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By D. S. Whitehead, M.A., Ph.D., A.M.I.Mech.E., A.F.R.Ae.S.,
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Summary. The Report describes an analyser which was developed for finding the power spectra of waveforms derived from vibrating axial compressor blades due to random excitation. The bandwidth of the instrument goes down to about 0·04 c.p.s. The standard deviation of the results is considered in some detail, and it is shown that long samples of data are required in order to get reasonably accurate results.

1. *Introduction.* The instrument to be described in this paper was designed for use in an investigation of the random vibration of axial compressor blades in a compressor under working conditions. The power spectrum of a signal obtained from strain gauges stuck onto the blades was required. The results and conclusions from the investigation are being reported separately (Ref. 9).

The power spectrum of the displacement of a lightly damped simple oscillator due to random excitation has the form shown in Fig. 1. This wave has exactly the same shape as a resonance curve corresponding to sinusoidal excitation, and the peak is centred on the natural frequency of the system (ω_0). The width of the peak at points where the power is half of its maximum value ($\Delta\omega$) is given by $\omega_0\delta/\pi$ where δ is the logarithmic decrement of the system.

In order to measure a power spectrum such as that shown in Fig. 1 experimentally, it is necessary, essentially, to pass the signal through a filter which only passes a narrow range of frequencies, and see how much comes through. In order to get a reasonable picture of the shape of the curve, the filter must have a pass band of not more than (say) $\Delta\omega/10$. If the logarithmic decrement of the blade is 0·03, this means that the filter pass band must be not more than about $\omega_0/1{,}000$. This demands a standard of resolution of an order of magnitude higher than that obtainable from any normal wave analyser.

2. *Description of Analyser.* It would be extremely difficult to build a tunable filter, with such a narrow pass band. The method used is therefore one of frequency changing, so that a simple low-pass filter can be used. The block diagram of the ideal system is shown in Fig. 2. The signal is fed into a multiplier, the other channel of which is fed from a standard oscillator. The oscillator is tuned to the frequency (ω_1) at which the power spectrum is required; this will normally be near the blade natural frequency ω_0. For each component of the input signal with a frequency ω, the output of the multiplier will contain frequencies of $(\omega+\omega_1)$ and $(\omega-\omega_1)$. This signal is then taken to a low-pass filter which only passes signals with frequency less than, say, $\omega_0/1{,}000$. The filter output therefore contains

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only those components of the original signal with frequencies between $(\omega_1 + \omega_0/1{,}000)$ and $(\omega_1 - \omega_0/1{,}000)$, as is required. This signal is then squared, integrated, and measured, so as to obtain its mean square value.

Although multipliers for analogue computors can be obtained, there were none readily available for this investigation, and so the ideal arrangement of Fig 2 was modified to the arrangement shown in Fig. 3. The first multiplier has been replaced by a gating circuit which is fed from the oscillator *via* a circuit which transforms the sine wave into a square wave. The gating circuit acts as a switch which reverses the sign of the signal every half cycle of the oscillator waveform. This means that instead of multiplying by a sine wave, the signal is multiplied by a square wave, which takes the values + 1 and − 1 on alternate half cycles. One advantage of this system is that the output is now independent of the amplitude of the oscillator waveform. However, since a square wave contains all the odd harmonics as well as the fundamental frequency, the system would then also respond to frequencies of $3\omega_1$, $5\omega_1$, $7\omega_1$, etc. The signal must therefore first be passed through an input filter which allows ω_1 to go through, but stops $3\omega_1$, $5\omega_1$, $7\omega_1$, etc.

The other change that has been made in Fig. 3 is that instead of squaring the output from the low-pass filter, it is passed to a full wave rectifier. A reading is then obtained which is proportional to the square root of the power spectrum. This may be roughly thought of as an 'amplitude spectrum'.

The circuit diagram of the analyser is shown in Fig. 4. It includes five d.c. amplifiers (A1 to A5), each with a gain of about 15,000. These have been developed by members of the Automatic Control group at the Cambridge University Engineering Laboratory. The input filter is a single RC circuit between two cathode followers. The signal then passes to a transformer phase splitter and then to the gating circuit, consisting of three double diodes, via further cathode followers. The other bi-phase input to the gating circuit comes from A1, which is used to provide the square wave in conjunction with a standard square wave unit. The output from the gating circuit is fed into the low-pass filter, which consists of two simple time lags in series, using the amplifiers A2 and A3. A4 is used as a phase inverter for the full wave rectifier, and A5 is used for the integrator. The output is then read on M1. Another meter, M2, is used for setting up the amplifiers and setting the zero of the instrument.

3. *Theoretical Considerations relating to Power Spectrum Measurement.* The theory of power spectrum measurement has been extensively studied, and the list of references given is far from being complete. The performance of the idealized system shown in Fig. 2 is analysed in the Appendix, but it is not claimed that any fundamentally new results are derived.

It is shown that the output of the analyser is directly proportional to the power spectrum of the input at the oscillator frequency ω_1. It is also shown that when the low-pass filter consists of two time lags in series with a time constant T', then the standard deviation of the reading, expressed as a fraction of the reading, is $\sigma_n = \sqrt{(5T'/T)}$ where T is the time over which the integration is taken.

There are two equal contributions to the variance (σ_n^2) of the result. One of these arises from variations between different samples of data, and the other arises from the random phase angle of the oscillator waveform at the start of the run. This second contribution could be eliminated by using the twin channel arrangement shown in Fig. 5, in which the second channel is fed by an oscillator waveform in quadrature with that supplied to the first channel. This would have the effect of dividing the standard deviation of the results by $\sqrt{2}$ and the maximum possible amount of information about the power spectrum would then be obtained from each sample. It would also have the practical advantage that an analysis of any given sample of data should be exactly repeatable.

