R. & M. No. 3354

MINISTRY OF AVIATION

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

The Estimation of Stresses around Unreinforced Holes in Infinite Elastic Sheets

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LONDON: HER MAJESTY'S STATIONERY OFFICE
1964
EIGHT SHILLINGS NET

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Communicated by the Deputy Controller Aircraft (Research and Development), Ministry of Aviation

*Reports and Memoranda No. 3354**
October, 1962

Summary.

A practical procedure for the investigation of the state of stress around an arbitrary doubly-symmetric hole in infinite elastic sheets in tension and shear is presented. The associated problem of conformally mapping an arbitrary region on to the unit-circle is solved using an iterative technique.

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* Replaces R.A.E. Report No. Structures 283—A.R.C. 24,479.

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1. *Introduction.*

The problem of the determination of stress distributions and in particular of stress-concentration factors for arbitrarily shaped holes is one of great importance in design. Considerable progress in analysis has followed the introduction of the complex-variable theory (Muskhelishvili[1, 2], Savin[3], Green and Zerna[4], Goodier[5] and others[6 to 8]). However the class of holes for which exact solutions have been found is not of great practical interest, although approximations to some practical holes using 'curved-square' and 'ovaloid' shapes have been esteemed by Savin[3].

Difficulties arise in the mapping of practical contours in reproducing accurately the profile and curvature distribution at all points of the boundary, particularly in those regions associated with high stress. A practical mapping procedure must be capable of a close approximation not only to the profile but to its curvature. Exact agreement with the desired profile may require an infinite number of terms in the mapping function, as, for example, when a curvature discontinuity is present. Generally it is found necessary to take 30 or 40 terms to avoid errors in curvature which produce significant, or even serious, errors in the stress. Kikukawa[6 to 8] has shown that the disparity in stress estimation can be important when a mapping function is extended from 2 to 6 terms for 'rounded polygonal' contours. In fact, considerably more terms have to be taken than are used by Kikukawa if undue distortions of the stress distribution are to be avoided.

Since an approximate mapping is necessarily deficient in some respect, a standard of approximation is required in order that unnecessary computation may be avoided. In Appendix I a controlled curvature variation is introduced into a known problem as a small perturbation of the local curvature and the influence on the stress concentration observed. On the basis of this example a tentative proposal for assessing the adequacy of an approximate mapping is introduced.

The problem of determining the stress field outside a given hole reduces to the determination of an adequate mapping technique, because once this has been obtained, the stress distribution is readily determined as an 'exact' analysis. The formal analysis of this elastic problem is well known[1 to 8] and its algebraic development is straightforward using one matrix inversion and some series formulation.

The most direct of alternative methods of solution replaces the biharmonic equation for the Airy stress function by its finite-difference equivalent (Ref. 9, p. 483). Incomplete satisfaction of the boundary-value problem for the stress function can be used to find approximations to it such as, for example, the expansion of the stress function as a series of biharmonic functions with the boundary conditions satisfied only approximately (by collocation, or least squares) or in the limit (by Fourier analysis)[9]. Similarly rigorous solution of the boundary conditions and Ritz-type approximation of the stress function using energy considerations is possible[9]. Division of the sheet into elements and the use of the matrix methods recently introduced[10] has not been explored fully for problems of local stress concentration. Finally exact solutions of problems by semi-inverse methods is unlikely to be useful for the title problem.

The procedure proposed in this report is considered superior to any of the above for the following reasons. The stress state is solved exactly for a contour which differs only slightly from the desired hole. The distance between the given hole and the approximation to it which is analysed is never serious and may easily be allowed for. The arithmetic effort required in solution is smaller for a given accuracy. Finally the method is quite general and independent of the shape of the hole.

2. *The Mapping Problem.*

For any given hole in an infinite sheet, there exists an unknown mapping function

$$z = m(\zeta) \tag{1}$$

which exactly maps the boundary C of the hole on to the unit-circle in the ζ-plane. Given the geometry of the hole, we seek the appropriate form of $m(\zeta)$.

The boundary C may or may not have sharp corners. Where there is a sharp corner with material projecting into the hole, the stresses around it are small and modifications to the profile of C near and at the corner do not introduce significant changes in other regions of higher stress. If there is a sharp corner forming a notch in the material, the stresses at the corner itself are theoretically infinite. Such corners are of little interest and for practical corners a small radius of curvature may be presumed. Thus we may assume that the tangent to C is continuously turning at all points of the boundary. This implies that the transformation (1) is free from singularities on the boundary and the mapping function $m(\zeta)$ may be expressed in the form of a power series in ζ,

$$z = \sum_{m=0}^{M} b_m \zeta^{1-m} \tag{2}$$

where M may be finite or infinite.

The parametric form of the profile C which corresponds to the unit circle in the ζ-plane is given by

$$z(\theta) = x(\theta) + iy(\theta) = \sum_{m=0}^{M} b_m e^{(1-m)i\theta}. \tag{3}$$

The curvature κ at any point is given by the expression

$$\kappa = \{\dot{x}\ddot{y} - \ddot{x}\dot{y}\}\{\dot{x}^2 + \dot{y}^2\}^{-3/2} \tag{4}$$

where dots denote differentiation with respect to θ.

For a transformation of the type (2) with finite M, the curvature κ is seen to vary continuously around the contour C, so that such a transformation cannot be exact for those important profiles with curvature discontinuities (e.g. rounded slots). In such cases an approximation C′ to C, based on a finite value of M, may be obtained which is nevertheless suitable for use in stress estimation. The disparity between C and C′ may be made as small as we please by allowing M to increase sufficiently and provided the b_m are suitably chosen. The disparity is greatest in the neighbourhood of a discontinuity in the curvature of C (cf. Gibbs phenomenon). In those problems for which the stress concentration is expected to occur in this region, particular care has to be taken to avoid unsatisfactory oscillations in the curvature of C′ which induce errors in the stress estimates. This means, in practice, choosing a large value of M.

