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SOME CONTRIBUTIONS TO JET-FLAP THEORY AND TO THE THEORY OF SOURCE-FLOW FROM AEROFOILS

By

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Some Contributions to Jet-Flap Theory and to the Theory of Source-Flow from Aerofoils

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SUMMARY

The paper presents a theoretical study of the thrust, lift and moment on an aerofoil due to a two-dimensional jet of air ejected from the trailing edge at an angle τ to the main stream. It is rigorously proved that in subsonic compressible flow the ideal thrust of the jet (assumed not to mix with the main stream) is independent of the exit angle τ. The theory for the lift and moment is developed for incompressible flow only. It is not rigorous, being based on the assumption that jets of equal momentum and at equal values of τ have essentially the same influence on the main stream. The theory is in satisfactory agreement with the few experimental values available.

Two appendices have been added to the paper; the first written in April, 1955, was added to clarify the paper and to answer some criticisms of its contents, while the second, dated November, 1956, was added to show the relation between the author's theory and that given later by Spence.

1. Introduction

Some recent reports[1,2] have drawn attention to the fact that the circulation about an aerofoil can be controlled by ejecting air from the trailing edge at an angle to the main stream. Because of the asymmetry of the resulting flow the jet induces a circulation about the aerofoil, and hence there is an induced lift which is additional to the component arising from the momentum flux of the jet itself. To this extent the jet is similar to a flap, although the term "jet-flap" which has been applied to it[1] is perhaps too restricted, since other types of control, e.g., a split flap, or a spoiler, may, in certain circumstances, be the more appropriate analogy[3].

While it it not surprising that the jet induces a lift - in general any asymmetric disturbance of the flow will do this - it is remarkable that, provided the jet does not cause flow separation from the aerofoil, and does not mix with the main stream, the thrust on the aerofoil is independent of τ, the angle of ejection[1]. Thus, ideally, the lift is obtained without loss in forward thrust. The proof of this result given in Ref. 1 is based on D'Alembert's paradox, which is not appropriate to the open contour of the aerofoil plus jet, to which it is there applied. A rigorous proof of the independence of the thrust and ejection angle for compressible flow is given in the next section.

In contrast to the paradoxical result on thrust, the induced lift, while easily predicted qualitatively, is much more difficult to determine quantitatively. The theory presented in this paper is not rigorous but involves one or two assumptions which appear reasonable. Also the agreement obtained with experiment provides further justification of these assumptions.

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The jet is assumed to be a distinct stream of fluid separated from the main stream by two vortex sheets. The theory for the lift and moment is based on an equation for the flow about an aerofoil behind which extends a vortex sheet, given in Ref. 4, where it was developed for application to unsteady aerofoil theory. In the present application the two vortex sheets produced by the jet are replaced by a single vortex sheet, the strength of which is shown to be proportional to C_J/R, where C_J is the jet momentum coefficient defined by equation (13) below, and R is the radius of curvature of the jet. Unfortunately the curvature cannot be determined until the vortex strength is known. This situation leads to an integral equation for the vortex strength, which, if C_J is small enough, can be solved by iteration. In the particular case when the velocity of the ejected fluid is the same as the main stream velocity, so that no vortex sheets arise, the theory is exact. Such flow is termed "source-type" flow to distinguish it from the "jet-type" flow which occurs at higher ejection velocities. A fundamental assumption of the theory is that a jet of given momentum and ejection angle has essentially the same effect on the main stream as a source-type flow having the same momentum and ejection angle.

The main conclusions are that (i) the lift coefficient C_L is proportional to $\tau\sqrt{C_J}$ and (ii) that the lift due to the jet acts approximately at the mid-chord point. The conclusion that at constant τ, C_L is proportional to $\sqrt{C_J}$, is in agreement with the few experimental results so far available (see Figs. 7, 8 and 9), but the dependence of C_L on τ has not yet been investigated experimentally. The author's theory is in good agreement with experiment for values of C_J less than 0.5 with $\tau = 31.4°$ (the lowest value of τ yet investigated). However as C_J and τ are increased there is an increasing discrepancy between theory and experiment, probably mainly due to turbulent mixing in the wake (at high C_J) and loss of circulation due to some flow separation near the trailing edge (at high τ).

The law $C_L \propto \sqrt{C_J}$ was also derived by an unconvincing argument in Ref. 1, based on an analogy with a mechanical flap. The reasoning is essentially as follows. The characteristics of a mechanical flap are functions of two independent variables, namely Ec, the flap chord, and τ_0 the flap deflection angle, while it is reasonable to suppose that the jet-flap characteristics are likewise dependent on the two variables, C_J and τ. Hence any two relations of the form $F(Ec, \tau_0) = f(C_J, \tau)$ may be taken as establishing "similarity" between jet and mechanical flaps. One of these relations is taken to be that the "lift" on the jet, $C_J \sin\tau$, is equal to the lift on the flap, which is a known function $F(Ec, \tau_0)$. Thus

$$C_J = \frac{1}{\sin\tau} F(Ec, \tau_0) \; . \qquad \ldots(1)$$

The other relation is that the ratio of the total lift to the lift on the flap in the mechanical system, is equal to the ratio of the total lift to the "lift" in the jet on the jet-flap system, i.e., that

$$\mathcal{M}_F(Ec, \tau_0) = \mathcal{M}_J(C_J, \tau) \; , \qquad \ldots(2)$$

in which $\mathcal{M}_F$ is a function known from classical aerofoil theory, while the form of $\mathcal{M}_J$ is unknown. Eliminating Ec from (1) and (2), we obtain

$$\mathcal{M}_J = \mathcal{M}(C_J, \tau, \tau_0) \; , \qquad \ldots(3)$$

where/

where $\mathcal{M}$ is a known function. Equation (3) is merely another form of the definition of "similarity" given by (1) and (2), and is valueless unless combined with some theory or plausible hypothesis giving τ_0 as a function of C_J and τ. Such theory or hypothesis must take into account the basis of equation (3), namely the definition of similarity. In Ref. 1 τ_0 is tacitly assumed to depend on τ only; it is then stated that the ratio τ/τ_0 "at this stage can only be guessed at", which of course begs the question completely. To given values of Ec and τ_0 correspond fixed values of F and $\mathcal{M}_F$, and hence from (1) and (2), fixed values of $C_J \sin \tau$ and $\mathcal{M}_J$. If the jet-flap theory were known it would then be possible to find fixed values of C_J and τ. However without this theory we can still say that given values of Ec and τ_0 determine fixed values of C_J and τ. It is not possible to impose a third relationship between τ and τ_0. The argument given in Ref. 1 is not a theory, despite the fact that the final result is in fair agreement with experiment. (This criticism was subsequently modified; see §2.1 of Appendix I.)

In view of the criticisms given above it is only fair to state that in the writer's opinion the authors responsible for Refs. 1, and 2 and other reports from the N.G.T.E., Pyestock, on the same topic have done most valuable work in attracting attention to and elucidating the physical principles of this neglected method of circulation control.

2. List of Symbols

x, y	the physical plane
z	$= x + iy$
n, s	distances measured normal to and along a streamline respectively
(q, θ)	velocity vector in polar co-ordinates
ρ	density
∞	as a suffix to denote undisturbed stream values
U	$= q_\infty$ in the main stream
M	local Mach number
(ϕ, ψ)	plane of equipotentials (ϕ = constant) and streamlines (ψ = constant), for zero circulation
w	$= \phi + i\psi$
η, γ	elliptic co-ordinates defined by equations (18) and (21)
p	pressure
V	velocity in jet at infinity
h	width of jet at infinity
H	width of stream of accelerated fluid upstream at infinity
c	chord distance
C_T, C_J, C_Q	thrust, moment and mass coefficients defined by equations (12), (13) and (14)
C_L, C_D	lift and drag coefficients
C_m	moment coefficient about mid-chord point
C_{m_0}	value of C_m at $C_L = 0$
τ	angle between main stream and jet flow at the trailing edge

a,/

α, α_0 incidence and no-lift angle respectively

δ defined by equation (51)

σ defined by equation (27)

K defined by equation (30).

3. The Ideal Thrust of a Two-Dimensional Jet

We now calculate the thrust on an aerofoil due to a two-dimensional jet leaving the trailing edge at some angle to the main stream. Two principal assumptions are made, namely (1) that the jet causes no flow separation and consequent form drag, and (2) that the jet is an irrotational stream separated from the main stream by two vortex sheets. The flow pattern is shown in Fig. 1. The fluid which is ejected from the trailing edge CC' is assumed to enter the aerofoil at the leading edge BB'. (The case when there is a source within the aerofoil is deduced from the present case below.) The fluid which passes through the aerofoil can thus be regarded as flowing in an infinite channel. The mass flow in this channel is constant but the momentum flux is, in general, subject to a rapid increase somewhere within the aerofoil. The velocity magnitude is continuous across $A_\infty B$ and $A'_\infty B'$, but in general discontinuous across CD_∞ and $C'D'_\infty$; the pressure is continuous across each of those lines.

The forces acting on the aerofoil are obtained by integrating the pressures acting on both the external and internal surfaces of the profile. In particular if T is the thrust force on the aerofoil (acting parallel to the undisturbed flow), it follows from Fig. 1 that,

$$T = \left\{ \int_{BFC} - \int_{B'F'C'} + \int_{B'E'C'} - \int_{BEC} \right\} p \sin\theta \, ds, \qquad \ldots(4)$$

where θ is the flow direction on the profile measured from the undisturbed flow direction, s is distance measured on the aerofoil surface, and p is the pressure.

From Euler's momentum theorem applied to the channel in Fig. 1,

$$\int_{A_\infty BFCD_\infty} p \sin\theta \, ds - \int_{A'_\infty B'F'C'D'_\infty} p \sin\theta \, ds + H(p_\infty + \rho_\infty U^2) - h(p_\infty + \rho_\infty V^2) = 0, \qquad \ldots(5)$$

where H, h are the widths of the jet at $A_\infty A'_\infty$ and $D_\infty D'_\infty$ respectively, U, V are the jet velocities at $A_\infty A'_\infty$ and $D_\infty D'_\infty$ respectively, and p_∞, ρ_∞ are the fluid pressure and density in the undisturbed flow.

It is now necessary to obtain a result corresponding to (5) for the "channels" of infinite width which lie on each side of the jet. Consider the flow in the channel shown in Fig. 2. The wall $G_\infty F_\infty$ is straight, so that the momentum theorem yields

$$-\int_{G'_\infty F'_\infty} p \sin\theta \, ds + H_0(p_0 + \rho_0 U_0^2) - (H_0 - b)(p_1 + \rho_1 U_1^2) = 0, \qquad \ldots(6)$$

where/

where the suffices o and 1 denote conditions at $G_\infty G'_\infty$ and $F_\infty F'_\infty$ respectively, and H_o, $H_o - b$ are the corresponding channel widths. From continuity of mass

$$\rho_o H_o U_o = \rho_1 (H_o - b) U_1 . \qquad \ldots(7)$$

It is easily deduced from Bernoulli's theorem (cf. any account of linear perturbation theory) that

$$\rho_1 = \rho_o \{1 - M_o^2 \delta\} + O(\delta^2) ,$$

and

$$p_1 = p_o - \rho_o U_o^2 \delta + O(\delta^2) ,$$

where δ is defined by

$$U_1 = U_o(1 + \delta)$$

and M_o is the Mach number at $G_\infty G'_\infty$. Substitution of these expansions in (6) and (7) leads to

$$\int_{G'_\infty F'_\infty} p \sin \theta \, ds = b p_o + O(\delta^2) .$$

If we now let H_o tend to infinity, δ tends to zero, and

$$\int_{G'_\infty F'_\infty} p \sin \theta \, ds = b p_o , \qquad \ldots(8)$$

which gives the drag on a wall caused by the flow of an infinite stream past it. (Two obvious applications of this result yields D'Alembert's Paradox for a closed body.)

