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Notes on the Analysis of Stability in Accelerated Motion

By S. B. GATES, F.R.S. and A. W. THORPE, B.A.

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Summary.

Linear differential equations whose coefficients are functions of the independent variable are now assuming importance in aeronautics in the discussion of the motion following a small disturbance in a specified accelerated motion. In such problems the undisturbed state is often a transition motion in a limited time interval between two steady motions and we are concerned to see that the disturbed motion does not exceed tolerable bounds in a limited time. The extension of classical stability theory to such problems involves some logical difficulties and very great mathematical ones, since such equations are seldom soluble algebraically.

For these reasons an indirect attack on the problem is made here by seeking to establish upper bounds to the solution of second-order equations, which are those most commonly occurring.

The subject is introduced by a study of the equation

$$\ddot{x} + b(t)\,\dot{x} + c(t)\,x = 0\,.$$

The theory is then applied to a simple problem of pitching motion in an airstream of varying velocity.

Finally a system of two second-order equations involving two variables x and y is discussed from this angle. This system is not tractable in its most general form, but the special cases that yield to treatment are those which have occurred in some recent problems.

This analysis should be useful in the examination of any problem to which it can be applied; the exploration of its range has hardly begun. It is, of course, open to the objection that the gap between the bound and the solution can in general only be found by numerical integration. Some surveys by numerical integration to compare with the bound analysis will be the quickest way of assessing this method as a tool for general use.

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1. *Introduction.*

With the growth of thrust/weight ratio, aircraft of all kinds now spend more time than formerly in manoeuvres with fairly large accelerations: missiles, for instance, spend their whole life in such conditions. Classical stability theory, with its postulate of a steady motion from which disturbances are measured, is strictly speaking inapplicable to the study of response to disturbances from a specified accelerated motion, though it is quite commonly used in such problems for want of a more exact analysis. Whenever the conventional theory is stretched to cover such cases the underlying argument is that in passing through a range of speeds the effect of a disturbance at any one speed will be the same as if the aircraft were in equilibrium at that speed. This sort of argument runs through much current aerodynamic work as the following examples will show:

(1) Conventional take-off and landing has always been treated thus, with apparently good results. In recent work on jet-borne aircraft we have had to apply the rough method to the much more drastic case of jet-borne acceleration from rest to airborne speed, and *vice versa*.

(2) The trajectory of a rocket propelled model after separation is a curved decelerating path under gravity and drag at zero lift. It is common practice now to disturb the model at several points in this path and analyse its motion on the classical assumptions.

(3) The trajectory of a homing missile is a complicated path which may involve very large accelerations. There has recently been a good deal of argument as to whether it is at all valid to apply conventional ideas of stability to such a case.

It is clear that the simple theory would not have been overworked to this extent if it had been at all easy to extend its datum from a steady motion to an accelerated one. Clearly, too, it is time now to make what headway we can against the difficulties of this extension if only to learn more precisely how far and in what ways the simple theory may be taken out of its strict context.

Lighthill[1] in 1945 attempted the direct solution of a particular problem of this kind. The subject was resumed in 1953 in connection with Gott's adaptation of the classical theory to the problem of the stability of a missile guided by proportional navigation[2,3]. Notes on various aspects of the subject have been contributed by Collar[4,5], Relf[6] and Grensted[7].

The formidable difficulties of the acceleration problem need hardly be laboured. In general terms the components U, V . . . of the undisturbed motion and its control C are given functions of the time t. After a disturbance u_0, v_0 . . . at time t_0 the components become $U + u$, $V + v$, . . . where u, v . . . are small quantities of the first order and C is unchanged. The problem is to find u, v . . . in terms of u_0, v_0. . . .

The disturbed motion being assumed small, the differential equations will be linear, and usually of the second order with two or more variables, but the coefficients will be now not constants but functions of t. The exponential solution of the classical theory, in which all the variables and their time derivatives have the same modes, gives place to one in which all these quantities have in general different modes. The solution in general depends on the time of the initial disturbance, and cannot be stated in terms of damping and period, which are themselves consequences of the exponential solution.

In such circumstances the very definition of stability may seem open to debate. Mathematically it seems logically sound to take this over from the classical theory and to say simply that the datum motion is a stable one when u, v, . . . and their time derivatives all ultimately decay, at whatever point of the datum motion the disturbance occurs. There are, however, definite practical objections to this view. In the classical theory the datum motion is invariable with time, and so it is practically useful as well as mathematically correct to define stability in relation to the limit $t \to \infty$. But the datum accelerated motion which we now want to study will usually be limited to a finite time interval. In a simple case it may be a controlled motion in which the speed increases linearly from V_0 to V_1. We are quite uninterested in what would happen, after a disturbance, if the datum speed were allowed to increase without limit. The practical problem is to ensure that after every small disturbance which may occur between V_0 and V_1 the components of the disturbed motion remain within tolerable limits when the datum speed reaches V_1. This line of argument would lead us to investigate bounds of the variables in finite intervals of time, and to define stability as an affair of tolerable bounds.

In these notes we have chosen this line of investigation because it seems at the moment the only way round the insuperable difficulties of a frontal attack. It is well known that the equation

$$\ddot{x} + b(t)\,\dot{x} + c(t)\,x = 0 \tag{1}$$

is only integrable in very special cases, and as far as we know there is no general theorem regarding its zeros which leads to a discrimination between its oscillatory and its non-oscillatory solutions. Anything in the nature of an exact solution of the systems of equations referred to above, or even of a close discrimination between the types of solution that they may yield, seems therefore very remote. It is possible on the other hand to construct bounds for x and $\dot{x}$ in the above equation, and to analyse similarly some cognate systems of equations with two variables, in such a way or to throw some limited light on the stability problems which they represent. The germ of the ideas developed below appears to be due to Polya[8]. Some simple examples have been studied by Bain[9].

PART I

The General Second-Order Equation

2. *Development of the Analysis of Bounds.*

2.1. *Theory for c Positive.*

The object of this discussion is to look for bounds to the solution of equation (1) and to determine where possible how the solution behaves as $t \to \infty$.

Consider the function

$$f(t) = px^2 + q\dot{x}^2$$

where p and q are disposable functions of t, and x satisfies (1). If $p = qc$, then

$$f = p\left(x^2 + \frac{\dot{x}^2}{c}\right)$$

$$\dot{f} = \dot{p}x^2 + \left(\frac{\dot{p}}{p} - H\right)\frac{p}{c}\,\dot{x}^2$$

where

$$H = \frac{\dot{c}}{c} + 2b\,.$$

Now suppose that $c > 0$. If we can find p to satisfy the conditions:

$$\left.\begin{aligned} p &> 0 \\ \dot{p} &\leqslant 0 \\ \frac{\dot{p}}{p} &\leqslant H \end{aligned}\right\} \tag{2}$$

then f is always positive and $\dot{f}$ is never positive.

In these conditions it follows that

$$px^2 < f < f_0$$

$$\frac{p}{c}\dot{x}^2 < f < f_0$$

where f_0, the value of f when $t = 0$, is $p_0\,(x_0{}^2 + \dot{x}_0{}^2/c_0)$. Thus if $x_B{}^2$, $\dot{x}_B{}^2$ are the bounds of x^2, $\dot{x}^2$ respectively we have

$$\left.\begin{aligned} x_B{}^2 &= \frac{p_0}{p}\left(x_0{}^2 + \frac{\dot{x}_0{}^2}{c_0}\right) \\ \dot{x}_B{}^2 &= cx_B{}^2. \end{aligned}\right\} \tag{3}$$

It is clear from (2) that the admissible forms of p depend on H. We shall consider three cases:

(a) $H > 0$ always

(b) $H < 0$ always

(c) H changes sign.

2.1.1. *Bounds for* $H > 0$.—The conditions are satisfied when p is any positive constant, which can be taken to be unity without loss of generality. This case therefore gives

$$\left.\begin{aligned} f &= x^2 + \frac{\dot{x}^2}{c} \\ \dot{f} &= -\frac{H}{c}\dot{x}^2 \\ x_B{}^2 &= x_0{}^2 + \frac{\dot{x}_0{}^2}{c_0} \\ \dot{x}_B{}^2 &= cx_B{}^2. \end{aligned}\right\} \tag{4}$$

We note that $x_B{}^2$ is constant and $\dot{x}_B{}^2$ is proportional to c. Thus x, but not necessarily $\dot{x}$, must remain finite.

We also note that $f = x^2$ and $\dot{f} = 0$ when $\dot{x} = 0$. Thus when the solution is oscillatory (*see* Fig. 1a) the curve of f against t, while descending steadily on the whole, flattens out and coincides with each peak of x^2. The peaks of x^2, and thus the successive peaks and troughs of x, form a series descending to zero. If, however, the time interval between successive zeros of x continually decreases, $\dot{x}$ does not necessarily tend to zero with x. This behaviour occurs when $c \to \infty$ with t.

A more direct argument leads to some of these asymptotic conclusions. $\dot{f}$ must $\to 0$ as $t \to \infty$, for otherwise f would ultimately become negative. Then if c remains finite, $\dot{x}_\infty = 0$ and $\ddot{x}_\infty = 0$. It then follows from the original equation that $x_\infty = 0$.

The conclusion is that when c and H are both positive, x has a finite bound and ultimately decays, while $\dot{x}$ has a bound that is proportional to c and also ultimately decays if c remains finite. In the excepted case, which must be oscillatory in the limit, $\dot{x} \to \infty$ with t.

2.1.2. *Bounds for* $H < 0$.—The conditions are satisfied by putting

$$\frac{\dot{p}}{p} - H = \epsilon$$

where ϵ is any function of t that is always negative.

This gives

$$p = \exp\left(\int_0^t (H+\epsilon)dt\right)$$

and so

$$f = \exp\left(\int_0^t (H+\epsilon)dt\right)\left(x^2 + \frac{\dot{x}^2}{c}\right)$$

$$\dot{f} = \exp\left(\int_0^t (H+\epsilon)dt\right)\left\{(H+\epsilon)x^2 + \epsilon\frac{\dot{x}^2}{c}\right\}$$

$$x_B{}^2 = \exp\left(-\int_0^t (H+\epsilon)dt\right)\left(x_0{}^2 + \frac{\dot{x}_0{}^2}{c_0}\right)$$

$$\dot{x}_B{}^2 = cx_B{}^2.$$

The narrowest bounds are clearly given by $\epsilon = 0$ in which case we have

$$\left.\begin{aligned} f &= \exp\left(\int_0^t H\,dt\right)\left(x^2 + \frac{\dot{x}^2}{c}\right) \\ \dot{f} &= H\exp\left(\int_0^t H\,dt\right)x^2 \\ x_B{}^2 &= \exp\left(-\int_0^t H\,dt\right)\left(x_0{}^2 + \frac{\dot{x}_0{}^2}{c_0}\right) \\ \dot{x}_B{}^2 &= cx_B{}^2. \end{aligned}\right\} \qquad (5)$$

It follows from (5) that if the solution is oscillatory the curve of f has the same sort of staircase descent as in case (a) but it flattens out at the zeros of x and has no such simple relation to the the curve of $\dot{x}$ as it has to the curve of x in case (a) (*see* Fig. 1b).

In this case the conclusions are much less definite than for $H > 0$. The x^2 and $\dot{x}^2$ bounds vary respectively as $\exp\left(-\int_0^t H\,dt\right)$ and $c\exp\left(-\int_0^t H\,dt\right)$. If $\int_0^\infty H\,dt$ is finite and equal to $-K$, the asymptotic values of the bounds are respectively $(x_0{}^2+x_0{}^2/c_0)\exp K$ and $c_\infty(x_0{}^2+x_0{}^2/c_0)\exp K$; and both are finite if c_∞ is finite. In this case the form of f shows that x_∞ is zero, and $\dot{x}_\infty$ will also be zero. If, on the other hand, $\int_0^\infty H\,dt$ is infinite and c_∞ finite, both bounds rise to infinity, and nothing can be concluded from the form of f as to the actual values of x_∞ and $\dot{x}_\infty$.

If we are interested in the behaviour of the system over a finite interval of time only then in most cases $\int_0^{t_1} H\,dt$ will be finite and a finite bound can be found for x. Whether the system is acceptable or not will depend on a physical assessment of the magnitude of this bound.

It may be said in general that the response to a disturbance x_0, $\dot{x}_0$ will be greater when $H < 0$ than when $H > 0$ since the x bound, instead of being constant, now rises as $\exp\left(-\int_0^t H\,dt\right)$ and if c is the same in both cases the $\dot{x}$ bound will have a correspondingly greater variation.

2.1.3. *Bounds when H changes sign.*—In this case the conditions are satisfied by $p = \exp\left(\int \phi\,dt\right)$, where ϕ is any function for which $\phi \leqslant 0$, $\phi \leqslant H$ always. Thus we may put $\phi = 0$ in any interval in which H is positive, and $\phi = H$ in any interval in which H is negative. Suppose for instance that H changes sign from positive to negative at $t = t_1$. The results are then:

$0 < t < t_1$

$$f = x^2 + \frac{\dot{x}^2}{c}$$

$$\dot{f} = -\frac{H}{c}\dot{x}^2$$

$$x_B^2 = x_0^2 + \frac{\dot{x}_0^2}{c_0}$$

$$\dot{x}_B^2 = c x_B^2$$

$t > t_1$

$$f = \exp\left(\int_{t_1}^t H\,dt\right)\left(x^2 + \frac{\dot{x}^2}{c}\right)$$

$$\dot{f} = H \exp\left(\int_{t_1}^t H\,dt\right) x^2.$$

Hence

$$x^2 < \exp\left(-\int_{t_1}^t H\,dt\right)\left(x_1^2 + \frac{\dot{x}_1^2}{c_1}\right)$$

$$< \exp\left(-\int_{t_1}^t H\,dt\right)\left(x_0^2 + \frac{\dot{x}_0^2}{c_0}\right)$$

from the first interval.

Thus

$$x_B^2 = \exp\left(-\int_{t_1}^t H\,dt\right)\left(x_0^2 + \frac{\dot{x}_0^2}{c_0}\right)$$

and

$$\dot{x}_B^2 = c x_B^2.$$

When H changes sign from negative to positive at $t = t_1$ the results are, by a similar argument:

$0 < t < t_1$

$$f = \exp\left(\int_0^t H\,dt\right)\left(x^2 + \frac{\dot{x}^2}{c}\right)$$

$$\dot{f} = H \exp\left(\int_0^t H\,dt\right) x^2$$

$$x_B^2 = \exp\left(-\int_0^t H\,dt\right)\left(x_0^2 + \frac{\dot{x}_0^2}{c_0}\right)$$

$$\dot{x}_B^2 = c x_B^2.$$

$\underline{t > t_1}$

$$f = \exp\left(\int_t^{t_1} H\,dt\right)\left(x^2 + \frac{\dot{x}^2}{c}\right)$$

$$\dot{f} = -\frac{H}{c}\exp\left(\int_0^{t_1} H\,dt\right)\dot{x}^2$$

$$x_B{}^2 = \exp\left(-\int_0^{t_1} H\,dt\right)\left(x_0{}^2 + \frac{\dot{x}_0{}^2}{c_0}\right)$$

$$\dot{x}_B{}^2 = c x_B{}^2.$$

At $t = t_1$, f is continuous, with $\dot{f} = 0$, and the bounds also have continuity of slope. The remarks of section (a) apply in general to an interval in which H is positive, and those of section (b) to an interval in which H is negative.

Any number of intervals may be treated in the same way with results as sketched in Fig. 2. The curve of f descends continuously, flattening out when $H = 0$. The curve of x_B rises discontinuously, with flats when $H > 0$ and continuity of slope at $H = 0$. In two successive flats the value of x_B is multiplied by $\exp\left(-\int H/2\,dt\right)$, the integral being taken over the interval (for which H is negative) which separates them. The curve of $\dot{x}_B$ depends on the function c. It is continuous in slope, but not in curvature, at $H = 0$.

2.2. *Possibility of Closer Bounds for $\dot{x}$.*

In this analysis we have considered an equation for x

$$\ddot{x} + b\dot{x} + cx = 0 \tag{1}$$

and deduced bounds for both x and $\dot{x}$. An alternative approach to a bound for $\dot{x}$ would be to differentiate this equation and hence obtain a second-order equation in $\dot{x}$. We can then treat this in the same way as the original equation provided the coefficient of $\dot{x}$ in the new equation is positive.

We obtain on differentiation

$$\ddot{u} + \left(b - \frac{\dot{c}}{c}\right)\dot{u} + \left(c + \dot{b} - \frac{b\dot{c}}{c}\right) u = 0$$

where $u = \dot{x}$. If $c + \dot{b} - b\dot{c}/c > 0$ we can obtain bounds for u as we did for x. For example, if H from the new equation is always positive we have

$$\dot{x}_B = u_B = \sqrt{\left(u_0{}^2 + \frac{\dot{u}_0{}^2}{C_0}\right)} = \sqrt{\left(\dot{x}_0{}^2 + \frac{\ddot{x}_0{}^2}{C_0}\right)}$$

where

$$C = \left(c + \dot{b} - \frac{b\dot{c}}{c}\right).$$

We may, using the original equation (1), express the bound in terms of the initial values of x and $\dot{x}$

$$\dot{x}_B = \sqrt{\left\{\dot{x}_0{}^2 + \frac{(b_0\dot{x}_0 + c_0x_0)^2}{C_0}\right\}}$$

and hence,

$$\mu = \frac{\dot{x}_B}{\sqrt{(\dot{x}_0{}^2 + c_0x_0{}^2)}} = \sqrt{\left\{\frac{\dot{x}_0{}^2 + \dfrac{(b_0\dot{x}_0 + c_0x_0)^2}{C_0}}{\dot{x}_0{}^2 + c_0x_0{}^2}\right\}}.$$

Thus for this case μ is not a function of time but is dependent on the initial conditions. If H is not always positive μ will depend on the initial conditions in the same way but will be a function of time too.

In this way we may or may not obtain closer bounds for $\dot{x}$ than from our first equation. We shall never obtain decreasing bounds so that if the bounds derived from the original equation decrease the bounds obtained by this method will be worse.

Examples of the use of this method are given in Sections 4.1 and 4.2.

2.3. *The Case of c Negative.*

In this case putting $p = 1$ we have

$$f = x^2 + \frac{\dot{x}^2}{c}$$

$$\dot{f} = -\frac{H}{c}\dot{x}^2$$

so that $f \leqslant x^2$ and $\dot{f} \geqslant$ or $\leqslant 0$ according as $H >$ or < 0. This case, therefore, yields a *lower bound* for x.

If $H > 0$, f is an increasing function which is less than x^2 except at $\dot{x} = 0$, where it is tangential to x^2. It follows that x^2 cannot have a maximum and so not more than one minimum. Therefore x^2 cannot be oscillatory and must ultimately increase without limit.

If $H < 0$, f is a decreasing function which is less than x^2 except at $\dot{x} = 0$ where it is tangential. It follows again that x^2 cannot have a maximum and so not more than one minimum. Therefore x^2 cannot be oscillatory but need not ultimately increase without limit. (*See* Fig. 3.)

Thus if c is positive we may, for $H < 0$, *prove* the decay of x but if $H < 0$ x *may* increase. If c is negative and $H > 0$ we may *prove* the divergence of x but if $H < 0$ x *may* decrease.

2.4. *Summary, with Dynamical Interpretation of Results.*

2.4.1. *General formulae.*—In summarising these results it is useful to introduce the integral symbol $\int^*$ to indicate that the integral is to be taken only for negative values of the integrand.

It appears then that, in general, if $c > 0$

$$f = \left(x^2 + \frac{\dot{x}^2}{c}\right) \exp\left(\int_0^{t*} H\,dt\right) \tag{6}$$

$$\dot{f} = \left(H_- x^2 - H_+ \frac{\dot{x}^2}{c}\right) \exp\left(\int_0^{t*} H\,dt\right)$$

$$\lambda \equiv \frac{x_B}{\sqrt{\left(x_0^2 + \dfrac{\dot{x}_0^2}{c_0}\right)}} = \exp\left(-\tfrac{1}{2}\int_0^{t*} H\,dt\right) \tag{7}$$

$$\mu \equiv \frac{\dot{x}_B}{\sqrt{(\dot{x}_0^2 + c_0 x_0^2)}} = \sqrt{\left(\frac{c}{c_0}\right)} \exp\left(-\tfrac{1}{2}\int_0^{t*} H\,dt\right) \tag{8}$$

where

$$H = \frac{\dot{c}}{c} + 2b\,.$$

At a given time H will be positive, i.e. H_+, or negative, i.e. H_-. In the formula for f only the appropriate one of these values is to be taken.

It follows from the expressions for λ and μ that x_B cannot exceed $\sqrt{(x_0^2+\dot{x}_0^2/c_0)}$ and $\dot{x}_B$ cannot exceed $\sqrt{(\dot{x}_0^2+c_0x_0^2)}$ if $H > 0$ and $c < c_0$.

2.4.2. *Dynamical interpretation.*—We can now consider the dynamical interpretation of these results. If the equation discussed above represents the motion of a system with displacement x, then b and c are respectively the resistance coefficient and the stiffness coefficient per unit mass (or moment of inertia), these being variable with time.

The energy E of the system (per unit mass or moment of inertia) is given by

$$E = \tfrac{1}{2}(\dot{x}^2+cx^2) \qquad \text{at time } t$$

and the energy E_0 of the disturbance x_0, $\dot{x}_0$ is

$$E_0 = \tfrac{1}{2}(\dot{x}_0^2+c_0x^2).$$

Equation (6) may therefore be written

$$f = \frac{2E}{c}\exp\left(\int_0^{t*} H\,dt\right)$$

and so

$$\frac{E}{c} < \frac{E_0}{c_0}\exp\left(-\int_0^{t*} H\,dt\right).$$

In particular,

$$\frac{E}{c} < \frac{E_0}{c_0} \qquad \text{when } H \text{ is always positive}$$

$$\frac{E}{c} < \frac{E_0}{c_0}\exp\left(-\int_0^{t} H\,dt\right) \qquad \text{when } H \text{ is always negative.}$$

It is to be noted that these relations yield bounds, not to the energy itself, but to energy/stiffness coefficient. When, however, b, c are constant, so that $H = 2b$, $c = c_0$, we have bounds to the energy itself:

$$E < E_0, \qquad b > 0$$

$$E < E_0\exp(-2bt),\ b < 0.$$

Further, equations (7) and (8) may now be written

$$x_B = \sqrt{\left(\frac{2E_0}{c_0}\right)}\,\lambda$$

$$\dot{x}_B = \sqrt{(2E_0)}\,\mu$$

where

$$\lambda = \exp\left(-\tfrac{1}{2}\int_0^{t*} H\,dt\right)$$

$$\mu = \sqrt{\left(\frac{c}{c_0}\right)}\exp\left(-\tfrac{1}{2}\int_0^{t*} H\,dt\right).$$

These expressions show that the functions λ, μ determine the *shapes* of the bounds x_B, $\dot{x}_B$, while their *amplitudes* are determined by the energy imparted to the system by the disturbance and the stiffness coefficient at that instant.

