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for a Cropped Delta Wing with  
Subsonic Leading Edges Oscillating  
in a Supersonic Air Stream

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# Calculation of Derivatives for a Cropped Delta Wing with Subsonic Leading Edges Oscillating in a Supersonic Air Stream

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*Summary.*—The lift, pitching moment and full-span constant-chord control hinge-moment are derived for a cropped delta wing describing harmonic plunging and pitching oscillations of small amplitude and low-frequency parameter in a supersonic air stream. It is assumed that (a) the wing has subsonic leading edges, (b) the wing is sufficiently thin and the Mach number sufficiently supersonic to permit the use of linearised theory.

Expressions for the various derivative coefficients are obtained for a particular delta wing of aspect ratio 1.8 and taper ratio 1/7; these are evaluated and tabulated for Mach numbers 1.1, 1.15, 1.2, 1.3, 1.4, 1.5, 1.6 and 1.944.

1. *Introduction.*—This report considers theoretical derivatives of lift, pitching moment and hinge moment corresponding to slow plunging and pitching oscillations of a cropped delta wing with subsonic leading edges. Values of these derivatives are determined for comparison with experimental data to be obtained at supersonic speeds. Measurements of the oscillatory derivatives for a wing of aspect ratio 1.8 and taper ratio 1/7 with a full-span constant-chord flap (Fig. 1) are in progress at the National Physical Laboratory for Mach numbers up to  $M = 1.8$ .

In a previous paper the author<sup>1</sup> (1955) determines theoretical derivatives for a cropped delta wing with constant-chord flap, when the flap alone is oscillating. In conjunction with Ref. 1, the present theory yields approximate expressions for the complete set of theoretical derivatives for low-frequency longitudinal oscillations of the wing-flap combination.

In a supersonic air stream the loading on part of a thin plane cropped delta wing of the plan-form shown in Fig. 1 is identically that on the complete triangular wing performing the same motion. Over the remaining part of the wing the velocity potential has been calculated by Eyyvard's<sup>2</sup> (1950) method (section 3).

Formulae for the aerodynamic coefficients and derivatives are obtained in section 4; the accuracy of the results is discussed in section 5.

2. *General Supersonic Theory.*—The governing equation and the boundary condition are made non-dimensional and, under the assumption of simple harmonic motion, a transformation is made which, for low frequency, reduces the unsteady problem to a steady one at the particular Mach number  $\sqrt{2}$  (e.g., Watson<sup>1</sup> (1955)).

2.1. *Governing Equation.*—The perturbation velocity potential,  $\phi$ , satisfies

$$\frac{\partial^2 \phi}{\partial t^2} + 2U \frac{\partial^2 \phi}{\partial x \partial t} + U^2 \frac{\partial^2 \phi}{\partial x^2} = a^2 \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right), \quad \dots \quad (2.1)$$

where  $a$  = velocity of sound,  $U$  = velocity of air stream.

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since  $\omega = vU/\bar{c}$  and  $x = \bar{c}\beta X$ . From this result the upwash on the transformed wing is given by

$$\begin{aligned} W &= \left( \frac{\partial \Phi}{\partial Z} \right)_{z=0} = \bar{c} \left( \frac{\partial \Phi}{\partial z} \right)_{z=0} = \bar{c} \left( \frac{\partial \phi}{\partial z} \right)_{z=0} \exp \left\{ -ivT + iv \left( \beta + \frac{1}{\beta} \right) X \right\} \\ &= -\alpha_0 \bar{c} U \left[ (1 - ivh) + iv \left( 2\beta + \frac{1}{\beta} \right) X \right] \end{aligned}$$

on neglecting squares of  $v$ . Thus the upwash on the transformed wing may be written in the form

$$W = W_1 + W_2, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (2.7)$$

where  $W_1 = -\alpha_0 \bar{c} U (1 - ivh)$  and  $W_2 = -qX$ , where  $q = iv\alpha_0 \bar{c} U \{2\beta + (1/\beta)\}$ . Since  $W_1$  is a constant, it may be regarded as the upwash when the transformed wing is at a constant incidence and since  $W_2 = -qX$ , where  $q$  is a constant,  $W_2$  may be regarded as the upwash on the transformed wing when it is pitching about the axis  $X = Z = 0$  with a constant angular velocity  $q$ . The condition  $W = W_1$  will be called the uniform incidence case and  $W = W_2$  the uniform pitching case, both being governed by (2.5), the quasi-steady equation corresponding to  $M = \sqrt{2}$ .

Let  $\Phi(X, Y)$  denote the value of  $\Phi$  at the point  $(X, Y)$  on the upper surface of the wing and  $(\partial\Phi/\partial X)_{z=0}$  the corresponding value of  $\partial\Phi/\partial X$ ; on the lower surface of the wing,  $\Phi = -\Phi(X, Y)$  and  $\partial\Phi/\partial X = -(\partial\Phi/\partial X)_{z=0}$ .

The pressure difference across the wing in the positive  $z$ -direction or the lift per unit area is

$$\begin{aligned} P &= 2\rho_0 \left( \frac{\partial \phi}{\partial t} + U \frac{\partial \phi}{\partial x} \right)_{z=0} \\ &= \frac{2\rho_0 U}{\bar{c}} \left( \frac{\partial \phi}{\partial T} + \frac{1}{\beta} \frac{\partial \phi}{\partial X} \right)_{z=0} \\ &= \frac{2\rho_0 U}{\beta \bar{c}} \left\{ \left( \frac{\partial \Phi}{\partial X} \right)_{z=0} - \frac{iv}{\beta} \Phi(X, Y) \right\} \exp \left\{ ivT - iv \left( \beta + \frac{1}{\beta} \right) X \right\}. \end{aligned}$$

Since terms of order  $v^2$  are being neglected the lift per unit area may be written as

$$P = \frac{2\rho_0 U}{\beta \bar{c}} e^{ivT} \left[ \left\{ 1 - iv \left( \beta + \frac{1}{\beta} \right) X \right\} \left( \frac{\partial \Phi}{\partial X} \right)_{z=0} - \frac{iv}{\beta} \Phi(X, Y) \right]. \quad \dots \quad \dots \quad (2.8)$$

