provided at the control desk. The meters on the desk are first switched to read the output of amplifier 3 which is the voltage representing the displacement x in the equation. Any drift can then be seen as a static displacement of the meter needle from centre zero. The meter is then re-centred by adjusting a potentiometer corresponding to the degree of freedom under examination. The potentiometer controls a direct current flowing to the summing point of the first amplifier in the chain thus introducing a compensating d.c. component into the equation.

4.2. *The Coefficient Unit.*

The circuit used to represent the coefficients A, B, C, D, E is shown in Fig. 7. This is a push-pull unit consisting of a pair of 100 kilohms potentiometers on a single spindle, tracked to within 0·2 per cent. The 1 megohm tapping resistors are matched to within 1 per cent and are connected, as is one side of the potentiometer supply line, through the contacts of a socket. These contacts are of a type spring-loaded to be normally closed until a special plug with separated double pins is inserted. The contacts are then forced apart and can be connected, through the pins, to external circuits. In this way a plug wired to reverse the leads to the summing unit is used to change the sign of the coefficient in the equation after the coefficient number has been set.

4.3. *The Coefficient Balancing Bridge.*

Each of the 720 coefficients in the complete set of equations is set up against a bridge network which is moved along the rows of coefficients.

The circuit of the bridge is given in Fig. 8. Before a coefficient can be set up all amplifiers must be locked by use of the switch at the control desk. This short-circuits the output of the amplifiers. The bridge and a supply battery are then connected to the coefficient by plugging into the spring-loaded socket which is part of the coefficient unit. A number is then set up on the bridge by operating push-buttons and the current output of the bridge is balanced to zero by manually adjusting the current output of the coefficient potentiometers. The null point is indicated by a centre-zero meter preceded by a d.c. transistor amplifier which is also supplied from a battery. By this system the total current output from a coefficient unit is correctly set irrespective of slight differences in the tapping resistors, and the accuracy of the ratio between one coefficient and another depends only on the stability of the single bridge unit. With the transistor amplifier (Fig. 9) the sensitivity is such that a change of 0·1 per cent in the setting of a coefficient number is shown on the meter as a change of about 5 per cent of full scale. The amplifier protects the meter from overload when the bridge is initially widely out of balance.

4.4. *Matrix Analysing Circuit.*

The circuit used to check that the determinants of all the principal minors of the matrix of A or $E+CV^2$ coefficients are positive is shown in Fig. 10. A three-position switch at the control desk, when in the 'A matrix' position, rearranges the normal circuits of the machine by locking all except the first amplifier in each degree of freedom. When changed to the '$E+CV^2$ matrix' position this switch also operates relays which remove all the A coefficient potentiometers from the output of the first amplifier and replace them with the CV^2 and E potentiometers. When this switch is in its central position all circuits remain as for normal representation of a problem.

The circuit which results when two degrees of freedom are switched to the 'A matrix' position in this way is shown in Fig. 10. In this position only the A terms are taken into account.

It is a necessary requirement for successful summation that the feedback across the high-gain summing amplifiers should be negative. This is achieved by making the amplifiers sign-reversing between the input and the output. Consider a small disturbance applied at the input of amplifier 1_{11} in Fig. 10. The summation of currents at the input of amplifier 1_{11} is approximately

$$A_{11} x_1 + A_{12} x_2 .$$

As the input admittance of the amplifier is very low the input current is negligible and, therefore, the summation of feedback currents must equal the disturbance current. The summation of currents at the input of amplifier 1_{22} (again neglecting the input current) is:

$$A_{22} x_2 + A_{21} x_1$$

which must equate to zero. Therefore the summation of feedback currents at the input of amplifier 1_{11} may be written:

$$\frac{A_{11} A_{22} - A_{12} A_{21}}{A_{22}} x_1 .$$

If the numerator of this expression is negative then the total feedback current will be in the same direction as the small current flowing into the amplifier input point. This will, therefore, be regenerative and the amplifier output voltage x_1 will increase to overload. For stability the numerator, which is the determinant of the A matrix, must be positive and, in this case, a disturbance applied to the system will disappear when the cause is removed.

The circuit of Fig. 10 is extended to include the complete 12×12 matrix present in the full set of equations. In this case all the possible feedback paths across each of the twelve amplifiers are summed at its input point. If any one of these summations should result in a total regenerative feedback across an amplifier its appropriate output meter at the desk will indicate overload. This is a considerable aid in tracing an incorrectly set coefficient or an error in the calculated coefficients.

4.5. *Non-linear Circuit.*

The non-linear circuit is shown in Fig. 11. This circuit is built on to a plug and, when inserted into a coefficient socket, modifies the normal ouput circuit of the amplifier to that coefficient only, as shown in the diagram. No current passes through the coefficient potentiometers until the amplifier voltage x exceeds the bias voltage X_B and $-X_B$. When this happens the diodes will conduct and a voltage difference will appear across the potentiometer setting s, giving a feedback current through the tapping resistors, so that for:

$$\begin{aligned} -X_B < x < \; & X_B \text{ total output} = 0 \\ x < -& X_B \text{ total output} = s(x + X_B) \\ x > \; & X_B \text{ total output} = s(x - X_B) . \end{aligned}$$

The 'dead-space' level depends on the voltage of the bias batteries and should be checked before use.

4.6. *Airspeed Control Unit.*

As already discussed, the airspeed V in the equations is adjustable simultaneously in all degrees of freedom. In the simulator V and V^2 are represented by the potentiometer and tapping resistor circuits shown in Figs. 4 and 5. As the circuits are push-pull throughout, each of the degree of freedom units has six ten-turn potentiometers associated with it for this purpose. For ease of control the seventy-two potentiometers are brought together at the control desk and driven together by an electric motor as can be seen from Fig. 12.

The input potentiometers to amplifiers 4 and 5 in Fig. 5 are 4000 ohms with 1 megohm tapping resistors; but the feedback potentiometers across amplifier 5 are 50 000 ohms which are tapped into a chain of resistors across the output of the amplifier to give an overall gain of unity when the potentiometer is fully in. The chains of resistors across the outputs of amplifiers 4 and 5 shown in Fig. 13 are also used to double the effective range of the airspeed setting 'V'. A switch at the desk operates a relay in each rack which changes over the tapping points of the feedback resistors as shown in the diagram. Switching to the 'x2' position connects the feedback lines to the centre tappings on the two chains of resistors. In the fourth amplifier this has the effect of doubling the output swing but, in the case of the fifth amplifier which represents V^2, the output is quadrupled. There is thus some danger that this amplifier may reach overload quickly when the scale is doubled and its output should be kept under observation when switched to this condition.

The control system is shown in Figs. 14 and 15. The motor, which drives the banks of potentiometers, has two field windings wound in opposition on the field yoke. The direction of rotation is therefore dependent on which of the windings is energised. In addition to the potentiometers the motor also drives a brush around the central disc in the diagram. This, the 'Tens' brush, also drives the 'Units' brush through a ten-to-one gear and the 'Hundreds' brush through an intermittent gear[3] which operates only as the 'Tens' passes from ninety to zero. The numbered peripheral contacts on each disc are wired as shown in Fig. 15 to the rows of press-buttons on the control desk. When a button on the 'Hundreds' row is pressed the corresponding contact on the disc is connected to the brush on the 'Tens' but disconnected from the remaining press buttons in the row. If the brush should at this time be to the right or left of the number which has been pressed, the base lead to one of the two transistors in the motor circuit will be connected through the wiring of the press-buttons and the rotating brush to the negative supply voltage. This will cause the motor to rotate towards the number which has been pressed. When this number has been reached on the 'Hundreds' disc the transistor base connection is made through the numbered contact to the 'Tens' brush. Here the process is repeated and the connection made through a pressed number to the 'Units' brush. At the 'Units' brush, however, a press-button and resistor are introduced into the base connection. The resistor has the effect of slowing the motor by controlling the transistor current while the press-button gives manual control of fine setting. The usual practice when investigating flutter conditions is to make a number of coarse trial settings to approximate to the flutter speed before attempting an exact reading. Also wired to the peripheral contacts on each disc are the indicator lights on the front panel of the control desk from which the final setting can be read.

4.7. *The Damping Control Unit.*

A method of controlling and measuring the intrinsic damping in the equations at speeds below the critical is given in Section 2.2. The unit is shown in Fig. 16 and the circuits used in Fig. 13. In addition to the capacitor units needed to perform integration amplifiers 2 and 3 are provided with parallel resistive feedback. This circuit consists of two 50 000 ohms potentiometers across the output of the amplifier from which are tapped the push-pull feedback resistors. These can be 1 megohm or 10 megohms depending on the damping range required. When the potentiometers are set at centre zero the effective feedback resistance R_f is infinity and can then be reduced to the lowest value of 1 megohm by setting the switches and potentiometer position. Normally the wiring of the change-over switch S shown on the output of amplifiers 2 and 3 is such that the effect of the feedback is to reduce the damping of the oscillating degree of freedom. Thus a normally damped oscillation can be controlled to a setting where the amplitude remains constant. This setting is then a measure of the amount of damping originally present. Operation of the switch S changes the sign of all damping coefficients present in the equations and, at the same time, of the control feedback resistors. An initially convergent oscillation would then become divergent and the control would approach the setting for zero damping from the opposite end of its range as a large initial setting is needed to give positive damping. The final setting should be the same for the same equations, of course, when only a single degree of freedom is switched on. There are, however, initial errors due to the losses in the amplifiers. These are not so large as to negate the value of the method in obtaining damping curves. If a positively damped problem including several degrees of freedom is observed on the control desk meters it can be seen that adjustment of the damping control (with switch S in the '+' position) will cause one of the modes to remain at constant amplitude while the rest decay. This mode is the one with minimum value of b (*see* Section 2.2.), with switch S in the '−' position a large setting of damping control is gradually reduced until one of the modes remains constant while the rest are positively damped. This mode is the one with the largest value of b in the original problem.

4.8. *The Altitude Control Unit.*

The altitude control unit is shown in Fig. 17 and the circuit is included in Fig. 13. This is the circuit used to adjust the value of $\sqrt{\sigma}$, the density ratio, in the set of equations (3). In Fig. 13 two 50 000 ohms rheostats are used as feedback resistors from the output chain of amplifier 4 to its input points. There are thus twenty-four rheostats mounted together as shown in Fig. 17 to control $\sqrt{\sigma}$ simultaneously throughout the set of equations. The normal setting is with maximum feedback resistance in which is marked 1·00 on the control scale, indicating unity density ratio. The resistors used are measured and adjusted to give an overall accuracy of better than 0·1 per cent.

4.9. *The Resonator Unit.*

This unit is an oscillator built into the control desk and designed to supply a sinusoidal function as the right-hand side {f} of the set of equations. It can, therefore, be used to examine the response of the system represented by the equations to the application of a sinusoidally varying force. Separate controls for each degree of freedom permit accurate adjustment of the amplitude of the forcing function applied to each of the twelve equations. These controls can be seen on the right-hand panel of the control desk (Fig. 3) with a switch above each which can be used to change the sign of the forcing function.

The circuit (Fig. 18) is built up on two push-pull amplifiers of the same type as those used in the rest of the machine. Similar components are used for the time-constant circuits and, therefore, the frequency range is the same. The oscillation would be damped, due to the finite gain of the amplifiers and losses in the components, were it not for the non-linear feedback circuit across amplifier 1. This provides just sufficient negative damping to maintain the oscillation at a constant amplitude. The pairs of potentiometers across the outputs of the two amplifiers are ganged and can be adjusted from the desk to give a smooth control of frequency. These are ten-turn potentiometers which can be set to 0·1 per cent. The feedback capacitors $\bar{C}$ are switched to give three frequency ranges.

Excluding the non-linear damping feedback, the summation of currents at the input of amplifier 1 is:

$$-\bar{C}\,\dot{x}_1+\frac{S}{R}x_2=0$$

and, at the input to amplifier 2:

$$\bar{C}\,\dot{x}_2+\frac{S}{R}x_1=0$$

or

$$\left(\frac{R\bar{C}}{S}\right)^2\ddot{x}_1+x_1=0\,.$$

This represents an oscillation of circular frequency $S/R\bar{C}$ where $R\bar{C}$ is the time constant.

4.10. *Frequency and Phase Measurement.*

The frequency of the oscillation in each degree of freedom is measured by timing the period of one complete cycle. For the low frequency range used in the flutter simulator this proves a quick and accurate method.

A block diagram of the system is shown in Fig. 19. The switch S_1 is used to select the sinusoidal output voltage of any one of the twelve degrees of freedom or of the resonator unit built into the control desk. This voltage operates the pulse unit which provides a sharp pulse synchronised to the incoming sine wave.

The first pulse received by the electronic switch opens it, permitting the 1000 cycles per second signal from a tuning-fork oscillator to reach the register unit, which begins to count. The second pulse received by the electronic switch closes it. It then remains closed irrespective of any further pulses which reach it. The register unit will then have timed the period of a single incoming sine wave in thousandths of a second. This reading is held for a predetermined time and then cleared by the auto reset. The instrument will then start on its counting cycle again.

Phase is measured by switching in the input change-over unit. The first pulse received by the electronic switch will then be due to the signal from switch S_1 and the second pulse from a separate source through switch S_2. The register unit will then record the time difference t_2 between the two pulses. If the time for one complete cycle is t_1 the phase difference between the two sources will be $\frac{t_2}{t_1} \times 360$ deg. The auto re-set unit can be disconnected by operating an external switch. Resetting is then manually controlled by a push-button. The instrument can be used as a stop-watch by operating a switch which connects the tuning-fork oscillator directly to the register unit through a push-button. The unit will then count for as long as the push-button is held down.

The circuit of the pulse unit is shown in Fig. 20. The input level is set to match the output level of the amplifiers in the computing circuits. A relay between the cathodes of the valves oscillates in sympathy with the incoming signal. Its moving contact is connected to a capacitor which charges to a controlled voltage when the contact is on one side and discharges through a resistor when the contact moves to the other side. It thus produces a sharp pulse once in each cycle of the incoming signal. This pulse is used to operate the electronic switch.

The circuit of the electronic switch is given in Fig. 21. Initially the thyratron T_2 is passing current while T_1 is not. The first pulse from the pulse unit fires T_1. This initiates a large voltage pulse through the capacitor between the cathodes of the two thyratrons. This has the effect of switching T_2 off, allowing the time base contacts to close. The register unit then begins to count the pulses received from the 1000 c/s oscillator. The second pulse to arrive at the input of the electronic switch fires T_2, which opens the relay contacts and stops the count. The register unit will then have recorded the time between two pulses and, hence, the period of one cycle of the incoming wave-form. The thyratrons are re-set by opening the re-set contacts which switches T_1 off while ensuring that T_2 remains fired.

Fig. 22 is the circuit diagram of the register unit. This consists of four Dekatrons with suitable coupling circuits. The first Dekatron is driven at 1000 cycles per second from an external oscillator through the electronic switch. The second, third and fourth Dekatrons count in ascending powers of ten, actuated by pulses from the cathode circuit of the preceding counter. In this way, after a count is completed, the period of one cycle of the incoming signal is indicated in seconds to three decimal places.

4.11. *Input and Output Panels.*

The circuit for the right-hand side input panel of the control desk is shown in Fig. 23. As they are all identical, only two of the twelve input sockets and associated components are shown. Each input socket can be used to apply an external signal, through a change-over switch and ten-turn potentiometer, to an individual degree of freedom rack. A four-position switch shown at the bottom of the diagram enables the operator to choose the form of external force to be applied to all the racks simultaneously. The most frequently used is 'step' which is a voltage applied through a press-button on the airspeed control box. The voltage step appears across all twelve potentiometers when the button is pressed unless a separate external source has been plugged into an individual socket. If the switch is turned to 'Sine' the output of the resonator unit is applied to the twelve potentiometers. 'Random' is intended to refer to an internal noise source* and 'External' connects a socket through the switch to the potentiometers, so that a common external source may be used.

*Although it was originally intended to provide an internal random function generator, it has proved more convenient to use external sources in practice.

The output panel circuit is shown in Fig. 24. The output of one of the five amplifiers in each of the twelve racks can be switched to this panel simultaneously. The choice of amplifier is made by using the five-position switch on the centre panel of the control desk. At the top of Fig. 24 appears one of the twelve circuits associated with each of the degree of freedom racks. A sign-changing switch and a pair of ten-turn potentiometers provide control of the output from each rack. An output socket is provided for each pair of potentiometers and their outputs are all passed through 1 megohm resistors to the input of a summing amplifier located behind the panel.