When c_0 is small the energy imparted to the system by the disturbance is almost wholly kinetic and so

$$\left.\begin{aligned} x_B &= \frac{\dot{x}_0}{\sqrt{c_0}}\lambda \\ \dot{x}_B &= \dot{x}_0\mu \end{aligned}\right\} c_0 \text{ small.}$$

When c_0 is large the energy of the disturbance is almost wholly potential, and so

$$\left.\begin{aligned} x_B &= x_0\lambda \\ \dot{x}_B &= \sqrt{(c_0)}x_0\mu \end{aligned}\right\} c_0 \text{ large.}$$

One further point of general interest may be mentioned here. It often happens that the functions b, c occur in the form

$$b(t) = b^*B(t)$$

$$c(t) = c^*C(t)$$

where the constants b^*, c^* respectively represent the general levels of the resistance coefficient and the stiffness coefficient.

It follows from the form of H that λ, μ are independent of c^*, while the amplitude factors are of course independent of b^*.

2.4.3. *Formulae for λ, μ in terms of b, c.*—It may be useful to express λ, μ in terms of b and c, as follows. Suppose that the time intervals in which H is negative are

$$t_1 t_2,\, t_3 t_4,\, \ldots,\, t_{2u-1} t_{2u},\, \ldots$$

so that the nth positive interval is $t_{2n-2}t_{2n-1}$ and the nth negative interval is $t_{2n-1}t_{2n}$.

Now

$$-\int \frac{H}{2}\,dt = \log\frac{1}{\sqrt{c}} - B$$

where

$$B = \int b\,dt$$

and so

$$-\int_{t_{2n-1}}^{t_{2n}} \frac{H}{2}\,dt = \log\sqrt{\left(\frac{c_{t_{2n-1}}}{c_{t_{2n}}}\right)} - B_n$$

where

$$B_n = \int_{t_{2n-1}}^{t_{2n}} b\,dt.$$

Thus

$$\exp\left(-\int_{t_{2n-1}}^{t_{2n}} \frac{H}{2}\,dt\right) = \sqrt{\left(\frac{c_{t_{2n-1}}}{c_{t_{2n}}}\right)}\exp\left(-B_n\right) \text{ for the interval } t_{2n-1}t_{2n}.$$

It follows that in the nth positive interval $t_{2n-2}t_{2n-1}$, λ is constant and given by

$$\lambda = \exp\left\{-\left(\int_{t_1}^{t_2}+\int_{t_3}^{t_4}+\ldots+\int_{t_{2n-3}}^{t_{2n-2}}\right)\frac{H}{2}\,dt\right\}$$

$$= \sqrt{\left(\frac{c_{t_1}c_{t_3}\ldots c_{t_{2n-3}}}{c_{t_2}c_{t_4}\ldots c_{t_{2n-2}}}\right)}\exp\left(-\sum_1^{n-1}B_r\right)$$

and μ is varying and given by

$$\mu = \sqrt{\left(\frac{c}{c_0}\right)}\,\lambda,$$

where c goes from $c_{t_{2n-2}}$ to $c_{t_{2n-1}}$.

In the nth negative interval $t_{2n-1}t_{2n}$, λ and μ are both varying:

$$\lambda = \sqrt{\left(\frac{c_{t_1}c_{t_3}\ldots c_{t_{2n-3}}}{c_{t_2}c_{t_4}\ldots c_{t_{2n-2}}}\right)}\exp\left(-\sum_1^{n-1}B_r\right)\sqrt{\left(\frac{c_{t_{2n-1}}}{c}\right)}\exp\left(-\int_{t_{2n-1}}^{t}b\,dt\right)$$

$$\mu = \sqrt{\left(\frac{c_{t_1}c_{t_3}\ldots c_{t_{2n-1}}}{c_0c_{t_2}c_{t_4}\ldots c_{t_{2n-2}}}\right)}\exp\left(-\sum_1^{n-1}B_r\right)\exp\left(-\int_{t_{2n-1}}^{t}b\,dt\right)$$

where c goes from $c_{t_{2n-1}}$ to $c_{t_{2n}}$.

These formulae apply when H is positive in the first interval $0t_1$. Corresponding formulae when H is negative in the first interval can easily be constructed.

3. *Application to Equations with Coefficients Varying Linearly with Time.*

Some simple examples of this analysis will now be given.

3.1. *Constant Coefficients.*

The trivial case when b, c are constants is of interest because, of course, the exact solution is known.

Then $H = 2b$, and so

$$\lambda = \mu = 1 \qquad \text{when } b > 0$$

$$\lambda = \mu = \exp(-bt) \qquad \text{when } b < 0.$$

These may be written in full

$$\left.\begin{aligned} x_B &= \sqrt{\left(x_0^2+\frac{\dot{x}_0^2}{c}\right)} \\ \dot{x}_B &= \sqrt{(\dot{x}_0^2+cx_0^2)} \end{aligned}\right\} b > 0$$

$$\left.\begin{aligned} x_B &= \sqrt{\left(x_0^2+\frac{\dot{x}_0^2}{c}\right)}\exp(-bt) \\ \dot{x}_B &= \sqrt{(\dot{x}_0^2+cx_0^2)}\exp(-bt) \end{aligned}\right\} b < 0.$$

3.1.1. *Periodic solution.*—The exact solution of $\ddot{x} + b\dot{x} + c = 0$, if it is oscillatory with exponential roots $\alpha \pm i\beta$, is

$$x = \sqrt{\left\{\left(1 + \frac{\alpha^2}{\beta^2}\right)\left(x_0{}^2 + \frac{\dot{x}_0{}^2}{c}\right) - \frac{2\alpha}{\beta^2}\,x_0\dot{x}_0\right\}}\ \exp\,(\alpha t)\,\sin\,(\beta t + \epsilon_1)$$

$$\dot{x} = \sqrt{\left\{\left(1 + \frac{\alpha^2}{\beta^2}\right)(\dot{x}_0{}^2 + cx_0{}^2 - 2\alpha x_0\dot{x}_0)\right\}}\ \exp\,(\alpha t)\,\sin\,(\beta t + \epsilon_2)$$

where $\alpha = -b/2$, $\alpha^2 + \beta^2 = c$, and ϵ_1, ϵ_2 are constants. If, therefore, x_E, $\dot{x}_E$ are the enveloping values of this oscillation we have

$$x_E = \sqrt{\left\{\left(1 + \frac{\alpha^2}{\beta^2}\right)\left(x_0{}^2 + \frac{\dot{x}_0{}^2}{c}\right) - \frac{2\alpha}{\beta^2}\,x_0\dot{x}_0\right\}}\ \exp\left(-\frac{b}{2}\,t\right)$$

$$\dot{x}_E = \sqrt{\left\{\left(1 + \frac{\alpha^2}{\beta^2}\right)(\dot{x}_0{}^2 + cx_0{}^2 - 2\alpha x_0\dot{x}_0)\right\}}\ \exp\left(-\frac{b}{2}\,t\right).$$

It is interesting to compare the bounds x_B, $\dot{x}_B$ with the actual envelope x_E, $\dot{x}_E$ of the motion. The above expressions show that in all cases x_B/x_E and $\dot{x}_B/\dot{x}_E$ vary as $\exp\,(\frac{1}{2}|b|t)$. In this particular case, therefore, we have an answer to a question which is basic to this analysis: how much too big are the bounds ? The uncertainty appears as a factor: the bounds multiply the envelopes by the increasing function

$$\exp\,(\tfrac{1}{2}|b|t).$$

We still have to account for the differences, as regards x_B and x_E, or $\dot{x}_B$ and $\dot{x}_E$, in the quantities under the radical. This can be seen geometrically in Fig. 4. There ABCDE is a portion of a damped oscillation x, the trough CDE being reflected as CD′E. The envelope x_E touches the x curve to the right of the peaks B,D′. The curve of $\sqrt{f}$ touches the x curve at the peaks B,D′. Thus x_E, $\sqrt{f}$ intersect twice between the peaks B,D′. The oscillation can be started by a disturbance x_0, $\dot{x}_0$ at any point of ABCD. Suppose it starts at the point K. Then the initial value of x_E is OQ and the constant value of x_B is OP; these are the expressions under the radicals in x_E,x_B. Also since $\dot{x}_B = \sqrt{c}\,x_B$ and $\dot{x}_E = \sqrt{c}\,x_E$, the constant value of $\dot{x}$ is $\sqrt{c}$.OP and the initial value of $\dot{x}_E$ is $\sqrt{c}$.OQ.

It is clear from the diagram that x_E, $\sqrt{f}$ tend to coincide as α/β, or the damping factor per period, becomes small; in an undamped oscillation these curves would be the straight line joining the peaks. Thus when α/β is small we have simply

$$x_E = \sqrt{\left(x_0{}^2 + \frac{\dot{x}_0{}^2}{c}\right)}\ \exp\left(-\frac{b}{2}\,t\right)$$

$$\dot{x}_E = \sqrt{(\dot{x}_0{}^2 + cx_0{}^2)}\ \exp\left(-\frac{b}{2}\,t\right).$$

The above argument for a decreasing oscillation applies unchanged for an increasing oscillation if we simply reverse the sign of t. Thus in all cases we have

$$x_B/x_E = \dot{x}_B/\dot{x}_E = \exp\,(\tfrac{1}{2}|b|t)$$

for slowly decreasing or increasing oscillations.

3.1.2. *Aperiodic solution.*—If the solution of $\ddot{x} + b\dot{x} + cx = 0$ is aperiodic we can no longer define an envelope so that we can no longer obtain a factor of uncertainty in the same sense. We can, however, investigate the extreme values of the solution and see how these compare with the bounds we have previously deduced.

Our original equation can be reduced by the substitution $\tau = bt$ to

$$x'' + x' + Cx = 0 \text{ if } b > 0$$

and by the substitution $\tau = -bt$ to

$$x'' - x' + Cx = 0 \text{ if } b < 0$$

where $C = c/b^2$ and a dash denotes differentiation w.r.t. τ.

The solution of $x'' + x' + Cx = 0$ can be evaluated for various initial conditions and in Fig. 5 the values of x/x_0 are plotted against τ for various values of $h = x_0'/x_0 = \dot{x}_0/bx_0$, for $C = \frac{1}{4}$. We see that if $h > 0$ the solution increases initially from unity to a maximum value and then decreases asymptotically to zero. The greatest value occurs at the maximum. If $0 > h > 0 \cdot 5$ the solution decreases asymptotically to zero and the greatest value is unity occurring initially. For values of h less than $-0 \cdot 5$ the solution decreases and becomes negative and has a minimum. For values of h between $-0 \cdot 5$ and about $-2 \cdot 4$ the value at the minimum will be less than 1 and the greatest value will occur initially. For large negative values of h the minimum will be the greatest value. There are then three regions of h, in which the greatest deviation occurs at the maximum, the initial value and the minimum. The solutions for other values of C are similar.

When b is positive, the predicted bound is $x_B/x_0 = 1/x_0 \sqrt{(x_0^2 + \dot{x}_0^2/c)} = 1/x_0 \sqrt{(x_0^2 + x_0'^2/C)} = \sqrt{(1 + h^2/C)}$. We may divide the extreme values of the solution by this and plot them against h as in Fig. 6. The curves of this figure may be calculated, without computing the complete solution, by the method of Appendix I.

We see that if an initial disturbance occurs in x only ($h = 0$) the bound is always equal to the initial value but if there is disturbance in $\dot{x}$ as well the bound of x will never be reached. The amount of the discrepancy increases as C decreases and when C is very small the bound is a gross over-estimate of the extreme value except for disturbances that contain very little velocity.

The corresponding solutions for $\dot{x}/\dot{x}_0 = x'/x_0'$ are shown in Fig. 7. Here the solution has a maximum when h is small and negative, and a maximum when h is small and positive, while for large positive and negative values of h the greatest value occurs initially. The bound for $\dot{x}/\dot{x}_0$ is $\sqrt{(1 + C/h^2)}$ and if the extreme values are divided by this we obtain the curves of Fig. 8.

The $\dot{x}$ bounds are always reached when the disturbance is entirely in $\dot{x}$ ($h \to \infty$). If there is a disturbance in x too the extreme always falls short of the bound. However, as C decreases, the amount by which the bound overestimates the extreme decreases. For very small C the bound is a good approximation to the extreme except for disturbances that contain very little velocity.

Comparison of Figs. 6 and 8 shows that the bound for $\dot{x}$ is close to the extreme when the bound for x is much greater than the extreme, and *vice versa*.

We must not forget that however close the extremes are to the bounds the variables decrease from the extremes so that for most of the time the overestimate in taking the bound for the actual value is much greater than the ratio of the bound to the extreme.

If b is negative the bounds increase with time like $\exp(-bt) = \exp \tau$ and we must consider the ratio of the solution to the bound as a function of time. Although the solution diverges this ratio is bounded and tends to zero as $t \to \infty$ so that we may discuss its extreme values in the same way as for the stable equation. A rough examination of this analysis suggests that the conclusions would be similar to those for the case of b positive.

3.2. *c Constant; Linear Variation of b.*

In this case we have

$$b = b_0 (1+\delta t)$$

$$c = c_0 .$$

Then $\lambda = \mu$, and $H = 2b_0 (1+\delta t)$, vanishing at $t = \tau = -1/\delta$. The solutions for λ, μ are as follows:

b	sign of H	λ, μ	range
$b_0 > 0,\ \delta > 0$	+	1	throughout
$b_0 > 0,\ \delta < 0$	+	1	$0 < t < \tau$
	−	$\exp\left\{-b_0(t-\tau) - \frac{b_0\delta}{2}(t^2-\tau^2)\right\}$	$t > \tau$
$b_0 < 0,\ \delta < 0$	−	$\exp\left(-b_0 t - \frac{b_0\delta}{2} t^2\right)$	$0 < t < \tau$
	+	$\exp\left(-b_0\tau - \frac{b_0\delta}{2}\tau^2\right)$	$t > \tau$
$b_0 < 0,\ \delta > 0$	−	$\exp\left(-b_0 t - \frac{b_0\delta}{2} t^2\right)$	throughout

Fig. 9 shows diagrammatically the shape of the curves.

There are two notable features of these results. First, λ, μ are independent of c_0. Second, λ, μ coincide with the results for b constant so long as b is positive. Whenever b becomes negative, trouble occurs.

3.3. *b Constant; Linear Variation of c.*

Consider next

$$b = b_0$$

$$c = c_0(1+\epsilon t) .$$

When ϵ is negative the solution stops when $c = 0$, at $t = T = -1/\epsilon$. In this case we have

$$H = \frac{\epsilon}{1+\epsilon t} + 2b_0$$

and so $H = \epsilon + 2b_0$ at $t = 0$ and decreases hyperbolically to $2b_0$ at $t = \infty$ when $\epsilon > 0$, and to $-\infty$ at $t = T$ when $\epsilon < 0$.

It follows from this that the sign changes in H are:

<u>$\epsilon > 0$</u>

$$b_0 > 0, \qquad H+$$

$$-\frac{\epsilon}{2} < b_0 < 0, \qquad H+-$$

$$b_0 < -\frac{\epsilon}{2}, \qquad H-$$

<u>$\epsilon < 0$</u>

$$b_0 > -\frac{\epsilon}{2}, \qquad H+-$$

$$b_0 < -\frac{\epsilon}{2}, \qquad H-$$

The sign changes occur at $t = \tau = -\left(\frac{1}{\epsilon} + \frac{1}{2b_0}\right)$.

The solutions may now be tabulated as follows:

ϵ	b_0	H	λ	μ	range
positive	$b_0 > 0$	+	1	$\sqrt{\left(\frac{c}{c_0}\right)}$	$0 < t < \infty$
positive	$-\frac{\epsilon}{2} < b_0 < 0$	+	1	$\sqrt{\left(\frac{c}{c_0}\right)}$	$0 < t < \tau$
positive	$-\frac{\epsilon}{2} < b_0 < 0$	−	$\sqrt{\left(\frac{c_\tau}{c}\right)} \exp\{-b_0(t-\tau)\}$	$\sqrt{\left(\frac{c_\tau}{c_0}\right)} \exp\{-b_0(t-\tau)\}$	$t > \tau$
positive	$b_0 < -\frac{\epsilon}{2}$	−	$\sqrt{\left(\frac{c_\tau}{c}\right)} \exp(-b_0 t)$	$\exp(-b_0 t)$	$0 < t < \infty$
negative	$b_0 > -\frac{\epsilon}{2}$	+	1	$\sqrt{\left(\frac{c}{c_0}\right)}$	$0 < t < \tau$
negative	$b_0 > -\frac{\epsilon}{2}$	−	$\sqrt{\left(\frac{c_\tau}{c}\right)} \exp\{-b_0(t-\tau)\}$	$\sqrt{\left(\frac{c_\tau}{c}\right)} \exp\{-b_0(t-\tau)\}$	$\tau < t < T$
negative	$b_0 < -\frac{\epsilon}{2}$	−	$\sqrt{\left(\frac{c_\tau}{c}\right)} \exp(-b_0 t)$	$\exp(-b_0 t)$	$0 < t < T$

The sketches of Fig. 10 show diagrammatically the shapes of the curves in the several cases.

Notable features of these results are:

(1) λ, μ always depend on b_0, becoming steadily worse as b_0 decreases. It is the sum of b_0 and half the slope of c/c_0 that determines the character of the curves.

(2) When c is increasing, μ gives more trouble than λ, and when c is decreasing, λ gives more trouble than μ.

(3) λ, μ are never the same as would be obtained on the assumption that c is constant at time t.

3.4. *Linear Variation of both Coefficients.*

In this case

$$b = b_0 (1+\delta t)$$

$$c = c_0 (1+\epsilon t)$$

and to limit the work we shall take b_0 to be positive. We then have

$$H = \frac{\epsilon}{1+\epsilon t} + 2b_0(1+\delta t)$$

and H_0 the initial value of H is $2b_0 + \epsilon$. Also

$$\int \frac{H}{2}\,dt = \log\sqrt{c} + b_0\left(t + \frac{\delta}{2}t^2\right).$$

It will be convenient to consider H in the form

$$H = x - y$$

where

$$x = 2b_0(1+\delta t) \qquad \text{a straight line}$$

$$y = -\frac{\epsilon}{1+\epsilon t} \qquad \text{a rectangular hyperbola.}$$

The zeros of H occur at the intersection of these curves.

3.4.1. *Consider first* $\epsilon > 0$.

(1) *If* $\delta > 0$, then $H > 0$ and the solution is

$$\lambda = 1$$

$$\mu = \frac{c}{c_0}.$$

This has already been sketched (Fig. 10).

(2) *If* $\delta < 0$ the geometry for H is sketched in Fig. 11a. The solution is therefore

$$0 < t < \tau, \qquad H > 0, \qquad \lambda = 1, \qquad \mu = \sqrt{\left(\frac{c}{c_0}\right)}$$

$t > \tau,\qquad H < 0\qquad \lambda = \exp\left(-\int_\tau^t \frac{H}{2}\,dt\right)$

$$= \sqrt{\left(\frac{c_\tau}{c}\right)} \exp\left\{-b_0(t-\tau) - \frac{b_0\delta}{2}(t^2-\tau^2)\right\}$$

$$\mu = \sqrt{\left(\frac{c}{c_0}\right)}\,\lambda$$

$$= \sqrt{\left(\frac{c_\tau}{c_0}\right)} \exp\left\{-b_0(t-\tau) - \frac{b_0\delta}{2}(t^2-\tau^2)\right\}$$

where $t = \tau$ is the positive root of

$$2b_0(1+\delta t)(1+\epsilon t) + \epsilon = 0.$$

This solution is sketched in Fig. 12a.

3.4.2. *Consider next* $\epsilon < 0$. The geometry for the zeros of H is sketched in Fig. 11b and there are several cases to consider.

(1) *If* $H_0 > 0$, *or* $b_0 > -\epsilon/2$, as at P, then H changes sign once, at A, and the solution is

$0 < t < \tau,\qquad H > 0,\qquad \lambda = 1,\qquad \mu = \sqrt{\left(\frac{c}{c_0}\right)}$

$t > \tau,\qquad H < 0,\qquad \lambda = \sqrt{\left(\frac{c_\tau}{c}\right)} \exp\left\{-b_0(t-\tau) - \frac{b_0\delta}{2}(t^2-\tau^2)\right\}$

$$\mu = \sqrt{\left(\frac{c_\tau}{c_0}\right)} \exp\left\{-b_0(t-\tau) - \frac{b_0\delta}{2}(t^2-\tau^2)\right\}.$$

The solution is sketched in Fig. 12b.

(2) *If* $H_0 < 0$, *or* $b_0 < -\epsilon/2$ as at Q, then H has two sign changes, at B_1, B_2, or none, according as the slope of x is greater or less than the slope of the tangent from Q to the hyperbola y. The critical slope is found as follows. The equation for $H = 0$ can be written

$$\delta\epsilon t^2 + (\delta+\epsilon)t + \frac{H_0}{2b_0} = 0. \qquad (9)$$

This has equal roots if

$$(\delta+\epsilon)^2 = 2\delta\epsilon\frac{H_0}{b_0}$$

or

$$\left(\frac{\delta}{\epsilon}\right)^2 + 2\frac{\delta}{\epsilon}\left(1-\frac{H_0}{b_0}\right) + 1 = 0.$$

This has two negative roots in δ/ϵ since $b_0 > 0$ and $H_0 < 0$, and for the critical $\delta = \bar{\delta}$ we want the less of these.

