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Some Convergence Problems in the Numerical Solution of the Navier-Stokes Equations

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Some Convergence Problems in the Numerical Solution of the Navier-Stokes Equations

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Summary. The convergence of numerical solutions of the Navier–Stokes equations for steady two-dimensional flow is examined and convergence criteria for both ψ and ζ are obtained for a rectangular mesh. The criterion for ψ is shown to be less stringent, in general, than that for ζ. A new method of solution, based on the process used to obtain the convergence criteria, is derived. This method widens the range over which convergence can be obtained and can also be used to accelerate the convergence rate.

1. *Introduction.* The Navier-Stokes equations for the flow of a viscous incompressible fluid past a body in two dimensions are considered in the form

$$\nabla^2\zeta = \frac{1}{\nu}\left(\frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x}\right) \tag{1}$$

$$\nabla^2\psi = \zeta \tag{2}$$

where ψ is the stream function, ζ the vorticity and ν the kinematic viscosity.

For the numerical solution of these equations the field of flow is replaced by a rectangular mesh at the discrete points of which values of ζ and ψ are calculated by finite difference approximations to the Equations (1) and (2). We take the mesh length in the x-direction as being n, and that in the y-direction as pn, where p is a constant. The simplest finite difference approximations to Equations (1) and (2) are then

$$\zeta_0 = \frac{1}{2(1+p^2)}\{p^2(a+c) + (b+d)\} - \frac{p}{8(1+p^2)\nu}\{(A-C)(d-b) - (a-c)(D-B)\} \tag{3}$$

$$\psi_0 = \frac{1}{2(1+p^2)}\{p^2(A+C) + (B+D)\} - \frac{n^2p^2}{2(1+p^2)}\zeta_0 \tag{4}$$

where ζ_0 is the value of ζ at the centre of a diamond with diagonals $2n$ and $2pn$, the small letters representing values of ζ, and the capitals values of ψ, at the corresponding mesh points as shown in Fig. 1. The method of solution is one of reiteration. Assumed values of ψ and ζ are placed at each mesh point and these are progressively improved at each point in turn by recalculation from the surrounding values using Equations (3) and (4).

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The convergence of the numerical process on a square mesh has been examined by Thom and Apelt[1], who obtained a convergence criterion for ζ. The analysis was based on a simplified representation of the process: a finite disturbance was applied to ζ only at a point 0 in a settled field and its effect allowed to spread over the four neighbouring points. From the disturbed values at these points the value of ζ at 0 was recalculated. The process was convergent if the new value was nearer the correct value than the original disturbed value. It was assumed that if ζ converged so also would ψ, this always having happened in practice.

In general, when starting from an assumed solution both ψ and ζ are considerably disturbed from their correct values. The convergence analysis based solely on a disturbance of ζ only gives a valid criterion if ψ is undisturbed; in some cases ψ may be incorrect by a large margin and whilst the ζ convergence criterion is satisfied the routine process may still diverge. The ζ criterion gives no information on this at all. It must be remembered that the criterion is based on a simple representation of the problem and as such cannot be expected to predict accurately when divergence will occur, despite this it has been found to give quite a good indication. Having established that divergence occurs, the problem remains of finding a method whereby convergence can be obtained. Thom and Apelt have shown that there is a limiting mesh size beyond which the numerical process is divergent. A reduction of the mesh size will frequently effect convergence but this greatly increases the number of field points and when using a computer often severely taxes the storage capacity of the machine. In any case the process is laborious and the solution on a coarse mesh may be sufficiently accurate. The alternative procedure is to apply only part of the change in the function values in the divergent region. This weighting process is largely a matter of trial and error. With a computer a considerable amount of machine time may be wasted in using say a fixed half movement, when convergence could be obtained with a larger movement. A technique is demanded which enables the movement to be calculated to give an optimum rate of convergence. Apelt[2] has developed a device depending on the convergence criterion for ζ. The method derived in the present work is similar but is based on a more exact theory dependent on the convergence of both ψ and ζ. In the cases where it has been used this has given quite a rapid convergence rate but for hand calculation the method is not really suitable owing to the complicated nature of some of the numerical factors involved.