It is also shown in the Appendix that there is no correlation between errors obtained from runs with the same sample of data, using different reference frequencies, provided that these frequencies are not so close together that they can be contained within the same frequency pass band of the filter.

The result quoted for the standard deviation in the ideal system is obtained under the assumption that the input waveform is Gaussian. If it is assumed that the waveform at the output from the low-pass filter is Gaussian, then this result follows directly from an analysis by Jacobs[6]. Jacobs' analysis also applies to the actual system used with a full wave rectifier instead of a squaring device (Fig. 3). This shows that the output is then proportional to the square root of the power spectrum, and the standard deviation lies in the range

$$\tfrac{1}{2}\sqrt{\left(\frac{5T'}{T}\right)} < \sigma_n < \tfrac{1}{2}\sqrt{\left\{\frac{5T'(\pi-2)}{T}\right\}}$$

provided that (T'/T) is small. This puts a narrow bracket on the value of σ_n and the lower limit (which corresponds to the intuitive idea that taking the square root of the output should correspond to halving the standard deviation) can be taken as a reasonable estimate of the accuracy achieved. For much of the work done with the analyser T was 180 sec and T' was $0{\cdot}22$ sec. This gives $\sigma_n = 0{\cdot}039$ or an estimated standard deviation of nearly 4 per cent. This result shows that long samples of data have to be analysed, in order to get reasonably accurate results, and it usually becomes essential to record the data on magnetic tape, so that an analysis using the whole of the available data can be carried out at each frequency.

When the experiment was being planned, an alternative approach to the problem was considered. Instead of measuring the power spectrum directly it would be possible to measure the autocorrelation function for the signal. For a random vibration waveform this has the form shown in Fig. 6; it has the same shape as a decaying free vibration with the same damping factor as in the original system. Since the power spectrum is the Fourier transform of the autocorrelation function, the power spectrum could be derived from this. The power spectrum measurement was chosen since it was thought to be considerably easier to achieve experimentally. However the autocorrelation function measurement appears at first sight to have theoretical advantages concerned with the accuracy of the measurement. For a random vibration waveform the standard deviation (σ) of the autocorrelation measurement is given by

$$\frac{\sigma^2}{(\overline{x^2})^2} = \frac{2\pi}{\omega_0 T\delta} = \frac{2}{T\Delta\omega}$$

where $\Delta\omega$ is the width of the power spectrum peak shown in Fig. 1 at the half maximum power points. This may be compared with the standard deviation given by the ideal analyser (Fig. 2) which is given by

$${\sigma_n}^2 = \frac{5T'}{T} = \frac{6{\cdot}44}{T\Delta\omega'}$$

where $\Delta\omega'$ is the width of the frequency pass band of the analyser, measured at the points where the power is down by a factor of 2. Since $\Delta\omega'$ must be made much less than $\Delta\omega$ it is clear that for a given sample length (T) the autocorrelation measurement is much more accurate. However, it is known that a good estimate of the autocorrelation function does not necessarily lead to a good estimate of the power spectrum (Refs. 1 and 6). During the Fourier transform process the errors are greatly magnified. The amount of useful information obtained from a sample of given length therefore appears to be about the same in each case.

4. *Conclusion.* After an initial period of development the analyser described in the Paper has worked well.

The analyser was tested using a second oscillator to provide the input, in which case the output of the analyser follows the characteristic of the low-pass filter.

Apart from the fundamental scatter of results due to variations between different samples, the main source of error was due to small variations in the speed of the tape recorder used to store the data.

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NOTATION

g	Power spectrum function for low-pass filter
h	Impulse response function for low-pass filter
t	Time
x	Input signal
A, B, C	Functions defined by Equation (2)
E	Denotes expectation of any quantity
G	Power spectrum of input signal
I	Integral defined by Equation (7)
T	Integration time
T'	Time constant of low-pass filter
V	Output of analyser
Y	Frequency response function of low-pass filter.
α	Phase angle
δ	Logarithmic decrement
ν	Variable with dimensions of frequency
σ	Standard deviation of output
τ	Variable with dimensions of time
ϕ	Autocorrelation function of input signal
ψ	Autocorrelation function of filter
ω	Angular frequency
ω_0	Natural frequency of compressor blade
ω_1, ω_2	Oscillator frequency
$\Delta\omega$	Bandwidth of input signal (measured between the half power points)
$\Delta\omega'$	Bandwidth of low-pass filter (measured between the half power points)
Ω	Cut-off frequency of low-pass filter

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APPENDIX

1. *Performance of Idealized System.* Consider the system shown in Fig. 2. When the input signal and the output from the oscillator are multiplied together the result is

$$x(t)\sin(\omega_1 t+\alpha).$$

If $h(\tau)$ is the weighting function for the low-pass filter, the output from the filter is

$$\int_0^\infty h(\tau)x(t-\tau)\sin(\omega_1 t-\omega_1\tau+\alpha)\,d\tau.$$

The output from the squarer is the square of this, and by using two variables τ_1, and τ_2 it may be written as a double integral as follows:

$$\int_0^\infty\int_0^\infty h(\tau_1)h(\tau_2)x(t-\tau_1)x(t-\tau_2)\sin(\omega_1 t-\omega_1\tau_1+\alpha)\sin(\omega_1 t-\omega_1\tau_2+\alpha)\,d\tau_1\,d\tau_2.$$

