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Summary. Using Fourier integrals, a quasi-steady solution is obtained to the title problem where the direction of motion, shape and size of the loading distribution do not vary with time. The loads are, moreover, assumed equally applied to both surfaces in such a way that the motion takes place in two dimensions. A numerical example is considered where the applied loading is distributed discontinuously according to a step function and is travelling with a velocity not greater than that of the shear wave.

The corresponding solution for plane stress is obtained by changing the value of one of the elastic constants and it is then an aid in the study of further problems such as the rapidly moving crack.

1. *Introduction.* A renewed attention, stimulated by problems arising in many branches of the applied sciences, is being given to the investigation of elastic waves. In particular, there have been recent and notable contributions (*e.g.*, Sneddon[1], Craggs[2]) in the theory of two dimensional waves in an elastic half space and this is of special interest in geophysics and soil mechanics. The engineer, however, interested in impact and rapidly moving cracks, is frequently concerned with a more difficult problem in so far as the medium has now two or more boundaries which introduce reflected wave systems.

Here we are concerned with an aspect of this latter type of problem where applied loads are travelling uniformly along the bounding surfaces of an infinite elastic plate. The mathematical and computational difficulties are now more severe and so advantage is taken of the appreciable simplification obtained by confining attention to the quasi-steady state. Thus, the direction of motion, shape and size of the loading distributions are assumed not to vary with time and, moreover, the loads are assumed equally applied to both surfaces in such a way that the motion takes place in two dimensions. All transient disturbances arising from the initial loading applications are ignored. The solution for plane stress is obtained by changing the value of one of the elastic constants and

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it is then an aid in the study of further and more difficult problems. Examples of current interest are those encountered during the study of a crack moving rapidly along a bounded flat sheet, or along the generator of a thin-walled cylinder where the consequent release of stress may be regarded, to a first approximation, as a discontinuous distribution of applied loading which travels along an otherwise free edge.

The method of solution is similar to that adopted by Filon[3] in 1903 to solve the corresponding static problem. For this method, the special case of a simple harmonic wave is studied as a preliminary to the derivation of the Fourier integrals which, in their turn, enable the boundary conditions to be satisfied. The analysis is simplified by taking advantage of some results, listed here, of the recent theory of generalised functions, *e.g.*, Lighthill[4], 1959. The investigation is concluded by considering a numerical example where the applied loading is distributed discontinuously according to a step function.

It is recalled that in an unbounded elastic solid the stress waves can be propagated with two different velocities—either with that of the dilation wave or, at about half this velocity, with that of the shear wave. When there is a bounding surface an elastic surface wave, or Rayleigh[5] wave, may also occur and this wave satisfies conditions of zero normal and tangential stress at the bounding surface while travelling slightly slower than the shear wave. Now, in the numerical example mentioned above, it is found that the peak stresses in the interior of the plate increase in magnitude as the applied loads travel faster along the bounding surfaces until, at coincidence with the Rayleigh wave velocity, there is a condition of resonance where infinitely large stresses occur. A similar phenomenon has been previously noted, *e.g.*, by Craggs[2] and by Ang[6], the latter suggesting that this is a limiting velocity for the propagation of cracks in an elastic medium. It is interesting to note, however, that Irwin[7] suggests that in a real solid the limiting velocity is reduced by other considerations to about one half this value. A strange feature of the phenomenon, hitherto unmentioned, is that there are distributions of the applied loading for which this resonance does not occur.

When the applied loads are travelling faster than the Rayleigh wave velocity the stresses in the interior again have finite values but the quasi-steady solution is no longer unique because of the presence of a free wave. This free wave travels with the same velocity as the applied loads and it satisfies the conditions of zero normal and tangential stress at the bounding surfaces; it can therefore be added to the quasi-steady solution in an arbitrary manner. Such waves were studied by Lamb[8] in 1916.

The present investigation is concerned chiefly with loads travelling at a velocity not greater than that of the shear wave. It is interesting to note, however, that at each of the higher velocities there are an infinite number of these free waves, but there are additional physical requirements in the interior of the plate which arise, for example, from the fact that shear wave disturbances now travel slower than the applied loading. Uniqueness of solution (presumably) returns when the applied loads are travelling faster than the velocity of the 'long waves' which themselves travel a little slower than the dilatation wave.

On completion of the manuscript, the author's attention was drawn to a recent (1958) paper by Fulton and Sneddon[12] who consider this amongst other plate problems. Their attention is, however confined to velocities less than that of the Rayleigh wave.

2. *Fundamental Equations.* The fundamental equations are well known, but it is convenient to have the list of them which is given below.

A two dimensional rectangular Cartesian co-ordinate system $x'oy'$, Fig. 1, is chosen with the ox' axis contained in the central plane and with the oy' axis normal to this. The thickness of the plate is denoted by $2h$.

When there are no body forces, the two dimensional equations of motion in terms of the stress components σ_x, σ_y and τ_{xy} are

$$\frac{\partial \sigma_x}{\partial x'} + \frac{\partial \tau_{xy}}{\partial y'} = \rho \frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial \tau_{xy}}{\partial x'} + \frac{\partial \sigma_y}{\partial y'} = \rho \frac{\partial^2 v}{\partial t^2}, \tag{1}$$

where u, v are the displacement components respectively in the directions of the x' and y' axes, t denotes time and ρ is the material density. The relationship between stress and strain is the same as in static elasticity

$$\left.\begin{aligned} \sigma_x = \lambda\left(\frac{\partial u}{\partial x'} + \frac{\partial v}{\partial y'}\right) + 2\mu \frac{\partial u}{\partial x'}, \quad \sigma_y = \lambda\left(\frac{\partial u}{\partial x'} + \frac{\partial v}{\partial y'}\right) + 2\mu \frac{\partial v}{\partial y'}, \\ \tau_{xy} = \mu\left(\frac{\partial u}{\partial y'} + \frac{\partial v}{\partial x'}\right), \end{aligned}\right\} \tag{2}$$

where $\lambda = E\nu/(1+\nu)(1-2\nu)$ and $\mu = E/2(1+\nu)$ are Lamé's elastic constants and ν is Poisson's ratio. (The equations for plane stress are obtained by replacing the constant λ by $\lambda^* = E\nu/(1-\nu^2)$.)

For Lamé's solution of the equations of motion we introduce the two potentials $\varphi(x', y', t)$ and $\psi(x', y', t)$ so that the components of displacement are given by

$$u = \frac{\partial \varphi}{\partial x'} + \frac{\partial \psi}{\partial y'}, \quad v = \frac{\partial \varphi}{\partial y'} - \frac{\partial \psi}{\partial x'}. \tag{3}$$

Equations (2) for the stress components can now be rewritten

$$\left.\begin{aligned} \sigma_x = \lambda\nabla^2\varphi + 2\mu\left(\frac{\partial^2 \varphi}{\partial x'^2} + \frac{\partial^2 \psi}{\partial x' \partial y'}\right), \quad \sigma_y = \lambda\nabla^2\varphi + 2\mu\left(\frac{\partial^2 \varphi}{\partial y'^2} - \frac{\partial^2 \psi}{\partial x' \partial y'}\right), \\ \tau_{xy} = \mu\left(2\frac{\partial^2 \varphi}{\partial x' \partial y'} - \frac{\partial^2 \psi}{\partial x'^2} + \frac{\partial^2 \psi}{\partial y'^2}\right) \end{aligned}\right\} \tag{4}$$

where ∇^2 is Laplace's operator

$$\nabla^2 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}. \tag{5}$$

When Equations (3) and (4) are substituted into the Equations (1) of motion, it is found that the functions φ and ψ must satisfy the wave equations

$$\nabla^2\varphi = \frac{1}{c_1^2}\frac{\partial^2 \varphi}{\partial t^2}, \quad \nabla^2\psi = \frac{1}{c_2^2}\frac{\partial^2 \psi}{\partial t^2}, \tag{6}$$

where c_1 and c_2 are respectively the dilatation and shear wave velocities given by

$$c_1^2 = (\lambda + 2\mu)/\rho, \quad c_2^2 = \mu/\rho. \tag{7}$$

For disturbances moving with uniform velocity c parallel to the x' axis, and in the positive direction, the dependent variables are all functions of $x = x' - ct$ and $y = y'$. For the moment, it is assumed that the solution is periodic with respect to x and this is most conveniently done by introducing the factor $\exp(-2\pi i \xi x)$, the wavelength L being accordingly