2. *Derivation of the Convergence Criteria.* The effect on the four neighbouring points of a disturbance applied simultaneously to both ζ and ψ at a mesh point 0 will now be considered and criteria for the convergence of ζ and ψ derived.

We assume that the values of ζ and ψ are held fixed at the settled values at the outer points $E, L, F, \ldots. K$ as shown in Fig. 2. Finite disturbances δ and ϵ are applied to the values of ζ and ψ respectively at 0, so that ζ_0 becomes $\zeta_0 + \delta$ and ψ_0 is $\psi_0 + \epsilon$. The values of ζ and ψ at the mesh points A, B, C and D are then recalculated and finally, from these, new values of ζ and ψ are obtained at 0. The order of operation is to recalculate ζ first and then ψ at A, B, C and D.

The disturbed values are denoted by a dash and are given by

$$\left.\begin{aligned} \zeta_A' &= \zeta_A + \frac{p^2}{2(1+p^2)}\,\delta - \frac{p(H-E)}{8(1+p^2)\nu}\,\delta + \frac{p(h-e)}{8(1+p^2)\nu}\,\epsilon \\ \zeta_D' &= \zeta_D + \frac{1}{2(1+p^2)}\,\delta - \frac{p(G-H)}{8(1+p^2)\nu}\,\delta + \frac{p(g-h)}{8(1+p^2)\nu}\,\epsilon \end{aligned}\right\} \tag{5}$$

with similar expressions for ζ_B' and ζ_C' respectively; also for ψ

$$\left.\begin{aligned} \psi_A' &= \psi_A + \frac{p^2}{2(1+p^2)}\,\epsilon - \frac{n^2p^2}{2(1+p^2)}\,(\zeta_A' - \zeta_A) \\ \psi_D' &= \psi_D + \frac{1}{2(1+p^2)}\,\epsilon - \frac{n^2p^2}{2(1+p^2)}\,(\zeta_D' - \zeta_D) \end{aligned}\right\} \tag{6}$$

with similar expressions for ψ_B' and ψ_C' respectively. Recalculating the values of ζ and ψ at 0 using these disturbed values we obtain on reduction:

$$\begin{aligned} \zeta_0' = \zeta_0 &+ \frac{(p^4+1)}{2(1+p^2)^2}\,\delta + \frac{p(p^2-1)}{16(1+p^2)^2\nu}\,[(\overline{E-F}+\overline{G-H})\delta - (\overline{e-f}+\overline{g-h})\epsilon] - \\ &- \frac{p^2\delta}{128(1+p^2)^3\nu^2}\,[p^2(\overline{E-G}+\overline{F-H})^2 + (\overline{L-N})(\overline{E-G}+\overline{F-H}) + \\ &+ (\overline{E-G}-\overline{F-H})^2 + p^2(K-M)(\overline{E-G}-\overline{F-H})] + \\ &+ \frac{p^2\epsilon}{128(1+p^2)^3\nu^2}\left[\begin{aligned} &p^2\,(\overline{e-g}+\overline{f-h})(\overline{E-G}+\overline{F-H}) + (L-N)(\overline{e-g}+\overline{f-h}) + \\ &+ (\overline{e-g}-\overline{f-h})(\overline{E-G}-\overline{F-H}) + p^2(\overline{K-M})(\overline{e-g}-\overline{f-h}) \end{aligned}\right] \end{aligned} \tag{7}$$

