R. & M. No. 3065
(17,783)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

# The Effect of Wing-Tailplane Aerodynamic Interaction on Tail Flutter

*By*
D. E. WILLIAMS, B.A.



LONDON: HER MAJESTY'S STATIONERY OFFICE
1958
PRICE 8s. 6d. NET

# The Effect of Wing-Tailplane Aerodynamic Interaction on Tail Flutter

*By*
D. E. Williams, B.A.

Communicated by the Director-General of Scientific Research (Air), Ministry of Supply

*Reports and Memoranda No.* 3065*
*February,* 1955

*Summary.*—When aerodynamic coefficients are calculated for tailplane flutter calculations it is usual to neglect the aerodynamic effect on the tailplane of the disturbance due to the wing and to assume that the tailplane oscillates in a steady stream. In this report a two-dimensional theory, which includes the effect of the wing-tailplane aerodynamic interaction, is developed for any wing and tailplane in the same horizontal plane. To represent this effect the standard derivatives are modified and additional derivatives are introduced. Calculations for a particular system show that the change in the standard derivatives is small but that the additional derivatives are comparable in size with the standard derivatives. The additional derivatives are used to investigate the effect of the wing motion on tailplane-elevator flutter, and it is shown that the aerodynamic interaction has little effect on the flutter of the binary system considered.

1. *Introduction.*—It is a basic assumption in flutter theory that the aerodynamic forces on an oscillating surface arise only from the disturbance in the flow due to the motion of the surface itself. This assumption is valid when only one surface is oscillating in a steady stream, but it is not valid when there are two such surfaces, each lying in the disturbed flow due to the other. The forces on each surface are then affected by the motion of the other surface; strictly speaking, even the presence of a second surface at rest affects the forces on the first.

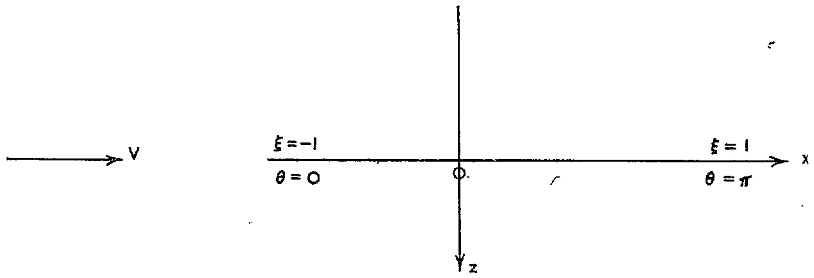
In this report an approximate evaluation is made of the aerodynamic forces which act on a tailplane when it is oscillating in the wake of an oscillating wing. The effect of the tailplane on the wing forces is also considered. For a complete attack on the problem we should need to deal with a finite wing and a finite tailplane, but here the simpler two-dimensional problem of wing-tailplane interaction is discussed. As an additional simplification the tailplane is assumed to lie in the same horizontal plane as the wing.

The method used is an extension of the method given by Lyon[1]. The vorticity distributions over the wing and tailplane are assumed to be given by infinite series of trigonometric functions with coefficients that are left as unknowns. The downwash over the wing and tailplane is evaluated for each distribution. The boundary condition to be satisfied is that near the surface the resultant airflow velocity must be tangential to the surface. From the wing and tailplane boundary conditions we can find the unknown coefficients in the vorticity distributions and so find the pressure distribution over each surface. The pressure on the tailplane is linearly dependent on the degrees of freedom of the wing and tailplane, and the expressions for the aerodynamic forces on the tailplane thus involve two sets of derivatives, one in respect of the wing motion and the other in respect of the tailplane motion. Similarly there are two sets of derivatives for the wing.

* R.A.E. Report Struct. 176, received 3rd August, 1955.

The theory is developed for any wing and tailplane in the same horizontal plane. No general formulae can be given for the derivatives, but tailplane derivatives are evaluated for one particular system, for which it is shown that the effect of the presence of the wing on the standard tailplane derivatives is small, but that the additional derivatives, tailplane forces due to wing motion, are of an appreciable size. These derivatives are used to assess the effect of the wing-tailplane aerodynamic interaction on the binary elevator flutter of the system, the two degrees of freedom being elevator rotation and the fundamental aircraft mode; the effect of the aerodynamic interaction is shown to be small.

## 2. *Theory for one Surface.*—2.1. *List of Symbols*




| | | |
|---|---|---|
| $V$ | | Velocity of the undisturbed flow |
| $c$ | | Chord |
| $l$ | | Semi-chord |
| $x$ | $=$ | $l\xi = -l\cos\theta$ |
| $\dfrac{p}{2\pi}$ | | Frequency |
| $\omega$ | $=$ | $pl/V$ = reduced frequency $= \frac{1}{2}$ . frequency parameter $= \frac{1}{2}\nu$ |
| $C(\omega)$ | $=$ | $\dfrac{K_1(i\omega)}{K_0(i\omega)+K_1(i\omega)}$, where $K_0$ and $K_1$ are modified Bessel Functions of the second kind |
| $V\Gamma\, e^{ipt}$ | | Bound vorticity |
| $VE\, e^{ipt}$ | | Free vorticity |
| $VW\, e^{ipt}$ | | Downwash velocity |
| $lZ\, e^{ipt}$ | | Downward displacement |
| $\rho$ | | Air density |

2.2. *A Summary of Flutter Derivative Theory for One Surface.*—Let the bound vorticity distribution over a wing be given by

$$\Gamma = a_0 \cot\frac{\theta}{2} + \sum_{1}^{\infty} a_n \sin n\theta \qquad \text{(2.2.1)}$$

and let the downwash over the wing induced by this vorticity be given by

$$W = \sum_{0}^{\infty} b_n \cos n\theta. \qquad \text{(2.2.2)}$$

If $E$ is the free vorticity it can be shown that

$$E = -\,i\omega\,\mathrm{e}^{-i\omega\xi}\int_{-1}^{\xi}\mathrm{e}^{i\omega u}\,\Gamma(u)\,du \qquad |\xi| \leqslant 1$$

$$= -\,i\omega\,\mathrm{e}^{-i\omega\xi}\int_{-1}^{+1}\mathrm{e}^{i\omega u}\,\Gamma(u)\,du \qquad \xi \geqslant 1. \quad \ldots \quad (2.2.3)$$

The downwash and the vorticity are connected by the equation

$$2\pi W(\xi) = \int_{-1}^{\infty}\frac{\Gamma + E}{\xi - u}\,du. \quad \ldots \quad (2.2.4)$$

If we substitute for $E$ we get

$$2\pi W(\xi) = \int_{-1}^{+1}\frac{\Gamma(u)}{\xi - u}\,du - i\omega\int_{-1}^{+1}\frac{\mathrm{e}^{-i\omega u}}{\xi - u}\,du\int_{-1}^{u}\mathrm{e}^{i\omega v}\,\Gamma(v)\,dv$$

$$- i\omega\int_{1}^{\infty}\frac{\mathrm{e}^{-i\omega u}}{\xi - u}\,du\int_{-1}^{+1}\mathrm{e}^{i\omega v}\,\Gamma(v)\,dv$$

and in Appendix II, section 1, it is shown that this can be reduced to the simpler form

$$2\pi W(\xi) = \int_{-1}^{+1}\frac{\Gamma(u)}{\xi - u}\,du - i\omega\,\mathrm{e}^{-i\omega\xi}\int_{-\infty}^{\xi}\mathrm{e}^{i\omega x}\,dx\int_{-1}^{+1}\frac{\Gamma(y)}{x - y}\,dy\,. \quad \ldots \quad (2.2.5)$$

We do not substitute the series (2.2.1) in this equation and integrate term by term, because we cannot evaluate the integrals which occur, but we replace the series by a series of linear functions of $\cot\frac{1}{2}\theta$, $\sin n\theta$ which give integrals that can be evaluated.

If we substitute $\Gamma_0' = 2[\operatorname{cosec}\theta - (1 - C)\cot\frac{1}{2}\theta]$, we get a value of $W(\xi)$ which depends only on $\omega$, and so by a suitable choice of the constant $C(\omega)$ we get $W_0'(\xi) = 0$.

If we substitute

$$\Gamma_0'' = 2[\cot\tfrac{1}{2}\theta - \operatorname{cosec}\theta + i\omega\sin\theta]$$

we get $\quad W_0'' = 1\,, \quad \ldots \quad (2.2.6)$

and for $\quad \Gamma_0 = \Gamma_0' + \Gamma_0'' = 2[C\cot\frac{1}{2}\theta + i\omega\sin\theta]$

we get $\quad W_0 = W_0' + W_0'' = 1\,.$

If $\quad \Gamma_1 = \cot\frac{1}{2}\theta - 2\sin\theta + i\omega[\sin\theta + \frac{1}{2}\sin 2\theta]$

$$W_1 = \tfrac{1}{2} + \cos\theta$$

and if $\quad \Gamma_n = -\,2\sin n\theta + i\omega\left[\dfrac{\sin(n+1)\theta}{n+1} - \dfrac{\sin(n-1)\theta}{n-1}\right], \qquad n \geqslant 2$

$$W_n = \cos n\theta.$$

The series (2.2.2) for $W$ can now be written in terms of the quantities $W_n$,

$$W = b_0 W_0 + b_1(W_1 - \tfrac{1}{2}W_0) + \sum_2^\infty b_n W_n$$

$$= (b_0 - \tfrac{1}{2}b_1) W_0 + \sum_1^\infty b_n W_n \qquad \text{(2.2.7)}$$

and so the vorticity distribution is

$$\Gamma = (b_0 - \tfrac{1}{2}b_1) \Gamma_0 + \sum_1^\infty b_n \Gamma_n . \qquad \text{(2.2.8)}$$

If we compare this equation with equation (2.2.1) we get the following relations between the $a$ and $b$ coefficients:

$$\left.\begin{aligned} a_0 &= 2Cb_0 + (1 - C)b_1 \\ a_1 &= 2i\omega b_0 - 2b_1 - i\omega b_2 \\ a_2 &= \tfrac{1}{2}i\omega b_1 - 2b_2 - \tfrac{1}{2}i\omega b_3 \\ a_n &= \frac{i\omega}{n} b_{n-1} - 2b_n - \frac{i\omega}{n} b_{n+1} \end{aligned}\right\} \qquad \text{(2.2.9)}$$

The boundary conditions are that the air-stream velocity adjacent to the wing must be tangential to the wing and that there must be smooth flow at the trailing edge. The trailing-edge condition is already satisfied because each $\Gamma_n$ is zero there, and the flow is tangential over the wing if

$$W = \frac{\partial Z}{\partial \xi} + i\omega Z . \qquad \text{(2.2.10)}$$

From equation (2.2.10) we can find the $b$ coefficients; equations (2.2.9) then give the $a$ coefficients and hence the pressure distribution $p = \rho V \Gamma$.