Applying (8) to the two regions outside the jet shown in Fig. 1, we have by subtraction that

$$\int_{A_\infty PECD_\infty} p \sin \theta \, ds - \int_{A'_\infty B'E'C'D'_\infty} p \sin \theta \, ds = p_\infty (h - H) . \qquad \ldots(9)$$

Subtracting equations (5) from (9), and making use of the continuity of the pressure across $A_\infty B$, $A'_\infty B'$, CD_∞ and $C'D'_\infty$, we find

$$\left\{ \int_{BEC} - \int_{BFC} + \int_{B'F'C'} - \int_{B'E'C'} \right\} p \sin \theta \, ds = \rho_\infty (HU^2 - hV^2) ,$$

and/

and hence from (4)

$$T = \rho_\infty(hV^2 - HU^2) . \qquad ...(10)$$

The thrust is thus independent of the angle of ejection, τ. From continuity of mass $HU = hV$, so that equation (10) can be written in the form

$$C_T = C_J - 2C_Q , \qquad ...(11)$$

where C_T, C_J and C_Q are thrust, momentum and mass coefficients defined by

$$C_T = \frac{T}{\frac{1}{2}\rho_\infty cU^2} , \qquad ...(12)$$

$$C_J = \frac{\rho_\infty hV^2}{\frac{1}{2}\rho_\infty cU^2} = \frac{2hV^2}{cU^2} , \qquad ...(13)$$

$$C_Q = \frac{\rho_\infty hV}{\rho_\infty cU} = \frac{hV}{cU} , \qquad ...(14)$$

c being the aerofoil chord.

Two special cases of (11) are of some interest:-

1. Jet derived from a source within the aerofoil.

In this case $H = 0$ in (10), and (11) reduces to

$$C_T = C_J , \qquad ....(15)$$

a result first given in Ref. 1.

2. Jet joining main stream smoothly so that no vortex sheets occur.

In this case the velocity of the jet is the same as that of the main stream; in particular $V = U$, so that from (13) and (14) $C_J = 2C_Q$. Equation (11) then yields $C_T = 0$, whereas if the fluid comes from a source on or within the aerofoil, from (15)

$$C_T = 2C_Q . \qquad ...(16)$$

This last equation gives the thrust coefficient due to a source on an aerofoil. Similarly a sink on an aerofoil gives rise to a "sink-drag" of amount $2C_Q$, a result that at least for incompressible flow is quite well-known[5].

It/

It is important to distinguish between the character of the flows giving rise to (15) and (16). In both cases the fluid comes from a source within the aerofoil, but in the case of the jet it does not turn the corners at the end of the jet channel exit. This case is shown in Fig. 3(a). The jet separates at points B and D and in general emerges at a different speed to the local flow, so that BF_∞ and DG_∞ are vortex sheets. When the thrust is given by (16), the flow will appear as in Fig. 3(b), in which the streamlines HF_∞ and EG_∞ bounding the emitted fluid are not vortex sheets. Incidentally it has been assumed in Fig. 3(b) that the circulation is such to make the trailing edge E a stagnation point, but of course this affects the lift only. The position of the stagnation point H will be a function of C_Q.

As far as thrust is concerned the result for the "jet-type" of flow (Fig. 3(a)+) can be derived from the result (16) for the "source-type" of flow (Fig. 3(b)+) simply by replacing $2C_Q$ by C_J. Later in the paper plausible reasons are given for adopting the same procedure when calculating the lift.

4. The Basic Transformations

In the remainder of this paper we confine our attention to incompressible flow, although the results obtained can be extended to subcritical subsonic flow by a fairly obvious application of linear perturbation theory. Before calculating the lift acting on the profile due to a jet-type of flow, we consider a particular case of source-type flow. The solution for jet-type flow is then deduced from this case by replacing $2C_Q$ by C_J, and adding a further term which arises from the velocity distribution induced on the profile by the vortex sheets bounding the jet. This additional term does not, of course, arise in the source-type of flow, and must be calculated separately. This is discussed in more detail in Section 7.

The particular case of source-type flow we consider arises when points H and E coincide with points B and E respectively in Fig. 3(b). The complete z-plane $(z = x + iy)$ is shown in Fig. 4(a), in which the aerofoil is shown at the zero-lift position. The problem will be solved if the relation between the no-lift angle, α_0, and C_Q can be determined, for if the aerofoil is placed at an incidence α, from thin aerofoil theory the lift coefficient will be given by

$$C_L = 2\pi(\alpha - \alpha_0) . \qquad \ldots(17)$$

This method assumes that both α and C_Q are small enough to permit the linear superposition of their effects.

Let ϕ and ψ be the equipotential and stream functions respectively, then the w-plane $(w = \phi + i\psi)$ for the no-lift case is shown in Fig. 4(b). The fluid issuing from the aerofoil is confined between the streamlines $\psi = 0$ and $\psi = Q$, and the aerofoil is assumed to be thin so that its length in the w-plane can be taken to be Uc. The origin of the w-plane has been selected to make $\phi = -\frac{1}{2}Uc$ at the front stagnation point and $\phi = \frac{1}{2}Uc$ at the point where the emitted fluid leaves the aerofoil.

The/

+The terms "source-type" flow and "jet-type" flow are used throughout this report to indicate the absence or presence respectively of vortex sheets separating the ejected fluid from the main stream. Both types of flow can be regarded as originating from a source within the aerofoil.

The w-plane is mapped into the upper half of the t-plane (Fig. 4(c)) by

$$\frac{dw}{dt} = A\left(t - \frac{t_o^2}{t}\right),$$

where A is a constant and t_o is a real constant. Thus

$$w = \tfrac{1}{2}At^2 - At_o^2 \log t + B + iC , \qquad \ldots(18)$$

where B and C are real constants. Since $\psi = Q$ when $t > 0$, and $\psi = 0$ when $t < 0$ on the real axis in the t-plane, it follows from (18) that $C = Q$, and $A = -Q/\pi t_o^2$. If the origin of the t-plane is now selected so that $\phi = -\frac{1}{2}Uc$ when $t = \pm 1$, and $\phi = \frac{1}{2}Uc$ when $t = \pm t_o$, the constants in (18) can be calculated. It is found that

$$w = \frac{1-t^2}{1-t_o^2}\left(Uc - \frac{Q}{\pi}\log t_o\right) + \frac{Q}{\pi}\log t + iQ - \tfrac{1}{2}Uc , \qquad \ldots(19)$$

where Q and t_o are related by

$$\frac{1}{2\pi}\left(\frac{Q}{Uc}\right) = t_o^2\,(1 - t_o^2 + t_o^2 \log t_o^2)^{-1} . \qquad \ldots(20)$$

The t-plane is mapped into the ζ-plane shown in Fig. 4(d) by

$$t = -\cosh \tfrac{1}{2}\zeta , \quad \zeta = \eta + i\gamma . \qquad \ldots(21)$$

The aerofoil surface is given by $\eta = 0$, so that if $\gamma = \pi + \delta$ at $t = t_o$ it follows from (21) that

$$t_o = \sin \tfrac{1}{2}\delta . \qquad \ldots(22)$$

It should be noticed that in the ζ-plane the central streamline of the fluid emerging from the aerofoil lies along $\gamma = \pi$, while the streamlines $\psi = 0$ and $\psi = Q$ lie on each side of $\gamma = \pi$. An equation giving the values of $\log U/q + i\theta$, where (q,θ) is the velocity vector in polar-co-ordinates, at any point in the ζ-plane, in terms of the boundary conditions, θ given on $\eta = 0$ (the aerofoil), and the jump in $\log U/q$ (if any) given on $\gamma = \pi$, has been derived for application to unsteady aerofoil theory in a previous report[4]. Its application to the present problem is given in the next section. In this application it is necessary to know the value of η at $\phi = \frac{1}{2}Uc$ on the streamline $\gamma = \pi$, that is at a point midway between D and D' of Fig. 4(a). If the value of η in question is σ, then $\phi = \frac{1}{2}Uc$ at $\zeta = \sigma + i\pi$, and (19) and (20) yield

Uc/

$$Uc = \frac{1 + \sinh^2 \frac{1}{2}\sigma}{1 - t_0^2} \left\{ Uc - \frac{Q}{\pi} \log t_0 \right\} + \frac{Q}{\pi} \log \left(- \sinh \frac{\sigma}{2} \right). \qquad \ldots(23)$$

It is shown below that $\sinh^2(\frac{1}{2}\sigma) = O(t_0^2)$; also from (19) $\frac{Q}{Uc} = O(t_0^2)$. Hence ignoring terms $O(t_0^4 \log t_0)$, we deduce from (19) and (23) that

$$1 + \left(\frac{\sinh \frac{1}{2} \sigma}{t_0} \right)^2 + \log \left(\frac{\sinh \frac{1}{2} \sigma}{t_0} \right)^2 = 0 ,$$

the solution of which is

$$\sinh \tfrac{1}{2}\sigma \approx - 0.528 \, t_0 \, . \qquad \ldots(24)$$

Now by definition Q is the mass flow from the source, so that (cf. equation (14))

$$C_Q = \frac{Q}{Uc} \, . \qquad \ldots(25)$$

Finally, by ignoring second-order terms, we have from (20), (22), (24) and (25) that

$$\delta = \sqrt{\left(\frac{1}{\pi} \times 2C_Q \right)} , \qquad \ldots(26)$$

and

$$\left(\frac{\delta}{\sigma} \right)^2 \doteqdot 3.58 \, . \qquad \ldots(27)$$

5. The Theory of Aerofoils Behind which Extend Vortex Sheets

At this point it is convenient to interrupt the calculation commenced above to give an outline of the theory of aerofoils which have vortex sheets lying along their trailing-edge streamlines. It includes ordinary aerofoil theory as the special case when the strength of the vortex sheet vanishes.

The first step is to transform the w-plane into the ζ-plane as in the previous section. In the usual applications to aerofoil theory there is no source or sink within the aerofoil, and (19) and (21) reduce to

$$w = - \tfrac{1}{2} Uc \cosh \zeta \, . \qquad \ldots(28)$$

However/

However, regardless of the form of the transformation by which the ζ-plane is achieved, in this plane the boundary conditions are comparatively simple, the aerofoil lying on $\eta = 0$, $0 \leqslant \gamma \leqslant 2\pi$, and the vortex sheet on $\gamma = \pi$, $-\infty \leqslant \eta \leqslant 0$. It has been shown that in this plane[4]

$$f = \log \frac{U}{q} + i\theta = \frac{1}{2\pi} \int_0^{2\pi} \theta(\gamma^*) \cot \tfrac{1}{2}(\gamma^* + i\zeta) d\gamma^* + \frac{i \sinh\zeta}{2\pi} \int_{-\infty}^{0} \frac{K(\eta)\, d\eta}{\cosh \eta + \cosh \zeta}, \quad \ldots(29)$$

where K is the jump in $\log (U/q)$ across the vortex sheet, i.e.,

$$K = \left[\log \frac{U}{q} \right]_{\gamma=\pi+0}^{\gamma=\pi-0} = \left[\log \frac{q}{U} \right]_{\gamma=\pi-0}^{\gamma=\pi+0} . \quad \ldots(30)$$

With slight modifications and approximations this equation can be applied to the jet-type of flow being considered in this paper. These are:-

1. If the jet is relatively thin, i.e., C_Q is small, then as far as its aerodynamic effects on the aerofoil are concerned it can be regarded as a vortex sheet lying on the streamline $\gamma = \pi$ in $-\infty \leqslant \eta \leqslant \sigma$, $\eta = \sigma$ being the point on $\gamma = \pi$ where the jet leaves the aerofoil and becomes free to assume a curved shape.