Ideally the curvature variation of C should be used to derive the mapping coefficients b_m but since κ is a non-linear function of them, such a procedure is quite unsuitable. An alternative procedure which first fixes the b_m without direct reference to the curvature leaves it as a derived item so that a transformation must be obtained before its merit can be assessed. Such a restriction is unimportant in practice.

In the subsequent discussion of mapping procedures the profile C is assumed symmetric about the x- and y-axes, so that all the b_m are real and for odd values of m they are zero. These restrictions are far from necessary but are included in developing a computer programme and accordingly the arguments which follow require modification for the general contour.

2.1. *Correspondence of Position.*

Suppose that a set of $(K+1)$ points $P_0, P_1, \ldots, P_K$ is selected on the unit-circle in the ζ-plane. Because of the assumed double symmetry of C, these points may all be taken to lie in the first quadrant. Under the mapping given in equation (1) each point P_k $(k = 0, 1, \ldots K)$ exactly corresponds with a point Q_k $(k = 0, 1 \ldots, K)$ of C. In order to obtain the mapping coefficients b_m it is necessary to estimate the positions of the points Q_k on C.

Any procedure for calculating the b_m must allow for an initial ignorance of the points Q_k. Only tentative estimates of the Q_k, say R_k, can be made. The difficulty in selecting the R_k is shown in Fig. 1 where the corresponding arguments of the Q_k as the P_k move round a quadrant of the unit-circle are depicted for two practical contours. Any computational procedure must be capable of improving the correspondence of the R_k with the P_k so that ultimately the curve C′ which is obtained by the approximate mapping of equation (2) is an adequate representation of the given profile C. Such a procedure is described below.

3. *Melentiev's Iterative Procedure for Finding the Mapping Coefficients.*

An algebraic correspondence between the co-ordinates of the points P_k and Q_k is an unsuitable basis for determining the b_m because each point contributes two algebraic equations. Since errors in position of any of the points R_k introduce errors into two of the equations, it is preferable to follow Melentiev[11,12] and compute the b_m from the reduced variable

$$z/\zeta = u(\theta) + iv(\theta) = \sum_{m=0}^{M} b_m \zeta^{-m}, \tag{5}$$

where the value of M will depend on K.

The co-ordinates x, y are periodic in θ whereas the co-ordinates u, v (*see* Fig. 2) behave quite differently. As ζ goes round the unit-circle, say $\zeta = e^{i\theta}$, $u(\theta)$ and $v(\theta)$ are given by the equations

$$u(\theta) = \sum_{m=0}^{M} b_m \cos m\theta, \qquad v(\theta) = -\sum_{m=1}^{M} b_m \sin m\theta. \tag{6}$$

For typical profiles C, $u(\theta)$ remains positive around the contour, whilst $v(\theta)$ is small and may change sign. Small errors in the location of the R_k seriously influence the $v(\theta)$ while the $u(\theta)$ change only slightly.

The mapping coefficients b_m are now derived by assuming values of $u(\theta)$ appropriate to the set of points R_k on C. The function $v(\theta)$ is ignored in this derivation and is subsequently evaluated from the derived b_m so that the exact correspondence of the P_k is determined. These points do not in general lie on C but near neighbours of them, which *do* lie on C, can be selected as a new set of R_k and the process repeated until the discrepancy between the contours C, C′ is sufficiently small.

No general principle of convergence is available and it is advisable to examine the approximation in each case. The method is found to work satisfactorily in practice and is most rapid for holes for which the variation is $u(\theta)$ is small.

3.1. *Requirements for a Computer Programme.*

For a given set of points P_k corresponding to specific values of θ, and initial estimates of the $u(\theta)$ for them, the computer evaluates the b_m and the exact correspondence of the P_k, say R_k. It is necessary to be able to select a better set of approximations, say R'_k, of the R_k within the machine and to repeat the process as often as necessary. The convergence of the process must be tested at each stage and a basis of agreement between the derived profile C′ and C established within the machine.

To improve the R_k the profile C must be stored, digitally or functionally, within the computer. Numerical storage of C is undesirable if an analytical form, using only a few stored parameters, is available. Storage and access time are important since the computer process is repeated many times. Automatic procedures for correcting the R_k are provided for arc elements of C which are either circular or linear, but more general contours require special artifices for this correction. It is necessary only that the set of R'_k is, as a whole, closer to C than the R_k and precise correction on to C may be avoided, particularly with complicated forms of C.

After the process has been repeated a number of times with $K + 1$ points P_k, there is a stage in the iteration process beyond which attempts to identify C and C′ are meaningless since the curves C, C′ may be very close at the control points R_k but may be quite dissimilar in between. No reference is made to this region in the iteration cycle. The curvature distribution at the R_k may be inspected and if the variation is unsatisfactory, K may be increased. The process is repeated until the curvature distribution is acceptable. In general the operator knows the likely value of K necessary to give satisfactory results in the subsequent stress analysis and is able to preset the extent of the iterative process. Acceptability decisions using the curvature distribution as a standard are made outside the computer and are based on experience with similar profiles. Some indication of inaccuracies in the curvature variation which will not influence the subsequent stress analysis is provided in Appendix I.

3.2. *Algebraic Details of the Iterative Procedure.*

The $(K+1)$ points P_k are chosen to subdivide equally the first quadrant so that θ can take any of the values

$$\theta_k = k\pi/2K, \qquad k = 0, 1, \ldots, K. \tag{7}$$

The extreme values of u which correspond to positions on the x- and y-axis, namely u_0 and u_K, are known precisely since $v(\theta)$ is zero for $\theta = 0$ and $\theta = \pi/2$. The remaining $(K-1)$ functions u_k are estimated initially, using a rough and ready correspondence of position. No great accuracy is necessary in these estimations since the trial values of u_k rapidly improve as the iterative process continues.

