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Renewal Processes Arising in the Study of Multiplex Systems

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Summary.

The object of this report is to discuss the improvement of reliability of systems when redundancy is introduced in the form of so-called 'multiplexing' so that a given task is performed not by one suitably chosen set of components (referred to as a 'lane') but by a number of separate lanes operating independently in parallel.

Under the assumption that the failure of individual lanes can be described by a Poissonian Process it is shown that the Renewal Process representing the failure mechanism of a system composed of m lanes exhibits, for comparatively small time intervals, a dramatic improvement of reliability (from a small probability p for one lane to a much smaller probability p^m for the system) whilst for the large time intervals the improvement is less than proportional to m. Thus the multiplexing appears to achieve its maximum advantage only when, after a comparatively short period of operation, all the lanes are inspected and brought to their initial state by repair or replacement; without these precautions multiplexing is still useful but the law of diminishing returns operates then: by increasing the number of lanes, less and less is added to the asymptotic reliability of the system.

The discussion of the general case of m lanes is followed by a more detailed analysis of duplex and triplex systems and, in the closing section, a modified method of multiplexing (the 'majority vote') is described.

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1. *Introduction.*

One of the methods of improving the reliability of systems is the introduction of redundancy into their design. If a set of suitably chosen components (often referred to as a lane) is designed to perform a given task, then it is possible to have the same task performed by a number m of separate lanes operating independently in parallel; this is described as multiplexing.

The failure of the system will then occur only when all the lanes have failed; if the probability p of any one lane failure in a given time interval is comparatively small, the probability of the whole system failing in this time interval will be approximately p^m, where m is the degree of multiplexing, provided that the events of failure of individual lanes can be regarded as independent.

If the failure probability of an individual lane is of the order, say, of 10^{-3} the above procedure leads to dramatic improvement of reliability: the failure probability of a duplex system becomes 1,000 and of a triplex system 1,000,000 times smaller and this is achieved merely by doubling (or trebling) the weight (and the costs) of the system and by the inconvenience of a double (or triple) effort in its maintenance and inspection. However, it must be borne in mind that this relation holds only on the assumption that the system operates for a short time (for which the probability of a lane failure is small) and that, after a period of operation it is subjected to examination and, if necessary, repair so that all the lanes are in their initial state. It is clear that if the system is allowed to operate for a longer time without these preventive measures, then its reliability will deteriorate since any uncorrected failure in the lanes will decrease the redundancy. Thus the problem arises as to for what periods of operation the above argument can be applied and it is also interesting to know what happens to the system if it is allowed to run for longer periods without the individual lanes being inspected and corrected (as in the case of missiles or artificial satellites).

In order to obtain some information about the behaviour of the system we shall investigate the underlying Renewal Process. Suppose that we start with a new system at time zero. Due to the accumulated lane failures the system ceases to operate at a time τ_1, and it is replaced by a new system which again fails after a time τ_2. Thus the second failure occurs at the time $S_2 = \tau_1 + \tau_2$, the third at the time $S_3 = \tau_1 + \tau_2 + \tau_3$ etc. The random variables $S_1, S_2, \ldots, S_r, \ldots$ (the times up to the rth renewal) define the instants at which forced replacements are required. Apart from these unavoidable replacements it may be advisable to introduce some scheduled replacements, i.e. inspections followed by replacement or repair of failed lanes in order to meet some safety requirements or in order to minimize the 'losses' connected with system failures (clearly any test which shows that no lane has failed is to be treated as equivalent to a replacement). To discuss all these problems it is necessary to know the distribution of the random variables S_r and to find some properties of the distribution of N_τ (the numbers of renewals) in the time interval $(0, \tau)$. Of particular interest is the expectation of N_τ: this is a function $H(\tau)$ of τ, the so-called Renewal Function. In some cases var (N_τ) is needed as well as the Renewal Density, the derivative $h(\tau)$ of $H(\tau)$. In the discussion of the optimum replacement policies the age distribution of systems at time τ will be required.

Before proceeding any further it should be stressed that, whenever in future 'time' is referred to, this does not mean clock-time, but a variable τ which enables us to describe the state of the system and which, according to particular applications may be the distance travelled, number of revolutions, number of prescribed cycles of operation, the clock-time itself or an appropriate transformation of the clock-time into a variable τ, the ideal 'time'. For instance, the assumption that the failure rate is constant, i.e. that the expected number of failures in the 'time' unit does not change, should not be understood as a statement that the same number of failures should be expected in the first as in the second clock-hour of the flight. Experience teaches us that such an assumption would be inconsistent with the observed facts (cf. Ref. 4 Table 1); it means only that there exists a transformation (assumed to be known) of the clock-time into a variable τ for which the statement is true.

We shall assume that the system has m parallel lanes, that the failure events in the lanes are independent and that the age-specific failure rate $\rho(\tau)$ of the lanes (the Hazard) is constant and equal to λ, where λ has the dimension of the reciprocal of time. In other words for each lane the age-specific failure rate $\rho(\tau) = \lambda$, the survivor function $\mathscr{F}(\tau) = e^{-\lambda\tau}$, the failure probability density is given by $f(t) = \lambda e^{-\lambda\tau}$ and the failure probability distribution function $F(\tau) = 1 - e^{-\lambda\tau}$. The expected life (often described as the mean time between failures, MTBF) of a lane is equal to $\theta = 1/\lambda$. Hence in the case of a single-lane system the underlying Renewal Process is assumed to be Poissonian with the parameter λ, the expected number of failures $H(\tau) = \lambda\tau$, the variance $V(\tau) = \lambda\tau$ and the probability of exactly n failures in this time interval $(0, \tau)$ is given by $P_n(t) = e^{-\lambda\tau}(\lambda\tau)^n/n!$ $(n = 0, 1, 2, \ldots)$. The renewal density $h(\tau)$ is constant and equal to λ. (See Figs. 1-5, single lane system).

In the case of a multiplex system the Renewal Process exhibits a feature of ageing which is caused not by the deterioration of its components with age (since the age-specific failure rate of the lanes is assumed constant) but is solely due to redundancy diminishing with time. The process is no longer Poissonian although, for small τ, it can be regarded as locally Poissonian in a sense. The description of such processes can be obtained in a straightforward way by the application of standard techniques of the Renewal Theory; for the convenience of the reader the notations of D. R. Cox's book on Renewal Theory (Ref. 1) will be generally adhered to.

In the next section some limited results will be given for the general case of m lanes and this will be followed by a more detailed study of duplex $(m = 2)$ and triplex $(m = 3)$ systems. In the closing sections a slightly different method of multiplexing ('the majority vote') will be described and investigated.

2. *General Multiplex Systems.*

Before discussing multiplex systems let us assume that the time is measured in the units of θ, the MTBF of a lane. In other words we introduce a new variable $t = \lambda\tau = \tau/\theta$. This does not affect the generality, but simplifies the formulae; to revert to the time τ measured in original units it is enough to substitute again $\lambda\tau$ for t, bearing in mind that the change of scale affects the derivatives.

A system of m lanes fails in the time interval $(0, t)$ if all the m lanes fail so that:

$$\begin{aligned}\mathscr{F}(t) &= 1-(1-e^{-t})^m,\\ f(t) &= m(1-e^{-t})^{m-1}e^{-t},\\ F(t) &= (1-e^{-t})^m,\\ \rho(t) &= me^{-t}(1-e^{-t})^{m-1}[1-(1-e^{-t})^m]^{-1}.\end{aligned} \tag{1}$$

It is easily shown that $\rho(t)$ increases from 0 to 1 when t increases from zero to infinity.

The moments of the above distribution are well known (*see* e.g. Ref. 2), but for their rapid calculation notice that the Laplace Transform $f^*(s)$ of $f(t)$ is equal to

$$f^*(s) = m\int_0^\infty (1-e^{-t})^{m-1}e^{-t}e^{-st}dt$$

which, on the substitution $u = e^{-t}$, becomes

$$f^*(s) = m\int_0^1 (1-u)^{m-1}u^s du = mB(m, 1+s) = \Gamma(m+1)\Gamma(1+s)/\Gamma(m+1+s) \tag{2}$$

The characteristic function of $f(t)$ is equal to

$$\phi(w) = \Gamma(m+1)\Gamma(1-iw)/\Gamma(m+1-iw), \tag{3}$$

and the 'second characteristic function' is given by

$$\log\phi(w) = \log\Gamma(m+1) + \log\Gamma(1-iw) - \log\Gamma(m+1-iw). \tag{4}$$

Hence the cumulants of the $f(t)$ – distribution are equal to

$$\begin{aligned}\kappa_1 &= \Psi(m+1)-\Psi(1) &&= \sum_{\nu=1}^{m}\frac{1}{\nu}\\ \kappa_2 &= \Psi'(1)-\Psi'(m+1) &&= \sum_{\nu=1}^{m}\frac{1}{\nu^2}\\ \kappa_3 &= \Psi''(m+1)-\Psi''(1) &&= 2\sum_{\nu=1}^{m}\frac{1}{\nu^3}\\ &\dots &&\dots\\ &\dots &&\dots\\ \kappa_n &= (-1)^n[\Psi^{(n-1)}(1)-\Psi^{(n-1)}(m+1)] &&= \sum_{\nu=1}^{m}\frac{(n-1)!}{\nu^n}\end{aligned} \tag{5}$$

where Ψ is the Euler Psi—Function, the logarithmic derivative of the Gamma Function (cf. Ref. 3, p. 15 and p. 44).

Denoting the first three sums in the above formulae by A_m, B_m, C_m respectively, we obtain (cf. Ref. 1, p. 47) the following asymptotic expression for the expected number $H(t)$ of renewals in the interval $(0, t)$ for large t:

$$H(t) = \frac{t}{A_m} - \frac{A_m^2 - B_m}{2A_m^2}. \tag{6}$$

The variance $V(t)$ of the number of renewals in $(0, t)$ is, for large t, asymptotically equal to:

$$V(t) = \frac{B_m t}{A_m^3} + \frac{1}{12} + \frac{5B_m^2}{4A_m^4} - \frac{4C_m}{3A_m^3} \tag{7}$$

Thus, for large t, the ratio of the variance to the mean is approximately equal to B_m/A_m^2 which is less than 1 and approaches zero as m tends to infinity: multiplexing produces eventually an 'under-dispersion' which becomes more and more marked with increasing degree of multiplexing.