It is convenient to introduce the non-dimensional loading function,  $F$ , defined by

$$F = P\bar{c}\beta / (2\rho_0 U e^{ivT}), \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (2.9)$$

so that, from (2.8)

$$F = \left\{ 1 - iv \left( \beta + \frac{1}{\beta} \right) X \right\} \left( \frac{\partial \Phi}{\partial X} \right)_{z=0} - \frac{iv}{\beta} \Phi(X, Y). \quad \dots \quad \dots \quad \dots \quad (2.10)$$

When  $F$  has been determined over the transformed wing the aerodynamic coefficients are found as follows: because of the symmetry about  $Y = 0$ , the complex lift coefficient is

$$\begin{aligned} C_L &= \frac{L}{\frac{1}{2}\rho_0 U^2 S} = \frac{2\bar{c}^2 \beta \iint_{A+B+C} P dX dY}{\frac{1}{2}\rho_0 U^2 2s\bar{c}} = \frac{2\bar{c}\beta}{\rho_0 U^2 s} \iint_{A+B+C} P dX dY \\ &= \frac{4e^{ivT}}{U s} \iint_{A+B+C} F dX dY; \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (2.11) \end{aligned}$$

the complex pitching-moment coefficient about the axis through the apex is

$$\begin{aligned}
C_m &= \frac{\mathcal{M}}{\frac{1}{2}\rho_0 U^2 S \bar{c}} \\
&= -\frac{2\bar{c}\beta^2}{\rho_0 U^2 s} \iint_{A+B+C} X P \, dX \, dY \\
&= -\frac{4\beta}{U_s} e^{i\nu T} \iint_{A+B+C} X F \, dX \, dY ; \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (2.12)
\end{aligned}$$

the complex hinge-moment coefficient about  $x = \bar{c}h_0$  is

$$\begin{aligned}
C_H &= \frac{H}{\frac{1}{2}\rho_0 U^2 S_f c_f} \\
&= -\frac{2\bar{c}^3\beta^2}{\frac{1}{2}\rho_0 U^2 2sc_f^2} \iint_{B+C} \left(X - \frac{h_0}{\beta}\right) P \, dX \, dY \\
&= -\frac{2\bar{c}^3\beta^2}{\rho_0 U^2 sc_f^2} \frac{2\rho_0 U e^{i\nu T}}{\beta\bar{c}} \iint_{B+C} \left(X - \frac{h_0}{\beta}\right) F \, dX \, dY \\
&= -\frac{4\beta}{U_s} e^{i\nu T} \left(\frac{\bar{c}}{c_f}\right)^2 \iint_{B+C} \left(X - \frac{h_0}{\beta}\right) F \, dX \, dY \\
&= \left(\frac{\bar{c}}{c_f}\right)^2 \left[ C_m + \frac{4\beta}{U_s} e^{i\nu T} \iint_A X F \, dX \, dY + h_0 C_L - h_0 \frac{4e^{i\nu T}}{U_s} \iint_A F \, dX \, dY \right]. \quad \dots \quad (2.13)
\end{aligned}$$

The suffices of the integral signs refer to the areas of integration (Fig. 3),

$$\left. \begin{aligned}
A &\text{ denoting the area } 0 \leq X \leq X_0, 0 \leq Y \leq m_1 X \\
B &\text{ denoting the area } X_0 \leq X \leq X_1, 0 \leq Y \leq X_0(1 + m_1) - X \\
C &\text{ denoting the area } X_0 \leq X \leq X_1, X_0(1 + m_1) - X \leq Y \leq m_1 X_0
\end{aligned} \right\}, \quad \dots \quad (2.14)$$

where  $X_1 = c_0/\beta\bar{c}$ ,  $X_0 = (c_0 - c_f)/\beta\bar{c}$  and  $m_1 = \beta s/(c_0 - c_f)$ .

It is assumed that the Mach lines from the tips of the leading edges do not intersect on the wing and that the wing has subsonic leading edges, so that

$$(X_1 - X_0)/X_0 \leq m_1 \leq 1, \quad \dots \quad \dots \quad \dots \quad \dots \quad (2.15)$$

which is equivalent to

$$\left[1 + \left(\frac{c_f}{s}\right)^2\right]^{1/2} \leq M \leq \left[1 + \left(\frac{c_0 - c_f}{s}\right)^2\right]^{1/2}. \quad \dots \quad \dots \quad (2.16)$$

For most practical cropped delta plan-forms the restriction imposed by the lower limit of  $M$  is unimportant; for the particular wing of aspect ratio 1.8 and taper ratio 1/7, being tested at the N.P.L., equation (2.16) gives the condition  $1.038 \leq M \leq 1.944$ .

3. *Velocity Potential*.—Let  $\Phi_1$  be the solution of equation (2.5) subject to the boundary condition  $(\partial\Phi_1/\partial Z)_{Z=0} = W_1$  (the uniform incidence case) and  $\Phi_2$  the solution of equation (2.5)

subject to the boundary condition  $(\partial\Phi_2/\partial Z)_{Z=0} = W_2$  (the uniform pitching case). Then

$$\Phi = \Phi_1 + \Phi_2 \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (3.1)$$

is the solution of (2.5) subject to (2.7). Non-dimensional loading functions  $F_1, F_2$  are defined by

$$F_1 = \left\{ 1 - iv \left( \beta + \frac{1}{\beta} \right) X \right\} \left( \frac{\partial\Phi_1}{\partial X} \right)_{Z=0} - \frac{iv}{\beta} \Phi_1(X, Y) \quad \dots \quad \dots \quad (3.2)$$

and

$$F_2 = \left\{ 1 - iv \left( \beta + \frac{1}{\beta} \right) X \right\} \left( \frac{\partial\Phi_2}{\partial X} \right)_{Z=0} - \frac{iv}{\beta} \Phi_2(X, Y), \quad \dots \quad \dots \quad (3.3)$$

so that, by (2.10) and (3.1)

$$F = F_1 + F_2. \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (3.4)$$

Expressions for  $\Phi_1(X, Y)$  and  $F_1$  are derived in section 3.1 and those for  $\Phi_2(X, Y)$  and  $F_2$  in section 3.2. Because of the symmetry about the plane  $Y = 0$ , only the region  $Y \geq 0$  is considered. Under the condition (2.15) the flow in regions  $A$  and  $B$  (Fig. 3) is the same as that over the infinite triangular wing formed by producing the leading edges downstream; there are known solutions for the velocity potential of an infinite triangular wing with subsonic leading edges at uniform incidence or with uniform pitching (Ref. 3), so that  $\Phi_1(X, Y)$  and  $\Phi_2(X, Y)$  are known in regions  $A$  and  $B$ . The potentials  $\Phi_1(X, Y)$  and  $\Phi_2(X, Y)$  in region  $C$  are found by Evvard's method in the case of steady flow at Mach number  $\sqrt{2}$  (Ref. 2, equation (29)).