The equations which determine the b_m are taken to be

$$u_k = \sum_{m=0}^{M} b_m \cos m\theta_k, \qquad (k = 0, 1, \ldots, K) \tag{8}$$

where there must be at least as many b_m's as there are values of u_k. If we take $M = 2K$, with the θ_k equally spaced, these equations can be solved exactly without matrix inversion (*see*, for example, Whittaker and Robinson[13]) to give

$$\left.\begin{aligned} b_0 &= \frac{1}{4K}\sum_{k=0}^{4K-1} u_k, \\ b_m &= \frac{1}{2K}\sum_{k=0}^{4K-1} u_k \cos\frac{k\pi m}{2K}, \\ \text{and} \qquad b_{2K} &= \frac{1}{4K}\sum_{k=0}^{4K-1} (-1)^k u_k \end{aligned}\right\} \tag{9}$$

where the values of u_k for $k > K$ follow from the double symmetry of C so that

$$u_k = u_{2K-k} = u_{2K+k} = u_{4K-k} \quad (k = 1, 2, \ldots, K-1). \tag{10}$$

The expressions for the b_m can be written in terms of $u_0, u_1, \ldots, u_K$ in the form

$$\left.\begin{aligned} b_0 &= \frac{1}{2K}\{u_0 + 2u_1 + 2u_2 + \ldots + 2u_{K-1} + u_K\}, \\ b_{2m} &= \frac{1}{2K}\left\{2u_0 + \sum_{k=1}^{K-1} 4u_k \cos\frac{mk\pi}{K} + 2(-1)^m u_K\right\}, \quad k = 1, 2, \ldots, K-1 \\ \text{and} \qquad b_{2K} &= \frac{1}{2K}\{u_0 - 2u_1 + 2u_2 \ldots + (-1)^{K-1}2u_{K-1} + (-1)^K u_K\}, \end{aligned}\right\} \tag{11}$$

with all b_{2m+1} zero. The first estimate of the mapping function is obtained in the form

$$z = b_0\zeta + b_2\zeta^{-1} + \ldots + b_{2K}\zeta^{1-2K}. \tag{12}$$

By substitution of the b_m from equation (11) into the second of the identities in equation (6), the v_k are computed and the R_k found. The correction of R_k to points R'_k nearer to C is straightforward for linear elements. Let R_k be the point (x_k, y_k) and let R'_k be the point $(x_k + \delta x_k, y_k + \delta y_k)$ and let the line element of boundary have the equation

$$ax + by + c = 0, \tag{13}$$

where a and b are normalized so that

$$a^2 + b^2 = 1. \tag{14}$$

The foot of the normal from (x_k, y_k) on to this line has co-ordinates $(b^2x_k - aby_k - ac, a^2y_k - abx_k - bc)$, so that the required corrections δx_k, δy_k in the co-ordinates of R_k are given by

$$\left.\begin{aligned} \delta x_k &= -a(ax_k + by_k + c), \\ \delta y_k &= -b(ax_k + by_k + c). \end{aligned}\right\} \tag{15}$$

The corresponding change in u_k, say δu_k is then

$$\delta u_k = \cos\theta_k \delta x_k + \sin\theta_k \delta y_k. \tag{16}$$

For circular-arc elements, the foot of the normal on to the arc is not found so conveniently but the point midway between R_k and its inverse point with respect to the circle is an excellent approximation. Let (ξ, η) be the co-ordinates of the centre of the circular arc of radius ρ on to which $R_k(x_k, y_k)$ is to be corrected. Then the required corrections δx_k, δy_k in the co-ordinates of R_k are given by

$$\delta x_k = \lambda(x_k - \xi), \qquad \delta y_k = \lambda(y_k - \eta) \tag{17}$$

where

$$\lambda = \frac{\frac{1}{2}\rho^2}{(x_k - \xi)^2 + (y_k - \eta)^2} - \frac{1}{2}. \tag{18}$$

The change in u_k, δu_k is given by equation (16).

The discrepancy between C and C′ at the reference points R_k is measured by the non-dimensional quantity

$$\delta = \left[\sum_{k=0}^{K} \{(\delta x_k)^2 + (\delta y_k)^2\}\right]^{1/2} \tag{19}$$

with equal weight attaching to all points. In a typical iterative cycle, δ is reduced to about 10^{-5}, beyond which further iterations produce no significant changes in the curvature variation of C′.

If it is necessary to refine the interval for θ, K should be doubled, so that new values of the u_k can be formed at once by computing mid-interval values, e.g.

$$u_{k+1/2} = \tfrac{1}{2}\{u_k + u_{k+1}\} \tag{20}$$

and subsequently renumbering the suffices to run from 0 to $2K$. The process may now be restarted and continued until satisfactory approximation in the curvature is obtained.

The final form of the transformation which is used in the stress analysis is obtained as

$$z = \sum_{n=0}^{N} a_{2n}\zeta^{1-2n}, \tag{21}$$

where $a_0 = 1$, $a_{2n} = b_{2n}/b_0$ and N depends on the extent of the iteration performed as above. The z- and ζ-planes are brought into coincidence at infinity to preserve a stress field which is uninfluenced thereabouts by the presence of the hole.

4. *Analysis of the Elastic Problem.*

The use of complex-variable methods in the investigation of two-dimensional stress systems has been proposed many times (Muskhelishvili[1, 2], Green and Zerna[5], Stevenson[14], Milne-Thomson[15] and others). Milne-Thomson's treatment of the analysis is followed and reference to his book should be made for fuller development of the results used here.

Let $z = m(\zeta)$ be the mapping function which maps the region exterior to the unit-circle in the ζ-plane on to the region exterior to the hole of profile C in the z-plane, with z and ζ asymptotically equal for large ζ. The stress field at infinity for large z which is uninfluenced there by the presence of the hole, is identical with the stress field for large ζ.