Switch A is used to select any one of the twelve outputs which is then connected to a large centre-zero meter. The input to the meter may be direct or rectified through a bridge rectifier. A socket connected across the meter can be used to examine the output on an oscilloscope or recorder. When switch A is placed at the 'Sum' position the meter is across the output of the summing amplifier behind the panel. The pair of potentiometers G across the meter are then part of the feedback circuit of the summing amplifier and are used to control the gain of the summing circuit. Switch B can be used to select any one of the twelve outputs, which can then be adjusted so that the sum of the input voltages to the summing circuit becomes zero. In this way an accurate comparison of the amplitudes of the output from each of the racks can be made in turn.

5. *Concluding Remarks.*

Since its completion there has been an increasing tendency to use the analogue computer for investigations not strictly of flutter problems although associated with them. This has led to the necessity for constructing additional components and devising new arrangements which are described in the Appendices. The computer can conveniently be used for the investigation of resonance problems, problems where many changes may be necessary in the coefficients, problems in random response and where non-linear effects are involved. The analogue computer is a useful complement to the digital computer for general dynamic response investigations.

Improvements to the design would be to replace all valve circuits by amplifiers using silicon planar transistors and by using improved types of miniature sealed relays. Suitable components were not available when this construction was undertaken but should be considered for any future modifications or similar designs.

LIST OF SYMBOLS

A	Inertia coefficient
B	Aerodynamic-damping coefficient
b	Damping ratio
C	Aerodynamic-stiffness coefficient
$\bar{C}$	Capacitance
D	Structural-damping coefficient
E	Structural-stiffness coefficient
$\bar{E}$	Stiffness coefficient modified by non-linear circuit
f	External force
G	Amplifier gain
R	Resistance
V	Equivalent airspeed
X_B	Bias voltage
X_0, X_1	Voltages
x	Voltage representing displacement
λ	Complex root of the flutter solution
λ'	Time decay constant
λ_n''	Natural frequency (radians/sec)
σ	Ratio of air density at altitude to air density at ground level
ω	Circular frequency
τ_f	Damping control time constant

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APPENDIX A

Scaling Equations for the Simulator.

by Ll. T. Niblett

1. *Introduction.*

This Appendix describes a method of scaling equations so that the coefficients are in a suitable form for the solution of the problem by the simulator. Scaling can be done in a multitude of ways. The method given below is thought to be systematic and likely to result, with a little care on the part of the user, in sets of well conditioned coefficients.

The method is expounded using simple matrix algebra involving, in the main, only pre- and post-multiplication by and inversion of diagonal matrices. Such an exposition clarifies the effect of each step in the scaling process on the terms of the equation. Roman majuscules denote square matrices, Roman minuscules denote column matrices, Roman minuscules with circumflexes denote diagonal matrices and Greek minuscules denote scalars.

2. *Method of Scaling.*

2.1. *The Scaled and Unscaled Equations.*

The matrix equation represented by the electrical circuits of the simulator is

$$A_s \frac{d^2}{d\tau_s^2} x_s + (\sigma^{\frac{1}{2}} \upsilon_s B_s + D_s) \hat{k} \frac{d}{d\tau_s} x_s + (\upsilon_s^2 C_s + E_s) \hat{k}^2 x_s = f_s(\tau_s) . \tag{8}$$

Restrictions on the coefficients are:

(a) no element of A_s, B_s, C_s, D_s, E_s, the coefficient matrices, can have a modulus greater than 1000,

(b) σ, the density ratio, cannot be greater than unity

(c) υ_s, the airspeed, cannot be greater than 1·998 and should not be greater than 0·999. The nominal maximum of the lower range, 1·000, cannot be set up without recourse to the higher range,

(d) elements of matrix $\hat{k}$, which correspond to the 'frequency' constants, must be in the range 0 to 100.

Of these (b) is trivial as any alteration to σ can be nullified by an alteration to B_s alone and hereinafter the value of σ will be assumed to be unity. Within these restrictions the coefficients should be as large as possible so that the effect of 'zero errors' in the electrical circuits representing them is minimised.

It is assumed that the unscaled equation is in the form

$$A \frac{d^2}{d\tau^2} x + (^{\upsilon} B + D) \frac{d}{d\tau} x + (\upsilon^2 C + E) x = f(\tau) . \tag{9}$$

It does not matter whether the coefficients are dimensional or not but care should be taken to see that the dimensions are consistent.

If more than one equation is to be scaled and the equations are related in some way (e.g. a control surface with different amounts of mass balance) it is convenient to scale the case which has coefficients of largest moduli or, if not all the coefficients have their largest modulus in the same case, a fictitious case which includes the most critical values of the coefficients should be used to obtain scaling matrices. If these matrices are used for all the cases the number of coefficients to be changed in going from case to case will be minimised. The common scaling matrices should not be used where they would lead to all the simulator coefficients having values which are small fractions of their maxima.

2.2. *Scaling Process.*

2.2.1. *Matrix scaling.*

Stage 1.

The object of this is to scale the coefficients so that the nominal maximum value of v, v_{max}, is represented by 1000 on the simulator speed scale. This is done by replacing B by B_v, C by C_v and v by v_s where $v_s = v/v_{max}$, $B_v = (v_{max}B)$, $C_v = (v^2_{max}C)$. $10^3 v_s$ is a general reading of the simulator speed scale which is taken to run from 0 to 999.

Equation (9) can now be written as

$$A\frac{d^2}{d\tau^2}x+(v_s B_v+D)\frac{d}{d\tau}x+(v_s^2 C_v+E)x = f(\tau). \tag{10}$$

Stage 2.

The object of this is to bring the elements on the leading diagonal of A to 1000. This is done by multiplying the elements in the ith row and jth column by $10^3/(A_{ii}\,A_{jj})^{\frac{1}{2}}$. If A_a is the resulting matrix this operation is denoted in matrix algebra by $A_a = \hat{k}_a A \hat{k}_a$ where $\hat{k}_a$ is a diagonal matrix with a typical element, $(k_a)_i = (10^3/A_{ii})^{\frac{1}{2}}$. Matrices B_v, C_v, D, E are similarly pre- and post-multiplied by $\hat{k}_a$ to give B_a, C_a, D_a and E_a.

Equation (10) can now be written, after pre-multiplication throughout by $\hat{k}_a$ as:

$$A_a\frac{d^2}{d\tau^2}x_a(v_s B_a+D_a)\frac{d}{d\tau}x_a+(v_s^2 C_a+E_a)x_a = \hat{k}_a f(\tau) = f_a(\tau) \tag{11}$$

where $x_a = \hat{k}_a^{-1}x$ and a typical element of $\hat{k}_a^{-1}$, $(\hat{k}_a^{-1})_i$, is $(A_{ii}/10^3)^{\frac{1}{2}}$.

Stage 3.

This alters all the matrices but A in such a way that the modulus of the largest element after scaling in any of four corresponding columns of B, C, D, E is 10^3. To do this the jth columns of B and D are multiplied by the suitable $(k_e)_j$ (say) and the corresponding columns of C and E by $(k_e^2)_j$. The value of $(k_e)_j$ will usually be determined by E_{jj} and will be $(10^3/E_{jj})^{\frac{1}{2}}$ but as will be seen from the worked example this is not always the case. With B_e, D_e equal to $B_a\,\hat{k}_e$, $D_a\,\hat{k}_e$ and C_e, E_e equal to $C_a\,\hat{k}_e^2$, $E_a\,\hat{k}_e^2$, equation (11) can be written:

$$A_a\frac{d^2}{d\tau^2}x_a+(v_s B_e+D_e)\hat{k}_e^{-1}\frac{d}{d\tau}x_a+(v_s^2 C_e+E_e)\hat{k}_e^{-2}x_a = f_a(\tau). \tag{12}$$

Stage 4.

Equation (12) resembles equation (8) closely enough for the square matrices of equation (12) to be considered final. The coefficients should be examined, however, to see whether the elements of all corresponding rows, i, are in general, disparate in modulus with elements of the corresponding columns, $j(=i)$, and if so the elements of the rows and columns should be made more comparable in modulus by premultiplying the matrices by a suitable $\hat{k}_f$ and post-multiplying by $\hat{k}_f^{-1}$. This operation will leave the coefficients on the leading diagonal and the values of the determinants of the principal minors as they were. Whether this final scaling is needed or not depends on the particular equation. It will often be unnecessary. Equation (12) can be written, after pre-multiplication by $\hat{k}_f$,

$$A_s\frac{d^2}{d\tau^2}x_s+(v_s B_s+D_s)\hat{k}_e^{-1}\frac{d}{d\tau}x_s+(v_s^2 C_s+E_s)\hat{k}_e^{-2}x_s = f_f(\tau) \tag{13}$$

where $A_s = \hat{k}_f\,A_a\,\hat{k}_f^{-1}$, $\quad B_s = \hat{k}_f\,B_e\,\hat{k}_f^{-1}$ etc., $\quad x_s = \hat{k}_f\,x_a$ and $f_f = \hat{k}_f\,f_a$.

2.2.2. *Time scaling.*

$\hat{k}_e^{-1}$ can be regarded as the product of a scalar κ and the diagonal matrix of 'frequency' constants, $\hat{k}$, whose elements lie between 0 and 100. Also the time co-ordinate can be changed to τ_s where $d\tau_s/d\tau_a = \kappa$. Bearing these in mind, equation (13) multiplied by κ^{-2} is identical with equation (8) provided $\kappa^{-2} f_f(\tau) \equiv f_s(\tau_s)$.

With the scaling procedure adopted here the oscillations of the simulator display will generally be at a convenient frequency if the elements of $\hat{k}$ are of the order of 10 and κ should be chosen with this in mind. A further practical point is that some values of $(k)_j$ can be set up more accurately than others (cf. 9·99 × 1 and 1·01 × 10) and a value of κ which increases the number of $(k)_j$ which can be set up with higher accuracy should be preferred.

The change in time co-ordinate means that the time rate of change of x will be κ times the time rate of change of x_s and thus that a frequency observed on the simulator display must be multiplied by κ to obtain the frequency referred to the unscaled equations. Some of the 'zero' errors of the simulator can be cancelled if the final adjustments to the 'frequency' constants is made such that the frequency of oscillation of each degree of freedom in isolation is κ^{-1} times the corresponding frequency derived from the unscaled equations.

2.2.3. *Summary.*

$$
\begin{aligned}
A_s &= \hat{k}_f \hat{k}_a A \hat{k}_a \hat{k}_f^{-1} \\
B_s &= v_{\max} \hat{k}_f \hat{k}_a B \hat{k}_a \hat{k}_e \hat{k}_f^{-1} \\
C_s &= v_{\max}^2 \hat{k}_f \hat{k}_a C \hat{k}_a \hat{k}_e^2 \hat{k}_f^{-1} \\
D_s &= \hat{k}_f \hat{k}_a D \hat{k}_a \hat{k}_e \hat{k}_f^{-1} \\
E_s &= \hat{k}_f \hat{k}_a E \hat{k}_a \hat{k}_e^2 \hat{k}_f^{-1} \\
x_s &= \hat{k}_f \hat{k}_a^{-1} x \\
f_s(\tau_s) &= \hat{k}^{-2} \hat{k}_f \hat{k}_a f(\tau)
\end{aligned}
$$

3. *Null Columns in the C, D and E Matrices.*

Null columns in the C, D and E matrices can give rise to a low-frequency instability of the simulator when the equation considered has a null right-hand side. Ways in which this difficulty can be surmounted are given by Broadbent in an Appendix to Ref. 1.

3.1. The simplest method when an element on the leading diagonal of E is zero is to give it a fictitious non-zero value just sufficient to stabilise the simulator. The corresponding element of the A matrix is altered by an amount which exactly reacts the fictitious E when there is a maintained oscillation. If δA_{ii} and δE_{ii} are the increments in the elements,

$$\delta A_{ii} \frac{d^2 x_i}{d\tau^2} + \delta E_{ii} k_i^2 x_i = 0. \tag{14}$$

If it is thought that resort to this artifice is likely, provision should be made when scaling for the subsequent inclusion of δA_{ii}.

3.2. A transformation of co-ordinates is sometimes useful when one or more corresponding columns of the C, D and E matrices are null.

For the purposes of argument we will assume that the first columns are null. Partition the square matrices into the first column and a rectangular matrix denoted in the case of the A matrix, for example, by $(a_{ii},\bar{A})$ and partition off the first elements of the x and $\hat{k}$ matrices to give $\begin{Bmatrix}(x)_1\\ \bar{x}\end{Bmatrix}$ and $\begin{bmatrix}(k)_1\\ \bar{k}\end{bmatrix}$.

With the inclusion of the unit matrix KK^{-1} an equation of the type (8) can be written:

$$\left[(a_{i1},\bar{A})\frac{d^2}{d\tau^2}+[\upsilon(b_{i1},\bar{B})+(0,\bar{D})]\begin{bmatrix}(k)_1\\ \bar{k}\end{bmatrix}\frac{d}{d\tau}+[\upsilon^2(0,\bar{C})+(0,\bar{E})]\begin{bmatrix}(k)_1^2\\ \bar{k}^2\end{bmatrix}\right]KK^{-1}\begin{Bmatrix}(x)_1\\ \bar{x}\end{Bmatrix}=0 \quad (15)$$

Put

$$K=\begin{bmatrix}\upsilon\int d\tau(k)_1 & 0\\ 0 & I\end{bmatrix}$$

where K is an operational matrix,

I is a unit matrix and, as in equation (15)

0 represents null matrices.

Since K is a diagonal matrix $\hat{k}$ and K commute, i.e. $\hat{k}K = K\hat{k}$.

Also
$$K^{-1}=\begin{bmatrix}(k)_1^{-1}\upsilon^{-1}\dfrac{d}{d\tau} & 0\\ 0 & I\end{bmatrix}$$

Equation (15) can be written

$$\left[(0,\bar{A})\frac{d^2}{d\tau^2}+[\upsilon(a_{i1},\bar{B})+(0,\bar{D})]\begin{bmatrix}(k)_1\\ \bar{k}\end{bmatrix}\frac{d}{d\tau}+[\upsilon^2(b_{i1},\bar{C})+(0,\bar{E})]\begin{bmatrix}(k)_1^2\\ \bar{k}^2\end{bmatrix}\right]\begin{Bmatrix}(k)_1^{-1}\upsilon^{-1}\dfrac{d(x)_1}{d\tau}\\ \bar{x}\end{Bmatrix}=0 \quad (16)$$

This equation will not have zero roots provided υ is non-zero and b_{11} is positive. The unscaled matrices corresponding to equation (15) can be used at the start of the scaling process.

3.3. A further transformation is useful when at least two corresponding columns of D and E and one of C are null and the non-null column of C corresponding to the column null in D and E only is a scalar multiple of the column of B corresponding to the null column common to C, D and E. The type of equation considered is probably described more clearly by equation (17) in which the partitioning used in equation (15) has been extended.

$$\left[(a_{i1},a_{i2},\bar{A})\frac{d^2}{d\tau^2}+[\upsilon(b_{i1}(k)_1,\,b_{i2}(k)_2,\bar{B})+(0,0,\bar{D})]\begin{bmatrix}1\\1\\ \bar{k}\end{bmatrix}\frac{d}{d\tau}+\right.$$
$$\left.+[\upsilon^2(0,\chi\, b_{i1}(k)_2^3,\bar{C})+(0,0,\bar{E})]\begin{bmatrix}1\\1\\ \bar{k}^2\end{bmatrix}\right]KK^{-1}\begin{Bmatrix}(x)_1\\(x)_2\\ \bar{x}\end{Bmatrix}=0 \quad (17)$$

In this equation the column submatrices have been multiplied by elements of $\hat{k}$ or $\hat{k}^2$ where appropriate. In this case we put

$$K=\begin{bmatrix}(k)_1\upsilon\int d\tau & -\chi(k)_1^{-1}(k)_2^3\upsilon^2\int\int(d\tau)^2 & 0\\ 0 & (k)_2\upsilon\int d\tau & 0\\ 0 & 0 & I\end{bmatrix}$$

an operational matrix, which commutes with $\left\lceil 1 \; 1_{\bar{k}} \right\rceil$ and has an inverse,

$$K^{-1} = \begin{bmatrix} (k)_1^{-1}\,\upsilon^{-1}\dfrac{d}{d\tau} & \chi(k_1)^{-2}(k_2)^2 & 0 \\ 0 & (k)_2^{-1}\,\upsilon^{-1}\dfrac{d}{d\tau} & 0 \\ 0 & 0 & I \end{bmatrix}$$

Equation (17) can be written

$$\Big[(0,0,\bar{A})\frac{d^2}{d\tau^2} + [\upsilon(a_{i1}(k)_1, a_{i2}(k)_2, \bar{B}) + (0,0,\bar{D})] \left\lceil 1 \; 1_{\bar{k}} \right\rceil \frac{d}{d\tau}$$

$$+ [\upsilon^2(b_{i1}(k_1)^2, \{b_{i2} - \chi a_{i1}(k)_1^{-1}(k)_2\}(k)_2^2, \bar{C}) + (0,0,\bar{E})] \left\lceil 1 \; 1_{\bar{k}^2} \right\rceil \Big]$$

$$\left\{ \begin{matrix} (k)_1^{-1}\upsilon^{-1}\dfrac{d(x)_1}{d\tau} + \chi(k)_1^{-2}(k)_2^2(x)_2 \\ (k)_2^{-1}\upsilon^{-1}\dfrac{d(x)_2}{d\tau} \\ \bar{x} \end{matrix} \right\} = 0 \qquad (18)$$

so that the following substitutions can be made

$(0,0,\bar{A})$ for A,

$(a_{i1}, a_{i2}, \bar{B})$ for B,

and

$(b_{i1}, b_{i2} - \chi a_{i1}\,(k)_1^{-1}\,(k)_2, \bar{C})$ for C.