2. The strength of the sheet (which to first order is directly proportional to K) is the algebraic sum of the strengths of the <u>two</u> vortex sheets which separate the jet from the main flow. A formula for K is deduced in the next section. For the source-type of flow discussed in the previous section, $K = 0$.

Since $\lim_{z=0} q = U$, and $\lim_{z=0} \theta = 0$, it follows from (29) that $\lim_{z=0} f(z) = 0$. Now $z \to \infty$ implies that $w \to \infty$, and hence from equations (19) and (21), that $\zeta \to -\infty$; therefore

$$\lim_{\zeta=-\infty} f(\zeta) = 0 . \quad \ldots(31)$$

For the problem considered in this paper (29) becomes

$$f = \frac{1}{2\pi} \int_0^{2\pi} \theta(\gamma^*) \cot \tfrac{1}{2} (\gamma^* + i\zeta)\, d\gamma^* + \frac{i \sinh \zeta}{2\pi} \int_{-\infty}^{\sigma} \frac{K(\eta)\, d\eta}{\cosh \eta + \cosh \zeta}, \quad (32)$$

where from (31), θ and K must satisfy

$$\int_0^{2\pi} \theta d\gamma - \int_{-\infty}^{\sigma} K\, d\eta = 0 . \quad \ldots(33)$$

Near/

Near $\zeta = -\infty$, equation (32) can be expanded

$$f = \frac{1}{\pi} e^{\zeta} \left(\int_0^{2\pi} e^{-iy}\, \theta dy + \int_{-\infty}^{\sigma} K \cosh \eta \, d\eta \right) + \frac{1}{\pi} e^{2\zeta} \left(\int_0^{2\pi} e^{-2iy}\, \theta dy \right.$$

$$\left. - \int_{-\infty}^{\sigma} K \cosh 2\eta \, d\eta \right) + O(e^{3\zeta}) \, , \qquad \ldots(34)$$

while from (19) and (21)

$$e^{\zeta} = - \frac{Uc}{4w} \left(1 + k + O\left(\frac{1}{w^2}\right)\right) , \qquad \ldots(35)$$

where k is a term of order $t_0^2 \log t_0$. From the definition of f given in equation (29)

$$f = \log \frac{Ue^{i\theta}}{q} = \log \left(U \frac{dz}{dw} \right). \qquad \ldots(36)$$

Therefore if

$$a_n = \frac{1}{\pi} \int_0^{2\pi} e^{-iny}\, \theta dy - \frac{(-1)^n}{\pi} \int_{-\infty}^{\sigma} K \cosh n\eta d\eta \, , \qquad \ldots(37)$$

it follows from (34), (35) and (36) that

$$\frac{dw}{dz} = U \left\{ 1 + i \left(\frac{Uc}{4w}\right) a_1 (1 + k) - i \left(\frac{Uc}{4w}\right)^2 (1 + k)^2 \left(a_2 - \frac{i}{2} a_1^2 \right) \right\} + O\left(\frac{1}{w^3}\right). \qquad \ldots(38)$$

We have derived this expansion in order to calculate the forces and moment acting on the profile from Blasius' theorem, but before doing this we must discuss the possibility of a jump in pressure across the vortex sheet. There are two ways of approximating to the jet, which is a region bounded by two vortex sheets, across which the pressure is continuous. Either the jet can be replaced by a thin sheet across which both the pressure and velocity are discontinuous, (by removing the fluid within the jet), or by a single vortex sheet, being the algebraic sum of the two separate sheets, across which the velocity alone is discontinuous. Either method can be adopted, and they lead essentially to the same result, but the second method is adopted in this paper since it alone gives the correct result when the velocity in the jet is reduced to the point when source-type flow occurs.

If/

If the lift, drag and moment (about the origin in the z-plane) are denoted by L, D and M respectively, Blasius' theorem states that

$$D - iL = \frac{i}{2}\rho \int_C \left(\frac{dw}{dz}\right)^2 dz \,, \qquad \ldots(39)$$

and

$$M + iN = \tfrac{1}{2}\rho \int_C z\left(\frac{dw}{dz}\right)^2 dz \,, \qquad \ldots(40)$$

where N is a dummy symbol, and C is any contour enclosing both the aerofoil and vortex sheet, the pressure being continuous across the latter. Now since L must vanish, the aerofoil being in the no-lift position, it follows from (37), (38) and (39) that $a_1 = 0$, i.e.,

$$\int_0^{2\pi} \theta \cos \gamma \, d\gamma + \int_{-\infty}^{\sigma} K \cosh \eta \, d\eta = 0 \,, \qquad \ldots(41)$$

and

$$D = -\tfrac{1}{2}\rho c U^2 \int_0^{2\pi} \theta \sin \gamma \, d\gamma \,.$$

Hence

$$C_D = -\int_0^{2\pi} \theta \sin \gamma \, d\gamma \,, \qquad \ldots.(42)$$

C_D being the drag coefficient. From (37), (38), (40) and (41) it is found from the theorem on residues that

$$\frac{M}{\tfrac{1}{2}\rho c^2 U^2} = C_{m_0} = \tfrac{1}{4}\left(\int_0^{2\pi} \theta \cos 2\gamma \, d\gamma - \int_{-\infty}^{\sigma} K \cosh 2\eta \, d\eta\right) \,, \qquad \ldots(43)$$

where a second-order term depending on C_Q has been neglected. C_{m_0} is the moment coefficient at zero lift and is therefore independent of the origin of the z-plane. From thin aerofoil theory the moment coefficient about the mid-chord point is

$$C_m = \tfrac{1}{4}C_L + C_{m_0} \,. \qquad \ldots(44)$$

It/

It is convenient in the application of the theory to combine equations (33) and (41) in the form

$$\int_0^{2\pi} \theta\,(\cos y - 1)\,dy + \int_{-\infty}^{\sigma} K\,(\cosh \eta + 1)\,d\eta = 0\,, \qquad \ldots(45)$$

and similarly

$$\int_0^{2\pi} \theta\,(\cos 2y - 1)\,dy - \int_{-\infty}^{\sigma} K\,(\cosh 2\eta - 1)\,d\eta = C_{m_0}\,. \qquad \ldots.(46)$$

Two special cases of the theory are:-

1. Source-type of flow, in which case $K = 0$. Equations (45) and (46) become

$$0 = \int_0^{2\pi} \theta\,(\cos y - 1)\,dy \qquad \ldots(47)$$

$$C_{m_0} = \tfrac{1}{4}\int_0^{2\pi} \theta\,(\cos 2y - 1)\,dy\,, \qquad \ldots(48)$$

while (42) is unchanged.

2. Flow about a closed aerofoil, when $C_Q = 0$. In this case the drag must vanish, i.e., from (42)

$$\int_0^{2\pi} \theta \sin y\,dy = 0\,,$$

the other equations remaining unchanged.

6. The Lift and Moment Due to a Source-Type of Flow

We now return to the particular problem discussed in Section 4, the equations appropriate to which are (47) and (48). The distribution of θ as a function of y is shown in Fig. 5, which should be compared with Figs. 4(a) and 4(d). In arriving at Fig. 5 we have assumed the aerofoil to be essentially a flat plate at an angle of incidence σ_0, with a parallel wall duct making an angle $-\tau$ with the aerofoil chord taking the fluid from within the aerofoil to the trailing edge. Of course exit ducts will not in practice be as simple in shape as the one we have assumed, but the principal feature of a duct as far as the external flow is concerned will be the direction of the flow at its exit, and provided the final section is straight and several times longer than it is wide it is difficult to imagine that our model will introduce significant errors.

From/

From equation (47) and the distribution of θ shown in Fig. 5 we find that, ignoring terms $O(\delta^3)$,

$$\alpha_o = -\frac{2\tau\delta}{\pi} .$$

Hence from (17) and (26) we have that

$$C_L = 2\pi\alpha + 4\tau\sqrt{\frac{2C_Q}{\pi}} . \qquad \ldots(49)$$

Similarly from (33) (in which $K = 0$) and (46) it follows that

$$C_{m_o} = -\tau\sqrt{\frac{2C_Q}{\pi}} ,$$

and hence from (44) and (49)

$$C_m = \frac{\pi}{2}\alpha . \qquad \ldots(50)$$

Hence C_m is independent of C_Q, and the force due to the source-type stream leaving the aerofoil must act at the mid-chord point.

This now completes the theory of the source-type flow+. The theory was developed for two reasons. First and foremost it is a special case of the more general jet-type flow, so the solution for this latter case must degenerate to the one found above when $V = U$, or from (13) and (14), when $C_J = 2C_Q$. Second, it is an exact solution as far as first-order terms are concerned - this is not necessarily true of the solution given below for the jet-type flow - and therefore has an intrinsic value.

7. The Basic Assumptions in the Theory for Jet-Type Flow

As in Section 3 it will be assumed that the jet does not mix with the main flow, and that Bernoulli's theorem applies in the jet. Then, since the average pressure taken across the jet will not vary to any degree with distance along the jet (except possibly near the jet exit), it follows that the velocity V, and hence the coefficient C_J, will remain essentially constant along the jet. In any case it will be assumed that C_J is constant from the jet exit to the point at infinity.

Suppose now we have a number of thin jets entering uniform streams at the same angle τ. It is not unreasonable to say that their effects on the main stream will depend essentially on their momentum

coefficients/

+See Ref. 10 (written two years after the account given above) for an exact treatment of source-type flow.

coefficients, C_J, and that if two thin jets have the same momentum coefficients they will have approximately the same overall effect on the flow pattern. Now for source-type flow the momentum coefficient is equal to $2C_Q$, so that a jet with a momentum coefficient of C_J can be said to be equivalent in its influence on the main stream to a source-type of jet the mass coefficient of which satisfies

$$2C_Q = C_J .$$

Thus from the simple rule embodied in this equation and equation (26) it follows that the number δ appropriate to the jet-type of flow is given by

$$\delta = \sqrt{\left(\frac{C_J}{\pi}\right)} . \qquad \ldots(51)$$

The equation, $2C_Q = C_J$, is the basic assumption in the present method of dealing with jet-type flows. The author has been unable to find a theoretical argument for it, but the physical argument given above seems quite plausible. It is strengthened by the fact that if the rule is used to deduce the thrust of a jet-type of flow from that of a source-type flow the exact answer is obtained.

The theory given in Section 6 for source-type flow simply takes into account the direct effect of the aerofoil shape on the lift and moment. No vortex sheets occur and therefore the second integrals in (45) and (46) vanish. With jet-type flow not only does the aerofoil shape "directly" affect the lift and moment but it also has an "indirect" effect through the two vortex sheets extending behind the aerofoil. The induced velocities on the aerofoil surface due to these vortex sheets largely cancel out since the algebraic sum of the strengths of the sheets is quite small. However the resultant velocity distribution is still large enough to contribute substantially to the lift and moment forces. This problem is considered in the next Section.