$$L = 1/\xi \tag{8}$$

where ξ is an arbitrary constant. Writing

$$\beta_1 = \left(1 - \frac{c^2}{c_1^2}\right)^{1/2}, \quad \beta_2 = \left(1 - \frac{c^2}{c_2^2}\right)^{1/2}, \tag{9}$$

the wave Equations (6) simplify to

$$\frac{\partial^2 \varphi}{\partial y^2} = 4\pi^2 \beta_1^2 \xi^2 \varphi, \quad \frac{\partial^2 \psi}{\partial y^2} = 4\pi^2 \beta_2^2 \xi^2 \psi, \tag{10}$$

and Equations (4) for the stress components reduce to

$$\left.\begin{aligned} \sigma_x &= -\mu\left\{(1-\beta_2^2+2\beta_1^2)4\pi^2\xi^2\varphi + 4\pi i\xi\frac{\partial\psi}{\partial y}\right\}, \\ \sigma_y &= \mu\left\{(1+\beta_2^2)4\pi^2\xi^2\varphi + 4\pi i\xi\frac{\partial\psi}{\partial y}\right\}, \\ \tau_{xy} &= \mu\left\{-4\pi i\xi\frac{\partial\varphi}{\partial y} + (1+\beta_2^2)4\pi^2\xi^2\psi\right\}. \end{aligned}\right\} \tag{11}$$

3. *Satisfaction of Boundary Conditions.* The boundary conditions at the surfaces $y = \pm h$ are satisfied with the aid of Fourier integrals in a similar way that Filon[3] used for the static problem.

The motion is symmetrical with respect to the central plane $y = 0$ and so solutions of the wave Equations (10) which are suitable for the present purpose are

$$\varphi = \Phi \exp(-2\pi i\xi x)\cosh(2\pi\beta_1\xi y), \quad \psi = \Psi \exp(-2\pi i\xi x)\beta_2^{-1}\sinh(2\pi\beta_2\xi y) \tag{12}$$

where Φ and Ψ are constants to be determined. The factor β_2^{-1}, inserted into the expression for ψ, is convenient in the particular case when the applied loading distribution is travelling with the same velocity as the shear wave, *i.e.*, when $c = c_2$ and $\beta_2 = 0$. The normal and tangential stress components on the bounding surfaces of the plate at $y = \pm h$ are now obtained by substituting Equations (12) into Equations (11) to find that

$$\left.\begin{aligned} \sigma_y(x, h) &= \mu\{\Phi(1+\beta_2^2)\cosh(2\pi\beta_1\xi h) + \Psi 2i\cosh(2\pi\beta_2\xi h)\}4\pi^2\xi^2\exp(-2\pi i\xi x), \\ \tau_{xy}(x, h) &= -\tau_{xy}(x, -h) = \mu\{-\Phi 2i\beta_1\sinh(2\pi\beta_1\xi h) + \Psi(1+\beta_2^2)\beta_2^{-1}\sinh(2\pi\beta_2\xi h)\} \times \\ &\quad \times 4\pi^2\xi^2\exp(-2\pi i\xi x). \end{aligned}\right\} \tag{13}$$

Thus, when the constants Φ and Ψ are given the values

$$\left.\begin{aligned} \Phi &= \frac{1}{4\pi^2\mu}\chi(1+\beta_2^2)\beta_2^{-1}\sinh(2\pi\beta_2\xi h), \\ \Psi &= \frac{1}{4\pi^2\mu}2i\chi\beta_1\sinh(2\pi\beta_1\xi h), \end{aligned}\right\} \tag{14}$$

where χ is a new constant, then the stresses at $y = \pm h$ are given by

$$\left.\begin{aligned} \sigma_y(x, h) &= \chi\Omega(\xi)\xi^2\exp(-2\pi i\xi x), \\ \tau_{xy}(x, h) &= \tau_{xy}(x, -h) = 0, \end{aligned}\right\} \tag{15}$$

where $\Omega(\xi)$ is used as an abbreviation for

$$\Omega(\xi) = \{(1+\beta_2^2)^2\cosh(2\pi\beta_1\xi h)\sinh(2\pi\beta_2\xi h) - 4\beta_1\beta_2\cosh(2\pi\beta_2\xi h)\sinh(2\pi\beta_1\xi h)\}\beta_2^{-1}. \tag{16}$$

If the above equations are assumed to hold for all values of ξ, we may then write formally

$$\left.\begin{aligned}\varphi &= \int_{-\infty}^{\infty} \Phi(\xi) \exp(-2\pi i\xi x) \cosh(2\pi\beta_1\xi y)d\xi, \\ \Psi &= \int_{-\infty}^{\infty} \Psi(\xi) \exp(-2\pi i\xi x)\, \beta_2^{-1} \sinh(2\pi\beta_2\xi y)d\xi\end{aligned}\right\} \tag{17}$$

and, assuming that it is permissible to differentiate under the integral sign, the normal and tangential stress components at $y = \pm h$ are now given by

$$\left.\begin{aligned}\sigma_y(x, h) &= -g(x) = \int_{-\infty}^{\infty} \chi(\xi)\Omega(\xi)\xi^2 \exp(-2\pi i\xi x)d\xi, \\ \tau_{xy}(x, h) &= \tau_{xy}(x, -h) = 0,\end{aligned}\right\} \tag{18}$$

where $g(x)$ is the distribution of loading which is applied to the bounding surfaces, a positive value denoting a pressure. Using the well known Fourier integral relationships which state that if

$$f(\xi) = \int_{-\infty}^{\infty} g(x) \exp(2\pi i\xi x)dx, \tag{19a}$$

then, for suitable functions $g(x)$

$$g(x) = \int_{-\infty}^{\infty} f(\xi) \exp(-2\pi i\xi x)d\xi, \tag{19b}$$

we find after making comparison with Equations (18) that

$$\chi(\xi) = -\frac{f(\xi)}{\Omega(\xi)\xi^2}. \tag{20}$$

The function $g(x)$, which gives the distribution of loading, is called the Fourier transform (F.T.) of $f(\xi)$.

If Equations (14) and (20) are now substituted into Equations (17) then the following Fourier integral expressions are obtained for the potentials φ and ψ

$$\left.\begin{aligned}\varphi &= -\frac{(1+\beta_2^2)}{4\pi^2\mu\beta_2}\int_{-\infty}^{\infty} \frac{\sinh(2\pi\beta_2\xi h)\cosh(2\pi\beta_1\xi y)\exp(-2\pi i\xi x)f(\xi)}{\Omega(\xi)\xi^2}d\xi, \\ \psi &= -\frac{i\beta_1}{2\pi^2\mu\beta_2}\int_{-\infty}^{\infty} \frac{\sinh(2\pi\beta_1\xi h)\sinh(2\pi\beta_2\xi y)\exp(-2\pi i\xi x)f(\xi)}{\Omega(\xi)\xi^2}d\xi.\end{aligned}\right\} \tag{21}$$

Assuming again that it is permissible to differentiate under the integral sign, the displacement components u and v are found by substituting these two potentials into Equations (3) so that

$$\left.\begin{aligned}u &= \frac{i}{2\pi\mu\beta_2}\int_{-\infty}^{\infty}\begin{Bmatrix}(1+\beta_2^2)\sinh(2\pi\beta_2\xi h)\cosh(2\pi\beta_1\xi y) \\ -2\beta_1\beta_2\sinh(2\pi\beta_1\xi h)\cosh(2\pi\beta_2\xi y)\end{Bmatrix}\frac{\exp(-2\pi i\xi x)f(\xi)}{\Omega(\xi)\xi}d\xi, \\ v &= -\frac{\beta_1}{2\pi\mu\beta_2}\int_{-\infty}^{\infty}\begin{Bmatrix}(1+\beta_2^2)\sinh(2\pi\beta_2\xi h)\sinh(2\pi\beta_1\xi y) \\ -2\sinh(2\pi\beta_1\xi h)\sinh(2\pi\beta_2\xi y)\end{Bmatrix}\frac{\exp(-2\pi i\xi x)f(\xi)}{\Omega(\xi)\xi}d\xi.\end{aligned}\right\} \tag{22}$$