It has been suggested that, with the progress of miniaturization (and micro-miniaturization) it might be useful to design systems with a great number of independent lanes dispensing completely with their maintenance and inspection. This approach may be essential in the case of systems which (like missiles and artificial satellites) are put into operation but which, due to their specific nature, cannot be subjected to maintenance and inspection. The asymptotic formula for $H(t)$ shows that in a single-lane system $H(t) = t$, whilst in an m-lane system it is equal to $H(t) = t/A_m$ so that the expected number of failures decreases with multiplexing in the ratio $1/A_m$. From the values displayed below:

m	1	2	3	4	5	6 ...
$1/A_m$	1·0000	0·6667	0·5455	0·4800	0·4380	0·4081 ...

it is clear that the multiplexing procedure can yield useful results only for the initial value of $m = 2, 3, 4$ and that any increased multiplexing will add less and less to the system reliability; for large m the improvement from m to m + 1 is approximately in the ratio

$$\frac{\log (m+1)}{\log m}.$$

The situation is completely different for small values of t. To investigate the behaviour of the Renewal Process in the vicinity of zero, it is sufficient to assess the behaviour of the corresponding Laplace Transforms when s tends to infinity.

In our case [cf. Ref. 1, p. 46, formula (4)]

$$H^*(s) = \frac{f^*(s)}{s[1-f^*(s)]} = \frac{\Gamma(m+1)\Gamma(1+s)}{s[\Gamma(m+1+s)-\Gamma(m+1)\Gamma(1+s)]}. \tag{8}$$

When s tends to infinity $H^*(s)$ is asymptotically equal (for $m \geqslant 2$) to:

$$H^*(s) = \frac{m\,!}{s^{m+1}} - \frac{m(m+1)\,!}{2s^{m+2}} + o\left(\frac{1}{s^{m+2}}\right)$$

so that, for t in the vicinity of zero,

$$H(t) = t^m - \frac{m}{2}\,t^{m+1} + o\,(t^{m+1}). \tag{9}$$

Similarly $V(t) = \psi(t) - H(t) - [H(t)]^2$, where (cf. Ref. 1, pp. 55-56).

$$\psi^*(s) = \frac{2f^*(s)}{s[1-f^*(s)]^2} = \frac{2\Gamma(m+1)\Gamma(1+s)\Gamma(m+1+s)}{s[\Gamma(m+1+s)-\Gamma(m+1)\Gamma(1+s)]^2}, \tag{10}$$

and this, for s tending to infinity, is asymptotically equal (for $m \geqslant 2$) to

$$\psi^*(s) = \frac{2m\,!}{s^{m+1}} - \frac{m(m+1)\,!}{s^{m+2}} + o\left(\frac{1}{s^{m+2}}\right),$$

so that

$$\psi(t) = 2t^m - mt^{m+1} + o\,(t^{m+1})$$

and, for t in the vicinity of zero,

$$V(t) = t^m - \frac{m}{2} t^{m+1} + o(t^{m+1}). \tag{11}$$

The ratio of the variance to the mean is asymptotically equal to unity so that, for small values of t, our Renewal Process exhibits a Poissonian character.

The probability that no system failure occurs in $(0, t)$ is equal to

$$\mathscr{F}(t) = 1 - (1-e^{-t})^m,$$

and its Laplace Transform is equal to

$$\mathscr{F}^*(s) = \frac{1}{s}[1-f^*(s)] = \frac{1}{s}\left[1 - \frac{\Gamma(m+1)\Gamma(1+s)}{\Gamma(m+1+s)}\right].$$

For large s:

$$\mathscr{F}^*(s) = \frac{1}{s} - \frac{m!}{s^{m+1}} + \frac{m(m+1)!}{2s^{m+2}} + o\left(\frac{1}{s^{m+2}}\right)$$

so that, for t in the vicinity of zero,

$$P_0(t) = 1 - t^m + \frac{m}{2} t^{m+1} + o(t^{m+1}). \tag{12}$$

The probability of exactly n system failures $n \geqslant 1$ in the time interval $(0, t)$ is equal to (cf. Ref. 1, pp. 36-37):

$$P_n(t) = K_n(t) - K_{n+1}(t),$$

where $K_n(t)$ is the probability that the nth renewal takes place in the interval $(0, t)$. The Laplace Transform of $P_n(t)$ is equal to:

$$P_n^*(s) = \frac{1}{s}[f^*(s)]^n - \frac{1}{s}[f^*(s)]^{n+1}, \tag{13}$$

and, in our case,

$$P_n^*(s) = \frac{1}{s}\left[\frac{\Gamma(m+1)\Gamma(1+s)}{\Gamma(m+1+s)}\right]^n \frac{\Gamma(m+1+s) - \Gamma(m+1)\Gamma(1+s)}{\Gamma(m+1+s)}.$$

For s tending to infinity

$$P_n^*(s) = \frac{(m!)^n}{s^{mn+1}} - \frac{(m!)^n nm(m+1)}{2s^{mn+2}} + o\left(\frac{1}{s^{mn+2}}\right),$$

so that for t in the vicinity of zero,

$$P_n(t) = \frac{(m!)^n t^{mn}}{(mn)!} - \frac{(m!)^n nm(m+1) t^{mn+1}}{2(mn+1)!} + o(t^{mn+1}). \tag{14}$$

Notice that although our Renewal Process exhibits, for small t, the Poissonian property of the ratio of variance to the mean being approximately equal to 1, the corresponding probabilities $P_n(t)$ decrease (with increasing n) much quicker than for a Poissonian distribution with parameter t^m.

From (14) we have (for $m \geqslant 2$):

$$P_1(t) = t^m - \frac{m}{2} t^{m+1} + o(t^{m+1}), \tag{15}$$

$$P_2(t) = \frac{t^{2m}(m!)^2}{(2m)!} - \frac{t^{2m+1}(m!)^2 m(m+1)}{(2m+1)!} + o(t^{2m+1}). \tag{16}$$

Let us now assume that t is equal to a small fraction of the mean life θ of a singular lane (i.e. that t is of the order, say of 10^{-3}).

Since $P_2(t)$ is less than $\frac{t^{2m}}{6}$ it can be neglected; one can also, a fortiori, neglect $P_n(t)$ for $n > 2$. With a small relative error of the order of $\frac{m}{2}t$ we can assume that $P_0(t) = 1 - t^m$, $P_1(t) = t^m$, and that there is a zero probability of more renewals in the time interval $(0, t)$. For large m, t^m is very small so that multiplexing introduces a very great improvement of reliability provided that the time of operation is comparatively short, i.e. that it does not exceed a small fraction of the mean life (MTBF) of an individual lane. If this fraction is of the order say of 10^{-3} and if, after each period of operation, the lanes are restored to their initial state, multiplexing becomes an extremely effective method of increasing reliability.

Before discussing the reliability requirements let us revert to the time measured in ordinary units. We obtain from (9) the following approximate formula for the expected number of failures in the time interval $(0, \tau)$ if τ is a small number:

$$H(\tau) = (\lambda\tau)^m - \frac{m}{2}(\lambda\tau)^{m+1} + o(\tau^{m+1}) \tag{17}$$

while its derivative is equal to:

$$h(\tau) = m\lambda(\lambda\tau)^{m-1} - \frac{m(m+1)}{2}\lambda(\lambda\tau)^m + o(\tau^m). \tag{18}$$

Similarly, from (12),

$$\mathscr{F}(\tau) = 1 - (\lambda\tau)^m + \frac{m}{2}(\lambda\tau)^{m+1} + o(t^{m+1}) \tag{19}$$

and

$$f(\tau) = m\lambda(\lambda\tau)^{m-1} - \frac{m(m+1)}{2}\lambda(\lambda\tau)^m + o(\tau^m). \tag{20}$$

The requirements on the reliability of the system are usually given in the following form: 'The failure rate in one hour of operation should not surpass a number α where α is a small number having dimensions of reciprocal of time. These simple requirements can be interpreted in two ways according to whether they are viewed from the economic or from the safety aspect. In each case we obtain different answers when we interpret the 'failure rate' either as the instantaneous or the average failure rate. Thus the reliability requirements can be interpreted in one of the following four ways:

(I). The expected number of failures in any hour of operation should be less than α (and in any fraction $\Delta\tau$ of an hour should be less than $\alpha\Delta\tau$) which means that the condition:

$$h(\tau) < \alpha \tag{21}$$

should be satisfied.

(II). The expected number of failures in the interval $(0, \tau)$ divided by τ should be less than α. In other words:

$$H(\tau) < \alpha\tau. \tag{22}$$

(III). The age-specific failure rate should be less than α:

$$\rho(\tau) < \alpha \tag{23}$$

(IV). The average age-specific failure rate in the interval $(0, \tau)$ should be less than α:

$$\int_0^\tau \rho(u)du < \alpha\tau$$

or

$$-\log [\mathscr{F}(\tau)] < \alpha\tau. \tag{24}$$

Notice that for a single-lane system all these interpretations lead to the same result: the requirements are satisfied for all the values of τ if and only if $\lambda < \alpha$, i.e. if the MTBF of the lane is greater than a fixed number $1/\alpha$. Thus the possibility of complying with the requirements is, in this case, strictly dependent on the existing state-of-the-art, on whether we can design a lane having the required MTBF.

The situation is completely different in the case of multiplex system ($m > 1$). Even with high values of λ (i.e. with unreliable lanes) we can satisfy the above requirements in one of the two ways: (a) if m is fixed we can satisfy the above conditions provided the time of operation τ_α is appropriately short and provided that after such a period all the lanes are inspected and those out of order are restored to their initial state, (b) if the required time of operation τ_α is specified we can satisfy the conditions by increasing the number m of lanes.

In subsequent sections when discussing duplex and triplex systems we shall revert to the accurate solution of problems (a) and (b) under the above four interpretations of official requirements. Using the approximate expressions (17)-(20), and taking into account that, for small τ, $\rho(\tau)$ is approximately equal to (20) and $-\log [\mathscr{F}(\tau)]$ to (17), we easily find the following:

(a). If m is given we can find the highest admissible operating time τ_α for a given failure rate α. In order to satisfy the 'instantaneous' requirements as interpreted in (I) and (III) we have to take

$$\tau_\alpha = \frac{1}{\lambda}\left(\frac{\alpha}{m\lambda}\right)^{\frac{1}{m-1}} = \theta\left(\frac{\alpha\theta}{m}\right)^{\frac{1}{m-1}} \tag{25}$$

and in the case of 'average' requirements we have to take

$$\tau_\alpha = \frac{1}{\lambda}\left(\frac{\alpha}{\lambda}\right)^{\frac{1}{m-1}} = \theta\,(\alpha\theta)^{\frac{1}{m-1}} \tag{26}$$

the latter value being higher, as it should be.