When this is integrated from $t = 0$ to $t = T$, the output of the system, V, is obtained:

$$V = \tfrac{1}{2}\int_0^T dt\int_0^\infty\int_0^\infty h(\tau_1)h(\tau_2)x(t-\tau_1)x(t-\tau_2)\times$$
$$\times\{\cos(\omega_1\tau_2-\omega_1\tau_1)-\cos(2\omega_1 t-\omega_1\tau_1-\omega_1\tau_2+2\alpha)\}\,d\tau_1 d\tau_2. \qquad (1)$$

Suppose that the signal $x(t)$ is recorded on magnetic tape, so that the experiment can be repeated as many times as required, the integration being performed over exactly the same piece of signal. If in these experiments only the phase angle α is varied, then Equation (1) shows that the results will be of the form

$$V = A + B\cos 2\alpha + C\sin 2\alpha. \qquad (2)$$

This is illustrated in Fig. 7, and shows that there will be a mean value, A, together with a fluctuation with α of amplitude $\sqrt{(B^2+C^2)}$.

This fluctuation with α could be removed by using the twin channel arrangement shown in Fig. 5, where the second channel is fed from the oscillator with a voltage in quadrature with that fed to the first channel. Then the voltage at a point P (Fig. 5) is

$$\int_0^\infty\int_0^\infty h(\tau_1)h(\tau_2)x(t-\tau_1)x(t-\tau_2)\cos(\omega_1 t-\omega_1\tau_1+\alpha)\cos(\omega_1 t-\omega_1\tau_2+\alpha)\,d\tau_1\,d\tau_2$$

$$= \tfrac{1}{2}\int_0^\infty\int_0^\infty h(\tau_1)h(\tau_2)x(t-\tau_1)x(t-\tau_2)\{\cos(\omega_1\tau_2-\omega_1\tau_1)+\cos(2\omega_1 t-\omega_1\tau_1-\omega_1\tau_2+2\alpha)\}\,d\tau_1\,d\tau_2.$$

When this is added to the signal from the first channel the $\cos(2\omega_1 t-\omega_1\tau_1-\omega_1\tau_2+2\alpha)$ term goes out, and the output from the twin channel system after integration is

$$V = 2A.$$

Returning to the single channel analyser it may be supposed that a large number of analyses are carried out on different samples of signal with random values of α. Taking the expectation of Equation (2) gives

$$E(V) = E(A) + E\{B\cos 2\alpha\} + E\{C\sin 2\alpha\}$$
$$= E(A) + E(B)E(\cos 2\alpha) + E(C)E(\sin 2\alpha)$$

since the random variable α is independent of the input signal. Since the expectations of $\cos 2\alpha$ and $\sin 2\alpha$ are zero this gives

$$E(V) = E(A). \tag{3}$$

This will now be computed.

From equations (1) and (2)

$$A = \tfrac{1}{2}\int_0^T dt \int_0^\infty \int_0^\infty h(\tau_1)h(\tau_2)x(t-\tau_1)x(t-\tau_2) \cos(\omega_1\tau_2 - \omega_1\tau_1)\, d\tau_1\, d\tau_2. \tag{4}$$

Taking the expectation of this, the expectancy of $x(t-\tau_1)x(t-\tau_2)$ is given by the autocorrelation function of x, which will be written

$$E\{x(t-\tau_1)x(t-\tau_2)\} = \phi(\tau_2-\tau_1). \tag{5}$$

Hence

$$\begin{aligned} E(V) &= \tfrac{1}{2}\int_0^T dt \int_0^\infty \int_0^\infty h(\tau_1)h(\tau_2)\phi(\tau_2-\tau_1) \cos(\omega_1\tau_2 - \omega_1\tau_1)\, d\tau_1\, d\tau_2 \\ &= \frac{T}{2}\int_0^\infty \int_0^\infty h(\tau_1)h(\tau_2)\phi(\tau_2-\tau_1) \cos(\omega_1\tau_2 - \omega_1\tau_1)\, d\tau_1\, d\tau_2. \end{aligned} \tag{6}$$

Instead of evaluating this directly, it will be convenient to evaluate a more general double integral which will be required in the later development.

2. *Lemma.* Consider the integral

$$I(t', \omega_1, \omega_2) = \int_0^\infty \int_0^\infty h(\tau_1)h(\tau_2)\phi(t'+\tau_2-\tau_1) \exp\{i(\omega_2\tau_2 - \omega_1\tau_1)\}\, d\tau_1\, d\tau_2. \tag{7}$$

The Fourier transform of this with respect to t' will now be taken:

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} I(t', \omega_1, \omega_2)e^{-i\omega t'}dt' = \frac{1}{\pi}\int_{-\infty}^{+\infty} e^{-i\omega t'}dt' \int_0^\infty \int_0^\infty h(\tau_1)h(\tau_2)\phi(t'+\tau_2-\tau_1) \times$$
$$\times \exp\{i(\omega_2\tau_2 - \omega_1\tau_1)\}\, d\tau_1\, d\tau_2.$$

