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SOME DETAILS OF THE QUINTIC PROFILE FOR USE IN THE POLHAUSEN-TYPE OF BOUNDARY-LAYER CALCULATION

By

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Some Details of the Quintic Profile for Use in the Polhausen-Type of Boundary-Layer Calculation

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March, 1957

Summary

The Pohlhausen method for approximate calculation of the laminar boundary layer uses a quartic one-parameter family of velocity profiles. In this paper the general theory is developed for a quintic profile, satisfying an additional boundary condition at the wall. A detailed analysis is given for the case of a linearly retarded main-stream; good agreement is obtained upon comparison with the exact solution.

1. Introduction

Since the concept of the boundary layer was first developed by Prandtl more than half a century ago, many exact and approximate solutions of the steady incompressible laminar-boundary-layer equations have been obtained. One of the earliest of the approximate methods of solution was due to Pohlhausen[1] (1921). In this method the boundary-layer equations are not solved everywhere, but only at the wall, at the edge of the boundary layer, and on an average by expressing the momentum equation in the integral form due to von Kármán.

For a two-dimensional boundary layer, x is measured along the wall and y perpendicular to it. The velocities in the x and y directions are u and v respectively, p is the pressure ρ the density, μ the viscosity and $\nu = \mu/\rho$ the kinematic viscosity of the fluid. At the edge of the boundary layer $u \to u_1(x)$. Then it is well known (Goldstein[2], 1938) that the momentum integral equation may be written as

$$\frac{\tau_0}{\rho u_1^2} = \frac{d\delta_2}{dx} + \frac{1}{u_1}\frac{du_1}{dx}(\delta_1 + 2\delta_2) \; . \qquad \ldots(1.1)$$

We express the velocity u in the form

$$\frac{u}{u_1} = f(\eta) \; , \; \eta = \frac{y}{\delta} \; , \qquad \ldots(1.2)$$

where δ is the boundary layer thickness, assumed finite. Then in (1.1) the skin-friction τ_0, the displacement thickness δ_1, and the momentum thickness δ_2 are given by

$\frac{\tau_0}{\rho u_1^2}$ /

$$\left.\begin{aligned}\frac{\tau_o}{\rho u_1^2} &= \frac{\nu}{u_1\delta} f'(0)\\ \frac{\delta_1}{\delta} &= \int_0^1 (1-f)\,d\eta\\ \frac{\delta_2}{\delta} &= \int_0^1 f(1-f)\,d\eta\end{aligned}\right\} \qquad \ldots(1.3)$$

The boundary conditions are

$$f(0) = 0,\ f''(0) = -\Lambda,\ f'''(0) = 0,\ \ldots. \qquad \ldots(1.4)$$

where

$$\Lambda = \frac{\delta^2}{\nu}\frac{du_1}{dx}, \qquad \ldots(1.5)$$

and

$$f(1) = 1,\ f'(1) = f''(1) = f'''(1) = \ldots = 0\ . \qquad \ldots(1.6)$$

The idea used by Polhausen is to approximate to f as a polynomial in η, and to satisfy as many of the boundary conditions, (1.4) and (1.6), as possible. In particular, Pohlhausen considered the case of a quartic satisfying the first two conditions (1.4) and the first three conditions (1.6). In this paper we shall consider the case of a quintic, which satisfies in addition the third condition of (1.4). Upon satisfying all these conditions we find that

$$f(\eta) = \tfrac{1}{3}\eta\,(5 - 5\eta^3 + 3\eta^4) + \tfrac{1}{4}\Lambda\eta(1+\eta)\,(1-\eta)^3\ . \qquad \ldots(1.7)$$

Upon substitution into (1.1) and (1.3) a first-order differential equation results for $Z = \frac{\delta^2}{\nu} = \frac{\Lambda}{u_1'}$, taking the form

$$Z' = \frac{g(\Lambda)}{u_1} + Z^2\, u_1''\, h(\Lambda)\ , \qquad \ldots(1.8)$$

which may also be written as an equation for Λ,

$$\Lambda' = g(\Lambda)\frac{u_1'}{u_1} + \{\Lambda + \Lambda^2\, h(\Lambda)\}\frac{u_1''}{u_1'} \qquad \ldots(1.9)$$

In/

In this paper the general theory is worked out, and the universal functions, $g(\Lambda)$ and $h(\Lambda)$, tabulated for the case of a quintic velocity profile. A detailed solution is then obtained for the case of a linearly retarded main-stream velocity. It is found that the predicted displacement and momentum thicknesses, the skin-friction and the separation position, are nowhere in error by more than 6% when compared with the exact solution of Howarth[3] (1938).

