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A Method for Calculating the Lifting Forces on Wings (Unsteady Subsonic and Supersonic Lifting-Surface Theory)

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Summary.—A numerical method is given for calculating the lifting forces on oscillating wings of any plan-form. The principles and techniques of Multhopp's subsonic theory have been applied to the supersonic problem resulting in a single basic theory which embraces both subsonic and supersonic cases.

One of the most important features of the method is the careful choice of the points at which the lift and downwash distributions are measured. The position of these points in the chordwise direction depend upon whether the local leading and trailing edges are subsonic or supersonic.

Extensive use has been made of various interpolation functions which simplify the evaluation of the integrals required for both the downwash and the generalised forces. In the latter case it is shown that the continuous lift distribution can be replaced without loss of accuracy by a set of concentrated lift forces at the lift points. The lift distribution is expressed in terms of these discrete forces since for most purposes they are more convenient to use.

It is shown that control surfaces can be dealt with by using equivalent continuous deflections and downwash angles to replace the true discontinuous values. Simple expressions are given for these equivalent values, and these expressions are applicable to both subsonic and supersonic cases.

1. *Introduction.*—The problem of calculating the lift distribution on wings of finite span in both steady and unsteady motion has been the subject of many investigations. For wings travelling at supersonic speeds attention has been chiefly concentrated upon obtaining exact solutions (within the limitations of linearised potential flow) for certain simple plan-forms and downwash distributions. This exact approach, however, is unsuitable for dealing with complicated plan-forms and downwash distributions since it becomes extremely cumbersome.

Numerical methods have been used for investigations of the subsonic problem, and in particular Multhopp's lifting-surface theory (Ref. 1) and its developments (Refs. 2, 3, 4 and 5) have been very successful. In the present report the principles and techniques of Multhopp's subsonic theory have been applied to the supersonic problem, resulting in a single basic theory which embraces both subsonic and supersonic cases.

2. *The Integral Equation.*—The first problem is to find the integral equation which gives the downwash angle on the wing surface in terms of the lift distribution.

Consider a small element of wing with area $\delta x\ \delta y$ at the point (x, y) and let this element have a pressure difference $\Delta p(x, y, t)$ between its upper and lower surfaces for a short time interval δt at time t. This lift can be considered as due to a line vortex of strength $\Gamma(x, y, t)$ travelling with the element of wing at speed V. The strength of this vortex is found by equating the lift due to a moving vortex with the lift on the wing element.

$$\Delta p(x, y, t)\ \delta x\ \delta y = \rho V \Gamma(x, y, t)\ \delta y \quad \ldots \quad (1)$$

* Fairey Aviation Company Tech. Office Report 165, received 27th August, 1956.

Since the lift exists for a short time δt only, a starting vortex of equal and opposite strength is left in the wake at a distance $V\delta t$ behind the bound vortex, and these two vortices are connected by trailing vortices at the tips of the wing element. Thus a closed vortex loop is formed surrounding an area $V\delta t\delta y$ (*see* Fig. 1).

This vortex ring is identical to a doublet of strength equal to the product of the vortex strength and the area of the ring. The strength $\mu(x, y, t)$ of the doublet on the wing element is therefore

$$\mu(x, y, t) = \frac{\Delta p}{\rho}(x, y, t)\, \delta x\, \delta y\, \delta t\, . \qquad (2)$$

In order to find the downwash on the wing at point $(x_0 y_0)$ and time t_0 the combined effect of all the doublets (of strength given in equation (2)) at all points (xy) on the wing surface and all times t must be found. However, instead of using doublets it is more convenient to consider the effect of sources and obtain the doublet solution by differentiation.

Consider a unit source which comes into being at time t. When the speed of sound is infinite the velocity potential at a distance r from the source at time t_0 will be :

$$\left.\begin{aligned} \phi_s &= \frac{-1}{4\pi r} && \text{when } t < t_0 \\ &= 0 && \text{when } t_0 < t \end{aligned}\right\} . \qquad (3)$$

The reason for these two parts to the result is clearly that the source cannot have any effect before it exists. When the speed of sound is finite, however, the velocity potential given in equation (3) will be modified in some way. It can be shown from the work of Lomax, Heaslet and Fuller (Ref. 6) that the change is merely to modify the regions of equation (3). Clearly the source can affect only points within a sphere of radius $a(t_0 - t)$. Therefore equation (3) is modified to

$$\left.\begin{aligned} \phi_s &= \frac{-1}{4\pi r} && \text{when } r < a(t_0 - t) \\ &= 0 && \text{when } a(t_0 - t) < r \end{aligned}\right\} . \qquad (4)$$

Then if the wing is in the plane z the velocity potential at point $(x_0 y_0 z_0 t_0)$ can be found by using equations (2) and (4) as follows :

$$\phi(x_0 y_0 z_0 t_0) = \frac{-1}{4\pi}\frac{\partial}{\partial z}\int_S\int\left\{\int_{r<a(t_0-t)}\frac{1}{r}\frac{\Delta p}{\rho}(x, y, t)dt\right\}dx\, dy\, . \qquad (5)$$

The downwash on the wing surface is therefore

$$w(x_0 y_0 t_0) = \frac{-1}{4\pi}\lim_{z_0 \to z}\frac{\partial^2}{\partial z_0\, \partial z}\int_S\int\left\{\int_{r<a(t_0-t)}\frac{1}{r}\frac{\Delta p}{\rho}(x, y, t)dt\right\}dx\, dy\, . \qquad (6)$$

The following substitutions can be made

$$\begin{aligned} X &= x_0 - x, & \tau &= VT - X \\ Y &= y_0 - y & r &= \sqrt{(\tau^2 + Y^2 + Z^2} \\ Z &= z_0 - z & R &= \sqrt{(X^2 + (1 - M^2)(Y^2 + Z^2))} \text{ when } M < 1 \\ T &= t_0 - t & &= \sqrt{(X^2 - (M^2 - 1)(Y^2 + Z^2))} \text{ when } M > 1 \end{aligned} \qquad (7)$$

Then the condition $r < a(t_0 - t)$ becomes

$$\left.\begin{array}{ll} \dfrac{-X + MR}{1 - M^2} < \tau < \infty & \text{when } M < 1 \\ \left.\begin{array}{l} \dfrac{X - MR}{M^2 - 1} < \tau < \dfrac{X + MR}{M^2 - 1} \\ X > |Y|\sqrt{(M^2 - 1)} \end{array}\right\} & \text{when } M > 1 \end{array}\right] \quad . \qquad (8)$$

Then equation (6) becomes

$$\left.\begin{array}{l} w(x_0 y_0 t_0) = \dfrac{1}{4\pi}\displaystyle\int_S\int\left\{\lim_{z\to 0}\frac{\partial^2}{\partial Z^2}\int_{\frac{-X+MR}{1-M^2}}^{\infty}\frac{\Delta p}{\rho V}(x, y, t)\,\frac{d\tau}{\sqrt{(\tau^2 + Y^2 + Z^2)}}\right\}dx\,dy \quad \text{when } M < 1 \\ = \dfrac{1}{4\pi}\displaystyle\int_S\int_{x>|y|\sqrt{(M^2-1)}}\left\{\lim_{z\to 0}\frac{\partial^2}{\partial Z^2}\int_{\frac{X-MR}{M^2-1}}^{\frac{X+MR}{M^2-1}}\frac{\Delta p}{\rho V}(x, y, t)\,\frac{d\tau}{\sqrt{(\tau^2 + Y^2 + Z^2)}}\right\}dx\,dy \\ \qquad\qquad \text{when } M > 1 \end{array}\right\} . \qquad (9)$$

The harmonic time variation can now be introduced

$$\begin{aligned} \Delta p\,(x, y, t) &= \Delta p(x, y, t_0)\, e^{-i\varpi(t_0 - t)} \\ &= \Delta p(x, y, t_0)\, e^{\frac{-i\varpi}{v}(x+\tau)} \,. \end{aligned} \qquad (10)$$

In addition we will introduce the notation

$$\left.\begin{array}{l} l(x, y) = \dfrac{\Delta p}{\frac{1}{2}\rho V^2}(x, y, t_0) \\ \alpha(x_0 y_0) = \dfrac{w}{V}\, x_0 y_0 t_0) \end{array}\right\} . \qquad (11)$$

Then equation (9) will become

$$\alpha(x_0 y_0) = \frac{-1}{8\pi}\int_S\int\frac{l(x, y)}{Y^2}K(X, Y)\,dx\,dy\,, \qquad (12)$$

where the kernel $K(X, Y)$ is defined as

$$\left.\begin{array}{ll} K(X, Y) = -\,Y^2 e^{\frac{-i\varpi X}{V}}\displaystyle\lim_{z\to 0}\frac{\partial^2}{\partial Z^2}\int_{\frac{-X+MR}{1-M^2}}^{\infty}\frac{e^{\frac{-i\varpi\tau}{V}}\,d\tau}{\sqrt{(\tau^2 + Y^2 + Z^2)}} & \text{when } M < 1 \\ \text{when } M > 1 \left\{\begin{array}{l} = -\,Y^2 e^{\frac{-i\varpi X}{V}}\displaystyle\lim_{z\to 0}\frac{\partial^2}{\partial Z^2}\int_{\frac{X+MR}{M^2-1}}^{\frac{X+MR}{M^2-1}}\frac{e^{\frac{-i\varpi\tau}{V}}\,d\tau}{\sqrt{(\tau^2 + Y^2 + Z^2)}} \\ = 0 \text{ when } X < |\gamma|\sqrt{(M^2 - 1)} \end{array}\right. & \text{when } X > |\gamma|\sqrt{(M^2 - 1)} \end{array}\right\} . \qquad (13)$$

This kernel is fully discussed in Appendix I.

3. *The Lift and Downwash Points.*—The basis of the method presented in this report is the replacement of the integral equation (which contains the lift and downwash as functions) by a matrix equation (which contains the values of the lift and downwash at specified points). These lift and downwash points must be carefully chosen to ensure the maximum accuracy from an equation of given matrix order.

The distribution of these points along a chordwise wing section will be considered at first, and the analysis must extend much further than Multhopp's which was concerned only with the subsonic case. If the airspeed normal to the leading edge (in plan view) is subsonic, then the lift distribution will have an infinite peak of the type $1/\sqrt{(1+\xi)}$ at the leading edge ($\xi \rightarrow -1$). Also if the trailing edge is subsonic (in the same sense) then the lift distribution will have a zero value of the type $\sqrt{(1-\xi)}$ at the trailing edge ($\xi \rightarrow +1$). If, however, an edge is supersonic, the lift distribution at that edge will remain finite. These edge conditions will be taken into account by defining a function $f(\xi)$ with four alternative forms as follows :

$$\left.\begin{aligned} f(\xi) &= \sqrt{\left(\frac{1-\xi}{1+\xi}\right)} && \text{(subsonic LE, subsonic TE)} \\ &= \sqrt{\left(\frac{1}{1+\xi}\right)} && \text{(subsonic LE, supersonic TE)} \\ &= \sqrt{(1-\xi)} && \text{(supersonic LE, subsonic TE)} \\ &= 1 && \text{(supersonic LE, supersonic TE)} \end{aligned}\right\} \quad \ldots \quad (14)$$

A chordwise lift distribution $l_\lambda(\xi)$ will now be defined by multiplying the function $f(\xi)$ by a polynomial in ξ of order λ as follows :

$$l_\lambda(\xi) = (a_{\lambda 0} + a_{\lambda 1}\xi + a_{\lambda 2}\xi^2 + a_{\lambda 3}\xi^3 + \ldots + a_{\lambda\lambda}\xi^\lambda)\, f(\xi)\,. \quad \ldots \quad (15)$$

The downwash distribution due to this lift distribution can be calculated from two-dimensional (infinite aspect ratio) theory. Subsonic theory is used for this purpose when both edges are subsonic, supersonic theory when both edges are supersonic and sonic theory when one edge is subsonic and one edge supersonic. The detailed analysis is given in Appendix II and in all cases the downwash distribution is a polynomial in ξ of order λ.

Suppose that the coefficients $a_{\lambda 0}$, $a_{\lambda 1}$, etc., are chosen so that the lift distribution $l_\lambda(\xi)$ gives zero total lift, zero pitching moment, zero second moment, and so on up to zero $(\lambda - 1)$th moment. It is then found that the downwash polynomial becomes

$$\alpha_\lambda(\xi) = (a_{\lambda 0} - a_{\lambda 1}\xi + a_{\lambda 2}\xi^2 - a_{\lambda 3}\xi^3 + \ldots a_{\lambda\lambda}\xi^\lambda) F_\lambda\,. \quad \ldots \quad (16)$$

The factor F_λ depends on the type of function of $f(\xi)$ used and does not affect the argument.

The chordwise pressure distribution can be expressed as an infinite series as follows

$$l(\xi) = \sum_{\lambda=0}^{\infty} A_\lambda l_\lambda(\xi)\,. \quad \ldots \quad (17)$$

The downwash distribution (in two-dimensional flow) will then be

$$\alpha(\xi) = \sum_{\lambda=0}^{\infty} A_\lambda \alpha_\lambda(\xi)\,. \quad \ldots \quad (18)$$

Suppose, however, a shortened series is used consisting of only the first p terms. It is clear from the definition of the lift functions that this shortened series will give the same lift, pitching moment, second moment and so on up to the $(p-1)$th moment as the full infinite series. This shortened series and its corresponding two-dimensional downwash will be

$$\left.\begin{aligned} l(\xi) &= \sum_{\lambda=0}^{\infty} A_\lambda l_\lambda(\xi) \\ \alpha(\xi) &= \sum_{\lambda=0}^{\infty} A_\lambda \alpha_\lambda(\xi) \end{aligned}\right\} \quad \ldots \quad (19)$$

However, both the lift and downwash are going to be expressed in terms of the values at p points along the chord instead of in terms of the arbitrary coefficients A_λ as above. The values of lift and downwash between these points though must conform to the above series.