$$\begin{aligned} \psi_0' = \psi_0 &+ \frac{(p^4+1)}{2(1+p^2)^2}\,\epsilon - \frac{n^2p^2(p^4+1)}{2(1+p^2)^3}\,\delta + \frac{n^2p^3(p^2-1)}{16(1+p^2)^3}\left[\begin{aligned} &(\overline{e-f}+\overline{g-h})\epsilon - \\ &- (\overline{E-F}+\overline{G-H})\delta \end{aligned}\right] + \\ &+ \frac{n^2p^4\delta}{256(1+p^2)^4\nu^2}\,[p^2(\overline{E-G}+\overline{F-H})^2 + (\overline{L-N})(\overline{E-G}+\overline{F-H}) + \\ &+ (\overline{E-G}-\overline{F-H})^2 + p^2(\overline{K-M})(\overline{E-G}-\overline{F-H})] - \\ &- \frac{n^2p^4\epsilon}{256(1+p^2)^4\nu^2}\left[\begin{aligned} &p^2\,(\overline{e-g}+\overline{f-h})(\overline{E-G}+\overline{F-H}) + (\overline{L-N})(\overline{e-g}+\overline{f-h}) + \\ &+ (\overline{e-g}-\overline{f-h})(\overline{E-G}-\overline{F-H}) + p^2(\overline{K-M})(\overline{e-g}-\overline{f-h}) \end{aligned}\right]. \end{aligned} \tag{8}$$

If the mesh is not too coarse the following approximations can be made:

$$(E-F)+(G-H) \doteqdot 0, \quad (E-G)+(F-H) \doteqdot (L-N), \quad (E-G)-(F-H) \doteqdot (K-M)$$

and similarly for the ζ values. Using these approximations in Equations (7) and (8)

$$\begin{aligned} \zeta_0' \doteqdot \zeta_0 &+ \frac{(p^4+1)}{2(1+p^2)^2}\,\delta - \frac{p^2\delta}{64(1+p^2)^2\nu^2}\,[(E-G)^2 + (F-H)^2] + \\ &+ \frac{p^2\epsilon}{64(1+p^2)^2\nu^2}\,[(E-G)(e-g) + (F-H)(f-h)] \end{aligned} \tag{9}$$

$$\begin{aligned} \psi_0' \doteqdot \psi_0 &+ \frac{(p^4+1)}{2(1+p^2)^2}\,\epsilon - \frac{n^2p^2(p^4+1)}{2(1+p^2)^3}\,\delta + \frac{n^2p^4\delta}{128(1+p^2)^3\nu^2}\,[(E-G)^2 + (F-H)^2] - \\ &- \frac{n^2p^4\epsilon}{128(1+p^2)^3\nu^2}\,[(E-G)(e-g) + (F-H)(f-h)]. \end{aligned} \tag{10}$$

If, initially only ζ had been disturbed, for ζ to converge it is necessary that

$$\left|\frac{(p^4+1)}{2(1+p^2)^2}\,\delta - \frac{p^2\delta}{64(1+p^2)^2\nu^2}\,[(E-G)^2+(F-H)^2]\right| < |\delta|\,.$$

The second term on the left hand side of this inequality is always positive and the inequality is satisfied if

$$\frac{1}{\nu^2}\,[(E-G)^2+(F-H)^2] < 32\left(3p^2+4+\frac{3}{p^2}\right). \tag{11}$$

For a square mesh, $p = 1$, and the criterion given by Thom and Apelt[1] is obtained:

$$\frac{1}{\nu^2}\,[(E-G)^2+(F-H)^2] < 320\,. \tag{12}$$

If only ψ had initially been disturbed, for the convergence of ψ we require

$$\left|\frac{(p^4+1)}{2(1+p^2)^2}\,\epsilon - \frac{n^2p^4\epsilon}{128(1+p^2)^3\nu^2}\,[(E-G)(e-g)+(F-H)(f-h)]\right| < |\epsilon|\,.$$

This inequality is satisfied if

$$-\frac{64}{n^2}\left(p^2+4+\frac{1}{p^2}\right)\left(1+\frac{1}{p^2}\right) < \frac{1}{\nu^2}[(E-G)(e-g)+(F-H)(f-h)] < \frac{64}{n^2}\left(3p^2+4+\frac{3}{p^2}\right)\left(1+\frac{1}{p^2}\right) \tag{13}$$

With $p = 1$ this is

$$-\frac{768}{n^2} < \frac{1}{\nu^2}[(E-G)(e-g)+(F-H)(f-h)] < \frac{1280}{n^2}. \tag{14}$$