2. General Theory for a Quintic Velocity Distribution

The assumed velocity profile in the boundary layer is

$$\frac{u}{u_1} = f(\eta) = \tfrac{1}{3}\eta\,(5 - 5\eta^3 + 3\eta^4) + \tfrac{1}{4}\Lambda\eta\,(1+\eta)\,(1-\eta)^3 , \qquad \ldots(2.1)$$

satisfying the boundary conditions

$$\left.\begin{aligned} f(1) &= 1,\ f'(1) = f''(1) = 0 \\ f(0) &= 0,\ f''(0) = -\Lambda,\ f'''(0) = 0 \end{aligned}\right\} \qquad \ldots(2.2)$$

Then upon substitution from (2.1) into (1.3), the skin friction, the displacement thickness and the momentum thickness are given, after integration by

$$\frac{\tau_0}{\rho u_1^2} = \frac{\nu}{u_1\delta}\left\{\frac{5}{3} + \frac{1}{4}\Lambda\right\}, \qquad \ldots(2.3)$$

$$\frac{\delta_1}{\delta} = \frac{1}{3} - \frac{1}{60}\Lambda , \qquad \ldots(2.4)$$

$$\frac{\delta_2}{\delta} = \frac{775}{6237}\left\{1 - \frac{3}{248}\Lambda - \frac{423}{124000}\Lambda^2\right\} . \qquad \ldots(2.5)$$

It follows from (2.3) that separation takes place when

$$\Lambda = \Lambda_s = -\frac{20}{3} . \qquad \ldots(2.6)$$

Upon substitution for τ_0, δ_1, δ_2 from (2.3) to (2.5) into the momentum integral equation (1.1), a first-order linear differential equation for Λ results. This is

Λ'/

$$\Lambda' = g(\Lambda)\frac{u_1'}{u_1} + \{\Lambda + \Lambda^2 h(\Lambda)\}\frac{u_1''}{u_1'} , \qquad \ldots(2.7)$$

where primes note differentiation with respect to x, and the universal functions $g(\Lambda)$, $h(\Lambda)$ are given by

$$g(\Lambda) = \frac{4}{5} \cdot \frac{831600 - 165580\,\Lambda + 9816\,\Lambda^2 + 423\,\Lambda^3}{(20 - 3\Lambda)(1240 + 141\,\Lambda)} , \qquad \ldots(2.8)$$

$$h(\Lambda) = \frac{12}{5}\,\frac{250 + 141\,\Lambda}{(20 - 3\Lambda)(1240 + 141\,\Lambda)} . \qquad \ldots(2.9)$$

The equation may alternatively be expressed as an equation in

$$Z = \frac{\delta^2}{\nu} = \frac{\Lambda}{u_1'} , \qquad \ldots.(2.10)$$

namely

$$Z' = \frac{g(\Lambda)}{u_1} + Z^2 u_1'' h(\Lambda) . \qquad \ldots(2.11)$$

The functions $g(\Lambda)$, $h(\Lambda)$ and $\Lambda + \Lambda^2 h(\Lambda)$ are tabulated in Table I.

Now, as is true in the quartic case, since u_1 vanishes at the forward stagnation point of the flow past a cylindrical obstacle, it follows that $g(\Lambda)$ must be zero there if Z' is to remain finite. But the only zero of $g(\Lambda)$ occurs at $\Lambda = -35.7$, which is not a permissible value, since by (2.3) it corresponds to a negative skin friction. It follows that the flow near a stagnation point cannot be dealt with by this method, in much the same way as the ordinary Pohlhausen method cannot be used in certain problems for which Λ becomes too large. In problems in which the velocity increases away from a stagnation point, and decreases after the point of minimum pressure, the classical Pohlhausen quartic must be used at least in the region of accelerated flow.

3. The Boundary Layer with a Linearly Retarded Mainstream

In certain simple cases equation (2.7) simplifies sufficiently to be solved by direct integration. For example, if the main-stream velocity may be expressed as

$$u_1(x) = u_0\left(1 - \frac{x}{c}\right)^n , \qquad \ldots(3.1)$$

then/

then

$$\frac{u_1''}{u_1'} = \left(1 - \frac{1}{n}\right)\frac{u_1'}{u_1} , \qquad \ldots(3.2)$$

and so (2.7) becomes simply

$$\Lambda' = \frac{u_1'}{u_1}\left\{ g(\Lambda) + \left(1 - \frac{1}{n}\right)\left[\Lambda + \Lambda^2 h(\Lambda)\right] \right\} . \qquad \ldots(3.3)$$

This equation may be integrated directly to give

$$u_1(x) = u_o \exp\left\{ \int_o^\Lambda \frac{d\Lambda}{g(\Lambda) + \left(1 - \frac{1}{n}\right) [\Lambda + \Lambda^2 h(\Lambda)]} \right\} , \qquad \ldots(3.4)$$

and the complete solution for the boundary layer is easily obtained.