Let $\Omega(z)$, $\omega(z)$ be the complex stress functions so that the complex displacement $u + iv$ is given by

$$4G(u+iv) = S\int \Omega(z)dz - z\overline{\Omega}(\bar{z}) - \int \overline{\omega}(\bar{z})d\bar{z}, \tag{22}$$

where $\overline{\Omega}(\bar{z})$ is the complex-number conjugate to $\Omega(z)$, S is $(3-4\nu)$ for plane strain and $(3-\nu)/(1+\nu)$ for generalised plane stress and G is the shear modulus.

The stresses are given by the Kolosov equations

$$\left.\begin{aligned} \sigma_x + \sigma_y &= \Omega(z) + \overline{\Omega}(\bar{z}), \\ \sigma_y - \sigma_x + 2i\tau_{xy} &= \bar{z}\Omega'(z) + \omega(z). \end{aligned}\right\} \tag{23}$$

For a tension T applied at infinity making a direction α with the x-axis, the form of the stress functions $\Omega(z)$, $\omega(z)$ is, for large z,

$$\left.\begin{aligned} \Omega(z) &= \tfrac{1}{2}T + O(z^{-2}), \\ \omega(z) &= -\ Te^{-2i\alpha} + O(z^{-2}). \end{aligned}\right\} \tag{24}$$

Under the given transformation these equations remain valid when ζ replaces z.

The condition that the displacements given by equation (22) are single-valued enables the stress field to be developed in terms of a single function of ζ and by introducing the analytic continuation of $\Omega(\zeta)$ across the stress-free boundary C, it is possible to express the stress function $\omega(\zeta)$ in the form

$$m'(\zeta)\omega(\zeta) = -\frac{d}{d\zeta}\left[\overline{m}\left(\frac{1}{\zeta}\right)\Omega(\zeta)\right] + \frac{1}{\zeta^2}\,\overline{m}'\left(\frac{1}{\zeta}\right)\overline{\Omega}\left(\frac{1}{\zeta}\right) \tag{25}$$

for all ζ outside the hole. Further investigation of the properties of $\Omega(\zeta)$ shows that for $|\zeta| < 1$, the singularities of $m'(\zeta)\Omega(\zeta)$ are, at most, those of $m(\zeta)$, $m'(\zeta)$ together with a pole of order 2 at the origin, $\zeta = 0$.

4.1. *Algebraic Development of Stress Functions for a Given Mapping Function.*

Let

$$m(\zeta) = \sum_{n=0}^{N} a_{2n}\zeta^{1-2n} \tag{26}$$

which is the final form of the iterated mapping given in equation (21). The a_{2n} are all real and a_0 is unity. Then the required forms of $\overline{m}(1/\zeta)$ and $m'(\zeta)$ are given, respectively, by the equations

$$\overline{m}\left(\frac{1}{\zeta}\right) = \sum_{n=0}^{N} a_{2n}\zeta^{2n-1} \tag{27}$$

and

$$m'(\zeta) = \sum_{n=0}^{N} (1-2n)a_{2n}\zeta^{-2n}. \tag{28}$$

The function $m'(\zeta)$ has a pole of order $2N$ at the origin so that $m'(\zeta)\Omega(\zeta)$ may be written as polynominal of order $2N$ in $1/\zeta$. An important simplification, which enables the stress field to be expressed in terms of real coefficients only occurs when α is zero or $\pi/2$, which cases correspond to tension along the axes of symmetry of the hole. Thus $m'(\zeta)\Omega(\zeta)$ may be written in the form

$$m'(\zeta)\Omega(\zeta) = \sum_{n=0}^{N} A_{2n}\zeta^{-2n}, \tag{29}$$

where the A_{2n} are real. Only even coefficients occur in this expression since for the loadings considered the stress field is doubly symmetric about the x- and y-axes. A similar simplification occurs for the case of shear loading when the coefficients A_{2n} are purely imaginary. This case is discussed in Appendix II.

For tensile loading we may write

$$[m'(\zeta)]^{-1} = \sum_{n=0}^{N-1} b_{2n}\zeta^{-2n} + O(\zeta^{-2N}), \tag{30}$$

where the b_{2n} are obtained from the identity $[m'(\zeta)][m'(\zeta)]^{-1} = 1 + O(\zeta^{-2N})$, which gives, on comparing coefficients of ζ^{-2}, ζ^{-4}, etc.,

$$\begin{bmatrix} a_0 & 0 & 0, \ldots, & 0 \\ -a_2 & a_0 & 0, & 0 \\ -3a_4 & -a_2 & a_0, & 0 \\ \cdot\cdot & \cdot\cdot & & \\ -(2N-5)a_{2N-4}, & -(2N-7)a_{2N-6}, & & a_0 \end{bmatrix} \begin{bmatrix} b_2 \\ b_4 \\ b_6 \\ \\ b_{2N-2} \end{bmatrix} = \begin{bmatrix} a_2 \\ 3a_4 \\ 5a_6 \\ \\ (2N-3)a_{2N-2} \end{bmatrix}. \tag{31}$$

This solves at once, row by row, for the b_{2n}.