8. The Integral Equation for the Vorticity in the Jet

In order to calculate C_{m_0} and C_L in the case of jet-type flow it follows from equations (43) and (45) that we must first calculate the function $K(\eta)$ - a function which from its definition (30) is clearly proportional to the strength of the vortex sheet representing the jet. The first step is to obtain a relation between the curvature of the jet, its momentum, and the sum of the strengths of the two vortex sheets separating it from the main flow.

Consider conditions at a section EF of the jet bounded by the vortex sheets AB and CD shown in Fig. 6. Jet stream values will be distinguished by a suffix J. Continuity of pressure across the vortex sheet AB yields

$$A_1 + \tfrac{1}{2}\rho q_E^2 = A_2 + \tfrac{1}{2}\rho q_{JE}^2 ,$$

where A_1, A_2 are constants, and q_E, q_{EJ} are the velocities at E just outside and inside the jet respectively. Similarly at F

A_1/

$$A_1 + \tfrac{1}{2}\rho q_F^2 = A_2 + \tfrac{1}{2}\rho q_{JF}^2 ,$$

and hence by subtraction

$$(q_E + q_F)\,(q_E - q_F) = (q_{JE} + q_{JF})\,(q_{JE} - q_{JF}) .$$

we make a slight approximation in this equation to find

$$U(q_E - q_F) = V(q_{JE} - q_{JF}) , \qquad \ldots(52)$$

where V is the mean jet velocity at the section EF.

If n and s are distances measured normal to and along a streamline of the jet respectively, the absence of vorticity within the jet yields

$$\frac{\partial q}{\partial n} = \frac{q}{R} ,$$

where R is the radius of curvature of the streamline. Applying this equation to the mid-streamline of the jet we have approximately

$$q_{JE} - q_{JF} = \frac{hV}{R} , \qquad \ldots(53)$$

where H is the width of the jet at EF.

Now the sum of the strengths of the vortex sheets AB and CD is

$$\Gamma = (q_E - q_{JE}) + (q_{JF} - q_F)$$

$$= (q_E - q_F) - (q_{JE} - q_{JF}) ,$$

and hence from (52) and (53)

$$\Gamma = \frac{hV}{R}\left(\frac{V}{U} - 1\right) . \qquad \ldots(54)$$

The sum of the jumps in $\log \frac{q}{U}$ across the vortex sheets, namely K (see (30)) follows immediately from (54). To first order in K we find that

$$K = \frac{\Gamma}{U} = \frac{hV}{cU}\left(\frac{V}{U} - 1\right)\frac{c}{R} .$$

Hence/

Hence from (13) and (14)

$$K = (C_J - 2C_Q)\,\frac{c}{2R}\,, \qquad \ldots(55)$$

which we note satisfies the condition that K must vanish for source-type flow. To calculate K we thus need to know the shape of the jet. This can be calculated from equation (32).

The middle streamline of the jet is defined by $\zeta = \eta + i\pi$ in the ζ-plane introduced in Section 4. On this streamline $\phi \approx Us$, and ignoring a second-order term due to the source strength we use (28) to find

$$\frac{s}{c} = \tfrac{1}{2}\cosh\eta\,. \qquad \ldots(56)$$

Also on $\zeta = \eta + i\pi$ equation (32) yields[+]

$$\theta = -\frac{\sinh\eta}{2\pi}\left\{\int_0^{2\pi}\frac{\theta(y^*)\,dy^*}{\cos y^* + \cosh\eta} + \int_{-\infty}^{\sigma}\frac{K(\eta^*)\,d\eta^*}{\cosh\eta^* - \cosh\eta}\right\}. \qquad \ldots(57)$$

From (56) and (57) and an integration by parts it is found that

$$\frac{c}{2R} = \frac{c}{2}\frac{\partial\theta}{\partial s} = \frac{1}{2\pi\sinh\eta}\left\{\int_0^{2\pi}\frac{\sin y^*\,d\theta\,(y^*)}{\cos y^* + \cosh\eta} - \int_{-\infty}^{\sigma}K(\eta^*)\,\frac{\cosh\eta\cosh\eta^* - 1}{(\cosh\eta - \cosh\eta^*)^2}\,d\eta^*\right\}.$$

The integral equation for K now follows from (55). It is

$$K = \frac{(C_J - 2C_Q)}{2\pi\sinh\eta}\left\{\int_0^{2\pi}\frac{\sin y^*\,d\theta(y^*)}{\cos y^* + \cosh\eta} - \int_{-\infty}^{\sigma}K(\eta^*)\,\frac{\cosh\eta\cosh\eta^* - 1}{(\cosh\eta - \cosh\eta^*)^2}\,d\eta^*\right\}. \qquad \ldots(58)$$

This integral equation is not one of the standard types, but if $(C_J - 2C_Q)^2$ is small compared with $(C_J - 2C_Q)^2$ it can be solved by iteration. On this assumption the first approximation is

$$K = \frac{(C_J - 2C_Q)}{2\pi\sinh\eta}\int_0^{2\pi}\frac{\sin y^*\,d\theta\,(y^*)}{\cos y^* + \cosh\eta}\,, \qquad \ldots(59)$$

which if substituted in the last integral in (58) should yield a more accurate value of K and so on. However in view of the other approximations made in this paper we shall be content with the approximation (59). To use

equation/

[+]All improper integrals occurring in this report are to be given their "principal values"[8].

equation (59) is essentially to assume that the jet lies along the position that would be adopted by a source-type flow, for which (since we will use equation (51) in evaluating (59) the momentum coefficient equals C_J. This approximation is then, in a sense, consistent with the argument given in Section 7.

Evaluating the Stieltjes integral in (59) by substituting in the discontinuities in $\theta(y^*)$ shown in Fig. 5, we arrive at

$$K = \frac{(C_J - 2C_Q)}{2\pi \sinh \eta} \left\{ \frac{\pi \sin y_0}{\cos y_0 + \cosh \eta} + \frac{2\tau \sin \delta}{\cos \delta - \cosh \eta} \right\}, \qquad \ldots (60)$$

in which δ is given by equation (50).

9. The Lift and Moment Due to a Jet-Type of Flow

From Fig. 5, equations (45) and (60) we deduce that

$$2\pi\alpha_0 + 4\tau\delta + \frac{(C_J - 2C_Q)}{2\pi} \left\{ 2\tau \cot \frac{\delta}{2} \log \left(\frac{\cosh \sigma - \cos \delta}{\cosh \sigma - 1} \right) \right.$$

$$\left. + \pi \tan \tfrac{1}{2} y_0 \times \log \left(\frac{\cosh \sigma - 1}{\cosh \sigma + \cos y_0} \right) \right\} = 0 . \qquad \ldots (61)$$

From (27) and (51) we find that δ and σ are small first-order numbers. From (33) it is found that the same is true of y_0. Thus retaining only the highest order terms in (61):-

$$2\pi\alpha_0 + 4\tau\delta + \frac{2(C_J - 2C_Q)}{\pi\delta} \tau \log \left(\frac{\sigma^2 + \delta^2}{\sigma^2} \right) = 0 .$$

Hence from (17), (27) and (50)

$$C_L = 2\pi\alpha + \frac{2\tau}{\sqrt{\pi}} \frac{(C_J - 2C_Q)}{\sqrt{C_J}} \log (4.58) + \frac{4\tau}{\sqrt{\pi}} \sqrt{C_J} .$$

Therefore

$$C_L = 2\pi\alpha + \frac{4\tau}{\sqrt{\pi}} \sqrt{C_J} \left\{ 1 + 0.76 \left(1 - \frac{2C_Q}{C_J} \right) \right\} . \qquad \ldots (62)$$

In attempting to determine the moment coefficient from equations (46) and (60) we have a difficulty which arises in a similar manner in the theory of an aerofoil in harmonic motion. It is that the integral along the vortex sheet, namely

$\lim_{R=\infty}$ /

$$\lim_{R=\infty} \int_{-R}^{\sigma} K(\cosh 2\eta - 1)\, d\eta \,, \qquad \ldots(63)$$

is divergent, becoming logarithmical infinite in the limit. As in unsteady aerofoil theory this infinity arises from the mathematical simplifications introduced at various points in the theory, and does not have any real physical significance. In an exact theory the logarithmic infinities would cancel out since the sum of the coefficients of the logarithmic terms arising in the integration are identically equal to zero. We assume this is the case here, and consider only the finite part of the integral (63). We then find that this finite part is a term of second order, which can be ignored in comparison with the term arising from the first integral of equation (44). This latter integral has already been evaluated in Section 6. It was found that $C_{m_0} = -\delta\tau$, and hence from equation (50)

$$C_{m_0} = -\tau\sqrt{\frac{C_J}{\pi}} \,. \qquad \ldots(64)$$

From (44), (62) and (64)

$$C_m = \frac{\pi\alpha}{2} + \frac{0.76}{\sqrt{\pi}}\,\tau\left(1 - \frac{2C_Q}{C_J}\right). \qquad \ldots(65)$$

Finally, in the case $\alpha = 0$, we deduce from (65) and (62) that the centre of pressure must be a distance

$$\frac{\bar{x}}{c} = 0.19\,\frac{(1 - 2C_Q/C_J)}{1 + 0.76\,(1 - 2C_Q/C_J)} \,, \qquad \ldots(66)$$

forward of the mid-chord point.

One final comment on the theory remains to be made here. It is that equations (15), (62) and (64) include as special cases the exact solutions for source-type flow given by (16), (49) and (50) respectively.

10. Comparison with Experiment

As mentioned in the Introduction there is at the moment a lack of experimental data on the phenomena; however a little work on the subject is reported in Refs. 1 and 2.

In Figs. 7, 8 and 9 theoretical and experimental values of C_L are plotted against C_J. From (13) and (14) it is found that

$$C_Q = \sqrt{\frac{h}{2c}}\,\sqrt{C_J} \qquad \ldots(67)$$

In/

In the experiments reported in Ref. 1, $\tau = 55\frac{1}{2}°$, h = 0.0095" and c = 5.5"; hence $2C_Q = 0.059\sqrt{C_J}$. Equation (62) yields for this case

$$C_L = 3.85\sqrt{C_J} - 0.10 ,$$

which is the curve drawn in Fig. 7. The agreement is quite satisfactory considering the approximations made in the theory and the acknowledged crudity of the experiments.

For the experiments of Ref. 2, $\tau = 90°$, H = 0.018 and c = 8". Hence from (62) and (67)

$$C_L = 6.23\sqrt{C_J} - 0.18 ,$$

which is compared in Fig. 8 with the experimental values. The agreement is not so good in this case, but this could hardly be expected with such a large ejection angle. It is difficult to believe that there is not some loss in circulation due to separation of the flow at the trailing edge. A further possible cause of the discrepancy between experiment and theory is that the angle τ was not actually measured in the experiments, but deduced to be 90° from a specious argument based on the theory of Ref. 1. Even if the argument given is correct, on the accuracy of the figures given (one decimal place) it is only possible to deduce that $76° \leqslant \tau \leqslant 104°$.

In Ref. 2 it is found experimentally that $C_{m_0} \approx -0.14 \times 2\pi\sqrt{C_J}$. From (64) we find for this case that $C_{m_0} = -\frac{2\pi}{4}\sqrt{C_J}/\pi = -0.141 \times 2\pi\sqrt{C_J}$, so there is good agreement for this coefficient.