**3.1. Velocity Potential for Uniform Incidence.**—Since the flow in regions  $A$  and  $B$  is the same as that over the infinite triangular wing with subsonic leading edges  $Y = \pm m_1 X$ , then in those regions  $\Phi_1(X, Y)$  is the perturbation-velocity potential on the triangular wing at a uniform incidence  $-W_1/U$  in a stream with Mach number  $\sqrt{2}$ . This perturbation-velocity potential is known (Ref. 3, p. 302, equation (148)); in the present notation

$$\left( \frac{\partial\Phi_1}{\partial X} \right)_{Z=0} = - \frac{W_1 m_1^2 X}{E(k)(m_1^2 X^2 - Y^2)^{1/2}}, \quad \dots \quad \dots \quad \dots \quad (3.5)$$

where  $E(k)$  is the complete elliptic integral of the second kind with modulus  $k = (1 - m_1^2)^{1/2}$ . From (3.5) it follows that

$$\Phi_1(X, Y) = - \frac{W_1}{E(k)} (m_1^2 X^2 - Y^2)^{1/2}, \quad \dots \quad \dots \quad \dots \quad (3.6)$$

since  $\Phi_1(X, Y)$  is zero on  $Y = m_1 X$ . From (3.2), (3.5) and (3.6), the loading function in regions  $A$  and  $B$

$$F_1 = - \frac{W_1}{E(k)} \left[ \left\{ 1 - iv \left( \beta + \frac{1}{\beta} \right) X \right\} \frac{m_1^2 X}{(m_1^2 X^2 - Y^2)^{1/2}} - \frac{iv}{\beta} (m_1^2 X^2 - Y^2)^{1/2} \right]. \quad \dots \quad (3.7)$$

In region  $C$ , Evvard's method for steady flow at  $M = \sqrt{2}$  (Ref. 2, equation (29)) gives  $\Phi_1(X, Y)$  as a double integral over a rectangular region of the type  $S_1 + S_2$  (Fig. 4), namely,

$$\Phi_1(X, Y) = - \frac{1}{\pi\sqrt{2}} \iint_{S_1 + S_2} \frac{\left( \frac{\partial\Phi_1}{\partial Z} \right)_{Z=0} dr_0 ds_0}{(r - r_0)^{1/2} (s - s_0)^{1/2}}, \quad \dots \quad \dots \quad \dots \quad (3.8)$$

where

$$\left. \begin{aligned} r\sqrt{2} &= (X - Y) \\ s\sqrt{2} &= (X + Y) \end{aligned} \right\} \dots \dots \dots \dots \quad (3.9)$$

In the region  $S_2$  on the plan-form the upwash  $(\partial\Phi_1/\partial Z)_{Z=0}$  equals  $W_1$  (equation (2.7)), so that (3.8) may be written in the form

$$-\frac{\pi}{W_1} \Phi_1(X, Y) = \frac{1}{W_1\sqrt{2}} \iint_{S_1} \frac{\left(\frac{\partial\Phi_1}{\partial Z}\right)_{Z=0} dr_0 ds_0}{(r - r_0)^{1/2} (s - s_0)^{1/2}} + C(X, Y), \quad \dots \quad (3.10)$$

where

$$\begin{aligned} C(X, Y) &= \frac{1}{\sqrt{2}} \iint_{S_2} \frac{dr_0 ds_0}{(r - r_0)^{1/2} (s - s_0)^{1/2}} \\ &= \frac{1}{\sqrt{2}} \int_{s - m_1 X \sqrt{2}}^r \frac{dr_0}{(r - r_0)^{1/2}} \int_{r_0 \left(\frac{1 - m_1}{1 + m_1}\right)}^s \frac{ds_0}{(s - s_0)^{1/2}}. \end{aligned}$$

This repeated integral is easily evaluated and by use of (3.9) we obtain

$$C(X, Y) = \frac{2(m_1 X + Y)}{(1 - m_1^2)^{1/2}} \left\{ \sinh^{-1} u^{1/2} + u^{1/2} (1 + u)^{1/2} \right\}, \quad \dots \quad (3.11)$$

where

$$u = \frac{(1 - m_1)(m_1 X_0 - Y)}{(m_1 X + Y)} \dots \dots \dots \dots \quad (3.12)$$

If the leading edges of the wing are produced downstream so as to form an infinite triangular wing, then the flow in  $S_1$  remains unaffected and is precisely that investigated by Behrbohm<sup>4</sup> (1952). The upwash outboard of an infinite triangular wing at uniform incidence with subsonic leading edges  $Y = \pm m_1 X$ , as quoted in equation (12a) of Ref. 4, should read

$$W_0 = -\frac{\alpha U_\infty}{E(k)} \left\{ E(\phi, k) - \frac{|\eta|(1 - \eta^2)^{1/2}}{(\eta^2 - m_1^2)^{1/2}} \right\} \quad \text{for } m_1 < |\eta| \leq 1,$$

so that, in the present notation

$$\left(\frac{\partial\Phi_1}{\partial Z}\right)_{Z=0} = \frac{W_1}{E(k)} \left\{ E(\phi, k) - \frac{|Y|(X^2 - Y^2)^{1/2}}{X(Y^2 - m_1^2 X^2)^{1/2}} \right\} \quad \text{for } m_1 < \left|\frac{Y}{X}\right| \leq 1,$$

where  $E(\phi, k)$  is the incomplete elliptic integral of the second kind with modulus  $k = (1 - m_1^2)^{1/2}$  and argument  $\phi = \sin^{-1} \left[ \frac{(X^2 - Y^2)^{1/2}}{kX} \right]$ . In region  $S_1$ ,  $Y/X$  is negative so that

$$\left(\frac{\partial\Phi_1}{\partial Z}\right)_{Z=0} = \frac{W_1}{E(k)} \left\{ E(\phi, k) + \frac{Y(X^2 - Y^2)^{1/2}}{X(Y^2 - m_1^2 X^2)^{1/2}} \right\} \dots \dots \dots \quad (3.13)$$