Replacing the variable t' by

$$\tau = t' + \tau_2 - \tau_1$$

$$\begin{aligned} &\frac{1}{\pi}\int_{-\infty}^{+\infty} I(t', \omega_1, \omega_2)e^{-i\omega t'}dt' \\ &\quad = \frac{1}{\pi}\int_{-\infty}^{+\infty} \phi(\tau)e^{-i\omega\tau}d\tau \,.\int_0^\infty h(\tau_1)\exp\{-i(\omega+\omega_1)\tau_1\}\, d\tau_1 \,.\int_0^\infty h(\tau_2)\exp\{i(\omega+\omega_2)\tau_2\}\, d\tau_2 \end{aligned}$$

where the R.H.S. has been split up into the product of three independent integrals. The first of these is the power spectrum of the input signal (Ref. 7, Section 3.6)

$$G(\omega) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \phi(\tau)e^{-i\omega\tau}d\tau.$$

The second and third integrals may be expressed in terms of the frequency response for the filter $Y(iv)$ (Ref. 7, Section 5.2).

$$\int_0^\infty h(\tau_1)\exp\{-i(\omega+\omega_1)\tau_1\}\, d\tau_1 = Y(i\omega + i\omega_1)$$

and

$$\int_0^\infty h(\tau_2)\exp\{+i(\omega+\omega_2)\tau_2\}\, d\tau_2 = Y(-i\omega - i\omega_2).$$

Hence

$$\frac{1}{\pi}\int_{-\infty}^{+\infty} I(t', \omega_1, \omega_2)e^{-i\omega t'}dt' = G(\omega)Y(i\omega+i\omega_1)Y(-i\omega-i\omega_2).$$

Inverting the Fourier transform, this gives

$$I(t', \omega_1, \omega_2) = \tfrac{1}{2}\int_{-\infty}^{+\infty} G(\omega)Y(i\omega+i\omega_1)Y(-i\omega-i\omega_2)e^{i\omega t'}d\omega. \tag{8}$$

Now $Y(i\nu)$, the frequency response for the low-pass filter, is zero unless ν is small. This may be written

$$Y(i\nu) = 0 \text{ unless } |\nu| < \Omega$$

where Ω is a low frequency above which the filter cuts off all signals. Then in Equation (8)

$$Y(i\omega+i\omega_1) = 0 \text{ unless } |\omega+\omega_1| < \Omega$$

and

$$Y(-i\omega-i\omega_2) = 0 \text{ unless } |\omega+\omega_2| < \Omega.$$

Hence unless $|\omega_1-\omega_2| < 2\Omega$ there is no value of ω for which the integral in Equation (8) is not zero.

Therefore

$$I(t', \omega_1, \omega_2) = 0 \text{ unless } |\omega_1-\omega_2| < 2\Omega. \tag{9}$$

Note that this includes the case $\omega_2 = -\omega_1$.

The other case of interest is when $\omega_2 = \omega_1$. A power spectrum function for the filter $g(\nu)$ may be defined so that

$$g(\nu) = Y(i\nu)Y(-i\nu). \tag{10}$$

Hence from (8)

$$I(t', \omega_1, \omega_1) = \tfrac{1}{2}\int_{-\infty}^{+\infty} G(\omega)g(\omega+\omega_1)e^{i\omega t'}d\omega. \tag{11}$$

Now $g(\omega+\omega_1)$ is zero except for very small values of $(\omega+\omega_1)$, or for a small range of ω near $-\omega_1$, and over this small range $G(\omega)$ will not vary appreciably. $G(\omega)$ may therefore be put equal to $G(-\omega_1)$ or $G(\omega_1)$ (since G is an even function).

Hence

$$\begin{aligned} I(t', \omega_1, \omega_1) &= \tfrac{1}{2}G(\omega_1)\int_{-\infty}^{+\infty} g(\omega+\omega_1)e^{i\omega t'}d\omega \\ &= \tfrac{1}{2}G(\omega_1)e^{-i\omega_1 t'}\int_{-\infty}^{+\infty} g(\nu)e^{i\nu t'}d\nu. \end{aligned}$$

This may be written

$$I(t', \omega_1, \omega_1) = G(\omega_1)e^{-i\omega_1 t'}\psi(t') \tag{12}$$

where $\psi(t')$ is an autocorrelation function for the filter defined by

$$\psi(t') = \tfrac{1}{2}\int_{-\infty}^{+\infty} g(\nu)e^{i\nu t'}d\nu. \tag{13}$$

Equations (9) and (12) give the required values of the double integral.

3. *Average Reading of Analyser.* From Equations (6) and (7)

$$\begin{aligned} E(V) &= \frac{T}{4}\{I(0, \omega_1, \omega_1) + I(0, -\omega_1, -\omega_1)\} \\ &= \tfrac{1}{2}TG(\omega_1)\psi(0) \end{aligned} \tag{14}$$

from Equation (12).

For a low-pass filter which consists of two simple time lags of magnitude T' in series, the frequency response function is

$$Y(i\nu) = \left(\frac{1}{1+i\nu T'}\right)^2.$$

Hence from Equation (10)

$$g(\nu) = \left(\frac{1}{1+\nu^2 T'^2}\right)^2.$$

And from Equation (13)

$$\psi(t') = \tfrac{1}{2}\int_{-\infty}^{+\infty}\left(\frac{1}{1+\nu^2 T'^2}\right)^2 e^{i\nu t'}d\nu$$

$$= \frac{\pi}{4}\,\frac{T'+t'}{T'^2}\,e^{-t'/T'}. \tag{15}$$

Hence from Equation (14)

$$E(V) = \frac{\pi}{8}\,\frac{T}{T'}\,G(\omega_1). \tag{16}$$

This equation relates the average reading of the analyser to the power spectrum of the input.