Finally in Fig. 9 we have compared theory with some further N.G.T.E. experimental results (as yet unpublished). The agreement in $0 < C_J < 0.25$ could hardly be improved. There are probably four reasons for this success, namely (i) it appears these last experiments have been carried out much more carefully than the earlier ones, (ii) the value of τ (31.4°) was reasonably low, and the ideal flow - which is the basis of the theory - is more closely achieved, (iii) at high values of C_J turbulent mixing will be an important factor, and (iv) in any case the theory is only developed to first order in $\sqrt{C_J}$.

11. Final Comments

The "thrust hypothesis" of Ref. 1 has been rigorously established for compressible flow. A first-order theory for the lift in incompressible flow has been developed, which is in fair agreement with the few experimental results available. The lift theory is deduced from the exact theory for source-type flow by a plausible argument, although more consideration is desirable here. The theory for pitching moments is unfortunately vitiated by an inadequate explanation of a logarithmic singularity arising in the mathematics, and the author hopes to be able to investigate this further at a later date.

It is perhaps worth reporting that closer agreement with experiment can be achieved by allowing $C_J - 2C_Q$ to vary with distance (s) along the jet. The velocity of the jet is gradually reduced by viscosity and turbulence so that as $s \to \infty$, $V \to U$, and from (13), (14)

and/

and (55), K -> 0. Now for a jet discharged into still fluid the maximum velocity of the jet is known[7] to be inversely proportional to $\sqrt{s}$ where s is measured from some suitable origin. If we assume that this law is applicable to the velocity V - U of the present problem, then we find that

$$C_J - 2C_Q \propto \frac{(C_{J_o} - 2C_Q)}{\sqrt{s}}, \qquad \ldots(68)$$

where C_{J_o} is the value of C_J at the jet exit. (C_Q is constant along the jet by continuity.) The use of (68) in equation (55) leads to a modification of equation (60) and hence to a change in the coefficient of $(1 - 2C_Q/C_J)$ in equation (62). It was found that this coefficient was reduced, and that the amount of reduction depended on the origin selected for s. As this origin was varied from being some distance from the jet exit to very close to the exit the coefficient in question varied from 0.76 to zero. However the method is rather doubtful theoretically, as the use of (68) implies that Bernoulli's equation is not satisfied in the jet, which is incompatible with the derivation of (55).

The pressure distribution over the aerofoil has not been discussed in the report, but if it is required it can be readily deduced from equation (29). We find that putting $\eta = 0$ and integrating by parts

$$\log(1 - C_p) = \frac{2}{\pi}\int_0^{2\pi} \log \sin \tfrac{1}{2}(\gamma^* - \gamma)\, d\theta(\gamma^*) + \frac{\sin\gamma}{\pi}\int_{-\infty}^{\sigma} \frac{K(\eta)\, d\eta}{\cosh\eta + \cos\gamma} \qquad .(69)$$

where Cp is the pressure coefficient, $1 - (q/U)^2$. The Stieltjes integral can be evaluated immediately from Fig. 5 and equation (51), while the second integral can be calculated from equations (60), (27) and (51).

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APPENDIX I

16th April, 1955

The following Appendix clarifies some of the physical assumptions made above, about which some misunderstandings have arisen. The distinction between source-type flow and jet-type flow is clarified, and the model of the latter type of flow used above is discussed in some detail.

Some comments are also made on other theories of the jet flap.

1. Basic Assumptions made in Paper

1.1 Source-Type Flow

Source-type flow can be defined as flow in which a source on or within an aerofoil produces fluid at the same total head as the main stream fluid.

Consider the flow shown in Fig. 10. A source at E emits fluid which flows down the duct EFC and then into the main stream. Stagnation points will occur at points D and D', and streamlines DG_{∞}, $D'G'_{\infty}$ will separate the source fluid from the main stream. Continuity of pressure across $D'G'_{\infty}$, DG_{∞} implies continuity of velocity across these streamlines provided we have source-type flow. If the total head of the source fluid is not the same as that of the main stream fluid, only the pressure will be continuous across DG_{∞}, $D'G'_{\infty}$ and these streamlines will now be vortex sheets.

Returning to the case of source-type flow we notice that the positions of three stagnation points have to be determined to define a unique flow. The third stagnation point is at B, near the leading edge. If we first assume that D coincides with the trailing edge F (Joukowski's condition), then the positions of the other two stagnation points follow from (1) the value of the circulation, and (2) the source strength. At a given value of the circulation there is just one source strength that makes D' coincide with C at the duct exit, and this produces the flow pattern shown in Fig. 11.

The flow pattern of Fig. 10 is clearly physically unrealistic - separation would occur at points C and F - but that of Fig. 11 could be achieved in practice. For the flow shown in Fig. 11 any change in incidence would require a change in source strength to maintain the stagnation point D' at C; but this can be shown[10] to be a second order effect, which for small values of the source mass coefficient, C_Q, and the incidence α can be ignored.

Now it is possible[10] to develop an exact theory of source-type flows which allows for the duct shape and is independent of the magnitudes of C_Q and α. However as this exact theory takes many pages to develop rigorously the theory given in §6 was an approximate one, valid only to first order in C_Q and α. (This restriction is made clear after equation (17) of §6.)

To this order of accuracy it is shown in §6 that for the type of flow shown in Fig. 2,

$$C_L = 2\pi\alpha + 4\tau\sqrt{\frac{2C_Q}{\pi}}, \qquad \ldots(1)$$

where/

where τ is the angle a parallel walled duct makes with the chord line. The mass coefficient in (1) is the usual one defined by

$$C_Q = \frac{Q}{Uc} , \qquad ...(2)$$

where Q is the mass output of the source per second, U is the undisturbed main stream velocity, and c is the aerofoil chord. It is also shown in §6 that if C_m is the moment coefficient about the midchord point,

$$C_m = \frac{\pi}{2} \alpha . \qquad ...(3)$$

Equations (1) and (3) ignore aerofoil thickness effects.

These equations were derived by an application of Blasius' theorem, a procedure which has recently been claimed "invalid"[9]. There is no doubt at all that Blasius' theorem can be applied to problems in which sources or sinks occur on or near aerofoils (see, for example, p. 213 of Ref. 11). The method used in §6 may not be adequately explained. It is as follows.

First the function (dw/dz), where w is the complex stream function, is (correctly) expanded in the neighbourhood of infinity. The expansion is expressed in terms of the variable ζ, where ζ is defined by equations (19) to (21). Then the corresponding expansion of the relation between the ζ and w-planes is written down, essentially ignoring the terms in C_Q, and likewise for the relation between the w- and z-planes. This permits (dw/dz) to be expanded near infinity in the z-plane (not given in §6), and Blasius' theorem applied. Thus the method is equivalent to ignoring the effect of the source on the relation between the ζ and z-planes, but taking it into account in its effect on (dw/dz).

This is a common type of approximation in aerodynamics. For instance in the theory of flaps the effect of the flap on the velocity distribution is correctly calculated, but its effect on the mapping of the w-plane on to the z-plane is ignored, and this is known to result in error terms of only second order in flap deflection angle.

In any case the method used in §6 is fully justified by the fuller and much longer analysis given in Ref. 10, where it is shown that the relation between z and ζ near infinity is exactly of the form

$$e^{\zeta} = -\frac{aP}{Uz} \{1 + O(1/z^2)\} , \qquad ...(4)$$

where in the notation of §6, $P = \left(1 - \frac{Q}{4\pi a} \log t_o\right) \Big/ (1 - t_o^2)$.

With the aid of (4) instead of (35) of §6, Blasius' theorem can be applied to the problem without the approximations mentioned above.

1.2/

1.2 Jet-Type Flow

In jet-type flow the fluid produced by the source is at a different total head from that of the main stream flow - usually a much higher total head. In this type of flow, although the pressure must still be continuous across the surfaces FH_{∞}, CH'_{∞} (see Fig. 11), this is no longer true for the velocity, and so FH_{∞}, CH'_{∞} are now vortex sheets. The principal difference between the source-type flow and jet-type flow lies in the presence of these vortex sheets in the latter case.

There is another difference between the two flows which occurs right at the jet exit, and is worth noting. In source-type flow the tangents to the dividing streamlines FH_{∞}, CH'_{∞} at the separation points bisect the angles of the jet exit, as shown in Fig. 12. On the other hand in jet-type flow the different stagnation pressures in the two streams implies that there cannot be stagnation points in both streams at C and F, and hence the flow must separate tangentially to that surface on which the total head is greatest. In Fig. 12 it is assumed that the jet is at a greater total head than the main stream, and so the jet flow is tangential to EF and EC. In general the streamline curvature at the separation points will be infinite.

The author is of the opinion that the effects of this discontinuity in behaviour between the two types of flow at the jet exit are very localized, and play an insignificant part in determining the main characteristics of the flow. It is certainly difficult to believe that the lift and moment coefficients for jet-type flow do not tend continuously to their values given in (1) and (3) for source-type flow, as the total head of the jet tends to that of the main stream. This belief is tacitly assumed in §8.

Now the really significant difference between the two types of flow lies in the vortex sheets. These sheets exist simply because the total head of the jet differs from that of the main stream. In fact it is shown in §8 that at any point along the jet the algebraic sum of the strengths of the two sheets is given by

$$\Gamma = U\,(C_J - 2C_Q)\,\frac{c}{2R}, \qquad \ldots(5)$$

in which C_J is the moment coefficient

$$C_J \equiv \frac{2QV}{cU^2} = 2\,\frac{V}{U}\,C_Q, \qquad \ldots(6)$$

V is the velocity in the jet at infinity, and R is the mean radius of curvature of the jet. In the derivation of (5) it is assumed that C_J is constant along the jet, and that the jet width is small compared with R. For source-type flow, V = U,

$$C_J = 2\,C_Q, \qquad \ldots(7)$$

and hence from (5), $\Gamma = 0$.

Now/

Now to represent fully the effects of the vortex sheets we need to consider not only their algebraic sum but also their possible doublet contribution. This point is emphasized in Ref. 12, and apparently overlooked in §8. However even if the doublet contribution significantly affects the velocity distribution, this contribution will be very nearly symmetrical about the chord line, especially for small values of τ, and hence the effect on the lift and moment forces will largely cancel out.

The author's method of deducing the lift force in jet-type flow is based on one further assumption, namely that C_J and not C_Q is the significant parameter for such flows, and this seems to be generally accepted in Refs. 9 and 12. This means that, considering source-type flow as a special case of jet-type flow, we should write (1) in the form

$$C_L = 2\pi\alpha + 4\tau\sqrt{\frac{\bar{C}_J}{\pi}}\,, \qquad \ldots(8)$$

on making use of (7).

We start with this result for source-type flow. Suppose now the total head of the jet is increased above that of the main stream, C_J being kept constant, then jet-type flow results. By (5) vortex sheets now appear giving rise to an <u>additional</u> contribution to C_L. Hence instead of (8) we have

$$C_L = 2\pi\alpha + 4\tau\sqrt{\frac{\bar{C}_J}{\pi}} + f\,(C_J - 2C_Q)\,, \qquad \ldots(9)$$

where f is a function which remains to be determined. In §9 it was calculated to first order in $(C_J - 2C_Q)$ on the assumption that the jet shape remained unchanged during the increase of the total head of the jet. The result is

$$C_L = 2\pi\alpha + 4\tau\sqrt{\frac{\bar{C}_J}{\pi}}\left\{1 + 0.76\left(1 - \frac{2C_Q}{C_J}\right)\right\}. \qquad \ldots(10)$$

The method of deriving (10) shows that it can only be valid for small values of $(C_J - 2C_Q)$, the actual range of validity being best determined by comparison with experiment. Certainly (10) is exact in the limit $C_J \to 2C_Q$.