Whether it is better to transform the unscaled or scaled co-ordinates will depend on the particular problem. If the unscaled co-ordinates are transformed the Stage 1 scaling process must be modified.

4. *Worked Example.*

Assume that the unscaled coefficient matrices are the five below:

$$A = \begin{bmatrix} 168{\cdot}70 & -1{\cdot}3176 & -75{\cdot}125 & 0{\cdot}89253 \\ -1{\cdot}3176 & 19{\cdot}762 & 9{\cdot}7685 & -0{\cdot}81182 \\ -75{\cdot}125 & 9{\cdot}7685 & 71{\cdot}816 & -1{\cdot}4566 \\ 0{\cdot}89253 & -0{\cdot}81182 & -1{\cdot}4566 & 0{\cdot}26845 \end{bmatrix}$$

$$B = \begin{bmatrix} 10{\cdot}699 & -1{\cdot}9627 & -10{\cdot}150 & -0{\cdot}019980 \\ 0{\cdot}20989 & 1{\cdot}5880 & 2{\cdot}1900 & -0{\cdot}32336 \\ -3{\cdot}2196 & 1{\cdot}9849 & 8{\cdot}8188 & -0{\cdot}79072 \\ 0{\cdot}096595 & -0{\cdot}044954 & -0{\cdot}14660 & 0{\cdot}022192 \end{bmatrix}$$

$$C = \begin{bmatrix} 1{\cdot}9786 & -1{\cdot}0875 & -3{\cdot}4467 & 0{\cdot}82434 \\ 0{\cdot}28975 & -0{\cdot}11208 & -0{\cdot}40967 & -0{\cdot}0067028 \\ 0{\cdot}94653 & -0{\cdot}53969 & -1{\cdot}6363 & -0{\cdot}32075 \\ 0{\cdot}0094703 & -0{\cdot}0055338 & -0{\cdot}017153 & 0{\cdot}010432 \end{bmatrix}$$

$$D = 10^2 \begin{bmatrix} 0{\cdot}5 & 0 & 0 & 0 \\ 0 & 0{\cdot}16 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E = 10^4 \begin{bmatrix} 49{\cdot}257 & 0 & 0 & 0 \\ 0 & 13{\cdot}185 & 0 & 0 \\ 0 & 0 & 157{\cdot}91 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Stage 1: with v_{max} taken to be 500

$$B_v = 10^2 \begin{bmatrix} 53{\cdot}495 & -9{\cdot}8135 & -50{\cdot}75 & -0{\cdot}09990 \\ 1{\cdot}04945 & 7{\cdot}940 & 10{\cdot}950 & -1{\cdot}6168 \\ -16{\cdot}098 & 9{\cdot}9245 & 44{\cdot}094 & -3{\cdot}9536 \\ 0{\cdot}482975 & -0{\cdot}22477 & -0{\cdot}7330 & 0{\cdot}11096 \end{bmatrix}$$

$$C_v = 10^4 \begin{bmatrix} 49{\cdot}465 & -27{\cdot}1875 & -86{\cdot}1675 & 20{\cdot}6085 \\ 7{\cdot}24375 & -2{\cdot}8020 & -10{\cdot}24175 & -0{\cdot}16757 \\ 23{\cdot}66325 & -13{\cdot}49225 & -40{\cdot}9075 & -8{\cdot}01875 \\ 0{\cdot}2367575 & -0{\cdot}138345 & -0{\cdot}428825 & 0{\cdot}26080 \end{bmatrix}$$

Stage 2: $(\hat{k}_a)_i^2 = (10^3/A_{ii})$

$$\text{i.e. } \hat{k}_a = \begin{bmatrix} 2{\cdot}4347 & 7{\cdot}1135 & 3{\cdot}7316 & 61{\cdot}033 \end{bmatrix}$$

$$A_a = \begin{bmatrix} 1000{\cdot}0 & -22{\cdot}820 & -682{\cdot}55 & 132{\cdot}62 \\ -22{\cdot}820 & 1000{\cdot}0 & 259{\cdot}30 & -352{\cdot}46 \\ -682{\cdot}55 & 259{\cdot}30 & 1000{\cdot}0 & -331{\cdot}71 \\ 132{\cdot}62 & -352{\cdot}46 & -331{\cdot}71 & 1000{\cdot}0 \end{bmatrix}$$

$$B_a = 10^2 \begin{bmatrix} 317{\cdot}10 & -169{\cdot}96 & -461{\cdot}08 & -14{\cdot}845 \\ 18{\cdot}1757 & 401{\cdot}78 & 290{\cdot}66 & -701{\cdot}94 \\ -146{\cdot}26 & 263{\cdot}44 & 614{\cdot}00 & -900{\cdot}42 \\ 71{\cdot}769 & -97{\cdot}586 & -166{\cdot}94 & 413{\cdot}33 \end{bmatrix}$$

$$C_a = 10^4 \begin{bmatrix} 293{\cdot}21 & -470{\cdot}86 & -782{\cdot}85 & 3062{\cdot}3 \\ 125{\cdot}45 & -141{\cdot}79 & -271{\cdot}87 & -72{\cdot}751 \\ 214{\cdot}99 & -358{\cdot}15 & -569{\cdot}63 & -1826{\cdot}3 \\ 35{\cdot}181 & -60{\cdot}064 & -97{\cdot}665 & 971{\cdot}49 \end{bmatrix}$$

$$D_a = 10^2 \begin{bmatrix} 2{\cdot}9639 & 0 & 0 & 0 \\ 0 & 8{\cdot}0963 & 0 & 0 \\ 0 & 0 & 27{\cdot}850 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_a = 10^4 \begin{bmatrix} 291{\cdot}98 & 0 & 0 & 0 \\ 0 & 667{\cdot}19 & 0 & 0 \\ 0 & 0 & 2198{\cdot}9 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Stage 3: $\hat{k}_e^2 = 10^3 \lceil (C_a)_{11}^{-1}, (E_a)_{22}^{-1}, (E_a)_{33}^{-1}, (C_a)_{14}^{-1} \rfloor$

$$\text{i.e. } \hat{k}_e^2 = 10^{-4} \lceil 3{\cdot}4105 \quad 1{\cdot}4988 \quad 0{\cdot}45477 \quad 0{\cdot}32655 \rfloor$$

$$\hat{k}_e = 10^{-2} \lceil 1{\cdot}8453 \quad 1{\cdot}2243 \quad 0{\cdot}67437 \quad 0{\cdot}57145 \rfloor$$

$$\hat{k}_e^{-1} = 10^{2} \lceil 0{\cdot}54192 \quad 0{\cdot}81679 \quad 1{\cdot}4829 \quad 1{\cdot}8970 \rfloor$$

$$B_e = \begin{bmatrix} 585{\cdot}14 & -208{\cdot}08 & -310{\cdot}94 & -8{\cdot}4832 \\ 33{\cdot}540 & 491{\cdot}90 & 196{\cdot}01 & -401{\cdot}12 \\ -269{\cdot}89 & 322{\cdot}53 & 414{\cdot}06 & -514{\cdot}55 \\ 132{\cdot}44 & -119{\cdot}47 & -112{\cdot}58 & 236{\cdot}20 \end{bmatrix}$$

$$C_e = \begin{bmatrix} 1000{\cdot}0 & -705{\cdot}72 & -356{\cdot}01 & 1000{\cdot}0 \\ 427{\cdot}85 & -212{\cdot}51 & -123{\cdot}64 & -23{\cdot}757 \\ 733{\cdot}22 & -536{\cdot}80 & -259{\cdot}05 & -596{\cdot}38 \\ 119{\cdot}98 & -90{\cdot}024 & -44{\cdot}415 & 317{\cdot}24 \end{bmatrix}$$

$$D_e = \begin{bmatrix} 5{\cdot}4693 & 0 & 0 & 0 \\ 0 & 9{\cdot}9123 & 0 & 0 \\ 0 & 0 & 18{\cdot}782 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$E_e = \begin{bmatrix} 995{\cdot}80 & 0 & 0 & 0 \\ 0 & 1000{\cdot}0 & 0 & 0 \\ 0 & 0 & 1000{\cdot}0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The coefficients to be used on the simulator are given by A_a, B_e, C_e, D_e and E_e as Stage 4 scaling is unnecessary in this example.

Time scaling: choose $\kappa = 10$.

Thus the 'frequency' constants are $5{\cdot}42 \times 1$, $8{\cdot}17 \times 1$, $1{\cdot}48 \times 10$ and $1{\cdot}90 \times 10$ and the oscillations of the simulator display are at one tenth the frequency of those derived from the original equation.

APPENDIX B

The Measurement of Sub-critical Response on the Simulator

by C. A. K. Irwin

1. *Introduction.*

Sub-critical response measurements provide values of rates of decay and natural frequencies for systems being tested at air speeds below the critical flutter speed. To make measurements of sub-critical response a vector-response method may be used in which a sinusoidal excitation force of constant amplitude is applied to the system under test and the response of the system in terms of amplitude and phase with respect to the exciting force is recorded over a range of excitation frequencies. The response is plotted on an Argand diagram and at frequencies close to a resonance frequency of the system the response traces out an approximate arc of a circle. From this circle the damping and frequency in the appropriate root of the equations governing the motion of the oscillating system can be obtained[4,5,6].

Experience on the R.A.E. 12 degrees of freedom simulator used in conjunction with a phasemeter and oscillator indicates certain aspects of the setting-up and use of the equipment which are particularly important when vector-response measurements are made. These aspects are discussed in detail but it is assumed that the reader is familiar with the general principles and operation of the simulator.

2. *Scaling the Simulator Inputs and Outputs.*

Provision is made on the simulator for generating a forcing function of sinusoidal form at any chosen frequency over a wide range. This function is applied to the degrees of freedom at a common phase angle but in any selected proportions (+ or −). It is thus possible to represent a structure under distributed excitation provided that the exciting forces are of the same frequency and at the same phase angle. The internally-generated forcing function can be replaced by that from an external oscillator if required.

When the forcing has been applied for a sufficient time for the transients to decay to negligible amplitudes the degrees of freedom will oscillate at a common frequency but with differing phase angles. In general, the required response (e.g. the displacement at a point on the structure) will be the vectorial sum of a proportion of the generalised co-ordinates. The simulator is provided with a panel of potentiometers which enables chosen proportions of each generalised co-ordinate to be added together to form the required output signal.

It is necessary to scale the inputs (voltages proportional to the excitation forces) and the outputs (voltages proportional to acceleration, velocity or displacement) so that the forces applied to individual degrees of freedom and the responses therefrom are compatible with the scalings which have been applied to the A, B, C, D and E coefficients.

It has been shown in Appendix A that the unscaled equation

$$A \frac{d^2}{d\tau^2} x + (\upsilon B + D) \frac{d}{d\tau} x + (\upsilon^2 C + E)\, x = f(\tau)$$

can be reduced to the scaled form represented on the simulator as

$$A_s \frac{d^2}{d\tau_s^2} x_s + (\sigma^{\frac{1}{2}} \upsilon_s B_s + D_s)\, \hat{k} \frac{d}{d\tau_s} x_s + (\upsilon_s^2 C_s + E_s)\, \hat{k}^2 x_s = f_s(\tau_s)$$

when

$$\begin{aligned}
A_s &= \hat{k}_f \hat{k}_a A \hat{k}_a \hat{k}_f^{-1} \\
B_s &= \upsilon_{\max} \hat{k}_f \hat{k}_a B \hat{k}_a \hat{k}_e \hat{k}_f^{-1} \\
C_s &= \upsilon_{\max}^2 \hat{k}_f \hat{k}_a C \hat{k}_a \hat{k}_e^2 \hat{k}_f^{-1} \\
D_s &= \hat{k}_f \hat{k}_a D \hat{k}_a \hat{k}_e \hat{k}_f^{-1} \\
E_s &= \hat{k}_f \hat{k}_a E \hat{k}_a \hat{k}_e^2 \hat{k}_f^{-1} \\
x_s &= \hat{k}_f \hat{k}_a^{-1} x \\
f_s(\tau_s) &= \kappa^{-2} \hat{k}_f \hat{k}_a f(\tau)
\end{aligned}$$

σ is assumed unity and scalar $\upsilon_s = \upsilon/\upsilon_{\max}$

where $\hat{k}_a$ is a diagonal matrix with typical element, $(k_a)_i = (10^3/A_{ii})^{\frac{1}{2}}$,

$\hat{k}_e$ is a diagonal matrix with typical element $(k_e)_j$ such that the largest element in any four corresponding columns of B, C, D, E becomes 10^3 after B and D have been multiplied by $(k_e)_j$ and C and E by $(k_e^2)_j$; in many cases $(k_e)_j = (10^3/E_{jj})^{\frac{1}{4}}$,

$\hat{k}_f$ is a diagonal matrix introduced to compensate for any large disparity between the moduli of the elements of all corresponding rows, i, and of the elements of the corresponding columns, $j(= i)$. This scaling step is often unnecessary,

$\hat{k}_e^{-1}$ is the product of scalar κ and the diagonal matrix of 'frequency' constants κ*.

The input force to each degree of freedom will be scaled by every step in the scaling process which affects all the terms in the row, i.e. each step involving pre-multiplication. Thus, as above

$$f_s(\tau_s) = \kappa^{-2} \hat{k}_f \hat{k}_a f(\tau).$$

The scalar κ^{-2} is introduced by the change in time scale, $d\tau_s/d\tau = \kappa$.

The force applied to each degree of freedom must be factored by the appropriate $\kappa^{-2}(k_f)_i\,(k_a)_i$ and also by a 'modal function' which allows for the relative amplitude between the point in the system under test at which the generalised co-ordinate for the degree of freedom is assumed to act and the point at which the force is to be applied.

Applying this scale to each degree of freedom we obtain a set of force factors where, for degree of freedom i, say

$$\text{Force factor} = \kappa^{-2}(k_f)_i\,(k_a)_i\,[\text{modal function}].$$

In order to obtain maximum accuracy in setting the potentiometers the scaled forces may be set up as fractions of the largest scaled force. The last step leads to a set of final force factors where, for degree of freedom i, say

$$\text{Final force factor} = \frac{\text{force factor in d.o.f. } i}{\text{largest force factor}}.$$

---

*The possible frequency range settings are 1, 10 or 100 and if a problem set up on the simulator is such that the factors 1 or 100 are not used throughout the degrees of freedom in use the complete problem can be raised or lowered in frequency as appropriate. This facility is useful when it is necessary to match the frequency range of the problem on the simulator to that of some external apparatus. However, for best accuracy the basic frequency for the simulator should be as low as possible.

It should be noted that the final force factors are independent of κ and that the time scale settings can, therefore, be changed in the course of a test run without requiring changes in the final force factors.