3. *Theory for a Wing and Tailplane.*—3.1. *Notation.*—An extension of the method given in section 2 is used to deal with the problem of wing-tailplane interaction. The same symbols will be used for each surface, but those which belong to the wing will have a superscript $^{(1)}$ and those which belong to the tailplane will have the superscript $^{(2)}$. The superscript $^{(12)}$ will denote the effect of the wing on the tailplane; for example $W^{(12)}$ is the downwash which the wing vorticity induces over the tailplane. Similarly the superscript $^{(21)}$ will denote the effect of the tailplane on the wing.

3.2. *The Downwash Ahead of and Behind a Surface.*—The downwash which $\Gamma^{(1)}$ (*see* equation (2.2.8) ) induces in the wake of the wing can be written in the form

$$W = (b_0 - \tfrac{1}{2}b_1) W_0 + \sum_1^\infty b_n W_n , \qquad \text{(3.2.1)}$$

where $W_n$ is the downwash induced by $\Gamma_n^{(1)}$, but $W_n$ no longer has the simple form given above.

Expressions for the $W_n$ for $\xi > 1$ are derived in the Appendix and are

$$W_0 = 1 - C^{(1)}\sqrt{\left(\frac{\xi - 1}{\xi + 1}\right)} - i\omega^{(1)} \exp\,(-\,i\omega^{(1)}\,\xi)\left[C^{(1)}\int_1^{\xi}\frac{\exp\,(i\omega^{(1)}\,u)}{\sqrt{(u^2-1)}}\,du\right.$$

$$\left.+\,(1 - C^{(1)})\int_1^{\xi}\frac{u\exp\,(i\omega^{(1)}\,u)}{\sqrt{(u^2-1)}}\,du\right] \quad \ldots \quad \ldots \quad (3.2.2)$$

$$W_1 = \frac{1}{2\sqrt{(\xi^2-1)}}\{-\,\xi + \sqrt{(\xi^2-1)}\}\{1 - \xi + \sqrt{(\xi^2-1)}\} \quad \ldots \quad \ldots \quad (3.2.3)$$

$$W_n = \{-\,\xi + \sqrt{(\xi^2-1)}\}^n \,. \quad \ldots \quad \ldots \quad (3.2.4)$$

When $\xi$ lies in the range covered by the tailplane, each downwash function can be expressed as a Fourier cosine series in the tailplane-chord parameter $\theta^{(2)}$.

The series for $W_n$ will be

$$W_n^{(12)} = \sum_0^{\infty} w_{nm}^{(12)} \cos m\theta^{(2)}. \quad \ldots \quad \ldots \quad (3.2.5)$$

The downwash ahead of the tailplane due to $\Gamma^{(2)}$ can be calculated by using the formulae

$$W_0 = 1 - C^{(2)}\sqrt{\left(\frac{\xi - 1}{\xi + 1}\right)} - i\omega^{(2)} \exp\,(-\,i\omega^{(2)}\,\xi)\left[C^{(2)}\int_1^{-\xi}\frac{\exp\,(-\,i\omega^{(2)}\,u)}{\sqrt{(u^2-1)}}\,du\right.$$

$$\left.-\,(1 - C^{(2)})\int_1^{-\xi}\frac{u\exp\,(-\,i\omega^{(2)}\,u)}{\sqrt{(u^2-1)}}\,du\right],$$

$$W_1 = -\,\frac{1}{2\sqrt{(\xi^2-1)}}\{-\,\xi - \sqrt{(\xi^2-1)}\}\{1 - \xi - \sqrt{(\xi^2-1)}\}\,,$$

$$W_n = \{-\,\xi - \sqrt{(\xi^2-1)}\}^n\,.$$

Over the wing each of these expressions can be replaced by a Fourier series in the wing-chord parameter $\theta^{(1)}$

$$W_n^{(21)} = \sum_0^{\infty} w_{nm}^{(21)} \cos m\theta^{(1)}\,.$$

It has not been possible to give analytical expressions for the $w_{nm}$ coefficients and so they have been calculated numerically for a particular wing-tailplane system.

3.3. *Boundary Conditions.*—The boundary condition for the wing is

$$\frac{\partial Z^{(1)}}{\partial \xi^{(1)}} + i\omega^{(1)}\,Z^{(1)} = W^{(11)} + W^{(21)}, \quad \ldots \quad \ldots \quad (3.3.1)$$

*i.e.*, the combined velocity adjacent to the wing must be tangential to the wing. If we write

$$\frac{\partial Z^{(1)}}{\partial \xi^{(1)}} + i\omega^{(1)}Z^{(1)} = \sum_0^{\infty} C_n^{(1)} \cos n\theta^{(1)}, \quad \ldots \quad \ldots \quad (3.3.2)$$

the boundary condition becomes

$$\sum_0^\infty b_n^{(1)} \cos n\theta^{(1)} + (b_0^{(2)} - \tfrac{1}{2}b_1^{(2)}) \frac{W_0^{(21)}}{V} + \sum_1^\infty b_n^{(2)} \frac{W_n^{(21)}}{V} = \sum_0^\infty C_n^{(1)} \cos n\theta^{(1)}$$

$$\sum_0^\infty b_n^{(1)} \cos n\theta^{(1)} + (b_0^{(2)} - \tfrac{1}{2}b_1^{(2)}) \sum_0^\infty w_{0n}^{(21)} \cos n\theta^{(1)} + \sum_1^\infty b_m^{(2)} \sum_0^\infty w_{mn}^{(21)} \cos n\theta^{(1)}$$

$$= \sum_0^\infty C_n^{(1)} \cos n\theta^{(1)}$$

and so

$$b_n^{(1)} + (b_0^{(2)} - \tfrac{1}{2}b_1^{(2)}) w_{0n}^{(21)} + \sum_1^\infty b_m^{(2)} w_{mn}^{(21)} = C_n^{(1)}$$

$$b_n^{(1)} + b_0^{(2)} w_{0n}^{(21)} + b_1^{(2)} (w_{1n}^{(21)} - \tfrac{1}{2}w_{0n}^{(21)}) + \sum_2^\infty b_m^{(2)} w_{mn}^{(21)} = C_n^{(1)} . \quad \ldots \quad (3.3.3)$$

If we put

$$u_{mn}^{(21)} = w_{mn}^{(21)} \qquad m \neq 1$$

$$u_{1n}^{(21)} = w_{1n}^{(21)} - \tfrac{1}{2}w_{0n}^{(21)} ,$$

equation (3.3.3) becomes

$$b_n^{(1)} + \sum_{m=0}^\infty b_m^{(2)} u_{mn}^{(21)} = C_n^{(1)} . \quad \ldots \quad \ldots \quad \ldots \quad (3.3.4)$$

Let $B^{(1)} = [b_n^{(1)}]$, $B^{(2)} = [b_n^{(2)}]$, $C^{(1)} = [C_n^{(1)}]$ be infinite row matrices and let $U^{(21)} = [u_{mn}^{(21)}]$ be an infinite square matrix, then the boundary condition is

$$B^{(1)} + B^{(2)}U^{(21)} = C^{(1)} . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.3.5)$$

Similarly by considering the boundary condition for the tailplane we get

$$B^{(2)} + B^{(1)}U^{(12)} = C^{(2)} . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3.3.6)$$

Therefore

$$B^{(2)}(I - U^{(21)}U^{(12)}) = C^{(2)} - C^{(1)}U^{(12)} \quad \ldots \quad \ldots \quad \ldots \quad (3.3.7)$$

$$B^{(1)}(I - U^{(12)}U^{(21)}) = C^{(1)} - C^{(2)}U^{(21)} , \quad \ldots \quad \ldots \quad \ldots \quad (3.3.8)$$

where $I$ is the unit matrix.

Each $U$ matrix may be regarded as an 'influence' matrix.

The coefficients $u_{mn}$ tend rapidly to zero as $m,n \to \infty$ and so the matrix $[I - U^{(21)}U^{(12)}]$ tends to the form

$$\left[\begin{array}{c|c} P & O \\ \hline O & I \end{array}\right].$$

If for $n \geqslant N$ the term $u_{nn}$ is unity and $u_{in}$, $u_{nj}$ are zero to the accuracy to which we are working, we can take this finite square matrix of order $N$ and the corresponding finite $B$ and $C$ matrices and substitute these approximate matrices in the equation for $B$. We can then invert the approximate matrix $[I - U^{(21)}U^{(12)}]$ and solve for $B^{(2)}$.

For small vibrations each term of the $C^{(1)}$ matrix will be a linear function of the co-ordinates of the wing motion and each term of the $C^{(2)}$ matrix will be a linear function of the co-ordinates of the tailplane motion. Therefore $B^{(1)}$, $B^{(2)}$ and so $A^{(1)}$, $A^{(2)}$ (the infinite row matrices $[a_n^{(1)}]$, $[a_n^{(2)}]$) are each linear functions of the displacements of both surfaces. The aerodynamic force on each surface will thus involve two sets of derivatives, one in respect of the wing motion and the other in respect of the tailplane motion.

If the action of the wing on the tailplane is neglected, the influence matrix $U^{(12)}$ is zero; equation (3.3.7) then becomes $B^{(2)} = C^{(2)}$ and expresses the physically obvious fact that the forces on the tailplane are those given by the standard tailplane derivatives.

If the action of the wing on the tailplane is considered but the action of the tailplane on the wing is neglected, then $U^{(12)} \neq 0$, $U^{(21)} = 0$ and equation (3.3.7) becomes

$$B^{(2)} = C^{(2)} - C^{(1)}U^{(12)} . \qquad \qquad (3.3.9)$$

This equation shows that the set of tailplane derivatives in respect of the tailplane motion is the same as the set of standard derivatives, since they arise from the term $C^{(2)}$; there is also the additional set of derivatives in respect of the wing motion, arising from the term $C^{(1)}U^{(12)}$.