Suppose one additional term is admitted to each of these series. These terms $l_p(\xi)$ and $\alpha_p(\xi)$ each have p zero values along the chord. If therefore these points are chosen for the lift and downwash points, the lift and downwash values will be unaffected by one additional series term. It is clear then that the best lift points ξ_q will be given by $l_p(\xi_q) = 0$ which becomes

$$a_{p0} + a_{p1}\xi_q + a_{p2}\xi_r^2 + \ldots a_{pp}\xi_q^p = 0 . \qquad (20)$$

The downwash points ξ_r will be given by $\alpha_p(\xi_r) = 0$ or

$$a_{p0} - a_{p1}\xi_r + a_{p2}\xi_r^2 - a_{p3}\xi_r^3 + \ldots a_{pp}\xi_r^p = 0 . \qquad (21)$$

Equations (20) and (21) show quite obviously that the lift and downwash points are complementary, *i.e.*, the lift points are the same as the downwash points on a wing section in reverse flow.

The chordwise analysis in the previous subsonic theory (Refs. 1, 2 and 4) is therefore reduced to the special case in which $f(\xi) = \sqrt{(1 - \xi/1 + \xi)}$. The angular co-ordinates used by Multhopp are, however, unsuitable for dealing with the additional three chordwise cases, and for this reason the complete analysis has been kept in terms of ξ for this report.

In the subsonic solution the case $p = 1$ leads to the well-known result that the lift point is at the $\frac{1}{4}$-chord, and the downwash point is at the $\frac{3}{4}$-chord. The analysis for the other chordwise cases leads to the following interesting generalisation of this rule :

Condition of LE	Condition of TE	Lift Point	Downwash Point
Subsonic	Subsonic	$\frac{1}{4}c$	$\frac{3}{4}c$
Subsonic	Supersonic	$\frac{1}{3}c$	$\frac{2}{3}c$
Supersonic	Subsonic	$\frac{2}{5}c$	$\frac{3}{5}c$
Supersonic	Supersonic	$\frac{1}{2}c$	$\frac{1}{2}c$

Now the spanwise positions of the wing sections are considered. At the tip $\eta \to -1$ the lift distribution will tend to zero in the manner $\sqrt{(1 + \eta)}$. At the tip $\eta \to +1$ the lift distribution will tend to zero in the manner $\sqrt{(1 - \eta)}$. Therefore the product $\sqrt{(1 - \eta^2)}$ of these two functions must appear in the lift distribution. There is no difference between the subsonic and supersonic cases for this spanwise analysis and the position of the spanwise lift and downwash sections will be identical to those used by Multhopp. The spanwise analysis will however be given here in polynomial form for the sake of completeness.

A spanwise lift distribution $\gamma_\lambda(\eta)$ will be defined as

$$\gamma_\lambda(\eta) = (a_{\lambda 0} + a_{\lambda 1}\eta + a_{\lambda 2}\eta^2 + a_{\lambda 3}\eta^3 + \ldots + a_{\lambda\lambda}\eta^\lambda)\sqrt{(1 - \eta^2)}. \qquad (22)$$

The downwash distribution due to this lift distribution can be calculated from two-dimensional (zero aspect ratio) theory which is independent of Mach number. The detailed analysis is given in Appendix II and in all cases the downwash distribution is a polynomial in η of order λ.

Suppose that the coefficients $a_{\lambda 0}$, $a_{\lambda 1}$, etc., are chosen so that the lift distribution $\gamma_\lambda(\eta)$ gives zero total lift, zero rolling moment, zero second moment, and so on up to zero $(\lambda - 1)$th moment. It will be found that the odd or even terms of the polynomial of equation (22) disappear according to whether λ is even or odd. Then equation (22) will be of the form

$$\left.\begin{aligned} \gamma_\lambda(\eta) &= (a_{\lambda 0} + a_{\lambda 2}\eta^2 + a_{\lambda 4}\eta^4 + \ldots a_{\lambda\lambda}\eta^\lambda)\sqrt{(1 - \eta^2)} \quad (\lambda \text{ even}) \\ &= (a_{\lambda 1}\eta + a_{\lambda 3}\eta^3 + a_{\lambda 5}\eta^5 + \ldots a_{\lambda\lambda}\eta^\lambda)\sqrt{(1 - \eta^2)} \quad (\lambda \text{ odd}) \end{aligned}\right\} . \qquad (23)$$

With this definition of the coefficients $a_{\lambda 0}$, $a_{\lambda 1}$, etc., the downwash polynomial becomes

$$\left.\begin{aligned} \alpha_\lambda(\eta) &= (a_{\lambda 0} + a_{\lambda 2}\eta^2 + a_{\lambda 4}\eta^4 + \ldots a_{\lambda\lambda}\eta_\lambda)\, E_\lambda \qquad (\lambda \text{ even}) \\ &= (a_{\lambda 1}\eta + a_{\lambda 3}\eta^3 + a_{\lambda 5}\eta^5 + \ldots a_{\lambda\lambda}\eta^\lambda)\, E_\lambda \qquad (\lambda \text{ odd}) \end{aligned}\right\} \quad \ldots \quad (24)$$

Then the spanwise pressure distribution and its appropriate two-dimensional downwash can be expressed as an infinite series :

$$\left.\begin{aligned} l(\eta)\, c(\eta) &= \sum_{\lambda=0}^{\infty} A_\lambda \gamma_\lambda(\eta) \\ \alpha(\eta) &= \sum_{\lambda=0}^{\infty} A_\lambda \alpha_\lambda(\eta) \end{aligned}\right\} \quad \ldots \quad (25)$$

If this series is shortened to the first m terms then it is clear from the definition of the lift functions $\gamma_\lambda(\eta)$ that the lift distribution will still give the correct lift, rolling moment, second moment and so on up to the $(m-1)$th moment. This shortened series for the lift and its corresponding two-dimensional downwash will be

$$\left.\begin{aligned} l(\eta)\, c(\eta) &= \sum_{\lambda=0}^{m-1} A_\lambda \gamma_\lambda(\eta) \\ \alpha(\eta) &= \sum_{\lambda=0}^{m-1} A_\lambda \alpha_\lambda(\eta) \end{aligned}\right\} \quad \ldots \quad (26)$$

The lift and downwash are, however, to be defined by the values at m points rather than the arbitrary coefficients A_λ in the above equation. However, the values of lift and downwash between these points must conform to the above series.

If one additional term $\gamma_m(\eta)$ or $\alpha_m(\eta)$ is admitted to each of the above series, the values of lift and downwash will be unaffected at the m points η_n and η_ν given by $\gamma_m(\eta_n) = 0$ and $\alpha_m(\eta_\nu) = 0$. These conditions (for odd values of m) are given by

$$\left.\begin{aligned} a_{m1}\eta_n + a_{m3}\eta_n^{\,3} + a_{m5}\eta_n^{\,5} + \ldots + a_{mm}\eta_n^{\,m} = 0 \\ a_{m1}\eta_\nu + a_{m3}\eta_\nu^{\,3} + a_{m5}\eta_\nu^{\,5} + \ldots + a_{mm}\eta_\nu^{\,m} = 0 \end{aligned}\right\} \quad \ldots \quad (27)$$

It is clear, therefore, that the spanwise positions of the lift and downwash stations are identical.

This spanwise analysis is identical in principle to that of Multhopp who, however, used angular co-ordinates. The above polynomial form shows the general connection with the chordwise analysis more clearly.

4. *The Lift Distribution.*—Consider the lift along the chordwise strip first of all. The lift function $l_p(\xi)$ is zero at all p lift points. If this lift function is divided by the line tangential to it at the qth lift point a new function $h_q(\xi)$ is obtained :

$$h(\xi) = \frac{l_p(\xi)}{(\xi - \xi_q)(\partial l_p(\xi)/\partial \xi)_{\xi=\xi_q}} . \quad \ldots \quad (28)$$

This interpolation function is unity at $\xi = \xi_q$ and zero at all the other lift points. Clearly from the definition of $l_p(\xi)$ the interpolation function $h_q(\xi)$ can be written as

$$h_q(\xi) = (b_{q0} + b_{q1}\xi + b_{q2}\xi^2 + \ldots b_{q,p-1}\xi^{p-1})\, f(\xi) . \quad \ldots \quad (29)$$

This is a linear combination of the functions $l_0(\xi)$, $l_1(\xi) \ldots l_{p-1}(\xi)$ and so the series for the lift distribution (equation (19)) can be replaced exactly by a series of the p interpolation functions $h_q(\xi)$ as follows :

$$l(\xi) = \sum_{q=1}^{p} l(\xi_q) h_q(\xi) . \quad \ldots \quad (30)$$

The unknown coefficients in this equation are the actual values of the lift at the lift points instead of the arbitrary coefficients A_λ in equation (19).

In an exactly similar manner a spanwise interpolation function $g_n(\eta)$ can be obtained from $\gamma_m(\eta)$ as follows :

$$g_n(\eta) = \frac{\gamma_m(\eta)}{(\eta - \eta_n)(\partial\gamma_m(\eta)/\partial\eta)_{\eta=\eta_n}} \quad \ldots \quad (31)$$

This function is unity at $\eta = \eta_n$ and zero at all the other lift points. It can be written in polynomial form as follows:

$$g_n(\eta) = (b_{n0} + b_{n1}\eta + b_{n2}\eta^2 + \ldots b_{n,m-1}\eta^{m-1}) \sqrt{(1 - \eta^2)} \, . \quad \ldots \quad (32)$$

This is of course the same as Multhopp's spanwise interpolation function, and is a linear combination of the functions $\gamma_0(\eta)$, $\gamma_1(\eta) \ldots \gamma_{m-1}(\eta)$. Therefore the spanwise series for the lift distribution (equation (26)) can be replaced exactly by a series of the m interpolation functions $g_n(\eta)$ as follows :

$$l(\eta)\, c(\eta) = \sum_{n=-\frac{m-1}{2}}^{\frac{m-1}{2}} l(\eta_n) c(\eta_n) g_n(\eta) \, . \quad \ldots \quad (33)$$

The unknown coefficients in this equation are the actual values of the lift at the lift points instead of the arbitrary coefficients in equation (26).

Equations (30) and (33) can now be combined to give a double series for the variation of lift over the wing surface.

$$l(\xi, \eta) = \frac{1}{c(\eta)} \sum_{n=-\frac{m-1}{2}}^{\frac{m-1}{2}} c(\eta_n)\, g_n(\eta) \sum_{q=1}^{p} l(\xi_q, \eta_n)\, h_q(\xi) \quad \ldots \quad (34)$$

The unknown coefficients in this series are the lift values at the lift points and it should be noted that the interpolation function $h_q(\xi)$ can have any one of its four alternative forms at any spanwise station. This makes it possible to deal with wings having edges which are subsonic for part of the span and supersonic for the rest of the span.

Equation (34) can be used directly in the form given. However, it is convenient to take the analysis a step further. The pressure distribution is usually required merely in order to be able to calculate the total lift or pitching moment or some other generalised force. If the wing has a general deflection $z(x,y)$ downwards then the generalised force Q corresponding to the generalised co-ordinate q will be

$$-Q = \frac{1}{2}\rho V^2 \int_S \int l(x,y) \frac{\partial z}{\partial q}(x,y)\, dx\, dy \, . \quad \ldots \quad (35)$$

Substituting the lift distribution of equation (34) into this expression gives

$$-Q = \frac{1}{2}\rho V^2 b \sum_{\frac{-m-1}{2}}^{\frac{m-1}{2}} \sum_{1}^{p} l(\xi_q, \eta_n)\, c(\eta_n) \frac{1}{2}\int_{-1}^{1} g_n(\eta) \left\{ \frac{1}{2}\int_{-1}^{1} h_q(\xi) \frac{\partial z}{\partial q}(\xi, \eta)\, d\xi \right\} d\eta \quad \ldots \quad (36)$$

Considering the bracketed integral first of all we will define the average value of $h_q(\xi)$ as follows :

$$H_q = \frac{1}{2}\int_{-1}^{1} h_q(\xi)\, d\xi \, . \quad \ldots \quad (37)$$

Now it is shown in Appendix III that when $\partial z/\partial q(\xi, \eta)$ is a polynomial expression in ξ of order not greater than p, we have

$$\frac{1}{2}\int_{-1}^{1} h_q(\xi) \frac{\partial z}{\partial q}(\xi, \eta)\, d\xi = H_q \frac{\partial z}{\partial q}(\xi_q, \eta) \, . \quad \ldots \quad (38)$$

In the spanwise direction the average value of $g_n(\eta)$ can similarly be defined by

$$G_n = \frac{1}{2}\int_{-1}^{1} g_n(\eta)\, d\eta \,. \qquad (39)$$

Then in a similar fashion to the chordwise case it is found that when $\partial z/\partial q(\xi_q, \eta)$ is a polynomial expression in η of order not greater than m we have

$$\frac{1}{2}\int_{-1}^{1} g_n(\eta) \frac{\partial z}{\partial q}(\xi_q, \eta)\, d\eta = G_n \frac{\partial z}{\partial q}(\xi_q, \eta_n) \,. \qquad (40)$$

Then equation (36) becomes

$$-Q = \frac{1}{2}\rho V^2 b \sum_{\frac{-m-1}{2}}^{\frac{m-1}{2}} \sum_{1}^{p} H_q G_n\, c(\eta_n)\, l(\xi_q, \eta_n) \frac{\partial z}{\partial q}(\xi_q, \eta_n) \,. \qquad (41)$$

Now a set of non-dimensional quantities $P_{\substack{n\\q}}$ will be defined as follows :

$$P_{\substack{n\\q}} = H_q G_n \frac{c(\eta_n)}{\bar{c}}\, l(\xi_q, \eta_n) \,. \qquad (42)$$

Then in terms of $P_{\substack{n\\q}}$ equation (41) becomes

$$-Q = \frac{1}{2}\rho V^2 S \sum_{\frac{-m-1}{2}}^{\frac{m-1}{2}} \sum_{1}^{p} P_{\substack{n\\q}} \frac{\partial z}{\partial q}(\xi_q, \eta_n) \,. \qquad (43)$$

It should be emphasised that equation (43) (and also of course equation (41)) is correct only when $\partial z/\partial q\ (\xi, \eta)$ is a double polynomial in ξ and η of order not greater than p and m respectively. With this restriction, however, equation (43) is exactly what we should get if the pressure distribution were replaced by a set of discrete lift forces of magnitude $\frac{1}{2}\rho V^2 S P_{\substack{n\\q}}$ at the lift points. It is therefore preferable from an engineering point of view to express the lift in terms of the non-dimensional discrete lift forces $P_{\substack{n\\q}}$ rather than the values $l(\xi_q, \eta_n)$.