A scaling process somewhat similar to that applied to the simulator inputs must be applied to the outputs. As stated above, the outputs can be voltages proportional to acceleration, velocity or displacement; these voltages are obtained from the amplifiers to which the A, B and D or C and E coefficients apply respectively and are effectively scaled in the simulator inversely as the scalings applied to the columns of the matrices of coefficients. It is, therefore, necessary to apply the same column scalings to the outputs potentiometers as are applied to the appropriate coefficients.

The columns of the matrices of coefficients are scaled by all the steps involving post-multiplication which are appropriate. Thus, the acceleration voltages, which are obtained from the A amplifiers, must be multiplied by $(k_a)_j\,(k_f^{-1})_j$, and the velocity voltages coming from the B or D amplifiers must be multiplied by $v_{\max}(k_a)_j\,(k_e)_j\,(k_f^{-1})_j$ or $(k_a)_j\,(k_e)_j\,(k_f^{-1})_j$ respectively. Similarly, the displacement voltages from the C or E amplifiers must be multiplied by $v^2_{\max}(k_a)_j\,(k_e^2)_j\,(k_f^{-1})_j$ or $(k_a)_j\,(k_e^2)_j\,(k_f^{-1})_j$ respectively.

Provision is made for summing the outputs of the degrees of freedom in any chosen proportions. If the factors above are applied and the outputs are summed the resultant represents the motion at the reference point for the generalised co-ordinates. If the motion of some other point is required each factored output must be multiplied by the appropriate modal function giving a modified factor (where modal function = displacement of the chosen point/displacement at the reference point). It is most convenient to quote the modified factors in terms of the largest modified factor since they should be as large as possible but not greater than unity for maximum accuracy in setting.

Summarising, the factor to be applied to the displacement output from the E amplifier in each degree of freedom is given by

$$\text{Final displacement factor} = \frac{(k_a)_j\,(k_e^2)_j\,(k_f^{-1})_j\,[\text{modal function}]}{[\text{largest modified displacement factor}]}.$$

Similarly, for each degree of freedom for the output from the D amplifier,

$$\text{Final velocity factor} = \frac{(k_a)_j\,(k_e)_j\,(k_f^{-1})_j\,[\text{modal function}]}{[\text{largest modified velocity factor}]}$$

and for the output from the A amplifier,

$$\text{Final acceleration factor} = \frac{(k_a)_j\,(k_f^{-1})_j\,[\text{modal function}]}{[\text{largest modified acceleration factor}]}.$$

3. *Setting-up and Checking the Coefficients.*

3.1. *Coefficient Setting.*

In general this is straightforward but whereas the values of the direct A and E terms are usually large and equal the other terms are set relative to these two. This can lead to small values for some coefficients. Since the coefficients can be set to a maximum of *three* figures the settings for small numbers often use the last digit only. It has been found that after setting the D term it is possible to obtain an incorrect value for the coefficient of damping when a response measurement is made. As much as a factor of 2 increase in damping has been observed at zero windspeed. The coefficients involved are small (say 3 per cent of full range) and, hence, susceptible to small changes in the circuit. In a conventional test to determine flutter speeds on the simulator the values of the D terms are usually not known precisely and arbitrary values are often inserted to make the system initially stable. In sub-critical response tests, however, where values of damping may be plotted against windspeed, the zero windspeed case is important since it provides a datum for any comparisons with theory. It is, therefore, advisable to check the potentiometer setting by response measurements. The B and C terms are in general considerably larger than the D terms and so less susceptible to minor circuit variations. The signs of the terms are readily set by inserting a plug adjacent to each coefficient potentiometer.

3.2. *Excitation Amplitude.*

The excitation voltage is fed into all of the d.o.f. under test, and the proportions of the full voltage which are applied to each d.o.f. are arranged relative to one another (for scaling *see* above) so that the forcing simulates the application of the force at some chosen point on the oscillating system.

The proportions of the full voltage are obtained from the input potentiometers and the zero of each one should be checked when the settings are made. The sign of the forcing applied to each d.o.f. can be set by a switch over each potentiometer.

The overall amplitude of the excitation voltage must be chosen with care since if the voltage is too high overloading may occur and if too low then the noise level is significant. The problem of overloading is particularly acute when one or more d.o.f. are being forced to large amplitudes in one or more of their amplifiers and their contributions to a mode being examined are small although important. In this case the large amplitudes must determine the excitation force, unless the proportions of the inputs are altered, simulating the removal of the 'exciter' to some other part of the system being simulated.

If the input voltage is reduced too low two effects become important: firstly, the 50 c/s and 600 c/s noise becomes significant* (neither is steady in amplitude) and, secondly, d.c. drift in the amplifiers begins to affect the phasemeter.

3.3. *Simulator Outputs.*

An arrangement of potentiometers similar to that of the excitation inputs is provided at the output. However, four aspects of the setting-up must be noted. Firstly, there are phase switches on the outputs; these must be set: secondly, the potentiometer zero positions require checking before values are set: thirdly, there is a switch which selects which one of the *A*, *B*, *C*, *D*, or *E* amplifiers in each d.o.f. provides the voltage output. The voltage in the chosen amplifier is also displayed on a meter. A separate switch selects which d.o.f. is observed. The choice of amplifiers has a bearing on the scale-factors to be applied (*see* above). If the *E* amplifiers are used the d.c. drift in each d.o.f. is superimposed on the output (although it should be nominally balanced out); if the *A*, *B*, *C*, or *D* amplifiers are selected the d.c. drift is not present. On the other hand, the *A*. *B*. *C*. *D* amplifiers do not indicate non-oscillatory divergence directly and only the *E* amplifiers clearly show this in that the meter needles go hard over to one side. Fourthly, it is necessary to check the d.c. balance of each d.o.f. using the *E* amplifiers before testing commences.

The choice of which d.o.f. to observe depends upon the object of the test but if the intention is to simulate the resultant motion on some part of the system being simulated then all the relevant d.o.f. will require to be summed together. There is a switch provided for this purpose but individual output sockets are provided for each degree of freedom and summing up can be performed externally. Some drift can occur in the output summing amplifier.

3.4. *Frequency Constant Settings.*

The frequency of each d.o.f. is set using the frequency constant controls and the calibrated knobs are approximately correct. However, in response testing small variations between required and set values of frequency can make large differences in the appearance of response plots and can lead to difficulties when comparisons of frequency are made between theory and simulation. Hence, it is important to check the frequency in some way. A built-in frequency counter is provided on the simulator but it is difficult to make small changes and check them rapidly. A more satisfactory method is to use the oscillator and phasemeter to be used for the response testing and, treating each d.o.f. alone, at zero windspeed make a check of the quadrature between signals of force and displacement (*E* amplifier) or acceleration (*A* amplifier). An additional advantage of this approach is that any small frequency errors inherent in the oscillator are allowed for on the d.o.f.

It may be mentioned at this point that the frequency constant potentiometer setting has been found to have a bearing on the level of 50 c/s noise in a d.o.f.; the level is a minimum at high or low settings and a maximum at the middle of the range. This information could be useful in a marginal case with a small signal to noise ratio.

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*Noise at 600 c/s originates at the commutator of the d.c. generator, 600V supply. Steps have recently been taken to reduce the noise generated by this source.

The frequency drift between the external oscillator (a high stability model) and simulator was found to be very good on the d.o.f. tested. Some typical values are as follows:—

d.o.f. 2	$\frac{1}{4}$ per cent change in a working day
3	$\frac{1}{2}$ per cent change in a working day
4	less than $\frac{1}{4}$ per cent change in a working day.

4. *The Vector-response Measurement.*

An outline of the vector-response measurement technique is as follows. The input and output potentiometers are set to represent respectively the point of application of the excitation force to the system being simulated and the point at which the response of the system is to be measured. The required airspeed and altitude are selected. The oscillating voltage is applied and the response voltage measured in amplitude and phase with respect to this excitation voltage (with some phasemeters the measurement may actually be in terms of the response in-phase and in quadrature with the excitation). By making a series of measurements at small increments of excitation frequency over a frequency range including the resonance frequencies of the system and plotting the response on an Argand diagram a set of vector-response plots will be obtained. When the frequency is close to a resonance frequency the plotted response will trace out an arc of a circle. How closely the arc corresponds to a true arc of a circle depends upon the effect of adjacent resonance frequencies and their relative dampings, whether the point in the simulated system chosen for the application of the force is the best possible (and the best possible at one airspeed may not be the best at another due to changes in the couplings between the degrees of freedom) and, similarly, whether the point at which the response is measured is the most suitable*.

Each resonance frequency is determined by finding the point on the arc at which the rate of change of frequency with arc length is a minimum. Theoretically, with viscous damping a better approximation to the resonance frequency is obtained if the frequency at the point where $1/\omega.\ d\omega/ds$ is a minimum, where ω is the circular frequency and s is the distance along the curve. When the damping is small the two minima are practically identical. The frequency obtained is the undamped resonance frequency which, again when the damping is small, approximates closely to the damped frequency.

The damping associated with each resonance is obtained by constructing a circle to fit the curve in the vicinity of the resonance frequency and marking-off equal angles (θ) each side of the radius joining the centre to the resonance point. Then, if the off-resonance frequencies are ω_A and ω_B at angle θ each side of the resonance, and ω_B is greater than ω_A, it can be shown that

$$b \simeq g/2 = \frac{(\omega_B - \omega_A)}{2\omega_0} \cot \theta/2$$

where b is the ratio of the viscous damping present to the critical viscous damping (i.e. the viscous damping required to change the motion from periodic to aperiodic)

g is the hysteretic damping factor

and ω_0 is the resonance frequency.

The choice of the angle θ will be determined by the length of the curve which approximates to an arc of the circle but it should not exceed 90 degrees. When $\theta = 90$ degrees we have that

$$b \simeq \frac{(\omega_B - \omega_A)}{2\omega_0}.$$

---

*The locus of the response is strictly an arc of a circle only for a single degree of freedom and only when the damping is hysteretic. With viscous damping, which is the type represented in the simulator since the majority of the damping is aerodynamic in origin, the locus of the response describes a curve which is a close approximation to a circle. For most practical purposes the difference is negligible, *see* Ref. 5.

The interpretation of vector plots is discussed in considerable detail in Ref. 6 and the case where the damping cannot be considered low is covered.

5. *Further General Considerations.*

In using the simulator some further points should be borne in mind. When the simulator is 'unlocked' all channels in use should be inspected to ensure that they are functioning correctly – a defective 'locking' relay may otherwise pass unnoticed. The unrectified meters do not show the full amplitude of the oscillating voltages, particularly at frequencies which are high for the simulator. This means that care must be taken to check voltages for overloading. Using the phasemeter as a voltmeter can be a convenient method of checking.

The driving frequencies ω_{force} on the simulator will, in general, be lower than those in the original data, ω. Thus,

$$\omega = \kappa\, \omega_{force} .$$

The scaled airspeed v_s cannot be greater than 1·998 and should not be greater, in general, than 0·999.

APPENDIX C

The Representation of Multi-point Excitation of an Aircraft Structure

by W. D. Hicks

1. *Introduction.*

An aircraft may experience varying forces at many points on the structure simultaneously (such as in suggested methods of multi-point excitation in ground resonance testing or due to buffeting in flight) but the analytical prediction of the aircraft response due to multi-point forcing proves difficult. Analogue techniques for these predictions appear attractive and accordingly consideration has been given to facilitating investigations of this type on the R.A.E. flutter simulator.

On this simulator the elastic aircraft is generally represented by a finite number (not more than twelve) of modes of deformation, and it is then necessary for the input excitation forces to be distributed in controlled proportions amongst these modes.

In this Appendix, an arrangement of the twelve degree of freedom simulator is described in which six freedoms are used for the representation of the aircraft and the remaining six freedoms provide the channels for six forcing functions (without the need for additional equipment).

This should be adequate for preliminary investigations and if these prove successful the provision of subsidiary circuits for the forcing functions will be considered so that all twelve freedoms are available for aircraft representation.

2. *The Forcing Equation.*

The equation for the force applied to each degree of freedom is:

$$f_s(t) = \sum_1^n d_r^{(s)} P_r(t)\, a_r \qquad (19)$$

where $d_r^{(s)}$ is the ratio of displacement at a particular point r in the sth mode to the displacement at a reference point for the mode

$P_r(t)a_r$ is the force at r.

In the buffeting case,

$P_r(t)$ is the randomly varying pressure as measured at a point

a_r is the area over which the pressure is presumed to act.

3. *Representation of the Equation.*

The forces may be represented as a voltage. As an example, the pressures $P_r(t)$ in equation (19) are measured with analogue transducers on an aircraft structure in flight or in a wind tunnel and recorded on multi-channel magnetic tape. Each of these pressures is then played-back into the flutter simulator but has first to be passed through a potentiometer adjusted to represent a_r and thence to a bank of potentiometers representing $d_r^{(s)}$. Fig. 25 shows a layout for three pressure point measurements feeding into three degrees of freedom through the necessary potentiometers. An amplifier has to be included in each line to avoid loading errors. With this system the input to degree of freedom 2, for example, is:

$$f_2(t) = \sum_1^3 d_r^{(2)} P_r(t)\, a_r\,. \qquad (20)$$

4. *Re-arrangement of Electronic Flutter Simulator Components.*

Each degree of freedom rack contains a summing amplifier which, separated from all other components, has a gain dependent on the setting of the coefficient A_{jj}. This coefficient is normally the direct feedback inertia coefficient in the degree of freedom equation. For this application A_{jj} would be set to 1000, giving an overall gain of 1. If needed the gain of the unit could be increased proportionally by reducing the setting of A_{jj}. It should be noted that this gain is also sign-reversing. Each summing amplifier feeds through a potentiometer on the coefficient panels to the input of all other degrees of freedom. This potentiometer is the inertia coupling coefficient A_{sj}. All coupling coefficients are summed at the input of each degree of freedom, thus A_{sj} can be regarded as a term added to the right-hand side of the equation for a degree of freedom.

If the first six degrees of freedom of the flutter simulator are used to represent the flutter equations in the normal way the remaining six (Nos. 7 to 12) can be used to represent the force equations in the following way.

(1) All amplifiers except the summing amplifier to be withdrawn from each rack.

(2) The direct inertia coefficient A_{jj} ($j = 7$ to 12) to be set at 1000 or at some lower value if the multiplying action is required.

(3) The output of six channels of tape-recorded pressures to be plugged into the jack points 7 to 12 on the right-hand panel of the simulator control desk. The potentiometer units above each jack can be set to represent a_r in equation (19). Associated with each potentiometer unit is a change-over switch which can be used to compensate for the sign-reversing effect of the amplifier where this is important.

(4) The inertia coupling potentiometer A_{sj} on the first six coefficient panels can be set to represent $d_r^{(s)}$ in equation (19) ($s = 1$ to 6 and $j = r$).

This arrangement is shown in Fig. 26 and completely represents:

$$\sum_1^6 d_r^{(s)} P_r(t)\, a_r \qquad s = 1 \text{ to } 6$$

added to the right-hand side of each of the six flutter equations, i.e.:

$$[A\ddot{x}+(BV+D)\,\dot{x}+(CV^2+E)x] = [\sum_1^6 d_r^{(s)} P_r(t)\, a_r]\,.$$

5. *Remarks.*

Although the above process enables the analogue representation of response to multi-point excitation to be performed on the simulator, a feature that has not been discussed is the important question of scaling. In some applications only limited scaling is required; for example in the ground resonance problem absolute amplitude, as distinct from relative amplitude, is rarely required and hence only frequency scaling of the input forces is necessary. Provision for this is already made on the simulator. In the buffeting case, however, the measurement of absolute amplitude of response may be essential and detailed consideration must then be given to both amplitude and frequency scaling.

6. *Conclusion.*

This method has been described as applied to analogue recorded forces particularly but may be used for the linking of other forms of external forces with the flutter simulator if the need should arise. It is limited to a six degree of freedom system and six channels of simultaneous forces. An increase to a number greater than this would require the construction of additional equipment or considerable wiring alterations to the simulator components.

APPENDIX D

A Time-sequenced Input System for the Simulator.

by W. D. Hicks, T. Mossman

1. *Introduction.*

The equipment at present provided on the 12 degree of freedom flutter simulator permits the injection of a sine wave or pulse representing a forcing function added to the right-hand side of each of the flutter equations simultaneously in all degrees of freedom. The operator is able to control the proportion applied to each degree of freedom and also its sense (positive or negative). By using external equipment it is also possible to inject random forces, noise, etc. into any or all of the degrees of freedom, and chosen racks may be adjusted to act as sharing units for proportioning forces amongst the remaining degrees of freedom.