We are now able to compute the stress-function coefficients A_{2n} since, for large ζ, $m'(\zeta) = 1 + O(\zeta^{-2})$ so that from equation (24)

$$m'(\zeta)\omega(\zeta) = -Te^{-2i\alpha} + O(\zeta^{-2}), \quad \left(\alpha = 0, \frac{\pi}{2}\right) \tag{32}$$

and

$$A_0 = \tfrac{1}{2}T. \tag{33}$$

Equation (25) in association with equation (32) enables all the remaining coefficients A_{2n} to be found. Thus, from equations (27), (29), (30) we may write

$$\Omega(\zeta)\overline{m}\left(\frac{1}{\zeta}\right) = \sum_{n=1}^{N} c_{2n}\zeta^{2n-1} \sum_{n=0}^{N} A_{2n}\zeta^{-2n} + O(\zeta^{-1}) \tag{34}$$

where

$$c_{2n} = \sum_{j=0}^{N-n} a_{2(n+j)}b_{2j}. \tag{35}$$

Substitution into equation (25), noting equation (32), gives the following identity in ζ:

$$\sum_{n=1}^{N} A_{2n}\zeta^{2n-2} - \sum_{n=1}^{N}\left(\sum_{j=0}^{N-n} A_{2j}c_{2(n+j)}\right)(2n-1)\zeta^{2n-2} = \mu T + O(\zeta^{-2}), \tag{36}$$

where μ is -1 for tension parallel to the x-axis and 1 for tension parallel to the y-axis. On comparing coefficients in equation (36) for non-negative powers of ζ, the following matrix equation for the A_{2n} is derived.

$$\begin{bmatrix} (1-c_4), & -c_6, & -c_8, & \ldots, & -c_{2N} \\ -3c_6, & (1-3c_8), & -3c_{10}, & & 0 \\ -5c_8, & -5c_{10}, & (1-5c_{12}), & & \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & & \cdot\cdot \\ -(2N-5)c_{2N-2}, & -(2N-5)c_{2N}, & 0, & 1, & 0 \\ -(2N-3)c_{2N}, & 0, & 0, & 0, & 1 \end{bmatrix} \begin{bmatrix} A_2 \\ A_4 \\ A_6 \\ \cdot\cdot \\ A_{2N-4} \\ A_{2N-2} \end{bmatrix} = \begin{bmatrix} \mu + \frac{1}{2}c_2 \\ \frac{3}{2}c_4 \\ \frac{5}{2}c_6 \\ \cdot\cdot \\ \frac{2N-5}{2}c_{2N-4} \\ \frac{2N-3}{2}c_{2N-2} \end{bmatrix} \tag{37}$$

together with

$$A_{2N} = (N-\tfrac{1}{2})c_{2N}.$$

The tangential or hoop-stress distribution around the hole is found from the first of equations (23) by putting $\zeta = e^{i\theta}$ and noting that since the normal stress is zero,

$$\sigma_\theta = \sigma_n + \sigma_\theta = \sigma_x + \sigma_y = \Omega(\zeta) + \overline{\Omega}(\bar{\zeta}). \tag{38}$$

Substitution of the A_{2n} from equation (37) gives the following expression for σ_θ,

$$\sigma_\theta = \frac{\sum_{n=0}^{N} A_{2n} e^{-2ni\theta}}{\sum_{n=0}^{N} (1-2n) a_{2n} e^{-2ni\theta}} + \frac{\sum_{n=0}^{N} A_{2n} e^{2ni\theta}}{\sum_{n=0}^{N} (1-2n) a_{2n} e^{2ni\theta}}$$

which can be written in real terms in the form

$$\sigma_\theta = \frac{\sum_{n=0}^{N} (1-2n) A_{2n} a_{2n} + \sum_{r=1}^{N} \sum_{n=0}^{N-r} \{(1-2r-2n) A_{2n} a_{2(n+r)} + (1-2n) A_{2(r+n)} a_{2n}\} \cos 2r\theta}{\frac{1}{2} \sum_{n=0}^{N} (1-2n)^2 a_{2n}{}^2 + \sum_{r=1}^{N} \sum_{n=0}^{N-r} (1-2n)(1-2r-2n) a_{2n} a_{2(r+n)} \cos 2r\theta}. \tag{39}$$

On putting $T = 1$, equation (39) may be used to determine the stress-concentration factor for the hole as the maximum of σ_θ for variation of θ.

A Mercury Computer programme incorporating these algebraic solutions has been written in a form that allows for up to 92 coefficients a_{2n} in the mapping function. Only one matrix inversion is required and a typical 40 term mapping and elastic analysis takes about 8 minutes for both tensile-loading cases.

5. *Example.*

A 'rounded square' profile has been investigated whose profile, C, is formed by the circles centre $(\pm a, \pm a)$ and radius a and their common external tangents. The stresses around this hole have been evaluated under axial tension, which has been applied parallel to the x-axis.

A trial mapping with $m = 5$ was found to be unsatisfactory and m was increased to 10, 20 and 40 in turn. For $m = 40$ the boundary of the derived profile C′ was almost identical with C, but the curvature distribution showed an oscillation about the point of discontinuity as illustrated in Fig. 3. The amplitude and frequency of this irregularity in the curvature are not significant in the stress analysis.

The hoop-stress distribution is shown in polar form in Fig. 4. The stress-concentration factor is slightly greater than 3 and occurs at a point about 76° round the quadrant.

5.1. *Criticisms of the Method.*

The choice of uniform spacing of the θ_k, which was desirable in the interests of high-speed computing, leaves the correspondence of C, C′ at its worst in the quadrant, where best agreement is sought. However, as is shown in Appendix I, the discrepancy should not influence estimates of stress concentration. If in more extreme examples, it is desirable to allow greater influence on the choice of mapping coefficients to be exerted by a region such as the quadrant, two choices are open to the computor. Either M can be increased up to the limit of 92 or an algebraic correspondence of points within the critical region and their associated points on the unit-circle can be established. This is made possible by the correspondence of arguments which is established by a limited iteration of the mapping programme. Interpolation of other points can be made with confidence. This special device may be necessary for profiles with local radii of curvature of the order of $0 \cdot 01$ times the diameter of the hole.

When attempts are made to map regions like the rounded square illustrated above, difficulties always arise in attempting to map accurately the region of a curvature change. Away from such a discontinuity the mapping will be smooth and it is only in the neighbourhood of the discontinuity that the mapping may be poor. Sometimes the stress concentration occurs near this region, as in the example.