Equation (34) will therefore become

$$l(\xi, \eta) = \frac{\bar{c}}{c(\eta)} \sum_{\frac{-(m-1)}{2}}^{\frac{m-1}{2}} \sum_{1}^{p} P_{\substack{n\\q}} \frac{h_q(\xi)}{H_q} \frac{g_n(\eta)}{G_n} \,. \qquad (44)$$

If the pressure values at the lift points are required they can be obtained from equation (42) when the values of $P_{\substack{n\\q}}$ have been found. The only restriction on the use of the discrete lift forces when they have been found is when $\partial z/\partial q\ (x,y)$ is a discontinuous function (as for instance when a control surface hinge moment is being considered). The general question of discontinuities in both downwash and deflection, however, is dealt with in detail in a separate Section of this report.

5. *The Matrix Equation.*—Equation 12 is the integral equation which gives the downwash in terms of the lift. It will be put in the non-dimensional co-ordinates (ξ, η) and the downwash measured at the point (ξ_r, η_v) as follows :

$$\alpha(\xi_r, \eta_v) = \frac{-1}{8\pi b} \int_{-1}^{1} \frac{c(\eta)}{(\eta - \eta_v)^2} \int_{-1}^{1} l(\xi, \eta) K\left\{(x_{rv} - x), (y_v - y)\right\} d\xi\, d\eta \,. \qquad (45)$$

The co-ordinates of which K is a function have been left in the dimensional form for convenience. The pressure distribution given in equation (44) can now be substituted in equation (45). This gives

$$\alpha(\xi_r, \eta_\nu) = \sum_{\frac{-m-1}{2}}^{\frac{m-1}{2}} \sum_{1}^{p} \frac{-1}{2\pi A}$$

$$\left[\frac{1}{2G_n}\int_{-1}^{1}\frac{g_n(\eta)}{(\eta-\eta_\nu)^2}\left\{\frac{1}{2H_q}\int_{-1}^{1} h_q(\xi) K\left\{(x_{r\nu}-x),(y_\nu-y)\right\}d\xi\right\}d\eta\right] P_{\substack{n\\q}} \quad \ldots \quad (46)$$

This is a set of linear simultaneous equations which can be put into matrix form. In general the kernel K will be complex (for unsteady motion) and we will therefore define the following matrix element in complex form

$$A_{\substack{\nu n\\rq}} - i\nu B_{\substack{\nu n\\rq}} = \frac{-1}{2\pi A}\left[\frac{1}{2G_n}\int_{-1}^{1}\frac{g_n(\eta)}{(\eta-\eta_\nu)^2}\left\{\frac{1}{2H_q}\int_{-1}^{1} h_q(\xi) K\left\{(x_{r\nu}-x),(y_\nu-y)\right\}d\xi\right\}d\eta\right]. \quad (47)$$

Then the matrix equation corresponding to equation (46) will be

$$\left[\alpha_{\substack{\nu\\r}}\right] = \left[A_{\substack{\nu n\\rq}} - i\nu B_{\substack{\nu n\\rq}}\right]\left[P_{\substack{n\\q}}\right]. \quad \ldots \quad (48)$$

The problem involved next will be the evaluation of the elements of A and B from equation (47).

6. *The Chordwise Integration.*—The chordwise part of the integral involved in equation (47) is denoted

$$\bar{K}_{\substack{\nu\\rq}}(\eta) = \frac{1}{2H_q}\int_{-1}^{1} h_q(\xi)\, K\left\{(x_{r\nu}-x),(y_\nu-y)\right\}d\xi. \quad \ldots \quad (49)$$

We are concerned only with the values for $\eta = \eta_\nu$ and these will be denoted

$$\bar{K}_{\substack{\nu n\\rq}} = \frac{1}{2H_q}\int_{-1}^{1} h_q(\xi)\, K\left\{(x_{r\nu}-x),(y_\nu-y_n)\right\}d\xi. \quad \ldots \quad (50)$$

Fig. 2 shows how the kernel is zero over a considerable region in the supersonic case. In the chordwise direction therefore the kernel is finite only in the range $x < x_{r\nu} - |y_\nu - y_n|\sqrt{(M^2-1)}$. This restriction applies also to the subsonic case when $n = \nu$. We will therefore restrict the range of integration to a region denoted by $-1 < \xi < \xi_A$ (where ξ_A denotes either the edge of the Mach cone $x = x_{r\nu} - |y_\nu - y_n|\sqrt{(M^2-1)}$ or the trailing edge $\xi = 1$, whichever is the less).

Then equation (50) may be written

$$\bar{K}_{\substack{\nu n\\rq}} = \frac{1}{2H_q}\int_{-1}^{\xi_A} h_q(\xi)\, K\left\{(x_{r\nu}-x),(y_\nu-y_n)\right\}d\xi. \quad \ldots \quad (51)$$

We will now introduce a co-ordinate ζ which is -1 at the leading edge ($\xi = -1$) and is $+1$ at $\xi = \xi_A$:

$$\zeta = \frac{1-\xi_A+2\xi}{1+\xi_A}. \quad \ldots \quad (52)$$

Then equation (51) becomes

$$\bar{K}_{\substack{\nu n\\rq}} = \left(\frac{1+\xi_A}{2}\right)\frac{1}{2H_q}\int_{-1}^{1} h_q\left\{\left(\frac{1+\xi_A}{2}\right)\zeta - \left(\frac{1-\xi_A}{2}\right)\right\}K\left\{(x_{r\nu}-x),(y_\nu-y_n)\right\}d\zeta. \quad (53)$$

When $\zeta = -1$ the interpolation function h_q has values of the order $1/\sqrt{(1+\zeta)}$ or 1, and the kernel K has values of the order 1. Therefore the product $h_q K$ will have values of the order $1/\sqrt{(1+\zeta)}$ or 1.

When $\zeta = +1$ the interpolation function h_q has values of the order $\sqrt{(1-\zeta)}$ or 1, and the kernel K has values of the order 1 or $1/\sqrt{(1-\zeta)}$. Therefore the product $h_q K$ will have values of the order $\sqrt{(1-\zeta)}$ or 1 or $1/\sqrt{(1-\zeta)}$.

These end conditions are shown in the sketch graphs of Fig. 3, and we will define a function $k(\zeta)$ with six alternative values to cater for these various conditions.

$$\left.\begin{aligned} k(\zeta) &= \frac{1}{\sqrt{(1-\zeta)}} \quad \text{or} \quad \frac{1}{\sqrt{(1+\zeta)}} \quad \text{or} \quad \sqrt{(1-\zeta)} \\ \text{or} &\quad \sqrt{\left(\frac{1-\zeta}{1+\zeta}\right)} \quad \text{or} \quad 1 \quad \text{or} \quad \frac{1}{\sqrt{(1-\zeta^2)}} \end{aligned}\right\} \quad \ldots \qquad (54)$$

We can now approximate to the product $h_q K$ by the series

$$h_q\left\{\left(\frac{1+\xi_A}{2}\right)\zeta - \left(\frac{1-\xi_A}{2}\right)\right\} K\left\{x_{rv} - x), (y_v - y)\right\} = \left((a_0 + a_1\zeta + \ldots + a_{a-1}\zeta^{a-1}\right) k(\zeta) \, . \qquad (55)$$

The function $k(\zeta)$ must be picked very carefully for any particular case with the help of Fig. 3. We must now substitute equation (55) into equation (53) and integrate. To do this we will make use of the idea of the interpolation function which was so successfully used in an earlier Section of this report. We will define a function $\mu_\lambda(\zeta)$ as follows :

$$\mu_\lambda(\zeta) = (a_{\lambda 0} + a_{\lambda 1}\zeta + a_{\lambda 2}\zeta^2 + \ldots a_{\lambda\lambda}\zeta^\lambda)\, k(\zeta) \, . \quad \ldots \qquad (56)$$

The coefficients $a_{\lambda 0}$, $a_{\lambda 1}$, etc., are chosen so that the lift, first moment, second moment and so on up to the $(\lambda - 1)$th moment are all zero for the range $-1 < \zeta < 1$. Then by an analysis which is exactly the same as that used to develop the lift points we can determine a set of 'special' points ζ_b from the following equation :

$$a_{a0} + a_{a1}\zeta_b + a_{a2}\zeta_b^2 + \ldots + a_{aa}\zeta_b^a = 0 \, . \quad \ldots \qquad (57)$$

Then following the same procedure as that used to determine the spanwise and chordwise interpolation functions we can obtain an interpolation function $i_b(\zeta)$ which is unity at the 'special' point ζ_b and zero at all the other $(a - 1)$ 'special' points.

$$i_b(\zeta) = (b_{b0} + b_{b1}\zeta + b_{b2}\zeta^2 + \ldots + b_{b,a-1}\zeta^{a-1})\, k(\zeta) \, . \quad \ldots \qquad (58)$$

For four of the alternative values ($k(\zeta) = \sqrt{(1-\zeta/1+\zeta)}$ or $1/\sqrt{(1+\zeta)}$, $\sqrt{(1-\zeta)}$ or 1) the above interpolation function can be obtained from the values of $h_q(\xi)$ already determined by substituting ζ for ξ. For the fifth alternative ($k(\zeta) = 1/\sqrt{(1-\zeta)}$ $i_b(\zeta)$ can be obtained from $h_q(\xi)$ (the case involving $f(\xi) = 1/\sqrt{(1+\xi)}$) by substituting $-\zeta$ for ξ. Only in sixth and final alternative ($k(\zeta) = 1/\sqrt{(1-\zeta^2)}$ does additional work need to be done and this case is dealt with in Appendix IV.

Now equation (55) may be written

$$\begin{aligned} & h_q\left\{\left(\frac{1+\xi_A}{2}\right)\zeta - \left(\frac{1-\xi_A}{2}\right)\right\} K\left\{(x_{rv} - x), (y_v - y_n)\right\} \\ &= \sum_{b=1}^{a} h_q\left\{\left(\frac{1+\xi_A}{2}\right)\zeta_b - \left(\frac{1-\xi_A}{2}\right)\right\} K\left\{(x_{rv} - x_{bn}), (y_v - y_n)\right\} i_b(\zeta) \, . \quad \ldots \qquad (59) \end{aligned}$$

Then equation (53) will become

$$\bar{K}_{\substack{\nu n\\ rq}} = \sum_{b=1}^{a} \left(\frac{1+\xi_A}{2}\right) \frac{1}{H_q} h_q \left\{\left(\frac{1+\xi_A}{2}\right)\zeta_b - \left(\frac{1-\xi_A}{2}\right)\right\}$$

$$K\left\{(x_{r\nu} - x_{bn}), (y_\nu - y_n)\right\} \frac{1}{2}\int_{-1}^{1} i_b(\zeta)\, d\zeta \,. \qquad (60)$$

We can denote

$$I_b = \frac{1}{2}\int_{-1}^{1} i_b(\zeta)\, d\zeta \,. \qquad (61)$$

Then we have

$$\bar{K}_{\substack{\nu n\\ rq}} = \sum_{b=1}^{a} \left(\frac{1+\xi_A}{2}\right) \frac{I_b}{H_q} h_q \left\{\left(\frac{1+\xi_A}{2}\right)\zeta_b - \left(\frac{1-\xi_A}{2}\right)\right\} K\left\{(x_{r\nu} - x_{bn}), (y_\nu - y_n)\right\}. \qquad (62)$$

Furthermore, since equation (61) holds for polynomials of greater order than that involved in $i_b(\zeta)$ (*see* for example equations (37) and (38)) the expression above for $\bar{K}_{\substack{\nu n\\ rq}}$ holds true when equation (55) is replaced by the better approximation :

$$h_q \left\{\left(\frac{1+\xi_A}{2}\right)\zeta - \left(\frac{1-\xi_A}{2}\right)\right\} K\left\{(x_{r\nu} - x), (y_\nu - y_n)\right\}$$

$$= (a_0 + a_1\zeta + \ldots + a_{2a-1}\zeta^{2a-1})\, k(\zeta)\,. \qquad (63)$$

In general the value of a should be at least as great as the value of p.