4. *Correlation between Runs at Different Frequencies.* Consider the results of two different runs with the analyser, using the same piece of data, but with two different values of the reference frequency, ω_1 and ω_2. The question to be answered is: 'If the result obtained at one frequency ω_1 is found to be high, owing to the variation between different samples of data, is the result obtained on the same piece of data at a different frequency ω_2 also likely to be high?'.

It is therefore necessary to calculate the correlation of the deviation of the readings from their mean values. This is given by

$$E\{[V(\omega_1) - \overline{V(\omega_1)}]\,[V(\omega_2) - \overline{V(\omega_2)}]\}$$
$$= E\{V(\omega_1)\,.\,V(\omega_2)\} - \overline{V(\omega_1)}\,.\,\overline{V(\omega_2)}. \tag{17}$$

From Equation (2) the product of the two readings is

$$V(\omega_1)\,.\,V(\omega_2) = \{A(\omega_1) + B(\omega_1)\cos 2\alpha_1 + C(\omega_1)\sin 2\alpha_1\}\times$$
$$\times\{A(\omega_2) + B(\omega_2)\cos 2\alpha_2 + C(\omega_2)\sin 2\alpha_2\}.$$

Multiplying out the R.H.S. and taking the expectation of this, it is found that since α_1 and α_2 are random variables independent of each other and of the sample of data, all the terms involving α_1 and α_2 are zero. Hence

$$E\{V(\omega_1)\,.\,V(\omega_2)\} = E\{A(\omega_1)\,.\,A(\omega_2)\}. \tag{18}$$

From Equation (4) the product of the two A functions may be written as a multiple integral as follows

$$A(\omega_1)\,.\,A(\omega_2) = \tfrac{1}{4}\int_0^T\int_0^T dt_1 dt_2 \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)\times$$
$$\times\, x(t_1-\tau_1)x(t_1-\tau_2)x(t_2-\tau_3)x(t_2-\tau_4)\cos(\omega_1\tau_2-\omega_1\tau_1)\cos(\omega_2\tau_4-\omega_2\tau_3)d\tau_1 d\tau_2 d\tau_3 d\tau_4 \tag{19}$$

where the suffixes have been duplicated as required. The expectancy of this over a large number of samples is now taken. It is assumed that $x(t)$ belongs to a stationary, ergodic and Gaussian random process. Then $E[x(t_1-\tau_1)x(t_1-\tau_2)x(t_2-\tau_3)x(t_2-\tau_4)]$ can be expressed in terms of the autocorrelation function of x as follows (Ref. 7, Sections 2.13 and 4.3):

$$E[x(t_1-\tau_1)x(t_1-\tau_2)x(t_2-\tau_3)x(t_2-\tau_4)] = \phi(\tau_2-\tau_1)\phi(\tau_4-\tau_3) + \\ + \phi(t_2-t_1-\tau_3+\tau_1)\phi(t_2-t_1-\tau_4+\tau_2) + \phi(t_2-t_1-\tau_4+\tau_1)\phi(t_2-t_1-\tau_3+\tau_2). \quad (20)$$

When Equation (20) is substituted into Equation (19), the first term in Equation (20) gives an integral which splits up into two triple integrals of the form of Equation (6). This term therefore gives $E[V(\omega_1)] \,.\, E[V(\omega_2)]$. The second and third terms in Equation (20) give identical results, as can be seen by interchanging τ_3 and τ_4. Hence

$$E[A(\omega_1).A(\omega_2)] = E[V(\omega_1)] \,.\, E[V(\omega_2)] + \tfrac{1}{2}\int_0^T\int_0^T dt_1dt_2 \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4) \times \\ \times \phi(t_2-t_1-\tau_3+\tau_1)\phi(t_2-t_1-\tau_4+\tau_2)\cos(\omega_1\tau_2-\omega_1\tau_1)\cos(\omega_2\tau_4-\omega_2\tau_3)d\tau_1d\tau_2d\tau_3d\tau_4. \quad (21)$$

The two cos factors may be written

$$\cos(\omega_1\tau_2-\omega_1\tau_1)\cos(\omega_2\tau_4-\omega_2\tau_3) = \tfrac{1}{4}\,\mathrm{Exp}\, i(\omega_1\tau_2-\omega_1\tau_1+\omega_2\tau_4-\omega_2\tau_3) + \\ + \tfrac{1}{4}\,\mathrm{Exp}\, i(\omega_1\tau_2-\omega_1\tau_1-\omega_2\tau_4+\omega_2\tau_3) + \\ + \tfrac{1}{4}\,\mathrm{Exp}\, i(-\omega_1\tau_2+\omega_1\tau_1+\omega_2\tau_4-\omega_2\tau_3) + \\ + \tfrac{1}{4}\,\mathrm{Exp}\, i(-\omega_1\tau_2+\omega_1\tau_1-\omega_2\tau_4+\omega_2\tau_3). \quad (22)$$

When this is substituted into Equation (21) the τ integrations may be split up in pairs, and are of the form of Equation (7). Hence

$$E[A(\omega_1)A(\omega_2)] - E[V(\omega_1)] \,.\, E[V(\omega_2)] \\ = \tfrac{1}{4}\int_0^T\int_0^T \{I(t_1-t_2,\,\omega_1,\,-\omega_2)\,.\,I(t_1-t_2,\,-\omega_1,\,\omega_2) + \\ + I(t_1-t_2,\,\omega_1,\,\omega_2)\,.\,I(t_1-t_2,\,-\omega_1,\,-\omega_2)\}\,dt_1dt_2. \quad (23)$$

But considering only the case when ω_1 and ω_2 are both positive, Equation (9) shows that unless $|\,\omega_1-\omega_2\,| < 2\Omega$ all the I functions on the R.H.S. of Equation (23) are zero. Hence using Equations (17) and (18)

$$E\{[V(\omega_1)-\overline{V(\omega_1)}][V(\omega_2)-\overline{V(\omega_2)}]\} = 0 \text{ unless } |\,\omega_1-\omega_2\,| < 2\Omega. \quad (24)$$

This result shows that there is no correlation between the errors obtained from two runs on the same piece of data at different reference frequencies, unless the two reference frequencies are so close together that they can be contained within the same frequency pass band of the filter.