The following proposal is made for assessing the merit of a mapping: if the errors in curvature relative to nominal are small (say up to 20%) and occur over distances which are small compared with the local radius of curvature the stress concentration remains uninfluenced by them. This proposal is substantiated by an exact analysis of a problem of the type described in this report for which a controlled perturbation of the curvature is introduced.

6. *Conclusion.*

A method suitable for the automatic computation of the stress distribution around an unreinforced hole of doubly-symmetric profile has been developed. The computed stresses are reliable in most practical cases.

LIST OF SYMBOLS

x, y		Cartesian co-ordinates
z	$=$	$x + iy$
ζ		An auxiliary complex variable
a_n, b_m		Coefficients of mapping functions
θ	$=$	$\arg \zeta$
κ		The local curvature of contours
$u(\theta), v(\theta)$		The real and imaginary parts of the complex variable z/ζ
$\left.\begin{matrix}\xi, \eta, \lambda \\ a, b, c\end{matrix}\right\}$		Parameters associated with circle elements and line elements, *see* equations (13), (18)
δ		Error function for curves
u, v		Elastic displacements
G		Shear modulus
S		Elastic constant {*see* equation (22)}
σ_x, σ_y		Direct stresses in the z-plane
τ_{xy}		Shear stress in the z-plane
$\Omega(z), \omega(z)$		Complex stress potentials in the z-plane
$\Omega(\zeta), \omega(\zeta)$		Corresponding potentials in the ζ-plane
$m(\zeta)$		A typical mapping function
T		The tension applied at infinity
Q		The shear applied at infinity
α		The direction (relative to Ox) of the applied tension [In Appendix I, a small displacement parameter]
σ_n, σ_θ		Normal and tangential direct stresses referred to curvilinear co-ordinates (n, θ)
b_{2r}, c_{2r}		Auxiliary coefficients in elastic analysis
μ		A loading constant, *see* equation (36)
f		A stress-concentration factor (Appendix I)

REFERENCES

No.	Author(s)	Title, etc.
1	N. I. Mushkhelishvili	Praktische Lösung der fundamentalen Randwertaufgaben der Elastizitätstherorie in der Ebene für einige Berandungsformen. *Z.A.M.M.*, Vol. 13, pp. 264 to 282. 1933.
2	N. I. Mushkhelishvili	*Some basic problems of the mathematical theory of elasticity.* 3rd edition, Moscow-Leningrad. 1949. (Translated by J. R. M. Radok. Noordhoff. 1953.)
3	G. N. Savin	*Concentration of stress around holes.* Moscow-Leningrad 1951. (Translated by E. Gros and edited by W. Johnson. Pergamon. 1961.)
4	A. E. Green and W. Zerna ..	*Theoretical elasticity.* Oxford. 1954.
5	J. N. Goodier and P. G. Hodge, Jr.	*Elasticity and plasticity.* John Wiley. 1958.
6	M. Kikukawa	On a method of calculating stress concentration in an infinite plate with a hole of arbitrary profile. *Proc. Jap. Nat. Cong. App. Mech.*, Vol. 1, pp. 185 to 191. 1951.
7	M. Kikukawa	On plane stress problems in regions of arbitrary profiles. *Proc. Jap. Nat. Cong. App. Mech.*, Vol. 3, pp. 5 to 9. 1953.
8	M. Kikukawa	On a method of solving elastic problems of plates. *Proc. Jap. Nat. Cong. App. Mech.*, Vol. 4, pp. 149 to 152. 1954.
9	S. P. Timoshenko and J. N. Goodier	*Theory of elasticity.* 2nd Edition, McGraw Hill. 1951.
10	J. H. Argyris and S. Kelsey ..	*Energy theorems and structural analysis.* Butterworth. 1960.
11	P. V. Melentiev	Several new methods and devices for approximate computations (in Russian). Moscow-Leningrad. 1937.
12	L. V. Kantorovitch and V. I. Krylov	*Approximate methods of higher analysis.* Translation by Curtis D. Benster. Noordhoff. 1958.
13	Sir E. Whittaker and G. Robinson	*The calculus of observations.* 4th edition. Blackie. 1944.
14	A. C. Stevenson	Complex potentials in two-dimensional elasticity. *Proc. Roy. Soc.* A, Vol. 184, pp. 129 to 179. 1945.
15	L. M. Milne-Thomson	*Plane elastic systems.* Springer, Berlin. 1960.

APPENDIX I

The Effect of Perturbations of a Contour on the Hoop-Stress Distribution around it

The influence of small errors in the local curvature arising from the use of an insufficiently extensive mapping function can be estimated by the use of the mapping function

$$z = \zeta + \alpha\zeta^{-n} \tag{1}$$

where, to produce a doubly-symmetric contour, n is an integer of the form $(4p+3)$.

The curve C corresponding to the unit-circle in the ζ-plane has parametric form

$$x = \cos\theta + \alpha\cos n\theta, \qquad y = \sin\theta - \alpha\sin n\theta \tag{2}$$

which defines a hypotrochoid formed by rolling a circle of radius $1/n$ on the outside of a circle radius $n - 1/n$ and observing a point on the rolling circle distant α from its centre. As the rolling circle moves round the other, it will complete $(n+1)$ revolutions before returning to its original position. The locus C will, for small α, be a figure which is nearly circular and has a radius varying between $1 + \alpha$ and $1 - \alpha$, and which is unity precisely $2n + 2$ times. This figure has a curvature variation

$$\kappa = \frac{1 + n(n-1)\alpha\cos(n+1)\theta - n^3\alpha^2}{[1 - 2n\alpha\cos(n+1)\theta + n^2\alpha^2]^{3/2}}. \tag{3}$$

The mean curvature is $\kappa = 1$, a value which is attained at $2n + 2$ points of C. The wavelength of curvature variations is $\pi/(n+1)$, which figure is also the ratio of this length to the mean radius of curvature.