In terms of the velocity components u and v at the point 0, the Equations (12) and (14) become respectively

$$0 < \frac{n^2}{\nu^2}\,(u^2+v^2) < 40 \tag{15}$$

and

$$-24 < \frac{n^2}{\nu^2}\left[\,u\left(\frac{n^2}{4}\,\nabla^2 u\right)+v\left(\frac{n^2}{4}\,\nabla^2 v\right)\right] < 40\,. \tag{16}$$

If the mesh size is small, then the terms $\frac{n^2}{4}\,\nabla^2 u$ and $\frac{n^2}{4}\,\nabla^2 v$ will be small compared to u and v and it follows that the convergence criterion for ψ is less stringent than that for ζ. This fact has already been remarked by Thom and Apelt[1] in the particular cases of plane Poiseuille flow and plane shear flow.

With a simultaneous disturbance applied to both ψ and ζ at 0, it has not proved possible to find any simple convergence criteria independent of δ and ϵ, based on the Equations (9) and (10). However, as has been pointed out earlier, it has been found that the Equation (11) predicts fairly closely the regions where the numerical process fails to converge.

3. *Convergence Criteria in the Transformed Plane.* When a problem is being worked in the transformed plane, say the (α, β) plane, the Navier-Stokes equations take the form

$$\nabla_{\alpha\beta}^2\zeta = \frac{1}{\nu}\left\{\frac{\partial\psi}{\partial\alpha}\frac{\partial\zeta}{\partial\beta} - \frac{\partial\psi}{\partial\beta}\frac{\partial\zeta}{\partial\alpha}\right\}$$

$$\nabla_{\alpha\beta}^2\psi = \zeta/M^2$$

where M is the modulus of transformation. On obtaining the convergence criterion for ζ it is found to be identical with the inequality (11) for the physical plane. If it is assumed that the modulus of transformation at the points A, B, C, D, and 0 is approximately constant and equal to M_0, then the ψ criterion in the transformed plane is identical with (13) provided that the 'n' of (13) is replaced by m/M_0, where m is the mesh length in the transformed plane.

4. *The New Method of Combating Divergence.* The measures to be taken when the process is divergent will now be considered. The method used by Apelt in his study of the viscous flow past a circular cylinder at a Reynolds number of 40 (Ref. 2) was to apply only part of the change in the value of ζ at each point in the field. Using the convergence criterion for ζ he showed that if the new value of ζ at 0 was taken as intermediate between the original disturbed value and the recalculated value, *i.e.*,

$$\zeta_{0\text{ new}} = \frac{\zeta_{0\text{ recalculated}} + b\zeta_{0\text{ original}}}{1+b}$$

then the optimum convergence rate was obtained when

$$b = \frac{1}{256\nu^2}\left[(E-G)^2 + (F-H)^2\right] - \frac{1}{4} \tag{17}$$

using the square mesh. The method was found to be quite effective, one of the provisos being that if $\dfrac{n\sqrt{u^2+v^2}}{\nu}$ was too great then the rate of convergence was so slow as to be impracticable. In the case of the circular cylinder this method was found to break down at about five radii from the cylinder and beyond this a relaxational technique had to be used. This technique was unsuitable for use with an electronic computer. In the light of the present work on convergence the breakdown of Apelt's method can be explained: the maximum value of ψ increases greatly with distance from the cylinder in a direction normal to the stream and the magnitude of the error is also liable to increase —unless one has been extremely fortunate in choosing starting values for ψ and ζ—and even assuming that the value at only one point is in error, if this error is great enough then the right-hand sides of Equations (9) and (10) will exceed $\zeta_0 + \delta$ and $\psi_0 + \epsilon$ respectively and divergence will occur. To provide a more effective means of obtaining convergence it is necessary to take into account the variations of the ψ values as well as those of ζ.