If the complete interaction between wing and tailplane is considered, that is, neither $U^{(12)}$ nor $U^{(21)}$ zero, the two sets of derivatives obtained from equation (3.3.7) will differ to some extent from those given by equation (3.3.9). This effect may be regarded as due to the disturbance in the flow caused by the presence of the wing, as distinct from the wing motion. However, since the matrix $[I - U^{(12)}U^{(21)}]$ is approximately equal to the unit matrix $I$ the differences will be small, and the two sets of derivatives given by equation (3.3.9) will be approximations to the correct derivatives as given by equation (3.3.7). Similar considerations apply to equation (3.3.8).

3.4. *Numerical Methods.*—The integrals

$$I = \int_1^{\xi} \frac{e^{i\omega u}}{\sqrt{(u^2 - 1)}}\, du, \quad J = \int_1^{\xi} \frac{u\, e^{i\omega u}}{\sqrt{(u^2 - 1)}}\, du$$

are a little difficult to evaluate because of the singularity at $u = 1$. This difficulty can be overcome by making the substitution $u = \cosh t$. If $\xi = \cosh \alpha$ we now get

$$I = \int_0^{\alpha} e^{i\omega \cosh t}\, dt, \; J = \int_0^{\alpha} e^{i\omega \cosh t} \cosh t\, dt .$$

These integrals can be evaluated by the usual method of calculating the ordinates at evenly distributed points and then using the standard integration formulae, but there are some disadvantages. The values of $\xi$ for which we need to evaluate the integrals occupy a greater part of the whole $\xi$ range than the corresponding values of $\alpha$ do of the whole $\alpha$ range, and so to evaluate the integrals to a certain number of decimal places we need about twice as many ordinates for the $\alpha$ range as we do for the $\xi$ range.

Another disadvantage is that while the values of $\xi$ are evenly spaced and can be made to be ordinates for our integration formulae, the correspoinding values of $\alpha$ are not, and so if we take a fixed set of evenly distributed ordinates most of the integrals we need will have to be found by interpolation. In the numerical calculations (section 5) it was finally decided to divide the $\xi$ range into two parts, (1–1·5) and (1·5–$\xi$). Over the first part the integrals were evaluated in the second $(t,\alpha)$ form and over the second part they were integrated in the first $(u,\xi)$ form*.

* It appeared later that little was gained by this device and the integrals might well have been evaluated using the second form only.

Schwarz[2] has given equations relating the integrals $I$ and $J$ to new functions $He^{(1)}$, $He^{(2)}$ which he has introduced and evaluated for some values of the parameters (*see* Appendix II, section 5). However, since these values of the parameter were not sufficiently extensive, it was not possible to obtain values of $I$ and $J$ from the relations of Schwarz (interpolation would not have given sufficiently accurate results). It has been possible to evaluate one of each of the integrals $I$ and $J$ by both the numerical integration method and the Schwarz method; the results agree to within two units in the third figure. The integrals used are correct to two figures and the error is believed to be less than 1 per cent.

There are two possible methods of evaluating the Fourier coefficients $w_{nm}$ in the series (3.2.5) for the downwash velocity. The first method is to express them as integrals, as in the theory of Fourier series, and then to evaluate the integrals by numerical methods. The usual integration formulae are, however, not very suitable because of the rapid oscillation of the function $\cos nx$ in the integrand and a large number of stations is needed to give sufficient accuracy.

In the second method the amount of computation needed for reasonable accuracy is much less. This method is to assume a finite cosine series and to calculate the coefficients so that the series gives the correct downwash velocities at certain points, which involves the solution of a set of simultaneous equations. This second method was used in the numerical calculations of section 5.

4. *Formulae for Lift, Pitching Moment and Hinge Moment.*—

$$\gamma = a_0 \cot \tfrac{1}{2}\theta + \sum_1^\infty a_n \sin n\theta .$$

*Lift.*—The lift on an aerofoil is given by the formula

$$L = \int \rho V^2 \Gamma \, dx$$

$$= \rho V^2 l \int_0^\pi \Gamma \sin\theta \, d\theta ,$$

$$\frac{L}{\rho c V^2} = \tfrac{1}{2}\pi (a_0 + \tfrac{1}{2} a_1)$$

$$= \tfrac{1}{2}\pi \{ b_0(2C + i\omega) - b_1 C - \tfrac{1}{2} i\omega b_2 \} .$$

*Moment about the Leading Edge* $= M$ (nose-up positive)

$$M = -\int (x + l)\rho V^2 \Gamma \, dx ,$$

$$-\frac{M}{\rho c^2 V^2} = \tfrac{1}{4}\int_0^\pi (1 - \cos\theta) \sin\theta \, \Gamma(\theta) \, d\theta ,$$

$$= \tfrac{1}{8}\pi (a_0 + a_1 - \tfrac{1}{2} a_2)$$

$$= \tfrac{1}{8}\pi [2b_0(C + i\omega) - b_1(1 + C + \tfrac{1}{4} i\omega) + b_2(1 - i\omega) + b_3 \tfrac{1}{4} i\omega] .$$

*Elevator Moment—No Aerodynamic Balance.*—If the hinge is at $x_h = -l\cos\theta$, the elevator hinge moment $H$, (positive nose-up) is given by

$$-H = \int_{x_h}^{x_t} (x - x_h) \rho V^2 \Gamma \, dx ,$$

$$-\frac{H}{\rho l^2 V^2} = \int_0^\pi \sin\phi\,(\cos\theta - \cos\theta)\,\Gamma(\phi)\,d\phi,$$

$$-\frac{H}{\rho c^2 V^2} = \sum_0^\infty a_n t_n,$$

where $t_0 = \frac{1}{8}\Phi_8,$

$$t_1 = \tfrac{1}{16}\Phi_4,$$

$$t_2 = \tfrac{1}{8}[\Phi_4 - \Phi_7],$$

$$t_n = \tfrac{1}{16}\left[\frac{\sin(n+2)\theta}{(n+1)(n+2)} - \frac{2\sin n\theta}{n^2-1} + \frac{\sin(n-2)\theta}{(n-1)(n-2)}\right], \qquad n \geqslant 3$$

$$a_0 = 2Cb_0 + (1-C)b_1,$$

$$a_1 = 2i\omega b_0 - 2b_1 - i\omega b_2,$$

$$a_2 = \tfrac{1}{2}i\omega b_1 - 2b_2 - \tfrac{1}{2}i\omega b_3,$$

$$a_n = \frac{i\omega}{n}(b_{n-1} - b_{n+1}) - 2b_n,$$

and

$$\Phi_4 = 2\cos\theta\,(\pi - \theta) + (2/3)\sin\theta\,(2 + \cos^2\theta),$$

$$\Phi_8 = (\pi - \theta)(2\cos\theta - 1) + \sin\theta\,(2 - \cos\theta),$$

$$\Phi_7 = (\pi - \theta)(\tfrac{1}{2} + 2\cos\theta) + (1/6)\sin\theta\,(8 + 5\cos\theta + 4\cos^2\theta - 2\cos^3\theta).$$

5. *Results*.—5.1. *Geometry of this System*.—The derivatives tabulated below are for a wing and tailplane which lie in the same horizontal plane. The wing chord is twice the tailplane chord and the distance between the mid-chords is three times the wing chord. The elevator chord is 30 per cent of the tailplane chord.

5.2. *Definition of the Derivatives*.—The tailplane derivatives are defined by the equations:

$$\frac{L^{(2)}}{\rho c^{(2)} V^2} = L_{z1}{}^{(2)}\frac{z^{(1)}}{c^{(1)}} + L_{\alpha 1}{}^{(2)}\,\alpha^{(1)} + L_{z2}{}^{(2)}\frac{z^{(2)}}{c^{(2)}} + L_{\alpha 2}{}^{(2)}\,\alpha^{(2)} + L_{\beta 2}{}^{(2)}\,\beta^{(2)},$$

$$-\frac{M^{(2)}}{\rho\{c^{(2)}\}^2 V^2} = \left(-M_{z1}{}^{(2)}\right)\frac{z^{(1)}}{c^{(1)}} + \left(-M_{\alpha 1}{}^{(2)}\right)\alpha^{(1)} + \left(-M_{z2}{}^{(2)}\right)\frac{z^{(2)}}{c^{(2)}} + \left(-M_{\alpha 2}{}^{(2)}\right)\alpha^{(2)} + \left(-M_{\beta 2}{}^{(2)}\right)\beta^{(2)},$$

$$-\frac{H^{(2)}}{\rho\{c^{(2)}\}^2 V^2} = \left(-H_{z1}{}^{(2)}\right)\frac{z^{(1)}}{c^{(1)}} + \left(-H_{\alpha 1}{}^{(2)}\right)\alpha^{(1)} + \left(-H_{z2}{}^{(2)}\right)\frac{z^{(2)}}{c^{(2)}} + \left(-H_{\alpha 2}{}^{(2)}\right)\alpha^{(2)} + \left(-H_{\beta 2}{}^{(2)}\right)\beta^{(2)}.$$

The wing derivatives are defined by:

$$\frac{L^{(1)}}{\rho c^{(1)} V^2} = L_{z1}{}^{(1)}\frac{z^{(1)}}{c^{(1)}} + L_{\alpha 1}{}^{(1)}\,\alpha^{(1)} + L_{z2}{}^{(1)}\frac{z^{(2)}}{c^{(2)}} + L_{\alpha 2}{}^{(1)}\,\alpha^{(2)} + L_{\beta 2}{}^{(1)}\,\beta^{(2)},$$

$$-\frac{M}{\rho\{c^{(1)}\}^2V^2} = \left(-M_{z1}{}^{(1)}\right)\frac{z^{(1)}}{c^{(1)}} + \left(-M_{\alpha 1}{}^{(1)}\right)\alpha^{(1)} + \left(-M_{z2}{}^{(1)}\right)\frac{z^{(2)}}{c^{(2)}} + \left(-M_{\alpha 2}{}^{(1)}\right)\alpha^{(2)} + $$
$$+ \left(-M_{\beta 2}{}^{(1)}\right)\beta^{(2)}.$$