The stress function for an applied tension parallel to Ox is[5]

$$\Omega(\zeta) = \frac{\tfrac{1}{2}T(1+n\alpha\zeta^{-n-1}) - [1-(n-2)\alpha^2]^{-1}(\zeta^{-2}+(n-2)\alpha\zeta^{-n+1})}{1 - n\alpha\zeta^{-n-1}} \tag{4}$$

so that the stress-concentration factor for the hole is found by putting $T = 1$, $\zeta = e^{i\theta}$, and forming

$$f = \frac{1}{T}\{\Omega(\zeta) + \bar{\Omega}(\bar{\zeta})\} \tag{5}$$

which gives

$$f = \frac{1 - n^2\alpha^2 + 2\{1-(n-2)\alpha^2\}^{-1}[2\alpha\cos(n-1)\theta + \{(n^2-2n)\alpha^2 - 1\}\cos 2\theta]}{1 - 2n\alpha\cos(n+1)\theta + n^2\alpha^2}. \tag{6}$$

For $\theta = \pi/2$ and n of the form $4p + 3$ this reduces to

$$f = \frac{1 + n\alpha + 2\{1-(n-2)\alpha^2\}^{-1}(1+n\alpha-2\alpha)}{1 - n\alpha} \tag{7}$$

and

$$\kappa = \frac{1 + n^2\alpha}{(1-n\alpha)^2}. \tag{8}$$

From these two results a simple approximation is found;

$$f' \approx 3\left[\frac{n - 1 + 3\kappa}{n + 1 + \kappa}\right]$$

where κ is the curvature at the point $\theta = \pi/2$, and is given by (8). This shows that the error in f is proportional to the maximum error in slope of the boundary which is approximately $(n\alpha)$. When $\pi/(n+1)$ is small, say $0\cdot 2$, $n = 15$ say, and for $\kappa = 0\cdot 8$, the resulting estimate of f is $2\cdot 4\%$ lower than the value of 3 for $\kappa = 1$. A table of values of f for various n and κ at $\theta = \pi/2$ is given. The error in estimating f is not important, and for practical problems such as the example given in the text, the estimate available is adequate.

Values of f from equation (7)

$\kappa \rightarrow$	0·8	0·9	1	1·1	1·2	$\pi/(n+1)$
$n =$ 3	2·8073	2·9052	3	3·0920	3·1814	0·7854
7	2·8770	2·9391	3	3·0598	3·1184	0·3927
11	2·9121	2·9563	3	3·0431	3·0855	0·2618
15	2·9318	2·9661	3	3·0335	3·0667	0·1963
19	2·9444	2·9723	3	3·0274	3·0547	0·1571
23	2·9530	2·9766	3	3·0232	3·0463	0·1309
27	2·9594	2·9798	3	3·0201	3·0401	0·1121

APPENDIX II

Extension to Shear-Loading Case

If the plate is loaded in shear at infinity, ($\sigma_x = \sigma_y = 0$; $\tau_{xy} = Q$ say) the analysis given in the main text is modified as follows.

The coefficients occurring in the expansion of $m'(\zeta)\Omega(\zeta)$ are pure imaginary so that we may write

$$m'(\zeta)\Omega(\zeta) = 2iQ \sum_{n=0}^{N} A_{2n}\zeta^{-2n},$$

where the A_{2n} are real.

At infinity

$$\Omega(\zeta) = O(\zeta^{-2})$$

and

$$\omega(\zeta) = 2iQ + O(\zeta^{-2}),$$

so that A_0 is zero.

If a_{2n}, b_{2n}, c_{2n} have the same meaning as in the main text, the identification of positive powers of ζ and of the constant term in equation (25) leads to the following matrix equation for the A_{2n}:

$$\begin{bmatrix} 1+c_4, & c_6, & c_8, & \ldots, & c_{2N-2}, & c_{2N} \\ 3c_6, & 1+3c_8, & 3c_{10}, & \ldots, & 3c_{2N}, & 0 \\ 5c_8, & 5c_{10}, & 1+5c_{12}, & \ldots, & 0, & 0 \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & & \cdot\cdot & \cdot\cdot \\ (2N-3)c_{2N}, & 0, & 0, & \ldots, & 0, & 1 \end{bmatrix} \begin{bmatrix} A_2 \\ A_4 \\ A_6 \\ \cdot\cdot \\ A_{2N-2} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ \cdot\cdot \\ 0 \end{bmatrix}$$

with $A_{2N} = A_0 = 0$.

Proceeding as in the main text, the hoop-stress distribution around the hole is obtained in the form

$$\sigma_\theta = \frac{4Q \sum_{r=1}^{N} \sum_{j=0}^{N-r} \{(1-2r-2j)A_{2j}a_{2r+2j} - (1-2j)A_{2j+2r}a_{2j}\} \sin 2r\theta}{\sum_{r=0}^{N} (1-2r)^2 a_{2r}^{\;2} + 2 \sum_{r=1}^{N} \sum_{j=0}^{N-r} (1-2r-2j)(1-2j)a_{2r+2j}a_{2j} \cos 2r\theta}.$$

The stress-concentration factor in shear is $\sigma_{\max}/\sqrt{3}Q$ using the Mises-Hencky criterion, which is based on the strain energy of distortion of the material.