For the particular cases shown in (b) and (c) of Fig. 3 when $a = p$ equation (62) reduces to

$$\bar{K}_{\substack{\nu n\\ rq}} = K\{x_{r\nu} - x_{pn}), (y_\nu - y_n)\}\,. \qquad (64)$$

For the other cases equation (64) is a fairly near approximation for points which are not near to the singularity at $\xi = \xi_A$.

7. *The Spanwise Integration.*—If we substitute the definition of $\bar{K}_{\substack{\nu\\ rq}}(\eta)$ from equation (49) into the surface integral of equation (47) we will have

$$A_{\substack{\nu n\\ rq}} - i\nu B_{\substack{\nu n\\ rq}} = \frac{-1}{2\pi A}\left[\frac{1}{2G_n}\int_{-1}^{1} \frac{g_n(\eta)}{(\eta - \eta_\nu)^2} \bar{K}_{\substack{\nu\\ rq}}(\eta)\, d\eta\right]. \qquad (65)$$

First of all it should be noted from Fig. 2 that (in the supersonic case only) this spanwise integral does not usually extend over the full span. Therefore equation (65) will be written

$$A_{\substack{\nu n\\ rq}} - i\nu B_{\substack{\nu n\\ rq}} = \frac{-1}{2\pi A}\left[\frac{1}{2G_n}\int_{\eta_A}^{\eta_B} \frac{g_n(\eta)}{(\eta - \eta_\nu)^2} \bar{K}_{\substack{\nu\\ rq}}(\eta)\, d\eta\right]. \qquad (66)$$

Then it must be noted that strictly $\bar{K}_{\substack{\nu\\ rq}}(\eta)$ has a logarithmic singularity at $\eta = \eta_\nu$ and should be written

$$\bar{K}_{\substack{\nu\\ rq}}(\eta) = \bar{K}_{\substack{\nu\\ rq}}{}^*(\eta) + L_{\substack{\nu\\ rq}}(\eta - n_\nu)^2 \log|\eta - \eta_\nu|\,. \qquad (67)$$

The term $L_{\substack{\nu\\ rq}}$ is dealt with in Appendix V. We can now modify equation (66) to read

$$A_{\substack{\nu n\\ rq}} - i\nu B_{\substack{\nu n\\ rq}} = \frac{-1}{2\pi A}\left[\frac{1}{2G_n}\int_{\eta_A}^{\eta_B} \frac{g_u(\eta)}{(\eta - \eta_\nu)^2} \bar{K}_{\substack{\nu\\ rq}}{}^*(\eta)\, d\eta + \frac{L_{\substack{\nu\\ rq}}}{2G_n}\int_{\eta_A}^{\eta_B} g_n(\eta) \log|\eta - \eta_\nu|\, d\eta\right]. \qquad (68)$$

When $n \neq \nu$ there is no singularity in the second integral of equation (68) since

$$g_n(\eta) \log |\eta - \eta_\nu| = 0 \text{ when } \eta = \eta_\nu .$$

Therefore the form of the integral given by equation (66) can be used. We will write

$$g_n(\eta) \underset{rq}{\bar{K}_\nu}(\eta) = j_n(\eta) \underset{rq}{\bar{K}_{\nu n}} \quad \ldots \quad (69)$$

The function $j_n(\eta)$ has similar properties to $g_n(\eta)$ except that it applies only to values of η in the range $\eta_A < \eta < \eta_B$. It is unity when $\eta = \eta_n$ and zero at all the other spanwise stations in the range defined. It is thus an interpolation function, similar in many ways to $h_q(\xi)$, $g_n(\eta)$ and $i_b(\zeta)$. It differs from these, however, in that the stations used are not defined as special intervals in the range from η_A to η_B (except for the case when $\eta_A = -1$ and $\eta_B = +1$). No special theorems exist therefore which simplify the integration (with the exception of the special case defined in the last sentence). A further discussion of the function $j_n(\eta)$ will be given shortly. In the meantime, however, we can substitute equation (69) into equation (66) giving

$$\underset{rq}{A_{\nu n}} - i\nu \underset{rq}{B_{\nu n}} = \frac{-1}{2\pi A} \left\{ \frac{1}{2G_n} \int_{\eta_A}^{\eta_B} \frac{j_n(\eta)}{(\eta - \eta_\nu)^2} d\eta \right\} \underset{rq}{\bar{K}_{\nu n}} . \quad \ldots \quad (70)$$

When $n = \nu$ the alternative form (equation (68)) of equation (66) must be used since the second term has a true logarithmic singularity in it. We can write

$$g_\nu(\eta) \underset{rq}{\bar{K}_\nu}{}^*(\eta) = j_\nu(\eta) \underset{\nu q}{K_{\nu\nu}} \quad \ldots \quad (71)$$

(since $\underset{rq}{\bar{K}_{\nu\nu}}{}^* = \underset{rq}{\bar{K}_{\nu\nu}}$ from equation (67)).

Therefore we can write

$$\underset{rq}{A_{\nu\nu}} - i\nu \underset{rq}{B_{\nu\nu}} = \frac{-1}{2\pi A} \left[\left\{ \frac{1}{2G_\nu} \int_{\eta_A}^{\eta_B} \frac{j_\nu(\eta)}{(\eta - \eta_\nu)^2} d\eta \right\} \underset{rq}{\bar{K}_{\nu\nu}} + \left\{ \frac{1}{2G_\nu} \int_{\eta_A}^{\eta B} g_\nu(\eta) \log |\eta - \eta_\nu| d\eta \right\} \underset{rq}{L_\nu} \right] . \quad (72)$$

Now we must consider the interpolation function $j_n(\eta)$ in more detail. When $\eta_A = -1$ (*i.e.*, the range of integration extends to the port tip) the factor $\sqrt{(1 + \eta)}$ must be included in $j_n(\eta)$ so that it becomes zero in the proper manner at this tip. Similarly when $\eta_B = +1$ the factor $\sqrt{(1 - \eta)}$ must be included in $j_n(\eta)$.

When one of these limits of integration is not at the tip it is because the Mach cone crosses the leading edge at this point (*see* Fig. 2). If the limit is at a subsonic edge then the value of $j_n(\eta)$ must remain finite at this limit. This can be shown to be due to the singularities of both the kernel and the leading-edge pressure becoming coincident at this point. However, when the limit is at a supersonic leading edge the value of $j_n(\eta)$ must be zero at this limit. Typical examples of the spanwise integrand are shown in Fig. 4.

Therefore in general we can write the interpolation function $j_n(\eta)$ in the following form :

$$j_n(\eta) = (b_{n0} + b_{n1}\eta + b_{n2}\eta^2 + \ldots + b_{n,m^*-1}\eta^{m^*-1})\, e(\eta) . \quad \ldots \quad (73)$$

Where m^* is the number of spanwise stations falling within the range $\eta_A < \eta < \eta_B$, and the function $e(\eta)$ has various alternative values depending on the conditions at η_A and η_B as follows

$$\left. \begin{array}{l} e(\eta) = \sqrt{(1 - \eta^2)} \text{ or } \sqrt{(1 + \eta)} \text{ or } \sqrt{(1 - \eta)} \\ \quad \text{or } (\eta - \eta_A)\sqrt{(1 - \eta)} \text{ or } (\eta - \eta_A)(\eta_B - \eta) \text{ or } (\eta - \eta_A) \\ \quad \text{or } (\eta_B - \eta)\sqrt{(1 + \eta)} \text{ or } 1 \text{ or } (\eta_B - \eta) \end{array} \right\} . \quad \ldots \quad (74)$$

The numerical calculation of the function $j_n(\eta)$ is given in Appendix VI, which also gives the method for calculating the following integrals:

$$\left.\begin{aligned} J_{\underset{r}{\nu n}}{}^* &= \frac{1}{2G_n}\int_{\eta_A}^{\eta_B} \frac{j_n(\eta)}{(\eta - \eta_\nu)^2}\, d\eta \\ G_{\underset{r}{\nu\nu}}{}^* &= \frac{1}{2G_\nu}\int_{\eta_A}^{\eta_B} g_\nu(\eta) \log|\eta - \eta_\nu|\, d\eta \end{aligned}\right\} \qquad (75)$$

Equations (70) and (72) then become

$$\left.\begin{aligned} A_{\underset{rq}{\nu n}} - i\nu B_{\underset{rq}{\nu n}} &= \frac{-1}{2\pi A}\left[J_{\underset{r}{\nu n}}{}^* \bar{K}_{\underset{rq}{\nu n}}\right] \qquad \text{when } n \neq \nu \\ A_{\underset{rq}{\nu\nu}} - i\nu B_{\underset{rq}{\nu\nu}} &= \frac{-1}{2\pi A}\left[J_{\underset{r}{\nu\nu}}{}^* \bar{K}_{\underset{rq}{\nu\nu}} + G_{\underset{r}{\nu\nu}}{}^* L_{\underset{rq}{\nu}}\right] \end{aligned}\right\} \qquad (76)$$

We now have the elements of the downwash matrix and must proceed to solve the matrix equation.

8. *The Solution of The Matrix Equation.*—Equation (48) is the matrix equation which gives the downwash in terms of the loads. The solution of this equation will clearly be given by

$$\left[P_{\underset{q}{n}}\right] = \left[A_{\underset{rq}{\nu n}} - i\nu B_{\underset{rq}{\nu n}}\right]^{-1} \left[\alpha_{\underset{r}{\nu}}\right]. \qquad (77)$$

This will be written as follows :

$$\left[P_{\underset{q}{n}}\right] = \left[F_{\underset{qr}{n\nu}} + i\nu G_{\underset{qr}{n\nu}}\right] \left[\alpha_{\underset{r}{\nu}}\right]. \qquad (78)$$

By matrix manipulation we can find the following expressions for the real and imaginary parts of this inverse matrix. For convenience the brackets and suffixes have been omitted from the matrices in the next two equations.

$$\left.\begin{aligned} F &= (A + \nu^2 BA^{-1}B)^{-1} \\ G &= (AB^{-1}A + \nu^2 B)^{-1} \end{aligned}\right\}. \qquad (79)$$

When $\nu \to 0$ these become

$$\left.\begin{aligned} F &= A^{-1} \\ G &= A^{-1}BA^{-1} \end{aligned}\right\}. \qquad (80)$$

9. *The Generalised Forces.*—For flutter calculations we need to find the generalised force Q appropriate to a generalised co-ordinate q. Let the downward deflection of the wing be denoted z where

$$\left.\begin{aligned} z &= \bar{c}\left[Z_t\right]\left[q_t\right] \\ \frac{\partial z}{\partial x} &= \left[Z_t'\right]\left[q_t\right] \end{aligned}\right\}. \qquad (81)$$

Then the boundary condition for the downwash will be

$$\alpha = \left[Z_t' + i\nu Z_t\right]\left[q_t\right]. \qquad (82)$$

Let the generalised force Q_s (appropriate to the generalised co-ordinate q_s) be given by

$$\left[-Q_s\right] = V^2\left[c_{st}\right]\left[q_t\right] + V\left[b_{st}\right]\left[\dot{q}_t\right]. \qquad (83)$$

Then $[b_{st}]$ and $[c_{st}]$ are matrices of aerodynamic damping and stiffness coefficients respectively. These flutter coefficients can be found by substituting the boundary condition from equation (82) into the matrix equation (78) and then finding the generalised force from equation (43). This gives

$$\begin{bmatrix} c_{st} \end{bmatrix} = \tfrac{1}{2}\rho S\bar{c} \begin{bmatrix} Z_{sn} \\ q \end{bmatrix} \begin{bmatrix} F_{nv} \\ qr \end{bmatrix} \begin{bmatrix} Z_{vt}{}' \\ r \end{bmatrix} - \nu^2 \tfrac{1}{2}\rho S\bar{c} \begin{bmatrix} Z_{sn} \\ q \end{bmatrix} \begin{bmatrix} G_{nv} \\ qr \end{bmatrix} \begin{bmatrix} Z_{vt} \\ r \end{bmatrix} . \quad \ldots \quad (84)$$

$$\begin{bmatrix} b_{st} \end{bmatrix} = \tfrac{1}{2}\rho S\bar{c}^2 \begin{bmatrix} Z_{sn} \\ q \end{bmatrix} \begin{bmatrix} F_{nv} \\ qr \end{bmatrix} \begin{bmatrix} Z_{vt} \\ r \end{bmatrix} + \tfrac{1}{2}\rho S\bar{c}^2 \begin{bmatrix} Z_{sn} \\ q \end{bmatrix} \begin{bmatrix} G_{nv} \\ qr \end{bmatrix} \begin{bmatrix} Z_{vt}{}' \\ r \end{bmatrix} . \quad \ldots \quad (85)$$

10. *The Effect of Kinks in the Plan-form.*—The effect of discontinuities in the plan-form can be treated quite simply by using Multhopp's rounding-off rule. In the supersonic case, however, care must be exercised to see that the proper chordwise functions are used at these stations. This is because the local sweep angle of the leading or trailing edge will be altered by the rounding-off process.

As an example we can consider the central section ($n = 0$) of a swept wing. When the rounding-off rule is introduced the resultant modified wing will have no sweep angle on either the leading or trailing edge at this station. Therefore in supersonic flow the local leading and trailing edges will be supersonic and the downwash and lift points at this central section must be chosen with this in mind.