5.3. *Wing Derivatives for Wing Frequency Parameter* $\nu_1 = 0\cdot 6$

| | $U^{(12)} \neq 0 \quad U^{(21)} \neq 0$ | $U^{(12)} = 0 \quad U^{(21)} = 0$ |
|---|---|---|
| $L_{z(1)}$ | $+ 0\cdot 056 + 1\cdot 27i$ | $0\cdot 055 + 1\cdot 25i$ |
| $L_{\alpha(1)}$ | $2\cdot 20 \quad + 0\cdot 86i$ | $2\cdot 20 \quad + 0\cdot 85i$ |
| $L_{z(2)}$ | $0\cdot 014 + 0\cdot 007i$ | 0 |
| $L_{\alpha(2)}$ | $+ 0\cdot 03 \quad - 0\cdot 04i$ | 0 |
| $- M_{z(1)}$ | $- 0\cdot 059 + 0\cdot 32i$ | $- 0\cdot 057 + 0\cdot 31i$ |
| $- M_{\alpha(1)}$ | $+ 0\cdot 51 \quad + 0\cdot 45i$ | $0\cdot 51 \quad + 0\cdot 45i$ |
| $- M_{z(2)}$ | $0\cdot 003 + 0\cdot 002i$ | 0 |
| $- M_{\alpha(2)}$ | $+ 0\cdot 010 - 0\cdot 007i$ | 0 |

The first column gives the derivative when the complete interaction, the action of the wing on the tailplane and the action of the tailplane on the wing is considered. The second column gives the derivative when the interaction is neglected, it gives the standard wing derivatives.

The additional derivatives given in the first column are small and the others differ little from the standard derivatives. Since the effect of the tailplane pitch and vertical translation on the wing derivatives is small, it is assumed that the effect of the elevator rotation will also be small, though no calculations have been made for this case.

5.4. *Tailplane Derivatives for Tailplane Frequency Parameter* $\nu_2 = 0\cdot 3$; *Comparison of the Two Approximate Methods.*

| | $U^{(12)} \neq 0 \quad U^{(21)} \neq 0$ | $U^{(21)} = 0$ | $U^{(21)} = 0 \quad U^{(12)} = 0$ |
|---|---|---|---|
| $L_{z(1)}$ | $- 0\cdot 67 \quad + 0\cdot 50i$ | $- 0\cdot 66 \quad + 0\cdot 49i$ | 0 |
| $L_{\alpha(1)}$ | $+ 0\cdot 33 \quad + 1\cdot 5i$ | $+ 0\cdot 32 \quad + 1\cdot 5i$ | 0 |
| $L_{z(2)}$ | $0\cdot 11 \quad + 0\cdot 74i$ | $0\cdot 11 \quad + 0\cdot 73i$ | $0\cdot 11 \quad + 0\cdot 73i$ |
| $L_{\alpha(2)}$ | $+ 2\cdot 6 \quad + 0\cdot 20i$ | $+ 2\cdot 5 \quad + 0\cdot 20i$ | $2\cdot 5 \quad + 0\cdot 20i$ |
| $- M_{z(1)}$ | $- 0\cdot 17 \quad + 0\cdot 13i$ | $- 0\cdot 16 \quad + 0\cdot 12i$ | 0 |
| $- M_{\alpha(1)}$ | $+ 0\cdot 085 \quad + 0\cdot 37i$ | $+ 0\cdot 084 \quad + 0\cdot 37i$ | 0 |
| $- M_{z(2)}$ | $0\cdot 0090 + 0\cdot 18i$ | $0\cdot 0086 + 0\cdot 18i$ | $0\cdot 0086 + 0\cdot 18i$ |
| $- M_{\alpha(2)}$ | $+ 0\cdot 63 \quad + 0\cdot 17i$ | $+ 0\cdot 62 \quad + 0\cdot 17i$ | $0\cdot 62 \quad + 0\cdot 17i$ |

The first column gives the derivatives when the complete interaction is considered, the second column gives the tailplane derivatives when the action of the tailplane on the wing is neglected, and the third column gives the standard tailplane derivatives.

The change in the values of the standard derivatives is small. The action of the tailplane on the wing seems to have little effect on the additional derivatives and so the derivatives for $\nu_2 = 0 \cdot 5$, $0 \cdot 7$ given in section 5·5 have been calculated on the assumption that this action can be neglected. It is assumed that if the elevator derivatives were calculated by both methods the conclusions would be the same.

5.5. *Tailplane Derivatives in Respect of Wing Motion* (*using approximation* $U^{21} = 0$)

| | $\nu_2$ | | |
|---|---|---|---|
| | 0·3 | 0·5 | 0·7 |
| $L_{z(1)}$ | $-0 \cdot 66 \quad + 0 \cdot 49i$ | $0 \cdot 32 \quad + 1 \cdot 71i$ | $2 \cdot 50 \quad + 1 \cdot 1i$ |
| $L_{\alpha(1)}$ | $+ 0 \cdot 32 \quad + 1 \cdot 5i$ | $2 \cdot 0 \quad + 0 \cdot 95i$ | $2 \cdot 63 \quad + 1 \cdot 0i$ |
| $-M_{z(1)}$ | $-0 \cdot 16 \quad + 0 \cdot 12i$ | $0 \cdot 080 \quad + 0 \cdot 43i$ | $0 \cdot 62 \quad + 0 \cdot 27i$ |
| $-M_{\alpha(1)}$ | $+ 0 \cdot 084 \quad + 0 \cdot 37i$ | $0 \cdot 49 \quad + 0 \cdot 24i$ | $0 \cdot 66 \quad - 0 \cdot 25i$ |
| $-H_{z(1)}$ | $-0 \cdot 0059 + 0 \cdot 0046i$ | $0 \cdot 0029 + 0 \cdot 016i$ | $0 \cdot 022 + 0 \cdot 0096i$ |
| $-H_{\alpha(1)}$ | $+ 0 \cdot 0032 + 0 \cdot 013i$ | $0 \cdot 018 \quad + 0 \cdot 0087i$ | $0 \cdot 024 - 0 \cdot 0089i$ |

6. *Application to Elevator Flutter*.—6.1. *Preliminary Remarks*.—To obtain some idea of the importance of the additional derivatives which have been calculated, they have been used to investigate the effect on elevator flutter of the aerodynamic interaction between the wing and tailplane. The results of the investigation are given in a non-dimensional form, but the aircraft considered has the proportions of a large transport aircraft (*see* Fig. 1). The calculations were made with two degrees of freedom, elevator rotation and a normal mode of the whole aircraft (*see* Fig. 2). The normal mode is made up of pitch, vertical translation, parabolic wing flexure and parabolic fuselage flexure.

The wing is assumed to be rigid in torsion, so that its angle of incidence is constant over the span and equal to the slope of the fuselage at the root of the wing mid-chord axis. The tailplane is assumed rigid in torsion and flexure and its angle of incidence is everywhere equal to the slope of the fuselage at the root of the hinge line. The elevator is rigid in torsion and flexure.

The tailplane derivatives are two-dimensional derivatives appropriate to the 0·7-span position. The additional derivatives in respect of the wing motion are then assumed to have the values they would have, if at this section the wing and tailplane were part of a two-dimensional system. The dimensions of the aircraft have been chosen so that this two-dimensional system is the one considered earlier in the report, *i.e.*, the wing chord is twice the tailplane chord and the distance between the mid-chords is three times the wing chord.

6.2. *Notation*.—

$s$ Unit of length

$\rho$ Air density

$\rho_0$ Air density at sea level

| | | |
|---|---|---|
| $m$ | $=$ | $\rho_0 s^3 =$ unit of mass |
| $M$ | | Mass-balance/mass of elevator |
| $c$ | | Wing chord |
| $c_e$ | | Elevator chord |
| $\eta$ | | Non-dimensional wing-span parameter |
| $u$ | | Non-dimensional fuselage co-ordinate |
| $V_0$ | | Unit of speed |
| | $=$ | $\omega s$ |
| $V$ | | Flutter speed |
| $V_e$ | | Equivalent flutter speed |
| | $=$ | $V\sqrt{(\rho/\rho_0)}$ |
| $\omega$ | | Natural frequency of normal mode in radn/unit time |

6.3. *Data.*—The dimensions of the aircraft are shown in Fig. 1 in terms of a unit of length $s$ ft, the distance from the wing root to the wing tip. The chord of the full-span elevator is 30 per cent of the tailplane chord and there is no aerodynamic balance.

Let the unit of mass be $m = \rho_0 s^3$. If $\eta$ is the usual non-dimensional wing spanwise co-ordinate the mass per unit length of each wing is $3 \cdot 0232\, m(1 - \eta)/s$. The mass per unit length of each elevator is $0 \cdot 5176 mc_e/s^2$, where $c_e$ is the elevator chord. Let $u$ be a non-dimensional fuselage co-ordinate which is zero at the root of the wing mid-chord line and unity at the root of the tail-plane mid-chord line. The mass per unit length of the fuselage is $(4 \cdot 904 - 8 \cdot 734u)m/s$ for $u < 0$ and $(4 \cdot 904 - 4 \cdot 414u)m/s$ for $u > 0$. The mass of the tail unit excluding the elevators and mass-balance is $0 \cdot 1040m$ and acts at the root of the tailplane mid-chord axis.

The inertia axis of the wing coincides with the mid-chord line and the radius of gyration of a chordwise section of the wing about the mid-chord line is $\frac{1}{4}c$ where $c$ is the wing chord. The inertia axis of the elevator is one third of the elevator chord aft of the hinge line, i.e., $\bar{x}_e = 1/3c_e$, and the radius of gyration of a chordwise section of the elevator about the hinge line is $k_e = \sqrt{(1 \cdot 25)}\bar{x}_e = 1/3\sqrt{(1 \cdot 25)}c_e$. The mass-balance is evenly distributed at a distance $1/3c_e$ ahead of the hinge line.

The natural frequency of the normal mode is $3 \cdot 2$ c.p.s.