Similarly, by substituting Equations (21) into Equations (11), the stress components are found to be

$$\left.\begin{aligned} \sigma_x &= \frac{1}{\beta_2}\int_{-\infty}^{\infty}\left\{\begin{matrix}(1+\beta_2^2)(1-\beta_2^2+2\beta_1^2)\sinh(2\pi\beta_2\xi h)\cosh(2\pi\beta_1\xi y)\\ -4\beta_1\beta_2\sinh(2\pi\beta_1\xi h)\cosh(2\pi\beta_2\xi y)\end{matrix}\right\}\frac{\exp(-2\pi i\xi x)f(\xi)}{\Omega(\xi)}\,d\xi\,,\\ \sigma_y &= -\frac{1}{\beta_2}\int_{-\infty}^{\infty}\left\{\begin{matrix}(1+\beta_2^2)^2\sinh(2\pi\beta_2\xi h)\cosh(2\pi\beta_1\xi y)\\ -4\beta_1\beta_2\sinh(2\pi\beta_1\xi h)\cosh(2\pi\beta_2\xi y)\end{matrix}\right\}\frac{\exp(-2\pi i\xi x)f(\xi)}{\Omega(\xi)}\,d\xi\,,\\ \tau_{xy} &= \frac{2i\beta_1(1+\beta_2^2)}{\beta_2}\int_{-\infty}^{\infty}\left\{\begin{matrix}\sinh(2\pi\beta_2\xi h)\sinh(2\pi\beta_1\xi y)\\ -\sinh(2\pi\beta_1\xi h)\sinh(2\pi\beta_2\xi y)\end{matrix}\right\}\frac{\exp(-2\pi i\xi x)f(\xi)}{\Omega(\xi)}\,d\xi\,. \end{aligned}\right\} \quad (23)$$

The formal analysis above is in anticipation of the results of the theory of generalised functions which are listed later in this Paper. It is formally applicable for all values of the velocity c of the applied loading distribution, the results for special values such as $c = 0$, c_1 or c_2 being obtained by limiting processes.

We now turn to examine the question of free waves which can occur for certain values of c. They were first studied by Lamb[8] in 1916.

4. *Free Waves.* Referring back to Equations (15) and (16) we see that if there is a real value of ξ for which $\Omega(\xi) = 0$ then it is possible for waves to be present which satisfy conditions of zero normal and tangential stress along the bounding surfaces of the plate, *i.e.*, $\sigma_y(x, h) = \tau_{xy}(x, h) = 0$. These are now referred to as free waves and they occur for the real values of $\xi = \pm\xi_0$ which are the roots of the equation

$$(1+\beta_2^2)^2\cosh(2\pi\beta_1\xi_0 h)\sinh(2\pi\beta_2\xi_0 h) - 4\beta_1\beta_2\cosh(2\pi\beta_2\xi_0 h)\sinh(2\pi\beta_1\xi_0 h) = 0\,. \quad (24)$$

The free wave has wavelength $L_0 = 1/\xi_0$ and frequency c/L_0.

These free waves have a minimum velocity of propagation. Since

$$\beta_1 = \left(1-\frac{c^2}{c_1^2}\right)^{1/2}, \quad \beta_2 = \left(1-\frac{c^2}{c_2^2}\right)^{1/2}$$

and $c_1 > c_2$, it follows that $\beta_1 \geqslant \beta_2$ for real values and, therefore, for Equation (24) to have a real root it is necessary for the velocity c to have a value such that

$$4\beta_1\beta_2 \geqslant (1+\beta_2^2)^2\,. \quad (25)$$

The slowest non zero velocity $c = c_S$ is found to satisfy the equality and is, in fact, the smallest root to the equation

$$16\left(1-\frac{c_S^2}{c_1^2}\right)\left(1-\frac{c_S^2}{c_2^2}\right) = \left(2-\frac{c_S^2}{c_2^2}\right)^4 \quad (26)$$

which governs the propagation of Rayleigh surface waves in an elastic solid with a plane bounding surface. The ratio c_S/c_2 is dependent only upon Poisson's ratio ν and this relationship is plotted in Fig. 2. In the special case when $\nu = 0\cdot3$ we have $c_S/c_2 = 0\cdot9274$.

The minimum velocity for free waves is, from Equation (24), seen to correspond to waves of zero wavelength, *i.e.*, $\xi_0 = \infty$. Let us now consider the other extreme where $\xi_0 \to 0$, *i.e.*, the wavelength is very long. Equation (24) then simplifies to

$$(1+\beta_2^2)^2 = 4\beta_1^2 \quad (27)$$

in the limit when $\xi_0 = 0$. After substituting from Equations (7) and (9) this is found to be the same as

$$\frac{c^2}{c_1^2} = \frac{c_L^2}{c_1^2} = \frac{1-2\nu}{(1-\nu)^2}, \tag{28}$$

where the notation c_L is introduced to denote the velocity of long waves in the plate. The variation of the ratio c_L/c_1 with Poisson's ratio is also shown in Fig. 2.

We have just seen that the first real root of Equation (24) occurs when $c = c_S$ and that this corresponds to $\xi_0 = \pm\infty$. This root ξ_0 has a finite value for higher velocities and its variation is shown in Fig. 3. Furthermore, it is the only real root for $c_S \leqslant c \leqslant c_2$. When $c_2 < c$, however, then β_2 is imaginary so that on writing

$$\beta_2 = i\alpha_2, \quad c_2 < c \tag{29}$$

we find that Equation (24) can be rewritten as

$$(1-\alpha_2^2)^2 \cosh(2\pi\beta_1\xi_n h) \sin(2\pi\alpha_2\xi_n h) - 4\beta_1\alpha_2 \cos(2\pi\alpha_2\xi_n h) \sinh(2\pi\beta_1\xi_n h) = 0, \tag{30}$$

where $\xi_n = \pm\xi_0, \pm\xi_1, \pm\xi_2, \ldots$ means that there are now an infinite number of real roots and so it is possible for an infinite number of free waves to exist each travelling with the same velocity c. In the special case where $c = c_L$, *see* Equations (27) and (28), we find that the roots are determined from the equation

$$\tan(2\pi\alpha_2\xi_n h) = \tanh(2\pi\beta_1\xi_n h), \quad c = c_L. \tag{31}$$

We have already discussed the root $\xi_0 = 0$, but it must be noted that there are an infinite number of real roots to Equation (31), in fact for large integers n we have

$$\xi_n \doteqdot \pm(4n+1)/(8\alpha_2 h), \quad c = c_L. \tag{32}$$

The wave with the longest wavelength is, however, the one most likely to be excited during a vibration and it is usual to confine attention to this case. There are waves which have a (phase) velocity greater than c_L, *e.g.*, the special case when $c = c_1$ then $\beta_1 = 0$ and we have

$$\left.\begin{aligned} &\varphi = 2(-1)^n \exp(-2\pi i\xi_n x), \quad \psi = i(1-\alpha_2^2)\exp(-2\pi i\xi_n x)\alpha_2^{-1}\sin(2\pi\alpha_2\xi_n y), \\ &\xi_n = \pm n/(2\alpha_2 h), \quad n = 1, 2, 3, \ldots\ldots \end{aligned}\right\} \tag{33}$$

The mathematical consequence of real roots ξ_n to the equation $\Omega(\xi) = 0$ is the singularity which occurs in the integrands of Equations (21) to (23) when $\xi = \xi_n$. Thus, for velocities greater than the Reyleigh wave the integrals, in the ordinary sense, are multivalued with a consequent lack of uniqueness of solution which persists until $c = c_L$*. In such cases it is possible to attach definite values to the stress and displacement components only by introducing additional requirements.

5. *Notes on the Theory of Generalised Functions.* Attention has already been drawn to the fact that part of the afore-going analysis has been conducted on a simple formal basis. Justification for this and for further analysis presently required is, however, readily obtained from the recently developed theory of generalised functions described by Lighthill[4] in 1959. When generalised functions are used then the integrals of equations such as (21) to (23) are interpreted in a definite and consistent manner, notwithstanding the singularities which may occur in the integrands.

* It should be noted that there is some uncertainty as to whether a unique solution is obtained in the range $c_L < c < c_1$. The same question arises in the Pochhammer treatment of cylindrical bars, *e.g.*, Kolsky[9].

Moreover, the use of the theory enables formal differentiation to be carried out under the sign of the integral for most integrands arising from practical problems and it provides a simple technique for the evaluation of the asymptotic behaviour of the Fourier integral.

Acknowledging that the theory is quite recent, it is perhaps helpful to list here those definitions which enable the recognition of the required generalised functions and then to describe the most useful theorems dealing with their Fourier transforms. Full details, with proofs, are given in the reference already quoted.

Definition I—Good function

A good function of x is everywhere differentiable any number of times and it and all its derivatives are $0(|x|^{-N})$ as $|x| \to \infty$ for all N, e.g., $\exp(-x^2)$ is a good function.