Hence

$$\bar{\delta} = -\epsilon\left[1 - \frac{H_0}{b_0} + \sqrt{\left\{\left(1-\frac{H_0}{b_0}\right)^2 - 1\right\}}\right].$$

Thus *if* $\delta > \bar{\delta}$, H changes sign at τ_1, τ_2, the two positive roots of (9) and the solution is

$\underline{0 < t < \tau_1}$, $\quad H < 0$,
$$\lambda = \sqrt{\left(\frac{c_0}{c}\right)} \exp\left(-b_0 t - \frac{b_0\delta}{2} t^2\right)$$
$$\mu = \exp\left(-b_0 t - \frac{b_0\delta}{2} t^2\right)$$

$\underline{\tau_1 < t < \tau_2}$, $\quad H > 0$,
$$\lambda = \sqrt{\left(\frac{c_0}{c_{\tau_1}}\right)} \exp\left\{-b_0\tau_1 - \frac{b_0\delta}{2}\tau_1^2\right\}$$
$$\mu = \sqrt{\left(\frac{c}{c_{\tau_1}}\right)} \exp\left\{-b_0\tau_1 - \frac{b_0\delta}{2}\tau_1^2\right\}$$

$\underline{\tau_2 < t < T}$, $H < 0$,
$$\lambda = \sqrt{\left(\frac{c_0}{c_{\tau_1}}\right)} \exp\left\{-b_0\tau_1 - \frac{b_0\delta}{2}\tau_1^2\right\} \sqrt{\left(\frac{c_{\tau_2}}{c}\right)} \exp\left\{-b_0(t-\tau_2) - \frac{b_0\delta}{2}(t_2-\tau_2^2)\right\}$$
$$= \sqrt{\left(\frac{c_0 c_{\tau_2}}{c_{\tau_1} c}\right)} \exp\left\{-b_0[\{t-(\tau_2-\tau_1)\} + \frac{\delta}{2}\{t^2 - (\tau_2^2-\tau_1^2)\}]\right\}$$
$$\mu = \sqrt{\left(\frac{c_{\tau_2}}{c_{\tau_1}}\right)} \exp\left\{-b_0[\{t-(\tau_2-\tau_1)\} + \frac{\delta}{2}\{t^2 - (\tau_2^2-\tau_1^2)\}]\right\}.$$

The solution is sketched in Fig. 12c.

Finally, *if* $\delta < \bar{\delta}$, $H < 0$ and the solution is

$0 < t < T$
$$\lambda = \sqrt{\left(\frac{c_0}{c}\right)} \exp\left\{-b_0 t - \frac{b_0\delta}{2} t^2\right\}$$
$$\mu = \exp\left\{-b_0 t - \frac{b_0\delta}{2} t^2\right\}.$$

This is sketched in Fig. 12d.

This example has been worked out mainly to show the complications that occur in analysing the bounds when b, c are simple functions of t. It includes examples 3.2, 3.3 as particular cases, for $b_0 > 0$, when first ϵ and then δ is made to vanish.

As noted in the other examples, the results get consistently worse as the slope of b decreases. When the slope of c is positive, μ gives more trouble than λ; and *vice versa*.

4. *An Aerodynamic Pitching Problem.*

We next consider the application of these ideas to a simple aerodynamic problem. Suppose an aircraft is pivoted at its c.g. and free to rotate in pitch in an airstream of varying velocity: what is its motion after receiving a small disturbance from the trimmed position ? Alternatively, the aircraft may be taken to fly horizontally with varying speed under a control which keeps the lift equal to the weight. If V is the speed and x the angle of rotation, the static moment varies as xV^2 and the damping moment as $\dot{x}V$. The equation of motion is therefore

$$\ddot{x} + m_2 V\dot{x} + m_1 V^2 x = 0, \tag{10}$$

m_1, m_2 being constants, of which m_1 is taken to be positive. The aircraft receives the disturbance x_0, $\dot{x}_0$ at $t = 0$ when the speed is V_0.

We thus have

$$b = m_2 V$$

$$c = m_1 V^2$$

$$H = \frac{2}{V}(V + m_2 V^2)$$

and so

$$\int_0^t \frac{H}{2}\,dt = \log\frac{V}{V_0} + m_2 s$$

where s is the distance travelled (by the air or the aircraft) after the disturbance.

The following relations should also be noted:

$$2E_0 = \dot{x}_0^2 + m_1 V_0^2 x_0^2 .$$

λ, μ are independent of m_1, and $\mu = \dfrac{V}{V_0}\lambda$.

We shall consider three speed laws: hyperbolic, linear and exponential, as sketched in Fig. 13. The first two are chosen because the equation can then be formally integrated. No exact solution is known with the exponential speed law, but the case is interesting as representing roughly a practical course of transition from one steady speed to another in a finite time.

The analysis of each motion will conclude with some numerical examples. The numbers used are collected in Table 1.

4.1. *Hyperbolic Speed Variation.*

4.1.1. *Analysis of bounds.*—The speed law is

$$v = \frac{V}{V_0} = \frac{1}{1 + KV_0 t}$$

so that if K is positive V decays to zero as $t \to \infty$, and if K is negative $V \to \infty$ at $t = -1/KV_0$.

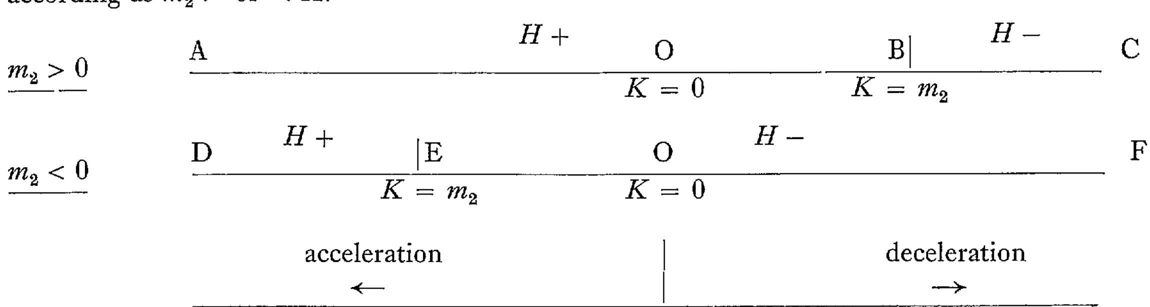
Thus we have

$$\dot{v} = -KV_0 v^2$$

$$c = m_1 V_0^2 v^2$$

$$H = 2V_0 v(m_2 - K).$$

In this example therefore H never changes sign during the motion but is positive or negative according as $m_2 >$ or $< K$.



We have

$$\int_0^t H\,dt = 2V_0(m_2-K)\int_0^t v\,dt$$

$$= 2V_0(m_2-K)\int_1^v \frac{v}{\dot{v}}\,dv$$

$$= -2\frac{m_2-K}{K}\int_1^v \frac{dv}{v}$$

$$= \log v^{2(1-m_2/K)}$$

and so

$$\exp\left(\int_0^t H\,dt\right) = v^{2(1-m_2/K)}$$

which is required when $m_2 < K$.

The results can now be written down from the general formulae.

<u>$m_2 > K$</u>

$$f = x^2 + \frac{\dot{x}^2}{m_1 V_0^2 v^2}$$

$$\dot{f} = -\frac{2}{m_1 V_0 v}(m_2-K)\dot{x}^2$$

$$\left.\begin{aligned} \lambda &= \frac{x_B}{\sqrt{\left(x_0^2+\dfrac{\dot{x}_0^2}{m_1 V_0^2}\right)}} = 1 \\ \mu &= \frac{\dot{x}_B}{\sqrt{(\dot{x}_0^2+m_1 V_0^2 x_0^2)}} = v\,. \end{aligned}\right\} \tag{11}$$

It will be seen from the sketch that this applies to the regions:

$m_2 > 0$ (*stable at constant speed*)

AO	(acceleration),	$v > 1$,	and so	$\mu \to \infty$
OB	(deceleration),	$v < 1$,	and so	$\mu \to 0$.

$m_2 < 0$ (*unstable at constant speed*)

DE	(acceleration),	$v > 1$,	and so	$\mu \to \infty$.

<u>$m_2 < K$</u>

$$f = v^{2(1-m_2/K)}\left(x^2 + \frac{\dot{x}^2}{m_1 V_0^2 v^2}\right)$$

$$\dot{f} = 2V_0 v(m_2-K)\,v^{2(1-m_2K)}x^2$$

$$\left.\begin{aligned} \lambda &= v^{m_2/K-1} \\ \mu &= v^{m_2/K}. \end{aligned}\right\} \tag{12}$$

This applies to the regions:

$\underline{m_2 > 0}$

$$\text{BC} \quad \text{(deceleration)}, \quad 0 < \frac{m_2}{K} < 1, \quad v < 1,$$

and so

$$\lambda \to \infty, \quad \mu \to 0.$$

$\underline{m_2 < 0}$

$$\text{EO} \quad \text{(acceleration)}, \quad \frac{m_2}{K} > 1, \quad v > 1$$

and so

$$\lambda \to \infty, \quad \mu \to \infty.$$

$$\text{OF} \quad \text{(deceleration)}, \quad \frac{m_2}{K} < 0, \quad v < 1$$

and so

$$\lambda \to \infty, \quad \mu \to \infty.$$

A broad conclusion from these results is that λ, or μ, or both, exceed unity in all cases except one, the region OB of the sketch. In this case the system is stable at constant speed, and has an initial deceleration which does not exceed $m_2 V_0^2$. In these conditions $\lambda = 1$ and $\mu \to 0$ with v.

4.1.2. *Exact solution when oscillatory.*—The hyperbolic equation is integrated in Appendix II where it is transformed to the constant coefficient equation

$$\frac{d^2x}{dy^2} + \left(\frac{m_2}{K} - 1\right)\frac{dx}{dy} + \frac{m_1}{K^2}x = 0 \tag{13}$$

by the substitution

$$y = \log(1 + KV_0 t).$$

The characteristic equation is therefore

$$u^2 + \left(\frac{m_2}{K} - 1\right)u + \frac{m_1}{K^2} = 0. \tag{14}$$

If the roots are complex, $-\alpha \pm i\beta$, so that

$$\alpha = \tfrac{1}{2}\left(\frac{m_2}{K} - 1\right)$$

$$\beta^2 = \frac{m_1}{K^2} - \tfrac{1}{4}\left(\frac{m_2}{K} - 1\right)^2,$$

the solution is

$$x = C\sqrt{\left(x_0^2 + \frac{\dot{x}_0^2}{m_1 V_0^2}\right)}\, v^\alpha \cos(\beta \log v + \epsilon_1)$$

$$\dot{x} = D\sqrt{(m_2 V_0^2 x_0^2 + \dot{x}_0^2)}\, v^{\alpha+1} \cos(\beta \log v + \epsilon_2)$$

where C, D, ϵ_1, ϵ_2 are constants and C, D become equal to unity when α/β is small.

The solution is oscillatory, the time interval between successive zeros increasing without limit in deceleration ($v < 1$, $K > 0$) and decreasing to zero in acceleration ($v > 1$, $K < 0$).

The enveloping curves are given by

$$\left.\begin{aligned} x_E &= C\sqrt{\left(x_0^2 + \frac{\dot{x}_0^2}{m_1 v_0^2}\right)}\, v^{\alpha} \\ \dot{x}_E &= D\sqrt{(m_1 V_0 x_0^2 + \dot{x}_0^2)}\, v^{\alpha+1}. \end{aligned}\right\} \qquad (15)$$

The asymptotic values are therefore:

$$\left.\begin{array}{ll} \alpha > 0, & x, \dot{x} \to 0 \\ -1 < \alpha < 0, & x \to \infty, \dot{x} \to 0 \\ \alpha < -1, & x, \dot{x} \to \infty \end{array}\right\} \text{in deceleration } (v < 1, K > 0)$$

$$\left.\begin{array}{ll} \alpha > 0, & x, \dot{x} \to \infty \\ -1 < \alpha < 0, & x \to 0, \dot{x} \to \infty \\ \alpha < -1, & x, \dot{x} \to 0 \end{array}\right\} \text{in acceleration } (v > 1, K < 0)$$

4.1.3. *Comparison of bounds with envelopes of oscillation.*—It is interesting to compare the bounds x_B, $\dot{x}_B$ of (11) and (12) with the envelopes x_E, $\dot{x}_E$ of (15). The bounds can be written

$$\left.\begin{aligned} x_B &= \sqrt{\left(x_0^2 + \frac{\dot{x}_0^2}{m_1 V_0^2}\right)} \\ \dot{x}_B &= v\sqrt{(m_1 V_0^2 x_0^2 + \dot{x}_0^2)} \end{aligned}\right\} m_2 > K$$

$$\left.\begin{aligned} x_B &= v^{2\alpha}\sqrt{\left(x_0^2 + \frac{\dot{x}_0^2}{m_1 V_0^2}\right)} \\ \dot{x}_B &= v^{2\alpha+1}\sqrt{(m_1 V_0^2 x_0^2 + \dot{x}_0^2)} \end{aligned}\right\} m_2 < K$$

In deceleration $\alpha > 0$ corresponds to $m_2 > K$ and $\alpha < 0$ to $m_2 < K$.

Hence for $\alpha > 0$ we have

$$x_B/x_E = \frac{1}{C} v^{-\alpha}, \qquad \dot{x}_B/\dot{x}_E = \frac{1}{D} v^{-\alpha}$$

and for $\alpha < 0$,

$$x_B/x_E = \frac{1}{C} v^{\alpha}, \qquad \dot{x}_B/\dot{x}_E = \frac{1}{D} v^{\alpha}.$$

Thus in all cases

$$x_B/x_E = \frac{1}{C} v^{-|\alpha|}, \qquad \dot{x}_B/\dot{x}_E = \frac{1}{D} v^{-|\alpha|}$$

and

$$x_B = x_E,\ \dot{x}_B = \dot{x}_E \qquad \text{when } \alpha = 0.$$

Hence, apart from the constants C, D the bounds multiply the envelopes by the increasing function $v^{-|\alpha|}$ $(v < 1)$.

In acceleration $\alpha > 0$ corresponds to $m_2 < K$ and $\alpha < 0$ to $m_2 > K$.

By a similar argument we find

$$x_B/x_E = \frac{1}{C} v^{|\alpha|}, \qquad \dot{x}_B/\dot{x}_E = \frac{1}{D} v^{|\alpha|}$$

$$x_B = x_E,\ \dot{x}_B = \dot{x}_E \qquad \text{when } \alpha = 0$$

and the bounds multiply the envelopes by the increasing function $v^{|\alpha|}$ $(v > 1)$.

Thus with this speed law, when the motion is oscillatory, the difference between the bounds and the envelopes is expressed by a *factor* of uncertainty. The bounds multiply the envelopes by an increasing function, $v^{|\alpha|}$ in acceleration and $v^{-|\alpha|}$ in deceleration. This result is analogous to that obtained in the analysis of the equation with constant coefficients.

4.1.4. *Aperiodic solution.*—If equation (14) has real roots $-\gamma$, $-\delta$ the solution is aperiodic:

$$x = Av^{\gamma} + Bv^{\delta}.$$

We cannot then define an envelope or discuss the factor of uncertainty in the same way. We can, however, see to some extent how the extreme values of the solutions compare with the bounds by adapting the analysis of Section 3.1.2.

It is convenient to transform (13) to

$$\frac{d^2x}{dz^2} + V_0(m_2 - K)\frac{dx}{dz} + m_1 V_0^2 x = 0 \tag{16}$$

by the further substitution

$$z = \frac{y}{KV_0}.$$

Equation (16) has, of course, the same initial and stationary values of x as equation (13). Its solution has also, by Section 3.1, the bounds (11), (12). And finally it may be verified that $(dx/dz)_0 = \dot{x}_0$.

We shall consider only the case $m_2 > K$, when $x \to 0$ as $t \to \infty$. In this case (*see* Section 3.1.2) Fig. 6 gives the ratio of the extreme values of x to its bound (11), in terms of the parameters

$$h = \frac{\left(\frac{dx}{dz}\right)_0}{V_0(m_2 - K)x_0} = \frac{1}{V_0(m_2 - K)}\frac{\dot{x}_0}{x_0}$$

$$C = \frac{m_1}{(m_2 - K)^2}.$$

The adaptation of Fig. 8 to give similar information about $\dot{x}$ appears to be much less simple.

4.1.5. *Numerical example.*—As an illustration the numerical solutions have been worked out for the following conditions:

$$V_0 = 200 \text{ ft/sec}$$

$$\text{initial acceleration} = g \quad \text{or} \quad -g.$$

The constants m_1 and m_2 have been chosen so that at a steady speed of 200 ft/sec the period is 3 seconds and the oscillation damps to half its amplitude in one cycle. This gives

$$m_1 V_0^2 = 4\cdot444, \qquad m_2 V_0 = 0\cdot4622, \qquad KV_0^2 = \mp 32\cdot2,$$

so that

$$m_1 = 0\cdot000\ 111\ 1, \qquad m_2 = 0\cdot002311, \qquad K = \mp 0\cdot000\ 805.$$

The solutions for x and $\dot{x}$ are plotted for the decelerated case in Fig. 14 and for the accelerated case in Fig. 15. The envelope of the curves and the bounds predicted by the Polya analysis are also shown. We see that the Polya bounds are pessimistic and in this example in the accelerating case they predict an increasing oscillation when in fact it decreases.

In this case we may use the method of Section 2.3 and try to obtain closer bounds. We obtain

$$\ddot{u} + (m_2+2K)\, V\dot{u} + (m_1+m_2K)\, V^2 u = 0$$

where $u = \dot{x}$. We deduce that if $m_1 + m_2K > 0$, $m_2 + K > 0$ the bound for u will be constant. As shown in Section 2.3 this constant value depends on the initial conditions. In our example we have

$$m_1 + m_2K = 0{\cdot}000\ 092\ 2 \qquad \text{and} \qquad m_2 + K = 0{\cdot}001\ 506$$

so that these conditions are satisfied and $\dot{x}$ has a constant bound.

In Fig. 16 the acceleration $\ddot{x}$ is plotted for the accelerated case and we see that although x and $\dot{x}$ are tending to die out the amplitude of $\ddot{x}$ is increasing. This is typical of this sort of equation; the decay of the variable itself is no guarantee of the decay of its derivatives.

4.2. *Linear Speed Variation.*

4.2.1. *Analysis of bounds.*—Write the speed law

$$v = \frac{V}{V_0} = 1 + nV_0 t,$$

n being a constant, positive for acceleration and negative for deceleration. In the latter case we shall consider only positive values of V.

Thus for $n > 0$, with acceleration nV_0^2, v goes from 1 to ∞ and for $n < 0$, with deceleration nV_0^2, v goes from 1 to 0.

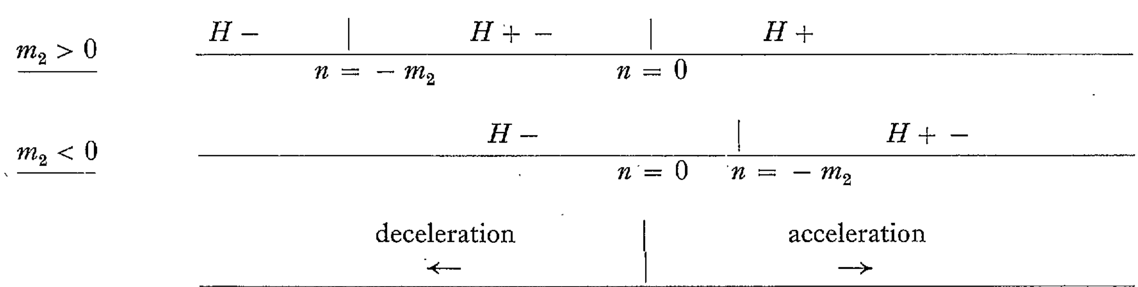
We then have

$$c = m_1 V_0^2 v^2$$

$$H = \frac{2V_0}{v}(m_2 v^2 + n)$$

so that $H = 0$ at $v = v_1$ where $v_1^2 = -n/m_2$.

Since v is always positive, H has the sign of $m_2v^2 + n$, and can therefore in certain cases change sign once during the motion. The signs of H can be summarised as follows:



To make a survey of the bounds we shall require the integral of H over certain intervals. We have

$$\int_T^t H\,dt = \int_{v_T}^{v} \frac{H}{\dot{v}}\,dv = \frac{2}{n}\int_{v_T}^{v}\left(m_2 v + \frac{n}{v}\right)dv$$

$$= \frac{m_2}{n}(v^2 - v_T^2) + \log\frac{v^2}{v_T^2}.$$

Then if $\tau = 0$

$$\int_0^t H\,dt = \frac{m_2}{n}(v^2-1) + \log v^2$$

and

$$\exp\left(\int_0^t H\,dt\right) = v^2 \exp\left\{\frac{m_2}{n}(v^2-1)\right\}.$$

This is required when H is negative throughout.

Again if $\tau = t_1$ where $H = 0$ and so $v_1^2 = -n/m_2$ we have

$$\int_{t_1}^t H\,dt = 1 - \frac{v^2}{v_1^2} + \log\frac{v^2}{v_1^2}$$

and

$$\exp\left(\int_{t_1}^t H\,dt\right) = \frac{v^2}{v_1^2}\exp\left(1 - \frac{v^2}{v_1^2}\right).$$

This is required when H changes sign from $+$ to $-$.

The f functions and the bounds can now be written down from the general formulae.

4.2.2. *Consider first the condition* $m_2 > 0$, the system being stable at constant speed. The sketch shows that there are three cases to consider.

(1) *Acceleration* $n > 0$.

$$f = x^2 + \frac{\dot{x}^2}{m_1 V_0^2 v^2}$$

$$\dot{f} = -\frac{2}{m_1 V_0 v^3}(m_2 v^2 + n)\,\dot{x}^2$$

$$\lambda = \frac{x_B}{\sqrt{\left(x_0^2 + \dfrac{\dot{x}_0^2}{m_1 V_0^2}\right)}} = 1$$

$$\mu = \frac{\dot{x}_B}{\sqrt{(\dot{x}_0^2 + m_1 V_0^2 x_0^2)}} = v \to \infty \text{ with } v.$$

(2) *Deceleration,* $-m_2 < n < 0$.

In this case H changes from $+$ to $-$ at $v^2 = v_1^2 = -n/m_2$.