A new method is now suggested for overcoming the problem of convergence which at the same time gives an optimum convergence rate. Using the same configuration as in obtaining the convergence criteria (11) and (13), we set the differences between the original disturbed values and the recalculated values of ζ and ψ such that

$$\left.\begin{aligned} \zeta_{\text{original}} - \zeta_0' &= \Delta = (\zeta_0+\delta) - \zeta_0' \\ \psi_{\text{original}} - \psi_0' &= \Gamma = (\psi_0+\epsilon) - \psi_0'. \end{aligned}\right\} \tag{18}$$

The values of Δ and Γ are known, and on substituting from Equations (18) in Equations (9) and (10) we obtain a pair of simultaneous equations in δ and ϵ. Solving these for δ and ϵ

$$\delta = \frac{4p^2 Y\Gamma + \left\{\left(\frac{p^4+4p^2+1}{2}\right) + \frac{2n^2p^4Y}{(1+p^2)}\right\}\Delta}{\frac{2p^2(p^4+4p^2+1)X}{(1+p^2)^2} + \frac{(p^4+4p^2+1)^2}{4(1+p^2)^2} + \frac{(3p^4+4p^2+3)p^4n^2Y}{(1+p^2)^3}} \tag{19}$$

$$\epsilon = \frac{\left\{\frac{p^4+4p^2+1}{2} + 4p^2X\right\}\Gamma + \left\{2p^4X - \frac{p^2(p^4+1)}{2}\right\}\frac{n^2\Delta}{(1+p^2)}}{\frac{2p^2(p^4+4p^2+1)X}{(1+p^2)^2} + \frac{(p^4+4p^2+1)^2}{4(1+p^2)^2} + \frac{(3p^4+4p^2+3)p^4n^2Y}{(1+p^2)^3}}, \tag{20}$$

where

$$X = \frac{1}{256\nu^2}[(E-G)^2 + (F-H)^2] \doteqdot \frac{1}{128\nu^2}[(A-C)^2 + (D-B)^2],$$

$$Y = \frac{1}{256\nu^2}[(E-G)(e-g) + (F-H)(f-h)] \doteqdot \frac{1}{128\nu^2}[(A-C)(a-c) + (D-B)(d-b)].$$

For the square mesh, with $p = 1$, Equations (19) and (20) are

$$\delta = \frac{4Y\Gamma + (3+n^2Y)\Delta}{3X + 1\cdot 25n^2Y + 2\cdot 25} \tag{21}$$

$$\epsilon = \frac{(3+4X)\Gamma + (X-0\cdot 5)n^2\Delta}{3X+1\cdot 25n^2Y+2\cdot 25}. \tag{22}$$

δ and ϵ can now be subtracted from the assumed original values of ζ and ψ to give the correct values.

The technique to be used in working over a field is as follows: at a point 0 (as in Fig. 1) the value of ζ is calculated from the surrounding points using Equation (3), this value of ζ is then used to recalculate the value of ψ from Equation (4). The values of Δ and Γ can then be obtained and δ and ϵ can be calculated from Equations (19) and (20). The new values of ζ and ψ are now the original values minus δ and ϵ respectively. Reiteration is then used over the whole field until the function values are settled to the required degree of accuracy.

Since steady flow has been postulated the divergence of the numerical process is not related to any problem of hydrodynamic stability, it may be that no solution of the problem being considered can exist, in which case divergence persists whatever preventative measures are taken. Because of the simplified representation used in formulating the problem whereby the new method has been derived, it must be realised that when working over a field containing a large number of mesh points it is still possible for divergence to occur; when this happens the mesh size has to be reduced and the method reapplied, the process being continued until convergence is obtained (provided that a solution does exist).

5. *Use of the Method for Accelerating Convergence.* When the straightforward iterative procedure using Equations (3) and (4) is convergent, the method derived in the previous section is still applicable and can be used to accelerate the convergence of the process.

As an example of the comparative rates of convergence of the several ways of attacking a numerical solution of the Navier-Stokes equations, a small field was settled using three methods:

(i) Reiteration using Equations (3) and (4) only.

(ii) The method used by Apelt.

(iii) The present 'δ, ϵ' method.