This upwash is a function of  $Y/X$  only or, by (3.9), a function of  $(s - r)/(s + r)$  only. Let

$$v = \frac{s_0 - r_0}{s_0 + r_0} \dots \dots \dots \dots \quad (3.14)$$

Then, in terms of current co-ordinates, the upwash is a function of  $v$  only and may be written in the form

$$\left(\frac{\partial\Phi_1}{\partial Z}\right)_{Z=0} = -\frac{W_1}{E(k)} L(v) \quad (\text{for } -1 \leq v < -m_1), \quad \dots \quad (3.15)$$

where

$$L(v) = - \left[ \frac{v(1-v^2)^{1/2}}{(v^2 - m_1^2)^{1/2}} + E(\phi, k) \right] \dots \dots \dots \dots \quad (3.16)$$

and

$$\phi = \sin^{-1} \frac{(1-v^2)^{1/2}}{k}$$

It is convenient to change the variables of integration in the integral (3.10) from  $(r_0, s_0)$  to  $(r_0, v)$ . Then from (3.15), equation (3.10) becomes

$$- \frac{\pi}{W_1} \Phi_1(X, Y) = - \frac{I}{E(k)} + C(X, Y), \dots \dots \dots \dots \dots \quad (3.17)$$

where

$$I = \sqrt{2} \iint_{S_1} \frac{r_0 L(v) dr_0 dv}{(1-v)^2 (r-r_0)^{1/2} (s-s_0)^{1/2}} \dots \dots \dots \quad (3.18)$$

The integral  $I$  does not appear to be integrable in terms of known functions. Accordingly, since the integrand of  $I$  is positive in  $S_1$ , the term  $(s-s_0)^{-1/2}$  in the integrand of  $I$  is expanded in the form

$$(s-s_0)^{-1/2} = s^{-1/2} \left[ 1 + \frac{s_0}{2s} + \frac{3}{8} \frac{s_0^2}{s^2} + \dots \right] \dots \dots \dots \quad (3.19)$$

and term-by-term integration gives successive approximations to  $I$ . Only the first two terms of this expansion are retained, as the effect on  $\Phi_1(X, Y)$  of omitting the remaining terms is in general small. Thus equation (3.18) becomes

$$I = I_1 + I_2, \dots \dots \dots \dots \dots \quad (3.20)$$

where

$$I_1 = \left(\frac{2}{s}\right)^{1/2} \iint_{S_1} \frac{r_0 L(v) dr_0 dv}{(1-v)^2 (r-r_0)^{1/2}} \dots \dots \dots \quad (3.21)$$

and, from (3.14),

$$I_2 = \frac{1}{2^{1/2} s^{3/2}} \iint_{S_1} \frac{r_0^2 L(v) (1+v) dr_0 dv}{(1-v)^3 (r-r_0)^{1/2}} \dots \dots \dots \quad (3.22)$$

In terms of  $(r_0, v)$ , the region  $S_1$  is defined by (Fig. 4)  $s - m_1 X_0 \sqrt{2} \leq r_0 \leq r$ ,  $-1 \leq v \leq -m_1$  so that (3.21) becomes

$$I_1 = \left(\frac{2}{s}\right)^{1/2} \int_{s-m_1 X_0 \sqrt{2}}^r \frac{r_0 dr_0}{(r-r_0)^{1/2}} \int_{-1}^{-m_1} \frac{L(v) dv}{(1-v)^2} \dots \dots \dots \quad (3.23)$$

The integration with respect to  $r_0$  is easily performed and it is found that

$$\begin{aligned} \int_{s-m_1 X_0 \sqrt{2}}^r \frac{r_0 dr_0}{(r-r_0)^{1/2}} &= \frac{2}{3} (r-s + m_1 X_0 \sqrt{2})^{1/2} (2r + s - m_1 X_0 \sqrt{2}) \\ &= \frac{2^{3/4}}{3} (m_1 X_0 - Y)^{1/2} (3X - Y - 2m_1 X_0) \dots \dots \dots \quad (3.24) \end{aligned}$$

by (3.9). The other integral in (3.23), where  $L(v)$  is given by (3.16), is an elliptic integral whose



value is

$$\int_{-1}^{-m_1} \frac{L(v) dv}{(1-v)^2} = \frac{1}{k^2} \{m_1 E(k) - (1-k^2)K(k)\}, \quad \dots \quad (3.25)$$

where  $K(k)$  is the complete elliptic integral of the first kind of modulus  $k = (1 - m_1^2)^{1/2}$ .

By (3.9) the factor  $(2/s)^{1/2}$  equals  $2^{3/4}/(X+Y)^{1/2}$ , so that (3.24) and (3.25) reduce (3.23) to

$$I_1 = \frac{2\sqrt{2}}{3} \frac{1}{k^2} \{m_1 E(k) - (1-k^2)K(k)\} \left(\frac{m_1 X_0 - Y}{X+Y}\right)^{1/2} (3X - Y - 2m_1 X_0). \quad (3.26)$$

Similarly, equation (3.22) may be written in the form

$$I_2 = \frac{1}{2^{1/2} s^{3/2}} \int_{s-m_1 X_0 \sqrt{2}}^r \frac{r_0^2 dr_0}{(r-r_0)^{1/2}} \int_{-1}^{-m_1} \frac{(1+v) L(v) dv}{(1-v)^3}, \quad \dots \quad (3.27)$$

where

$$\int_{s-m_1 X_0 \sqrt{2}}^r \frac{r_0^2 dr_0}{(r-r_0)^{1/2}} = \frac{2^{1/4}}{15} (m_1 X_0 - Y)^{1/2} (15X^2 - 10XY + 7Y^2 - 20m_1 X_0 X - 4m_1 X_0 Y + 12m_1^2 X_0^2), \quad \dots \quad (3.28)$$

$$\int_{-1}^{-m_1} \frac{(1+v) L(v) dv}{(1-v)^3} = \frac{1}{k^4} [\{4 - 3k^2 - (1 - m_1)^2 - m_1 k^2\} E(k) - 2(2 - 3k^2 + k^4)K(k)] \quad \dots \quad (3.29)$$

and

$$\frac{1}{2^{1/2} s^{3/2}} = \frac{2^{1/4}}{(X+Y)^{3/2}}. \quad \dots \quad (3.30)$$