Hence in the interval $v_1 < v < 1$ the results are as for case (1), and thereafter:

$$f = \frac{v^2}{v_1^2}\exp\left(1 - \frac{v^2}{v_1^2}\right)\left(x^2 + \frac{\dot{x}^2}{m_1 V_0^2 v^2}\right)$$

$$\dot{f} = -2 m_2 V_0 v \left(1 - \frac{v^2}{v_1^2}\right)\exp\left(1 - \frac{v^2}{v_1^2}\right) x^2$$

$$\lambda = \frac{v_1}{v}\exp\left\{-\tfrac{1}{2}\left(1 - \frac{v^2}{v_1^2}\right)\right\} \to \infty \text{ as } v \to 0$$

$$\mu = v_1 \exp\left\{-\tfrac{1}{2}\left(1 - \frac{v^2}{v_1^2}\right)\right\} \to \frac{v_1}{\sqrt{e}} \text{ as } v \to 0.$$

(3) *Deceleration*, $n < -m_2$.

H being negative throughout we have

$$f = v^2 \exp\left\{-\frac{m_2}{n}(1-v^2)\right\}\left(x^2 + \frac{\dot{x}^2}{m_1 V_0^2 v^2}\right)$$

$$\dot{f} = 2V_0 v(m_2 v^2 + n)\exp\left\{-\frac{m_2}{n}(1-v^2)\right\} x^2$$

$$\lambda = \frac{1}{v}\exp\left\{\frac{m_2}{2n}(1-v^2)\right\} \to \infty \text{ as } v \to 0$$

$$\mu = \exp\left\{\frac{m_2}{2n}(1-v^2)\right\} \to \exp\frac{m_2}{2n} \text{ as } v \to 0.$$

4.2.3. *Next consider the condition* $m_2 < 0$, the system now being unstable at constant speed. The sketch shows that there are two cases, according as $n <$ or $> -m_2$.

(1) $n < -m_2$. The solution is formally the same as case (3) above, so that

$$\lambda = \frac{1}{v}\exp\left\{\frac{m_2}{2n}(1-v^2)\right\}$$

$$\mu = \exp\left\{\frac{m_2}{2n}(1-v^2)\right\},$$

but we must distinguish between deceleration and acceleration. In deceleration ($n < 0$, $v < 1$), λ increases and $\to \infty$ as $1/v$, and μ decreases to the limit $\exp(m_2/2n)$. In acceleration up to $n = -m_2$, λ increases and $\to \infty$ as $(1/v)\exp\{(-m_2/2n)v^2\}$, and μ increases and $\to \infty$ as $\exp\{(-m_2/2n)v^2\}$.

(2) *Acceleration with* $n > -m_2$.

The solution is formally the same as case (2) above, and n is now positive and $v > 1$. $\lambda = 1$ up to $v = v_1$ or $\sqrt{(n/-m_2)}$, and thereafter increases to ∞ like

$$(1/v)\exp\{\tfrac{1}{2}(v^2/v_1^2)\}.$$

μ increases as v up to v_1 and thereafter increases to ∞ like

$$\exp\{\tfrac{1}{2}(v^2/v_1^2)\}.$$

Surveying the above results as a whole, it appears that λ, or μ, or both, exceed unity in every case except the first part of the motion in case 4.2.2 (2) when the system is stable at constant speed and the deceleration is less than $m_2 V_0^2$.

4.2.4. *Comparison with exact solution.*—The exact solution of the equation discussed above is derived formally in terms of confluent hypergeometric functions in Appendix II. This solution is of no practical use for finite values of t in important practical cases because the functions have not been tabulated for imaginary values of the argument. The asymptotic behaviour ($t \to \infty$) of the exact solution can, however, be obtained in a case of particular interest: when the motion at constant speed is oscillatory. The result is as follows:

If the constant speed solution is $\exp(-a \pm ib)t$, then x behaves like $t^{-1/2}\exp\{(-an/2)V_0 t^2\}$ and $\dot{x}$ behaves like $t^{1/2}\exp\{(-an/2)V_0 t^2\}$. Thus x, $\dot{x} \to 0$ or ∞ according as an is positive or negative.

The bounds derived above for $n > 0$ do not suggest this result in case 4.2.2(1) ($an > 0$) for in that case $\lambda = 1$ and μ increases as v or t. Thus in this particular case the $\dot{x}$ bound is too wide to be of practical use.

We may, however, try to narrow the $\dot{x}$ bound by the method of Section 2.2. The equation for u or $\dot{x}$ is

$$\ddot{u} + \left(m_2 V - \frac{2nV_0^2}{V}\right) \dot{u} + (m_1 V^2 - m_2 n V_0^2) u = 0 .$$

The model being stable at constant speed and accelerating, m_1, m_2 and n are all positive. Thus if the 'c' of this equation is positive we must have

$$m_1 - m_2 n > 0$$

and under this condition H reduces to

$$H = \frac{2V_0}{v^3(m_1 - m_2 n)} \{m_1 m_2 v^4 - n(m_2^2 + m_1)v^2 + 2m_2 n^2\}$$

where

$$v = \frac{V}{V_0} .$$

Since $m_1 m_2$ is positive, H must be positive in a range extending from a finite value of t to ∞.

Hence by the general theory the bound of $\dot{x}$ must be finite and constant as $t \to \infty$, provided $m_1 > m_2 n$.

4.2.5. *Numerical example.*—As an example we consider the same steady speed conditions as in Section 4.1. That is a period of 3 seconds and/or time to halve amplitude of 1 cycle at 200 ft/sec. We assume that the speed increases or decreases from this steady value with an acceleration of $\pm g$.

The values of λ and μ for $1g$ deceleration are plotted in Fig. 17. We see that as V decreases from 200 ft/sec λ is at first constant and μ decreases as V. As the speed decreases further λ increases to infinity as V tends to zero and μ continues to decrease to a value $0 \cdot 369$.

If there is an acceleration, λ is constant at unity and μ increases being equal to V/V_0. If we apply the method to the differentiated equation we find that in this example H does not change sign at all and the $\dot{x}$ bound is constant. The value of μ varies with the initial conditions but is always close to unity.

4.3. *Exponential Speed Variation.*

4.3.1. *Analysis of bounds.*—Suppose the speed changes exponentially from $V = V_0$ at $t = 0$ to $V = V_\infty$ at $t = \infty$, so that

$$V - V_\infty = (V_0 - V_\infty) \exp(-aV_0 t)$$

where a is a positive constant.

Writing $v = V/V_0$, $v_\infty = V_\infty/V_0$ we have

$$v - v_\infty = (1 - v_\infty) \exp(-aV_0 t)$$

and so

$$\dot{v} = aV_0(v_\infty - v)$$

where v goes from 1 to v_∞ as t goes from 0 to ∞, and v_∞ may have any value between 0 and ∞.

In this case, therefore,

$$c = m_1 V_0^2 v^2$$

$$\frac{\dot{c}}{c} = \frac{2\dot{v}}{v} = 2aV_0 \left(\frac{v_\infty}{v} - 1\right)$$

and

$$H = 2aV_0 \frac{kv^2 - v + v_\infty}{v}$$

where

$$k = \frac{m_2}{a}.$$

Since v is positive, H has the sign of $kv^2 - v + v_\infty$, and changes sign at the roots of

$$kv^2 - v + v_\infty = 0$$

that lie between 1 and v_∞.

This expression and hence H may have no changes of sign, or one, or two, between $v = 1$ and v_∞, depending on the values of v_∞ and k. The sign changes are determined in relation to v_∞ and k as follows.

Write

$$kv^2 - v + v_\infty \equiv k\left(v - \frac{1}{2k}\right)^2 - \frac{1}{4k} + v_\infty \equiv \psi \text{ (say)}.$$

Hence

$$\psi(1) = k - 1 + v_\infty$$

$$\psi(v_\infty) = kv_\infty^2$$

and ψ has a maximum when $k < 0$, and a minimum when $k > 0$, at $v = 1/2k$ with the value $v_\infty - 1/4k$.

Thus if k is negative there can be no stationary value in the range 1 to v_∞. $\psi(v_\infty)$ is always negative and there is one or no change of sign according as $k + v_\infty - 1 >$ or < 0.

If k is positive $\psi(v_\infty) > 0$ and if $k + v_\infty - 1 < 0$ there must be one sign change. If $k + v_\infty - 1 > 0$ there is no sign change or two. There are two if there is a minimum in the range 1 to v_∞ and the value at the minimum is negative, that is if

$$v_\infty - \frac{1}{4k} < 0.$$

If this lies in the range, then $1/2k$ lies between 1 and v_∞. This is impossible if $v_\infty > \frac{1}{2}$.

This analysis shows that the various sign sequences of H are separated by $k = 0$, $k + v_\infty - 1 = 0$ and $kv_\infty - \frac{1}{4} = 0$. These are shown on a v_∞, k diagram in Fig. 18.

If H is negative in an interval beginning with $t = t_1$, $v = v_1$ then

$$\int_{t_1}^{t} H\,dt = \int_{t_1}^{t} \left\{\frac{d}{dt}(\log c) + 2b\right\} dt$$

$$= \log \frac{v^2}{v_1^2} + 2\int_{t_1}^{t} b\,dt.$$

Also

$$\int_{t_1}^{t} b\,dt = m_2 V_0 \int_{v_1}^{v} \frac{v}{\dot{v}}\,dv$$

$$= k\int_{v_1}^{v} \frac{v}{v_\infty - v}\,dv$$

$$= -k(v-v_1) + \log\left(\frac{v - v_\infty}{v_1 - v_\infty}\right)^{-kv_\infty}.$$

Thus

$$\int_{t_1}^{t} H\,dt = \log\left[\frac{v^2}{v_1^2}\left(\frac{v - v_\infty}{v_1 - v_\infty}\right)^{-2kv_\infty}\right] - 2k(v-v_1)$$

and so

$$\exp\left(\int_{t_1}^{t} H\,dt\right) = \frac{v^2}{v_1^2}\left(\frac{v - v_\infty}{v_1 - v_\infty}\right)^{-2kv_\infty} \exp\{-2k(v-v_1)\}.$$

Expressions for $f, \dot{f}, x_B\ \dot{x}_B$ can now be constructed from the general formulae.

Expressions for $\lambda = x_B/\sqrt{(x_0^2 + \dot{x}_0^2/m_1V_0^2)}$ as a function of v, v_∞ and k are tabulated below for all the sign sequences of H. μ follows from the relation $\mu = \dot{x}_B/\sqrt{(\dot{x}_0^2 + m_1V_0^2x_0^2)} = v\lambda$.

Sign of H	λ	range in v
$+$	1	1 to v_∞
$-$	$\frac{1}{v}\left(\frac{v - v_\infty}{1 - v_\infty}\right)^{kv_\infty} \exp\{k(v-1)\}$	1 to v_∞
$+\ -$ changing at v_1	1	1 to v_1
	$\frac{v_1}{v}\left(\frac{v - v_\infty}{v_1 - v_\infty}\right)^{kv_\infty} \exp\{k(v-v_1)\}$	v_1 to v_∞
$-\ +$ changing at v_1	As for $H\ -$	1 to v_1
	$\frac{1}{v_1}\left(\frac{v_1 - v_\infty}{1 - v_\infty}\right)^{kv_\infty} \exp\{k(v_1-1)\}$	v_1 to v_∞
$+\ -\ +$ changing at v_1, v_2	As for $H + -$	1 to v_1 v_1 to v_2
	$\frac{v_1}{v_2}\left(\frac{v_2 - v_\infty}{v_1 - v_\infty}\right)^{kv_\infty} \exp\{k(v_2-v_1)\}$	v_2 to v_∞

4.3.2. *Numerical examples.*—From the preceding analysis we have seen that λ and μ, which determine the shape of the bounds of x and $\dot{x}$, are functions of $k = m_2/a$, v_∞ and v. These bounds have been plotted for various k for $v_\infty = 0{\cdot}2$ and $v_\infty = 2{\cdot}0$ in Figs. 19 to 22. Figs. 19, 20, for $v_\infty = 0{\cdot}2$, explore the section AB of Fig. 18. Figs. 21, 22 for $v_\infty = 2{\cdot}0$ explore the section CD.

Since a is a positive constant, positive k corresponds to a system which is stable in steady motion and negative k to an unstable system. When k is positive, λ and μ are always finite, λ is always less than 5 and μ always less than 1 when $v_\infty = 0 \cdot 2$. When $v = 2 \cdot 0$ λ is always less than 1 and μ less than 2. If $k < 0$ then λ and μ both increase without limit.

In order to present a clearer picture of the effects of change of speed these results have been used to plot λ and μ against time for constant m_2 and varying a. The values of m_2 chosen are appropriate to a model which halves or doubles its amplitude in 3 seconds when in a steady airstream at 200 ft/sec. The variation of speed with time for the various k are shown in Fig. 23.

The conclusions are:

(a) *Stable model decelerating*, $v_\infty = 0 \cdot 2$ (Figs. 24, 25).

For small decelerations (k large) λ is constant at unity and μ decreases slowly from unity to $0 \cdot 2$. As the deceleration increases λ remains constant at unity but μ decreases more rapidly to the value $0 \cdot 2$ until k reaches the value $1 \cdot 25$. When the deceleration increases beyond this, λ stays constant at unity for the first part of the motion and then rises gradually to a new value and stays constant again. μ still decreases but the value to which it falls begins to rise as the deceleration increases. This trend continues until k reaches $0 \cdot 8$. For larger decelerations than this λ rises from the beginning of the motion and its final value is higher for larger decelerations. The final value of μ continues to increase. When the deceleration is very large λ rises rapidly to the value 5 and remains constant and μ decreases little from unity.

(b) *Stable model accelerating*, $v_\infty = 2 \cdot 0$ (Fig. 26).

λ is always constant at unity. μ for $\dot{x}$ increases from $1 \cdot 0$ to $2 \cdot 0$ behaving in exactly the same way as v.

(c) *Unstable model decelerating*, $v_\infty = 0 \cdot 2$ (Figs. 27, 28).

μ always increases without limit but as the deceleration becomes larger the rate of increase becomes less. λ also increases without limit; the initial rate of increase is larger for larger decelerations but finally λ is lower for higher decelerations.

(d) *Unstable model accelerating*, $v_\infty = 2 \cdot 0$ (Figs. 29, 30).

Comparison of Figs. 29, 30 with Figs. 27, 28 shows that acceleration has opposite effects to deceleration on the λ, μ diagrams for an unstable model. μ now increases more steeply as the acceleration rises (Fig. 30) while λ increases at first less steeply and subsequently more steeply as the acceleration rises. If the acceleration is large enough to make $k < -1 \cdot 0$, λ is constant for some time after the disturbance.

PART II

Bound Analysis for Two Variables

5. *One second-order and one first-order equation.*

5.1. *Theory.*

So much for the analysis of a single second-order equation. The method can be extended in some cases to second-order equations with two variables x, y. The simplest system is that which is second order in x and first order in y:

$$\left.\begin{aligned} \ddot{x} + a\dot{x} + bx + cy &= 0 \\ \dot{y} + dy + e\dot{x} + fx &= 0 \end{aligned}\right\} \tag{17}$$

where a, b, c, d, e, f are functions of t.

Here we seek p, q, r, functions of t such that

$$F = px^2 + q\dot{x}^2 + ry^2 \tag{18}$$

is always positive, and $\dot{F}$ is always negative.

In this case the bounds are

$$x_B{}^2 = \frac{F_0}{p}, \qquad \dot{x}_B{}^2 = \frac{F_0}{q}, \qquad y_B{}^2 = \frac{F_0}{r} \tag{19}$$

where F_0 is the value of F at $t = 0$.

We find by differentiation that

$$\dot{F} = \dot{p}x^2 + (\dot{q}-2aq)\dot{x}^2 + (\dot{r}-2dr)y^2 + 2x\dot{x}(p-bq) - 2y\dot{x}(cq+er) - 2frxy$$

and so

$$\dot{F} = \dot{p}x^2 + (\dot{q}-2aq)\dot{x}^2 + (\dot{r}-2dr)y^2 \tag{20}$$

if

$$\left.\begin{aligned} p - bq &= 0 \\ cq + er &= 0 \\ fr &= 0\,. \end{aligned}\right\} \tag{21}$$

The method fails unless $f = 0$, for if $f \neq 0$ we must have

$$p = q = r = 0\,.$$

It follows from (21) that b and $\lambda = -c/e$ must be positive. (21) gives two relations between the three disposable functions p, q, r. We choose the third condition by equating to zero any one of the coefficients of x^2, $\dot{x}^2$, y^2 in $\dot{F}$. Consider these in turn.

<u>Case 1.</u> <u>$p = 1$</u>, so that $q = \dfrac{1}{b}$, $r = \dfrac{\lambda}{b}$.

Now

$$\dot{q} - 2aq = \dot{q}' \exp\left(\int 2a\,dt\right)$$

$$\dot{r} - 2dr = \dot{r}' \exp\left(\int 2d\,dt\right)$$

where

$$q' = q \exp\left(-\int 2a\,dt\right) = \frac{1}{b}\exp\left(-\int 2a\,dt\right)$$

$$r' = r \exp\left(-\int 2d\,dt\right) = \frac{\lambda}{b}\exp\left(-\int 2d\,dt\right).$$

Thus we have

$$F = x^2 + \frac{1}{b}\dot{x}^2 + \frac{\lambda}{b}y^2$$

$$\dot{F} = \dot{q}' \exp\left(\int 2a\,dt\right)\dot{x}^2 + \dot{r}' \exp\left(\int 2d\,dt\right)y^2.$$

Hence if b and λ are positive,
and

$$\frac{d}{dt}\left\{\frac{1}{b}\exp\left(-\int 2a\,dt\right)\right\} \quad \text{and} \quad \frac{d}{dt}\left\{\frac{\lambda}{b}\exp\left(-\int 2d\,dt\right)\right\}$$

are negative, all the conditions are satisfied and the bounds are

$$x_B^{\,2} = F_0, \qquad \dot{x}_B^{\,2} = bF_0, \qquad y_B^{\,2} = \frac{b}{\lambda}F_0$$

where

$$F_0 = x_0^{\,2} + \frac{1}{b_0}\dot{x}_0^{\,2} + \frac{\lambda_0}{b_0}y_0^{\,2}.$$

Note that if a and d are both positive the differential conditions reduce to $db/dt > 0$ and $(d/dt)\,(\lambda/b) < 0$.

Case 2. $\quad \dot{q} - 2aq = 0, \quad$ so that $\quad q = \exp\left(\int_0^t 2a\,dt\right), \quad p = bq, \quad r = \lambda q.$

In this case

$$\begin{aligned}\dot{r} - 2dr &= q\{\dot{\lambda} + 2(a-d)\lambda\}\\ &= q\exp\left(-\int 2(a-d)dt\right)\frac{d}{dt}\left\{\lambda\exp\left(\int 2(a-d)dt\right)\right\}\\ &= \exp\left(\int 2d\,dt\right)\frac{d}{dt}\left\{\lambda\exp\left(\int 2(a-d)dt\right)\right\}.\end{aligned}$$

Thus we have

$$F = \exp\left(\int_0^t 2a\,dt\right)(bx^2 + \dot{x}^2 + \lambda y^2)$$

$$\dot{F} = \frac{d}{dt}\left\{b\exp\left(\int 2a\,dt\right)\right\}x^2 + \exp\left(\int 2d\,dt\right)\frac{d}{dt}\left\{\lambda\exp\left(\int 2(a-d)dt\right)\right\}y^2.$$

Hence if b and λ are positive,
and

$$\frac{d}{dt}\left\{b\exp\left(\int 2a\,dt\right)\right\} \quad \text{and} \quad \frac{d}{dt}\left\{\lambda\exp\left(\int 2(a-d)dt\right)\right\}$$

are negative, all the conditions are satisfied and the bounds are

$$x_B^2 = \frac{\exp\left(-\int_0^t 2a\,dt\right)}{b} F_0, \quad \dot{x}_B^2 = \exp\left(-\int_0^t 2a\,dt\right) F_0, \quad y_B^2 = \frac{\exp\left(-\int_0^t 2a\,dt\right)}{\lambda} F_0$$

where

$$F_0 = b_0 x_0^2 + \dot{x}_0^2 + \lambda_0 y_0^2.$$

Case 3. $\dot{r} - 2dr = 0$, so that $r = \exp\left(\int_0^t 2d\,dt\right)$, $q = \frac{r}{\lambda}$, $p = \frac{b}{\lambda} r$.

In this case by similar reasoning we have

$$F = \exp\left(\int_0^t 2d\,dt\right)\left(\frac{b}{\lambda} x^2 + \frac{1}{\lambda}\dot{x}^2 + y^2\right)$$

$$\dot{F} = \frac{d}{dt}\left\{\frac{b}{\lambda}\exp\left(\int 2d\,dt\right)\right\} x^2 + \exp\left(\int 2a\,dt\right)\frac{d}{dt}\left\{\frac{1}{\lambda}\exp\left(\int 2(d-a)dt\right)\right\}\dot{x}^2.$$

Hence if b and λ are positive, and

$$\frac{d}{dt}\left\{\frac{b}{\lambda}\exp\left(\int 2d\,dt\right)\right\} \quad \text{and} \quad \frac{d}{dt}\left\{\frac{1}{\lambda}\exp\left(\int 2(d-a)dt\right)\right\}$$

are negative, all the conditions are satisfied and the bounds are

$$x_B^2 = \frac{\lambda}{b}\exp\left(-\int_0^t 2d\,dt\right)F_0, \quad \dot{x}_B^2 = \lambda\exp\left(-\int_0^t 2d\,dt\right)F_0, \quad y_B^2 = \exp\left(-\int_0^t 2d\,dt\right)F_0$$

where

$$F_0 = \frac{b_0}{\lambda_0}x_0^2 + \frac{\dot{x}_0^2}{\lambda_0} + y_0^2.$$

5.1.1. It can be shown as follows that the cases considered above are mutually exclusive. For the functions whose differential coefficients have to be negative are:

(1) $\frac{1}{b}\exp\left(-\int 2a\,dt\right)$, $\frac{\lambda}{b}\exp\left(-\int 2d\,dt\right)$

(2) $b\exp\left(\int 2a\,dt\right)$, $\lambda\exp\left(\int 2(a-d)dt\right)$

(3) $\frac{b}{\lambda}\exp\left(\int 2d\,dt\right)$, $\frac{1}{\lambda}\exp\left(\int 2(d-a)dt\right)$

and it will be seen that a function and its reciprocal occur in any two cases. Hence there is only one solution at most for given functions a, b, c, d, e.