The problem was worked on a computer, the modulus of the maximum change in function values after each iteration was recorded (here termed the residual) and the computation stopped when the residuals of both ψ and ζ were below unity. Both methods (i) and (ii) took approximately the same machine time and method (iii) a little over half this time. The results are shown in Figs. 3 and 4. It is seen that the first method required 21 iterations on ζ and 17 on ψ to reduce the residuals to unity, the second method required 20 on ζ and 21 on ψ, and the third only 12 on ζ and 11 on ψ. A significant improvement has thus been obtained in using the new method. The second method has not improved the convergence rate at all in this case, but it should be remembered that its major object was to make the process converge and not to provide an accelerating factor.

6. *Precautions to be Observed.* Some cautionary words are necessary at this stage. One of the main problems, and a matter of which little is known, is the convergence of ζ on a solid boundary. Only in a few cases is the vorticity ζ known at the boundary, when it is unknown an approximate value has to be calculated from the neighbouring values of ψ and ζ, using a formula such as that of Woods[3]. This process may also diverge, for some discussion on the problem the reader is referred to Thom[4] and Thom and Apelt[1]. The present new method is not strictly applicable on the line of mesh points nearest to the boundary and since the movement it gives is often quite large, a violent oscillation leading to divergence can under certain circumstances be set up in the values of successive approximations to ζ on the boundary. In particular, this form of divergence is liable to occur if a very small mesh size is being used and the initial assumed values are not very good. It is recommended that the Equations (3) and (4) in conjunction with either Woods' formula or Thom's two-point formula[5] be used on the first two or three mesh lines off the boundary and the new method employed out in the field. The effect of this is to limit the changes of ζ on the boundaries; an alternative method is to apply only part of the movement given by the boundary formula and is usually a matter of trial and error, this is not too difficult when working by hand but can be a complicated and time consuming operation on a computer. If the boundary values of ζ are given then the present method can be used immediately with considerable advantage.

Acknowledgement. The author wishes to thank Professor Thom for his valuable advice and criticism.

LIST OF SYMBOLS

(x, y)	The physical plane
(α, β)	The transformed plane
M	The modulus of transformation from the physical to the (α, β) plane
ψ	The stream function in viscous flow
ζ	The vorticity
δ	A finite disturbance to the value of ζ at a point
ϵ	A finite disturbance to the value of ψ at a point
∇^2	The Laplacian operator, $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$
ν	The kinematic viscosity
n	The distance between adjacent points of the mesh in the x-direction on which a numerical solution is obtained
p	The ratio of the mesh length in the y-direction to that in the x-direction
u, v	The components of the velocity parallel to the axes of the mesh used for the numerical solution

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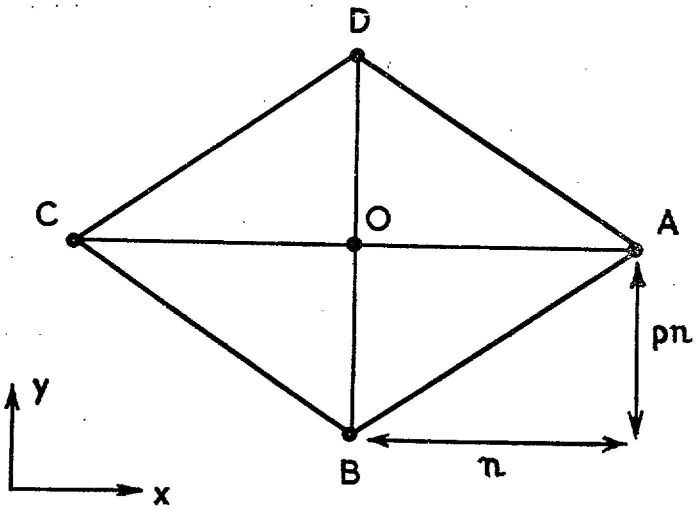

FIG. 1.