From (3.28), (3.29) and (3.30), equation (3.27) becomes

$$I_2 = \frac{\sqrt{2}}{15} \frac{1}{k^4} [\{4 - 3k^2 - (1 - m_1)^2 - m_1 k^2\} E(k) - 2(2 - 3k^2 + k^4)K(k)] \times \left(\frac{m_1 X_0 - Y}{X+Y}\right)^{1/2} (15X^2 - 10XY + 7Y^2 - 20m_1 X_0 X - 4m_1 X_0 Y + 12m_1^2 X_0^2). \quad (3.31)$$

Finally from (3.20), (3.26) and (3.31), the velocity potential  $\Phi_1(X, Y)$  from (3.17) may be written in the form

$$-\frac{\pi}{W_1} \Phi_1(X, Y) = C(X, Y) - 2\tau_1(m_1) \left(\frac{m_1 X_0 - Y}{X+Y}\right)^{1/2} (3X - Y - 2m_1 X_0) - 2\tau_2(m_1) \frac{(m_1 X_0 - Y)^{1/2}}{(X+Y)^{3/2}} (15X^2 - 10XY + 7Y^2 - 20m_1 X_0 X - 4m_1 X_0 Y + 12m_1^2 X_0^2), \quad (3.32)$$

where  $C(X, Y)$  is given by (3.11),

$$\tau_1(m_1) = \frac{\sqrt{2}}{3} \frac{1}{k^2 E(k)} \{m_1 E(k) - (1 - k^2)K(k)\} \quad \dots \quad (3.33)$$

and

$$\tau_2(m_1) = \frac{\sqrt{2}}{30} \frac{1}{k^4 E(k)} [4 - 3k^2 - (1 - m_1)^2 - m_1 k^2] E(k) - 2(2 - 3k^2 + k^4) K(k). \quad (3.34)$$

Differentiation of (3.32) with respect to  $X$  yields

$$\begin{aligned} -\frac{\pi}{W_1} \left( \frac{\partial \Phi_1}{\partial X} \right)_{z=0} &= \frac{2m_1}{(1 - m_1^2)^{1/2}} \sinh^{-1} u^{1/2} - \tau_1(m_1) \frac{(m_1 X_0 - Y)^{1/2}}{(X + Y)^{3/2}} (3X + 7Y + 2m_1 X_0) \\ &\quad - \tau_2(m_1) \frac{(m_1 X_0 - Y)^{1/2}}{(X + Y)^{5/2}} (15X^2 + 70XY - 41Y^2 + 20m_1 X_0 X \\ &\quad \quad \quad - 28m_1 X_0 Y - 36m_1^2 X_0^2), \quad \dots \quad (3.35) \end{aligned}$$

where  $u$  is defined by (3.12). The loading function,  $F_1$ , in region  $C$  is given by (3.2), (3.32) and (3.35), namely,

$$\begin{aligned} -\frac{\pi}{W_1} F_1 &= \left\{ 1 - i\nu \left( \beta + \frac{1}{\beta} \right) X \right\} \left[ \frac{2m_1}{(1 - m_1^2)^{1/2}} \sinh^{-1} u^{1/2} - \right. \\ &\quad - \tau_1(m_1) \frac{(m_1 X_0 - Y)^{1/2}}{(X + Y)^{3/2}} (3X + 7Y + 2m_1 X_0) - \\ &\quad - \tau_2(m_1) \frac{(m_1 X_0 - Y)^{1/2}}{(X + Y)^{5/2}} (15X^2 + 70XY - 41Y^2 + 20m_1 X_0 X - \\ &\quad \quad \quad \left. - 28m_1 X_0 Y - 36m_1^2 X_0^2) \right] \\ &\quad - \frac{i\nu}{\beta} \left[ C(X, Y) - 2\tau_1(m_1) \frac{(m_1 X_0 - Y)^{1/2}}{(X + Y)} (3X - Y - 2m_1 X_0) \right. \\ &\quad - 2\tau_2(m_1) \frac{(m_1 X_0 - Y)^{1/2}}{(X + Y)^{3/2}} (15X^2 - 10XY + 7Y^2 - 20m_1 X_0 X - \\ &\quad \quad \quad \left. - 4m_1 X_0 Y + 12m_1^2 X_0^2) \right] \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (3.36) \end{aligned}$$

where  $u$ ,  $C(X, Y)$ ,  $\tau_1(m_1)$ ,  $\tau_2(m_1)$  are defined by equations (3.12), (3.11), (3.33) and (3.34) respectively.

3.2. *Velocity Potential for Uniform Pitching.*—The derivation of the velocity potential in this case is similar to that performed in section 3.1. Here the perturbation-velocity potential  $\Phi_2(X, Y)$  on the triangular wing with subsonic leading edges,  $Y = \pm m_1 X$ , has to satisfy the steady condition  $W = W_2 = -qX$  in a stream of Mach number  $\sqrt{2}$ . By Ref. 3, p. 334,

$$\Phi_2(X, Y) = \frac{qX(m_1^2 X^2 - Y^2)^{1/2}}{e(m_1)}, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (3.37)$$

where  $e(m_1) = \frac{1 - 2m_1^2}{1 - m_1^2} E(k) + \frac{m_1^2}{1 - m_1^2} K(k); \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (3.38)$

this represents the perturbation-velocity potential in regions  $A$  and  $B$ . It follows from (3.37) that

$$\left(\frac{\partial\Phi_2}{\partial X}\right)_{z=0} = \frac{q}{e(m_1)} \frac{(2m_1^2 X^2 - Y^2)}{(m_1^2 X^2 - Y^2)^{1/2}} \dots \dots \dots \dots \dots \dots (3.39)$$

Thus from (3.3), (3.37) and (3.39) we obtain the loading function  $F_2$  in regions  $A$  and  $B$ . Since  $q$  is proportional to  $v$  and terms of order  $v^2$  are being neglected, then the loading function may be written in the form

$$F_2 = \left(\frac{\partial\Phi_2}{\partial X}\right)_{z=0};$$

so that, in regions  $A$  and  $B$ ,

$$\frac{1}{q} F_2 = \frac{1}{e(m_1)} \frac{(2m_1^2 X^2 - Y^2)}{(m_1^2 X^2 - Y^2)^{1/2}} \dots \dots \dots \dots \dots \dots (3.40)$$