Definition II—Fairly good function

A fairly good function of x is everywhere differentiable any number of times and such that it and all its derivatives are $0(|x|^N)$ as $|x| \to \infty$ for some N, *e.g.*, a polynomial is a fairly good function.

Definition III—Generalised function

A generalised function is a regular sequence of good functions.

Definition IV—Ordinary functions as generalised functions

If $f(x)$ is a function of x in the ordinary sense such that $(1+x^2)^{-N}f(x)$ is absolutely integrable from $-\infty$ to ∞ for some N, then it can be shown that $f(x)$ is also a generalised function.

Definition V—Properties of a generalised function

If two generalised functions $f(x)$ and $h(x)$ are defined by sequences $f_n(x)$ and $h_n(x)$, then their sum $f(x) + h(x)$ is defined by the sequence $f_n(x) + h_n(x)$. Also, the derivative $f'(x)$ is defined by the sequence $f_n'(x)$. Also, $f(ax+b)$ is defined by the sequence $f_n(ax+b)$. Also, $\varphi(x)f(x)$, where $\varphi(x)$ is a fairly good function is defined by the sequence $\varphi(x)f_n(x)$ of good functions. It should be noted, however, that there is no satisfactory definition for the product of two generalised functions.

As an example, the sequence $\exp(-x^2/n^2)$ of good functions defines the generalised function $I(x)$ such that

$$\int_{-\infty}^{\infty} I(x)F(x)dx = \int_{-\infty}^{\infty} F(x)dx$$

and so $I(x)$ can be denoted more simply by 1. Again, the sequence $\{\exp(-nx^2)\}(n/\pi)^{1/2}$ defines a generalised function $\delta(x)$, the Dirac delta function, such that

$$\int_{-\infty}^{\infty} \delta(x)F(x)dx = F(0)$$

for any good function $F(x)$. Furthermore, since $(1+x^2)^{-1}\,\text{sgn}\, x$ is absolutely integrable from $-\infty$ to ∞ it follows that sgn x is a generalised function, it can therefore be differentiated and the result is

$$\frac{d}{dx}\,\text{sgn}\, x = 2\delta(x). \tag{34}$$

Lighthill[4] defines x^{-1} as the odd generalised function which satisfies the equation $xf(x) = 1$ (the general solution of which is $f(x) = x^{-1} + C\delta(x)$, where C is a constant) and

$$x^{-m} = \frac{(-1)^{m-1}}{(m-1)!}\frac{d^{m-1}}{dx^{m-1}}(x^{-1}) \tag{35}$$

where m is any integer > 0. It is interesting to note that the generalised function x^{-m} equals the ordinary function x^{-m} in $0 < x < \infty$ and in $-\infty < x < 0$.

Useful results concerning the Fourier transforms (F.T.) of generalised functions are collected in the following theorems. If $f(\xi)$ is a generalised function with F.T. $g(x)$ then

Theorem I

The sequence $f_n(\xi)$ which defines $f(\xi)$ has F.T. $g_n(x)$ which also defines the generalised function $g(x)$.

Theorem II

The F.T. of $f(a\xi+b)$ is $|a|^{-1} \exp(2\pi ibx/a)g(x/a)$.

Theorem III

Fourier's inversion theorem for generalised functions, $f(x)$ is the F.T. of $g(-\xi)$.

Theorem IV

The F.T. of $f'(\xi)$ is $2\pi ixg(x)$.

Theorem V

If $f_y(\xi)$ is a generalised function of ξ for each value of the parameter y in $a \leqslant y \leqslant b$ and has F.T. $g_y(x)$, then the F.T. of $(\partial/\partial y)f_y(\xi)$ is $(\partial/\partial y)g_y(x)$ for $a \leqslant y \leqslant b$.

Theorem VI

If $f(\xi)$ has a finite number of singularities, $\xi = \xi_0, \xi_1, \ldots \xi_p, \ldots \xi_P$ and if $f(\xi) - \sum_{p=1}^{P} A_p(\xi-\xi_p)^{-m_p}$ is absolutely integrable for integral m_p, then $g(x)$, the F.T. of $f(\xi)$, satisfies

$$g(x) = -\pi i \operatorname{sgn} x \sum_{p=0}^{P} A_p \frac{(-2\pi ix)^{m_p-1}}{(m_p-1)!} \exp(-2\pi i\xi_p x)$$

as $|x| \to \infty$.

As an example, using the exponential sequence instanced previously we find that the F.T. of $\delta(\xi)$ is

$$\int_{-\infty}^{\infty} \delta(\xi) \exp(-2\pi i\xi x)d\xi = 1. \tag{36}$$

Using this result with Equation (34) and Theorem IV it can be shown that the F.T. of sgn ξ is

$$\int_{-\infty}^{\infty} \operatorname{sgn} \xi \exp(-2\pi i\xi x)d\xi = (\pi ix)^{-1}, \tag{37}$$

whereupon using Theorem III the F.T. of ξ^{-1} is found to be

$$\int_{-\infty}^{\infty} \xi^{-1} \exp(-2\pi i\xi x)d\xi = -\pi i \operatorname{sgn} x. \tag{38}$$

Furthermore, using this result with Equation (35) and Theorem IV, the F.T. of ξ^{-m} is

$$\int_{-\infty}^{\infty} \xi^{-m} \exp(-2\pi i\xi x)d\xi = -\pi i \frac{(-2\pi ix)^{m-1}}{(m-1)!} \operatorname{sgn} x \tag{39}$$

where m is any integer > 0.

Theorem VI is useful for the asymptotic estimation of the Fourier transforms which are later encountered; it is derived through a generalisation of the Riemann-Lebesque lemma together with the aid of Theorem II and Equation (39).

6. *Validity of Solutions.* Using the above notes in conjunction with an inverse method it is now a simple matter to demonstrate the mathematical validity of the solutions for φ and ψ which are given in Equation (21):

It does, however, remain to establish in each case the physical validity because the use of the theory of generalised functions does not preclude discontinuous displacement components u, v. Two further theorems, *e.g.*, Titchmarsh[10], are needed to examine this. These theorems, this time for ordinary functions, are:

Theorem VII

If $p(\xi, x)$ is continuous in the rectangle $a \leqslant \xi \leqslant b$, $c \leqslant x \leqslant d$, for all values of b, and the integral

$$P(x) = \int_a^\infty p(\xi, x) d\xi$$

converges uniformly with respect to x in the interval (c, d) then $P(x)$ is a continuous function of x in this interval.

Theorem VIII

The above integral is uniformly convergent with respect to x in the interval (c, d) if there is a positive function $q(\xi)$, independent of x, such that $|p(\xi, x)| \leqslant q(\xi)$ for all values of ξ and x and such that the integral

$$\int_a^\infty q(\xi) d\xi$$

is convergent.

Finally, the solution is subject to the limitations of the linear theory of elasticity and the limit of proportionality must not be exceeded. It follows therefore that the solution is not valid in the immediate neighbourhood of points of discontinuity in the applied loading distribution.

The mathematical validity of the solution is now discussed for various values of the velocity c which are less than, or equal to, the velocity of the shear wave.

6.1. *Velocity of Applied Loading such that* $0 < c < c_S$. Provided that $0 < c < c_S$, it is seen with the aid of the results of Section 4 that the functions

$$\frac{\sinh(2\pi\beta_2\xi h)\cosh(2\pi\beta_1\xi y)}{\Omega(\xi)}, \quad \frac{\sinh(2\pi\beta_1\xi h)\sinh(2\pi\beta_2\xi y)}{\Omega(\xi)} \tag{40}$$

with

$$\Omega(\xi) = \{(1+\beta_2^2)^2\cosh(2\pi\beta_1\xi h)\sinh(2\pi\beta_2\xi h) - 4\beta_1\beta_2\cosh(2\pi\beta_2\xi h)\sinh(2\pi\beta_1\xi h)\}\beta_2^{-1},$$

which occur in the integrands of Equations (21) are fairly good functions of ξ for $|y| \leqslant h$ (Definition II). Thus, both φ and ψ are generalised functions of x for $|y| \leqslant h$ (Definition V and Theorem I) provided that

$$f(\xi)\xi^{-2} \tag{41}$$

is also a generalised function. Reference to Section 5 shows that this last equation admits most of the distributions of applied loading which are of interest in practical applications and it follows that under these circumstances it is admissible (Theorems IV and V) to formally differentiate φ and ψ repeatedly with respect to x and y under the sign of the Fourier integral. In other words, the mathematical validity of the solution is established for the displacement components u, v and stress components σ_x, σ_y, τ_{xy} are just as given by Equations (22) and (23), moreover, the solution satisfies the fundamental equations listed in Section 2 together with the boundary conditions, Section 3.