5. *Standard Deviation of Results.* Consider a number of runs with the analyser, each on a different sample of data, but with the same reference frequency, ω_1. The standard deviation of these readings, is given by

$$\sigma^2 = E\{[V-\bar{V}]^2\} \\ = E\{V^2\} - [E(V)]^2. \quad (25)$$

From Equation (2)

$$E(V^2) = E\{(A+B\cos 2\alpha + C\sin 2\alpha)^2\} \\ = E(A^2) + \tfrac{1}{2}E(B^2) + \tfrac{1}{2}E(C^2) \quad (26)$$

since $E(\cos 2\alpha)$ and $E(\sin 2\alpha)$ are zero and $E(\cos^2 2\alpha)$ and $E(\sin^2 2\alpha)$ are $\tfrac{1}{2}$.

$E(A^2)$ can be obtained from Equation (23) by putting $\omega_2 = \omega_1$. From Equation (9) the first term is zero, giving

$$E(A^2) - \{E(V)\}^2 = \tfrac{1}{4}\int_0^T\int_0^T I(t_1-t_2,\, \omega_1,\, \omega_1) I(t_1-t_2,\, -\omega_1,\, -\omega_1)\, dt_1\, dt_2$$

$$= \tfrac{1}{4}G^2(\omega_1)\int_0^T\int_0^T \psi^2(t_1-t_2)\, dt_1\, dt_2 \tag{27}$$

from Equation (12).

Turning next to the calculation of $E(B^2+C^2)$ Equations (1) and (2) give

$$\begin{matrix} B \\ C \end{matrix} = \tfrac{1}{2}\int_0^T dt \int_0^\infty\int_0^\infty h(\tau_1)h(\tau_2)x(t-\tau_1)x(t-\tau_2) \begin{matrix} -\cos \\ +\sin \end{matrix}\, (2\omega_1 t - \omega_1\tau_1 - \omega_1\tau_2)\, d\tau_1\, d\tau_2 .$$

This may be written

$$-B + iC = \tfrac{1}{2}\int_0^T dt \int_0^\infty\int_0^\infty h(\tau_1)h(\tau_2)x(t-\tau_1)x(t-\tau_2) \exp\{i\omega_1(2t-\tau_1-\tau_2)\}\, d\tau_1\, d\tau_2 .$$

Multiplying the conjugate expression and duplicating suffices as required gives

$$\begin{aligned} B^2 + C^2 = \tfrac{1}{4}\int_0^T\int_0^T dt_1 dt_2 \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \\ \times\, h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4)x(t_1-\tau_1)x(t_1-\tau_2)x(t_2-\tau_3)x(t_2-\tau_4) \times \\ \times \exp\{i\omega_1(2t_1-\tau_1-\tau_2-2t_2+\tau_3+\tau_4)\}\, d\tau_1 d\tau_2 d\tau_3 d\tau_4 . \end{aligned}$$

Taking the expectation of this and using Equation (20) gives

$$\begin{aligned} E(B^2+C^2) = \tfrac{1}{4}\int_0^T\int_0^T \exp\{2iw_1(t_1-t_2)\} dt_1 dt_2 \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty h(\tau_1)h(\tau_2)h(\tau_3)h(\tau_4) \times \\ \times \{\phi(\tau_2-\tau_1)\phi(\tau_4-\tau_3) + \phi(t_2-t_1-\tau_3+\tau_1)\phi(t_2-t_1-\tau_4+\tau_2) + \\ + \phi(t_2-t_1-\tau_4+\tau_1)\phi(t_2-t_1-\tau_3+\tau_2)\} \times \\ \times \exp\{i\omega_1(-\tau_1-\tau_2+\tau_3+\tau_4)\}\, d\tau_1\, d\tau_2\, d\tau_3\, d\tau_4 . \end{aligned}$$

The τ integrations may be split up in pairs and are of the form of Equation (7). Hence

$$E(B^2+C^2) = \tfrac{1}{4}\int_0^T\int_0^T \exp\{2i\omega_1(t_1-t_2)\}\{I^2(0,\, \omega_1,\, -\omega_1) + 2I^2(t_1-t_2,\, \omega_1,\, \omega_1)\}\, dt_1 dt_2 .$$

From Equation (9) the first term in the bracket is zero, and the other term gives by Equation (12)

$$E(B^2+C^2) = \tfrac{1}{2}G^2(\omega_1)\int_0^T\int_0^T \psi^2(t_1-t_2) dt_1 dt_2 . \tag{28}$$

From Equations (25), (26), (27) and (28)

$$\sigma^2 = \tfrac{1}{2}G^2(\omega_1)\int_0^T\int_0^T \psi^2(t_1-t_2)\, dt_1\, dt_2 . \tag{29}$$

Putting $t = t_1 - t_2$ this may be reduced to a single integral (Ref. 7, Section 4.3)

$$\sigma^2 = G^2(\omega_1)\int_0^T (T-t)\psi^2(t)\, dt . \tag{30}$$

If the integration time T is large compared with the time constant of the filter, the t term in the $(T-t)$ factor may be neglected, and the top limit replaced by infinity. Hence

$$\sigma^2 = TG^2(\omega_1)\int_0^\infty \psi^2(t)\,dt. \tag{31}$$

If σ is divided by the mean reading, $E(V)$, to give a normalized standard deviation, σ_n, then

$$\sigma_n{}^2 = \left\{\frac{\sigma}{E(V)}\right\}^2 = \frac{4}{T}\int_0^\infty \left\{\frac{\psi(t)}{\psi(0)}\right\}^2 dt \tag{32}$$

from Equations (31) and (14).