In addition to this, however, it may be necessary in certain cases to designate a definite time sequence in the simulation of the force as applied to each of the degrees of freedom. An example of this arises in the representation of the forcing of an aircraft of slender planform entering a sharp edged gust.

If the planform is arbitrarily divided into a number of transverse sections (Fig. 27) and the lift increment due to entry into the gust is regarded as concentrated at a point in each section, then a representation of the growth of lift at each point, with the correct time sequence, provides a reasonable approximation to the gust forming function. If each point is equally spaced from the next by a distance d and airspeed is V, then the separation in time between the commencement of application of the force to each strip sequentially will be

$$t = \frac{d}{V}. \tag{21}$$

The rise and decay of the gust pressure at the point must be of controlled shape and, in application to the flutter simulator, this is represented as a controlled change of voltage. Six of the units of the flutter simulator can be used to represent six of the modes of deformation of the structure, while the remaining six units are used to proportion the sequential pulses appropriately amongst the modes.

2. *Block Diagram of Sequencing System.*

The system used is shown in block diagram form in Fig. 28. The time t in equation (21) has to be smoothly variable to represent changes in the airspeed V. A low-frequency generator of square waves is therefore used as the time base. The square waves are passed to a pulse-shaper which supplies a positive pulse of definite length for each positive-going edge of the square wave. These pulses drive a cascade of three binary frequency dividers. The two output points of each of these frequency dividers alternately take up one of two conditions, 1 or 0, but each binary unit operates at half the frequency of the preceding unit. The six output points, therefore, change for each input pulse from the pulse-shaper, providing an output pattern of voltages which does not repeat until the ninth input pulse, as shown in Fig. 29.

Seven 'AND' gates, each with three input lines, are connected to chosen trios of the six output points of the binaries in such a way that the first 'AND' gate operates for the first pattern in the pulse sequence, the second for the second pattern, and so on. Recognition of its appropriate pattern by an 'AND' gate results in the production of a pulse at its output point. This pulse operates an electronic switch which initiates the onset of a gust shape to be applied among the modes of the structure. This action continues for the first six input pulses, the seventh pulse operates the seventh 'AND' gate which, in turn, changes over an electronic switch which prevents the application of any further pulses to the binary system. The voltages representing gust forces are, meanwhile, left applied to the structure.

As the structure passes out of the gust the forces applied to each strip must be removed in the same order in which they were applied. An external pulse injected into the seventh electronic switch changes it over and the input pulses are applied to the binary units. This repeats the sequence of operations once more but removes the forces applied to the structure.

3. *Operation of the Equipment.*

A sketch of the front panel of the instrument is given in Fig. 30. Six output plugs on long leads are provided so that the output voltage shapes may be connected to the coefficient panels of six of the units of the flutter simulator. From the coefficient panels these voltages may be distributed among the six units which have been set up to represent six modes of the aircraft structure.

The instrument should be connected to a 24 volt d.c. supply and a suitable square wave generator connected to its input point. The frequency of the generator should be adjusted so that $\frac{1}{f} = t$ of equation (21). The 'SET' switch should now be placed in the on position. If the generator frequency is sufficiently low the row of indicator lights marked 1 to 6 in Fig. 30 will be seen to come on sequentially from right to left as shown by their numbering. When these lights are on the voltage shape is present at the six output plugs. When the press button on the panel is operated these lights will be extinguished in the same order. To repeat the complete pattern it is only necessary to operate the press-button.

Below the row of lights is a row of numbered potentiometers. These are used to control the rise time of the output voltage on each of the six output plugs individually. In the present system this is an exponential rise and the decay when each voltage is switched off is also exponential. Comparison and adjustment of these shapes is best made with the aid of a double-beam oscilloscope connected to each of the outputs in pairs.

4. *The Transistor Circuits.*

The bi-stable circuit of Fig. 31 is used for the three units of the pattern generator of Fig. 29. This basic circuit is extended as shown in Fig. 32 to control the switching of the indicator lights and the output voltage shaper. Fig. 33 shows the arrangement of diodes for the 'AND' gates. The circuit of Fig. 32 with a small relay replacing the indicator bulb and no output voltage circuit, is operated by the seventh of the incoming pulses to the pattern generator. This pulse then operates the relay which breaks the supply of square waves and stops any further action. Operation of the press-button causes a voltage pulse to be applied to this unit which changes over its state and allows the relay to close for a further sequence of pulses. The 'SET' switch S_1 ensures that all units are set to the same state before the pulse count begins.

5. *Conclusions.*

This equipment has been tested with the flutter simulator and works satisfactorily. The present arrangement permits only the application of exponentially rising and decaying voltage shapes to the modes, with control of the rise-time. If required, other circuits for the output voltages could be developed which would provide a variety of voltage shapes.

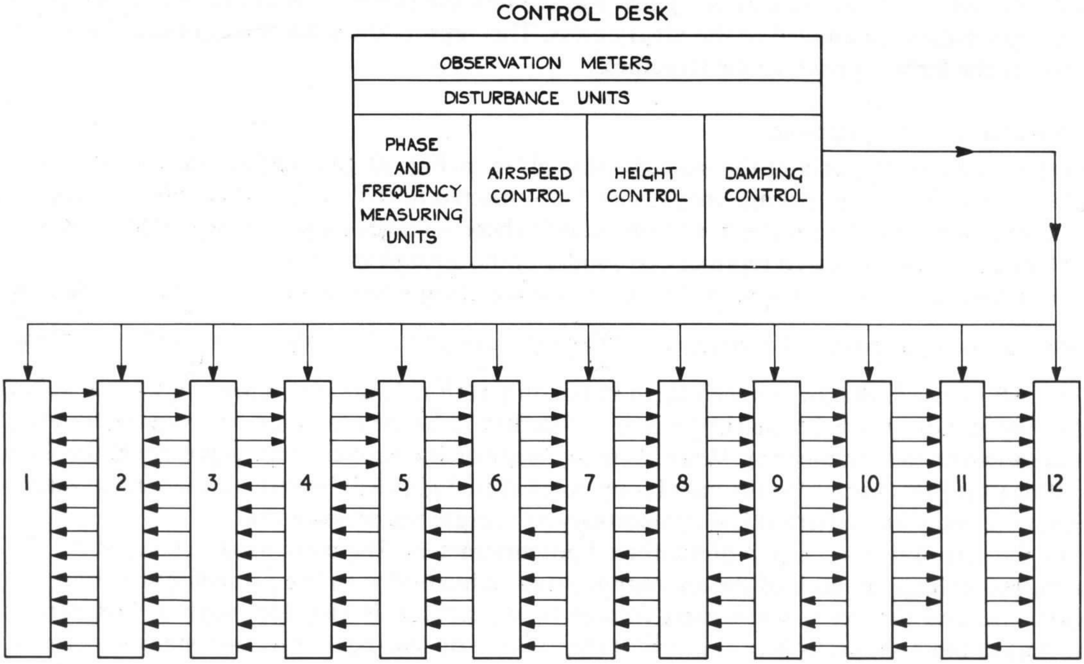


FIG. 1. Simulator block diagram.

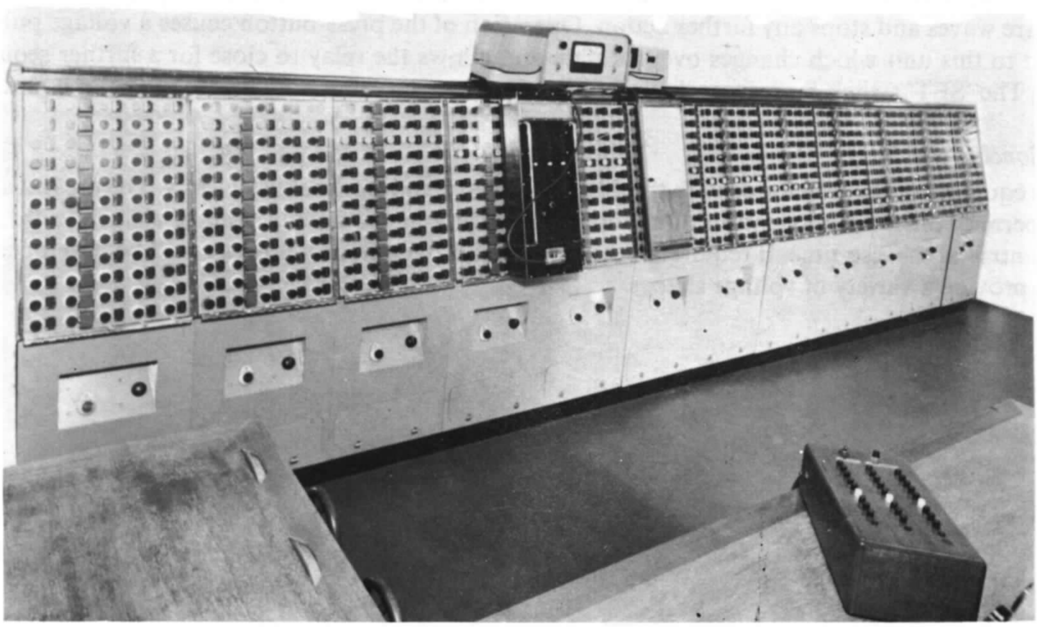


FIG. 2. The coefficient cabinets.

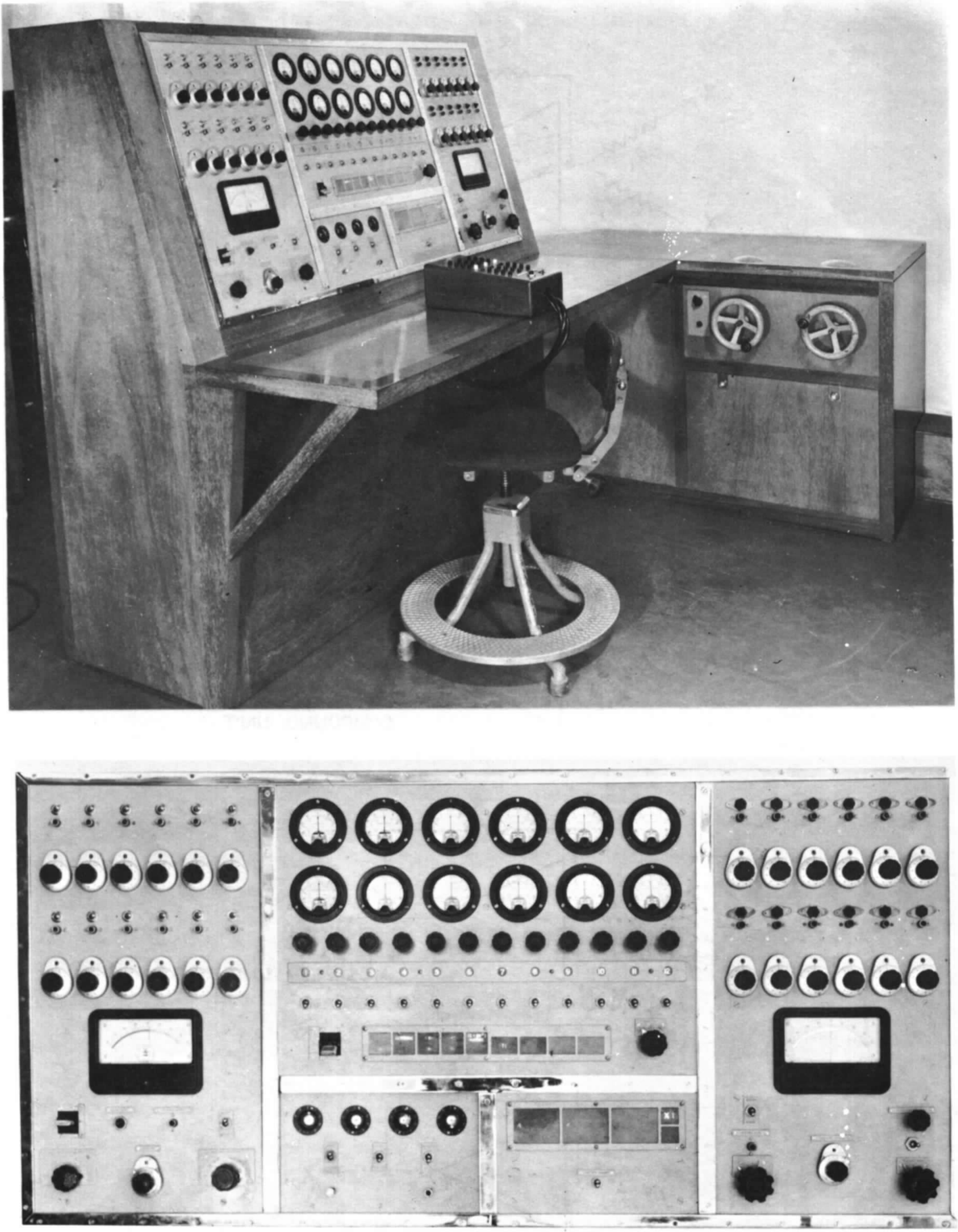


FIG. 3. Control desk and panels.

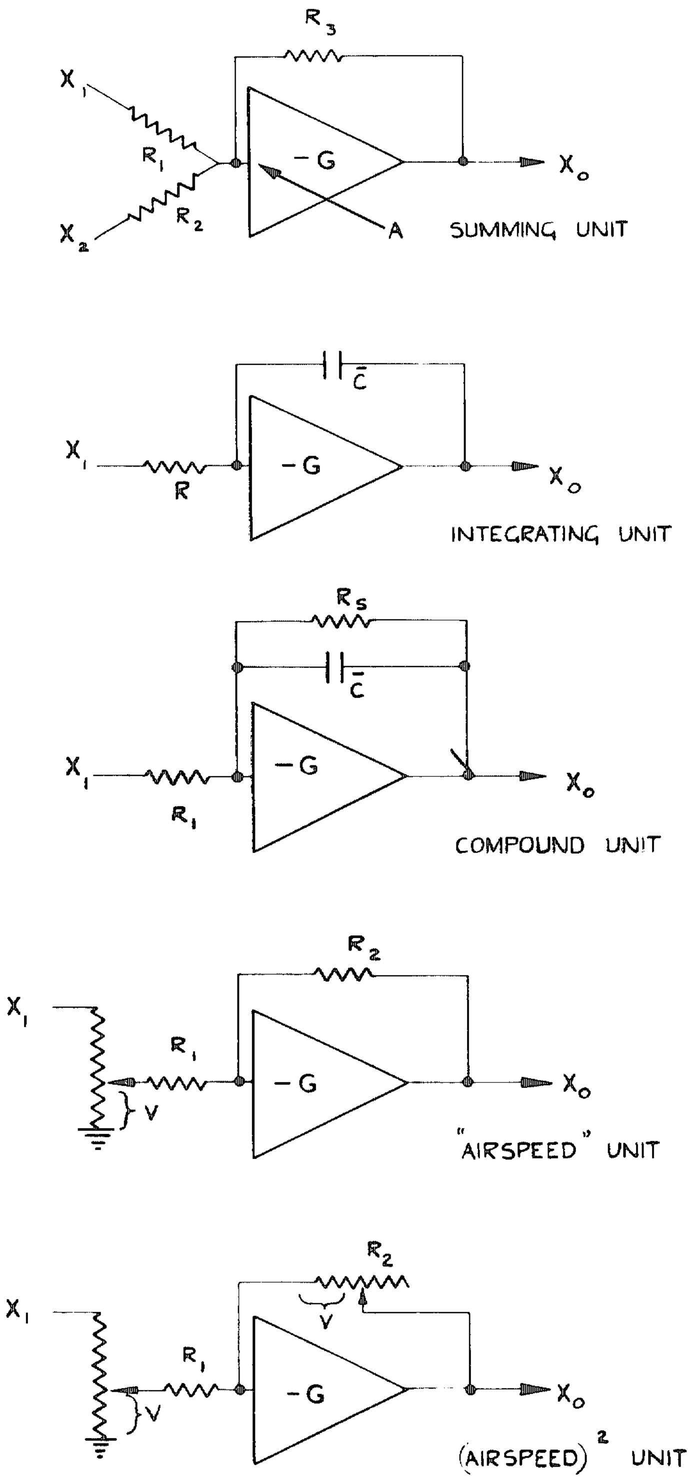


FIG. 4. Operational circuits.