When Theorems VII and VIII are applied to Equations (22) it is possible to determine the restrictions on $f(\xi)$ which are sufficient to ensure that the displacement components u and v are continuous functions of x and y. These restrictions are not listed here for, as demonstrated later, it is quite easy to check the continuity during the practical application of the equations.

6.2. *Velocity of Applied Loading such that $c = c_S$.* When the applied loads are travelling with a velocity which is equal to that of the Rayleigh wave, *i.e.*, $c = c_S$, then Equation (25) shows that

$$(1+\beta_2{}^2)^2 = 4\beta_1\beta_2$$

and hence

$$\Omega(\xi) = -4\beta_1\beta_2 \sinh(2\pi\overline{\beta_1-\beta_2}\xi h). \tag{42}$$

In conjunction with the integrands of Equations (21) we now note that

$$\frac{\sinh(2\pi\beta_2\xi h)\cosh(2\pi\beta_1\xi y)}{\sinh(2\pi\overline{\beta_1-\beta_2}\xi h)\cosh(4\pi\beta_2\xi h)}, \quad \frac{\sinh(2\pi\beta_1\xi h)\sinh(2\pi\beta_2\xi y)}{\sinh(2\pi\overline{\beta_1-\beta_2}\xi h)\cosh(4\pi\beta_2\xi h)}, \tag{43}$$

are fairly good functions of ξ for $|y| \leqslant h$ (Definition II). Thus, both φ and ψ are generalised functions of x for $|y| \leqslant h$ (Definition V and Theorem I) provided that

$$f(\xi)\xi^{-2}\cosh(4\pi\beta_2\xi h) \tag{44}$$

is also a generalised function. Unlike Equation (41), this last imposes a serious limitation upon the admissible functions for $f(\xi)$ and consequently upon the applied loading distributions $g(x)$. The limitation is remarkable in so far as it is difficult to account for from a physical basis ; its consequence is, however, readily demonstrated.

Consider, for example, the applied loading distribution

$$g(x) = \frac{d^2}{dx^2}\{k^2 + (x/h)^2\}^{-1}, \quad k > 0, \tag{45}$$

which is a continuous function of x for values of the constant $k > 0$ and is well behaved at $x = \infty$. Substituting this into Equation (19a) we find in the ordinary way that

$$f(\xi) = -\frac{4\pi^3 h\xi^2}{k}\exp(-2\pi k|\xi|h). \tag{46}$$

Substituting this in turn into Equation (44) we see that the requirement is for

$$-\frac{4\pi^3 h}{k}\exp(-2\pi k|\xi|h)\cosh(4\pi\beta_2\xi h) \tag{47}$$

to be a generalised function of ξ. This is only the case when $k \geqslant 2\beta_2$ (Definition IV) and the solution is then mathematically valid, *see* Section 6.1. With regard to the physical validity, application of Theorems VII and VIII shows that the displacement components u and v are continuous provided that $k > 2\beta_2$, a discontinuity occurring at the boundary surfaces $y = \pm h$ when $k = 2\beta_2$.

We now have a remarkable state of affairs. When the constant in Equation (45) for the applied loading distribution is $k < 2\beta_2$, i.e., $k < 0{\cdot}748$ for $\nu = 0{\cdot}3$, there results the condition of resonance within the plate as described in the Introduction. When, however, the constant $k > 2\beta_2$ the stresses are finite, the displacements are continuous, there is no resonance and the above solution is valid.

6.3. *Velocity of Applied Loads such that $c_S < c < c_2$.* It is shown in Section 4 that there is one real root $\xi = \pm\, \xi_0$ to the equation $\Omega(\xi) = 0$ whenever the velocity c lies between those of the Rayleigh and shear waves. The existence of such a root implies that there is a free wave which satisfies the conditions of zero normal and tangential stress along the bounding surfaces and which travels with the same velocity c. The quasi-steady solution is therefore no longer unique because this free wave can be added in an arbitrary manner.

Now, the quantity on the right-hand side of the equation

$$\operatorname*{Lim}_{\xi\to-\xi_0} \frac{\xi+\xi_0}{\Omega(\xi)} = \operatorname*{Lim}_{\xi\to\xi_0} \frac{\xi-\xi_0}{\Omega(\xi)} = \frac{1}{\Omega'(\xi_0)} \tag{48}$$

has a finite value, the prime denoting differentiation with respect to ξ. In virtue of this we note, in conjunction with the integrands of Equations (21), that

$$\frac{(\xi^2-\xi_1{}^2)\sinh(2\pi\beta_2\xi h)\cosh(2\pi\beta_1\xi y)}{\Omega(\xi)}, \quad \frac{(\xi^2-\xi_1{}^2)\sinh(2\pi\beta_1\xi h)\sinh(2\pi\beta_2\xi y)}{\Omega(\xi)}, \tag{49}$$

are fairly good functions of ξ for $|y| \leqslant h$ (Definition II). Thus, both φ and ψ are generalised functions of x for $|y| \leqslant h$ (Definition V and Theorem I) provided that

$$f(\xi)\xi^{-2}(\xi^2-\xi_0{}^2)^{-1} \tag{50}$$

is also a generalised function. Similarly, with Equation (41), this last admits the distributions of applied loading which are usually of interest in practical applications and it then follows, by the same arguments as in Section 6.1, that the solution is mathematically valid.

In the special case when $c = c_2$, the shear wave velocity, then $\beta_2 = 0$ from Equation (9). Equation (16) now simplifies to

$$\Omega(\xi) = 2\pi\xi h\cosh(2\pi\beta_1\xi h) - 4\beta_1\sinh(2\pi\beta_1\xi h) \tag{51}$$

and this still has only one real root at $\xi = \pm\,\xi_0$. The Equations (21) for the potentials φ and ψ simplify in their turn to

$$\left.\begin{aligned} \varphi &= -\frac{h}{2\pi\mu}\int_{-\infty}^{\infty}\frac{\cosh(2\pi\beta_1\xi y)}{\Omega(\xi)\xi}\exp(-2\pi i\xi x)f(\xi)d\xi\,, \\ \psi &= -\frac{i\beta_1 y}{\pi\mu}\int_{-\infty}^{\infty}\frac{\sinh(2\pi\beta_1\xi h)}{\Omega(\xi)\xi}\exp(-2\pi i\xi x)f(\xi)d\xi\,, \end{aligned}\right\} \tag{52}$$

where $\Omega(\xi)$ is given by Equation (51). It is now seen that the requirement for the mathematical validity of the solution is the same as that described by Equation (50).

7. *Applied Loading Distributed According to the sgn Function.* Having in mind the more difficult problems posed by moving cracks, let us now consider a numerical example where the applied loading $g(x)$ is distributed discontinuously along the bounding surfaces of the plate, *e.g.*,

$$g(x) = g_0 \operatorname{sgn} x \tag{53}$$

equal to g_0 for $x > 0$ and to $-g_0$ for $x < 0$, where g_0 is a constant. Substituting for $g(x)$ in Equation (19a) we find that

$$f(\xi) = g_0\int_{-\infty}^{\infty}\operatorname{sgn} x\exp(2\pi i\xi x)dx \tag{54}$$

which, although the integral does not exist in the ordinary sense, has a definite meaning when sgn x is regarded as a generalised function. In fact, from Equation (39) and Theorem III it is found that

$$f(\xi) = -\frac{g_0}{\pi i \xi}. \tag{55}$$

The potentials φ and ψ for this loading distribution are obtained by substituting this last equation into Equations (21) so that

$$\left.\begin{aligned} \varphi &= -\frac{i(1+\beta_2^2)g_0}{4\pi^3\mu}\int_{-\infty}^{\infty}\frac{\sinh(2\pi\beta_2\xi h)\cosh(2\pi\beta_1\xi y)\exp(-2\pi i\xi x)}{\Omega(\xi)\xi^3}\,d\xi,\\ \psi &= \frac{\beta_1 g_0}{2\pi^2\mu}\int_{-\infty}^{\infty}\frac{\sinh(2\pi\beta_1\xi h)\sinh(2\pi\beta_2\xi y)\exp(-2\pi i\xi x)}{\Omega(\xi)\xi^3}\,d\xi. \end{aligned}\right\} \tag{56}$$

These potentials provide a mathematically valid solution for the velocity range $0 < c \leqslant c_2$, $c \neq c_S$, because the functions

$$\frac{1}{\xi^3} \text{ and } \frac{1}{\xi^3(\xi^2-\xi_0^2)} \equiv \frac{1}{\xi_0^2}\left\{\frac{1}{2\xi_0^2(\xi-\xi_0)} + \frac{1}{2\xi_0^2(\xi+\xi_0)} - \frac{1}{\xi_0^2\xi} - \frac{1}{\xi^3}\right\}$$

obtained from Equations (41), (50) and (55) are generalised functions of ξ by virtue of Equation (35) and Definition V. It is seen from Equations (44) and (55), however, that when $c = c_S$, the Rayleigh wave velocity, then

$$\xi^{-3}\cosh(4\pi\beta_2\xi h)$$

is not a generalised function by virtue of Definitions I and III. The solution is then invalid and the plate suffers the resonance as described in the Introduction.