As in section 3.1, Evvard's method in the case of steady flow at Mach number  $\sqrt{2}$  (Ref. 1, equation (29)) is used to calculate  $\Phi_2(X, Y)$  in region  $C$ . Thus  $\Phi_2(X, Y)$  is given by (3.8) with  $\phi_1$  replaced by  $\phi_2$ ; since  $(\partial\Phi_2/\partial Z)_{z=0} = W_2 = -qX$  over  $S_2$ , it follows that

$$\frac{\pi}{q} \Phi_2(X, Y) = -\frac{1}{q\sqrt{2}} \iint_{s_1} \frac{\left(\frac{\partial\Phi_2}{\partial Z}\right)_{z=0} dr_0 ds_0}{(r-r_0)^{1/2} (s-s_0)^{1/2}} + D(X, Y), \dots \dots (3.41)$$

where

$$\begin{aligned} D(X, Y) &= \frac{1}{\sqrt{2}} \iint_{s_2} \frac{\left(\frac{s_0+r_0}{\sqrt{2}}\right) dr_0 ds_0}{(r-r_0)^{1/2} (s-s_0)^{1/2}} \\ &= \frac{1}{2} \int_{s-m_1 X \sqrt{2}}^r \frac{dr_0}{(r-r_0)^{1/2}} \int_{r_0 \frac{1-m_1}{1+m_1}}^s \frac{(s_0+r_0)}{(s-s_0)^{1/2}} ds_0. \end{aligned}$$

This integral is easily evaluated and by use of (3.9) we obtain

$$\begin{aligned} D(X, Y) &= \frac{(m_1 X + Y)}{(1-m_1^2)^{3/2}} \{(2-m_1^2)X + m_1 Y\} \{\sinh^{-1} u^{1/2} + u^{1/2} (1+u)^{1/2}\} \\ &\quad - \frac{2}{3} \frac{(2+m_1)}{(1-m_1^2)^{3/2}} (m_1 X + Y)^2 u^{1/2} (1+u)^{3/2}, \dots \dots (3.42) \end{aligned}$$

where  $u$  is defined by (3.12).

The upwash outboard of a uniformly pitching infinite triangular wing at  $M = \sqrt{2}$  as quoted by Behrbohm (Ref. 4, equation (12)) should read

$$W_0 = -\frac{qX}{e(m_1)} \left\{ \frac{1-2m_1^2}{1-m_1^2} E(\phi, k) + \frac{m_1^2}{1-m_1^2} F(\phi, k) - \frac{|\eta|(1-\eta^2)^{1/2}}{(\eta^2-m_1^2)^{1/2}} \right\} \quad \text{for } m_1 < |\eta| \leq 1,$$

so that, in the present notation

$$\left(\frac{\partial \Phi_2}{\partial Z}\right)_{Z=0} = -\frac{qX}{e(m_1)} \left\{ \frac{1-2m_1^2}{1-m_1^2} E(\phi, k) + \frac{m_1^2}{1-m_1^2} F(\phi, k) - \frac{|Y|(X^2-Y^2)^{1/2}}{X(Y^2-m_1^2 X^2)^{1/2}} \right\}$$

for  $m_1 < |Y/X| \leq 1$ ,

where  $F(\phi, k)$  is the incomplete elliptic integral of the second kind with modulus  $k = (1 - m_1^2)^{1/2}$  and argument  $\phi = \sin^{-1} [\{(X^2 - Y^2)^{1/2}\}/kX]$ . Since  $Y/X$  is negative in  $S_1$ , then in that region

$$\left(\frac{\partial \Phi_2}{\partial Z}\right)_{Z=0} = -\frac{qX}{e(m_1)} \left\{ \frac{1-2m_1^2}{1-m_1^2} E(\phi, k) + \frac{m_1^2}{1-m_1^2} F(\phi, k) + \frac{Y(X^2 - Y^2)^{1/2}}{X(Y^2 - m_1^2 X^2)^{1/2}} \right\}, \quad (3.43)$$

which in terms of the current co-ordinates  $(r_0, v)$  (equations (3.9) and (3.14)) may be written as

$$\left(\frac{\partial \Phi_2}{\partial Z}\right)_{Z=0} = \sqrt{2} \frac{qr_0 M(v)}{e(m_1)} \quad (\text{for } -1 \leq v < -m_1), \quad \dots \dots \dots (3.44)$$

where

$$M(v) = -\frac{1}{(1-v)} \left\{ \frac{1-2m_1^2}{1-m_1^2} E(\phi, k) + \frac{m_1^2}{1-m_1^2} F(\phi, k) + \frac{v(1-v^2)^{1/2}}{(v^2 - m_1^2)^{1/2}} \right\}. \dots (3.45)$$

and

$$\phi = \sin^{-1} \frac{(1-v^2)^{1/2}}{k}.$$

By changing the variables of integration in the integral of (3.41) from  $(r_0, s_0)$  to  $(r_0, v)$ , and by making use of (3.44), we have

$$\frac{\pi}{q} \Phi_2(X, Y) = -\frac{J}{e(m_1)} + D(X, Y), \quad \dots \dots \dots (3.46)$$

where  $J = 2 \iint_{S_1} \frac{r_0^2 M(v) dr_0 dv}{(1-v)^2 (r-r_0)^{1/2} (s-s_0)^{1/2}} \dots \dots \dots (3.47)$

It can readily be shown that the integrand of  $J$  is positive within  $S_1$ . As in section 3.1, an approximation to  $J$  is obtained by replacing the term  $(s - s_0)^{-1/2}$  in the integrand of  $J$  by the first two terms in the expansion (3.19). Thus (3.47) becomes

$$J = J_1 + J_2, \quad \dots \dots \dots (3.48)$$

where

$$J_1 = \frac{2}{s^{1/2}} \iint_{S_1} \frac{r_0^2 M(v) dr_0 dv}{(1-v)^2 (r-r_0)^{1/2}} \dots \dots \dots (3.49)$$

and, since  $s_0 = r_0(1+v)/(1-v)$  by (3.14),

$$J_2 = \frac{1}{s^{3/2}} \iint_{S_1} \frac{r_0^3 M(v) (1+v) dr_0 dv}{(1-v)^3 (r-r_0)^{1/2}} \dots \dots \dots (3.50)$$