5.1.2. *When* $b = 0$, the original equations are both of the first order if we put $\dot{x} = z$:

$$\dot{z} + az + cy = 0$$

$$\dot{y} + dy + ez = 0.$$

Thus x and case (1) disappear from the solution and we are left with cases (2) and (3) with $b = 0$. The necessary conditions are:

case (2), $$\lambda > 0, \frac{d}{dt}\left\{\lambda \exp\left(\int 2(a-d)dt\right)\right\} < 0$$

case (3), $$\lambda > 0, \frac{d}{dt}\left\{\frac{1}{\lambda} \exp\left(\int 2(d-a)dt\right)\right\} < 0.$$

5.2. *Application to a Problem of Jet Lift.*

Equations of the form discussed in this section occur in the theory of the motion of transition of a jet-borne aircraft between rest and airborne flight. Consider a simple transition motion in which the aircraft flies horizontally with acceleration n at no lift. The weight is supported by the jet and the drag is neglected. In considering the disturbed motion from this path there are two variables, x the attitude and y the incidence, and the speed is

$$V = V_0 + nt.$$

The usual static and damping moments in pitch are respectively proportional to V^2y and $V\dot{x}$. These are reinforced by two moments, proportional to x and $\dot{x}$, to provide stability in the hovering condition. The pitching-moment equation is therefore

$$\ddot{x} + (m_2V+m_3)\dot{x} + m_4x + m_1V^2y = 0$$

where m_1, m_2, m_3, m_4 are positive constants.

The forces acting in the disturbed motion are shown in Fig. 31, $n(W/g)$ being the component of the jet reaction supplying the acceleration n in the undisturbed motion. The acceleration $V(d/dt)(x-y)$ normal to the path is provided by lift proportional to V^2y and by $n(W/g)y$. The lift equation is therefore

$$\dot{y} + \left(lV + \frac{n}{V}\right)y - \dot{x} = 0$$

where l is a positive constant.

In this case therefore we have

$$a = m_2V + m_3$$
$$b = m_4$$
$$c = m_1V^2$$
$$d = lV + \frac{n}{V}$$
$$e = -1.$$

It follows that if $p = 1$, then

$$q = \frac{1}{m_4}$$
$$r = \frac{m_1V^2}{m_4}$$
$$\dot{r} - 2dr = -\frac{2lm_1V^3}{m_4}$$
$$\dot{q} - 2aq = -\frac{2(m_2V+m_3)}{m_4}.$$

Hence so long as $V > 0$, case (1) is applicable, with the results

$$F = x^2 + \frac{\dot{x}^2}{m_4} + \frac{m_1 V^2}{m_4} y^2$$

$$\dot{F} = -\frac{2(m_2 V + m_3)}{m_4} \dot{x}^2 - \frac{2lm_1 V^3}{m_4} y^2$$

$$x_B{}^2 = F_0, \qquad \dot{x}_B{}^2 = m_4 F_0, \qquad y_B{}^2 = \frac{m_4}{m_1 V^2} F_0$$

where

$$F_0 = x_0{}^2 + \frac{\dot{x}_0{}^2}{m_4} + \frac{m_1 V_0{}^2}{m_4} y_0{}^2 .$$

We also note that if $w = Vy$ is the velocity normal to the chord, then

$$w_B{}^2 = \frac{m_4}{m_1} F_0 .$$

These results are independent of n and therefore apply whether the motion is accelerating or decelerating so long as in the latter case the speed does not fall to zero. In all cases the bounds of x, $\dot{x}$ and w are constant, but the bound of y (the incidence) varies inversely as V. We should therefore expect no trouble in transition motions of this simple kind.

It is interesting to examine the effect of removing the automatic control $m_3\dot{x}$, $m_4 x$. Here, cases (2), (3) apply with $b = 0$. In case (2) we have, from $\dot{q} - 2aq = 0$,

$$q = \exp\left(\int_0^t 2m_2 V\,dt\right)$$

$$= \exp(2m_2 s)$$

where s is distance travelled in time t

$$r = m_1 V^2 q$$

$$\dot{r} - 2dr = 2m_1(m_2 - l)V^3 q .$$

If, therefore, $m_2 < l$ the results are

$$F = (\dot{x}^2 + m_1 V^2 y^2) \exp(2m_2 s)$$

$$\dot{F} = -2m_1(l - m_2)V^3 \exp(2m_2 s) y^2$$

$$\dot{x}_B{}^2 = \exp(-2m_2 s) F_0$$

$$y_B{}^2 = \frac{\exp(-2m_2 s)}{m_1 V^2} F_0$$

$$w_B{}^2 = \frac{\exp(-2m_2 s)}{m_1} F_0$$

where

$$F_0 = \dot{x}_0{}^2 + m_1 V_0{}^2 y_0{}^2 .$$

In case (3) we have from $\dot{r} - 2dr = 0$,

$$r = \exp\left\{\int_0^t 2\left(lV + \frac{n}{V}\right) dt\right\}$$

$$= \exp\left\{\int 2\left(lds + \frac{dV}{V}\right)\right\}$$

$$= \frac{V^2}{V_0^2} \exp(2ls)$$

$$q = \frac{r}{m_1 V^2}$$

$$\dot{q} - 2aq = \frac{2(l - m_2)}{m_1 V} r$$

$$= \frac{2(l - m_2)}{m_1 V_0^2} V \exp(2ls).$$

If, therefore, $m_2 > l$ the results are

$$F = \frac{\exp(2ls)}{m_1 V_0^2} (\dot{x}^2 + m_1 V^2 y^2)$$

$$\dot{F} = -\frac{2(m_2 - l)}{m_1 V_0^2} V \dot{x}^2 \exp(2ls)$$

$$\dot{x}_B{}^2 = m_1 V_0^2 \exp(-2ls) F_0$$

$$y_B{}^2 = \frac{V_0^2}{V^2} \exp(-2ls) F_0$$

$$w_B{}^2 = V_0^2 \exp(-2ls) F_0$$

where

$$F_0 = \frac{\dot{x}_0^2}{m_1 V_0^2} + y_0^2.$$

As before, these results apply whether the motion is accelerating or decelerating, so long as in the latter case the speed does not become negative. With this proviso the bounds of $\dot{x}$ and w always decrease. The bound of y decreases in acceleration but ultimately increases in deceleration as V approaches zero.

6. *Two Second-Order Equations.*

Finally we may sketch the bounds analysis for the general second-order equations with two variables:

$$\left.\begin{aligned} \ddot{x} + a\dot{x} + bx + c\dot{y} + dy &= 0 \\ \ddot{y} + e\dot{y} + fy + g\dot{x} + hx &= 0 \end{aligned}\right\} \qquad (22)$$

where $a, b \ldots$ are functions of t.

We now want to find p, q, r, s functions of t such that

$$F = px^2 + q\dot{x}^2 + ry^2 + s\dot{y}^2$$

is always positive and $\dot{F}$ is always negative, where x, y satisfy (22).

By differentiation, using equations (22)

$$\dot{F} = \dot{p}x^2 + \dot{r}y^2 + (\dot{q}-2aq)\dot{x}^2 + (\dot{s}-2es)\dot{y}^2 + \\ + 2(p-bq)x\dot{x} + 2(r-fs)y\dot{y} - 2(cq+gs)\dot{x}\dot{y} - \\ - 2qd\dot{x}y - 2sh\dot{y}x\,.$$

It is not easy to proceed unless $d = h = 0$, in which case the only coupling terms are damping ones.

With this simplification we have

$$\dot{F} = \dot{p}x^2 + \dot{r}y^2 + (\dot{q}-2aq)\dot{x}^2 + (\dot{s}-2es)\dot{y}^2 \tag{23}$$

provided p, q, r, s are chosen so that

$$\left.\begin{aligned} p - bq &= 0 \\ r - fs &= 0 \\ cq + gs &= 0\,. \end{aligned}\right\}$$

Since p, q, r, s must be positive, these relations show that

$$b, f, \qquad \text{and} \qquad \sigma = -\frac{c}{g} \tag{24}$$

must be positive.

We then have

$$\frac{p}{b} = \frac{q}{1} = \frac{r}{f\sigma} = \frac{s}{\sigma}$$

three relations between p, q, r, s.

We may choose the fourth condition by suppressing one of the four terms in (23). We get one type of solution by suppressing either x^2 or y^2, and a second type by suppressing either $\dot{x}^2$ or $\dot{y}^2$.

Type 1. Suppress x^2 by choosing $p = 1$, so that

$$q = \frac{1}{b}, \qquad r = \frac{f\sigma}{b}, \qquad s = \frac{\sigma}{b}.$$

Now put

$$\left.\begin{aligned} q' &= q \exp\left(-\int 2a\,dt\right) \\ s' &= s \exp\left(-\int 2e\,dt\right) \\ \dot{x}' &= \dot{x} \exp\left(\int a\,dt\right) \\ \dot{y}' &= \dot{y} \exp\left(\int e\,dt\right) \end{aligned}\right\}$$

and we have

$$F = x^2 + ry^2 + q'\dot{x}'^2 + s'\dot{y}'^2$$

$$\dot{F} = \qquad \dot{r}y^2 + \dot{q}'\dot{x}'^2 + \dot{s}'\dot{y}'^2\,.$$

Hence, if $\dot{r}$, $\dot{q}'$, $\dot{s}'$ are always negative, then $F < F_0$ and we get bounds for x, y, $\dot{x}$, $\dot{y}$ in the usual way. The conditions to be satisfied besides (24) are

$$\frac{d}{dt}\left(\frac{f\sigma}{b}\right), \qquad \frac{d}{dt}\left\{\frac{1}{b}\exp\left(-\int 2a\,dt\right)\right\}, \qquad \frac{d}{dt}\left\{\frac{\sigma}{b}\exp\left(-\int 2e\,dt\right)\right\} \text{ all negative.} \qquad (25)$$

If a, e are both positive it is sufficient to have

$$\frac{df}{dt}, \qquad \frac{d}{dt}\left(\frac{1}{b}\right), \qquad \frac{d}{dt}\left(\frac{\sigma}{b}\right) \text{ all negative.}$$

If we suppress y^2 in $\dot{F}$ by putting $r = 1$ we obtain a similar solution, the differential conditions being

$$\frac{d}{dt}\left(\frac{b}{f\sigma}\right), \qquad \frac{d}{dt}\left\{\frac{1}{f\sigma}\exp\left(-\int 2a\,dt\right)\right\}, \qquad \frac{d}{dt}\left\{\frac{1}{f}\exp\left(-\int 2e\,dt\right)\right\} \text{all negative.} \qquad (26)$$

Type 2. Suppress $\dot{x}^2$ by choosing $\dot{q} - 2aq = 0$ so that

$$q = \exp\left(\int 2a\,dt\right)$$

$$p = bq$$

$$r = f\sigma q$$

$$s = \sigma q.$$

It will be found that the substitutions

$$\left.\begin{aligned} b' &= b\exp\left(\int 2a\,dt\right) \\ (f\sigma)' &= (f\sigma)\exp\left(\int 2a\,dt\right) \\ \sigma' &= \sigma\exp\left(\int 2(a-e)dt\right) \\ \dot{x}' &= \dot{x}\exp\left(\int a\,dt\right) \\ \dot{y}' &= \dot{y}\exp\left(\int e\,dt\right) \end{aligned}\right\}$$

reduce F $\dot{F}$, to the forms

$$F = b'x^2 + \dot{x}'^2 + (f\sigma)'y^2 + \sigma'\dot{y}'^2$$

$$\dot{F} = \dot{b}'x^2 + (f\dot{\sigma})'y^2 + \dot{\sigma}'\dot{y}'^2.$$

Thus if $\dot{b}'$, $(f\dot{\sigma})'$, $\dot{\sigma}'$ are negative, $F < F_0$ and we get bounds for x, y, $\dot{x}$, $\dot{y}$ in the usual way. The conditions to be satisfied besides (24) are

$$\frac{d}{dt}\left\{b\exp\left(\int 2a\,dt\right)\right\}, \qquad \frac{d}{dt}\left\{f\sigma\exp\left(\int 2a\,dt\right)\right\}, \qquad \frac{d}{dt}\left\{\sigma\exp\left(\int 2(a-e)dt\right)\right\} \text{ all negative.} \qquad (27)$$

If we suppress $\dot{y}^2$ in $\dot{F}$ by choosing $\dot{s} - es = 0$ we obtain a similar solution, the differential conditions being

$$\frac{d}{dt}\left\{\frac{b}{\sigma}\exp\left(\int 2e\,dt\right)\right\}, \quad \frac{d}{dt}\left\{f\exp\left(\int 2e\,dt\right)\right\}, \quad \frac{d}{dt}\left\{\frac{1}{\sigma}\exp\left(\int 2(e-a)dt\right)\right\} \text{ all negative.} \qquad (28)$$

If the differential conditions (25) to (28) of the four solutions are examined it will be found, in any two cases, that the differential coefficients of one function and of its reciprocal are required to be negative. For example, in the first two solutions the function is $f\sigma/b$. Thus the solutions are mutually exclusive, and any set of functions $a, b, \ldots g$ will yield at most only one set of bounds, at least of the kind considered here.

If h, d are retained in equation (22) it is possible to proceed by writing $\dot{F}$ in the form

$$\dot{F} = \{\dot{p}x^2 - 2hsx\dot{y} + (\dot{s}-2es)\dot{y}^2\} + \{\dot{r}y^2 - 2dqy\dot{x} + (\dot{q}-2aq)\dot{x}^2\}$$

and introducing the conditions that the quantities within curly brackets are perfect squares, in addition to the conditions

$$p - bq = 0$$

$$r - fs = 0$$

$$cq + gs = 0.$$

However, the conditions under which bounds can be obtained now become too complicated for practical use.

7. *Conclusions.*

In classical stability theory the amplitudes of the motion do not explicitly occur: if the motion decays it is assumed that the amplitudes will always be small enough. It has, however, long been realised that in some flight problems this treatment is not adequate. Even if a complex system is stable, some types of disturbance may force one or more components of the motion to initial amplitudes that cannot be tolerated. Behind all such arguments lies the concept of bounds.

In the present analysis of the motion following a disturbance to a given accelerated motion we have in effect initiated a search for its bounds. This approach seems to us to be dictated by a combination of mathematical necessity with the practical logic of its application. In practical applications the datum motion persists for a short time only: the problem is to ensure that no disturbance occurring in that time produces intolerable amplitudes. Thus in any actual problem of this kind we start with a set of practical bounds. Even if the equation could be solved, we should be comparing the extremes of the motion with the practical bounds. This being generally impossible, the next best thing is to establish as narrowly as possible a set of theoretical bounds. If these turn out to be less than those practically required, a short cut has been found through the mathematical difficulties of the problem. The present paper is a preliminary sortie in this direction.

The particular method of bounds explored here has some very clear limitations. It does not, except in special cases, work for equations of higher degree than the second. It works only when certain relations between the functional coefficients of the equations are satisfied. And even when bounds are found, a basic and as yet unanswered question remains: how much too big are they in relation to the actual extremes of the motion ? Nevertheless we feel that the method has the right angle to the problem and could profitably be pursued in two directions.

(1) To attack the problem of excess in the bounds. This we think can only be done by making a survey of the numerical solutions of a number of widely assorted second-order equations, and examining the field of uncertainty thus obtained.

(2) Part II of this paper contains only a rough sketch of the application of the bound analysis to systems containing two variables. We need here a thorough classification of such systems for which bounds can be obtained, in order to see more clearly to what extent the method can be applied to practical current problems of accelerated flight.

Finally we may glance at the alternatives to this method. Analytically, since the frontal attack is so difficult, the only way open seems to be to divide the time range into intervals so small that in each the coefficients may be treated as constants, and then to construct the solution in steps, using the end condition of one interval as the initial conditions of the next. A matrix solution of this kind is worked out by Frazer, Duncan and Collar in Ref. 10. It is, however, laborious; it is of course blind in the sense that it is difficult to assess the nature of the solution until its salient points have been worked out; and it is impossible to assess the changes in character which arise from changing the parameters without computing many solutions. In particular, it is quite unsound in general to discuss one of the small intervals in terms of stability with constant coefficients: this will often lead to absurd results.

It may be then that a frontal attack is only possible by transferring it to the powerful computing machines now available. In this way we could obtain a wide range of solutions of some important practical problems, and then proceed by experience or by using the numerical survey as a basis for an approximate analytical approach. This is the sort of situation in which one is probably right in first working hard for a short cut.

NOTATION

b, c		Coefficients of the general second-order linear equation (Section 1)
x_B, $\dot{x}_B$		The bounds for x and $\dot{x}$ derived by the present analysis (Section 2.1)
x_E, $\dot{x}_E$		The envelopes of an oscillatory solution for x and $\dot{x}$ (Section 3.1.1)
H	$=$	$\frac{\dot{c}}{c} + 2b$ (Section 2.1)
λ	$=$	$x_B/\sqrt{(x_0^2 + \dot{x}_0^2/c_0)}$ (Section 2.4.1)
μ	$=$	$\dot{x}_B/\sqrt{(cx_0^2 + \dot{x}_0^2)}$ (Section 2.4.1)
		A dot denotes differentiation with respect to t
		A suffix 0 denotes the value at $t = 0$
$\int^*$		sign of integration when only negative values of the integrand are admitted.

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APPENDIX I

The Stationary Values of the Aperiodic Solution of $\ddot{x} + b\dot{x} + cx = 0$ when the Coefficients are Constant

We assume that b and c are positive. Then we have seen in Section 3.1.2 that the equation

$$\ddot{x} + b\dot{x} + cx = 0$$

may be reduced to

$$x'' + x' + Cx = 0$$

where $C = c/b^2$, $\tau = bt$ and a dash denotes differentiation w.r.t. τ.

If $C < \frac{1}{4}$ the solution is aperiodic and may be written

$$x = A \exp(-\alpha\tau) + B \exp(-\beta\tau)$$

where $\alpha + \beta = 1$, $\alpha\beta = C$ and the constants A, B are determined by the initial conditions. We shall assume $x = x_0$, $x' = Kx_0$ at $t = 0$. We have

$$\dot{x} = -\{A\alpha \exp(-\alpha\tau) + B\beta \exp(-\beta\tau)\}$$

so that

$$A = -\left(\frac{\beta + K}{\alpha - \beta}\right) x_0, \qquad B = \left(\frac{\alpha + K}{\alpha - \beta}\right) x_0.$$

Now since $C > 0$, α and β are both positive. It is convenient to take α to be the larger root so that we can be sure about the signs of the terms.

The solution for x has a stationary value when $\dot{x} = 0$, i.e. when

$$A\alpha \exp(-\alpha\tau) + B\beta \exp(-\beta\tau) = 0,$$

i.e. when

$$\exp\{(\alpha-\beta)\tau\} = -\frac{A\alpha}{B\beta} = \frac{\alpha(\beta + K)}{\beta(\alpha + K)}.$$

τ is real and positive when $\{\alpha(\beta+K)\}/\{\beta(\alpha+K)\} > 1$, i.e. when $K > 0$ or $K < -\alpha$. Now

$$\begin{aligned} x &= A \exp(-\alpha\tau) + B \exp(-\beta\tau) \\ &= \exp\left(-\frac{\tau}{2}\right)\left[B \exp\left\{\left(\frac{\alpha-\beta}{2}\right)\tau\right\} + A \exp\left\{-\left(\frac{\alpha-\beta}{2}\right)\tau\right\}\right] \end{aligned}$$

so the stationary value of x when $K > 0$ is given by

$$\begin{aligned} x_m &= x_0 \exp\left[-\frac{1}{2(\alpha-\beta)} \ln\left(\frac{\alpha(\beta+K)}{\beta(\alpha+K)}\right)\right]\left[\frac{(\alpha+K)}{(\alpha-\beta)}\sqrt{\left\{\frac{\alpha(\beta+K)}{\beta(\alpha+K)}\right\}} - \frac{(\beta+K)}{(\alpha-\beta)}\sqrt{\left\{\frac{\beta(\alpha+K)}{\alpha(\beta+K)}\right\}}\right] \\ &= x_0 \left\{\frac{\alpha(\beta+K)}{\beta(\alpha+K)}\right\}^{-1/\{2(\alpha-\beta)\}} \frac{\sqrt{\{(\alpha+K)(\beta+K)\}}}{\sqrt{(\alpha\beta)}} \\ &= x_0 \frac{\left(1 + \dfrac{K}{\alpha}\right)^{\alpha/(\alpha-\beta)}}{\left(1 + \dfrac{K}{\beta}\right)^{\beta/(\alpha-\beta)}}. \end{aligned}$$

Similarly when $K < -\alpha$

$$x_m = \frac{-x_0\left\{-\frac{K}{\alpha}-1\right\}^{\alpha/(\alpha-\beta)}}{\left\{-\frac{K}{\beta}-1\right\}^{\beta/(\alpha-\beta)}}.$$

The bound we deduced for x is

$$x_B = x_0\sqrt{\left(1+\frac{K^2}{C}\right)}$$

so that

$$\left|\frac{x_m}{x_B}\right| = \frac{\left|1+\frac{K}{\alpha}\right|^{\alpha/(\alpha-\beta)}}{\left|1+\frac{K}{\beta}\right|^{\beta/(\alpha-\beta)}\left\{1+\frac{K^2}{C}\right\}^{1/2}}.$$

For a stationary value of $\dot{x}$, $\ddot{x} = 0$ so that

$$\exp(\alpha-\beta)\tau = -\frac{A\alpha^2}{B\beta^2} = \frac{\alpha^2(\beta+K)}{\beta^2(\alpha+K)}$$

and

$$\dot{x}_m = -x_0\left\{\frac{\alpha^2(\beta+K)}{\beta^2(\alpha+K)}\right\}^{-1/\{2(\alpha-\beta)\}}\left\{\left(\frac{\alpha+K}{\alpha-\beta}\right)\beta\frac{\alpha}{\beta}\sqrt{\left(\frac{\beta+K}{\alpha+K}\right)}-\left(\frac{\beta+K}{\alpha-\beta}\right)\alpha\frac{\beta}{\alpha}\sqrt{\left(\frac{\alpha+K}{\beta+K}\right)}\right\}$$

which can be reduced to

$$\dot{x}_m = -x_0\frac{\left(1+\frac{K}{\alpha}\right)^{\alpha/(\alpha-\beta)}}{\left(1+\frac{K}{\beta}\right)^{\beta/(\alpha-\beta)}}\left(\frac{\beta}{\alpha}\right)^{1/\{2(\alpha-\beta)\}}\sqrt{(\alpha\beta)}.$$

Now the bound for $\dot{x}$ is

$$\dot{x}_B = x_0\sqrt{(K^2+C)} = x_0\sqrt{(\alpha\beta)}\sqrt{\left(1+\frac{K^2}{C}\right)}$$

so that

$$\left|\frac{\dot{x}_m}{\dot{x}_B}\right| = \left|\frac{x_m}{x_B}\right|\left(\frac{\alpha}{\beta}\right)^{1/\{2(\alpha-\beta)\}}.$$

The stationary values may or may not exceed the initial values. For these we have

$$\left|\frac{x_0}{x_B}\right| = \frac{1}{\sqrt{\left(1+\frac{K^2}{C}\right)}}$$

$$\left|\frac{\dot{x}_0}{\dot{x}_B}\right| = \frac{K}{\sqrt{(K^2+C)}} = \frac{K}{\sqrt{c}}\left|\frac{x_0}{x_B}\right|.$$

APPENDIX II

Exact Solution of the Pitching Equation for Hyperbolic Speed Variation

If the speed varies hyperbolically with time we have

$$\frac{V}{V_0} = \frac{1}{1+KV_0t}.$$

If we now write $y = \log(1+KV_0t) = \log V_0/V$ then the equation

$$\ddot{x} + m_2V\dot{x} + m_1V^2x = 0$$

reduces to

$$\frac{d^2x}{dy^2} + \left(\frac{m_2}{K} - 1\right)\frac{dx}{dy} + \frac{m_1}{K^2}x = 0$$

which is linear with constant coefficients and if the characteristic equation has complex roots $-\alpha \pm i\beta$ the solution is

$$\begin{aligned} x &= \exp(-\alpha y)\{A \sin \beta y + B \cos \beta y\} \\ &= v^\alpha \{A \sin(\beta \log v) + B \cos(\beta \log v)\} \end{aligned}$$

where

$$v = \frac{V}{V_0}$$

$$\alpha = \tfrac{1}{2}\left(\frac{m_2}{K} - 1\right)$$

$$\beta^2 = \frac{m_1}{K^2} - \tfrac{1}{4}\left(\frac{m_2}{K} - 1\right)^2.$$

Now since $\dot{x} = (dx/dv)\dot{v}$ and $\dot{v} = -KV_0v^2$ we have on differentiating x
$-\dot{x}/KV_0 = v^{\alpha+1}\{(\alpha A - \beta B)\sin(\beta \log v) + (\beta A + \alpha B)\cos(\beta \log v)\}$.