(b) If the required time of operation τ_α is fixed (and if it is a small fraction of the MTBF of the lanes) then we can satisfy the requirements by appropriately high multiplexing. To satisfy (II) and (IV) it is sufficient to choose

$$m > \frac{\log(\alpha\tau)}{\log(\lambda\tau)} \tag{27}$$

and to satisfy (I) and (II) the smallest m satisfying the inequality

$$m(\tau\lambda)^{m-1} < \alpha/\lambda$$

is to be found. (Notice that since $\tau\lambda$ is small the left-hand side of the inequality tends to 0 with increasing m).

In this section the behaviour of the Renewal Process of a multiple system was discussed for very large and for very small values of t. Clearly, between the short periods for which redundancy pays great dividends and the long periods for which it is less advantageous, there exist time lengths for which multiplexing can still be very useful. Thus it would be interesting to investigate for some specific values of m, the exact behaviour of the underlying Renewal Process for the whole range of positive values of t. This will be done in the subsequent sections for duplex and triplex systems.

3. *Duplex Systems.*

Putting $m = 2$ in formulae (1) we find:

$$\begin{aligned} \mathscr{F}(t) &= 2e^{-t} - e^{-2t}, \\ f(t) &= 2e^{-t} - 2e^{-2t}, \\ \rho(t) &= 1 - \frac{1}{2e^{t} - 1}, \end{aligned} \tag{28}$$

and from formulae (5):

$$A_2 = \frac{3}{2}, \qquad B_2 = \frac{5}{4}, \qquad C_2 = \frac{9}{8}.$$

In view of (6) and (7) we have asymptotically, for large t:

$$H(t) = \frac{2t}{3} - \frac{2}{9} + \mathrm{o}(1), \tag{29}$$

$$V(t) = \frac{10t}{27} + \frac{2}{81} + \mathrm{o}(1). \tag{30}$$

The exact values for any t are easily obtained. The Laplace Transform of $f(t)$ being equal to

$$f^*(s) = \frac{2}{1+s} - \frac{2}{2+s} = \frac{2}{(1+s)(2+s)}, \tag{31}$$

formula (8) yields

$$H^*(s) = \frac{2}{s^2(s+3)} = \frac{2}{3s^2} - \frac{2}{9s} + \frac{2}{9(3+s)},$$

which shows that

$$H(t) = \frac{2t}{3} - \frac{2}{9} + \frac{2}{9}e^{-3t}. \tag{32}$$

The shape of the $H(t)$-curve showing the way in which it approaches the line (29) is displayed in Fig. 1, and its derivative is given in Fig. 2, whilst Fig. 3 shows the $\rho(t)$-curve. Expanding $H(t)$ into Taylor Series we see that

$$H(t) = t^2 - t^3 + \frac{3}{4}t^4 + \mathrm{o}(t^4)$$

which agrees with formula (9).

From (10)

$$\psi^*(s) = \frac{4(s+1)(s+2)}{s^3(s+3)^2} = \frac{1}{27}\left[\frac{24}{s^3} + \frac{20}{s^2} - \frac{4}{s} - \frac{8}{(s+3)^2} + \frac{4}{s+3}\right]$$

and

$$\psi(t) = \frac{1}{27}\left[12t^2 + 20t - 4 - 8te^{-3t} + 4e^{-3t}\right].$$

Subtracting from this expression the sum $H(t) - [H(t)]^2$ we get

$$V(t) = \frac{10t}{27} + \frac{2}{81} - \frac{16}{27} te^{-3t} + \frac{2}{81} e^{-3t} - \frac{4}{81} e^{-6t}, \tag{33}$$

which again agrees with (30). For large values of t the ratio of the variance to the mean tends to 5/9. Expanding $V(t)$ into Taylor Series we see that

$$V(t) = t^2 - t^3 + 0(t^3)$$

which agrees with (11). For small values of t the ratio of the variance to the mean is approximately equal to unity. The shape of the $V(t)$ curve showing the way in which it approaches the line (30) is displayed in Fig. 4.

To evaluate the probabilities $P_n(t)$ notice that, in view of (13),

$$P_n{}^*(s) = \frac{1}{s}\left[\frac{2}{(1+s)(2+s)}\right]^n \left[1 - \frac{2}{(1+s)(2+s)}\right] = \frac{2^n(3+s)}{(1+s)^{n+1}(2+s)^{n+1}}. \tag{34}$$

Since the poles $s_1 = -1$ and $s_2 = -2$ lie to the left of the imaginary axis

$$P_n(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}P_n{}^*(s)ds$$

and, by the standard argument,

$$P_n(t) = \text{Sum of residues of } P_n{}^*(s) . e^{st} \text{ at } s_1 \text{ and } s_2 .$$

Expanding $P_n{}^*(s)e^{st}$ into the Laurent Series in the vicinity of $s_1 = -1$, we find:

$$P_n{}^*(s)e^{st} = 2^n e^{-t} \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{k=0}^{\infty} (-1)^k \binom{n+k}{k} [(1+s)^{k+r-n} + 2(1+s)^{k+r-n-1}].$$

The coefficient of $(1+s)^{-1}$ is equal to

$$2^n e^{-t} \sum_{r=0}^{n-1} \frac{t^r}{r!} (-1)^{n-1-r} \binom{2n-r-1}{n} + 2^{n+1}e^{-t} \sum_{r=0}^{n} \frac{t^r}{r!} (-1)^{n-r} \binom{2n-r}{r}$$

and, defining $\binom{l}{n} = 0$ whenever $l < n$, we have

$$\text{Residue of } P_n{}^* . e^{st} \text{ (at } s_1 = -1) = (-2)^n e^{-t} \sum_{r=0}^{h} (-1)^r \frac{t^r}{r!} \left[2\binom{2n-r}{n} - \binom{2n-r-1}{n}\right].$$

Similarly, expanding $P_n{}^*(s)e^{st}$ in the vicinity of $s_2 = -2$ we find:

$$P_n{}^*(s)e^{st} = 2^n e^{-2t}(-1)^{n+1} \sum_{r=0}^{\infty} \frac{t^r}{r!} \sum_{k=0}^{\infty} \binom{n+k}{k} [(2+s)^{k+r-n} + (2+s)^{k+r-n-1}]$$

and:

$$\text{Residue of } P_n{}^*(s)e^{st} \text{ (at } s_2 = -2) = -(-2)^n e^{-2t} \sum_{r=0}^{n} \left[\binom{2n-r}{n} + \binom{2n-r-1}{n}\right].$$

Hence

$$P_n(t) = (-2)^n e^{-t} \sum_{r=0}^{n} (-1)^r \frac{t^r}{r!} \left[2\binom{2n-r}{n} - \binom{2n-r-1}{n}\right] -$$

$$- (-2)^n e^{-2t} \sum_{r=0}^{n} \frac{t^r}{r!} \left[\binom{2n-r}{n} + \binom{2n-r-1}{n}\right]. \tag{35}$$

Consequently, for a few initial values of n,

$$P_0(t) = 2e^{-t} - e^{-2t},$$

$$P_1(t) = -e^{-t}(6-4t) + e^{-2t}(6+2t),$$

$$P_2(t) = e^{-t}(36-20t+4t^2) - e^{-2t}(36+16t+2t^2),$$

$$P_3(t) = -e^{-t}\left(240 - 128t + 28t^2 - \frac{8}{3}t^3\right) + e^{-2t}\left(240 + 112t + 20t^2 + \frac{4}{3}t^3\right),$$

$$P_4(t) = e^{-t}\left(1680 - 880t + 200t^2 - 24t^3 + \frac{4}{3}t^4\right) -$$

$$- e^{-2t}\left(1680 + 800t + 160t^2 + 16t^3 + \frac{2}{3}t^4\right),$$

$$P_5(t) = -32e^{-t}\left(378 - 196t + \frac{91}{2}t^2 - 6t^3 + \frac{11}{24}t^4 - \frac{1}{60}t^5\right) +$$

$$+ 32e^{-2t}\left(378 + 182t + \frac{77}{2}t^2 + \frac{9}{2}t^2 + \frac{7}{24}t^4 + \frac{1}{120}t^5\right).$$

It is easy to verify, by expanding the probabilities $P_n(t)$ into the Taylor Series, that formula (16) is approximately valid. The graphs of $P_n(t)$-functions for $n = 0, 1, 2, 3$ are shown in Fig. 6; when interpreting the results it should be borne in mind that the abscissae t correspond to the 'real' time τ/θ and that in most applications where θ is of the order of 10^3 or 10^4 and the period of operation (the period between the overhauls) is equal to a few hours, it is the initial part of the graph (between 0 and 0·01) which is of interest. Notice that, whilst for the single-lane system the maxima are situated at $t_n = n$, for the duplex system the maxima occur at t_n', where $t_0' = 0$, $t_1' = 1{\cdot}77603$, $t_2' = 3{\cdot}27663$, $t_3' = 4{\cdot}77496$. Thus, in comparison with the Poissonian distribution (cf. Fig. 5) that $P_n(t)$-curves are shifted to the right so that the maxima occur respectively 1·776, 1·683, 1·592 times later than for a Poissonian Distribution; this agrees, to some extent, with the fact that a given expected number of failures in a duplex system occurs approximately 3/2 times later.

There is no difficulty in obtaining other functions of Renewal Theory if they are required. The probability density function of the nth renewal time can be easily found by inverting its Laplace Transform

$$k_n^*(s) = [f^*(s)]^n = \frac{2^n}{(1+s)^n(2+s)^n},$$

and is equal to:

$$k_n(t) = (-2)^n \sum_{r=0}^{n-1} \frac{t^r}{r!}\binom{2n-2-r}{n-1}[e^{-2t} - (-1)^r e^{-t}]. \tag{36}$$

By integrating (36) from t to infinity we find $1 - K_n(t)$, the probability that the nth renewal time occurs not earlier than t, and subtracting two neighbouring values of these functions we can verify formula (35). This can be also done by expanding in terms of ζ the probability Generating Function of the probabilities $P_n(t)$, which is equal to

$$G(t, \zeta) = e^{-3t/2}\left\{\zeta(1+8\zeta)^{-1/2}\sinh\left[\frac{t}{2}(1+8\zeta)^{1/2}\right] + \cosh\left[\frac{t}{2}(1+8\zeta)^{1/2}\right]\right\}. \tag{37}$$

Reverting to the time measured in ordinary units we have from (32) the following formula for the expected number of failures in the time interval $(0, \tau)$:

$$H(\tau) = \frac{2}{3}\lambda\tau - \frac{2}{9} + \frac{2}{9}e^{-3\lambda\tau},$$

whilst its derivative, the renewal density, is given by:

$$h(\tau) = \frac{2}{3}\lambda - \frac{2}{3}\lambda e^{-3\lambda\tau}.$$

We also find from the formulae (28)

$$\rho(\tau) = \lambda - \frac{\lambda}{2e^{\lambda\tau} - 1}$$

and

$$-\log \mathscr{F}(\tau) = -\log(2e^{-\lambda\tau} - e^{-2\lambda\tau}).$$

The above formulae allow us to find, for different values of α, the accurate values of the highest admissible operation time τ_α discussed above in section 2. These values are given in Table 1.