2. <u>Other Theories of the Jet-Flap</u>

2.1 <u>The Mechanical Flap Analogy</u>

The author criticised the earlier presentation of the flap-analogy theory in §1, and now feels that this criticism although philosophically correct, does not do justice to the theory.

It will be recalled that the method is based on "similarity" being defined between the jet-flap and a mechanical flap in such a way that the lifts on the aerofoil and jet are made equal to the lifts on the fixed part of the flapped aerofoil and flap respectively. On this basis it is established that C_L can be written in the form

C_L/

$$C_L = \frac{\eta}{\tau} F (C_J, \tau) , \qquad \ldots(11)$$

where F is a known function, and η is that value of the deflection of the mechanical flap necessary to produce the imposed similarity. As pointed out in §1, (11) merely shifts the problem from that of determining C_L as a function of C_J and τ to that of determining η/τ as a function of C_J and τ.

However, from physical considerations, it seems reasonable to suppose that η/τ is only a slowly varying function of C_J and τ, and so (11) does simplify the problem in practice. The significant feature of the flap-analogy is to transform the problem from that of determining a rapidly varying function to that of determining a slowly varying one. If η/τ can be approximated to by a constant over a reasonable range of values of C_J and τ, then the method depends only on a single experiment to determine the value of η/τ, and so can be termed a "semi-empirical" theory.

2.2 The Theory of Reference 12

A rough draft of Ref. 12 was recently made available to the author. The method is to collapse the jet into a thin sheet by taking the limit $h \to 0$, $V \to \infty$, such that C_J remains finite. Then, on the assumption that τ is small, the problem is linearized by making the usual thin aerofoil approximations. This leads to an integral equation for the downwash on the jet, which appears to be related to that found in §8 for the strength of the vorticity in the jet. The method is much more direct than that given by the author, and when the basic integral equation has been solved, should give results valid for quite large values of C_J+.

Two remarks on the limitations of this theory are worth making. firstly the limit $h \to 0$ used in the theory invalidates it for small values of $(C_J - 2\, C_Q)$. Secondly the author found that in his approach to the problem this limit created mathematical difficulties at the jet exit that he was unable to surmount. The limit $h \to 0$ may be satisfactory well away from the trailing edge, but it seems to be an over-simplification in the neighbourhood of the trailing edge - especially as the answer depends so critically on the character of the flow at the trailing edge. It is to be hoped that this view is wrong[x].

2.3 The Theory of Reference 9

This theory closely follows that given by the author, except that the lift induced by the vortex sheets - the term $f(C_J - 2C_Q)$ of equation (9) - is calculated by an alternative method. This alternative method is to use the well-known theorem that the lift on an aerofoil behind which extends a vortex sheet is equal to $\rho U\Gamma$, where Γ is the total circulation around both the aerofoil and the sheet. Then with the aid of the author's result for the strength of the vortex sheet (equation (5)) it is deduced that the direct contribution to C_L from the circulation about the sheet alone is $(C_J - 2C_Q)\tau$. Although the author of Ref. 9 apparently appreciates that the vortex sheet also has an indirect contribution to make to C_L - by modifying the

circulation/

+It subsequently proved that Spence's integral equation (when transformed) was in essence identical to that given in §8 (see Appendix II), and consequently this last remark is not true.

x This hope was apprently justified.

circulation about the aerofoil - this is ignored and $(C_J - 2C_Q)\tau$ is taken to be the total contribution from the sheet, that is, it is identified as the function f in (9). The function f, of course, represents the effect of the sheet on the total circulation about both aerofoil and sheet. The method of calculating it given in §8, whatever its deficiencies is sound on this point.

A further curious anomaly to be found in Ref. 9 is the acceptance of the term $4\tau\sqrt{C_J}/\pi$ from §9 despite the criticism made in the same paper of its derivation from Blasius' theorem. No other derivation of this term has been given.

Another criticism made in §2 of Ref. 9 of the author's theory is on the question of whether to take the pressure continuous or discontinuous across the vortex sheet. Both methods are acceptable, but with the particular model adopted in §8 it was definitely preferable to take the pressure as being continuous - the author wished to avoid the awkward limit h -> 0 (see the second paragraph of §2.2 above).

APPENDIX II/

APPENDIX II

19th November, 1956

In a recent paper[13] Spence derived an integro-differential equation for the slope of a high speed sheet of air emerging from the trailing edge of an aerofoil. In this Appendix it is shown that this equation is identical in form with one given by the author in §8, a point which was overlooked by Spence. The author's derivation is more direct, being based from the start on a well-known solution possessing the appropriate type of mixed boundary conditions.

1. Spence's Method

Spence linearizes the problem from the start by applying the boundary conditions on $y = 0$, with the aerofoil in $0 \leq x < 1$, and the wake in $1 < x < \infty$. The strength of the vorticity in the wake, Γ, is (see §8)

$$\Gamma = \frac{Uc}{2R}(C_J - 2C_Q) \quad (1 < x) , \qquad \ldots(1)$$

where U is the undisturbed main stream velocity, c is the aerofoil chord (unity in the present case), R is the radius of curvature of the jet, C_Q is the mass coefficient, $C_Q \equiv$ mass flow in the jet/Uc, and C_J is the momentum coefficient, $C_J \equiv$ momentum in the jet/$\frac{1}{2}\rho cU^2$, where ρ is the density. In order to avoid the difficulties of the non-homogeneous flow Spence allows the jet width to tend to zero in such a way that C_J remains finite, but C_Q tends to zero. This gives

$$\Gamma = \tfrac{1}{2}UC_J \frac{\partial\theta_1}{\partial x} , \quad (1 < x < \infty) , \qquad \ldots(2)$$

where θ_1 is the slope of the jet.

Let the vorticity strength on the aerofoil be $Uf(x)$, then the downwash equation is

$$w(x) = -\frac{U}{2\pi}\int_0^1 \frac{f(\xi)}{\xi - x}\,d\xi - \frac{UC_J}{4\pi}\int_1^\infty \frac{\partial\theta_1/\partial\xi}{\xi - x}\,d\xi . \qquad \ldots(3)$$

If θ_a is the (known) average slope of the aerofoil surface, then $w(x) \approx -U\theta_a$ in $0 < x < 1$, and $w(x) \approx -U\theta$ in $1 < x < \infty$. Thus we arrive at the pair of simultaneous integro-differential equations

$$-\frac{1}{\pi}\int_0^1 \frac{f(\xi)}{\xi - x}\,d\xi - \frac{C_J}{2\pi}\int_1^\infty \frac{\partial\theta_1/\partial\xi}{\xi - x}\,d\xi \begin{array}{l} = -2\theta_a \quad (0 < x < 1) \\ = -2\theta \quad (1 < x < \infty) \end{array} .$$

Dr.A. E. Billington/

Dr. A. E. Billington of the Aeronautical Research Laboratories, Fisherman's Bend, Melbourne, Australia, reduced this pair of equations in two pages of algebra (too lengthy to give here: see Spence's paper[13]) to the single integro-differential equation

$$\theta_1 = -\frac{1}{\pi}\left(\frac{x-1}{x}\right)^{\frac{1}{2}}\left\{\int_0^1\left(\frac{\xi}{1-\xi}\right)^{\frac{1}{2}}\frac{\theta_a(\xi)}{\xi - x}\,d\xi - \frac{C_J}{4}\int_1^\infty\left(\frac{\xi}{\xi-1}\right)^{\frac{1}{2}}\frac{\partial\theta_1/\partial\xi}{\xi - x}\,d\xi\right\}, \qquad \ldots(4)$$

in which we have made some trifling changes in notation. Spence then solves this equation by an approximate Fourier series method. From this solution the lift, moment and pressure distribution are readily deduced.

2. Woods' Method

A more direct derivation of (4) can be obtained once it is realized that the mixed boundary conditions $-\ \theta_a$ given in $0 < x < 1$, Γ given in $1 < x < \infty$ – are just those occurring in the well-established theory for unsteady aerofoil motion.

Let the aerofoil and wake be transformed into a (η, γ)-plane such that the aerofoil lies on $\eta = 0,\ 0 < \gamma < 2\pi$, and the vortex sheet representing the jet lies on $\gamma = \pi,\ -\infty < \eta < 0$, then it is easily shown that (see Ref. 4)

$$\ln\frac{U}{q} + i\theta = \frac{1}{2\pi}\int_0^{2\pi}\theta_a(\gamma^*)\cot\tfrac{1}{2}(\gamma^* + i\eta - \gamma)\,d\gamma^*$$

$$+\ \frac{i\sinh(\eta + i\gamma)}{2U\pi}\int_{-\infty}^{0}\frac{\Gamma(\eta^*)\,d\eta^*}{\cosh\eta^* + \cosh(\eta + i\gamma)}, \qquad \ldots(5)$$

where (q, θ) is the velocity vector in polar co-ordinates at (η, γ).

The conformal mapping which takes $y = -0,\ 1 \geqslant x \geqslant 0$; $y = +0,\ 0 \leqslant x \leqslant 1$, on to $\eta = 0,\ 0 \leqslant \gamma \leqslant 2\pi$, and $y = 0,\ 1 \leqslant x \leqslant \infty$, on to $\gamma = \pi,\ 0 \geqslant \eta \geqslant -\infty$ is

$$z = x + iy = \tfrac{1}{2}\left(1 - \cosh(\eta + i\gamma)\right), \qquad \ldots(6)$$

although this transformation was not made in the derivation by the author, as it was more convenient to work in the (η, γ)-plane.

On the jet, $\gamma = \pi$, (5) yields

$$\theta_1 = -\frac{\sinh\eta}{2\pi}\left\{\int_0^{2\pi}\frac{\theta_a(\gamma^*)\,d\gamma^*}{\cos\gamma^* + \cosh\eta} + \frac{1}{U}\int_{-\infty}^{0}\frac{\Gamma(\eta^*)\,d\eta^*}{\cosh\eta^* - \cosh\eta}\right\}, \qquad \ldots(7)$$

which combined with (1) immediately yields an integro-differential equation for θ_1. Unable to find an exact solution of this equation, the author solved it approximately, only as far as the first step of the

Liouville-Neumann/

Liouville-Neumann iterative process - a solution valid only for small values of $C_J - 2C_Q$. In addition the theory was slightly complicated by the requirement imposed by the author that the solution remain valid in the limit $C_J \to 2C_Q$. When $C_J = 2C_Q$ we have a homogeneous source-type of flow for which the solution is known exactly[10]. While Spence's method of putting $C_Q = 0$ obviates the difficulty of the non-homogeneous flow, it cannot provide a correct solution for small values of C_J.