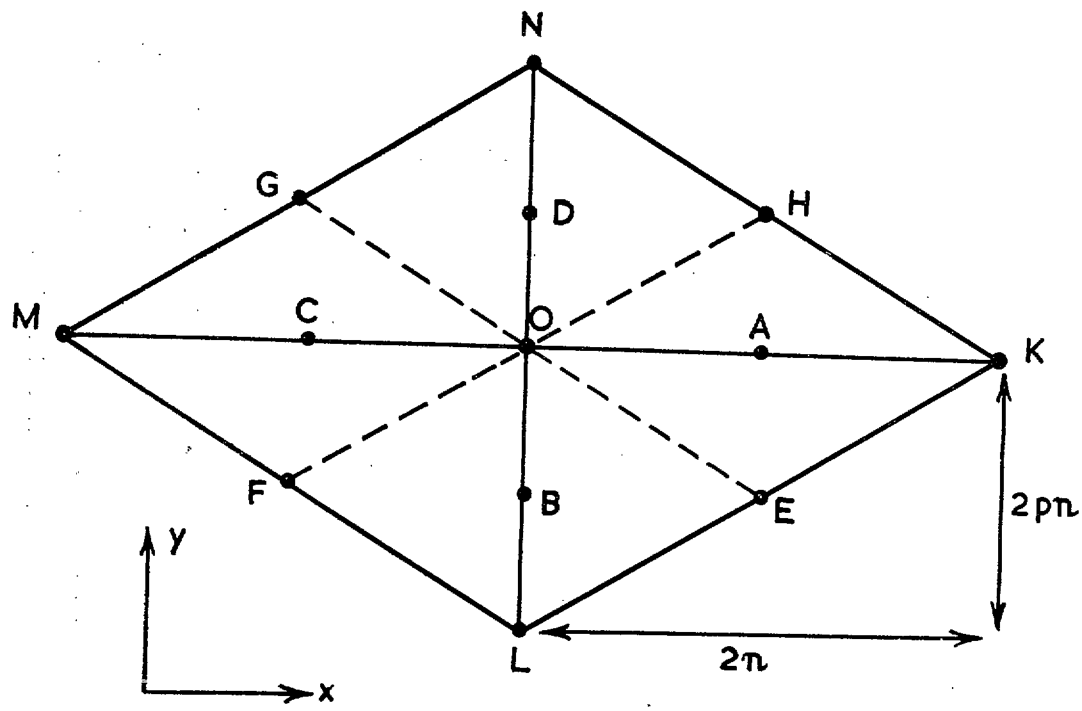

FIG. 2.

(83325) Wt. 67/1876 K.5 12/61 Hw.