TABLE 1

Highest admissible operation times

$\theta\alpha$	I τ_a/θ	II τ_a/θ	III τ_a/θ	IV τ_a/θ
0·00001	0·000005	0·000010	0·000005	0·000010
0·00002	0·000010	0·000020	0·000010	0·000020
0·00005	0·000025	0·000050	0·000025	0·000050
0·0001	0·000050	0·000100	0·000050	0·000100
0·0002	0·000100	0·000200	0·000100	0·000200
0·0005	0·000250	0·000500	0·000250	0·000500
0·001	0·000500	0·001001	0·000500	0·001001
0·002	0·001002	0·002004	0·001002	0·002004
0·005	0·002509	0·005025	0·002509	0·005024
0·01	0·005038	0·010101	0·005038	0·010099
0·02	0·010153	0·020410	0·010152	0·020406
0·05	0·025987	0·052668	0·025975	0·052617
0·1	0·054173	0·111448	0·054061	0·111006

It may be seen from the above table that, for small values of α (of the order of 0·1), the values of τ_α are practically equal in cases I and III and that they are about twice as high when the 'average' approach is adopted (cases II and IV); this agrees with formulae (25), (26). The most stringent are the requirements as interpreted in case III and when we are concerned with the safety aspects we should read τ_α from the column III of the table. Thus for example, if the mean life of the lanes is $\theta = 10{,}000$ hours and if the stipulated maximum failure rate is equal to $\alpha = 10^{-6}$ per hour, then $\theta\alpha = 0{\cdot}01$ and $\tau_\alpha = 0{\cdot}005038\ \theta = 50{\cdot}4$ hours; with the less demanding requirement of $\alpha = 10^{-5}$, we find $\theta\alpha = 0{\cdot}1$ and $\tau_\alpha = 0{\cdot}054061\ \theta = 540{\cdot}6$ hours.

The Renewal Process provides information on the probabilities of the occurrence of system failures and on the expected frequency of repairs and replacements which have to be undertaken if the system is to perform the required task. Apart from these unavoidable replacements (sometimes called 'service' replacements) it may be necessary to submit the system at regular times to inspection (and if necessary subsequent repair or part replacement) in order to satisfy the safety requirements discussed above. There is, however, another aspect of the problem. We can look upon a suitable replacement policy from the economic point of view and try to find the replacement times for which an appropriately defined loss function will attain a minimum. Let C be the total loss connected with the system failing during its operation, d the cost of dismantling the system (including the losses incurred by the system being inoperative during this time) and let a be the cost of the repair (or the replacement) of one faulty lane. In the interval $(0, t)$ the expected number of service replacements is $H(T)$ so that the total cost of these replacements is $H(T)\,(C+d+2a)$. Assuming that the system is inspected at time t the cost is $(d+a)$ if only one lane is found to be working and d if, on the inspection, both lanes are found to be in working condition. Denoting these two events by (E_1) and (E_2) respectively we have to evaluate the probabilities of these events. To assess them we have to find first the distribution of the age $u(t)$ of the system at time t. This distribution is given (cf. Ref. 1, pp. 61-62) by:

$$\text{Probability } [U_t = t] = \mathscr{F}(t),$$

$$\text{Probability density of } [U_t = x] = h(t-x)\,\mathscr{F}(x) \qquad \text{if} \qquad 0 \leqslant x < t.$$

The conditional probability of a system of age x having one lane in order is $2e^{-x}(1-e^{-x})/\mathscr{F}(x)$ and the conditional probability of a system of age x having both lanes in order is $e^{-2x}/\mathscr{F}(x)$. Thus the probability of event (E_1) is

$$2e^{-t}(1-e^{-t}) + \int_0^t 2e^{-x}(1-e^{-x})h(t-x)dx = \frac{2}{3}\,(1-e^{-3t}),$$

and the probability of event (E_2) is

$$e^{-2t} + \int_0^t e^{-2x}h(t-x)dx = \frac{1}{3}\,(1+2e^{-3t}).$$

Notice that these probabilities add to unity (as it should be) and that for large t they are in the proportion of 2:1.

The total cost per unit time connected with service replacements scheduled at time t is equal to:

$$L_t = \frac{1}{t}\,[H(t)\,(C+d+a) + \text{Prob}\,(E_1)\,.\,(d+a) + \text{Prob}\,(E_2)\,.\,d],$$

and, if we define

$$A = \frac{2}{9}\,C - \frac{2}{9}\,a - \frac{7}{9}\,d\,, \qquad\qquad B = \frac{2}{9}\,C - \frac{2}{9}\,a + \frac{2}{9}\,d,$$

we can write

$$L_t = \frac{2}{3}\,(C+d+a) - \frac{1}{t}\,(A - Be^{-3t}).$$

The derivative of this expression is

$$\frac{dLt}{dt} = \frac{1}{t^2}\,[A - Be^{-3t}(1+3t)]\,.$$

In most applications the value C will be much larger than both a and d, so that A and B will be positive numbers. For $t \to 0$ the derivative is negative since $A\text{-}B = -d$ is negative and for $t \to \infty$ the derivative is positive since the expression in the brackets tends to A. The expression in brackets increases from $-d$ to A, so that dL_t/dt has a unique zero at $t = t_{\min}$ and, for this value $t_{\min}$ the loss function L_t is a minimum. We regard $t_{\min}$ as the optimum replacement time. The minimum loss is then

$$L_{\min} = \frac{2}{3}(C+d+2a) - 3Be^{-3t_{\min}}.$$

It may be shown that the greater is C in comparison with a and d the smaller is the value of $t_{\min}$; with increasing costs of system failure we have to introduce more frequent replacements. If C is so large that a/C and d/C can be neglected in comparison with $\surd(a/C)$, $\surd(d/C)$ we obtain approximately:

$$t_{\min} = \surd(d/C) \qquad L_{\min} = 2\surd(Cd).$$

An example. If the losses connected with the system failing during its operation are 10,000 times greater than the costs of dismantling d and if the costs a are of the same order as the costs d, then the optimum replacement policy is to inspect the system after every $t_{\min} = 1/100$, i.e. after $\theta/100$ hours and if the lane mean life MTBF $= \theta = 1{,}000$ hours this means a replacement after every 10 hours of operation. The probability of finding a faulty lane will be approximately $0{\cdot}02$, whilst the probability that both are right is $0{\cdot}98$. The minimum expected loss per unit of time will be $L_{\min} = 200\, d$ (if time is measured in units of θ) and is equal to $0{\cdot}2d$ per one hour of operation.

4. *Triplex Systems.*

Putting $m = 3$ in formulae (1) we find

$$\mathscr{F}(t) = 1-(1-e^{-t})^3 = 3e^{-t} - 3e^{-2t} + e^{-3t},$$

$$f(t) = 3e^{-t} - 6e^{-2t} + 3e^{-3t},$$

$$\rho(t) = \frac{3-6e^{-t}+3e^{-2t}}{3-3e^{-t}+e^{-2t}}. \tag{38}$$

and from (5)

$$A_3 = \frac{11}{6}, \qquad B_3 = \frac{49}{36}, \qquad C_3 = \frac{251}{216}.$$

From (6) and (7) we get, for large t,

$$H(t) = \frac{6}{11}t - \frac{36}{121} + o(1). \tag{39}$$

$$V(t) = \frac{294}{1331}t + \frac{540}{14641} + o(1). \tag{40}$$

The exact expressions are easily obtained. The Laplace Transform of $f(t)$ being equal to:

$$f^*(s) = \frac{3}{1+s} - \frac{6}{2+s} + \frac{3}{3+s} = \frac{6}{(1+s)(2+s)(3+s)}, \tag{41}$$

we find:

$$H^*(s) = \frac{6}{s^2(s^2+6s+11)} = \frac{6}{11s^2} - \frac{36}{121s} + \frac{36s+150}{121\,[(s+3)^2+2]}$$

and this yields:

$$H(t) = \frac{6t}{11} - \frac{36}{121} + \frac{36e^{-3t}}{121}\cos(t\surd 2) + \frac{21\surd 2}{121}e^{-3t}\sin(t\surd 2), \tag{42}$$

which agrees with (39). The $H(t)$-curve is shown in Fig. 1 whilst Fig. 3 gives the graph of the $\rho(t)$-curve.

From (10)

$$\psi^*(s) = \frac{12(s^3+6s^2+11s+6)}{s^3(s^2+6s+11)^2}$$

or

$$\psi^*(s) = \frac{72}{121s^3} + \frac{588}{1331s^2} - \frac{2520}{14641s} + \frac{1800s + 6048}{1331(s^2+6s+11)} + \frac{2520s + 8652}{14641(s^2+6s+11)^2}.$$

Consequently,

$$\psi(t) = \frac{36t^2}{121} + \frac{588t}{1331} - \frac{2520}{14641} + e^{-3t}\left[\frac{2520}{14641}\cos(t\sqrt{2}) - \frac{345\sqrt{2}}{14641}\sin(t\sqrt{2})\right] +$$

$$+ te^{-3t}\left[\frac{162}{1331}\cos(t\sqrt{2}) - \frac{450\sqrt{2}}{1331}\sin(t\sqrt{2})\right],$$

which, on subtraction of the sum $H(t) + [H(t)]^2$ from the above, gives

$$V(t) = \frac{294t}{1331} + \frac{540}{14641} + e^{-3t}\left[\frac{756}{14641}\cos(t\sqrt{2}) - \frac{1374\sqrt{2}}{14641}\sin(t\sqrt{2}\right] -$$

$$- te^{-3t}\left[\frac{270}{1331}\cos(t\sqrt{2}) + \frac{702\sqrt{2}}{1331}\sin(t\sqrt{2})\right] -$$

$$- e^{-6t}\left[\frac{36}{121}\cos(t\sqrt{2}) + \frac{21\sqrt{2}}{121}\sin(t\sqrt{2})\right]. \tag{43}$$

This agrees with (40) and for large values of t the ratio of the variance to the mean is equal to 49/121. The $V(t)$-curve is shown in Fig. 4.