Now if $x = x_0$, $\dot{x} = \dot{x}_0$ when $t = 0$, i.e. $v = 1$

$$B = x_0$$

$$\beta A + \alpha B = -\frac{\dot{x}_0}{KV_0}$$

giving

$$A = -\frac{1}{\beta}\left(\alpha x_0 + \frac{\dot{x}_0}{KV_0}\right)$$

and

$$\alpha A - \beta B = -\frac{\alpha^2+\beta^2}{\beta}x_0 - \frac{\alpha}{\beta}\frac{\dot{x}_0}{KV_0}.$$

Thus

$$\begin{aligned} x &= v^\alpha\left\{x_0 \cos(\beta \log v) - \frac{1}{\beta}\left(\alpha x_0 + \frac{\dot{x}_0}{KV_0}\right)\sin(\beta \log v)\right\} \\ &= Pv^\alpha \cos(\beta \log v + \epsilon_1) \end{aligned}$$

$$\begin{aligned} \dot{x} &= KV_0v^{\alpha+1}\left\{\frac{\dot{x}_0}{KV_0}\cos(\beta \log v) + \left(\frac{\alpha^2+\beta^2}{\beta}x_0 + \frac{\alpha}{\beta}\frac{\dot{x}_0}{KV_0}\right)\sin(\beta \log v)\right\} \\ &= Qv^{\alpha+1}\cos(\beta \log v + \epsilon_2) \end{aligned}$$

where

$$P^2 = x_0^2\left(1+\frac{\alpha^2}{\beta^2}\right) + \frac{\dot{x}_0^{\,2}}{\beta^2 K^2 V_0^{\,2}} + \frac{2\alpha}{\beta^2 K V_0}\,x_0\dot{x}_0$$

$$Q^2 = K^2 V_0^{\,2}\left\{\frac{(\alpha^2+\beta^2)^2}{\beta^2}\,x_0^{\,2} + \frac{1}{K^2V_0^{\,2}}\left(1+\frac{\alpha^2}{\beta^2}\right)\dot{x}_0^{\,2} + \frac{2\alpha(\alpha^2+\beta^2)}{\beta^2}\,\frac{x_0\dot{x}_0}{KV_0}\right\}.$$

If α/β is small these reduce to

$$x = \sqrt{\left(x_0^{\,2} + \frac{\dot{x}_0^{\,2}}{m_1 V_0^{\,2}}\right)}\; v^{\alpha}\cos(\beta\log v + \epsilon_1)$$

$$\dot{x} = \sqrt{(m_1V_0^{\,2}x_0^{\,2}+\dot{x}_0^{\,2})}\,v^{\alpha+1}\cos(\beta\log v+\epsilon_1).$$

It is convenient then to write

$$x = C\sqrt{\left(x_0^{\,2} + \frac{x_0^{\,2}}{m_1 V_0^{\,2}}\right)}\; v^{\alpha}\cos(\beta\log v + \epsilon_1)$$

$$\dot{x} = D\sqrt{(m_1V_0^{\,2}x_0^{\,2}+\dot{x}_0^{\,2})}\,v^{\alpha+1}\cos(\beta\log v+\epsilon_2)$$

where

$$C^2\left(x_0^{\,2}+\frac{\dot{x}_0^{\,2}}{m_1V_0^{\,2}}\right) = P^2$$

$$D^2(m_1V_0^{\,2}x_0^{\,2}+\dot{x}_0^{\,2}) = Q^2.$$

The solution is an oscillation with the frequency varying logarithmically. The amplitudes of x and $\dot{x}$ vary in the following way:

$$\left.\begin{array}{ll} \alpha > 0 & x,\ \dot{x}\to 0 \\ -1<\alpha<0 & x\to\infty,\ \dot{x}\to 0 \\ \alpha < -1 & x,\ \dot{x}\to\infty \end{array}\right\}\ \text{as } t\to\infty\,.$$

The envelopes are for α/β small

$$x_E = v^{\alpha}\sqrt{\left(x_0^{\,2}+\frac{\dot{x}_0^{\,2}}{m_1V_0^{\,2}}\right)}$$

$$\dot{x}_E = v^{\alpha+1}\sqrt{(m_1V_0^{\,2}x_0^{\,2}+\dot{x}_0^{\,2})}\,.$$

If the characteristic equation has real roots, $-\gamma$, $-\delta$, the solution is

$$x = A\exp(-\gamma y) + B\exp(-\delta y)$$

$$= Av^{\gamma} + Bv^{\delta}$$

and on differentiation

$$\frac{\dot{x}}{-KV_0} = (\gamma A v^{\gamma+1} + \delta B v^{\delta+1})\,.$$

APPENDIX III

*Solution of the Pitching Equation for Linear Speed Variation**

The equation is

$$\ddot{x} + m_2(V_0+nt)\dot{x} + m_1(V_0+nt)^2x = 0.$$

This may be written

$$\ddot{x} + 2(p+qt)\dot{x} + \lambda(p+qt)^2x = 0$$

where

$$p = \frac{m_2}{2}V_0, \qquad q = \frac{m_2}{2}n, \qquad \lambda = \frac{4m_1}{m_2{}^2}.$$

Change the dependent variable from x to z by the substitution

$$x = \exp(-T)z$$

where

$$T = pt + \tfrac{1}{2}qt^2 + \frac{c}{2}(t-m)^2$$

c and m being disposable constants.

The result is

$$\ddot{z} - 2c(t-m)\dot{z} + \{(\lambda-1)(p+qt)^2 + c^2(t-m)^2 - q - c\}z = 0.$$

If c and m are chosen so that the coefficient of z is constant we have

$$\left.\begin{aligned} c &= \pm\sqrt{(1-\lambda)}q \\ m &= -\frac{p}{q} \end{aligned}\right\} \tag{1}$$

and the coefficient of z is $-(q+c)$ or $-c\{1\pm 1/\sqrt{(1-\lambda)}\}$.

The equation is now

$$\ddot{z} - 2c(t-m)\dot{z} - c\left(1 \pm \frac{1}{\sqrt{(1-\lambda)}}\right)z = 0$$

where c, m are given by (1).

Now change the independent variable by the substitution

$$c(t-m)^2 = u.$$

The result is

$$u\frac{d^2z}{du^2} + (\tfrac{1}{2}-u)\frac{dz}{du} - \alpha z = 0 \tag{2}$$

where

$$\alpha = \tfrac{1}{4}\left(1 \pm \frac{1}{\sqrt{(1-\lambda)}}\right). \tag{3}$$

---

* The properties of the confluent hypergeometric function used in this analysis are taken from a very useful paper by Webb and Airey[11].

The solution of (2) is

$$z = AM(\alpha, \tfrac{1}{2}, u) + Bu^{1/2} M(\alpha + \tfrac{1}{2}, \tfrac{3}{2}, u) = \overline{M}(\alpha, \tfrac{1}{2}, u)$$

in terms of the confluent hypergeometric function

$$M(\alpha, \gamma, u) = 1 + \frac{\alpha}{1.\gamma} u + \frac{\alpha(\alpha+1)}{1.2.\gamma(\gamma+1)} u^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{1.2.3\gamma(\gamma+1)(\gamma+2)} u^3 + \ldots$$

A and B are constants.

Thus the solution of the original equation is

$$x = \exp\left[-\{pt + \tfrac{1}{2}qt^2 + \frac{c}{t}(t-m)^2\}\right] \overline{M}\{\alpha, \tfrac{1}{2}, c(t-m)^2\} \tag{4}$$

where

$$\overline{M} = AM\{\alpha, \tfrac{1}{2}, c(t-m)^2\} + Bc^{1/2}(t-m)\, M\{\alpha + \tfrac{1}{2}, \tfrac{3}{2}, c(t-m)^2\}$$

and c, m, α are given by (1) and (3).

It appears that there are two values of c and α, associated with $\pm\sqrt{(1-\lambda)}$. But from the relation

$$M(\alpha, \gamma, u) = M(\gamma - \alpha, \gamma, -u) \exp u$$

connecting M's of positive and negative arguments u and $-u$ it can be shown that

$$\exp\left\{-\frac{\sqrt{(1-\lambda)}}{2} q(t-m)^2\right\} M\left\{\tfrac{1}{4}\left(1 + \frac{1}{\sqrt{(1-\lambda)}}\right), \tfrac{1}{2}, \sqrt{(1-\lambda)}q(t-m)^2\right\}$$

$$= \exp\left\{-\frac{\sqrt{(1-\lambda)}}{2} q(t-m)^2\right\} M\left\{\tfrac{1}{4}\left(1 - \frac{1}{\sqrt{(1-\lambda)}}\right), \tfrac{1}{2}, -\sqrt{(1-\lambda)}q(t-m)^2\right\}$$

and similarly for the second M function involved in $\overline{M}$.

It follows that in (4) the complete solution is given by choosing either of the two values of c given by (1).

The solution at constant speed ($n = 0$) is $\exp(-\mu t)$ where μ is given by

$$\mu^2 - 2p\mu + \lambda p^2 = 0$$

or

$$\frac{\mu}{p} = 1 \pm \sqrt{(1-\lambda)}.$$

Now

$$pt + \tfrac{1}{2}qt^2 + \frac{c}{2}(t-m)^2 = pt + \tfrac{1}{2}qt^2 \pm \frac{\sqrt{(1-\lambda)}}{2} q\left(t + \frac{p}{q}\right)^2$$

$$= \{1 \pm \sqrt{(1-\lambda)}\}\, p\left(t + \frac{q}{2p}t^2\right) \pm \sqrt{(1-\lambda)}\frac{p^2}{2q}$$

$$= \{1 \pm \sqrt{(1-\lambda)}\}\, p\left(t + \frac{n}{2V_0}t^2\right) + \text{constant}.$$

It follows that the exponential factor in (4) is

$$\exp\left\{-\mu\left(t + \frac{n}{2V_0}t^2\right)\right\} \quad \text{or} \quad \exp\left(-\frac{\mu s}{V_0}\right)$$

where μ is *one* root of the constant-speed solution

and s is the distance travelled by the airstream in time t.

We are particularly interested in the case where the constant speed solution is oscillatory. If $\mu = a + ib$ the exponential factor is of the form $\exp(-as/V_0)\cos(bs/V_0)$. When the speed is increasing s increases without limit and so the period $\to 0$ and the damping $\to \infty$ or $-\infty$ according as $a >$ or < 0. When the speed is decreasing s is a maximum at $V = 0$. Hence the period $\to \infty$ at $V = 0$, while if the damping is positive at constant speed it increases up to $V = 0$, and if the damping is negative at constant speed it increases negatively up to $V = 0$.

This, however, is only one factor in the solution for x. The M functions in (4) have an imaginary argument and a complex parameter α, and as they have not been tabulated in such ranges we do not know the form of x for finite t. We can, however, examine the asymptotic values of x since the asymptotic forms of the M functions are known.

When u is large

$$M(\alpha, \gamma, u) \propto u^{\alpha-\gamma} \exp u$$

and

$$u^{1-\gamma} M(\alpha-\gamma+1, 2-\gamma, u) \propto u^{-\alpha}.$$

In our case u is proportional to it^2, $\gamma = \frac{1}{2}$,

$$\alpha = \tfrac{1}{4}\left(1 \pm \frac{i}{\sqrt{(\lambda-1)}}\right)$$

$$\alpha - \gamma = \tfrac{1}{4}\left(-1 \mp \frac{i}{\sqrt{(\lambda-1)}}\right).$$

Hence

$$u^{-\alpha} \propto t^{-1/2}\, t^{i/\{2\sqrt{(\lambda-1)}\}}$$

$$\propto t^{-1/2} \exp\left[i \log t/\{2\sqrt{(\lambda-1)}\}\right]$$

and

$$u^{\alpha-\gamma} \exp u \propto t^{-1/2} \exp\left[i \log t/\sqrt{(\lambda-1)}\right] \exp(it^2).$$

Both functions therefore behave as $t^{-1/2} \exp(i\phi)$ where ϕ is a real function of t.

The asymptotic form of x is therefore

$$t^{-1/2} \exp\left(-(a+ib)\frac{nt^2}{2V_0}\right) \exp(i\phi)$$

and its behaviour depends on $t^{-1/2} \exp\{-(an/2V_0)t^2\}$. Thus $x \to 0$ or ∞ according as $an >$ or < 0.

By differentiation we find that $\dot{x}$ depends on $t^{1/2} \exp\{-(an/2V_0)t^2\}$ with the same result.

TABLE 1

Pitching Model—Values Used in Numerical Examples

General

Period at steady speed of 200 ft/sec	3 sec
Time to halve amplitude at steady speed of 200 ft/sec	3 sec
m_1	$0{\cdot}000\,111$ ft^{-2}
m_2	$0{\cdot}00231$ ft^{-1}

Hyperbolic speed variation $V = V_0/(1 + KV_0t)$.

$K = \pm\, 0{\cdot}000\,805$ ft^{-1} corresponding to initial acceleration of $\mp\, g$.

Linear speed variation $V = V_0(1 + KV_0t)$.

$K = \pm\, 0{\cdot}000\,805$ ft^{-1} corresponding to acceleration of $\pm\, g$.

Exponential speed variation $(V - V_\infty) = (V_0 - V_\infty) \exp(-aV_0t)$.

$v_\infty = V_\infty/V_0 = 2{\cdot}0$

$\frac{m_2}{a}$	a (ft^{-1})	initial acceleration	
		ft/sec^2	g
2	0·00116	46·2	1·44
1	0·00231	92·4	2·87
0·5	0·00462	184·9	5·74

$v_\infty = V_\infty/V_0 = 0{\cdot}2$

$\frac{m_2}{a}$	a (ft^{-1})	initial deceleration	
		ft/sec^2	g
2	0·00116	37·0	1·15
1	0·00231	74·0	2·30
0·5	0·00462	147·9	4·59

The unstable case with the sign of m_2 changed has also been considered.

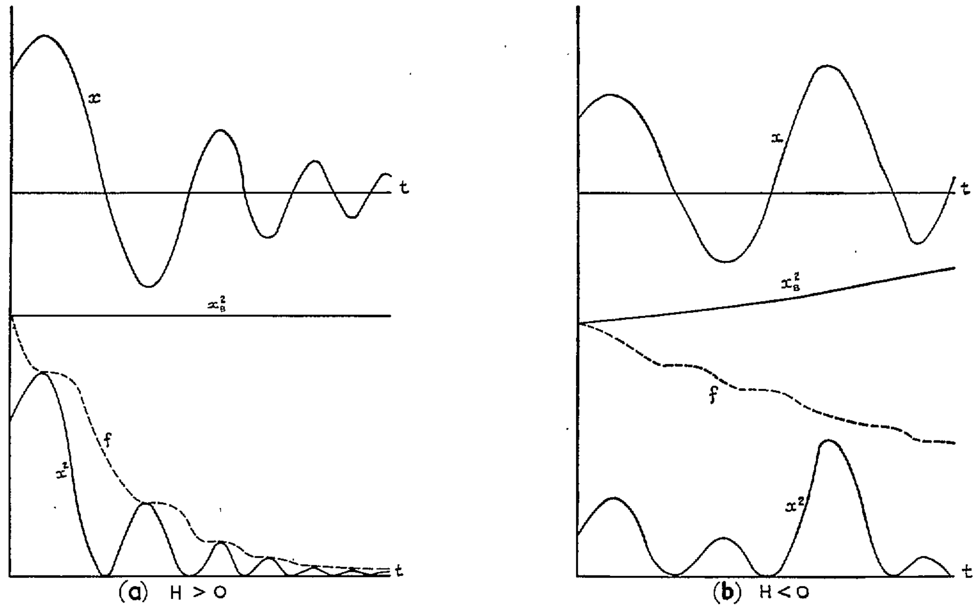

FIG. 1a and b. Sketch of analysis when motion is oscillatory.

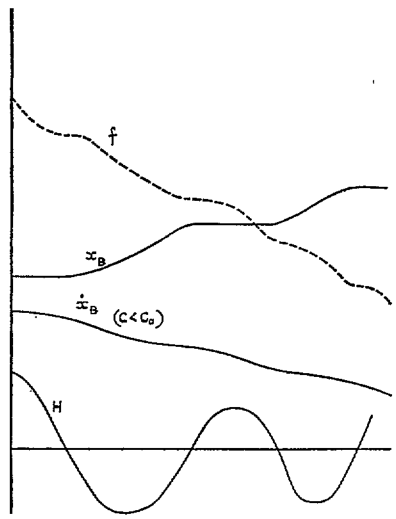

FIG. 2. Sketch of analysis when H changes sign.

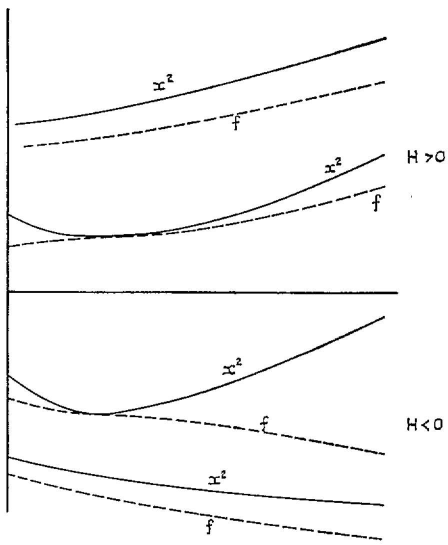

FIG. 3. Sketches of analysis $c < 0$.

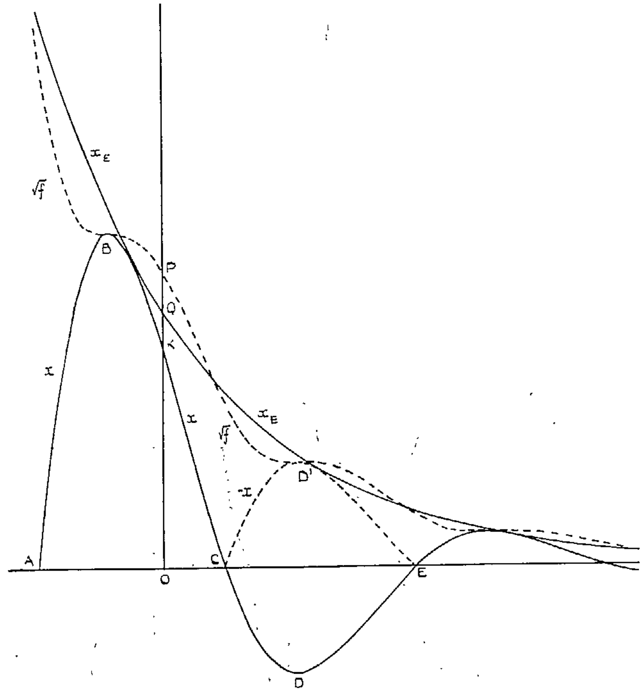

FIG. 4. Relation between envelope and $\sqrt{f}$.

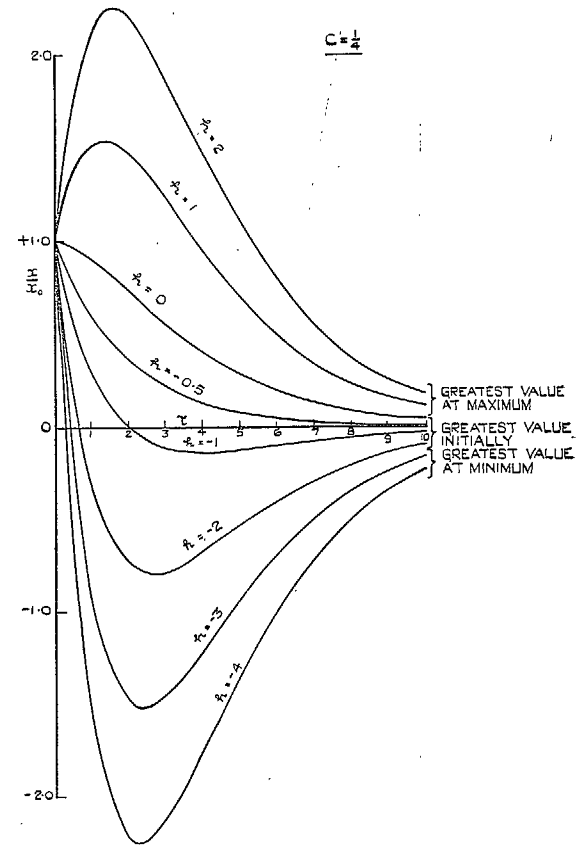

FIG. 5. Aperiodic solutions of equation with constant coefficients—Displacement.