With the aid of Theorems VII and VIII it can be confirmed that the potentials φ and ψ of Equations (56) provide continuous displacement components u and v. Let us, for example, examine the continuity of the component u with respect to x when $c_S < c \leqslant c_2$. Substituting Equation (55) into the first of Equations (22) we obtain

$$u = -\frac{g_0}{2\pi^2\mu\beta_2}\int_{-\infty}^{\infty}\left\{\begin{aligned}&(1+\beta_2^2)\sinh(2\pi\beta_2\xi h)\cosh(2\pi\beta_1\xi y)\\ &-2\beta_1\beta_2\sinh(2\pi\beta_1\xi h)\cosh(2\pi\beta_2\xi y)\end{aligned}\right\}\frac{\exp(-2\pi i\xi x)}{\Omega(\xi)\xi^2}\,d\xi.$$

This integrand is discontinuous at $\xi = 0,\ \pm\xi_0$ but, on using Equations (39) and (48), it can be rewritten

$$\begin{aligned} u = &-\frac{g_0}{2\pi^2\mu\beta_2}\int_0^{\infty}\left[\left\{\begin{aligned}&(1+\beta_2^2)\sinh(2\pi\beta_2\xi h)\cosh(2\pi\beta_1\xi y)\\ &-2\beta_1\beta_2\sinh(2\pi\beta_1\xi h)\cosh(2\pi\beta_2\xi y)\end{aligned}\right\}\frac{1}{\Omega(\xi)\xi^2} -\right.\\ &-\left\{\frac{(1+\beta_2^2)-2\beta_1^2}{(1+\beta_2^2)^2-4\beta_1^2}\right\}\frac{\beta_2}{\xi^2} -\\ &\left.-\left\{\begin{aligned}&(1+\beta_2^2)\sinh(2\pi\beta_2\xi_0 h)\cosh(2\pi\beta_1\xi_0 y)\\ &-2\beta_1\beta_2\sinh(2\pi\beta_1\xi_0 h)\cosh(2\pi\beta_2\xi_0 y)\end{aligned}\right\}\frac{2}{\Omega'(\xi_0)(\xi^2-\xi_0^2)\xi_0}\right]\cos(2\pi\xi x)\,d\xi +\\ &+\frac{g_0}{\mu}x\,\mathrm{sgn}\,x +\\ &+\frac{g_0}{\pi\mu\beta_2}\left\{\begin{aligned}&(1+\beta_2^2)\sinh(2\pi\beta_2\xi_0 h)\cosh(2\pi\beta_1\xi_0 y)\\ &-2\beta_1\beta_2\sinh(2\pi\beta_1\xi_0 h)\cosh(2\pi\beta_2\xi_0 y)\end{aligned}\right\}\frac{\sin(2\pi\xi_0 x)\,\mathrm{sgn}\,x}{\Omega'(\xi_0)\xi_0^2} \end{aligned}$$

where the integrand is now continuous. For $|y| \leqslant h$, Theorem VIII shows that the integral converges uniformly with respect to x because $|\cos(2\pi\xi x)| \leqslant 1$. It follows now from Theorem VII that u is a continuous function of x. This remains true when $c = c_2$. Further application of these theorems shows that, in addition to the displacement components, the stress components σ_x, σ_y, τ_{xy} obtained by substituting Equation (55) into Equation (23) are all continuous functions of x and y in the interior of the plate.

In connection with the effect of the reflected waves it is of interest to examine the distribution of the direct stress component σ_y along the central plane $y = 0$. Substituting Equation (55) into the second of Equations (23) we find that this is given by

$$\sigma_y(x, 0) = \frac{g_0}{\pi i \beta_2} \int_{-\infty}^{\infty} \left\{ \frac{(1+\beta_2^2)^2 \sinh(2\pi\beta_2\xi h) - 4\beta_1\beta_2 \sinh(2\pi\beta_1\xi h)}{\Omega(\xi)\xi} \right\} \exp(-2\pi i \xi x) d\xi . \tag{57}$$

Numerical values were calculated by Dr. K. I. McKenzie and Mr. G. G. Pope (both of the Royal Aircraft Establishment), to whom the author is indebted, for several values of the velocity c in the range $0 \leqslant c \leqslant c_2$. The calculations were carried out with the aid of the Auto-code programme for the Ferranti Mercury computer. It was decided to consider separately the ranges of integration $|\xi| \leqslant \xi'$ and $|\xi| \geqslant \xi'$ where ξ' was chosen so that the hyperbolic sine is indistinguishable from the hyperbolic cosine to an accuracy of less than one half of one per cent. The finite integral was then evaluated using Filon's[11] method and, on approximating the integrands, the infinite integrals were expressed explicitly in terms of elementary functions. In all the calculations, Poisson's ratio was taken as $\nu = 0 \cdot 3$ and this corresponds to a Rayleigh wave velocity $c_S = 0 \cdot 9274 c_2$.

The results of the calculations for $c < c_S$ are shown plotted in Fig. 4. The asymptotic behaviour as $|x| \to \infty$ is easily found, by using Theorem VI and noting that the only singularity in the integrand occurs at $\xi = 0$, to be

$$\sigma_y(x, 0) = -g(x) = -g_0 \operatorname{sgn} x , \qquad \text{as } |x| \to \infty . \tag{58}$$

It is interesting to note that there is little difference from the static stress distribution (given by $c = 0$) for velocities of the applied loading which are less than $0 \cdot 6 c_2$.* However, high peak stresses occur for the faster velocities and their magnitudes increase beyond all limits as c approaches the Rayleigh wave velocity c_S. When $c = c_S$ there is the resonance described in the Introduction.

The results of the calculations for $c_S < c \leqslant c_2$ are shown plotted in Fig. 5. Because of the singularities at $\xi = \pm \xi_0$ the solution is now no longer unique and, in fact, Equation (57) can be rewritten in the form

$$\sigma_y(x, 0) = A_1 \cos(2\pi\xi_0 x) + A_2 \sin(2\pi\xi_0 x) + \\ + \frac{g_0}{\pi i \beta_2} \int_{-\infty}^{\infty} \left\{ \begin{matrix} (1+\beta_2^2)^2 \sinh(2\pi\beta_2\xi h) \\ - 4\beta_1\beta_2 \sinh(2\pi\beta_1\xi h) \end{matrix} \right\} \frac{\exp(-2\pi i \xi x)}{\Omega(\xi)\xi} d\xi \tag{59}$$

where the constants A_1 and A_2 are quite arbitrary. Now, the asymptotic behaviour as $|x| \to \infty$ is controlled by the singularities in the integrand and, on using Theorem VI, it is found to be

$$\sigma_y(x, 0) = A_1 \cos(2\pi\xi_0 x) + A_2 \sin(2\pi\xi_0 x) - g_0 \operatorname{sgn} x - \\ - \frac{2g_0}{\beta_2} \left\{ \begin{matrix} (1+\beta_2^2)^2 \sinh(2\pi\beta_2\xi_0 h) \\ - 4\beta_1\beta_2 \sinh(2\pi\beta_1\xi_0 h) \end{matrix} \right\} \frac{\cos(2\pi\xi_0 x)}{\Omega'(\xi_0)\xi_0} \operatorname{sgn} x, \text{ as } |x| \to \infty . \tag{60}$$

---

* This is near the limiting velocity of propagating cracks in a real solid[7].