Expanding $H(t)$ and $V(t)$ into the Taylor Series at $t = 0$ we find:

$$H(t) = t^3 - \frac{3}{2}t^4 + o(t^4),$$

$$V(t) = t^3 - \frac{3}{2}t^4 + o(t^4),$$

which is a special case of (9) and (11).

To evaluate the probabilities $P_n(t)$ notice that, in view of (13),

$$P_n^*(s) = \frac{1}{s}\left[\frac{6}{(1+s)(2+s)(3+s)}\right]^n\left[1 - \frac{6}{(1+s)(2+s)(3+s)}\right] = \frac{6^n(s^2+6s+11)}{(1+s)^{n+1}(2+s)^{n+1}(3+s)^{n+1}}. \tag{44}$$

Since the poles $s_1 = -1$, $s_2 = -2$, $s_3 = -3$ of $P_n^*(s)e^{st}$ are all situated to the left of the imaginary axis, we have

$$P_n(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{st}P_n^*(s)ds$$

and, again by the standard argument,

$P_n(t) =$ Sum of the residues of $e^{st}P_n^*(s)$ at s_1, s_2, s_3.

Expanding $e^{st}P_n^*(s)$ in the vicinity of $s_1 = -1$ into the Laurent Series we find

$$e^{st}P_n^*(s) = 6^n e^{-t}\sum_{r=0}^{\infty}\frac{t^r}{r!}\sum_{k=0}^{\infty}(-1)^k\binom{n+k}{k}\frac{1}{2^{n+1}}\sum_{l=0}^{\infty}(-1)^l\frac{(1+s)^{r+k-1}}{2^l}[6(1+s)^{-n-1} +$$

$$+ 4(1+s)^{-n} + (1+s)^{-n+1}]\binom{n+l}{l}$$

and the coefficient of $(1+s)^{-1}$ is equal to:

$$(-6)^n e^{-t}\frac{1}{2^{2n+1}}\sum_{r=0}^{n}\frac{t^r}{r!}2^r(-1)^r\sum_{k=0}^{n-r}2^k\binom{n+k}{k}\left[6\binom{2n-r-k}{n}-\right.$$
$$\left.-8\binom{2n-r-k-1}{n}+4\binom{2n-r-k-2}{n}\right].$$

Similarly we obtain as the residue at $s_2 = -2$:

$$-6^n e^{-2t}\sum_{r=0}^{n}\frac{t^r}{r!}(-1)^r\sum_{k=0}^{n-r}(-1)^k\binom{n+k}{k}\left[3\binom{2n-r-k}{n}-\right.$$
$$\left.-2\binom{2n-r-k-1}{n}+\binom{2n-r-k-2}{n}\right],$$

and as the residue at $s_3 = -3$:

$$6^n\frac{1}{2^{n+1}}e^{-3t}\sum_{r=0}^{n}\frac{t^r}{r!}\sum_{k=0}^{n-r}\frac{1}{2^k}\binom{n+k}{k}\left[2\binom{2n-r-k}{n}+\binom{2n-r-k-2}{n}\right].$$

Consequently, for a few initial values of n:

$$P_0(t) = 3e^{-t}-3e^{-2t}+e^{-3t},$$

$$P_1(t) = (-21+9t)e^{-t}+(12+18t)e^{-2t}+(9+3t)e^{-3t},$$

$$P_2(t) = (495-207t+27t^2)\frac{e^{-t}}{2}-(360+72t+54t^2)e^{-2t}+(225+81t+9t^2)\frac{e^{-3t}}{2},$$

$$P_3(t) = (-6453+2700t-432t^2+27t^3)\frac{e^{-t}}{2}+(1728+2808t+216t^2+108t^3)e^{-2t}+$$
$$+(2997+1134t+162t^2+9t^3)\frac{e^{-3t}}{2}. \tag{45}$$

The graphs of $P_n(t)$ functions, for $n = 0, 1, 2, 3$ are shown in Fig. 7. The maxima for a triplex system are situated for the abscissae

$$t_0' = 0, \qquad t_1' = 2\cdot2818, \qquad t_2' = 4\cdot11499, \qquad t_3' = 5\cdot94586,$$

i.e. 2·2818, 2·0575, 1·98195 time later than for a Poissonian distribution (cf. Fig. 5); this can be compared with the fact that a given expected number of failures in a triplex system occurs 11/6 times (1·833 . . . times) later.

Other functions of the Renewal Theory can be also found. The probability density function of the nth renewal can be obtained by inverting its Laplace Transform

$$k_n(s) = \frac{6^n}{(1+s)^n(2+s)^n(3+s)^n};$$

it is equal to

$$k_n(t) = 6^n\sum_{r=0}^{n-1}\frac{t^r}{r!}\sum_{k=0}^{n-r-1}\binom{n-1+k}{n-1}\binom{2n-2-r-k}{n-1}\left[\frac{(-1)^{n+r+1}e^{-t}}{2^{2n-r-k-1}}+\right.$$
$$\left.+(-1)^{r+k+1}e^{-2t}+\frac{1}{2^{n+k}}e^{-3t}\right]$$

and can be used in verifying formulae (45).

Reverting again to the time measured in ordinary units we have from (42)

$$H(t) = \frac{6\lambda\tau}{11} - \frac{36}{121} + \frac{36e^{-3\lambda\tau}}{121}\cos(\tau\lambda\sqrt{2}) + \frac{21\sqrt{2}}{121}e^{-3\lambda\tau}\sin(\tau\lambda\sqrt{2})$$

and its derivative is equal to

$$h(t) = \frac{6\lambda}{11} - \frac{6\lambda}{11}e^{-3\lambda\tau}\cos(\tau\lambda\sqrt{2}) - \frac{9\sqrt{2}\lambda}{11}e^{-3\lambda\tau}\sin(\tau\lambda\sqrt{2})$$

The graph of the $h(t)$-function, for $\lambda = 1$ is given in Fig. 2.

If we accept the requirements as interpreted on p. 7 under (III) we have to find a time interval $(0, \tau_\alpha)$ such that for all τ of this interval $\rho(\tau) < \alpha$. These values are given below in Table 2 under the appropriate column. If we accept the reliability requirements interpreted on p. 7 under (I) we have to find a time interval $(0, \tau_\alpha)$ such that for all τ of this interval $h(\tau) < \alpha$. However, in the case of a triplex system the renewal density is not any longer an increasing function. From its derivative:

$$h'(\tau) = 3\lambda^2\sqrt{2}\,e^{-3\lambda\tau}\sin(\tau\lambda\sqrt{2})$$

it is clear that it increases from $h(0) = 0$ to an absolute maximum

$$h\left(\frac{\pi}{\lambda\sqrt{2}}\right) = \frac{6\lambda}{11}(1+e^{-3\pi/\sqrt{2}}) = 0\cdot5468066\ldots\lambda, \text{ then decreases to}$$

$$h\left(\frac{2\pi}{\lambda\sqrt{2}}\right) = \frac{6\lambda}{11}(1-e^{-6\pi/\sqrt{2}}) = 0\cdot545454\ldots\lambda, \text{ etc. oscillating}$$

around the value $6\lambda/11$ with rapidly decreasing amplitude. These oscillations are, however, so small that their existence can be neglected; they could not be displayed in Fig. 2.

The $h(\tau)$-line crosses the straight line $y = \frac{6\lambda}{11}$ for the first time at τ_0, where

$$\tau_0 = \frac{1}{\lambda\sqrt{2}}\arctan\left(-\frac{\sqrt{2}}{3}\right) = \frac{\theta}{\sqrt{2}}\cdot\arctan(-0\cdot4714045) = 1\cdot9099953\theta$$

and between $\tau = 0$ and this value τ_0 the function $h(\tau)$ is increasing and its converse can be found. The values of τ_α satisfying the requirements I are given in Table 2 under the appropriate column. Col. II gives the highest admissible operation times for the requirements $H(\tau) < \alpha\tau$, discussed previously under II.

As in the case of the duplex system, it may be seen from the table overleaf that for small values of $\alpha\theta$ the values of τ_α are practically equal in cases I and III, being about $\sqrt{3} = 1\cdot732\ldots$ higher when the 'average' approach is adopted (case II); this agrees with formulae (25), (26). The most stringent are the requirements as interpreted in case III and when we are concerned with safety aspects we should again read τ_α from the column III of the table.

An example. Let us assume, as in the example discussed for the duplex system, that the mean life of the lanes is 10,000 hours and that the stipulated maximum failure rate is $\alpha = 10^{-6}$ per hour. Then $\alpha\theta = 0\cdot01$ and from the table $\tau_\alpha = 0\cdot061373\,\theta = 613\cdot7$ hours. Thus passing from the duplex to the triplex system we can safely operate our system without inspection and repair for a period

TABLE 2

Highest admissible operation times

$\alpha\theta$	I τ_a/θ	II τ_a/θ	III τ_a/θ
0·00001	0·001829	0·003188	0·001829
0·00002	0·002589	0·004500	0·002589
0·00005	0·004099	0·007116	0·004099
0·0001	0·005807	0·010087	0·005807
0·0002	0·008233	0·014290	0·008232
0·0005	0·013080	0·022745	0·013080
0·001	0·018600	0·032398	0·018600
0·002	0·026513	0·046295	0·026513
0·005	0·042598	0·074762	0·042597
0·01	0·061379	0·108385	0·061373
0·02	0·089238	0·159074	0·089211
0·05	0·149796	0·273012	0·149590
0·1	0·228897	0·431128	0·227824

about 12 times longer. Alternatively we can use less reliable (and perhaps cheaper) lanes: if $\theta = 2{,}000$ hours, $\theta\alpha = 0{\cdot}002$ and $\tau_\alpha = 0{\cdot}026513\ \theta = 53{\cdot}0$ hours. This means that with components five times less reliable we can achieve by triplexing the same safety and operate for slightly longer time.