6.4. *Results.*—On Figs. 3, 4 and 5 flutter curves are given of $V_e/V_0$ against the mass-balance parameter $M$, for sea level and heights of approximately 33,000 and 45,000 ft. The parameter $M$ is unity for static balance. For each height two flutter curves are given, the one obtained when the aerodynamic interaction between the wing and tailplane is considered and the other obtained when this interaction is neglected.

The additional derivatives are only available for three frequency parameters, $0 \cdot 3$, $0 \cdot 5$ and $0 \cdot 7$. With these derivatives we can only get a part of the sea-level curve. We can, however, investigate the effect on various regions of the curve by carrying out the calculations for different heights. The effect on the nose can be seen from the 33,000-ft curve and the effect on the lowest flutter speeds can be seen from the 45,000-ft curve.

In the binary flutter calculations for 33,000 ft and 45,000 ft, flutter occurred for large positive mass-balance giving a second branch of the curve for both sets of derivatives. This second branch is not shown on the graphs because the mass-balance is so large as to be of no practical importance. It is further believed that the aircraft mode used is not appropriate to large mass-balance, which would induce a node nearer to itself, and that this second branch would disappear if a suitable mode or if more aircraft modes were used.

The aerodynamic interaction between the wing and tailplane has little effect on the binary system except on the upper part of the curve for low-frequency parameters, and this is unimportant from a practical point of view.

7. *Conclusions.*—Flutter derivatives have been calculated for a particular two-dimensional wing-tailplane configuration oscillating in a steady stream. The changes in the values of the standard derivatives due to the presence of the other surface is small. Additional derviatives have been introduced to represent the forces on each surface due to the motion of the other. The additional tailplane derivatives are comparable in size to the standard derivatives representing the effect of the tailplane motion, but the additional wing derivatives are small. This difference can only in part be accounted for by the difference in size of the wing and the tailplane, and it would seem that most of the change in pressure is due to the vortex wake of the wing and not to a change in the strength of the bound vorticity. In practice the vortex wake will die out because of the viscosity of the air. In the theory used, all viscosity effects have been neglected and the wake persists undiminished in strength. The effect of the wake is thus overestimated and the additional derivatives calculated on this theory are much larger than the derivatives which would be given by a theory which included viscosity effects.

The use of these additional derivatives in flutter calculations will give only a qualitative estimate of the change which the wing-tailplane interaction will induce in the critical flutter speed of the system. When they are applied to a two-dimensional system the change will be overestimated, for the reasons given above, but when the same derivatives are applied to a three-dimensional system it cannot be said with any certainty whether the effect will be overestimated or underestimated. An investigation into the three dimensional problem would be of interest particularly if the rolling up of the wake were considered.

The effect of the wing-tailplane aerodynamic interaction on the elevator flutter of the binary system considered is small in those respects that are of practical importance.

## REFERENCES


| *No.* | *Author* | *Title, etc.* |
|---|---|---|
| 1 | H. M. Lyon .. .. .. | Aerodynamical derivatives of flexural-torsional flutter derivatives of a wing of finite span. R. &. M. 1900. July, 1939. |
| 2 | L. Schwarz .. .. .. | Untersuchung einiger mit den Zylinderfunktionen nullter Ordnung verwandter Funktionen. L.F.F. 20, pp. 341 to 372. 1943. A.R.C. 8699. June, 1945. |

## APPENDIX I

*Determination of $W_0''(\xi)$, $W_n(\xi)$ for $|\xi| > 1$*

1. *Expression of the Downwash in Terms of a Basic Integral.*—The equation connecting the downwash with the bound and free vorticities is

$$2\pi W(\xi) = \int_{-1}^{\infty} \frac{\Gamma + E}{\xi - u}\, du$$

<u>$W_0''(\xi)$.</u>

When
$$\Gamma_0'' = 2[\cot \tfrac{1}{2}\theta - \operatorname{cosec}\theta + i\omega \sin\theta],$$

we can show by using equation (2.2.3) that

$$E_0'' = -\,2i\omega \sin\theta, \qquad u < 1$$
$$= 0. \qquad u > 1$$

Hence
$$\Gamma_0'' + E_0'' = 2[\cot \tfrac{1}{2}\theta - \operatorname{cosec}\theta], \qquad u < 1$$
$$= 0. \qquad u > 1$$

$$2\pi W_0''(\xi) = 2\int_0^{\pi} \frac{\cot\frac{1}{2}\theta - \operatorname{cosec}\theta}{\cos\theta + \xi} \sin\theta\, d\theta,$$

$$\pi W_0''(\xi) = \int_0^{\pi} \frac{\cos\theta}{\cos\theta + \xi}\, d\theta = \pi - \xi\pi I_0,$$

*i.e.*,
$$W_0''(\xi) = 1 - \xi I_0,$$

where
$$\pi I_n = \int_0^{\pi} \frac{\cos n\theta}{\cos\theta + \xi}\, d\theta$$

<u>$W_1(\xi)$</u>

In the same way it can be shown that

$$\Gamma_1 + E_1 = \cot\tfrac{1}{2}\theta - 2\sin\theta, \qquad |u| < 1$$
$$= 0. \qquad u > 1$$

Therefore
$$2\pi W_1(\xi) = \int_0^{\pi} \frac{\cot\frac{1}{2}\theta - 2\sin\theta}{\cos\theta + \xi} \sin\theta\, d\theta,$$

$$= \int_0^{\pi} \frac{\cos\theta + \cos 2\theta}{\cos\theta + \xi}\, d\theta.$$

$$W_1(\xi) = \tfrac{1}{2}(I_1 + I_2).$$

<u>$W_n(\xi)$</u> $n \geqslant 2$

$$\Gamma_n + E_n = -\,2\sin n\theta,$$

and so

$$2\pi W_n = \int_0^\pi \frac{-2 \sin n\theta \sin \theta}{\cos \theta + \xi} \, d\theta,$$

$$= \int_0^\pi \frac{\cos (n + 1)\theta - \cos (n - 1)\theta}{\cos \theta + \xi} \, d\theta,$$

$$W_n = \tfrac{1}{2}\{I_{n+1} - I_{n-1}\}.$$

2. *Evaluation of $I_n$*

$$\pi I_n = \int_0^\pi \frac{\cos n\theta}{\cos \theta + \xi} \, d\theta ,$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \frac{e^{in\theta}}{\cos \theta + \xi} \, d\theta ,$$

$$= \frac{1}{i} \int_C \frac{z^n}{z^2 + 2\xi z + 1} \, dz ,$$

where $C$ is the unit circle and $z = e^{i\theta}$.

The integrand has simple poles at $z = \alpha, \beta$ where

$$\alpha = -\xi + \sqrt{(\xi^2 - 1)},$$

$$\beta = -\xi - \sqrt{(\xi^2 - 1)},$$

and

$$\beta = \frac{1}{\alpha}.$$

The residue at $\alpha$ is

$$\frac{\alpha^n}{\alpha - \frac{1}{\alpha}} = \frac{\{-\xi + \sqrt{(\xi^2 - 1)}\}^n}{2\sqrt{(\xi^2 - 1)}}$$

and at $\beta$ the residue is

$$\frac{\beta^n}{\beta - \frac{1}{\beta}} = -\frac{\{-\xi - \sqrt{(\xi^2 - 1)}\}^n}{2\sqrt{(\xi^2 - 1)}}.$$

Since $\alpha\beta = 1$, one pole lies inside the unit circle and one lies outside, or both lie on the circle. If $\xi > 1$, $\alpha$ lies inside $C$; if $\xi < -1$, $\beta$ lies inside $C$; and if $|\xi| < 1$, both poles lie on $C$.

By Cauchy's Theorem we get for $\xi > 1$

$$\pi I_n = 2\pi i \;\frac{1}{i}\; \frac{\{-\xi + \sqrt{(\xi^2 - 1)}\}^n}{2\sqrt{(\xi^2 - 1)}},$$

*i.e.*,

$$I_n = \frac{\{-\xi + \sqrt{(\xi^2 - 1)}\}^n}{\sqrt{(\xi^2 - 1)}}$$

and similarly for $\xi < -1$

$$I_n = -\frac{\{-\xi - \sqrt{(\xi^2 - 1)}\}^n}{\sqrt{(\xi^2 - 1)}}.$$

If $|\xi| < 1$, put $\xi = -\cos\phi$; the poles are then at $z = e^{\pm i\phi}$ and the residues at these points are

$$\pm\frac{e^{\pm in\phi}}{2i\sin\phi}.$$

We now integrate round the unit circle with small semi-circular indentations at $e^{\pm i\phi}$. As the radii of these indentations tend to zero the integrals about them tend to

$$-\pi i\left\{\pm\frac{e^{\pm in\phi}}{2i\sin\phi}\right\} = -\pm\pi\frac{e^{\pm in\phi}}{2\sin\phi}.$$

The sum of these integrals is

$$-\pi i\frac{\sin n\phi}{\sin\phi}$$

and so

$$P\int_{-\pi}^{\pi}\frac{e^{in\theta}}{\cos\theta+\xi}\,d\theta = \pi\frac{\sin n\phi}{\sin\phi}.$$

*i.e.*,

$$I_n = \frac{\sin n\phi}{\sin\phi}.$$

3. $W_0''(\xi)$, $W_n(\xi)$ for $|\xi| > 1$

$\xi > 1$.

$$W_0''(\xi) = 1 - \frac{\xi}{\sqrt{(\xi^2-1)}},$$

$$W_1(\xi) = \frac{1}{2\sqrt{(\xi^2-1)}}\{-\xi+\sqrt{(\xi^2-1)}\}\{1-\xi+\sqrt{(\xi^2-1)}\},$$

$$W_n(\xi) = \{-\xi+\sqrt{(\xi^2-1)}\}^n.$$

$\xi < -1$.

$$W_0''(\xi) = 1 + \frac{\xi}{\sqrt{(\xi^2-1)}},$$

$$W_1(\xi) = -\frac{\pi}{2\sqrt{(\xi^2-1)}}\{-\xi-\sqrt{(\xi^2-1)}\}\{1-\xi-\sqrt{(\xi^2-1)}\},$$

$$W_n(\xi) = \{-\xi-\sqrt{(\xi^2-1)}\}^n.$$

## APPENDIX II

*The Calculation of $W_0'(\xi)$ for $|\xi| > 1$*

1. *General Theory.*—The downwash equation is

$$2\pi W(\xi) = \int_{-1}^{+1} \frac{\Gamma(u)\,du}{\xi - u} - i\omega \int_{-1}^{+1} \frac{e^{-i\omega u}}{\xi - u}\,du \int_{-1}^{u} e^{i\omega v}\,\Gamma(v)\,dv$$

$$- i\omega \int_{1}^{\infty} \frac{e^{-i\omega u}}{\xi - u}\,du \int_{-1}^{+1} e^{i\omega v}\,\Gamma(v)\,dv$$

$$= \int_{-1}^{+1} \frac{\Gamma(u)\,du}{\xi - u} - i\omega \iint_S \frac{e^{-i\omega(u-v)}}{\xi - u}\,\Gamma(v)\,du\,dv,$$

where the integral is taken over the area shaded in the Fig. 1.