11. *The Treatment of Control Surfaces.*—The only feature which distinguishes a control-surface mode from an elastic mode or body freedom is that the deflection and downwash angle are discontinuous functions instead of being expressible as polynomials in x and y. In Ref. 5 the equivalent slope techniques of Falkner and De Young were generalised to give equivalent continuous deflections and downwash angles for the subsonic case.

The results obtained for the subsonic case can be generalised into results applicable to all four types of chordwise condition. A proof of these expressions will not be developed in this report, but an analysis is given in Appendix VII which shows that the subsonic results given in Ref. 5 can be developed into the form given in equations (86) and (87).

In the chordwise direction the equivalent continuous deflections and downwash angles are given by

$$\left. \begin{aligned} \frac{\partial z_e}{\partial q}(\xi_q) &= \frac{1}{2H_q}\int_{-1}^{1} h_q(\xi)\,\frac{\partial z}{\partial q}(\xi)\,d\xi \\ \alpha_e(\xi_r) &= \frac{1}{2H_{p+1-r}}\int_{-1}^{1} h_{p+1-r}(-\,\xi)\alpha(\xi)\,d\xi \end{aligned} \right\} . \quad \ldots \quad (86)$$

The function $h_{p+1-r}(-\,\xi)$ is of course simply the reverse-flow version of the function $h_q(\xi)$, and equation (86) (which applies to all four types of chordwise function) is really just a generalisation of the various reverse-flow theorems which occur in aerodynamics. The expressions are applicable to any type of discontinuous deflection or downwash angle whether due to a control surface mode or not. It can be seen that if $\partial z/\partial q(\xi)$ or $\alpha(\xi)$ in equation (86) already exhibits the continuous properties which are required (*i.e.*, if they are polynomials in ξ of order not greater than p) then the equivalent and true values will be identical. This can be seen by comparing equation (86) with equation (38).

The spanwise version of equation (86) is

$$\left. \begin{aligned} \frac{\partial z_a}{\partial q}(\eta_n) &= \frac{1}{2G_n}\int_{-1}^{1} g_n(\eta)\,\frac{\partial z}{\partial q}(\eta)\,d\eta \\ \alpha_e(\eta_\nu) &= \frac{1}{2G_\nu}\int_{-1}^{1} g_\nu(\eta)\,\alpha(\eta)\,d\eta \end{aligned} \right\} . \quad \ldots \quad (87)$$

This is a general form of the spanwise equations given in Ref. 5 and it can be seen that the equivalent deflection and downwash angle are given by identical formulae because of the spanwise symmetry of the equations.

NOTATION

x, y, z	Rectangular co-ordinates system attached to the wing, origin at the vertex
x	Positive rearwards from the vertex
y	In the starboard direction
z	Positive downwards
p	Pressure } of the flow about the wing
ρ	Density } of the flow about the wing
a	Speed of sound } of the flow about the wing
u, v, w	Perturbation velocity components, in the direction x, y, z respectively
X, Y, Z, T, τ, r, R	*See* equation (7)
V	Velocity of undisturbed flow relative to wing
M	(V/a) Mach number
ϖ	Circular frequency
$l(x, y)$	Non-dimensional load distribution defined by equation (11)
$\alpha(x_0 y_0)$	Wing incidence at $(x_0 y_0) = w/V$
$K(X, Y)$	Kernel function defined by equation (13)
$\xi = \dfrac{2(x - x_{1/2c_\eta})}{c_{(\eta)}}$, $\eta = \dfrac{2y}{b}$	Non-dimensional wing co-ordinates related to the inducing wing section
$c(\eta)$	Local wing chord
$f(\xi)$	Chordwise lift distribution having edge conditions defined by equation (14)
$l_\lambda(\xi)$	$f(\xi) \times$ polynomial in ξ of order λ
$l(\xi)$	Chordwise pressure distribution defined by equation (17)
$\alpha(\xi)$	Chordwise downwash angle distribution defined by equation (18)
ξ_q	Points at which the lift is evaluated
ξ_r	Points at which the downwash is evaluated
$\gamma_\lambda(\eta)$	Spanwise lift distribution in polynomial form defined by equation (23)
$\alpha_\lambda(\eta)$	Spanwise distribution of downwash incidence
$_\nu, _n$	Suffices numerating the spanwise stations, $_\nu$ giving the pivotal station, $_n$ the inducing station
$h_q(\xi)$	Chordwise interpolation function defined by equations (28) and (29)
$g_n(\eta)$	Spanwise interpolation function defined by equation (31) and (32)
Q	Generalised component of force due to mode $z(x, y)$
q	Generalised co-ordinate associated with mode $z(x, y)$

H_q	$=$	$\frac{1}{2}\int_{-1}^{1} h_q(\xi)\,d\xi$ (suffix $_q$ refers to lift points)
G_n	$=$	$\frac{1}{2}\int_{-1}^{1} g_n(\eta)\,d\eta$
b		Wing span
$\bar{c}$		Mean chord of wing
$\underset{q}{P_n}$		Non-dimensional quantity defined by equation (42)
m		Total number of spanwise stations in Multhopp's theory
S		Wing area
A		Aspect ratio
$\underset{rq}{A_{\nu n}} - i\nu \underset{rq}{B_{\nu n}}$		Matrix elements in equation giving downwash in terms of pressure (equation (47))
$\underset{rq}{\bar{K}_{\nu n}}$		Chordwise integral in expression for downwash (equation (49))
ξ_A		Edge of Mach cone $x = x_{r\nu} - \lvert y_\nu - y_n \rvert \sqrt{(M^2 - 1)}$ or the trailing edge $\xi = 1$, whichever is the less
ζ	$=$	$\frac{1 - \xi_A + 2\xi}{1 + \xi_A}$, a co-ordinate which is -1 at L.E. ($\xi = -1$) and $+1$ for $\xi = \xi_A$
ζ_b		'Special' points used for the chordwise integration (equation (57))
$k(\zeta)$		Function defining the end conditions of the product $h_q K$ (*see* equation (54))
$i_b(\zeta)$		Chordwise interpolation function for integrating the product $h_q K$
I_b	$=$	$\frac{1}{2}\int_{-1}^{1} i_b(\zeta)\,d\zeta$
$\underset{rq}{\bar{K}_\nu}{}^*(\eta)$		Regular part of $\underset{rq}{\bar{K}_\nu}(\eta)$ for $\eta = \eta_\nu$
$\underset{rq}{L_\nu}$		Coefficient of logarithmic singularity part of $\underset{rq}{\bar{K}_\nu}(\eta)$ when $\eta = \eta_\nu$ (*see* equation (67) and Appendix V)
$j_n(\eta)$		Spanwise interpolation function similar to, $g_n(\eta)$ but only for η in the range $\eta_A < \eta < \eta_B$ (*see* Fig. 2)
$e(\eta)$		Expression in $j_n(\eta)$ dependent on the boundary conditions at η_A and η_B (equation (74))
$\underset{r}{J_{\nu n}}{}^*$, $\underset{r}{G_{\nu\nu}}{}^*$		Functions defined by equation (75)
$\left[\underset{qr}{F_{n\nu}} + i\nu \underset{qr}{G_{n\nu}}\right]$	$=$	$\left[\underset{rq}{A_{\nu n}} - i\nu \underset{rq}{B_{\nu n}}\right]^{-1}$ (*see* equation (77))
$\left[Z_t\right]$		Vertical displacement of wing in mode t
$\left[b_{st}\right]$		Matrix of aerodynamic damping coefficients
$\left[c_{st}\right]$		Matrix of aerodynamic stiffness coefficients
ν		Frequency parameter

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APPENDIX I

The Kernel Function of the Integral Equation

The kernel $K(X, Y)$ is defined by equation (13). It is not possible to give a single analytical solution to the integral involved and therefore various cases are treated separately.

First of all when $Y = 0$ we have for all frequencies and for both subsonic and supersonic speeds

$$\left.\begin{aligned} K(X, 0) &= 2\cos\frac{\varpi X}{V} - 2i\sin\frac{\varpi X}{V} \quad \text{when } X > 0 \\ &= 0 \qquad\qquad\qquad\qquad\qquad\;\; \text{when } X < 0 \end{aligned}\right\} \qquad \text{(A.1)}$$

Secondly we can consider the case of limiting small frequency ($\varpi \to 0$) for all values of Y. This leads to

$$\left.\begin{aligned} K(X, Y) &= \left[1 + \frac{X}{R}\right] - \frac{i\varpi}{V}\left[X + \frac{X^2 + Y^2}{R}\right] \quad \text{when } M < 1 \\ &= \left[\frac{2X}{R}\right] - \frac{i\varpi}{V}\left[\frac{2(X^2 + Y^2)}{R}\right] \quad \text{when } M > 1 \text{ and } X > |Y|\sqrt{(M^2 - 1)} \end{aligned}\right\} \qquad \text{(A.2)}$$

These small frequency results have been evaluated by expanding the exponential in the integral to the first two terms of a power series. In the subsonic case it is not possible to take this series analysis any further because the semi-infinite range of integration leads to infinite values for the third and succeeding terms of the series. However, in the supersonic case the range of integration is finite and therefore this difficulty does not apply and it is possible to evaluate the kernel as a power series of the frequency. Of course there is a limitation to the practical use of this series at the lower supersonic speeds because as $M \to 1$ the upper limit of integration tends to infinity which makes the series converge very slowly indeed. This series (which applies for $M > 1$ and $X > |Y|\sqrt{(M^2 - 1)}$) is as follows :

$$\begin{aligned} K(X, Y) &= \left[\frac{2X}{R}\right] - \frac{i\varpi}{V}\left[\frac{2(X^2 + Y^2)}{R}\right] \\ &\quad - \frac{1}{\underline{|2}}\left(\frac{\varpi}{V}\right)^2 \left[\frac{2X}{R}\left(X^2 + \frac{2M^2 - 1}{M^2 - 1}Y^2\right) + Y^2 \sinh^{-1}\frac{2XR}{Y^2(M^2 - 1)}\right] \\ &\quad + \frac{1}{\underline{|3}}\, i \left(\frac{\varpi}{V}\right)^3 \left[\frac{2}{R}\left(X^4 + \frac{3M^4 - 1}{(M^2 - 1)^2}X^2Y^2 + \frac{-3M^2 + 2}{M^2 - 1}Y^4\right)\right. \\ &\quad \left. + 3XY^2 \sinh^{-1}\left(\frac{2XR}{Y^2(M^2 - 1)}\right)\right] + \ldots \text{etc.} \end{aligned} \qquad \text{(A.3)}$$

Any number of terms of this series may be evaluated but the expressions involved become more and more cumbersome as the expansion proceeds. It will therefore be necessary to evaluate $K(X, Y)$ numerically in most cases when $Y \neq 0$ and $\varpi \neq 0$. However, equation (13) as it stands is not in a suitable form for numerical integration. We must therefore perform the differentiation and write the exponentials as trigonometric functions.

Then for $M < 1$ we have :

$$\begin{aligned} K(X, Y) &= \left[\frac{MY^2(MX + R)}{R(X^2 + Y^2)} \cos\left(\frac{\varpi M(MX - R)}{V(1 - M^2)}\right)\right. \\ &\quad \left. + Y^2 \cos\left(\frac{\varpi X}{V}\right) \int_{\frac{-X+MR}{1-M^2}}^{\infty} \frac{\cos\left(\frac{\varpi\tau}{V}\right) d\tau}{(\tau^2 + Y^2)^{3/2}} - Y^2 \sin\left(\frac{\varpi X}{V}\right) \int_{\frac{-X+MR}{1-M^2}}^{\infty} \frac{\sin\left(\frac{\varpi\tau}{V}\right) d\tau}{(\tau^2 + Y^2)^{3/2}}\right] \\ &\quad - i\left[-\frac{MY^2(MX + R)}{R(X^2 + Y^2)} \sin\left(\frac{\varpi M(MX - R)}{V(1 - M^2)}\right)\right. \\ &\quad \left. + Y^2 \cos\left(\frac{\varpi X}{V}\right) \int_{\frac{-X+MR}{1-M^2}}^{\infty} \frac{\sin\left(\frac{\varpi\tau}{V}\right) d\tau}{(\tau^2 + Y^2)^{3/2}} + Y^2 \sin\left(\frac{\varpi X}{V}\right) \int_{\frac{-X+MR}{1-M^2}}^{\infty} \frac{\cos\left(\frac{\varpi\tau}{V}\right) d\tau}{(\tau^2 + Y^2)^{3/2}}\right]. \end{aligned} \qquad \text{(A.4)}$$

For $M = 1$ we have (for $X > 0$)

$$K(X, Y) = \left[\frac{2Y^2}{X^2 + Y^2} \cos\left(\frac{\varpi(X^2 + Y^2)}{2VX}\right)\right.$$

$$+ Y^2 \cos\left(\frac{\varpi X}{V}\right) \int_{\frac{(Y^2 - X^2)}{2X}}^{\infty} \frac{\cos\left(\frac{\varpi\tau}{V}\right) d\tau}{(\tau^2 + Y^2)^{3/2}} - Y^2 \sin\left(\frac{\varpi X}{V}\right) \int_{\frac{(Y^2 - X^2)}{2X}}^{\infty} \frac{\sin\left(\frac{\varpi\tau}{V}\right) d\tau}{(\tau^2 + Y^2)^{3/2}}\Bigg]$$

$$- i \left[\frac{2Y^2}{X^2 + Y^2} \sin\left(\frac{\varpi(X^2 + Y^2)}{2VX}\right)\right.$$

$$\left. + Y^2 \cos\left(\frac{\varpi X}{V}\right) \int_{\frac{Y^2 - X^2}{2X}}^{\infty} \frac{\sin\left(\frac{\varpi\tau}{V}\right) d\tau}{(\tau^2 + Y^2)^{3/2}} + Y^2 \sin\left(\frac{\varpi X}{V}\right) \int_{\frac{Y^2 - X^2}{2X}}^{\infty} \frac{\cos\left(\frac{\varpi\tau}{V}\right) d\tau}{(\tau^2 + Y^2)^{3/2}}\right]. \quad .. \quad \text{(A.5)}$$