The simplest example is that of the linearly retarded mainstream, for which $n = 1$. This will be considered in some detail, as the exact solution is well known. With $n = 1$, and therefore $u_1 = u_o\left(1 - \frac{x}{c}\right)$, equation (3.4) becomes

$$\frac{x}{c} = 1 - \exp\left\{ \int_o^\Lambda \frac{d\Lambda}{g(\Lambda)} \right\} . \qquad \ldots(3.5)$$

The relevant integrations have been carried out, and the results for τ_o are shown in Table 2. We note that τ_o is 3% low at $x/c = 0$, and the error increases with x/c until it reaches a maximum of 6% at about $x/c = 0.076$. The error then decreases, until at about $x/c = 0.11$ the skin friction is predicted correctly. When $x/c > 0.11$ the predicted skin friction is too great, and the distance to separation $x_s = 0.127\ c$ is too great by 6%. It could equally be shown that the predicted δ_1 is fairly uniformly about 1% high and δ_2 about 3% low. The accuracy is therefore most satisfactory.

References/

References

No.	Author(s)	Title, etc.
1	K. Pohlhausen	Zur Integration der Differentialgleichung der laminaren Grenzschicht. Zeits.fur.ang.Math.und Mech.1, p. 252, (1921).
2	S. Goldstein (Ed.)	Modern Developments in Fluid Dynamics. Vol. 1. O.U.P. (1938).
3	L. Howarth	On the Solution of the Laminar Boundary Layer Equations. Proc. Roy. Soc. A, 164. p. 542. (1938).

TABLE 1/

TABLE 1

The Functions $g(\Lambda)$, $h(\Lambda)$, $\Lambda + \Lambda^2 h(\Lambda)$

3Λ	$g(\Lambda)$	$h(\Lambda)$	$\Lambda + \Lambda^2 h(\Lambda)$
-20	149.76	-0.1380	-12.800
-19	128.07	-0.1140	-10.907
-18	111.52	-0.0955	-9.439
-17	98.46	-0.0808	-8.260
-16	87.88	-0.0686	-7.284
-15	79.13	-0.0583	-6.458
-14	71.76	-0.0495	-5.744
-13	65.48	-0.0417	-5.117
-12	60.06	-0.0348	-4.557
-11	55.32	-0.0286	-4.051
-10	51.15	-0.0228	-3.587
-9	47.45	-0.0175	-3.158
-8	44.14	-0.0125	-2.756
-7	41.18	-0.0077	-2.375
-6	38.50	-0.0031	-2.012
-5	36.07	0.0014	-1.663
-4	33.87	0.0059	-1.323
-3	31.86	0.0103	-0.990
-2	30.03	0.0149	-0.660
-1	28.35	0.0194	-0.331
0	26.83	0.0242	0
1	25.44	0.0292	0.337
2	24.18	0.0344	0.682
3	23.04	0.0400	1.040
4	22.03	0.0460	1.415
5	21.15	0.0526	1.813
6	20.39	0.0599	2.240
7	19.77	0.0681	2.704
8	19.30	0.0775	3.190
9	18.93	0.0883	3.795
10	18.92	0.1011	4.456
11	19.09	0.1164	5.232
12	19.59	0.1354	6.166
13	20.55	0.1595	7.328
14	22.17	0.1914	8.834
15	24.84	0.2357	10.892
16	29.31	0.3018	13.918
17	37.34	0.4116	18.883
18	54.24	0.6305	28.698
19	106.56	1.2861	57.919
20	∞	∞	∞

TABLE 2/

TABLE 2

Comparisons with Howarth's Results

x/c	$\dfrac{\nu^{\frac{1}{2}} \left(\dfrac{\partial u}{\partial y}\right)_0}{u_1 \, (-u_1')^{\frac{1}{2}}}$ Curle	Howarth
0.0232	1.837	1.884
0.0435	1.155	1.203
0.0610	0.825	0.866
0.0761	0.612	0.648
0.0890	0.456	0.484
0.0999	0.333	0.346
0.1090	0.231	0.231

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