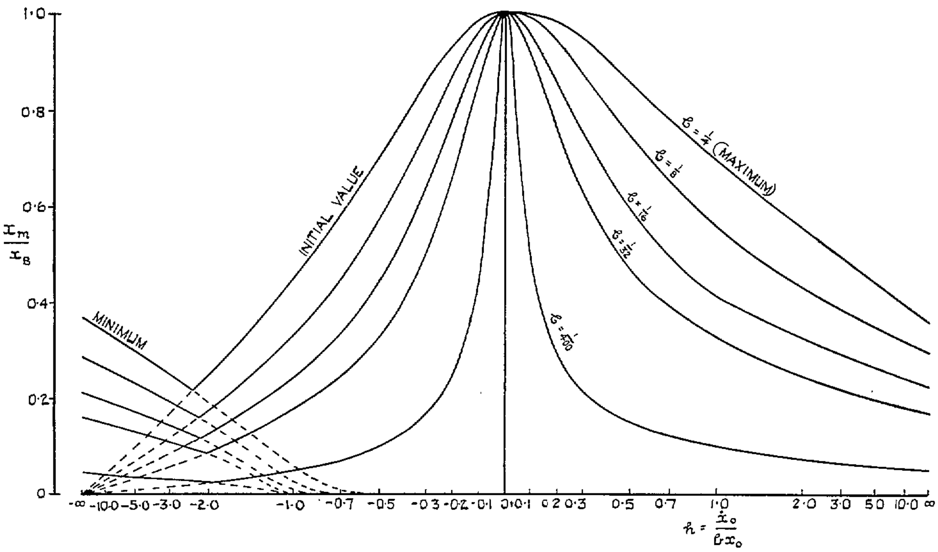

FIG. 6. Equation with constant coefficients. Comparison of extreme values and bounds for x.

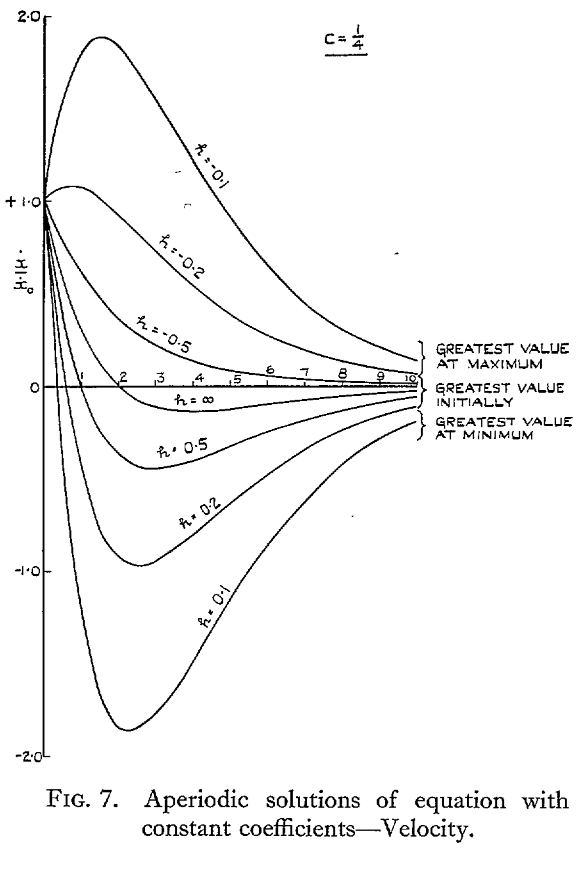

FIG. 7. Aperiodic solutions of equation with constant coefficients—Velocity.

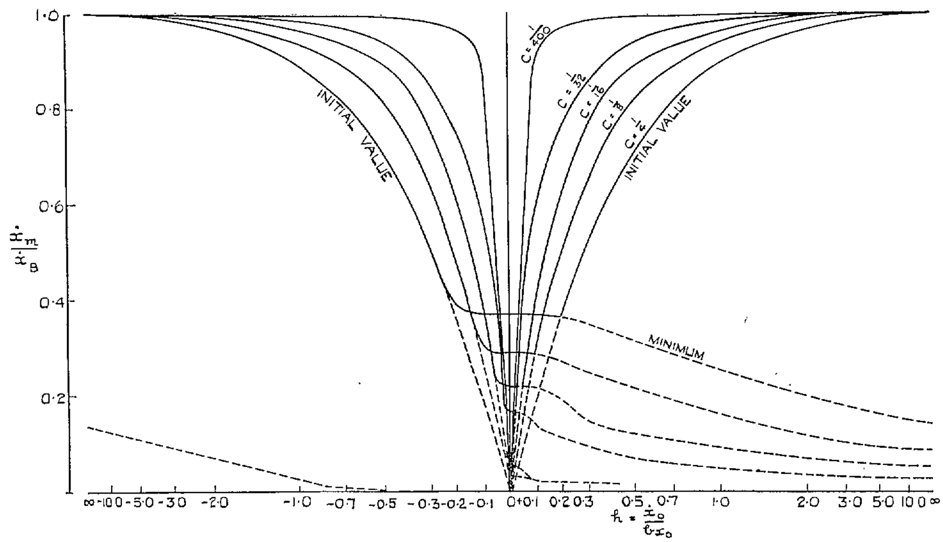

FIG. 8. Equation with constant coefficients. Comparison of extreme values and bounds for $\dot{x}$.

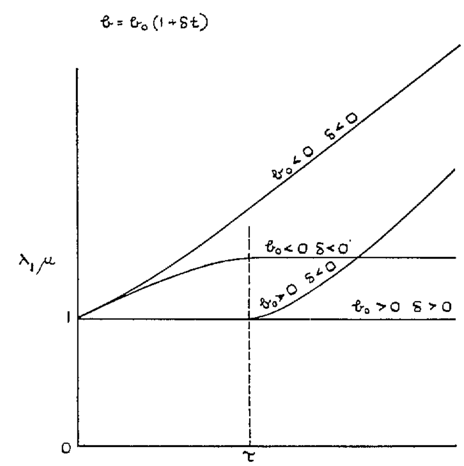

FIG. 9. Shape of bounds—b varying linearly, c constant.

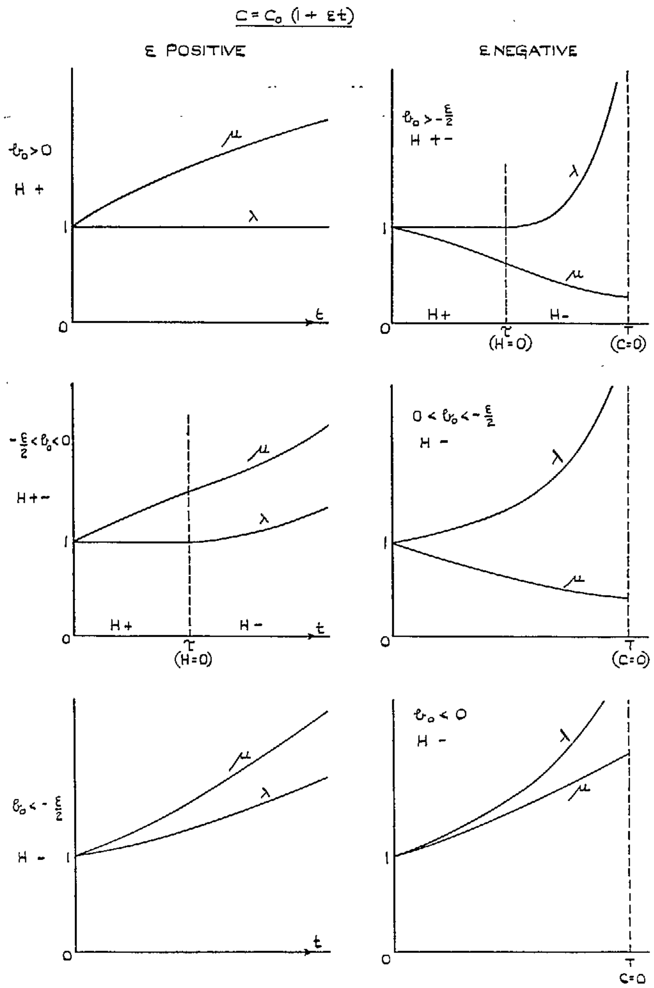

Fig. 10. Shape of bounds—b constant, c varying linearly.

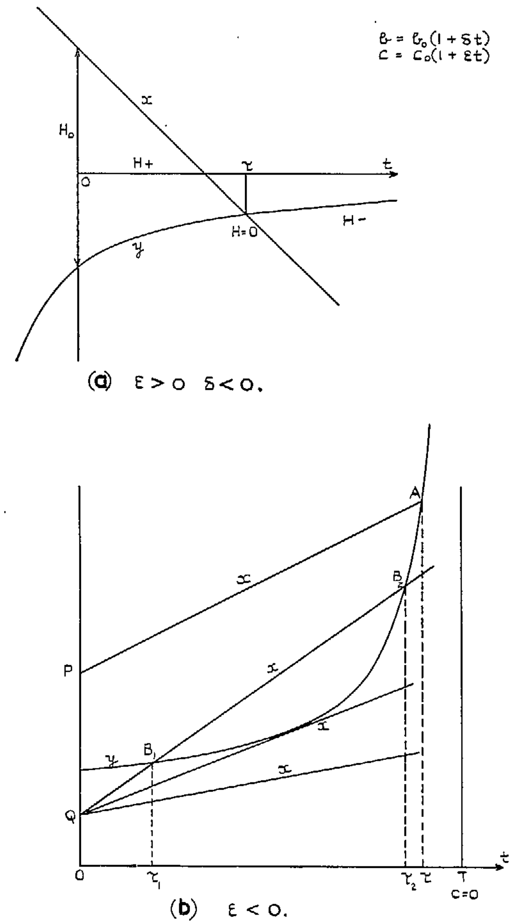

Fig. 11a and b. Geometry for H—both coefficients varying linearly.

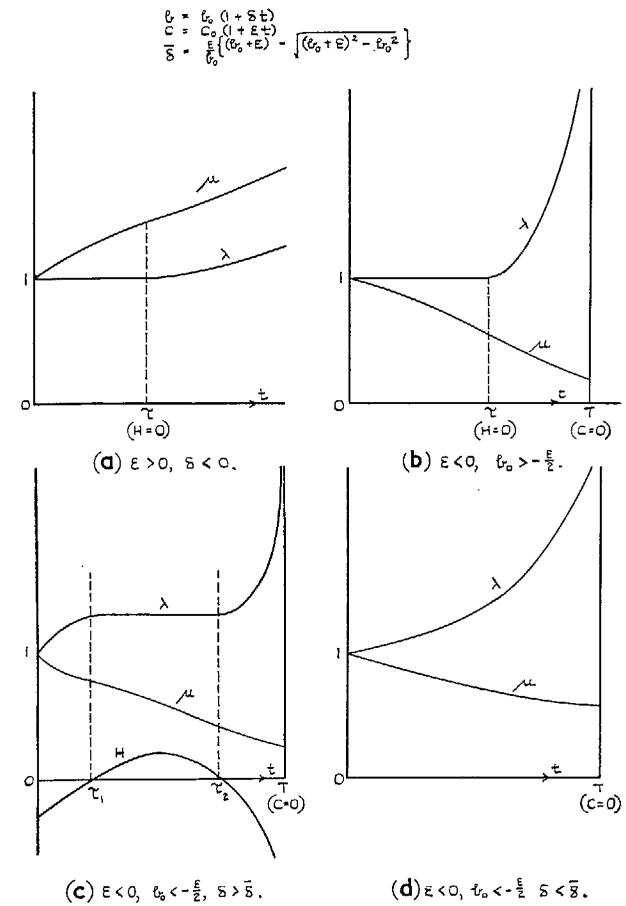

FIG. 12a to d. Shape of bounds—both coefficients varying linearly.

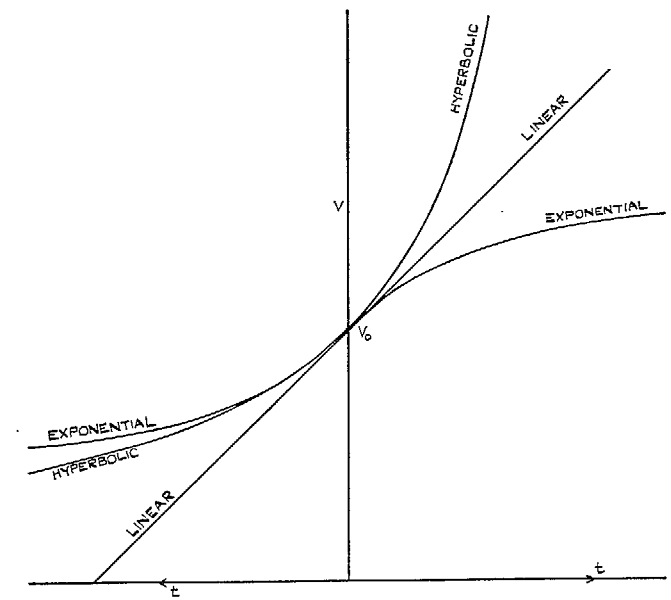

FIG. 13. Pitching model—laws of speed variation.

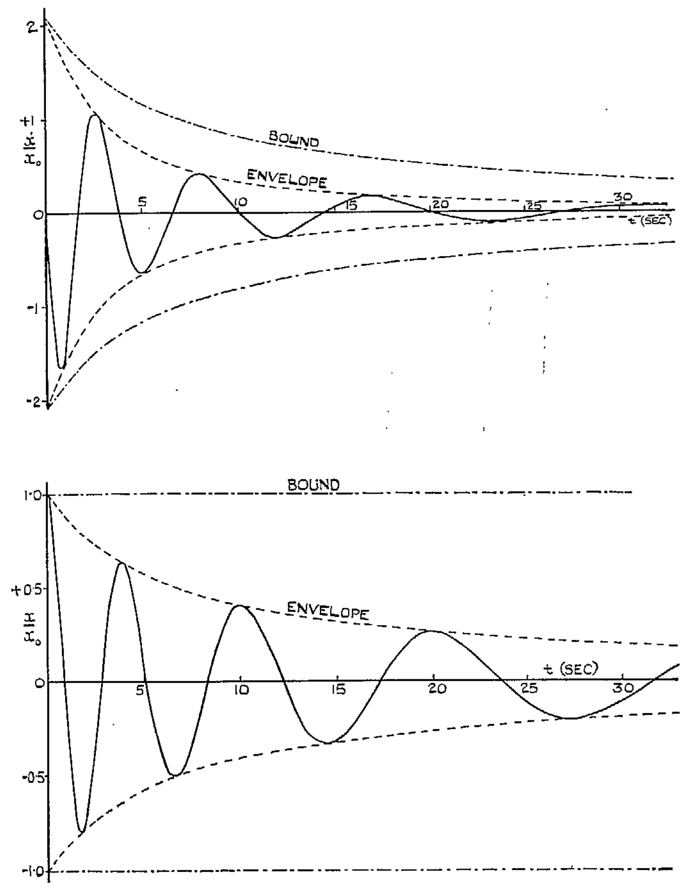

FIG. 14. Pitching model with hyperbolic decrease of speed. Comparison of bounds and exact solution.

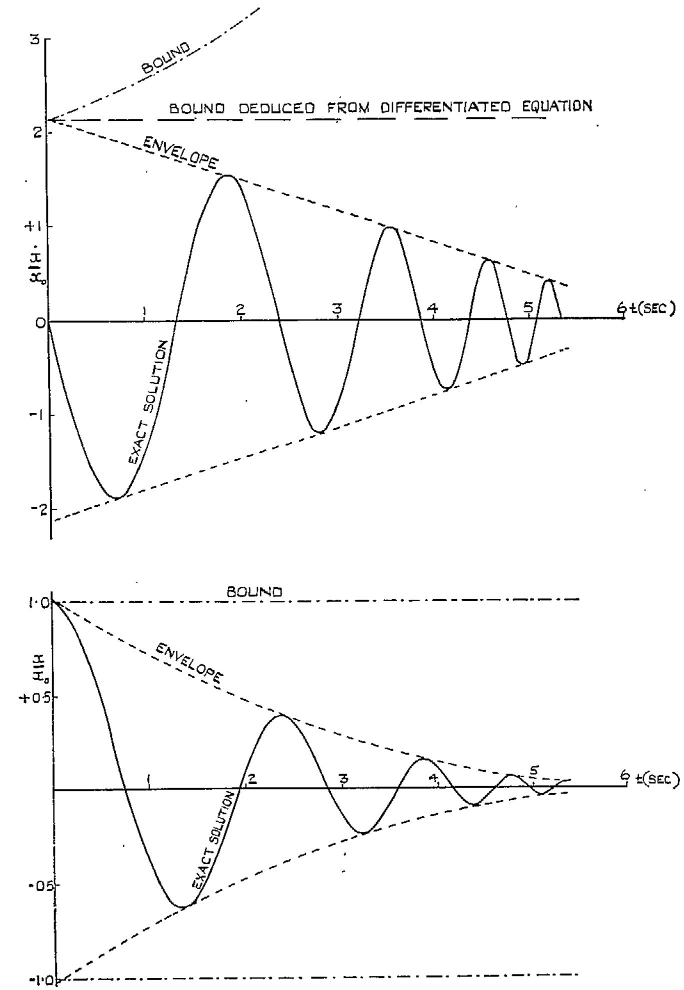

FIG. 15. Pitching model with speed increasing hyperbolically. Comparison of bounds and exact solution.

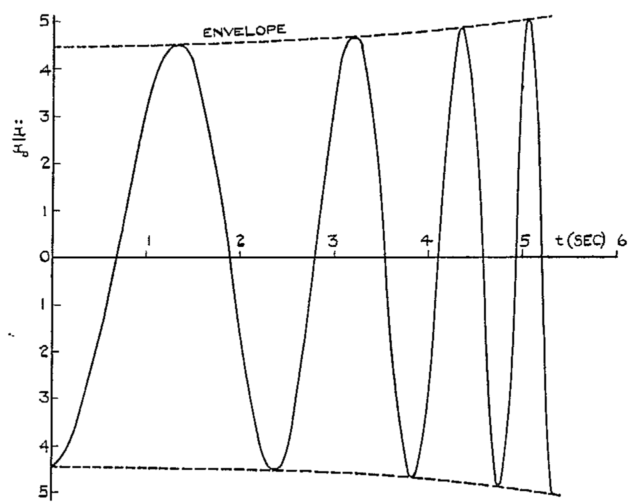

FIG. 16. Pitching model with speed increasing hyperbolically. Time history of acceleration.

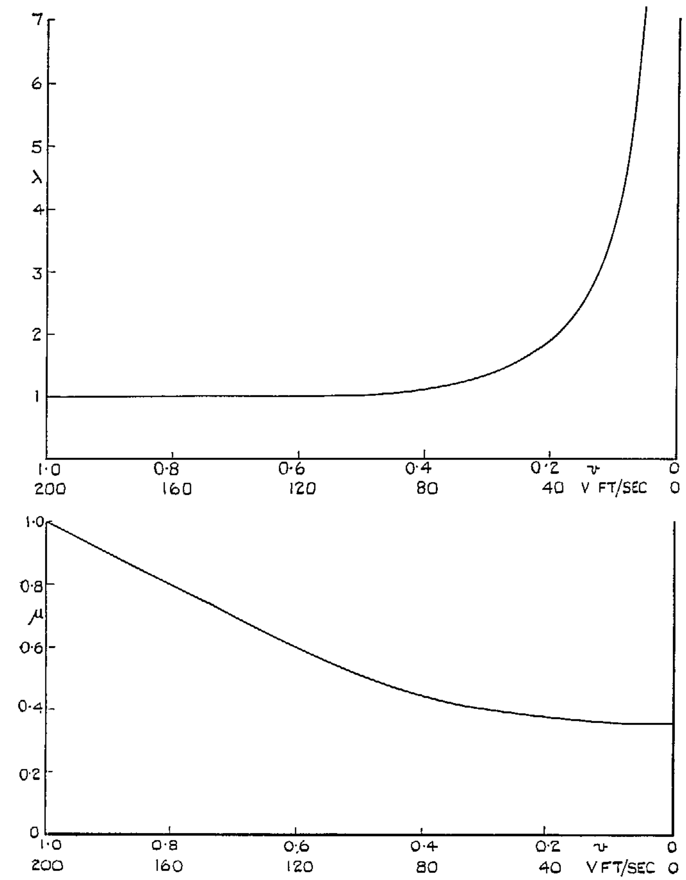

FIG. 17. Pitching model with linear deceleration—shape of bounds.

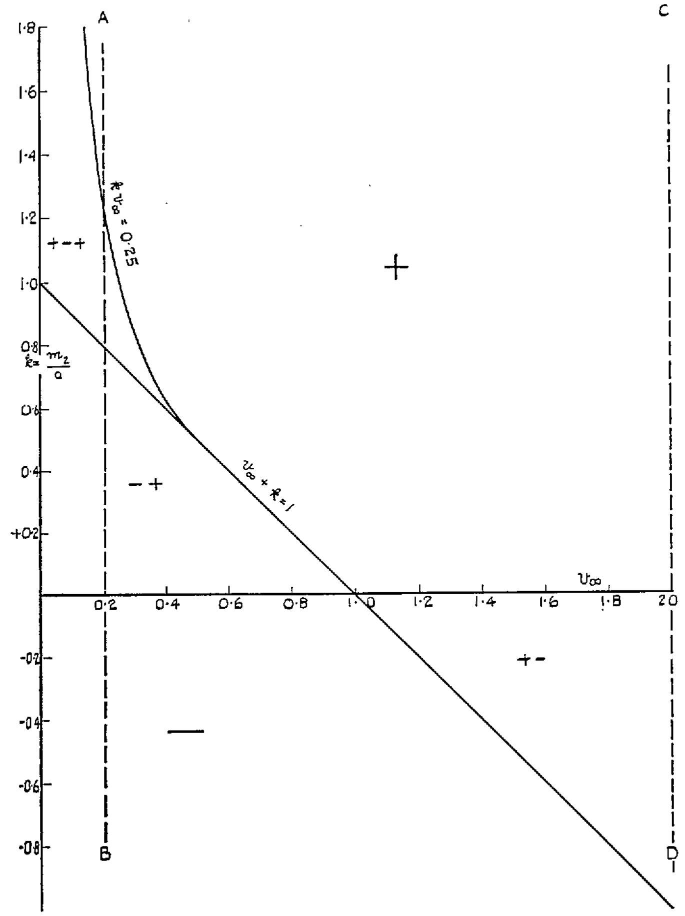

FIG. 18. Pitching model with exponential speed variation. Sign variations of H.