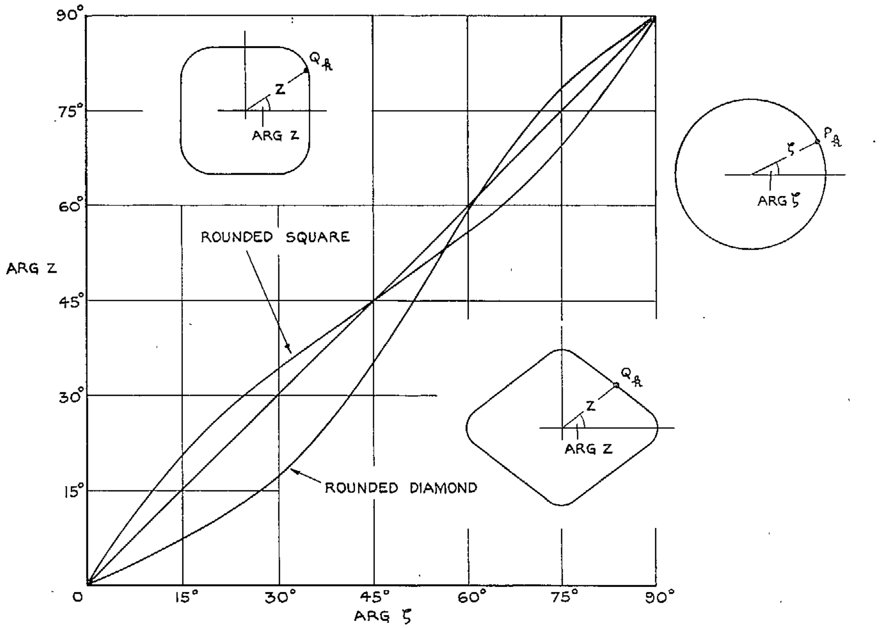

FIG. 1. Variation of arg z with arg ζ for mapping of two doubly-symmetric figures shown on to the unit-circle in the ζ-plane.

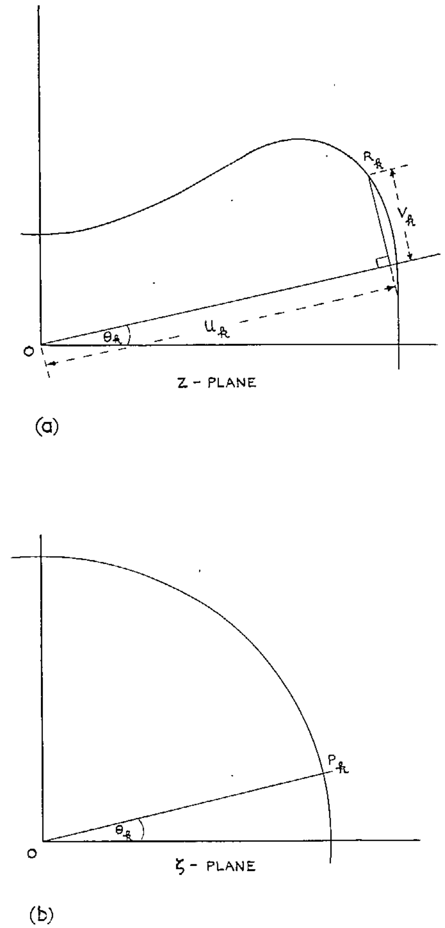

FIG. 2a and b. Co-ordinate system used in the Melentiev process.

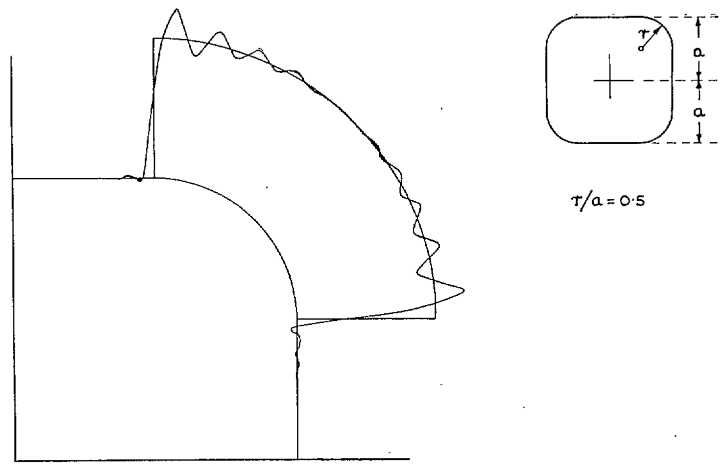

FIG. 3. Variation of curvature of approximate mapping around quadrant of rounded square.

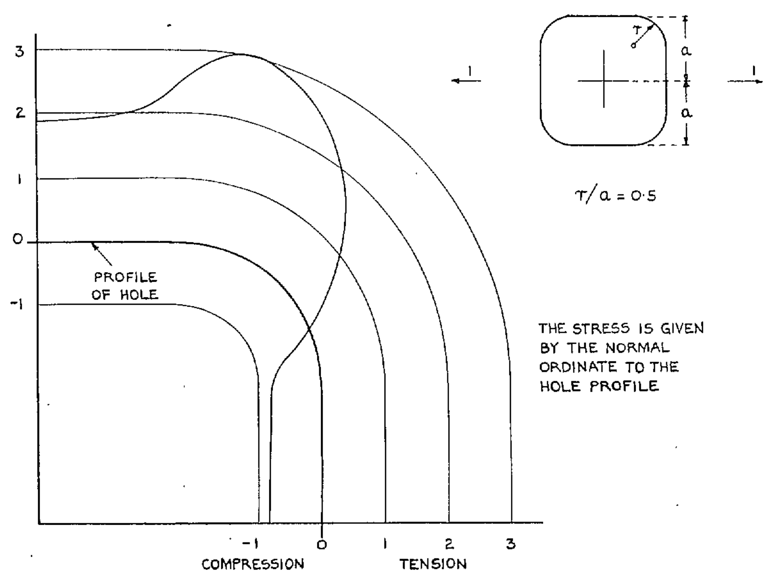

FIG. 4. Hoop-stress distribution around edge of rounded square hole in infinite plate under tension.

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