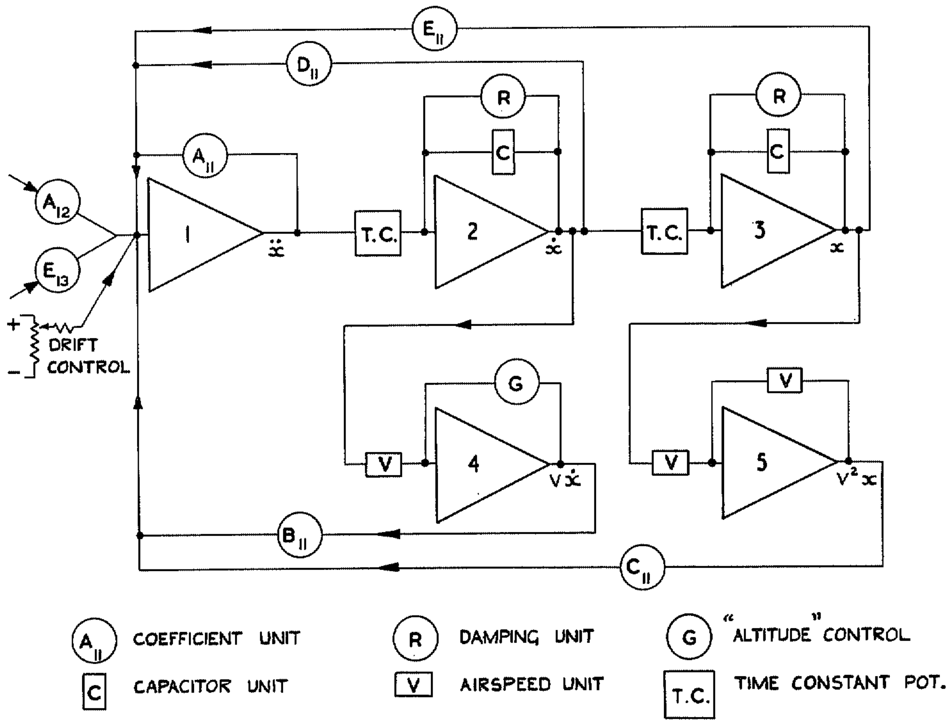


FIG. 5. A degree-of-freedom unit.

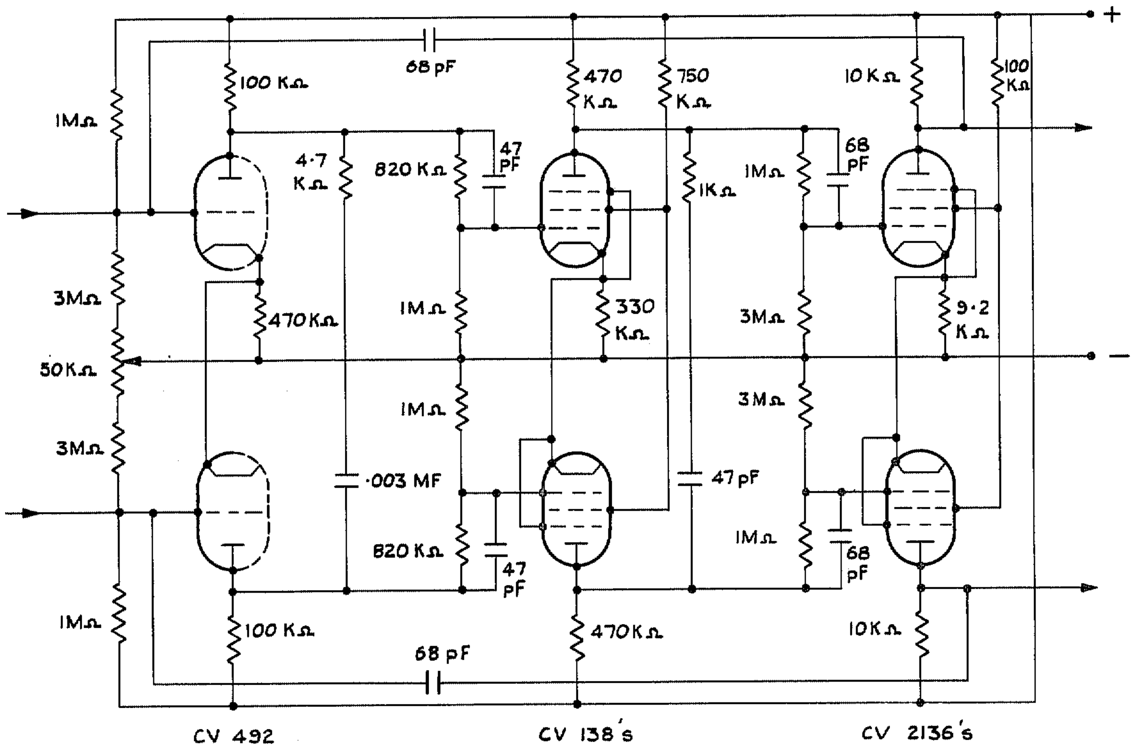


FIG. 6. D.C. amplifier.

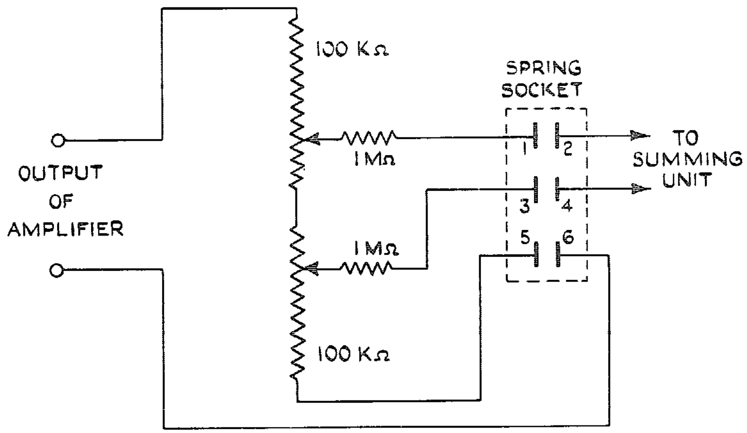


FIG. 7. Coefficient unit.

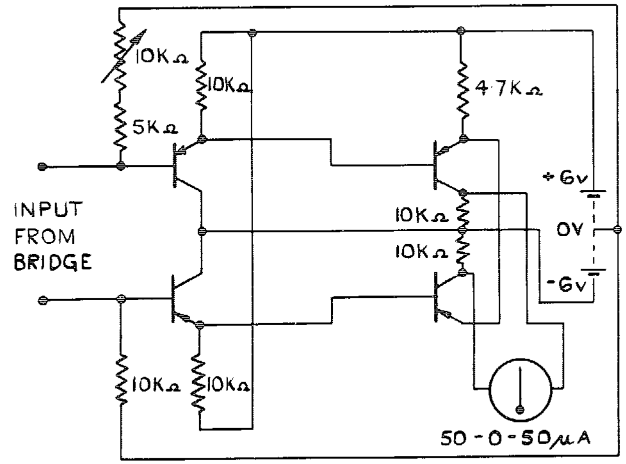


FIG. 9. Null balance amplifier.

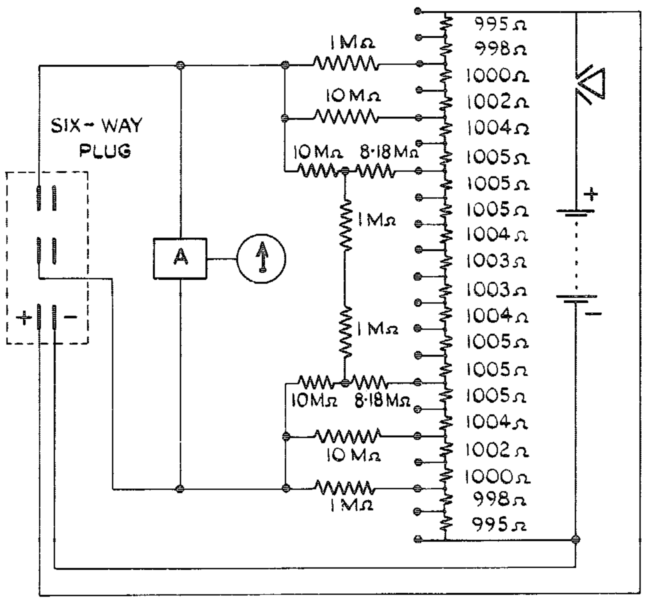


FIG. 8. Coefficient bridge.

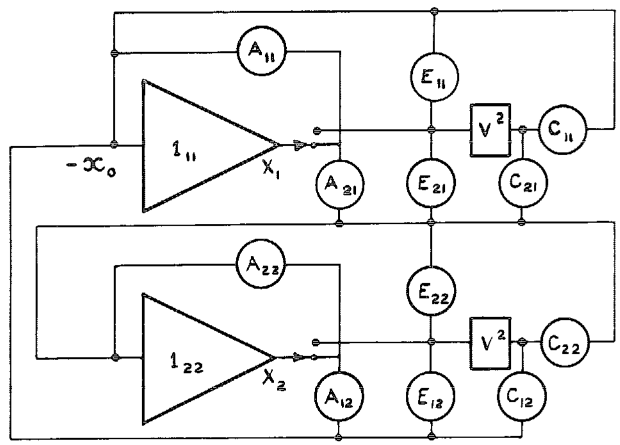


FIG. 10. Matrix analysis circuits.

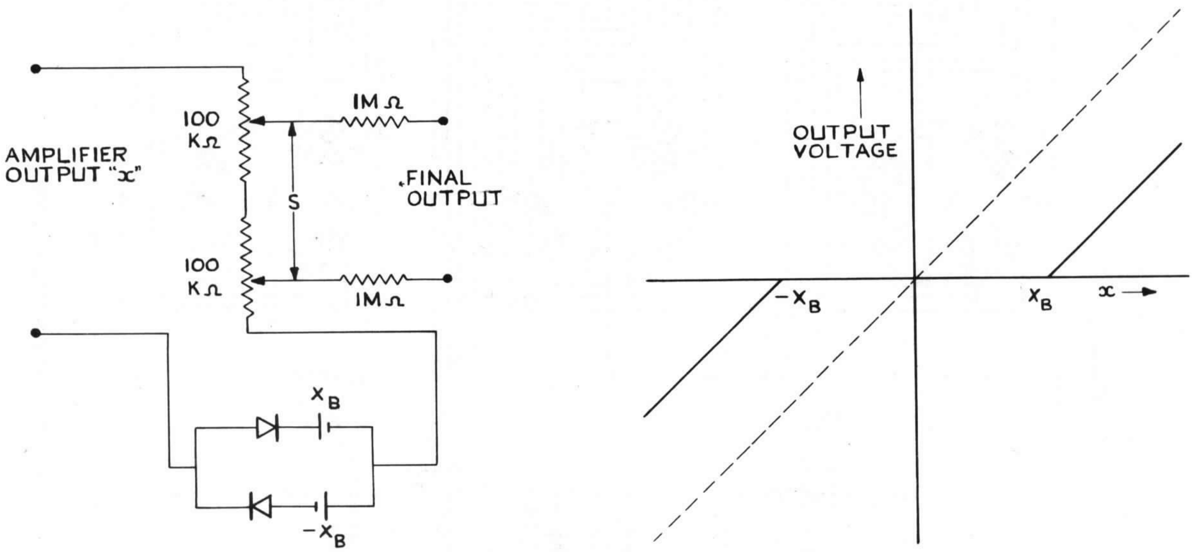


FIG. 11. Non-linear circuit.

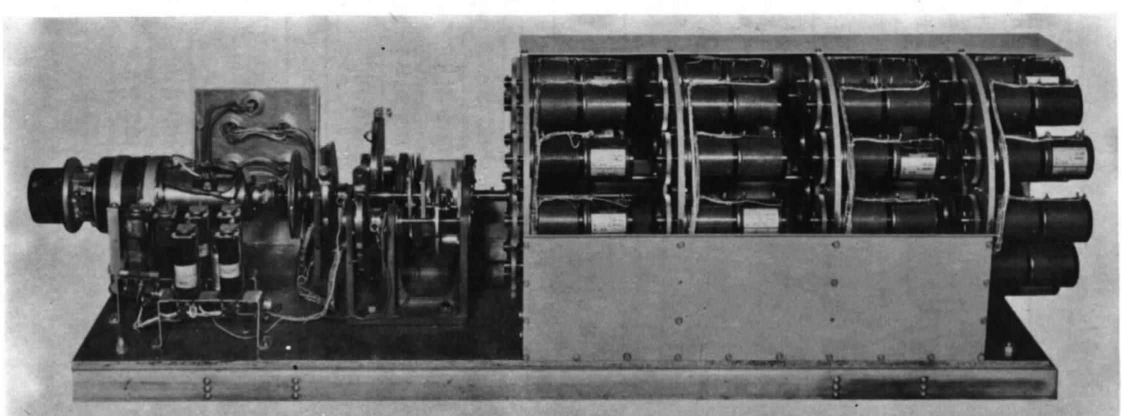


FIG. 12. Airspeed unit.

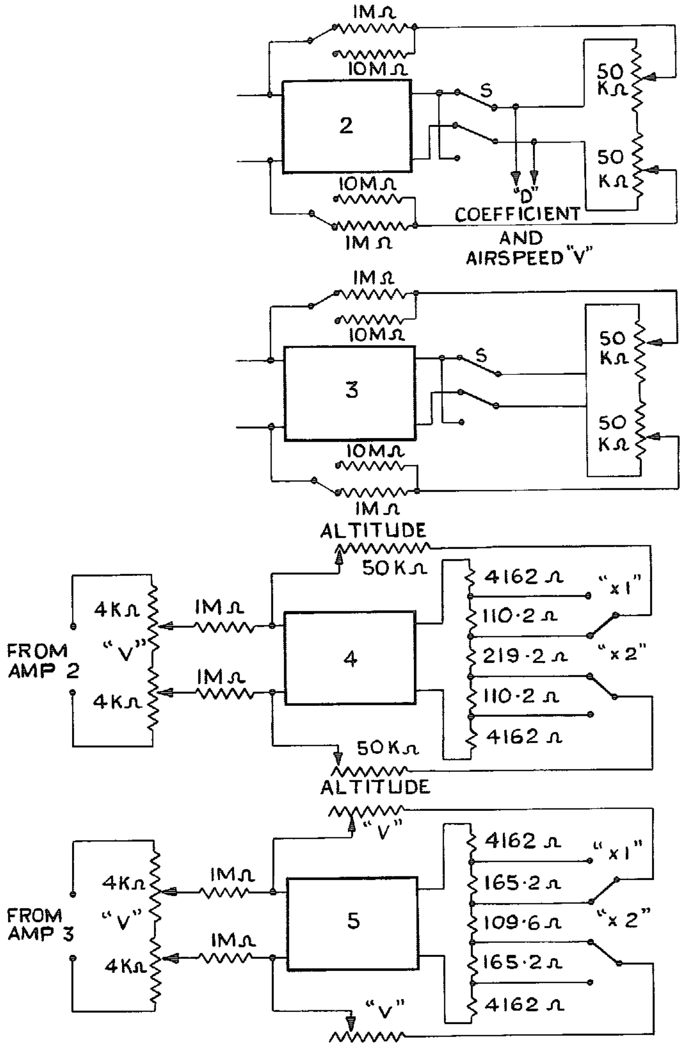


FIG. 13. Airspeed, altitude, damping circuits.