As for the corresponding integrals in section 3.1, (3.49), (3.50) may be written as a product of two integrals. Thus (3.49) becomes

$$J_1 = \frac{2}{s^{1/2}} \int_{s-m_1 X_0 \sqrt{2}}^r \frac{r_0^2 dr_0}{(r-r_0)^{1/2}} \int_{-1}^{-m_1} \frac{M(v) dv}{(1-v)^2}, \quad \dots \dots \dots (3.51)$$

where the first integral is given by (3.28) and the second integral, where  $M(v)$  is given by (3.45), is an elliptic integral whose value is

$$\int_{-1}^{-m_1} \frac{M(v) dv}{(1-v)^2} = \frac{15}{\sqrt{2}} e(m_1) \sigma_1(m_1), \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (3.52)$$

where

$$\sigma_1(m_1) = \frac{\sqrt{2}}{30k^6 e(m_1)} [\{k^2(2 - k^2) + (1 - m_1)^2(1 - 2k^2)\}E(k) - \{2k^2 + (1 - m_1)^2\}(1 - k^2)K(k)]. \quad \dots \quad \dots \quad (3.53)$$

By (3.9) the factor  $2/s^{1/2}$  equals  $2^{3/4}/(X + Y)^{1/2}$ , so that (3.28) and (3.52) reduce (3.51) to

$$J_1 = 2e(m_1) \sigma_1(m_1) \left( \frac{m_1 X_0 - Y}{X + Y} \right)^{1/2} (15X^2 - 10XY + 7Y^2 - 20m_1 X_0 X - 4m_1 X_0 Y + 12m_1^2 X_0^2). \quad \dots \quad \dots \quad \dots \quad (3.54)$$

Similarly, equation (3.50) may be written in the form

$$J_2 = \frac{1}{s^{3/2}} \int_{s - m_1 X_0 \sqrt{2}}^r \frac{r_0^3 dr_0}{(r - r_0)^{1/2}} \int_{-1}^{-m_1} \frac{(1+v) M(v) dv}{(1-v)^3}, \quad \dots \quad \dots \quad \dots \quad \dots \quad (3.55)$$

where

$$\int_{s - m_1 X_0 \sqrt{2}}^r \frac{r_0^3 dr_0}{(r - r_0)^{1/2}} = \frac{(m_1 X_0 - Y)^{1/2}}{2^{1/4} 35} (35X^3 - 35X^2Y + 49XY^2 - 9Y^3 - 70m_1 X_0 X^2 - 28m_1 X_0 XY - 22m_1 X_0 Y^2 + 84m_1^2 X_0^2 X + 36m_1^2 X_0^2 Y - 40m_1^3 X_0^3), \quad (3.56)$$

and

$$\int_{-1}^{-m_1} \frac{(1+v) M(v) dv}{(1-v)^3} = (35\sqrt{2})e(m_1) \sigma_2(m_1), \quad \dots \quad \dots \quad \dots \quad (3.57)$$

where

$$\sigma_2(m_1) = \frac{\sqrt{2}}{420k^8 e(m_1)} [\{k^2(32 - 34k^2 + 3k^4) + (1 - m_1)^2(4 - 4m_1 - 3k^2) \times (1 - 2k^2)\}E(k) - \{2k^2(16 - 9k^2) + (1 - m_1)^2(4 - 4m_1 - 3k^2)\} \times (1 - k^2)K(k)], \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (3.58)$$

and

$$\frac{1}{s^{3/2}} = \frac{2^{3/4}}{(X + Y)^{3/2}}. \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (3.59)$$

From (3.56), (3.57) and (3.59), equation (3.55) becomes

$$J_2 = 2e(m_1) \sigma_2(m_1) \frac{(m_1 X_0 - Y)^{1/2}}{(X + Y)^{2/3}} (35X^3 - 35X^2Y + 49XY^2 - 9Y^3 - 70m_1 X_0 X^2 - 28m_1 X_0 XY - 22m_1 X_0 Y^2 + 84m_1^2 X_0^2 X + 36m_1^2 X_0^2 Y - 40m_1^3 X_0^3). \quad (3.60)$$

Finally from (3.48), (3.54) and (3.60), the velocity potential  $\Phi_2(X, Y)$  from (3.46) may be written in the form

$$\begin{aligned} \frac{\pi}{q} \Phi_2(X, Y) = & D(X, Y) - 2\sigma_1(m_1) \left( \frac{m_1 X_0 - Y}{X + Y} \right)^{1/2} (15X^2 - 10XY + 7Y^2 - 20m_1 X_0 X \\ & - 4m_1 X_0 Y + 12m_1^2 X_0^2) - 2\sigma_2(m_1) \frac{(m_1 X_0 - Y)^{1/2}}{(X + Y)^{3/2}} (35X^3 - 35X^2 Y \\ & + 49XY^2 - 9Y^3 - 70m_1 X_0 X^2 - 28m_1 X_0 XY - 22m_1 X_0 Y^2 + 84m_1^2 X_0^2 X \\ & + 36m_1^2 X_0^2 Y - 40m_1^3 X_0^3), \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (3.61) \end{aligned}$$

where  $D(X, Y)$  is given by (3.42),  $\sigma_1(m_1)$  by (3.53) and  $\sigma_2(m_1)$  by (3.58). By differentiation of (3.61) with respect to  $X$  we obtain  $(\partial\Phi_2/\partial X)_{Z=0}$ . Since  $q$  is proportional to  $v$  and terms of order  $v^2$  are being neglected, the loading function,  $F_2$ , equals  $(\partial\Phi_2/\partial X)_{Z=0}$  so that, on performing the differentiation of (3.61) with respect to  $X$ , we obtain for  $F_2$  in region  $C$ ,

$$\begin{aligned} \frac{\pi}{q} F_2 = & \frac{2}{(1 - m_1^2)^{3/2}} [\{m_1(2 - m_1^2)X + Y\} \sinh^{-1} u^{1/2} + (1 - m_1 - m_1^2)(m_1 X + Y) \\ & \times u^{1/2} (1 + u)^{1/2}] - \sigma_1(m_1) \frac{(m_1 X_0 - Y)^{1/2}}{(X + Y)^{3/2}} (45X^2 + 50XY - 27Y^2 - 20m_1 X_0 X \\ & - 36m_1 X_0 Y - 12m_1^2 X_0^2) - \sigma_2(m_1) \frac{(m_1 X_0 - Y)^{1/2}}{(X + Y)^{5/2}} (45X^2 + 130XY + 181Y^2 \\ & + 20m_1 X_0 X + 68m_1 X_0 Y + 36m_1^2 X_0^2), \quad \dots \quad \dots \quad \dots \quad \dots \quad (3.62) \end{aligned}$$

where  $u$ ,  $\sigma_1(m_1)$ ,  $\sigma_2(m_1)$  are defined by (3.12), (3.53) and (3.58) respectively.