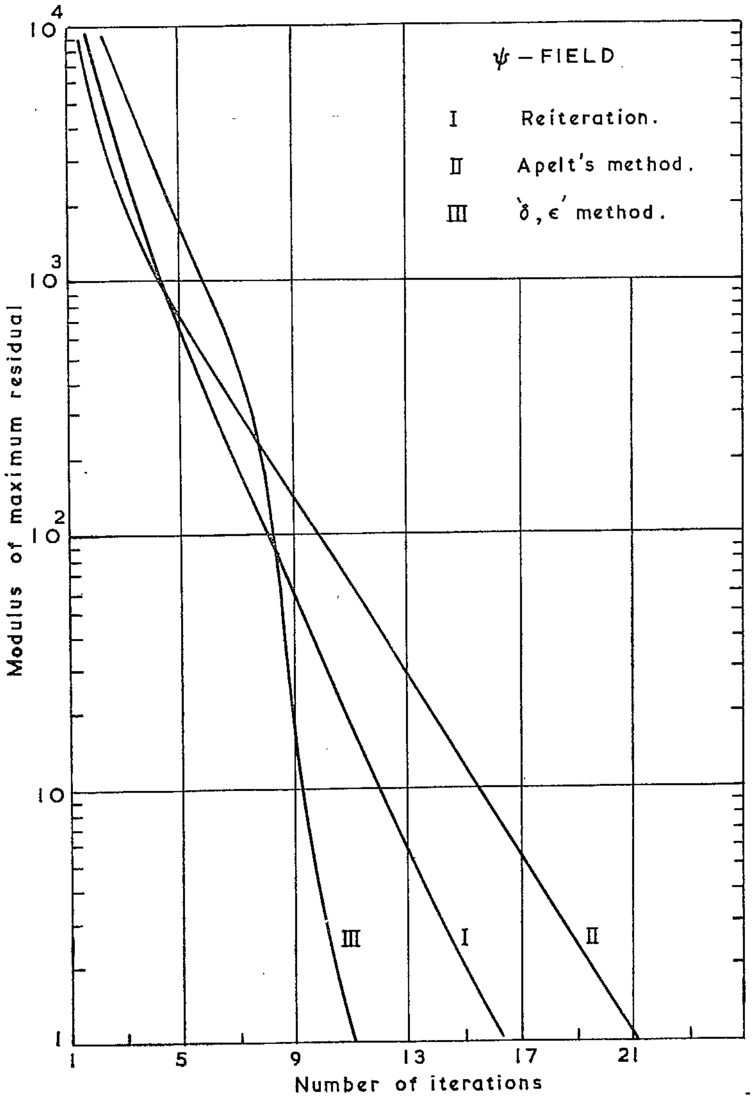

FIG. 3.

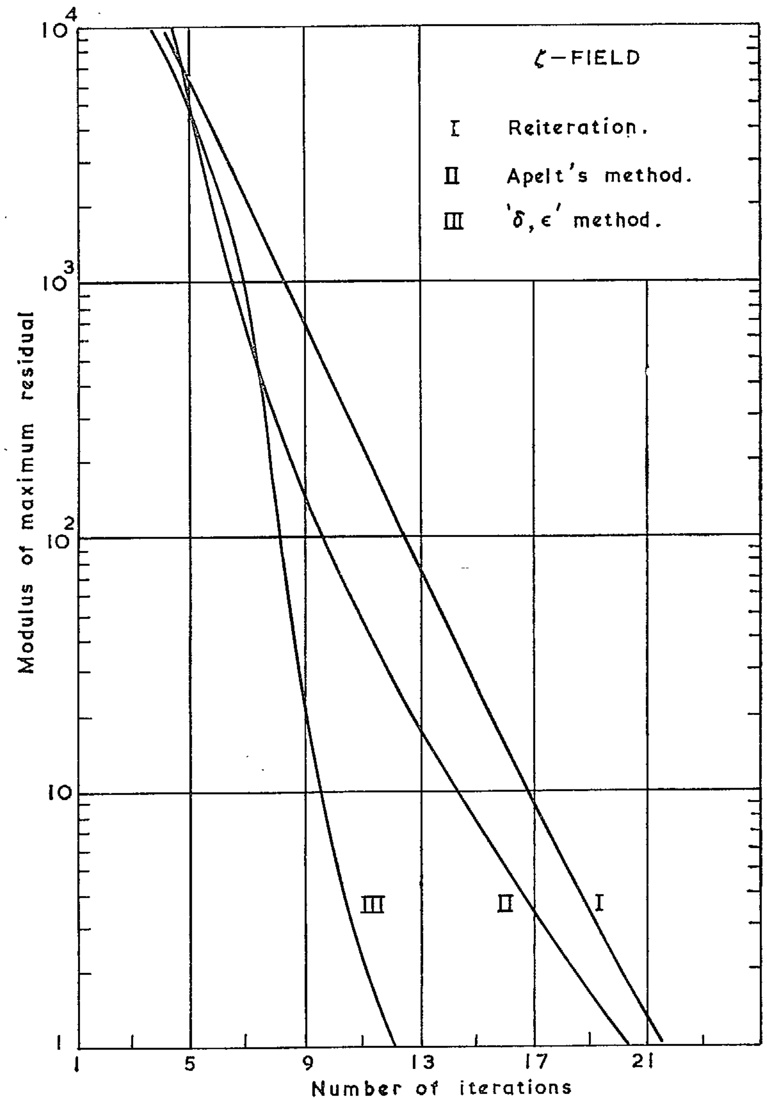

FIG. 4.

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