Finally for $M > 1$ we have (for $X > |Y| \sqrt{(M^2 - 1)}$) :

$$K(X, Y) = \left[\frac{2M^2Y^2X}{R(X^2 + Y^2)} \cos\left(\frac{\varpi M^2 X}{V(M^2 - 1)}\right) \cos\left(\frac{\varpi MR}{V(M^2 - 1)}\right)\right.$$

$$+ \frac{2MY^2}{X^2 + Y^2} \sin \frac{\varpi M^2 X}{V(M^2 - 1)} \sin\left(\frac{\varpi MR}{V(M^2 - 1)}\right)$$

$$+ Y^2 \cos\left(\frac{\varpi X}{V}\right) \int_{\frac{X - MR}{M^2 - 1}}^{\frac{X + MR}{M^2 - 1}} \frac{\cos\left(\frac{\varpi\tau}{V}\right) d\tau}{(\tau^2 + Y^2)^{3/2}} - Y^2 \sin\left(\frac{\varpi X}{V}\right) \int_{\frac{X - MR}{M^2 - 1}}^{\frac{X + MR}{M^2 - 1}} \frac{\sin\left(\frac{\varpi\tau}{V}\right) d\tau}{(\tau^2 + Y^2)^{3/2}}\Bigg]$$

$$- i \left[\frac{2M^2Y^2X}{R(X^2 + Y^2)} \sin\left(\frac{\varpi M^2 X}{V(M^2 - 1)}\right) \cos\left(\frac{\varpi MR}{V(M^2 - 1)}\right)\right.$$

$$- \frac{2MY^2}{X^2 + Y^2} \cos\left(\frac{\varpi M^2 X}{V(M^2 - 1)}\right) \sin\left(\frac{\varpi MR}{V(M^2 - 1)}\right)$$

$$\left. + Y^2 \cos\left(\frac{\varpi X}{V}\right) \int_{\frac{X - MR}{M^2 - 1}}^{\frac{X + MR}{M^2 - 1}} \frac{\sin\left(\frac{\varpi\tau}{V}\right) d\tau}{(\tau^2 + Y^2)^{3/2}} + Y^2 \sin\left(\frac{\varpi X}{V}\right) \int_{\frac{X - MR}{M^2 - 1}}^{\frac{X + MR}{M^2 - 1}} \frac{\cos\left(\frac{\varpi\tau}{V}\right) d\tau}{(\tau^2 + Y^2)^{3/2}}\right]. \quad .. \quad \text{(A.6)}$$

Equations (A.4), (A.5) and (A.6) all contain the two integrals

$$Y^2 \int \left[\cos\left(\frac{\varpi\tau}{V}\right) \Big/ (\tau^2 + Y^2)^{3/2}\right] d\tau \text{ and } Y^2 \int \left[\sin\left(\frac{\varpi\tau}{V}\right) \Big/ (\tau^2 + Y^2)^{3/2}\right] d\tau$$

with various limits of integration. These must be evaluated numerically by some suitable method. This is possible even in the case $M \leqslant 1$ (which involves an infinite limit) since the integrand tends rapidly to zero as this limit is approached.

The equations in this Appendix give the real and imaginary parts of $K(X, Y)$ separately in a form suitable for substitution in equation (62). However, the equations have been given in dimensional form involving ϖ/V rather than $\nu = \varpi\bar{c}/V$. For the non-dimensional form the values of X, Y and R must be given in terms of $\bar{c}$.

APPENDIX II

Two-dimensional Lift and Downwash

In Section 3 of this report the downwash due to various two-dimensional lift distributions is given. These are all derived from two-dimensional aerofoil theory and there are five cases to consider :

(1) The chordwise case when both edges are subsonic can be derived from two-dimensional subsonic steady theory.

(2) The chordwise case when both edges are supersonic can be derived from two-dimensional supersonic steady theory.

(3) The chordwise case when the leading edge is subsonic and the trailing edge is supersonic can be derived from two-dimensional sonic indicial lift theory when the distance travelled from the step in the downwash approaches infinity (*see* Ref. 6). This theory is used because the lift at sonic speeds continually increases with time and this never reaches a steady state.

(4) The chordwise case when the leading edge is supersonic and the trailing edge is subsonic is derived by a theory which is more difficult to justify than those above. In the other sonic case above (3) there is an additional second-order lift distribution which tends to zero as the distance travelled increases. This additional lift is zero at the leading edge and finite at the trailing edge. If therefore we use this additional lift term in reverse flow (so that the zero pressure is at the trailing edge) we will have the right sort of distribution required. Since this type of chordwise condition is not very common (occurring in the main on swept-forward wings) no attempt has been made to offer a better justification of this case.

(5) The spanwise case is independent of Mach number and can be derived from slender-wing theory.

The equations for these five cases are as follows :—

$$\alpha(\xi_0) = \frac{-\sqrt{(1-M^2)}}{4\pi}\int_{-1}^{1}\frac{l(\xi)\,d\xi}{(\xi_0-\xi)} \quad \text{(LE subsonic, TE subsonic}$$

$$\alpha(\xi_0) = \frac{\sqrt{(M^2-1)}}{4}\int_{-1}^{1} l(\xi)\,\delta(\xi_0-\xi)\,d\xi \quad \text{(LE supersonic, TE supersonic)} \qquad \ldots \quad (A.7)$$

$$= \frac{\sqrt{(M^2-1)}}{4}\,l(\xi_0)$$

(since the Dirac function $\delta(\xi_0-\xi)$ is defined by $\int l(\xi)\,\delta(\xi_0-\xi)\,d\xi = l(\xi_0)$) $\ldots$ (A.8)

$$\alpha(\xi_0) = \frac{1}{4\sqrt{(2s)}}\int_{-1}^{\xi_0}\frac{l(\xi)\,d\xi}{\sqrt{(\xi_0-\xi)}} \quad \text{(LE subsonic, TE supersonic)} \quad \ldots \quad \ldots \quad (A.9)$$

(where s = number of half chords travelled).

$$\alpha(\xi_0) = \frac{\sqrt{(2s)}}{3}\frac{\partial}{\partial\xi}\int_{\xi_0}^{1}\frac{l(\xi)\,d\xi}{\sqrt{(\xi-\xi_0)}} \quad \text{(LE supersonic, TE subsonic)} \quad \ldots \quad \ldots \quad (A.10)$$

$$\alpha(\eta_0) = \frac{-1}{4\pi}\int_{-1}^{1}\frac{\gamma(\eta)\,d\eta}{(\eta_0-\eta)^2} \quad \text{(spanwise distribution)} \quad \ldots \quad \ldots \quad \ldots \quad (A.11)$$

It can be shown by substitution of the various distributions $l_\lambda(\xi)$ and $\gamma_\lambda(\eta)$ from Section 3 of this report into the appropriate equation above that the corresponding downwash will be a polynomial in ξ (or η) of order λ.

APPENDIX III

The Integration of the Interpolation Functions

Consider the chordwise functions first of all. We have (*see* equation (15)) a loading function :

$$l_\lambda(\xi) = (a_{\lambda 0} + a_{\lambda 1}\xi + a_{\lambda 2}\xi^2 + a_{\lambda 3}\xi^3 + \ldots + a_{\lambda\lambda}\xi^\lambda) f(\xi) \,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{(A.12)}$$

Suppose that we define a polynomial function $l_\mu{}^*(\xi)$ using the same coefficients :

$$l_\mu{}^*(\xi) = (a_{\mu 0} + a_{\mu 1}\xi + a_\mu{}^1\xi^2 + \ldots + a_{\mu\mu}\xi^\mu) \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{(A.13)}$$

Then because of the way in which the coefficients in $l_\lambda(\xi)$ are defined (no lift, no pitching moment, etc.) we can show that these two functions are orthogonal :

$$\frac{1}{2}\int_{-1}^{1} l_\lambda(\xi)\, l_\mu{}^*(\xi)\, d\xi = 0 \qquad \text{when } \lambda \neq \mu \,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{(A.14)}$$

Now consider the interpolation function $h_q(\xi)$ which is derived from $l_\lambda(\xi)$ (*see* equation 29) :

$$h_q(\xi) = (b_{q0} + b_{q1}\xi + b_{q2}\xi^2 + \ldots + b_{q,\,p-1}\xi^{p-1}) f(\xi) \,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{(A.15)}$$

Suppose we define a polynomial function $h_s{}^*(\xi)$ using the same coefficients

$$h_s{}^*(\xi) = (b_{s0} + b_{s1}\xi + b_s{}^1\xi^2 + \ldots b_{s,\,p-1}\xi^{p-1}) f(\xi_s) \,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{(A.16)}$$

This new function is unity at $\xi = \xi_s$ and zero at all the other stations. Then it can be shown that because of the orthogonality of the two functions $l_\lambda(\xi)$ and $l_\mu{}^*(\xi)$ (from which $h_q(\xi)$ and $h_s{}^*(\xi)$ are derived) a similar relation holds between $h_q(\xi)$ and $h_s{}^*(\xi)$ as follows :

$$\left.\begin{aligned} \frac{1}{2}\int_{-1}^{1} h_q(\xi)\, h_s{}^*(\xi)\, d\xi &= H_q \quad \text{when } s = q \\ &= 0 \quad \text{when } s \neq q \end{aligned}\right\} . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{(A.17)}$$

Then if the deflection $\dfrac{\partial z}{\partial q}(\xi)$ is a polynomial in ξ (of the appropriate order) it can be expressed in terms of the new function $h_s{}^*(\xi)$ as follows :

$$\frac{\partial z}{\partial q}(\xi) = \sum_{s=1}^{p} \frac{\partial z}{\partial q}(\xi_s)\, h_s{}^*(\xi) \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{(A.18)}$$

Then the chordwise integral required in Section 4 of this report is

$$\frac{1}{2}\int_{-1}^{1} h_q(\xi)\, \frac{\partial z}{\partial q}(\xi)\, d\xi = \sum_{s=1}^{p} \frac{\partial z}{\partial q}(\xi_s)\, \frac{1}{2}\int_{-1}^{1} h_q(\xi)\, h_s{}^*(\xi)\, d\xi \,. \quad \ldots \quad \ldots \quad \ldots \text{(A.19)}$$

Substituting equation (A.17) in this expression gives the required result (*see* also equation (38)).

$$\frac{1}{2}\int_{-1}^{1} h_q(\xi)\, \frac{\partial z}{\partial q}(\xi)\, d\xi = H_q \frac{\partial z}{\partial q}(\xi_q) \,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{(A.20)}$$

An exactly similar analysis may be made in the spanwise direction leading to the result

$$\frac{1}{2}\int_{-1}^{1} g_n(\eta)\, \frac{\partial z}{\partial q}(\eta)\, d\eta = G_n \frac{\partial z}{\partial q}(\eta_n) \,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \text{(A.21)}$$

Chordwise Distribution

Position of 'p' Loading and 'p' Downwash Points, together with the Equivalent Interpolation Function

Subsonic Leading Edge. Supersonic Trailing Edge

$p = 1$

Loading point	Downwash point
$\xi = -0\cdot3333$	$\xi = 0\cdot3333$
$h_1(\xi) = \dfrac{0\cdot81650}{\sqrt{(1+\xi)}}$	$H_1(\xi) = 1\cdot15470$

$p = 2$

Loading points	Downwash points
$\xi_1 = -\,0\cdot76883$	$\xi_1 = -\,0\cdot48311$
$\xi_2 = 0\cdot48311$	$\xi_2 = 0\cdot76883$
$h_1(\xi) = \dfrac{1}{\sqrt{(1+\xi)}}(0\cdot18554 - 0\cdot38405\xi)$	$H_1(\xi) = 0\cdot44343$
$h_2(\xi) = \dfrac{1}{\sqrt{(1+\xi)}}(0\cdot74788 + 0\cdot97276\xi)$	$H_2(\xi) = 0\cdot59910$

$p = 3$

Loading points	Downwash points
$\xi_1 = -\,0\cdot88612$	$\xi_1 = -\,0\cdot73900$
$\xi_2 = -\,0\cdot12561$	$\xi_2 = +\,0\cdot12561$
$\xi_3 = 0\cdot73900$	$\xi_3 = 0\cdot88612$
$h_1(\xi) = \dfrac{1}{\sqrt{(1+3)}}(-\,0\cdot02535 - 0\cdot16748\xi + 0\cdot27304\xi^2)$	$H_1(\xi) = 0\cdot22331$
$h_2(\xi) = \dfrac{1}{\sqrt{(1+3)}}(0\cdot93125 - 0\cdot20922\xi - 1\cdot42210\xi^2)$	$H_2(\xi) = 0\cdot47708$
$h_3(\xi) = \dfrac{1}{\sqrt{(1+3)}}(0\cdot10446 + 0\cdot949533 + 0\cdot93853\xi^2)$	$H_3(\xi) = 0\cdot31951$