3. The Equivalence of the Two Integro-Differential Equations

The equivalence of (4) and (7) is easily shown. First from (5) the assumption that θ vanishes at infinity, i.e., at $\eta = -\infty$, gives

$$\int_0^{2\pi} \theta_a(\gamma^*)\, d\gamma^* - \frac{1}{U}\int_{-\infty}^{0} \Gamma(\eta^*)\, d\eta^* = 0 . \qquad \ldots(8)$$

If (8) is now multiplied by $\sinh\eta/\{2\pi(\cosh\eta + 1)\}$, and added to (7) there results

$$\theta_1 = -\frac{\sinh\eta}{2\pi}\left\{\int_0^{2\pi} \frac{\theta(\gamma^*)\,(1-\cos\gamma^*)d\gamma^*}{(1+\cosh\eta)\,(\cos\gamma^* + \cosh\eta)} + \frac{1}{U}\int_{-\infty}^{0} \frac{\Gamma(\eta^*)\,(1+\cosh\eta^*)d\eta^*}{(1+\cosh\eta)\,(\cosh\eta^* - \cosh\eta)}\right\} . \qquad \ldots(9)$$

On the aerofoil $(\eta = 0)$ and jet $(\gamma = \pi)$ equation (6) gives $x = \frac{1}{2}(1 - \cos\gamma)$, $x = \frac{1}{2}(1 + \cosh\eta)$, by which (9) is transformed into

$$\theta = \frac{1}{\pi}\left(\frac{x-1}{x}\right)^{\frac{1}{2}}\left\{\int_0^1 \frac{\frac{1}{2}\{\theta(\gamma^*) + \theta(2\pi - \gamma^*)\}}{x - x^*}\left(\frac{x^*}{1-x^*}\right)^{\frac{1}{2}}dx^* + \frac{C_J}{4}\int_1^{\infty} \frac{\partial\theta_1/\partial x^*}{x^* - x}\left(\frac{x^*}{x^*-1}\right)^{\frac{1}{2}}dx^*\right\}, \qquad \ldots(10)$$

on eliminating Γ by (2). As $\frac{1}{2}\{\theta(\gamma^*) + \theta(2\pi - \gamma^*)\}$ is the average of the slopes on the upper and lower surfaces of the aerofoil it is immediately obvious that equations (4) and (10) are identical.

This means that equation (4) is simply an alternative form of equation (7). Although the author derived equation (8) in §5, he did not combine it with (7) to produce the form (9), which exactly corresponds with (4). Equation (9) is clumsier than (7), and in the (η,γ)-plane at least has no advantages over (7).

4. Conclusions

The method of dealing with the jet-flap given by Spence[13] leads to exactly the same integro-differential equation for the jet slope (excepting a trifling transformation of variables) as that derived earlier by the author using hodograph methods. The general expression given for the lift by Spence (his equation (105)) is also exactly the same as that derived by the author, but this is not acknowledged.

However Spence has carried the numerical work of solving the integral equation, and deriving values for the lift and moment to a stage which renders obsolete this part of the author's work.

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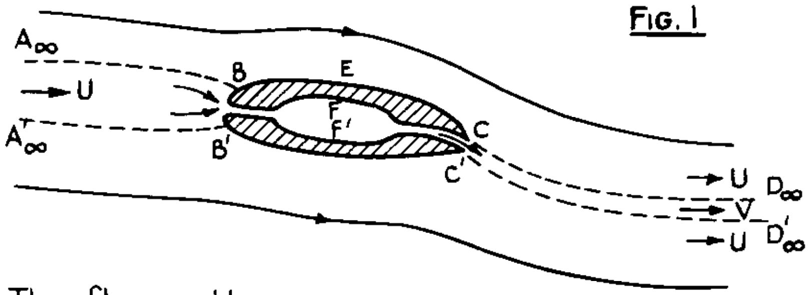
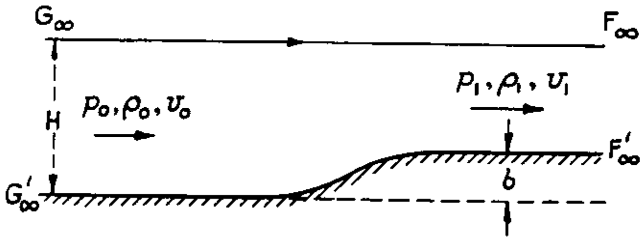
FIGS. 1 - 4

FIG. 1



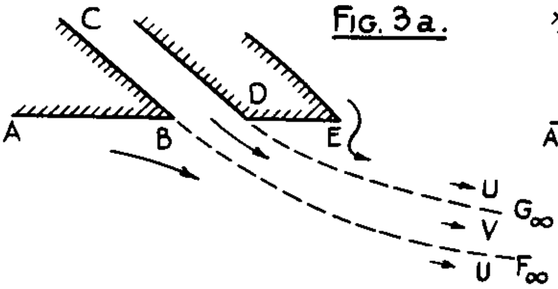
The flow pattern

FIG 2.



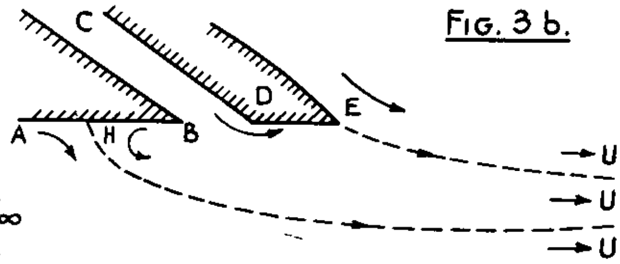
Flow in a channel

FIG. 3 a.



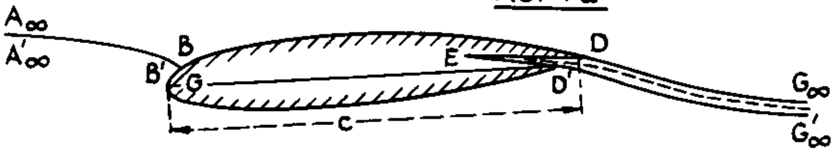
Jet - type flow

FIG. 3 b.



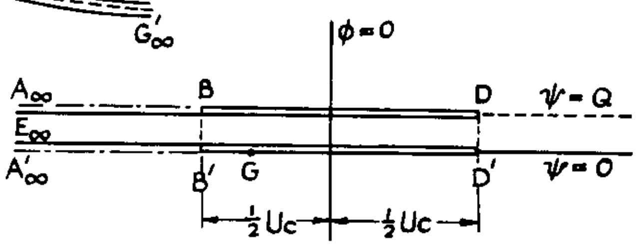
Source - type flow

FIG. 4 a



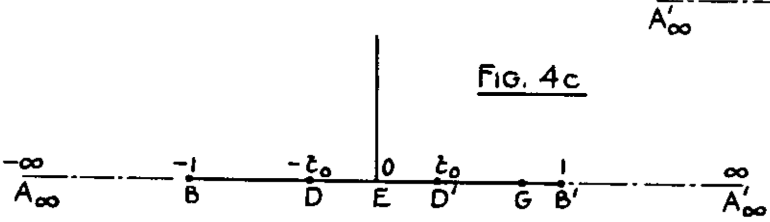
z - plane

FIG. 4 b.



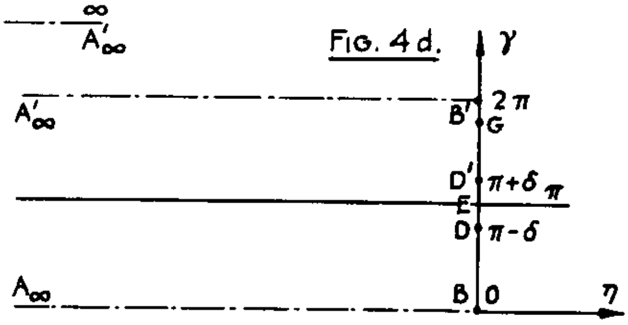
w - plane

FIG. 4c



t - plane

FIG. 4 d.

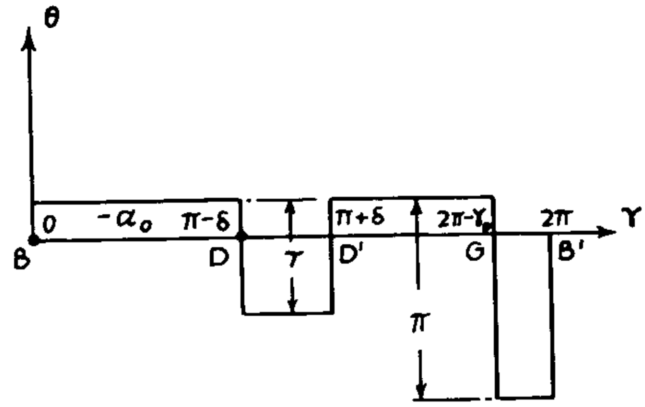


ζ - plane

The conformal transformations

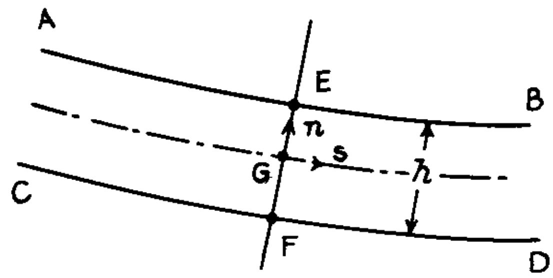
FIGS. 5, 6

FIG. 5.



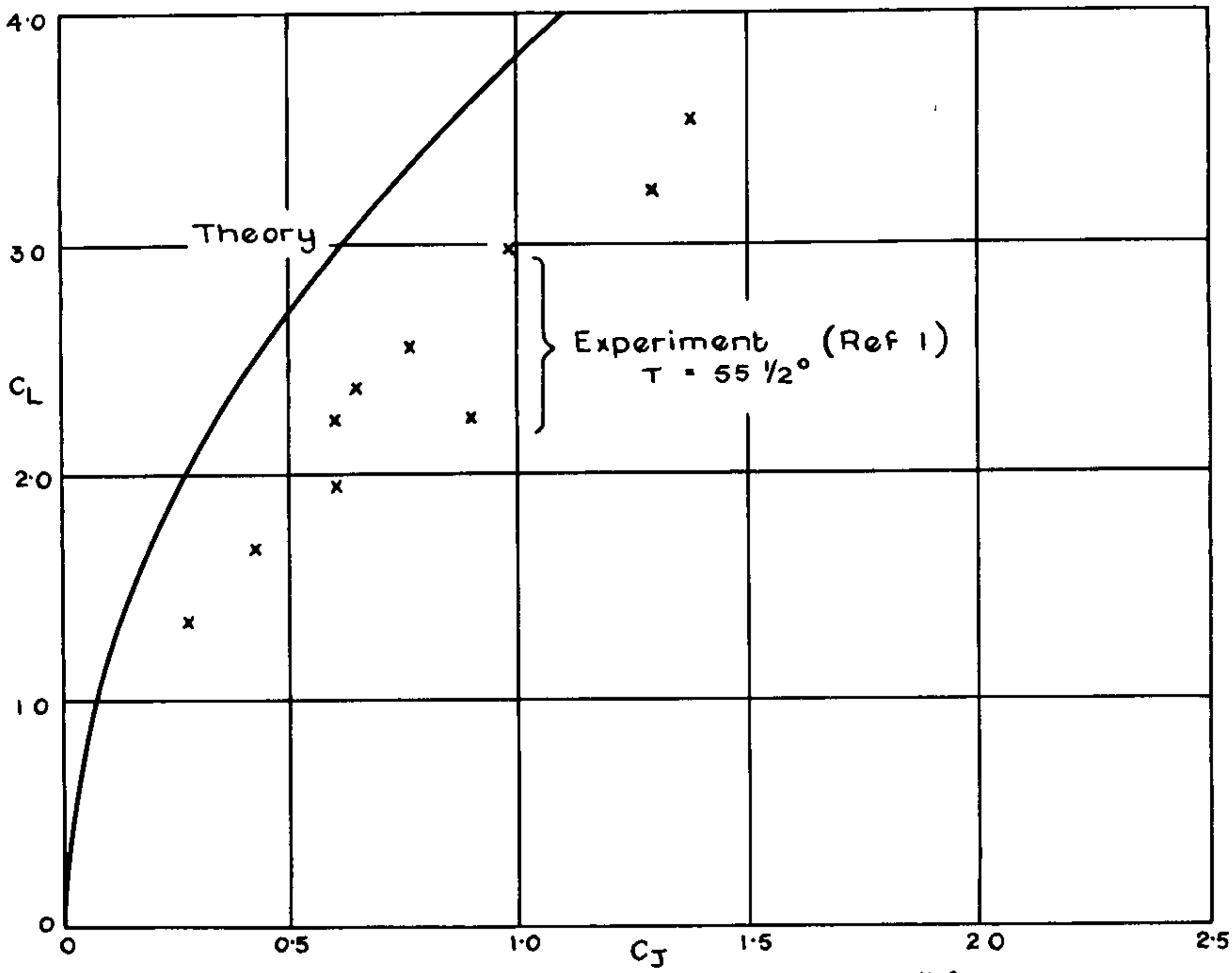
Distribution of θ.