For a filter consisting of two simple time lags T' in series, ψ is given by Equation (15). Hence

$$\sigma_n{}^2 = \frac{4}{T}\int_0^\infty \left(1+\frac{t}{T'}\right)^2 e^{-2t/T'}dt = 5\frac{T'}{T}. \tag{33}$$

This result is the same as would be obtained by passing white noise through the filter and taking the R.M.S. value of the output (Ref. 7, Section 4.3).

It is interesting to note that there are two equal contributions to the variance σ^2: one from the fluctuations between samples of data, and one from the random phasing of the oscillator waveform. By using a twin channel analyser the second contribution is eliminated, and the variance is halved. Thus the effect of using the twin channel arrangement is exactly the same as doubling the length of data and using the single channel analyser.

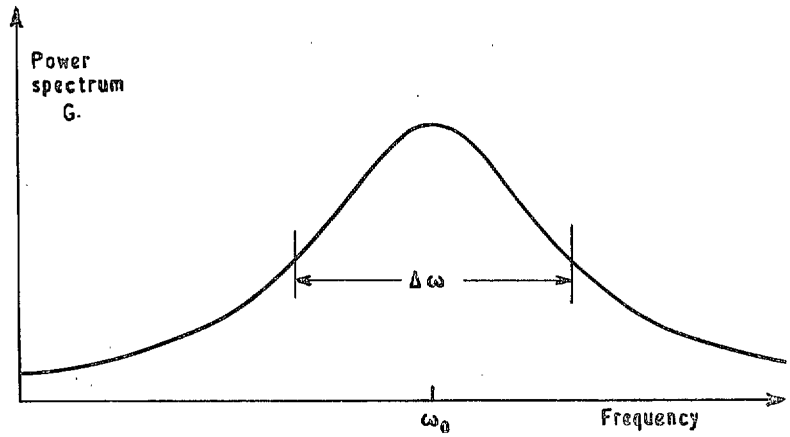

FIG. 1. Power spectrum of random vibration waveform.

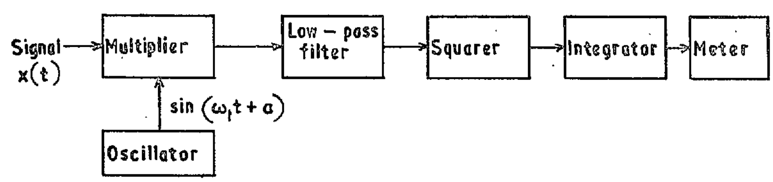

FIG. 2. Arrangement of ideal analyser.

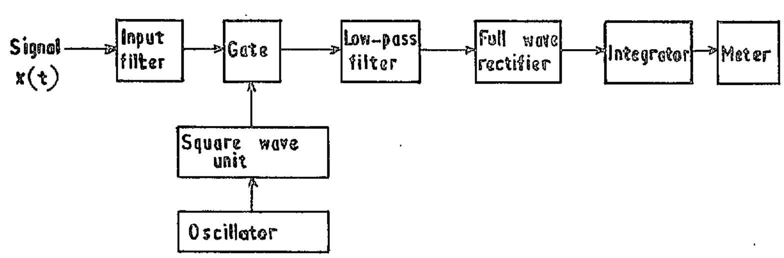

FIG. 3. Arrangement of actual analyser.