Chordwise Distribution

Position of 'p' Loading and 'p' Downwash Points together with the Equivalent Interpolation Function

Supersonic Leading and Trailing Edges

$p = 1$

Loading point	Downwash point
$\xi = 0 \cdot 0000$	$\xi = 0 \cdot 0000$
$h_1(\xi) = 1 \cdot 0000$	$H_1(\xi) = 1 \cdot 0000$

$p = 2$

Loading points	Downwash points
$\xi_1 = -0 \cdot 57735$	$\xi_1 = -0 \cdot 57735$
$\xi_2 = 0 \cdot 57735$	$\xi_2 = 0 \cdot 57735$
$h_1(\xi) = (0 \cdot 50000 - 0 \cdot 86603\xi)$	$H_1(\xi) = 0 \cdot 5000$
$h_2(\xi) = (0 \cdot 50000 + 0 \cdot 86603\xi)$	$H_2(\xi) = 0 \cdot 5000$

$p = 3$

Loading points	Downwash points
$\xi_1 = -0 \cdot 77460$	$\xi_1 = -0 \cdot 77460$
$\xi_2 = 0 \cdot 00000$	$\xi_2 = 0 \cdot 00000$
$\xi_3 = 0 \cdot 77460$	$\xi_3 = 0 \cdot 77460$
$h_1(\xi) = (-0 \cdot 64550\xi + 0 \cdot 83333\xi^2)$	$H_1(\xi) = 0 \cdot 27778$
$h_2(\xi) = (1 \cdot 00000 - 1 \cdot 66667\xi^2)$	$H_2(\xi) = 0 \cdot 4444$
$h_3(\xi) = (0 \cdot 64550\xi + 0 \cdot 83333\xi^2)$	$H_3(\xi) = 0 \cdot 27778$

APPENDIX IV

An Additional Interpolation Function

Equation (58) gives the general interpolation function $i_b(\zeta)$. When $k(\zeta) = 1/\sqrt{(1 - \zeta^2)}$ it is easier to use the angular co-ordinates used by Multhopp (as for $k(\zeta) = \sqrt{(1 - \zeta/1 + \zeta)}$ or $\sqrt{(1 - \zeta^2)}$) rather than the polynomial form which is forced upon us in the other cases.

Substitute :

$$\zeta = -\cos\phi \, . \qquad \text{(A.22)}$$

Then $\mu_\lambda(\zeta)$ in equation (56) can be written :

$$\mu_\lambda(\zeta) = \frac{\cos\lambda\phi}{\sin\phi} . \qquad \text{(A.23)}$$

The special points ζ_b then become :

$$\phi_b = \frac{\pi(2b - 1)}{2a} . \qquad \text{(A.24)}$$

Then the function $i_b(\zeta)$ becomes :

$$i_b(\phi) = \frac{1}{a}\frac{\sin\phi_b}{\sin\phi} + \frac{1}{a}\sum_{\lambda=1}^{a-1}\left[\sin(\lambda+1)\phi_b - \sin(\lambda-1)\phi_b\right]\frac{\cos\lambda\phi}{\sin\phi} , \qquad \text{(A.25)}$$

and I_b is given by :

$$I_b = \frac{1}{2}\int_0^\pi i_b(\phi)\sin\phi \, d\phi ,$$

$$= \frac{\pi}{2a}\sin\phi_b . \qquad \text{(A.26)}$$

APPENDIX V

The Logarithmic Singularity

The logarithmic singularity has been treated for the low-frequency subsonic case (Refs. 1 and 2). The supersonic case will be dealt with now. For small frequencies we have in non-dimensional co-ordinates :

$$\left.\begin{aligned} \bar{K}_{\underset{rq}{\nu}}(\eta) &= \frac{1}{2H_q}\int_{-1}^{\xi_r-\left(\frac{b}{c_\nu}\right)(|\eta_\nu-\eta|)\sqrt{(M^2-1)}} h_q(\xi)\,\frac{2(\xi_r-\xi)}{\sqrt{\{(\xi_r-\xi)^2-(M^2-1)\left(\frac{b}{c_\nu}\right)^2(\eta_\nu-\eta)^2\}}}\,d\xi \\ &\quad - \frac{i\varpi}{V}\frac{c_\nu}{2}\frac{1}{2H_q}\int_{-1}^{\xi_r-\left(\frac{b}{c_\nu}\right)|\eta_\nu-\eta|\sqrt{(M^2-1)}} h_q(\xi)\,\frac{2\left\{(\xi_r-\xi)^2+\left(\frac{b}{c_\nu}\right)^2(\eta_\nu-\eta)^2\right\}}{\sqrt{\{(\xi_r-\xi)^2-(M^2-1)\left(\frac{b}{c_\nu}\right)^2(\eta_\nu-\eta)^2\}}}\,d\xi \end{aligned}\right\} . \quad \text{(A.27)}$$

We will now expand $h_q(\xi)$ as a Taylor series from the point ξ_r as follows :

$$h_q(\xi) = h_q(\xi_r) - (\xi_r-\xi)\frac{\partial h_q}{\partial \xi}(\xi_r) + \tfrac{1}{2}(\xi_r-\xi)^2\frac{\partial^2 h_q}{\partial \xi^2}(\xi_r) + \ldots \quad \text{(A.28)}$$

Substituting this in the expression for $\bar{K}_{\underset{rq}{\nu}}(\eta)$ we have :

$$\begin{aligned} \bar{K}_{\underset{rq}{\nu}}(\eta) &= \frac{1}{2H_q}h_q(\xi_r)\int_{\left(\frac{b}{c_\nu}\right)|\eta_\nu-\eta|\sqrt{(M^2-1)}}^{\xi_r+1} \frac{2(\xi_r-\xi) - i\nu\left(\frac{c_\nu}{\bar{c}}\right)\left\{(\xi_r-\xi)^2+\left(\frac{b}{c_\nu}\right)^2(\eta_\nu-\eta)^2\right\}}{\sqrt{\{(\xi_r-\xi)^2-(M^2-1)\left(\frac{b}{c_\nu}\right)^2(\eta_\nu-\eta)^2\}}}\,d(\xi_r-\xi) \\ &\quad - \frac{1}{2H_q}\frac{\partial h_q}{\partial \xi}(\xi_r)\int_{\left(\frac{b}{c_\nu}\right)|\eta_\nu-\eta|\sqrt{(M^2-1)}}^{\xi_r+1} \\ &\quad \frac{2(\xi_r-\xi)^2 - i\nu\left(\frac{c_\nu}{\bar{c}}\right)\left\{(\xi_r-\xi)^3+\left(\frac{b}{c_\nu}\right)^2(\xi_r-\xi)(\eta_\nu-\eta)^2\right\}}{\sqrt{\{(\xi_r-\xi)^2-(M^2-1)\left(\frac{b}{c_\nu}\right)^2(\eta_\nu-\eta)^2\}}}\,d(\xi_r-\xi) + \ldots \text{etc.} \end{aligned} \quad \text{(A.29)}$$

When these integrals are evaluated it is found that most of the terms are regular and could be expanded as a power series of η. However, the remaining terms are of the form $(\eta_\nu - \eta)^2 \log|\eta_\nu - \eta|$ and $(\eta_\nu - \eta)^4 \log|\eta_\nu - \eta|$ and $(\eta_\nu - \eta)\log|\eta_\nu - \eta|$ and so on. Only the first of these terms causes an important singularity at $\eta = \eta_\nu$ and so $\bar{K}_{\underset{rq}{\nu}}(\eta)$ can be written (*see* equation (67)) as follows :

$$\bar{K}_{\underset{rq}{\nu}}(\eta) = \bar{K}_{\underset{rq}{\nu}}^{*}(\eta) + L_{\underset{rq}{\nu}}(\eta - \eta_\nu)^2\log|\eta - \eta_\nu| . \quad \text{(A.30)}$$

The term $\bar{K}_{\underset{rq}{\nu}}^{*}(\eta)$ is the regular part and the singular term $L_{\underset{rq}{\nu}}$ is found to be :

$$L_{\underset{rq}{\nu}} = \frac{1}{2H_q}\frac{\partial h_q}{\partial \xi}(\xi_r)(M^2-1)\left(\frac{b}{c_\nu}\right)^2 + i\nu\frac{1}{2H_q}h_q(\xi_r)(M^2+1)\left(\frac{b^2}{2\bar{c}c_\nu}\right) . \quad \text{(A.31)}$$

The subsonic low frequency case can of course be evaluated in a similar manner and leads to exactly the same result. The result differs slightly from that of Garner (Ref. 2) in its unsteady part because the true lift and downwash (instead of the transformed values) are being used in this report.

The above expression applies only to the small frequency case. A further analysis (on the same lines) is required for the finite frequency solution.

APPENDIX VI

The Function $j_n(\eta)$

Equation (73) gives the series expression for the function $j_n(\eta)$. The coefficients b_{n0}, b_{n1}, etc., must be chosen to make $j_n(\eta)$ be zero at all the spanwise stations in the range except station η_n. Since the stations will not in general have any special mathematical significance a purely numerical method must be found for evaluating them. We will consider a third-order system (taking in points η_1, η_2 and η_3) as an example. In matrix notation :

$$\begin{bmatrix} j_1(\eta) \\ j_2(\eta) \\ j_3(\eta) \end{bmatrix} = \begin{bmatrix} b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \\ b_{30} & b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} e(\eta) \\ \eta e(\eta) \\ \eta^2 e(\eta) \end{bmatrix} \quad \ldots \quad (A.32)$$

Then by definition :

$$\begin{bmatrix} b_{10} & b_{11} & b_{12} \\ b_{20} & b_{21} & b_{22} \\ b_{30} & b_{31} & b_{32} \end{bmatrix} \begin{bmatrix} e(\eta)_1 , & e(\eta_2) , & e(\eta_3) \\ \eta_1 e(\eta_1) , & \eta_2 e(\eta_2) , & \eta_3 e(\eta_3) \\ \eta_1^2 e(\eta_1) , & \eta_2^2 e(\eta_2) , & \eta_3^2 e(\eta_3) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \ldots \quad (A.33)$$

This can be solved to give the b's which can be then substituted into equation (A.32) as follows :

$$\begin{bmatrix} j_1(\eta) \\ j_2(\eta) \\ j_3(\eta) \end{bmatrix} = \begin{bmatrix} \frac{1}{e(\eta_1)} , & 0 , & 0 \\ 0 , & \frac{1}{e(\eta_2)} , & 0 \\ 0 , & 0 , & \frac{1}{e(\eta_3)} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ \eta_1 & \eta_2 & \eta_3 \\ \eta_1^2 & \eta_2^2 & \eta_3^2 \end{bmatrix}^{-1} \begin{bmatrix} e(\eta) \\ \eta\, e(\eta) \\ \eta^2 e(\eta) \end{bmatrix} \quad \ldots \quad (A.34)$$

The advantage of this form is that the matrix inversion required is independant of the end conditions which determine $e(\eta)$. Furthermore, since a number of such matrices require inversion for a calculation on a given wing, it is found that the most convenient method of inversion will be by submatrices. This method of inversion enables some of the required (small order) inversions to be obtained as stages in the calculation of the larger ones.

The values of $\underset{r}{J_{\nu n}}^*$ are then :

$$\begin{bmatrix} \underset{r}{J_{\nu 1}} \\ \underset{r}{J_{\nu 2}} \\ \underset{r}{J_{\nu 3}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2G_1\, e(\eta_1)} , & 0 , & 0 \\ 0 , & \frac{1}{2G_2\, e(\eta_2)} , & 0 \\ 0 , & 0 , & \frac{1}{2G_3\, e(\eta_3)} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ \eta_1 & \eta_2 & \eta_3 \\ \eta_2^2 & \eta_1^2 & \eta_3^2 \end{bmatrix}^{-1} \begin{bmatrix} \int_{\eta_A}^{\eta_B} \frac{e(\eta)\, d\eta}{(\eta - \eta_\nu)^2} \\ \int_{\eta_A}^{\eta_B} \frac{\eta\, e(\eta)\, d\eta}{(\eta - \eta_\nu)_2} \\ \int_{\eta_A}^{\eta_B} \frac{\eta^2\, e(\eta)\, d\eta}{(\eta - \eta_\nu)^2} \end{bmatrix} . \quad (A.35)$$

When the integration extends to both tips, the formulae of Multhopp can be used and we have

$$\left.\begin{aligned} \underset{r}{J_{\nu n}}^* &= \frac{-(m+1)^2}{2(1-\eta_\nu^2)} && \text{when } n = \nu \\ &= \frac{2}{(\eta_n - \eta_\nu)^2} && \text{when } |n - \nu| = 1, 3, 5 \\ &= 0 && \text{when } |n - \nu| = 2, 4, 6 \end{aligned}\right\} . \quad \ldots \quad (A.36)$$

The values of $\underset{r}{G_{\nu\nu}}^*$ can be calculated in a similar manner. (*See* equation (75).)