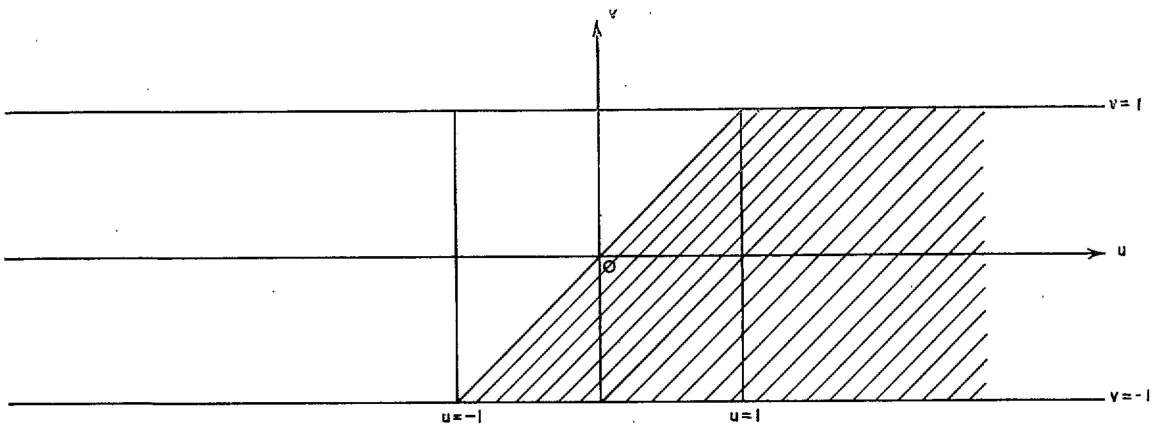



FIG. 1.

If we make the substitution $x = u - v - \xi$, $y = v$, the region is transformed into the region shown in Fig. 2

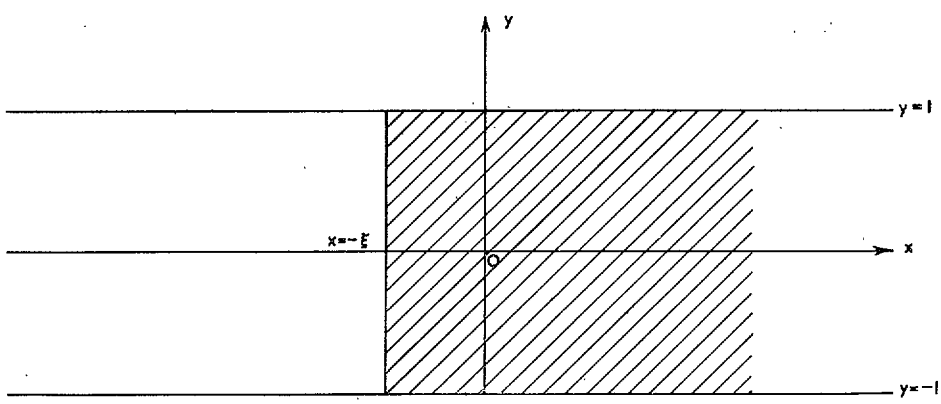



FIG. 2.

and the double integral becomes

$$\int_{-\xi}^{\infty} dx \int_{-1}^{+1} \frac{e^{-i\omega(x+\xi)}}{-x-y}\,\Gamma(y)\,dy.$$

The downwash equation is now

$$2\pi W(\xi) = \int_{-1}^{+1} \frac{\Gamma(u)}{\xi - u}\, du - i\omega\, e^{-i\omega\xi} \int_{-\xi}^{\infty} e^{-i\omega x}\, dx \int_{-1}^{+1} \frac{\Gamma(y)}{-x-y}\, dy.$$

If we write

$$G(\xi) = \int_{-1}^{+1} \frac{\Gamma(u)}{\xi - u}\, du,$$

we get

$$2\pi W(\xi) = G(\xi) - i\omega\, e^{-i\omega\xi} \int_{-\xi}^{\infty} e^{-i\omega x}\, G(-x)\, dx,$$

$$= G(\xi) - i\omega\, e^{-i\omega\xi} \int_{-\infty}^{\xi} e^{i\omega u}\, G(u)\, du.$$

When

$$\Gamma_0' = 2\{\operatorname{cosec}\theta - (1 - C)\cot \tfrac{1}{2}\theta\},$$

$$G(\xi) = 2\int_{0}^{\pi} \frac{\operatorname{cosec}\theta - (1 - C)\cot\frac{1}{2}\theta}{\cos\theta + \xi} \sin\theta\, d\theta,$$

$$= 2\int_{0}^{\pi} \frac{C - (1 - C)\cos\theta}{\cos\theta + \xi}\, d\theta,$$

$$= -2(1 - C)\pi + 2\pi\{C + (1 - C)\xi\} I_0.$$

If

$$|\xi| < 1 \qquad G(\xi) = -2(1 - C)\pi$$

$$\xi > 1 \qquad = -2(1 - C)\pi + \frac{2\pi}{\sqrt{(\xi^2 - 1)}}\{C + \xi(1 - C)\}$$

$$\xi < -1 \qquad = -2(1 - C)\pi - \frac{2\pi}{\sqrt{(\xi^2 - 1)}}\{C + \xi(1 - C)\}.$$

2. *The Downwash $W_0'(\xi)$ for $|\xi| < 1$.*—The downwash equation is

$$2\pi W(\xi) = G(\xi) - i\omega\, e^{-i\omega\xi} \int_{-\infty}^{\xi} e^{i\omega u}\, G(u)\, du.$$

$G(u)$ is defined differently in the ranges $(-\infty, -1)$, $(-1, 1)$ and so we shall consider the integrals over these ranges separately. In the range $(-\infty, -1)$

$$G(u) = -2\pi\left\{(1 - C) + \frac{C + u(1 - C)}{\sqrt{(u^2 - 1)}}\right\}$$

and so

$$\int_{-\infty}^{-1} e^{i\omega u}\, G(u)\, du = -2\pi \int_{-\infty}^{-1} e^{i\omega u}\left\{(1 - C) + \frac{C + u(1 - C)}{\sqrt{(u^2 - 1)}}\right\} du,$$

$$= -2\pi \int_{0}^{\infty} e^{-i\omega\cosh t}\{(1 - C)\sinh t + C - (1 - C)\cosh t\}\, dt,$$

$$= -\, 2\pi\int_0^\infty e^{-i\omega\cosh t}\,[C - (1 - C)\,e^{-t}]\,dt,$$

$$= -\, 2\pi C\int_0^\infty e^{-i\omega\cosh t}\,dt + 2\pi\,(1 - C)\int_0^\infty e^{-i\omega\cosh t - t}\,dt.$$

The first integral is $K_0(i\omega)$. To evaluate the second integral, consider

$$I = \int_0^\infty e^{-t - x\cosh t}\,dt,$$

where $x$ is real and positive.

$$I = \int_0^\infty e^{-x\cosh t}\,(\cosh t - \sinh t)\,dt,$$

$$= K_1(x) - \left[-\frac{1}{x}\,e^{-x\cosh t}\right]_0^\infty$$

$$= K_1(x) - \frac{e^{-x}}{x}.$$

The integral

$$\int_0^\infty e^{-t - z\cosh t}\,dt$$

is uniformly convergent, and so is an analytic function of $z$, for $R(z) \geqslant 0$. The function

$$K_1(z) - \frac{e^{-z}}{z}$$

is analytic in the $z$ plane cut from 0 to $-\infty$.

For $z$ real, $z \geqslant 0$ we have

$$\int_0^\infty e^{-t - z\cosh t}\,dt = K_1(z) - \frac{e^{-z}}{z}$$

and so by analytic continuation it must hold for $z$ complex for $R(z) \geqslant 0$.

If we put $z = i\omega$ we get

$$\int_0^\infty e^{-t - i\omega\cosh t}\,dt = K_1(i\omega) - \frac{e^{-i\omega}}{i\omega}.$$

In the range $(-\,1, 1)$

$$G(u) = -\,2\pi\,(1 - C)$$

and so

$$\int_{-1}^{\xi} G(u)\,e^{i\omega u}\,du = -\,2\pi\,(1 - C)\int_{-1}^{\xi} e^{i\omega u}\,du$$

$$= -\,\frac{2\pi\,(1 - C)}{i\omega}\,[e^{i\omega\xi} - e^{-i\omega}]\,.$$

Therefore

$$W_0'(\xi) = -(1-C) - i\omega\, e^{-i\omega\xi}\Big[-\frac{(1-C)}{i\omega}\{e^{i\omega\xi} - e^{-i\omega}\} - CK_0(i\omega) +$$

$$+ (1-C)\Big\{K_1(i\omega) - \frac{e^{-i\omega}}{i\omega}\Big\}\Big]$$

$$= -i\omega\, e^{i\omega\xi}\{(1-C)\,K_1(i\omega) - CK_0(i\omega)\}\,.$$

$$W_0'(\xi) = 0 \text{ if}$$

$$(1-C)K_1(i\omega) - CK_0(i\omega) = 0,$$

*i.e.*,

$$C = \frac{K_1(i\omega)}{K_0(i\omega) + K_1(i\omega)}\,.$$

With this value of $C$

$$\int_{-\infty}^{-1} e^{i\omega u}\, G(u)\, du = -2\pi\,(1-C)\frac{e^{-i\omega}}{i\omega}\,.$$