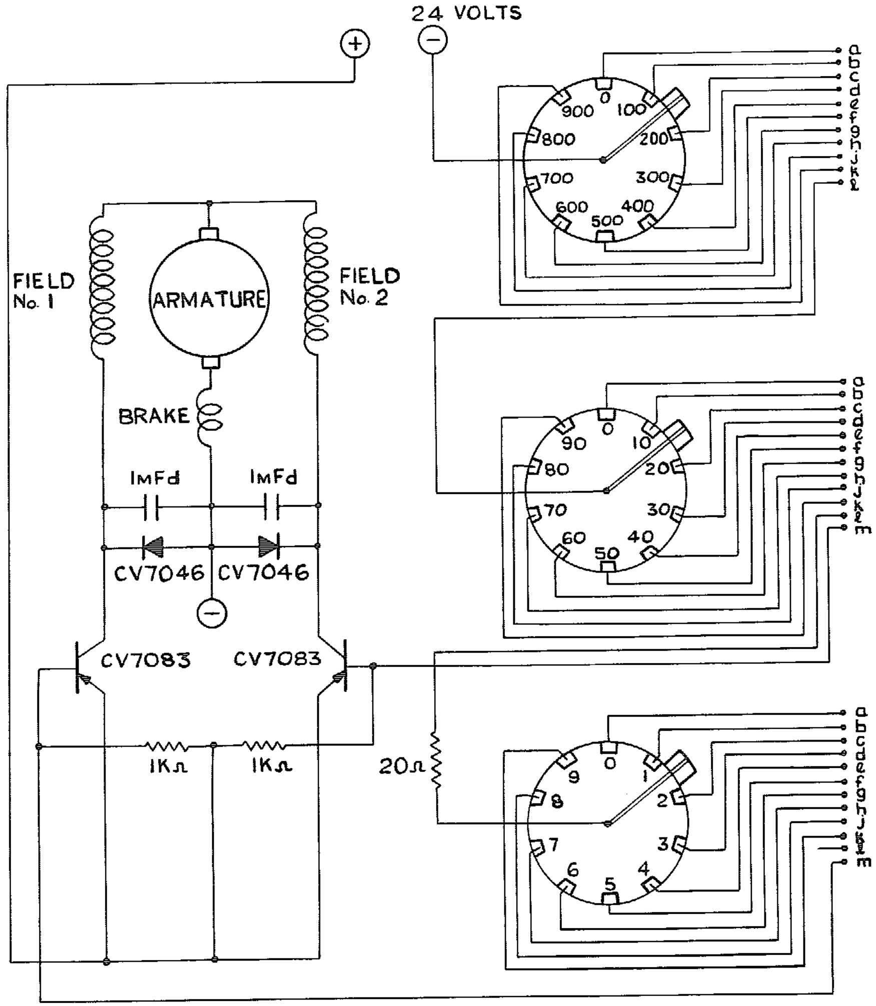


FIG. 14. Motor control system.

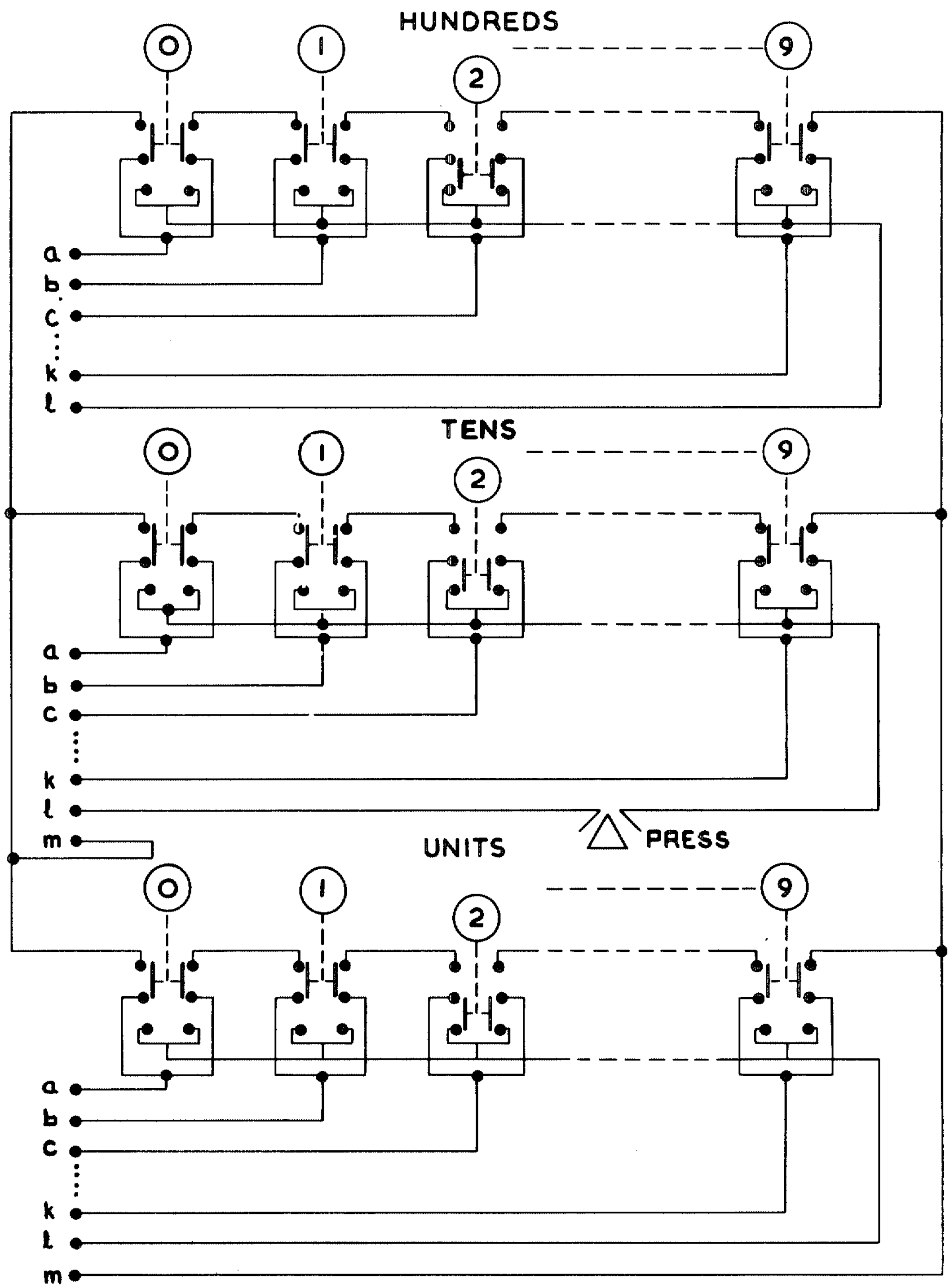


FIG. 15. Airspeed control switches.

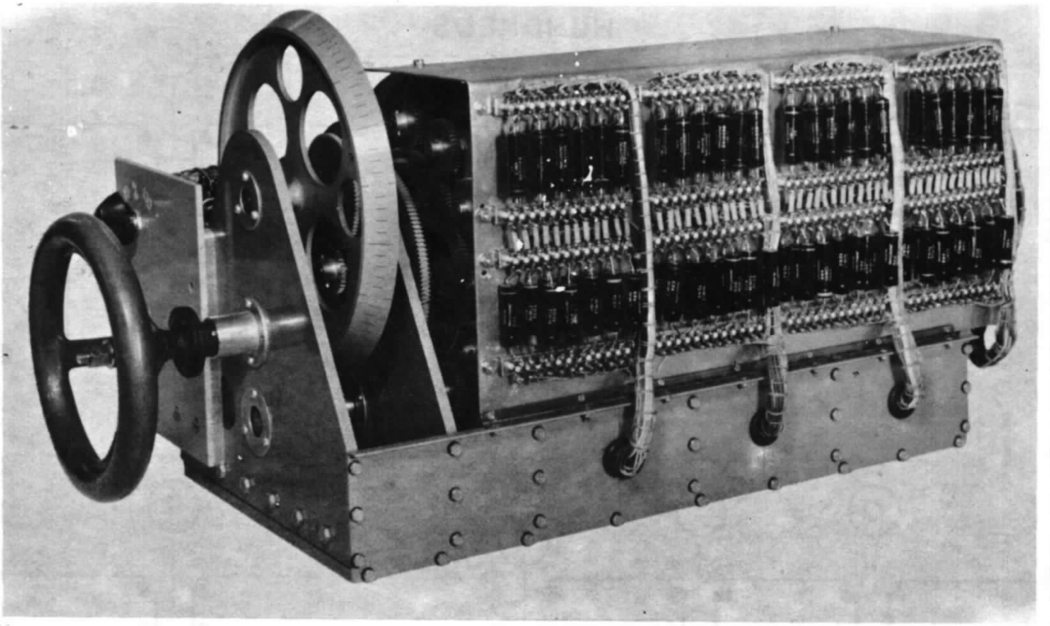


FIG. 16. Damping control unit.

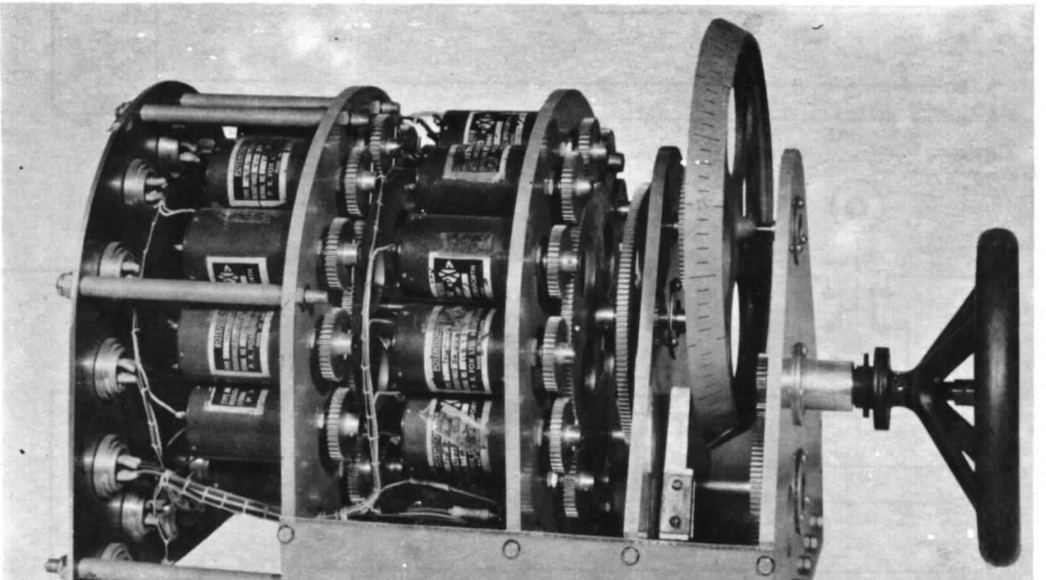


FIG. 17. Altitude control unit.

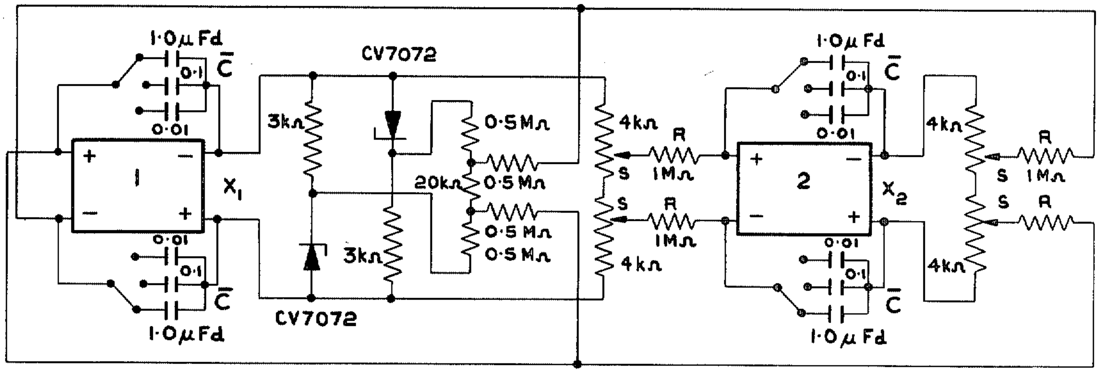


FIG. 18. Resonator unit.

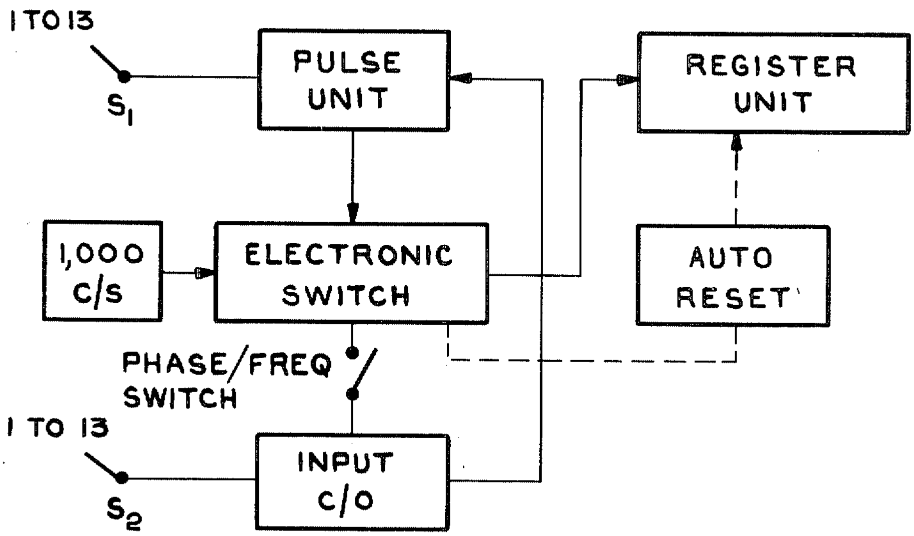


FIG. 19. Frequency/phase unit.

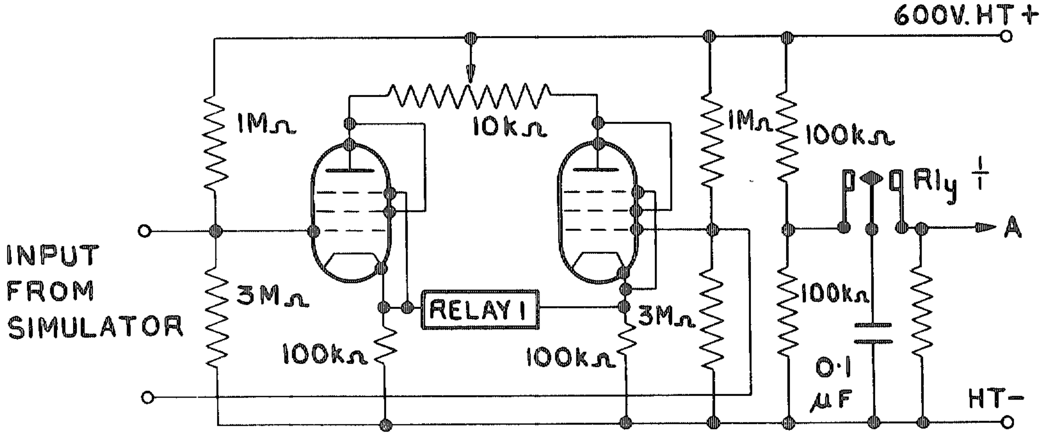


FIG. 20. Pulse unit.

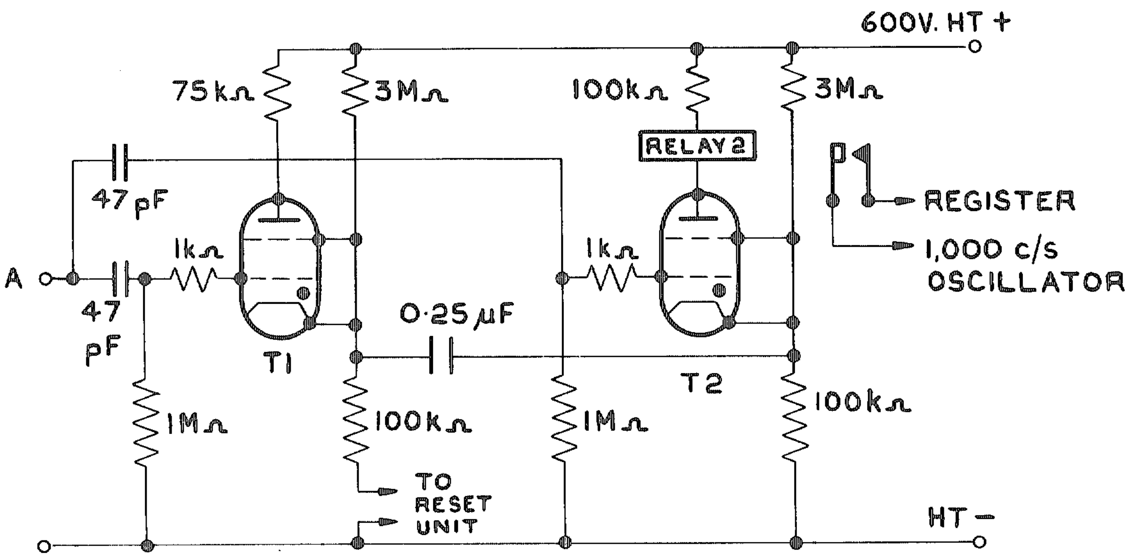


FIG. 21. Electronic switch.