As in the case of duplex systems, we can evaluate the optimum replacement time. The expected number of failures in the interval $(0, t)$ is $H(t)$ so that the total cost of service replacements is $H(t)(C+d+3a)$. If the operating system is dismantled at the time t, then the cost of bringing it to the initial state is $(d+2a)$, $(d+a)$ or d according to whether one, two or all three lanes are found in working condition. As before by denoting these three events by E_1, E_2, E_3 respectively we have to evaluate their probability. By the same argument as that used in the discussion of the duplex system we have

$$\text{Prob }(E_1) = 3e^{-t}(1-e^{-t})^2 + \int_0^t 3e^{-x}(1-e^{-x})^2 h(t-x)dx$$

$$= \frac{6}{11} - \frac{6}{11} e^{-3t} \cos (t\sqrt{2}) - \frac{9\sqrt{2}}{11} e^{-3t} \sin (t\sqrt{2})$$

$$\text{Prob }(E_2) = 3e^{-2t}(1-e^{-t}) + \int_0^t 3e^{-2x}(1-e^{-x})h(t-x)dx$$

$$= \frac{3}{11} - \frac{3}{11} e^{-3t} \cos (t\sqrt{2}) + \frac{12\sqrt{2}}{11} e^{-3t} \sin (t\sqrt{2})$$

$$\text{Prob }(E_3) = e^{-3t} + \int_0^t e^{-3t}h(t-x)dx =$$

$$= \frac{2}{11} + \frac{9}{11} e^{-3t} \cos (t\sqrt{2}) - \frac{3\sqrt{2}}{11} e^{-3t} \sin (t\sqrt{2}).$$

Again the sum of these three probabilities is equal to unity, as it should be, and, for large t, the probabilities of finding 1, 2 or 3 lanes in working condition are in proportion 6:3:2. If

$$A = \frac{1}{121}(-36C+85d+57a),$$

$$B = \frac{1}{121}(36C+36d-57a),$$

$$D = \frac{1}{121}(21C+21d-3a)$$

then the loss function is equal to

$$L_t = (C+d+3a)\frac{6}{11} + \frac{1}{t}[A + Be^{-3t}\cos(t\sqrt{2}) + D\sqrt{2}\,e^{-3t}\sin(t\sqrt{2})]$$

and its derivative is given by

$$\frac{dL_t}{dt} = \frac{1}{t^2}[(2D-3B)te^{-3t}\cos t\sqrt{2} - (B+3D)te^{-3t}\sqrt{2}\sin(t\sqrt{2}) -$$
$$- A - Be^{-3t}\cos(t\sqrt{2}) - D\sqrt{2}\,e^{-3t}\sin t\sqrt{2}].$$

The expression in the brackets is equal to $-A-B = -d$ for $t = 0$ and tends to $-A$ when t tends to infinity; if C is large in comparison with a and d this value $-A$ is positive. Thus the derivative attains zero at least once, and the lowest value $t_{\min}$ for which this happens determines the minimum of L_t. As before, for large C (in comparison with a and d) we have approximately:

$$t_{\min} \approx \sqrt[3]{(d/2C)} \qquad \text{and} \qquad L_{\min} \approx 3\sqrt[3]{(Cd^2/4)}.$$

An example. If, as in the previous example discussed at the end of Section 3, $C = 10{,}000\ d$, then the optimum replacement policy is to inspect the system after every $t_{\min} = 1/20{,}000)^{1/3} = 0{\cdot}03684$, i.e. after $0{\cdot}03684\ \theta$ hours and if $\theta = 1{,}000$ hours, this means a replacement after every $36{\cdot}84$ hours in operation. The probability of finding a faulty lane is now greater and equal approximately $0{\cdot}11$. The minimum expected loss per unit of time will be $L_{\min} = 40{\cdot}716\ d$ which corresponds to $0{\cdot}0407\ d$ per one hour of operation. Thus, in comparison with a duplex system, the time between replacements is $3\frac{1}{2}$ times longer and the loss per one hour of operation is about 5 times smaller.

5. *'Majority Vote' Systems.*

The models described above in sections 2-4 apply to cases in which the failure to operate in a lane eliminates it automatically. In many applications, however, it is necessary to consider not only those failures which make the lane inoperative, but also failures consisting in faulty functioning of a lane. The system must then be designed in such a way that the faulty lane should be eliminated if its functioning disturbs the proper functioning of the remaining lanes; by a 'majority vote' the offending lane is recognised as such and is eliminated by an appropriate device. In a model discussed here we shall assume, for simplicity's sake, that the monitoring device which compares the performance of different lanes never fails and that the time needed for the elimination of a faulty lane is infinitely small. It is clear that as long as at least two lanes are working properly the system will operate properly; if, however, one of the remaining two lanes become faulty then it is impossible to decide which of them should be eliminated and it is the $(m-1)$st failure in the set of m lanes which becomes critical.

A system fails in the time interval $(0, t)$ if all the m lanes fail or if a set of $(m-1)$ lanes fails with only one lane working properly, so that (for $m \geqslant 2$)

$$\mathscr{F}(t) = 1 - (1-e^{-t})^m - m(1-e^{-t})^{m-1}e^{-t}$$

$$f(t) = m(m-1)e^{-2t}(1-e^{-t})^{m-2}$$

$$F(t) = (1-e^{-t})^m + m(1-e^{-t})^{m-1}e^{-t}$$

$$\rho(t) = \frac{m(m-1)e^{-2t}(1-e^{-t})^{m-2}}{1-(1-e^{-t})^m - m(1-e^{-t})^{m-1}e^{-t}}. \tag{46}$$

It is easily shown that, for $m > 2$, $\rho(t)$ increases from 0 to 2 when t increases from 0 to infinity and that, for $m = 2$, it is constant and equal to 2.

The Laplace Transform $f^*(s)$ of $f(t)$ is equal to:

$$\int_0^\infty m(m+1)e^{-2t}(1-e^{-t})^{m-2}e^{-st}dt = \int_0^1 m(m-1)u(1-u)^{m-2}u^s du = \frac{\Gamma(m+1)\Gamma(2+s)}{\Gamma(m+1+s)}, \tag{47}$$

and the characteristic function of $f(t)$ is given by

$$\phi(w) = \Gamma(m+1)\Gamma(2-iw)/\Gamma(m+1-iw), \tag{48}$$

so that the 'second characteristic function' has the form:

$$\log \phi(w) = \log \Gamma(m+1) + \log \Gamma(2-iw) - \log \Gamma(m+1-iw). \tag{49}$$

Hence the cumulants of $f(t)$-distribution are equal to:

$$\kappa_1 = -[\Psi(2) - \Psi(m+1)] = \sum_{\nu=2}^{m} \frac{1}{\nu}$$

$$\kappa_2 = \Psi'(2) - \Psi'(m+1) = \sum_{\nu=2}^{m} \frac{1}{\nu^2}$$

$$\kappa_3 = -[\Psi''(2) - \Psi''(m+1)] = 2\sum_{\nu=2}^{m} \frac{1}{\nu^3}$$

$$\vdots$$

$$\kappa_n = (-1)^n [\Psi^{(n-1)}(2) - \Psi^{(n-1)}(m+1)] = (n-1)!\sum_{\nu=2}^{m} \frac{1}{\nu^n} \tag{50}$$

Denoting the first three sums in the above formulae by A_m', B_m', C_m' respectively we obtain the following asymptotic expressions for the expected number of renewals in the interval $(0, t)$ and for their variance:

$$H(t) = \frac{t}{A_m'} - \frac{A_m'^2 - B_m'}{2A_m'^2} + o(1), \tag{51}$$

$$V(t) = \frac{B_m't}{A_m'^2} + \frac{1}{12} + \frac{5B_m'^2}{4A_m'^4} - \frac{4C_m'}{3A_m'^3} + o(1). \tag{52}$$

Since for $m = 2$ the hazard $\rho(t) = \text{const} = 2$, the underlying Renewal Process is Poissonian with $\lambda' = 2$. Thus, for $m = 2$, $H(t) = V(t) = 2t$ precisely. For the m-lane system $(m > 2)$ the expected number of renewals is asymptotically equal to t/A_m' so that the expected number of failures, in a long time interval, decreases with multiplexing in the proportion A_2'/A_m'. From the values given below:

m	2	3	4	5	6
A_m'	0·500000	0·833333	1·083333	1·283333	1·450000
A_2'/A_m'	1·000000	0·600000	0·461538	0·389610	0·344828

it is clear that for large t the multiplexing procedure gives useful results for the initial values $m = 3, 4, 5, \ldots\ldots$ only and that the more advanced multiplexing improves the asymptotic system reliability less and less.

For small values of t the situation is again completely different. The Laplace Transform $f^*(s)$ of $f(t)$ is equal to (47) and the Laplace Transform of the Renewal Function $H(t)$ is given by

$$H^*(s) = \frac{f^*(s)}{s[1-f^*(s)]} = \frac{\Gamma(m+1)\Gamma(2+s)}{s[\Gamma(m+1+s)-\Gamma(m+1)\Gamma(2+s)]}. \tag{53}$$

For large values of s:

$$H^*(s) = \frac{m!}{s^m} - \frac{m!(m-1)(m+2)}{2s^{m+1}} + o\left(\frac{1}{s^{m+1}}\right),$$

so that, for small values of t,

$$H(t) = mt^{m-1} - \frac{(m-1)(m+2)}{2}t^m + o(t^m). \tag{54}$$

By a similar argument it is easily shown that, for small t,

$$V(t) = mt^{m-1} - \frac{(m-1)(m+2)}{2}t^m + o(t^m) \tag{55}$$

so that, for small t, the ratio of the variance to the mean is near to unity.