For the purpose of presenting the results in Fig. 5 the constants A_1 and A_2 are given the values

$$\left.\begin{aligned} A_1 &= \frac{2g_0}{\beta_2}\left\{\begin{matrix}(1+\beta_2{}^2)^2 \sinh(2\pi\beta_2\xi_0 h) \\ -\,4\beta_1\beta_2 \sinh(2\pi\beta_1\xi_0 h)\end{matrix}\right\}\frac{1}{\Omega'(\xi_0)\xi_0}, \\ A_2 &= 0 \end{aligned}\right\} \tag{61}$$

so that $\sigma_y(x, 0) \to -g_0$ as $x \to \infty$. It is seen in the Figure that this gives a peak of stress near the origin and larger fluctuating stresses in the wake.

NOTATION

c	Uniform velocity with which the applied loading distribution travels along the bounding surfaces of the plate
c_1	Velocity of dilatation waves in an unbounded medium $= \{(\lambda+2\mu)/\rho\}^{1/2}$
c_2	Velocity of shear waves in an unbounded medium $= (\mu/\rho)^{1/2}$
c_L	Velocity of long waves in an infinite plate
c_S	Velocity of Rayleigh surface waves
$2h$	Thickness of plate
$f(\xi)$	Has Fourier transform $g(x)$, *see* Equations (19)
$g(x)$	Distribution of applied loading
g_0	A constant
L	Wavelength
$O(\|x\|^N)$	Of order at most $\|x\|^N$
t	Time
u, v	Displacement components respectively in the directions of the ox' and oy' axes
x, y	Rectangular Cartesian co-ordinate system moving with the same uniform velocity c, and in the same direction, as the applied loading distribution. Note, $x = x' - ct$; $y = y'$
x', y'	Rectangular Cartesian co-ordinate system with axes respectively in and normal to the central plane of the plate. The axis of ox' is taken in the same direction as that of the motion of the applied loading distribution
α_2	*See* Equation (29)
β_1, β_2	Abbreviations introduced by Equation (9)
$\delta(x)$	Dirac delta function
λ, μ	Lamé's elastic constants
ν	Poisson's ratio, taken as $\nu = 0 \cdot 3$ in numerical example
ξ	Independent variable
ξ_n	The value of ξ which satisfies Equation (24) for $n = 0, 1, 2, \ldots$
ρ	Density
σ_x, σ_y, τ_{xy}	Stress components
φ, ψ	Lamé's potentials
Φ, Ψ	Constants defined by Equation (11)
$\chi(\xi)$	Defined by Equation (20)
$\Omega(\xi)$	Defined by Equation (16)

REFERENCES

No.	Author	Title, etc.
1	I. N. Sneddon	The stresses produced by a pulse of pressure moving along the surface of a semi-infinite solid. *R.C. Circ. mat. Palermo.* Vol. 2. 1952. p. 57.
2	J. W. Craggs	On two-dimensional waves in an elastic half-space. *Proc. Camb. Phil. Soc.* Vol. 56. 1960. p. 269.
3	L. N. G. Filon	On an approximate solution for the bending of a beam of rectangular cross-section under any system of load, with special reference to points of concentrated or discontinuous loading. *Phil. Trans.* A. Vol. 201. 1903. p. 63.
4	M. J. Lighthill	*An introduction to Fourier analysis and generalised functions.* Cambridge University Press, London. 1959.
5	Lord Rayleigh	On waves propagated along the plane surface of an elastic solid. *Proc. Lond. Math. Soc.* Vol. 17. 1885. p. 4.
6	D. D. Ang	Elastic waves generated by a force moving along a crack. *J. Math. Phys.* Vol. 38. 1960. p. 246.
7	G. R. Irwin	Fracture. Encyclopaedia of Physics, Springer Verlag, Berlin. Vol. 6. 1958. p. 579.
8	H. Lamb	On waves in an elastic plate. *Proc. Roy. Soc.* A. Vol. 93. 1916–17. p. 114.
9	H. Kolsky	*Stress waves in solids.* Oxford: Clarendon Press. 1953. p. 63.
10	E. C. Titchmarsh	*Theory of functions.* Oxford University Press. 1932. pp. 22 and 25.
11	L. N. G. Filon	On a quadrature formula for trigonometric integrals. *Proc. Roy. Soc. Edinb.* Vol. XLIX. 1928–29. p. 38.
12	J. Fulton and I. N. Sneddon ..	The dynamic stresses produced in a thick plate by the action of surface forces. *Proc. Glasgow Math. Association.* Vol. III. 1958. p. 153.

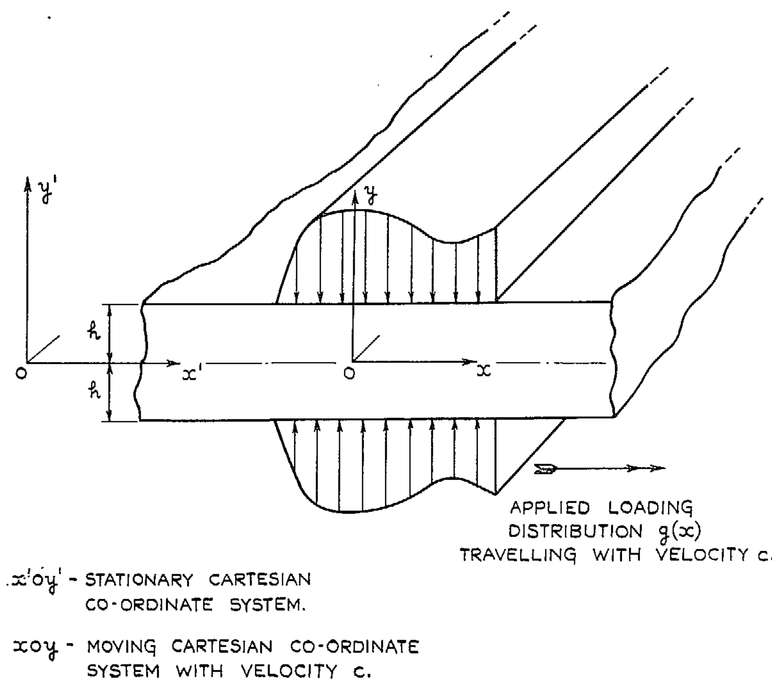

FIG. 1. Co-ordinate systems.

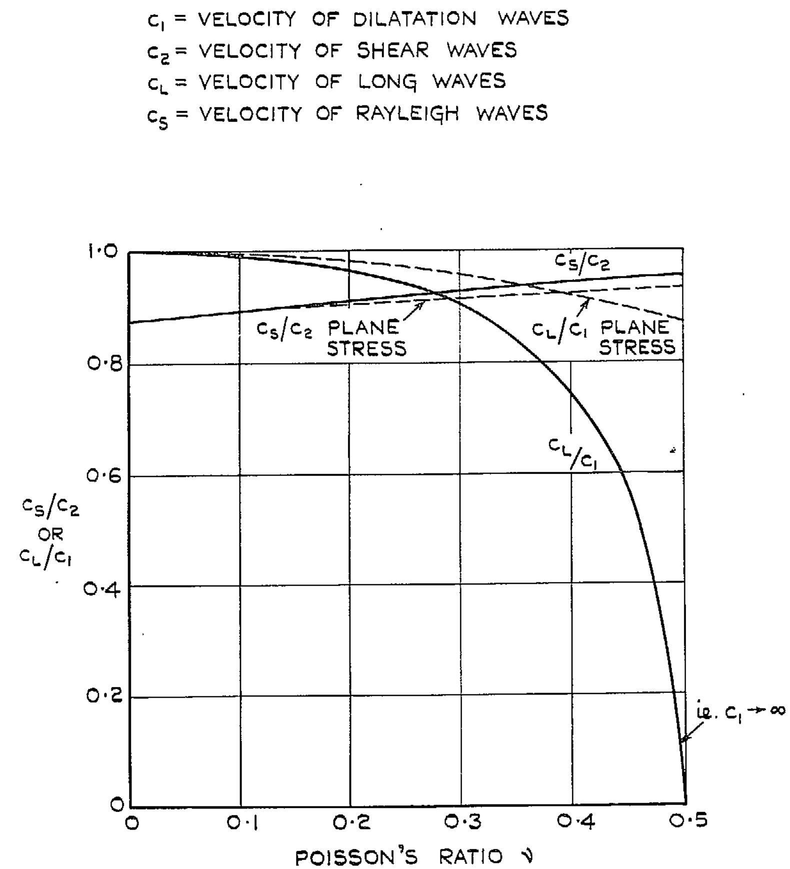

FIG. 2. Variation of Rayleigh and long wave velocities with Poisson's ratio.