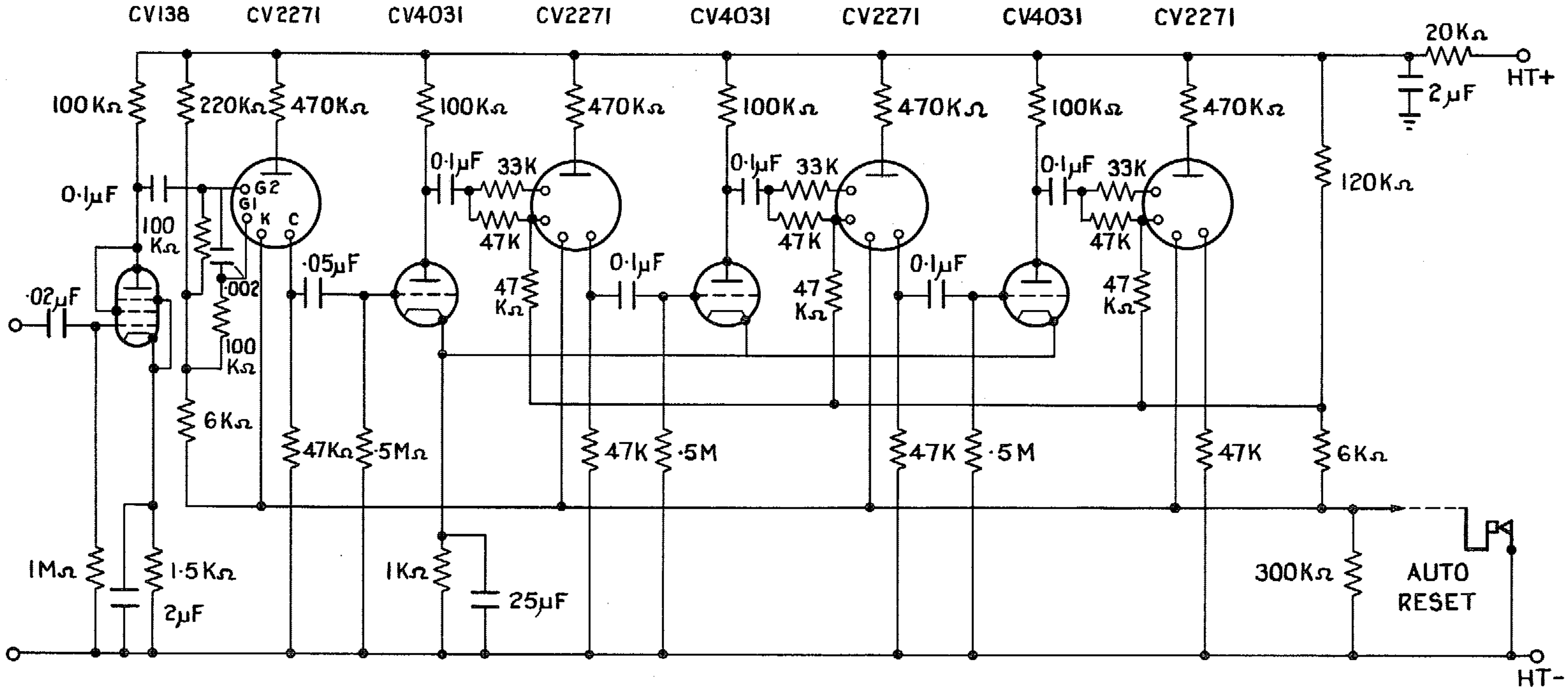


FIG. 22. Register unit.

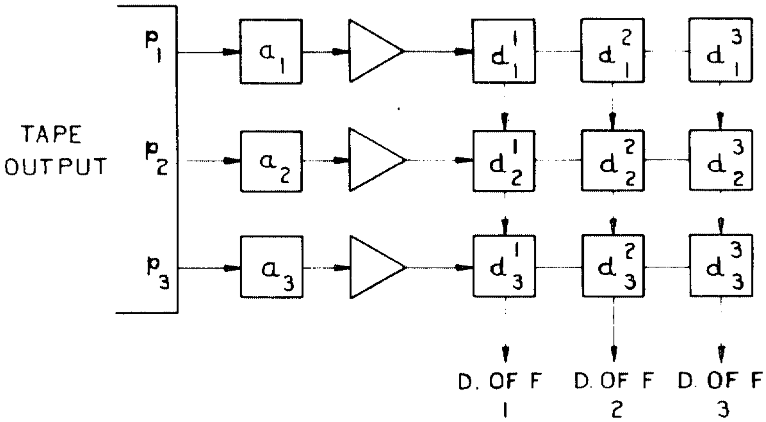


FIG. 25. Potentiometer network.

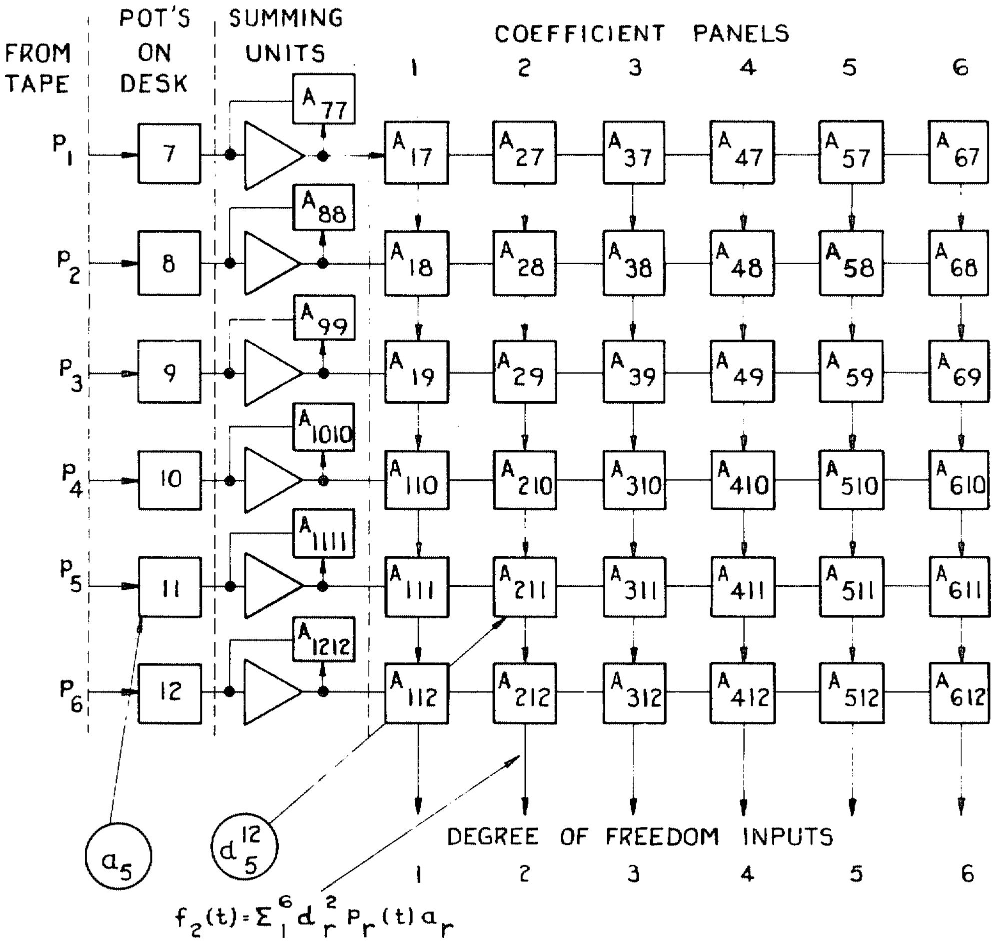


FIG. 26. Arrangement of simulator components.

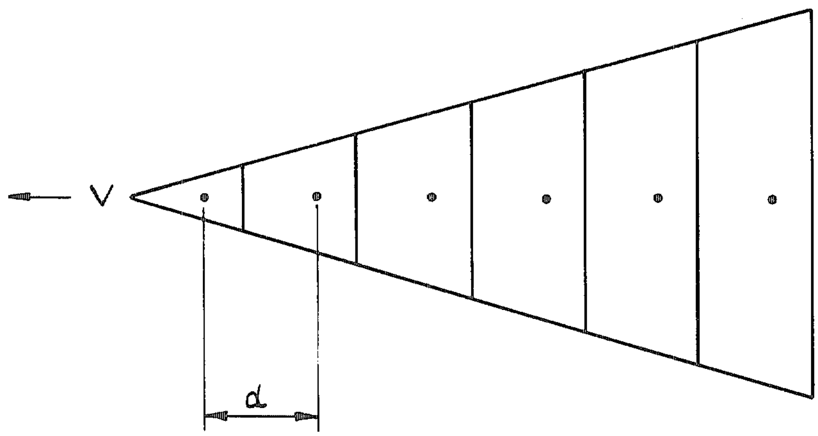


FIG. 27. Segmented structure.

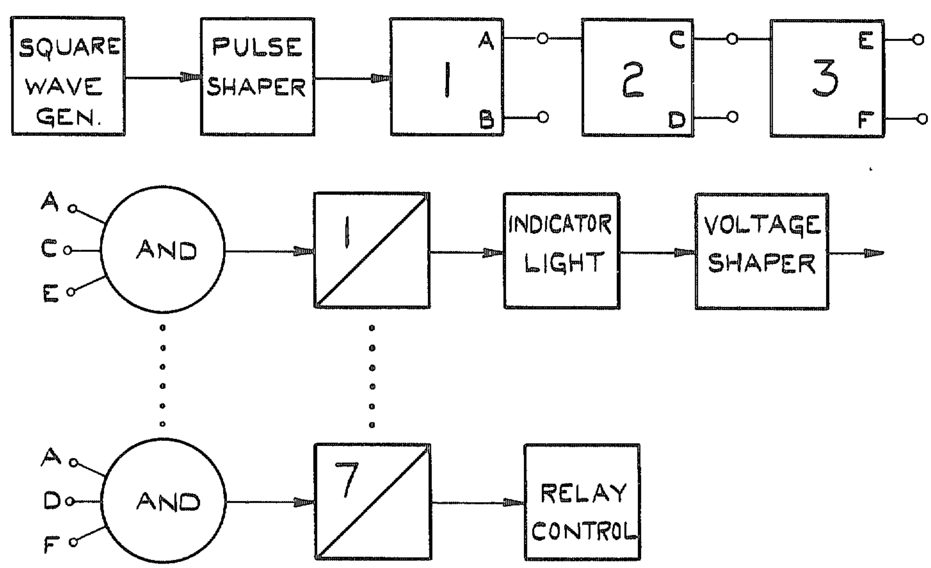
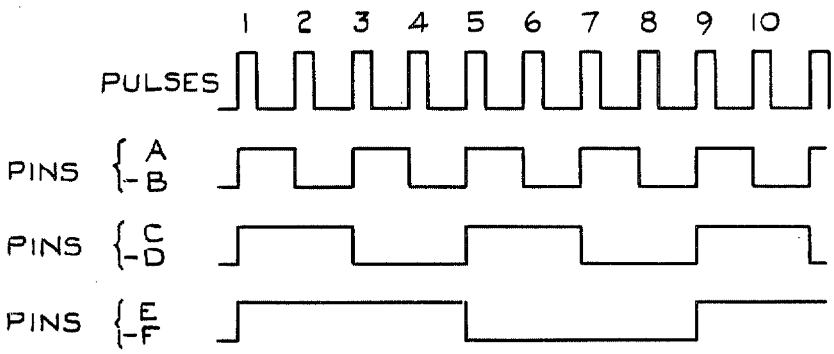


FIG. 28. Block diagram of sequencing system.



PULSE No.	"AND" CONNECTIONS		
1	A	C	E
2	B	C	E
3	A	D	E
4	B	D	E
5	A	C	F
6	B	C	F
7	A	D	F
8	B	D	F
9	A	C	E

FIG. 29. Pulse sequence.

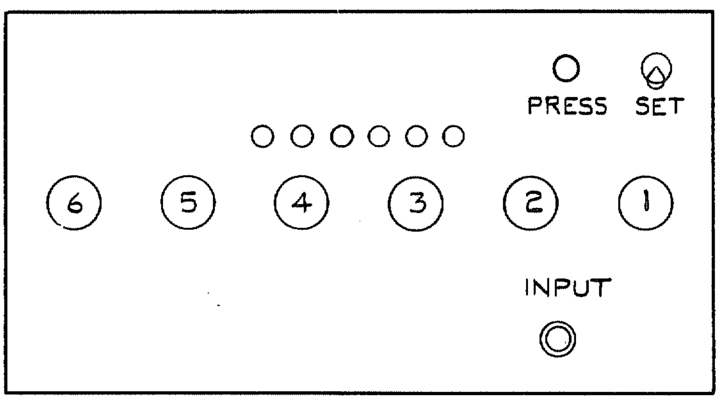


FIG. 30. Control panel.

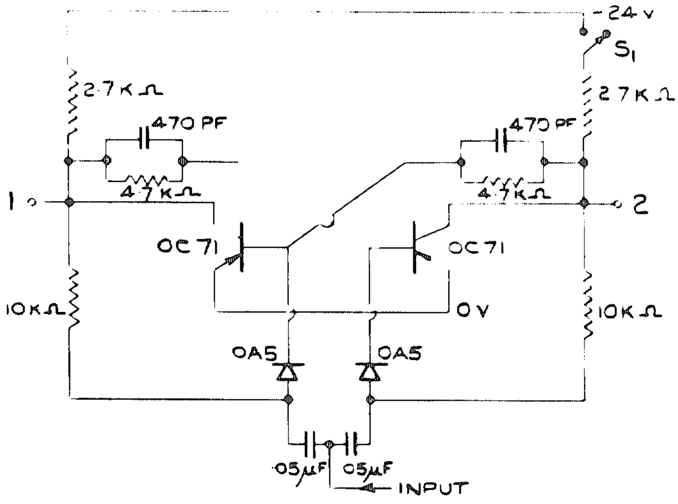


FIG. 31. Bi-stable unit.

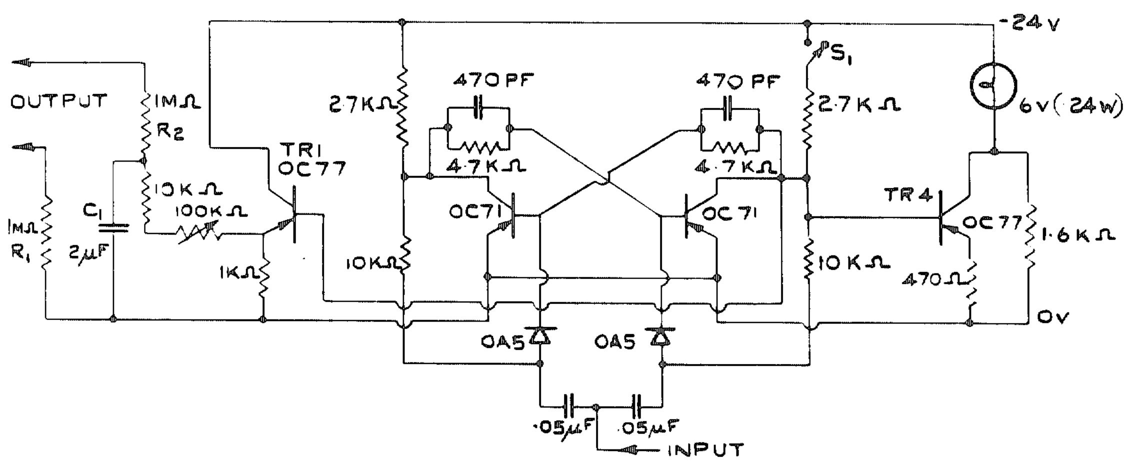


FIG. 32. Control circuit.

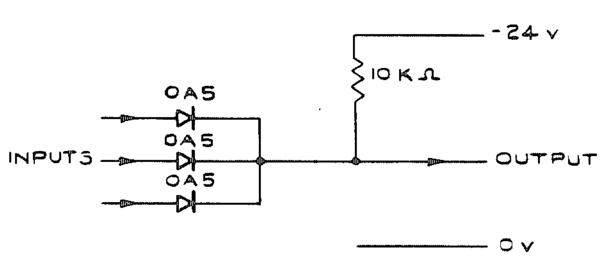


FIG. 33. 'And' circuit.

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