In a similar way it may be shown that, for small t,

$$P_0(t) = 1 - mt^{m-1} + \frac{(m-1)(m+2)}{2}t^m + o(t^m) \tag{56}$$

and for $n \geqslant 1$

$$P_n(t) = \frac{(m!)^n}{(mn-n)!}t^{mn-n} - \frac{n(m-1)(m+2)(m!)^n}{2(mn-n+1)!}t^{mn-n+1} + o(t^{mn+1-n}) \tag{57}$$

so that

$$P_1(t) = mt^{m-1} - \frac{(m-1)(m+2)}{2}t^m + o(t^m),$$

$$P_2(t) = \frac{(m!)^2}{(2m-2)!}t^{2m-2} - \frac{(m-1)(m+2)(m!)^2}{(2m-1)}t^{2m-1} + o(t^{2m-1})$$

Here again, the probability of two, three or more renewals in the time interval $(0, t)$ can be neglected. If $t = \alpha$, i.e. if the time of operation is a small fraction α of the lane MTBF, we can assume that

$$P_0(t) = 1 - m\alpha^{m-1} \qquad P_1(t) = m\alpha^{m-1}$$

As before, we can obtain exact formulae for various values of m. Thus, if $m = 3$, we have:

$$\mathscr{F}(t) = 3e^{-2t} - 2e^{-3t},$$

$$f(t) = 6e^{-2t} - 6e^{-3t},$$

$$\rho(t) = 2 - \frac{2}{3e^{t} - 2}.$$

The Laplace Transform of $f(t)$ is equal to:

$$f^*(s) = -\frac{6}{s^2 + 5s + 6}$$

and that of $H(t)$ is given by:

$$H^*(s) = \frac{6}{s^2(s+5)} = \frac{6}{5s^2} - \frac{6}{25s} + \frac{6}{25(s+5)}$$

and

$$H(t) = \frac{6t}{5} - \frac{6}{25} + \frac{6}{25}e^{-5t}; \qquad h(t) = \frac{6}{5} - \frac{6}{5}e^{-5t}.$$

By an argument similar to that used in previous sections we can find:

$$V(t) = \frac{78}{125}t + \frac{6}{125} + \frac{6}{625}e^{-5t} - \frac{144}{625}te^{-5t} - \frac{36}{625}e^{-10t}.$$

It can be easily verified that, for small t, formulae (54) and (55) are satisfied since, expanding $H(t)$ and $V(t)$ in the Taylor Series, we have:

$$H(t) = 3t^2 - 5t^3 + o(t^3) \qquad \text{and} \qquad V(t) = 3t^2 - 5t^3 + o(t^3).$$

To evaluate the probabilities $P_n(t)$ notice that:

$$P_n{}^*(s) = \frac{6^n(s+5)}{(s+2)^{n+1}(s+3)^{n+1}}.$$

As before by finding the residues at the poles $s_1 = -1$, $s_2 = 2$, we obtain:

$$P_n(t) = (-6)^n e^{-2t} \sum_{r=0}^{n} (-1)^r \frac{t^r}{r!}\left\{3\binom{2n-r}{n} - \binom{2n-r-1}{n}\right\} -$$
$$- (-6)^n e^{-3t} \sum_{r=0}^{n} \frac{t^r}{r!}\left\{2\binom{2n-r}{n} + \binom{2n-r-1}{n}\right\}.$$

For a few initial values of n:

$$P_0(t) = 3e^{-2t} - 2e^{-3t},$$

$$P_1(t) = -6e^{-2t}(5-3t) + 6e^{-3t}(5+2t),$$

$$P_2(t) = 36e^{-2t}\left(15 - 8t + \frac{3t^2}{2}\right) - 36e^{-3t}(15+7t+t^2),$$

$$P_3(t) = -216e^{-2t}\left(50 - 26t + \frac{11}{2}t^2 - \frac{t^3}{3}\right) + 216e^{-3t}\left(50 + 24t + \frac{9}{2}t^2 + \frac{t^3}{3}\right).$$

Reverting to the time τ measured in ordinary units we have

$$H(\tau) = \frac{6}{5}\lambda\tau - \frac{6}{25} + \frac{6}{25}e^{-5\lambda\tau} \qquad \text{and} \qquad h(\tau) = \frac{6}{5}\lambda - \frac{6}{5}\lambda e^{-5\lambda\tau}.$$

If we want to find an interval $(0, \tau_\alpha)$ in which $h(\tau) < \alpha$ it is sufficient to find a value τ_α such that $h(\tau_\alpha) = \alpha$; in view of the monotonic increase of $h(\tau)$ this condition will be satisfied for all $\tau < \tau_\alpha$.

Clearly

$$\tau_\alpha = -\frac{\theta}{5}\log\left(1-\frac{5\theta\alpha}{6}\right).$$

Example. If $\theta = 10\,000$ hours and if the maximum failure rate is fixed as $\alpha = 10^{-6}$, then $\theta\alpha = 10^{-2}$ and $\tau_\alpha = -\frac{10\,000}{5}\log(1-5.10^{-2}/6) = 16{\cdot}7$ hours; with less exacting requirement $\alpha = 10^{-5}$, we find the admissible operation time $\tau_\alpha = 174$ hours.

There is no difficulty in constructing a table of admissible operation times similar to Table 1 and 2. Following the argument of previous sections it would be also easy to establish formulae for the optimum replacement times.

For $m = 4$:

$$f(t) = 12e^{-2t}(1-e^{-t})^2 = 12e^{-2t} - 24e^{-3t} + 12e^{-4t}$$

$$f^*(s) = \frac{12}{s+2} - \frac{24}{s+3} + \frac{12}{s+4} = \frac{24}{(s+2)(s+3)(s+4)}$$

$$H^*(s) = \frac{24}{s^2(s^2+9s+26)} = \frac{12}{13s^2} - \frac{54}{169s} + \frac{54s+330}{169(s^2+9s+26)}.$$

Hence

$$H(t) = \frac{12}{13}t - \frac{54}{169} + \frac{54}{169}e^{-9t/2}\cos\left(\frac{t\sqrt{23}}{2}\right) + \frac{174}{169\sqrt{23}}e^{-9t/2}\sin\left(\frac{t\sqrt{23}}{2}\right),$$

$$h(t) = \frac{12}{13} - \frac{12}{13}e^{-9t/2}\cos\left(\frac{t\sqrt{23}}{2}\right) - \frac{108}{13\sqrt{23}}e^{-9t/2}\sin\left(\frac{t\sqrt{23}}{2}\right).$$

For small values of t the expansion into the Taylor Series yields

$$H(t) = 4t^3 + o(t^3) \qquad h(t) = 12t^2 + o(t^2)$$

which agrees with formula (54).

Again, there should be no difficulty in finding, by the same arguments as those used in preceding sections, the formulae for the probabilities $P_n(t)$, for the admissible operation time and for the optimum replacement policy.

Other, more complicated systems in which r lanes out of m must operate properly to make the system operative can be investigated by the same method. These 'r out of m'-systems have similar properties: their failure probabilities have characteristic functions which are Beta functions, i.e. the ratios of two Gamma functions, their 'second characteristic function' is the difference of logarithms of Gamma functions and the cumulants are simple finite sums of reciprocals of powers of consecutive integers. The Laplace Transforms required in these investigations are all rational functions of s with no poles on the positive side of the imaginary axis: this makes the application of Tauberian theorems simple and the behaviour of these functions for small and large values of t can be easily assessed from the behaviour of corresponding Laplace Transforms for s tending to infinity and to zero respectively. The exact formulae can also be easily obtained although the appropriate calculations could be in some cases cumbersome.

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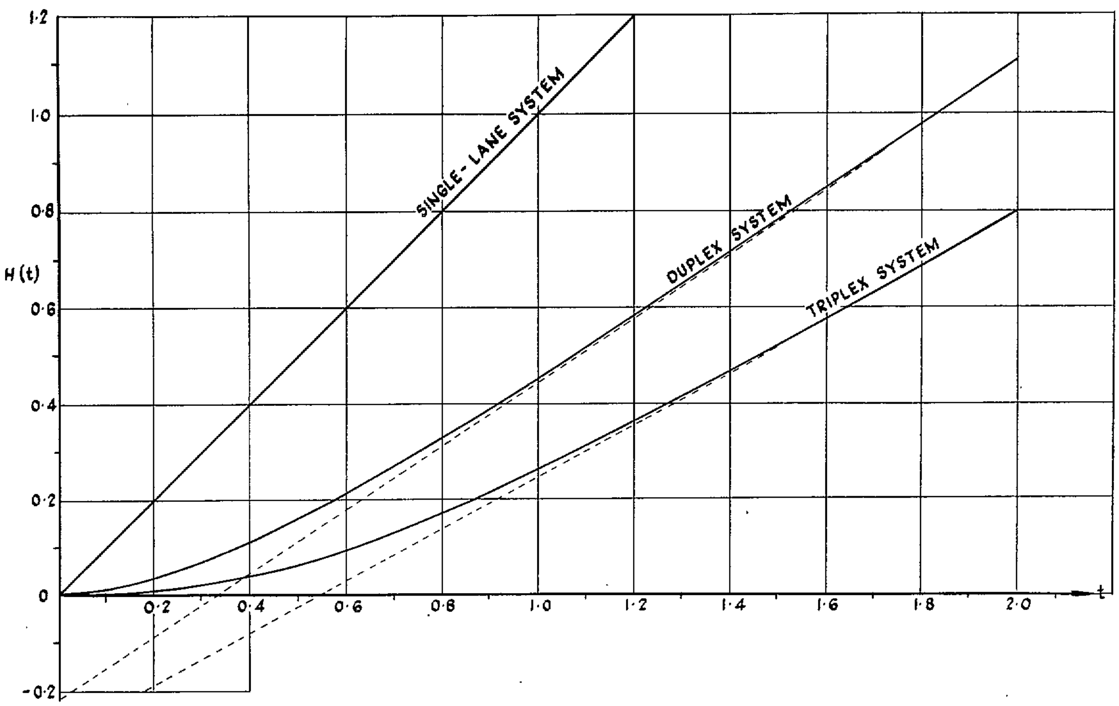

FIG. 1. Renewal function. Expected number of renewals in the interval $(0, t)$.

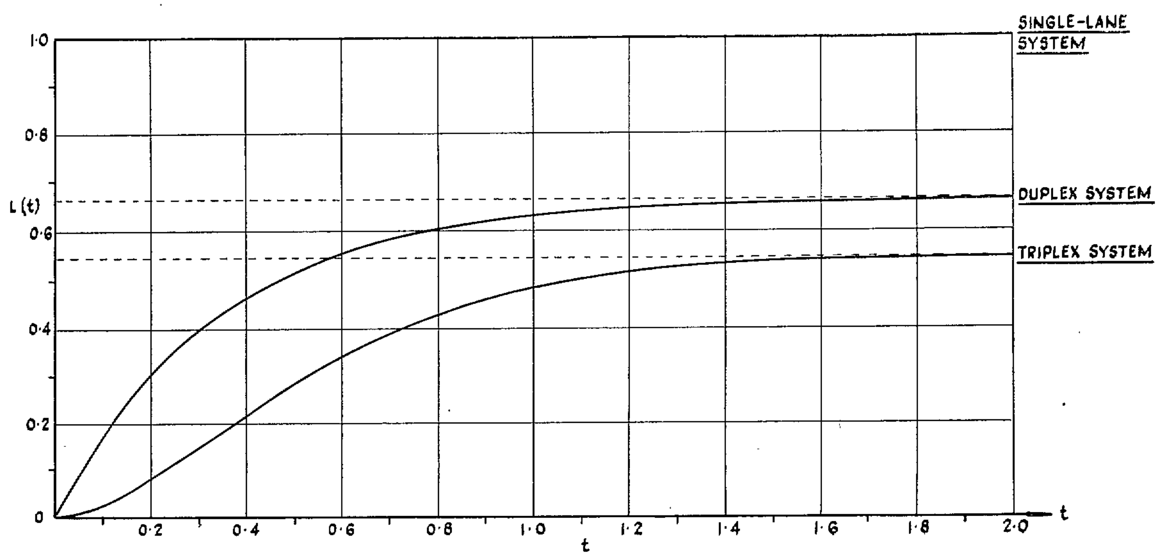

FIG. 2. Renewal density.