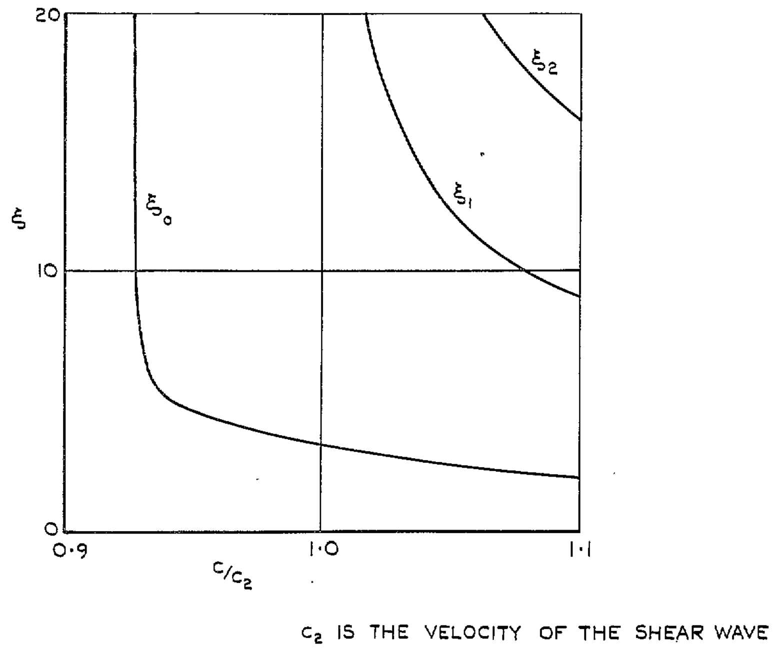

FIG. 3. Variation of the root ξ_0 with the velocity c.

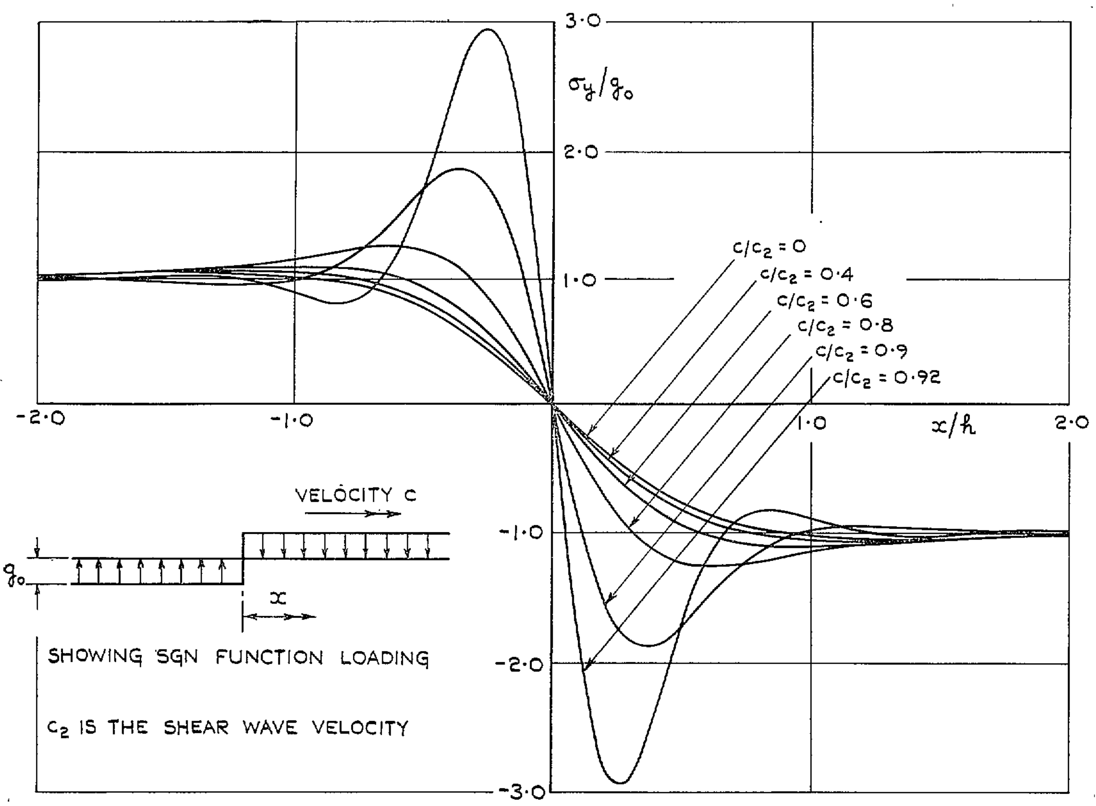

FIG. 4. Distribution of direct stress σ_y along the central plane of the plate for sgn function loading ($c < c_S$)

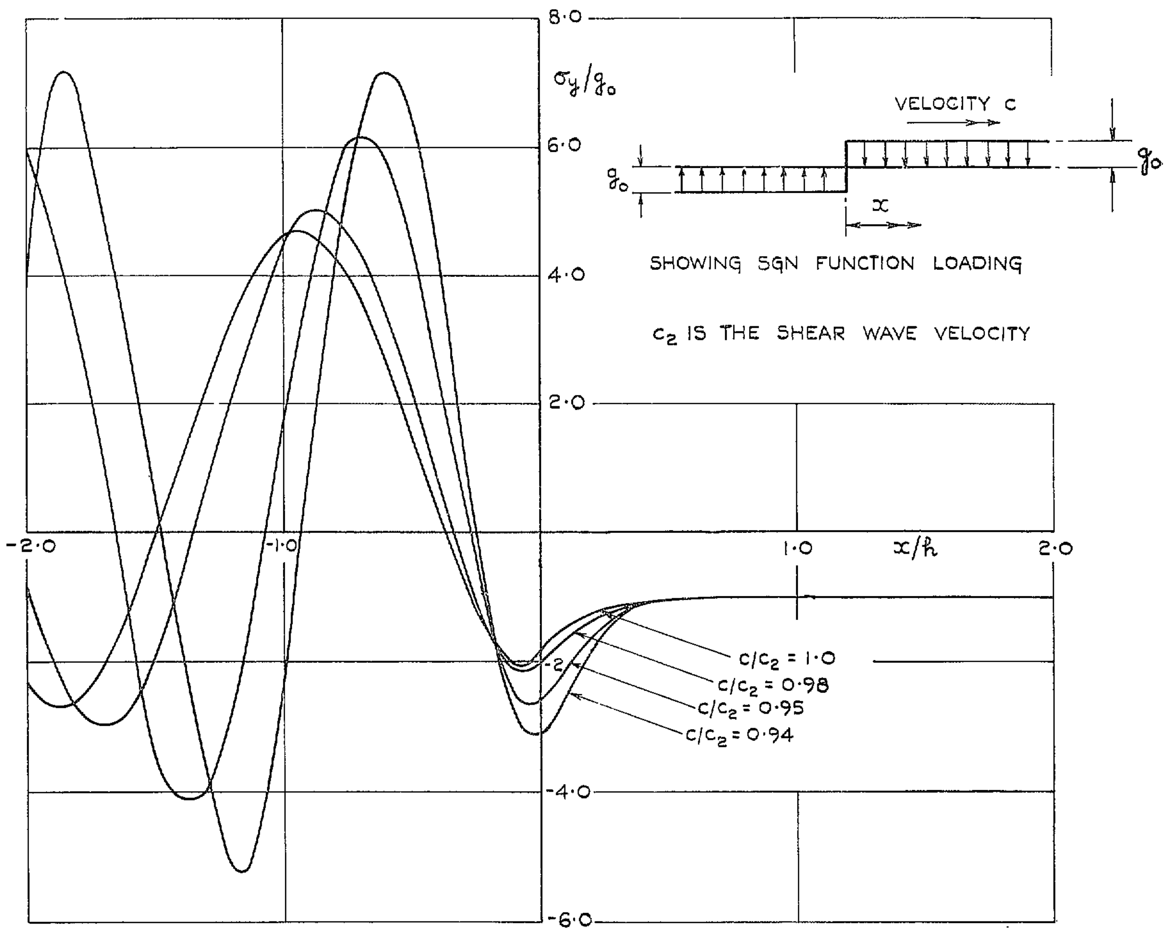

Fig. 5. Distribution of direct stress σ_y along the central plane of the plate for sgn function loading ($c_S < c \leqslant c_2$).

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