3. <u>$W_0'(\xi)$ for $\xi > 1$</u>

$$2\pi W(\xi) = G(\xi) - i\omega\, e^{-i\omega\xi}\int_{-\infty}^{\xi} e^{i\omega u}\, G(u)\, du$$

$$= G(\xi) - i\omega\, e^{-i\omega\xi}\Big\{2\pi\,(C-1)\,\frac{e^{-i\omega}}{i\omega} + \int_{-1}^{+1} e^{i\omega u}\, G(u)\, du + \int_{1}^{\xi} e^{i\omega u}\, G(u)\, du\Big\}\,,$$

$$\int_{-1}^{+1} e^{i\omega u}\, G(u)\, du = 2\pi(C-1)\int_{-1}^{+1} e^{i\omega u}\, du$$

$$= 2\pi\,(C-1)\,\frac{e^{i\omega} - e^{-i\omega}}{i\omega}\,,$$

$$\int_{+1}^{\xi} e^{i\omega u}\, G(u)\, du = 2\pi\int_{1}^{\xi}\Big\{(C-1) + \frac{C + u\,(1-C)}{\sqrt{(u^2-1)}}\Big\}e^{i\omega u}\, du$$

$$= 2\pi\,(C-1)\,\frac{(e^{i\omega\xi} - e^{i\omega})}{i\omega} + 2\pi CI + 2\pi\,(1-C)J,$$

where

$$I = \int_1^{\xi}\frac{e^{i\omega u}}{\sqrt{(u^2-1)}}\, du, \qquad J = \int_1^{\xi}\frac{u\, e^{i\omega u}}{\sqrt{(u^2-1)}}\, du\,.$$

$$W_0'(\xi) = (C-1) + \frac{C + \xi\,(1-C)}{\sqrt{(\xi^2-1)}} - (C-1)\, e^{-i\omega(\xi+1)}$$

$$-(C-1)\{e^{i\omega - \overline{\xi+1}} - e^{-i\omega\overline{\xi+1}}\}$$

$$- (C-1)\{1 - e^{+i\omega\overline{1-\xi}}\} - \{CI + (1-C)J\}i\omega\, e^{-i\omega\xi}$$

$$= \frac{C + \xi\,(1-C)}{\sqrt{(\xi^2 - 1)}} - \{CI + (1-C)J\}i\omega\, e^{-i\omega\xi}.$$

$$W_0''(\xi) = 1 - \frac{\xi}{\sqrt{(\xi^2-1)}}\,.$$

Therefore

$$W_0(\xi) = 1 - C\sqrt{\left(\frac{\xi-1}{\xi+1}\right)} - i\omega\, e^{-i\omega\xi}\{CI + (1-C)J\}.$$

4. $\underline{W_0'(\xi) \text{ for } \xi < -1}$

$$2\pi W(\xi) = G(\xi) - i\omega\, e^{-i\omega\xi}\int_{-\infty}^{\xi} e^{i\omega u}\, G(u)\, du,$$

$$= G(\xi) - i\omega\, e^{-i\omega\xi}\left\{2\pi\,(C-1)\frac{e^{-i\omega}}{i\omega} + \int_{-1}^{\xi} e^{i\omega u}\, G(u)\, du\right\},$$

$$= G(\xi) - 2\pi\,(C-1)\, e^{-i\omega(1+\xi)} - i\omega\, e^{-i\omega\xi}\int_{-1}^{\xi} e^{i\omega u}\, G(u)\, du.$$

Now

$$\int_{-1}^{+\xi} e^{i\omega u}\, G(u)\, du = -2\pi\int_{-1}^{+\xi}\left\{(1-C) + \frac{C + u\,(1-C)}{\sqrt{(u^2-1)}}\right\} e^{i\omega u}\, du,$$

$$= 2\pi\int_{1}^{\xi'}\left\{(1-C) + \frac{C - u\,(1-C)}{\sqrt{(u^2-1)}}\right\} e^{i\omega u}\, du,$$

where

$$\xi' = -\xi$$

$$= -\frac{2\pi\,(1-C)}{i\omega}\{e^{-i\omega\xi'} - e^{-i\omega}\} + 2\pi CP - 2\pi(1-C)\,Q,$$

where

$$P = \int_{+1}^{+\xi'}\frac{e^{-i\omega u}}{\sqrt{(u^2-1)}}\, du, \qquad Q = \int_{+1}^{\xi'}\frac{u\, e^{-i\omega u}}{\sqrt{(u^2-1)}}\, du.$$

Therefore

$$W_0'(\xi) = -(1-C) - \frac{C + \xi\,(1-C)}{\sqrt{(\xi^2-1)}} - (C-1)\, e^{-i\omega(1+\xi)}$$

$$- (1-C)\{e^{-i\omega(\xi+\xi')} - e^{-i\omega(1+\xi)}\} - i\omega\, e^{-i\omega\xi}\,[CP - (1-C)Q],$$

$$= -\frac{C + \xi\,(1-C)}{\sqrt{(\xi^2-1)}} - i\omega\, e^{-i\omega\xi}\{CP - (1-C)Q\}\,,$$

$$W''_0(\xi) = 1 + \frac{\xi}{\sqrt{(\xi^2-1)}}\,,$$

$$W_0(\xi) = 1 - C\sqrt{\left(\frac{\xi - 1}{\xi + 1}\right)} - i\omega\, e^{-i\omega\xi}\, [CP - (1 - C)Q].$$

5. *The integrals I, J, P, Q.*—Scharz[2] has introduced two new functions

$$He^{(1)}(\lambda,x) = \int_0^x H_0^{(1)}(\lambda u)\, e^{iu}\, du$$

$$He^{(2)}(\lambda,x) = \int_0^x H_0^{(2)}(\lambda u)\, e^{iu}\, du,$$

where $H_0^{(1)}$, $H_0^{(2)}$ are Hankel functions of order zero, and using them he has been able to evaluate the integrals $I$ and $J$. He gives the following expressions for the integrals:

$$I = \int_1^\xi \frac{e^{i\omega x}}{\sqrt{(x^2 - 1)}}\, dx = -\frac{\pi\sqrt{(\xi^2 - 1)}}{4\xi} \begin{vmatrix} He^{(1)}\left(\frac{1}{\xi}, \omega\xi\right) & He^{(2)}\left(\frac{1}{\xi}, \omega\xi\right) \\ H_0^{(1)}(\omega) & H_0^{(2)}(\omega) \end{vmatrix}$$
$$+ J_0(\omega) \log\{\xi + \sqrt{(\xi^2 - 1)}\}$$

and

$$J = \int_1^\xi \frac{x\, e^{i\omega x}}{\sqrt{(x^2 - 1)}}\, dx = \frac{\pi i\sqrt{(\xi^2 - 1)}}{4\xi} \begin{vmatrix} He^{(1)}\left(\frac{1}{\xi}, \omega\xi\right) & He^{(2)}\left(\frac{1}{\xi}, \omega\xi\right) \\ H_0^{(1)\prime}(\omega) & H_0^{(2)\prime}(\omega) \end{vmatrix}$$
$$- iJ_0'(\omega) \log\{\xi + \sqrt{(\xi^2 - 1)}\},$$

where $J_0(\omega)$ is the Bessel function of order zero.

The functions $He^{(1)}(\lambda,x)$, $He^{(2)}(\lambda,x)$ have been tabulated for $x = 0(0\cdot02)2(0\cdot1)5$ for $\lambda = 0(0\cdot1)1$.

The integrals $P$ and $Q$ can be found by calculating $I$ and $J$ and then taking the conjugate values.

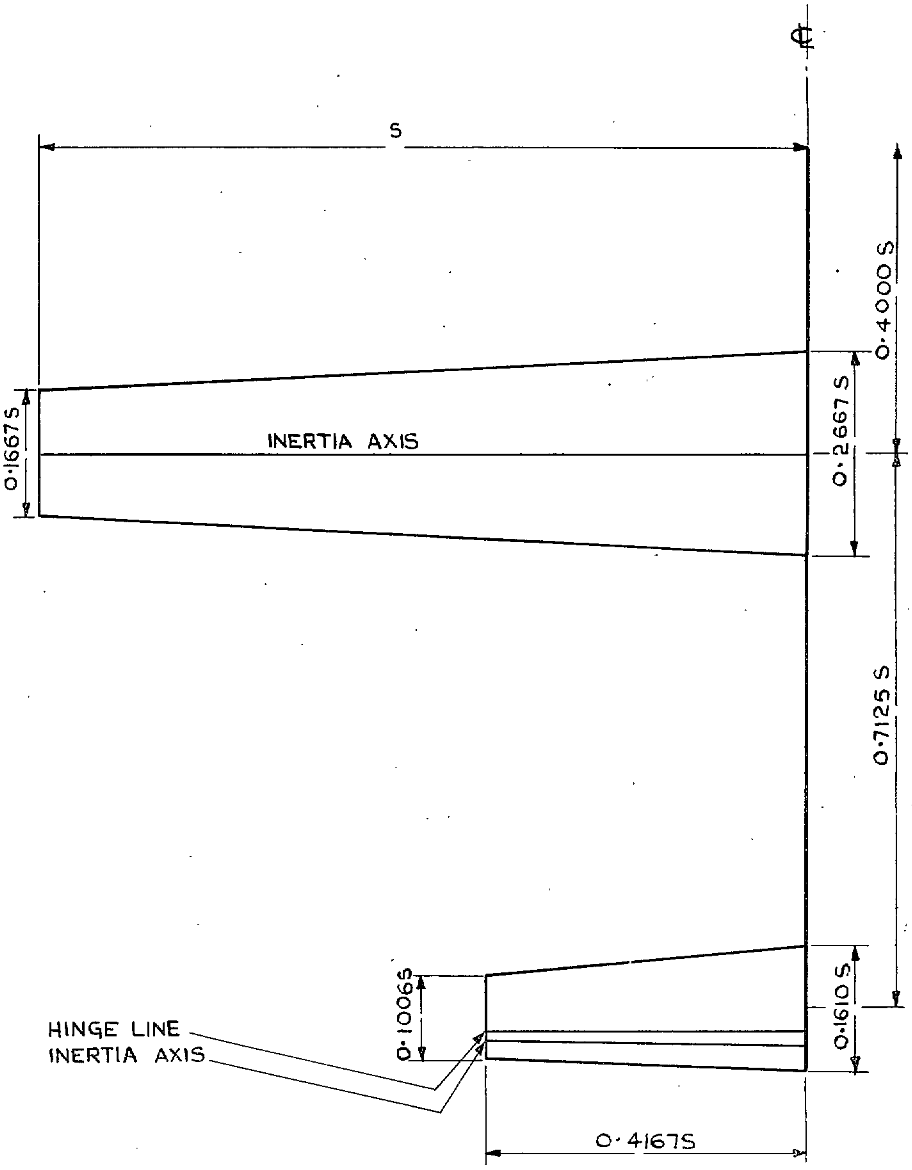



FIG. 1. Plan of aircraft (The unit of length is $S = 60$ ft).