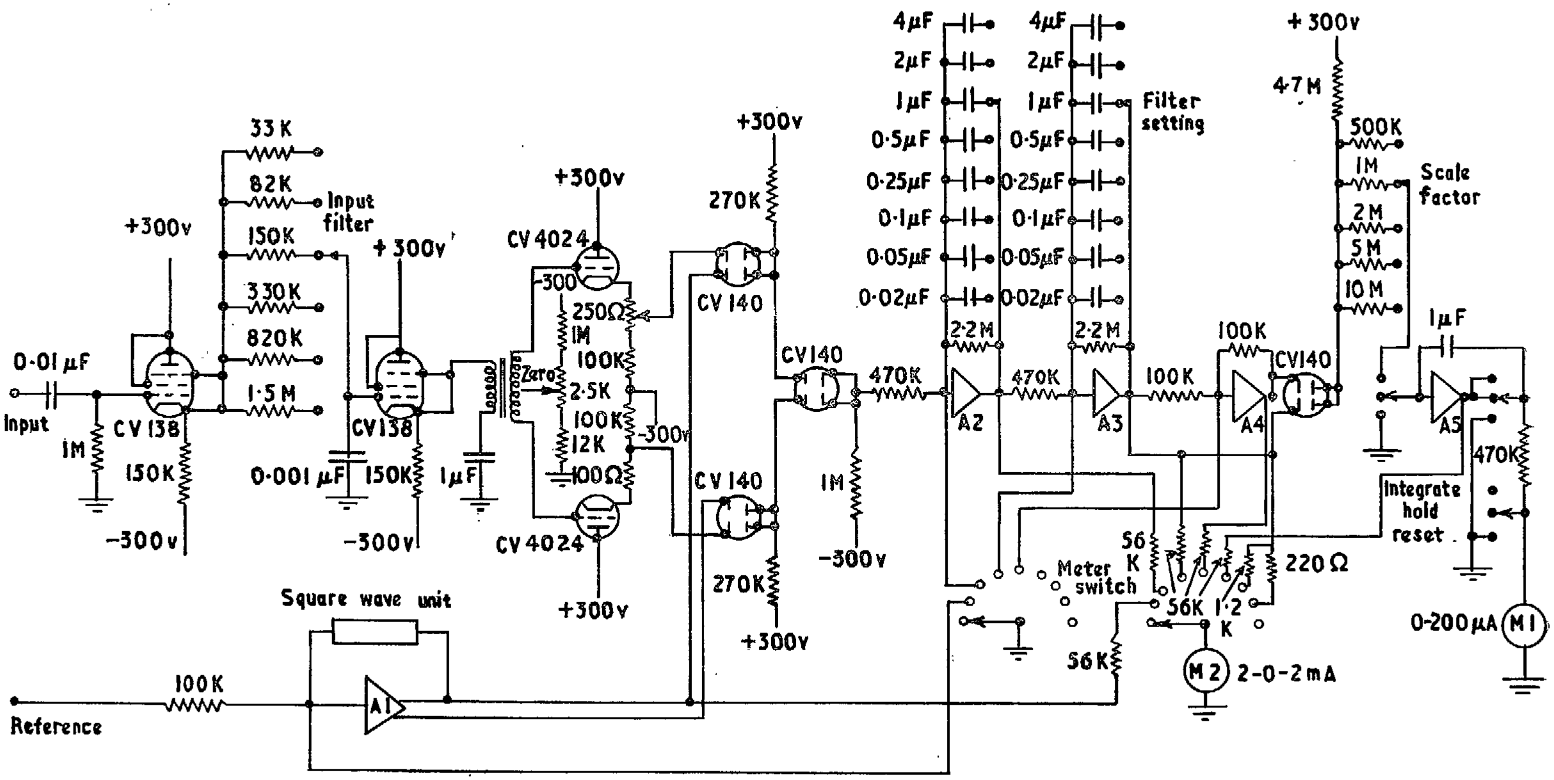

Fig. 4. Vibration wave analyser.

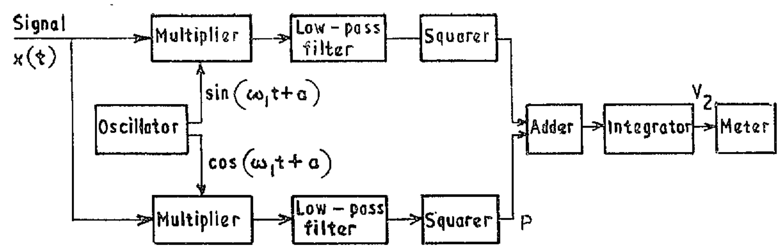

FIG. 5. Twin channel analyser.

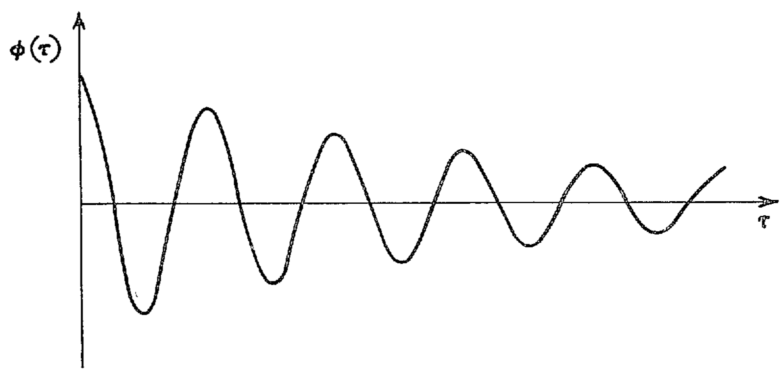

FIG. 6. Autocorrelation function for random vibration waveform.

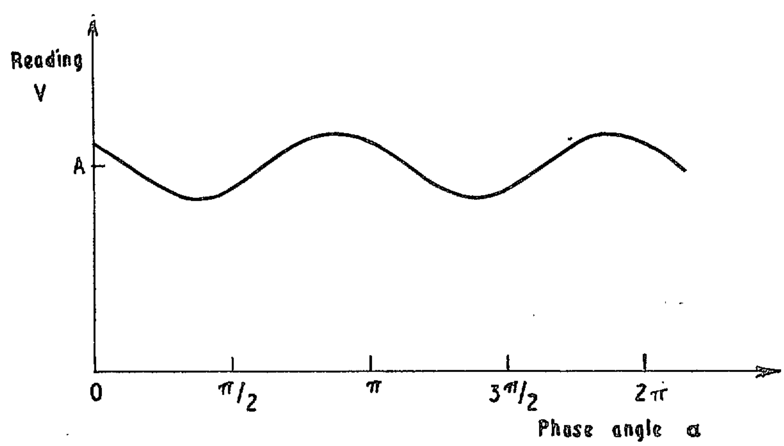

FIG. 7. Variation of reading with oscillator phase angle.

(83851) Wt. 67/1876 K.5 3/62 Hw.

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