FIG 6.



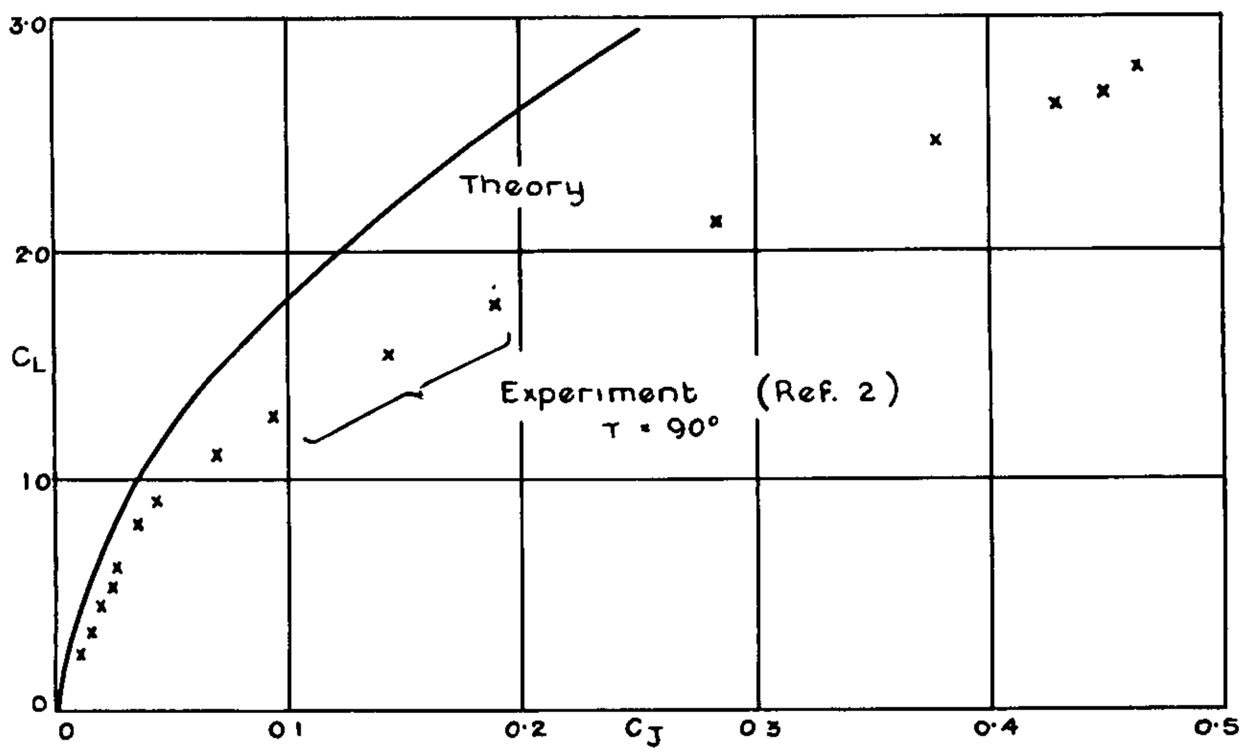
The Jet.

Fig. 7.



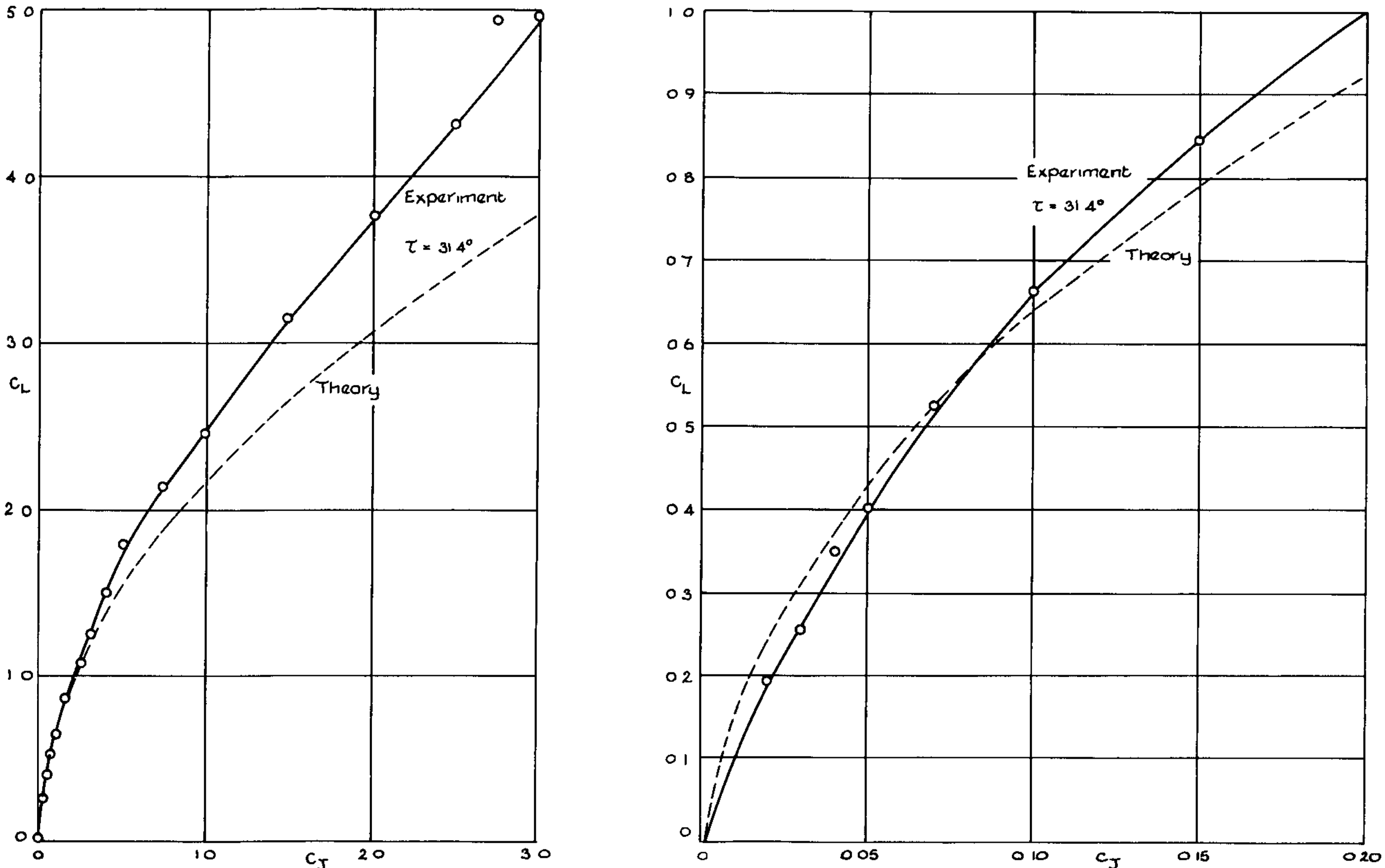
Variation of C_L with C_J, $\tau = 55\frac{1}{2}°$

Fig. 8.



Variation of C_L with C_J, $\tau = 90°$

FIG 9



Variation of C_L with C_J, τ = 31 4°

FIGS 10 - 12.

FIG. 10.

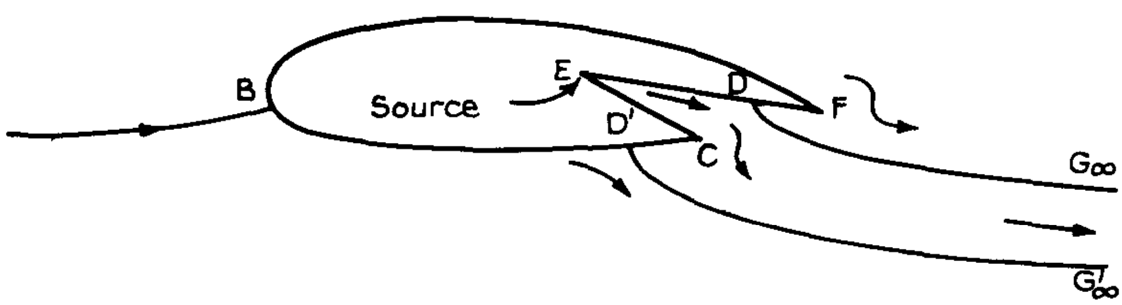


FIG. 11.

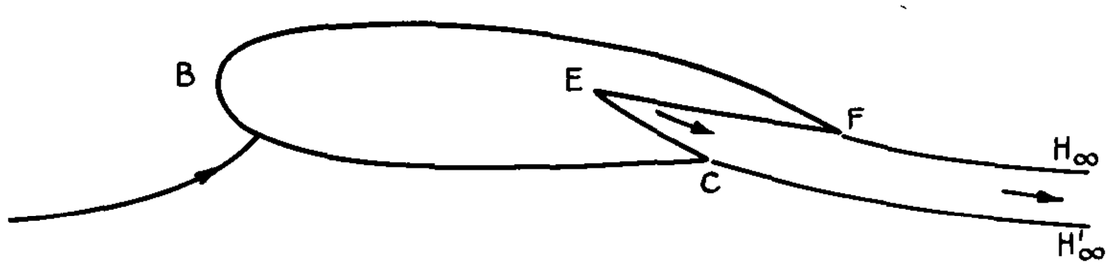
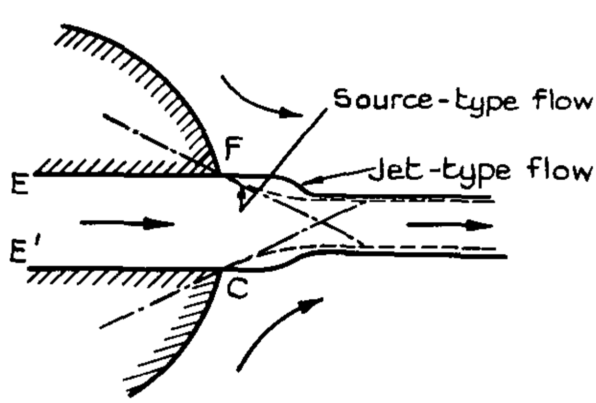


FIG. 12.



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