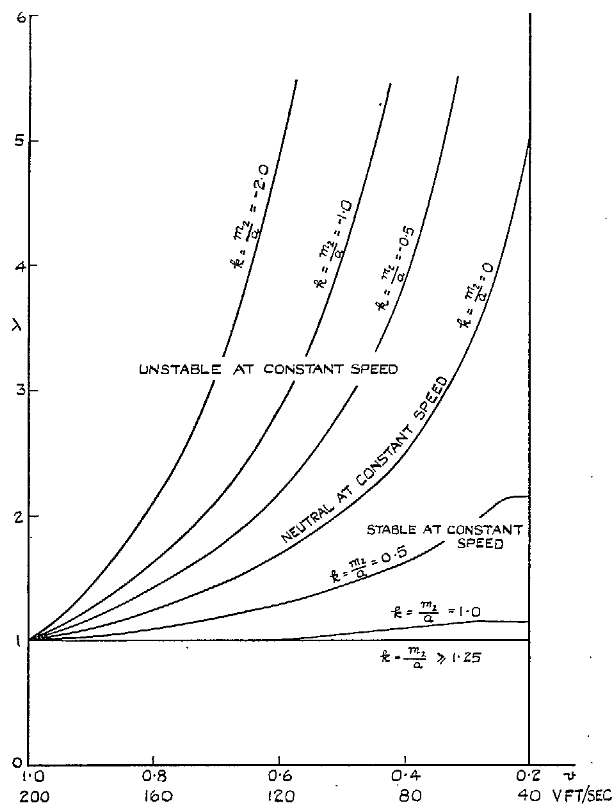

FIG. 19. Pitching model for exponential speed variation. x bounds for deceleration to $v_\infty = 0 \cdot 2$.

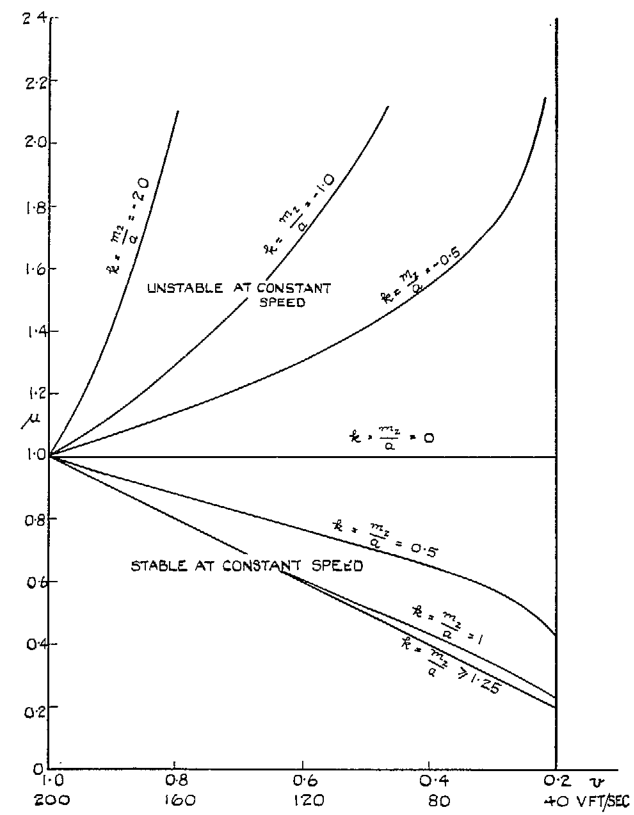

FIG. 20. Pitching model with exponential speed variation. $\dot{x}$ bounds for deceleration to $v_\infty = 0 \cdot 2$.

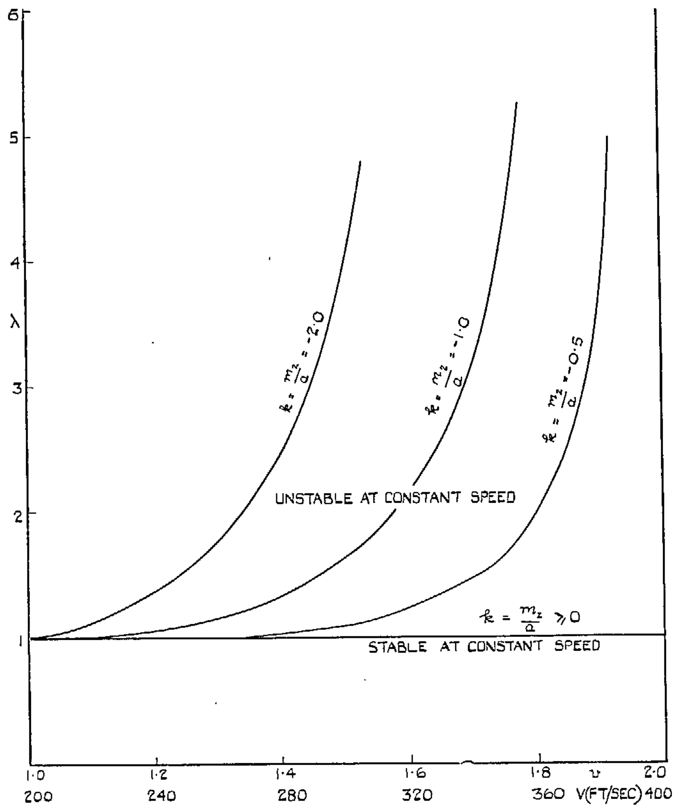

FIG. 21. Pitching model with exponential speed variation. x bounds for acceleration to $v_\infty = 2\cdot0$.

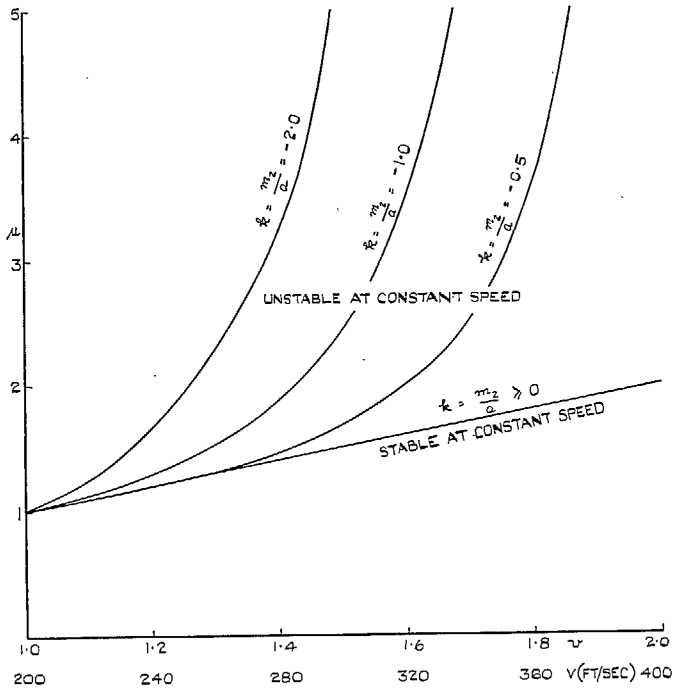

FIG. 22. Pitching model with exponential speed variation. $\dot{x}$ bounds for acceleration to $v_\infty = 2\cdot0$.

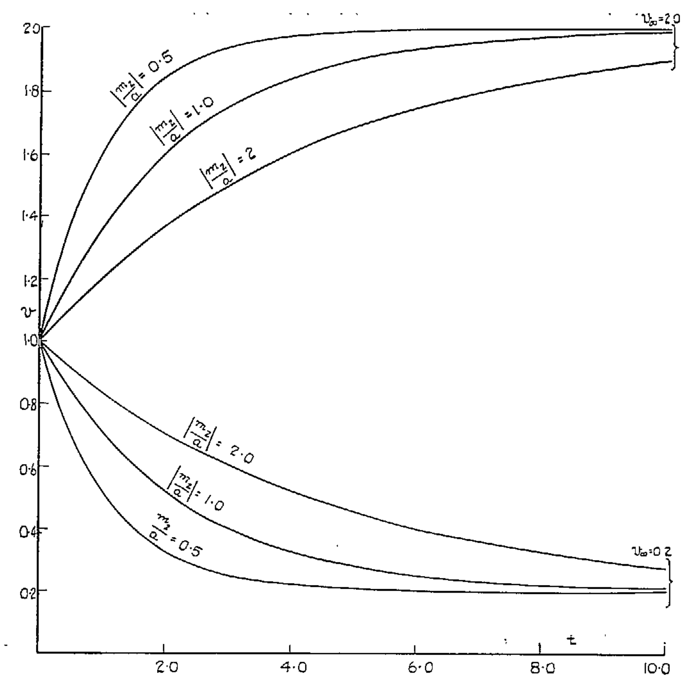

FIG. 23. Pitching model with exponential speed variation. Relation between speed and time for $v_\infty = 0 \cdot 2$ and $2 \cdot 0$, $|m_2|$ constant.

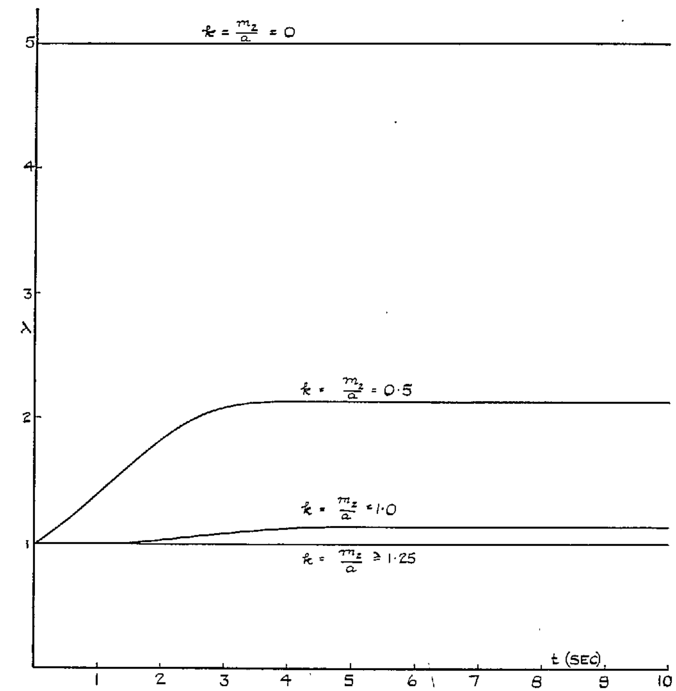

FIG. 24. Pitching model with exponential speed variation. x bounds for deceleration to $v_\infty = 0 \cdot 2$, $m_2 > 0$.

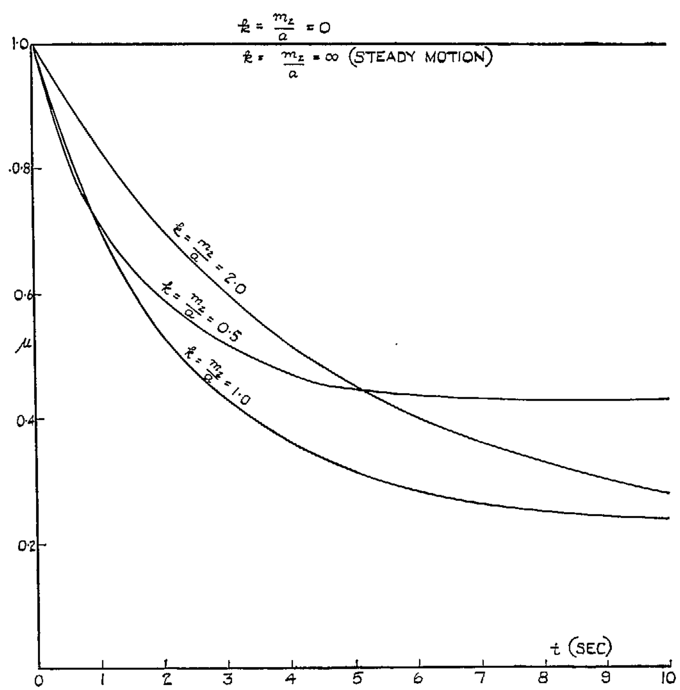

Fig. 25. Pitching model with exponential speed variation. $\dot{x}$ bounds for deceleration to $v_\infty = 0{\cdot}2$, $m_2 > 0$.

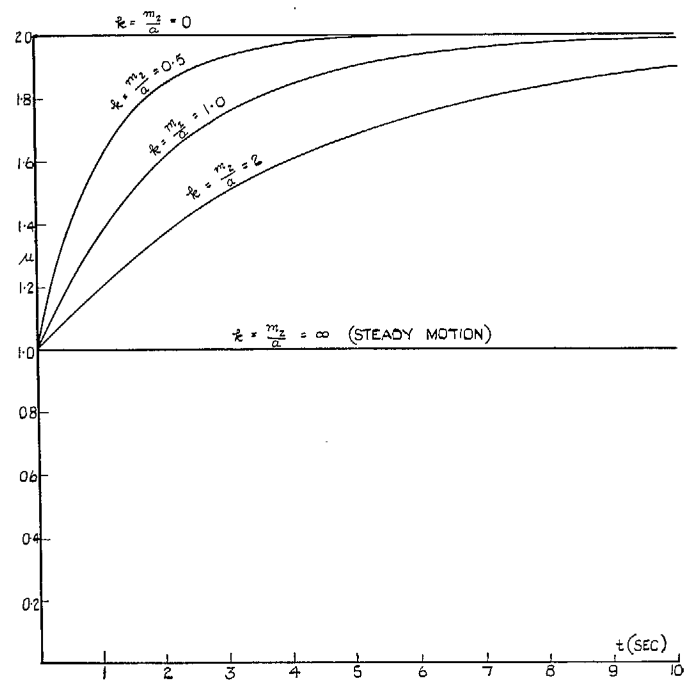

Fig. 26. Pitching model with exponential speed variation. $\dot{x}$ bounds for acceleration to $v_\infty = 2{\cdot}0$, $m_2 > 0$.

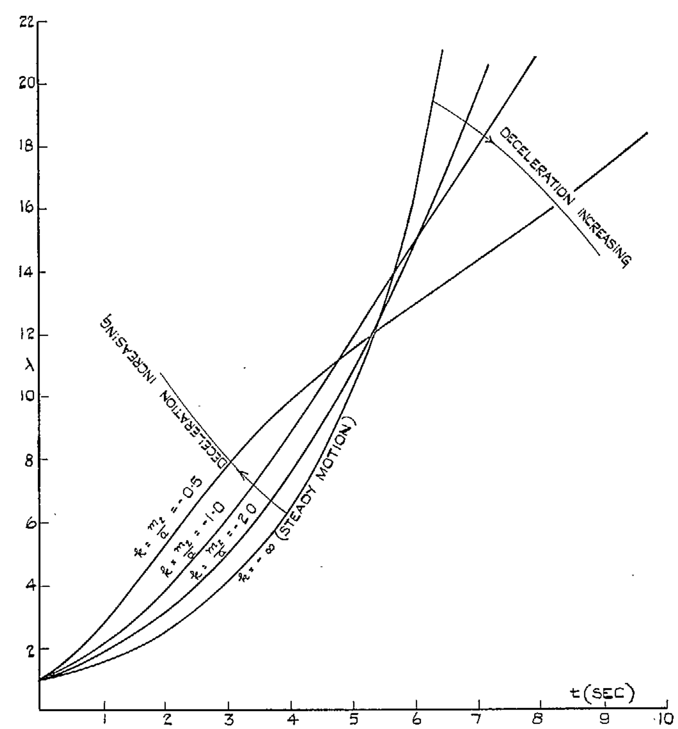

FIG. 27. Pitching model with exponential speed variation. x bounds for deceleration to $v_\infty = 0{\cdot}2$, $m_2 < 0$.

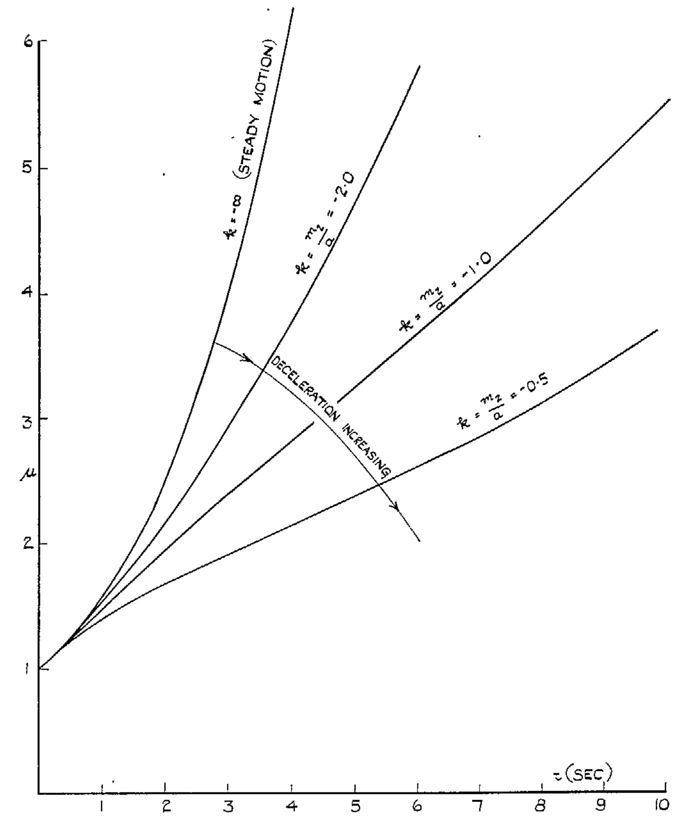

FIG. 28. Pitching model with exponential speed variation. $\dot{x}$ bounds for deceleration to $v_\infty = 0{\cdot}2$, $m_2 < 0$.

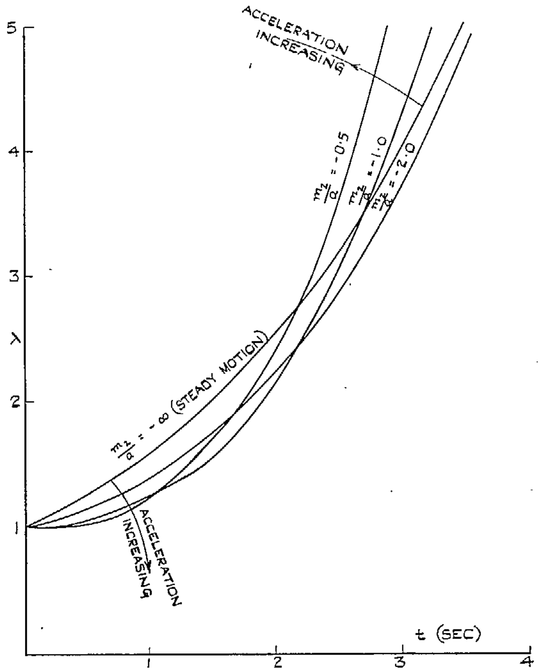

FIG. 29. Pitching model with exponential speed variation. x bounds for acceleration to $v_\infty = 2{\cdot}0$, $m_2 < 0$.

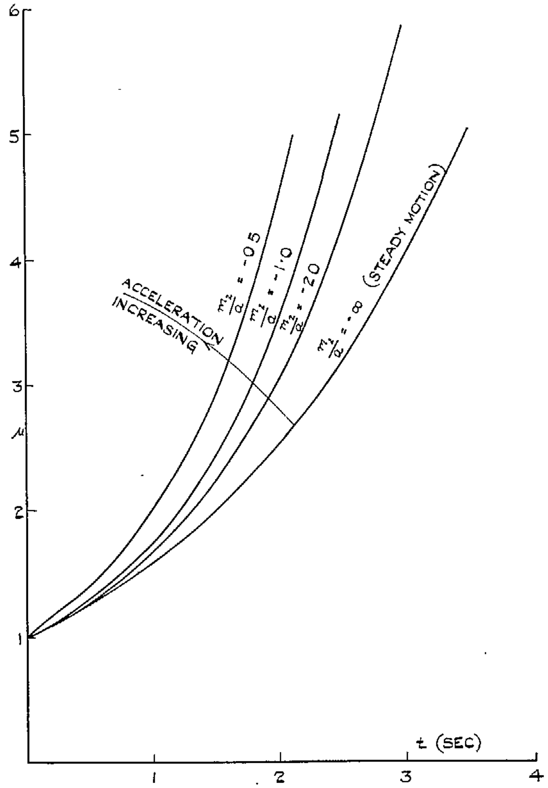

FIG. 30. Pitching model with exponential speed variation. $\dot{x}$ bounds for acceleration to $v_\infty = 2{\cdot}0$, $m_2 < 0$.

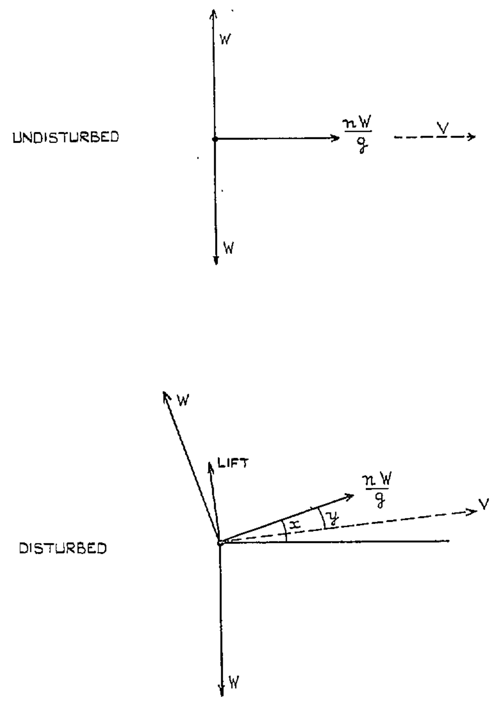

FIG. 31. Forces acting in jet-borne transition motion.

(90847) Wt. 66/2301 K.5 3/65 Hw.

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