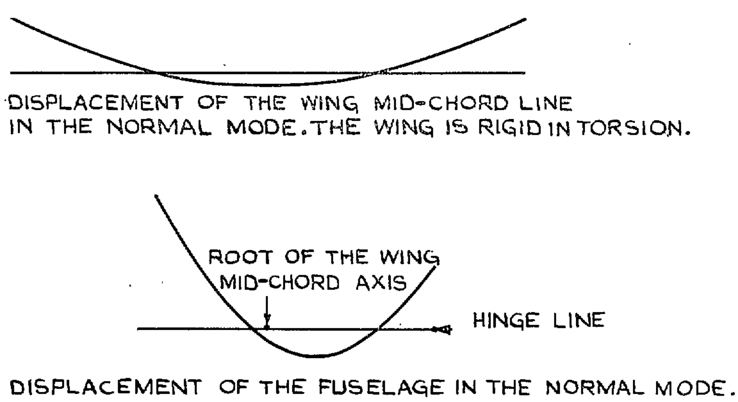



FIG. 2. The normal mode.

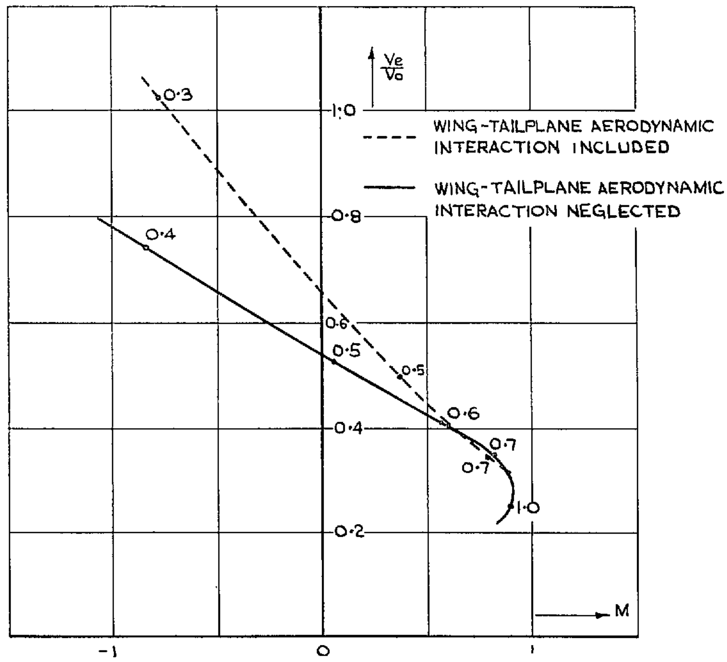



FIG. 3. Flutter curve—Sea level.

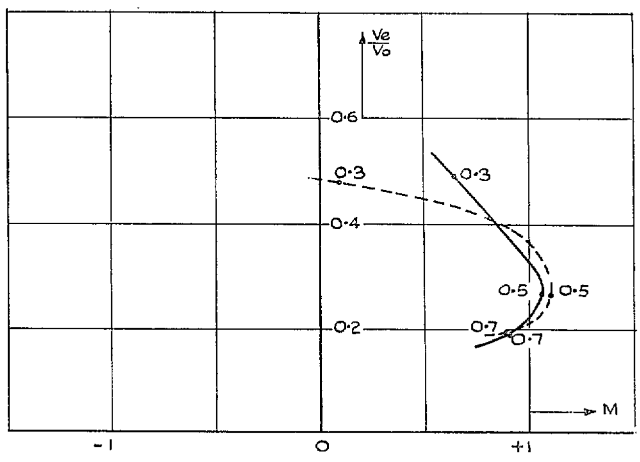



FIG. 4. Flutter curve—33,000 ft.

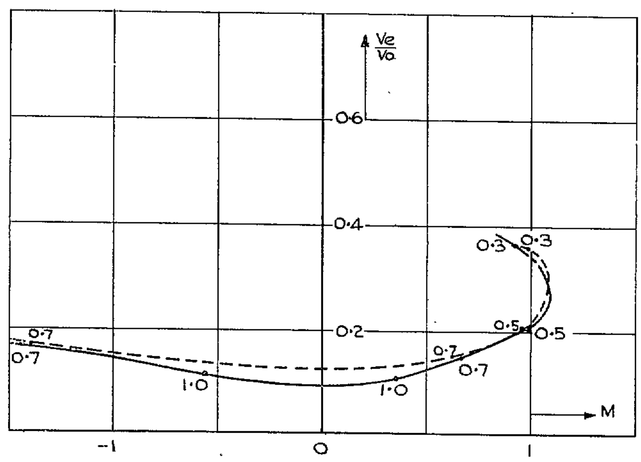



FIG. 5. Flutter curve—45,000 ft.

(41872) Wt. 52 K7 10/57 F. M. & S.



PRINTED IN GREAT BRITAIN

# Publication of the Aeronautical Research Council

## ANNUAL TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL (BOUND VOLUMES)

1939 Vol. I. Aerodynamics General, Performance, Airscrews, Engines. 50*s.* (52*s.*)
Vol. II. Stability and Control, Flutter and Vibration, Instruments, Structures, Seaplanes, etc. 63*s.* (65*s.*)

1940 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Icing, Stability and Control Structures, and a miscellaneous section. 50*s.* (52*s.*)

1941 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Stability and Control Structures. 63*s.* (65*s.*)

1942 Vol. I. Aero and Hydrodynamics, Aerofoils, Airscrews, Engines. 75*s.* (77*s.*)
Vol. II. Noise, Parachutes, Stability and Control, Structures, Vibration, Wind Tunnels. 47*s.* 6*d.* (49*s.* 6*d.*)

1943 Vol. I. Aerodynamics, Aerofoils, Airscrews. 80*s.* (82*s.*)
Vol. II. Engines, Flutter, Materials, Parachutes, Performance, Stability and Control, Structures. 90*s.* (92*s.* 9*d.*)

1944 Vol. I. Aero and Hydrodynamics, Aerofoils, Aircraft, Airscrews, Controls. 84*s.* (86*s.* 6*d.*)
Vol. II. Flutter and Vibration, Materials, Miscellaneous, Navigation, Parachutes, Performance, Plates and Panels, Stability, Structures, Test Equipment, Wind Tunnels. 84*s.* (86*s.* 6*d.*)

1945 Vol. I. Aero and Hydrodynamics, Aerofoils. 130*s.* (132*s.* 9*d.*)
Vol. II. Aircraft, Airscrews, Controls. 130*s.* (132*s.* 9*d.*)
Vol. III. Flutter and Vibration, Instruments, Miscellaneous, Parachutes, Plates and Panels, Propulsion. 130*s.* (132*s.* 6*d.*)
Vol. IV. Stability, Structures, Wind Tunnels, Wind Tunnel Technique. 130*s.* (132*s.* 6*d.*)

**Annual Reports of the Aeronautical Research Council—**

1937 2*s.* (2*s.* 2*d.*) 1938 1*s.* 6*d.* (1*s.* 8*d.*) 1939–48 3*s.* (3*s.* 5*d.*)

**Index to all Reports and Memoranda published in the Annual Technical Reports, and separately—**

April, 1950 - - - - R. & M. 2600 2*s.* 6*d.* (2*s.* 10*d.*)

**Author Index to all Reports and Memoranda of the Aeronautical Research Council—**

1909—January, 1954 R. & M. No. 2570 15*s.* (15*s.* 8*d.*)

**Indexes to the Technical Reports of the Aeronautical Research Council—**

| | | |
|---|---|---|
| December 1, 1936—June 30, 1939 | R. & M. No. 1850 | 1*s.* 3*d.* (1*s.* 5*d.*) |
| July 1, 1939–June 30, 1945 | R. & M. No. 1950 | 1*s.* (1*s.* 2*d.*) |
| July 1, 1945—June 30, 1946 | R. & M. No. 2050 | 1*s.* (1*s.* 2*d.*) |
| July 1, 1946—December 31, 1946 | R. & M. No. 2150 | 1*s.* 3*d.* (1*s.* 5*d.*) |
| January 1, 1947—June 30, 1947 | R. & M. No. 2250 | 1*s.* 3*d.* (1*s.* 5*d.*) |

**Published Reports and Memoranda of the Aeronautical Research Council—**

| | | |
|---|---|---|
| Between Nos. 2251–2349 | R. & M. No. 2350 | 1*s.* 9*d.* (1*s.* 11*d.*) |
| Between Nos. 2351–2449 | R. & M. No. 2450 | 2*s.* (2*s.* 2*d.*) |
| Between Nos. 2451–2549 | R. & M. No. 2550 | 2*s.* 6*d.* (2*s.* 10*d.*) |
| Between Nos. 2551–2649 | R. & M. No. 2650 | 2*s.* 6*d.* (2*s.* 10*d.*) |
| Between Nos. 2651–2749 | R. & M. No. 2750 | 2*s.* 6*d.* (2*s.* 10*d.*) |

*Prices in brackets include postage*

HER MAJESTY'S STATIONERY OFFICE

York House, Kingsway, London W.C.2; 423 Oxford Street, London W.1; 13a Castle Street, Edinburgh 2; 39 King Street, Manchester 2; 2 Edmund Street, Birmingham 3; 109 St. Mary Street, Cardiff; Tower Lane, Bristol, 1; 80 Chichester Street, Belfast, *or through any bookseller.*

S.O. Code No. 23-3065