APPENDIX VII

Equivalent Deflections and Downwash Angles

In the chordwise case with subsonic leading and trailing edges we can put $h_q(\xi)$ in angular co-ordinates as follows :

$$\left.\begin{aligned} h_q(\phi) &= \frac{2}{2p+1}\sum_{\lambda=0}^{p-1}\left[\sin(\lambda+1)\phi_q - \sin\lambda\phi_q\right]\left[\frac{\cos(\lambda+1)\phi + \cos\lambda\phi}{\sin\phi}\right] \\ H_q &= \frac{\pi}{2p+1}\sin\phi_q \end{aligned}\right\}. \quad \text{.. (A.37)}$$

Substituting these results into equation (86) we have for the equivalent deflection :

$$\frac{\partial z_e}{\partial q}(\phi_q) = \sum_{\lambda=0}^{p-1}\left[\frac{\sin(\lambda+1)\phi_q - \sin\lambda\phi_q}{\sin\phi_q}\right]\left[\frac{1}{\pi}\int_0^{\pi}\frac{\partial z}{\partial q}(\phi)\left\{\cos(\lambda+1)\phi + \cos\lambda\phi\right\}d\phi\right]. \quad \text{(A.38)}$$

We can also put :

$$h_{p+1-r}(\phi) = \frac{2}{2p+1}\sum_{\lambda=0}^{p-1}\left[\sin(\lambda+1)\phi_r - \sin\lambda\phi_r\right]\left[\frac{-\cos(\lambda+1)\phi + \cos\lambda\phi}{\sin\phi}\right]. \quad \text{.. (A.39)}$$

Then the equivalent downwash angle becomes :

$$\alpha_e(\phi_r) = \sum_{\lambda=0}^{p-1}\left[\frac{\sin(\lambda+1)\phi_r + \sin\lambda\phi_r}{\sin\phi_r}\right]\left[\frac{1}{\pi}\int_0^{\pi}\alpha(\phi)\left\{-\cos(\lambda+1)\phi + \cos\lambda\phi\right\}d\phi\right]. \quad \text{(A.40)}$$

These two expressions ((A.38) and (A.40)) are those developed in Ref. 5. A similar substitution in the chordwise case shows that equation (87) and the spanwise results of Ref. 5 are identical. The spanwise substitution is of course :

$$\left.\begin{aligned} g_n(\theta) &= \frac{2}{m+1}\sum_{\lambda=0}^{m-1}\left[\sin(\lambda+1)\theta_n\right]\left[\sin(\lambda+1)\theta\right] \\ G_n &= \frac{\pi}{2(m+1)}\sin\theta_n \end{aligned}\right\}. \quad \text{.. .. .. .. (A.41)}$$

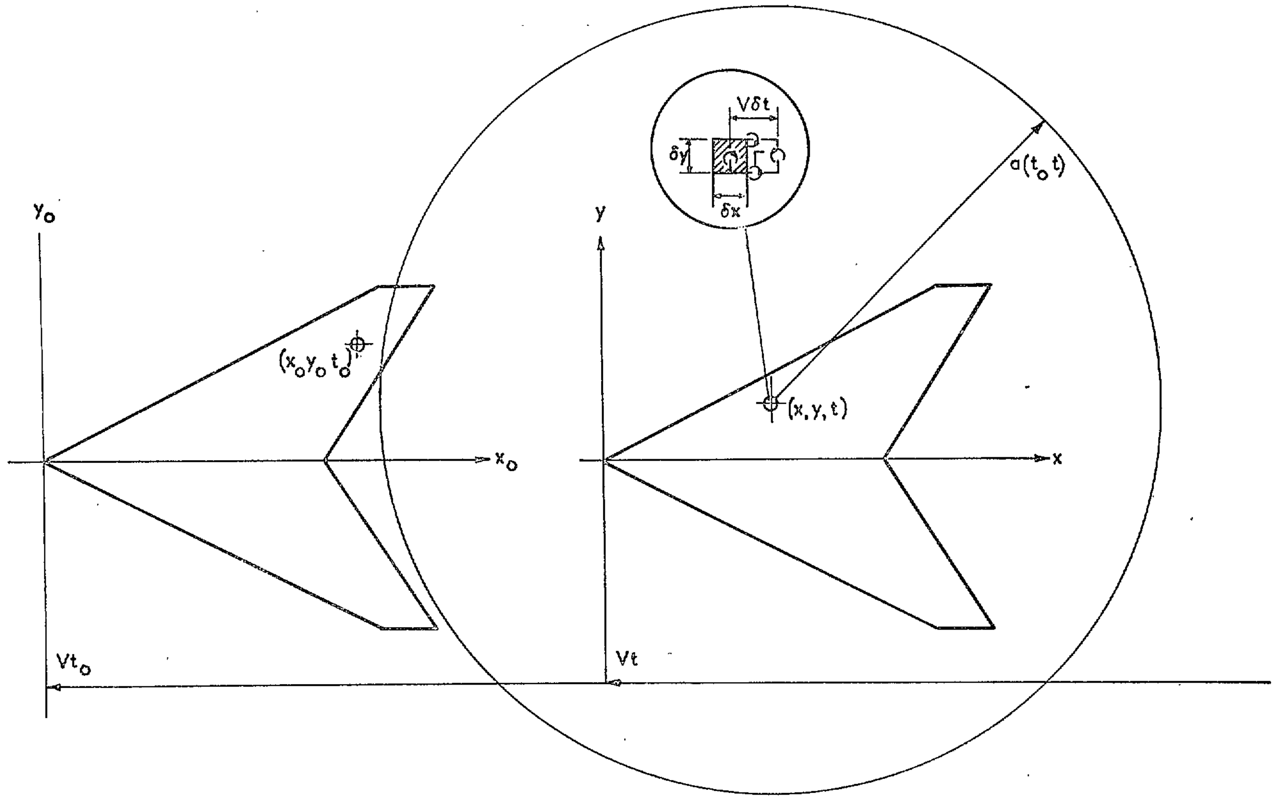

FIG. 1. Co-ordinates.

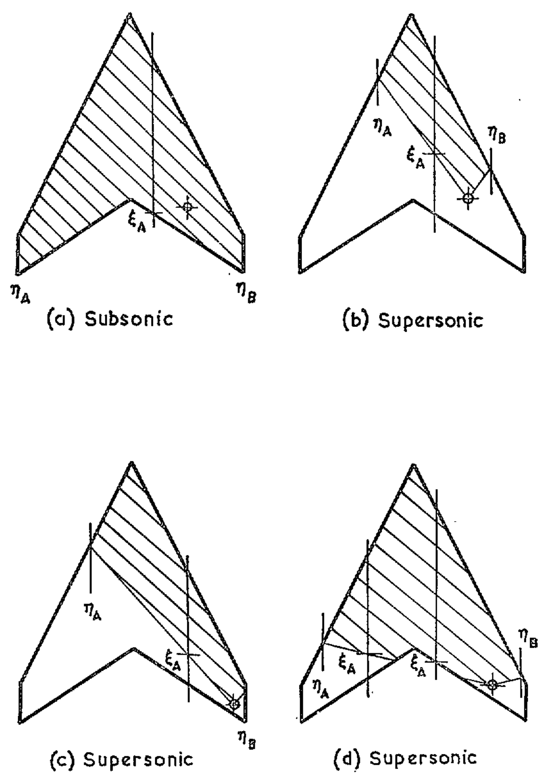

FIG. 2. Areas for integration.

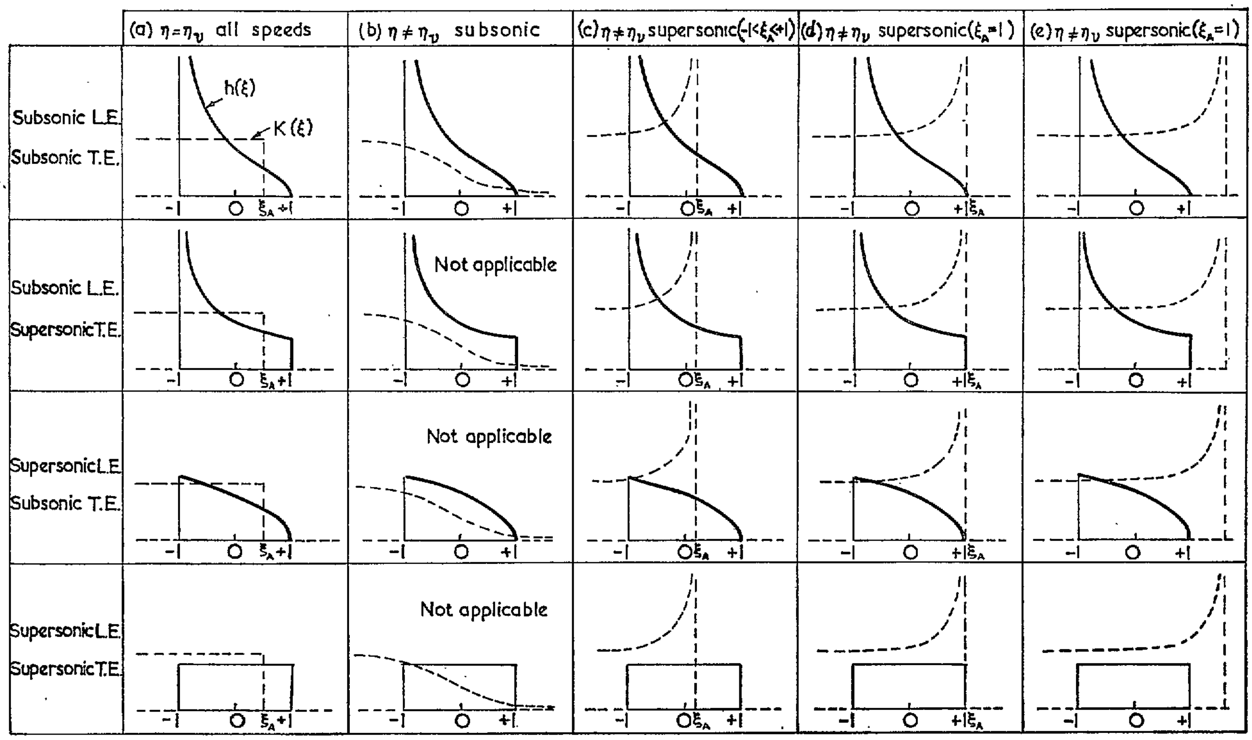

FIG. 3. Chordwise integration.

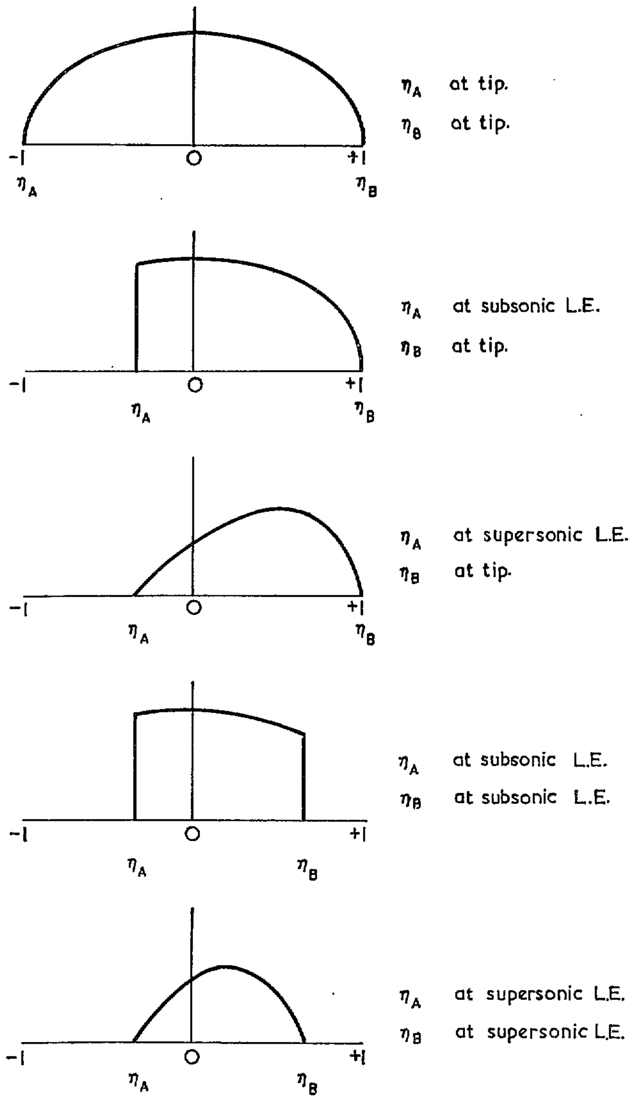

FIG. 4. Spanwise integrands.

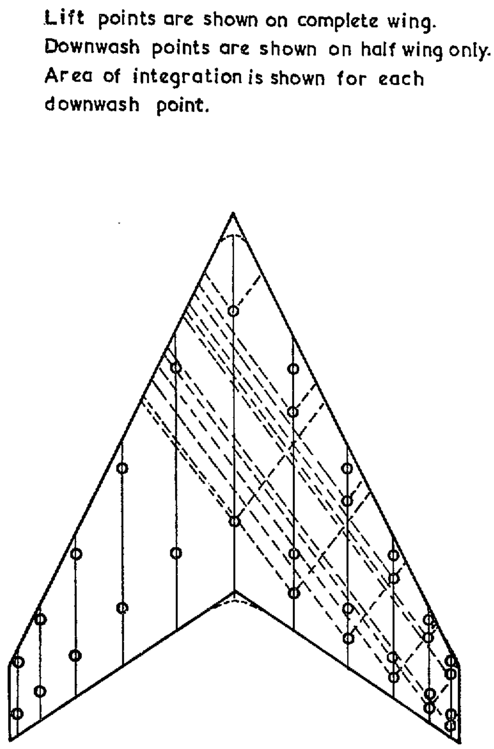

FIG. 5. Lay-out of points on a wing for $M = 1{\cdot}667$ ($p = 2$; $m = 11$).

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