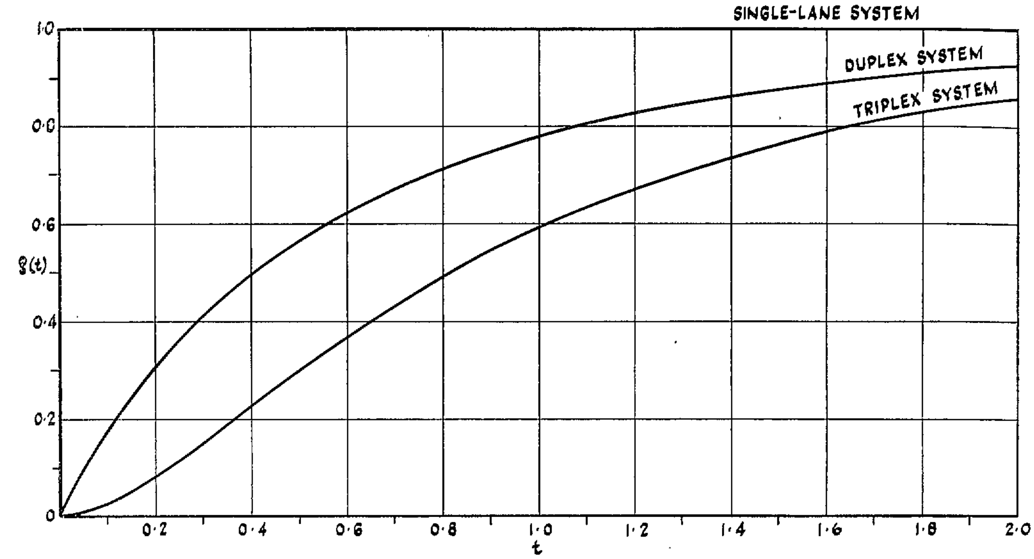

FIG. 3. Age-specific failure rate (hazard) $\rho(t)$ as a function of t.

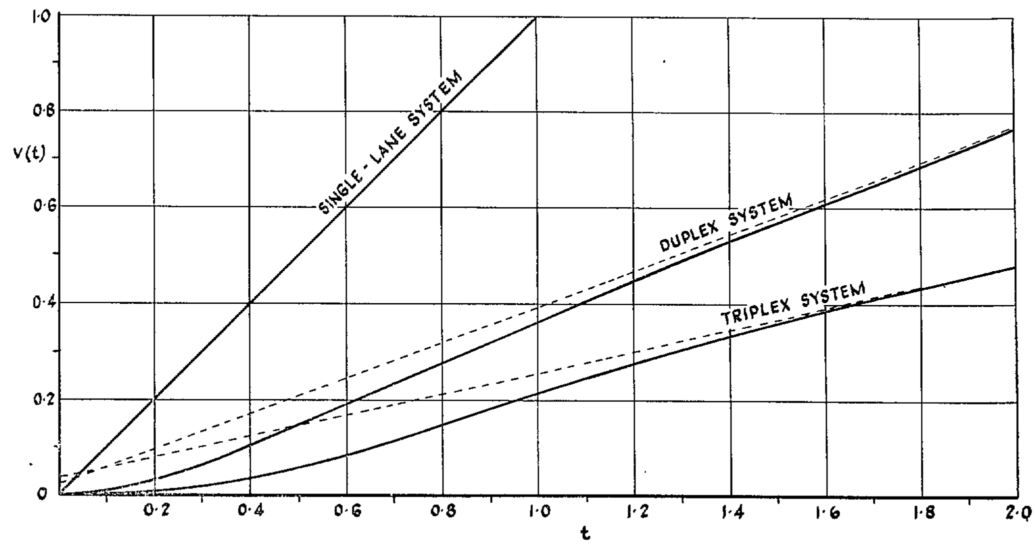

FIG. 4. Variance of the numbers of renewals in (0, t).

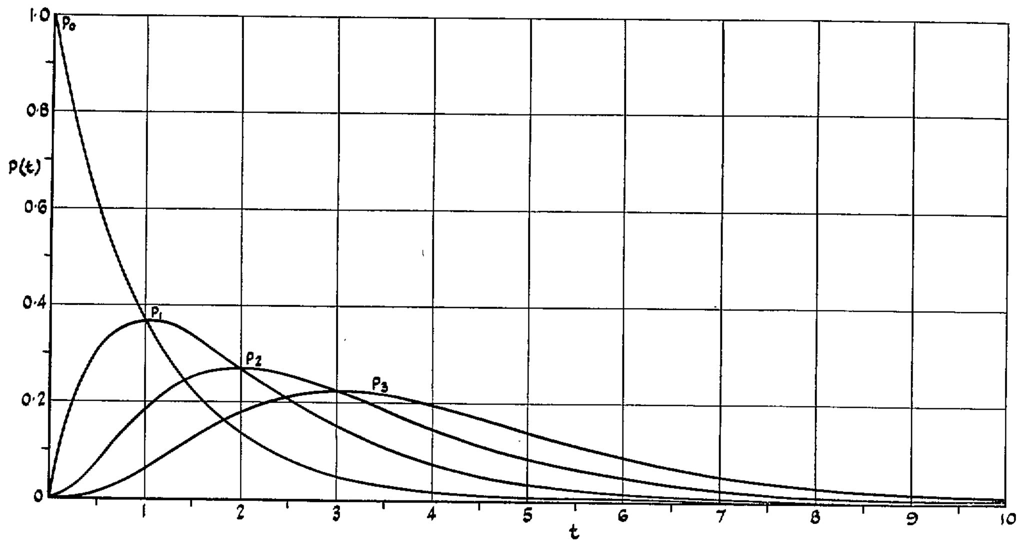

FIG. 5. Distribution of number of renewals—single lane system.

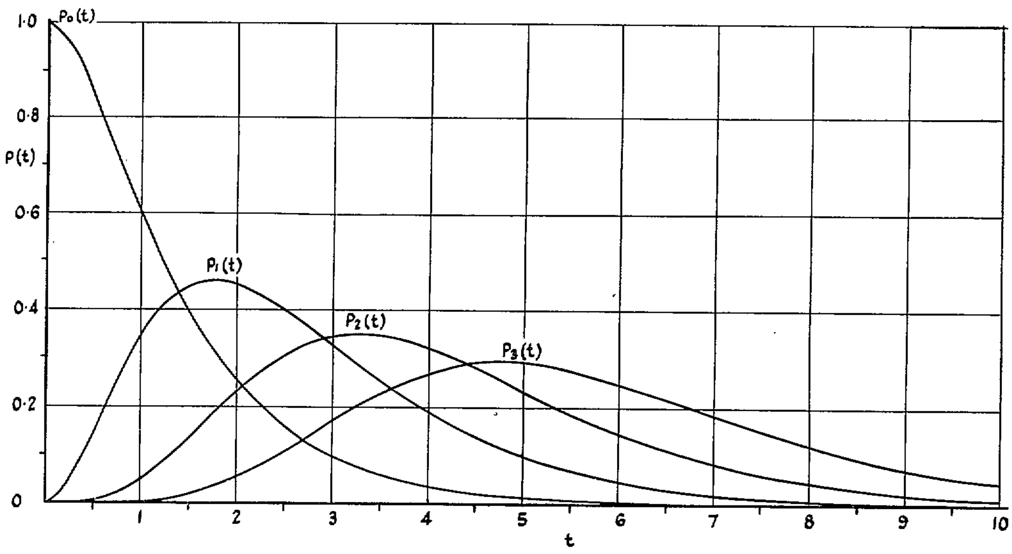

FIG. 6. Distribution of number of renewals—duplex system.

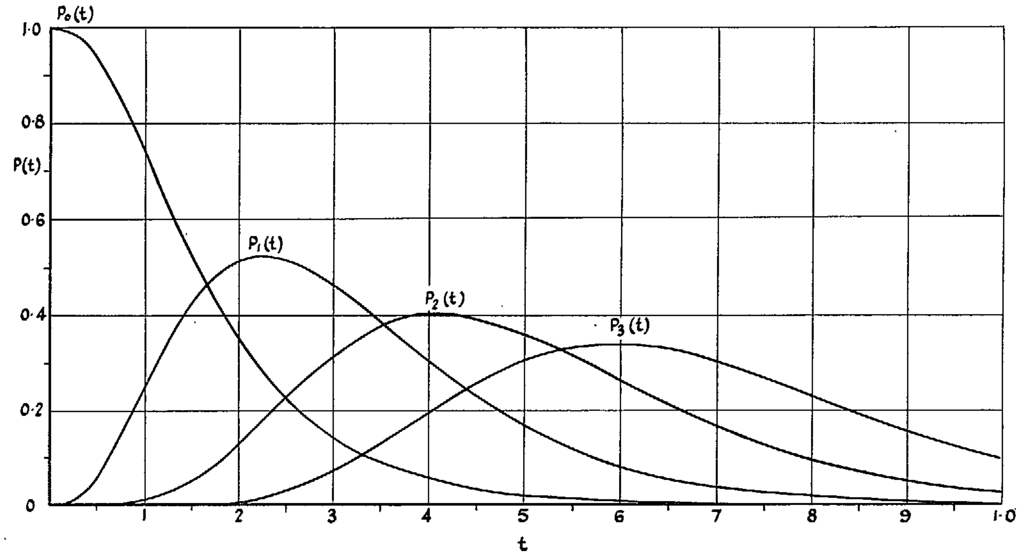

FIG. 7. Distribution of number of renewals—triplex system.

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