To summarise, the loading function,  $F$ , is given by (3.4), where, in regions  $A$  and  $B$ ,  $F_1$ ,  $F_2$  are given by equations (3.7), (3.40) respectively, and, in region  $C$ ,  $F_1$ ,  $F_2$  are given by equations (3.36), (3.62) respectively.

4. *Aerodynamic Coefficients and Derivatives.*—In section 3, expressions have been given for the loading function,  $F$ , in the regions  $A$ ,  $B$  and  $C$  (Fig. 3), so that the aerodynamic coefficients follow from the integrations in (2.11), (2.12) and (2.13). Expressions for the lift, pitching moment and hinge moment are obtained in section 4.1. The aerodynamic derivatives in section 4.2 then follow directly from these coefficients.

4.1. *Calculation of Aerodynamic Coefficients.*—In order to calculate the aerodynamic coefficients, (2.11), (2.12) and (2.13), it is necessary to evaluate the integrals

$$\iint_{A+B+C} (1, X) F dX dY, \quad \iint_A (1, X) F dX dY,$$

where  $(1, X)$  denotes 1 or  $X$ . Since the expression for  $F$  in regions  $A$  and  $B$ , given by equations (3.4), (3.7) and (3.40), differs from the expression for  $F$  in region  $C$ , given by (3.4), (3.36) and

(3.62), then the integrals to be evaluated are split up into the six integrals

$$\iint_{A+B} (1, X) F dX dY, \iint_C (1, X) F dX dY, \iint_A (1, X) F dX dY,$$

or more conveniently into the twelve integrals

$$\left. \begin{aligned} & \iint_{A+B} (1, X) F_1 dX dY, \iint_{A+B} (1, X) F_2 dX dY, \iint_C (1, X) F_1 dX dY, \\ & \iint_C (1, X) F_2 dX dY, \iint_A (1, X) F_1 dX dY, \iint_A (1, X) F_2 dX dY \end{aligned} \right\} \dots \quad (4.1)$$

It is indicated in Appendix A how the integrals (4.1) reduce to a linear combination of the integrals in Appendices B, C, D, E, F. We thus obtain the following twelve results in a form suitable for calculation:

$$- \frac{\pi}{W_1} \iint_{A+B} F_1 dX dY = \frac{\pi}{E(k)} \left[ m_1^2 P_1 - iv \left\{ m_1^2 P_2 \left( \beta + \frac{1}{\beta} \right) + \frac{P_4}{\beta} \right\} \right]; \quad \dots \quad (4.2)$$

$$- \frac{\pi}{W_1} \iint_{A+B} X F_1 dX dY = \frac{\pi}{E(k)} \left[ m_1^2 P_2 - iv \left\{ m_1^2 P_3 \left( \beta + \frac{1}{\beta} \right) + \frac{P_5}{\beta} \right\} \right]; \quad \dots \quad (4.3)$$

$$\frac{\pi}{q} \iint_{A+B} F_2 dX dY = \frac{\pi}{e(m_1)} (m_1^2 P_2 + P_4); \quad \dots \quad (4.4)$$

$$\frac{\pi}{q} \iint_{A+B} X F_2 dX dY = \frac{\pi}{e(m_1)} (m_1^2 P_3 + P_5); \quad \dots \quad (4.5)$$

$$\begin{aligned} - \frac{\pi}{W_1} \iint_C F_1 dX dY = & \left[ \frac{2m_1}{(1 - m_1^2)^{1/2}} Q_1 - \tau_1(m_1) (3S_{0,-1} - 4S_{1,-2} + 6m_1 X_0 S_{0,-2}) \right. \\ & - \tau_2(m_1) (15S_{0,-1} - 40S_{1,-2} - 96S_{2,-3} + 60m_1 X_0 S_{0,-2} \\ & \left. + 240m_1 X_0 S_{1,-3} - 180m_1^2 X_0^2 S_{0,-3}) \right] - iv \left[ \left( \beta + \frac{1}{\beta} \right) \left\{ \frac{2m_1}{(1 - m_1^2)^{1/2}} Q_3 - \right. \right. \\ & - \tau_1(m_1) (3S_{0,0} - S_{1,-1} - 4S_{2,-2} + 3m_1 X_0 S_{0,-1} + 10m_1 X_0 S_{1,-2} \\ & - 6m_1^2 X_0^2 S_{0,-2}) - \tau_2(m_1) (15S_{0,0} - 25S_{1,-1} - 136S_{2,-2} \\ & - 96S_{3,-3} + 45m_1 X_0 S_{0,-1} + 340m_1 X_0 S_{1,-2} + 336m_1 X_0 S_{2,-3} \\ & \left. - 240m_1^2 X_0^2 S_{0,-2} - 420m_1^2 X_0^2 S_{1,-3} + 180m_1^3 X_0^3 S_{0,-3}) \right\} \\ & + \frac{2}{\beta} \left\{ \frac{(m_1 Q_3 + Q_2 + R_1)}{(1 - m_1^2)^{1/2}} - \tau_1(m_1) (3S_{0,0} + 4S_{1,-1} - 6m_1 X_0 S_{0,-1}) \right. \\ & - \tau_2(m_1) (15S_{0,0} + 40S_{1,-1} + 32S_{2,-2} - 60m_1 X_0 S_{0,-1} \\ & \left. \left. - 80m_1 X_0 S_{1,-2} + 60m_1^2 X_0^2 S_{0,-2}) \right\} \right]; \quad \dots \quad (4.6) \end{aligned}$$

$$\begin{aligned}
-\frac{\pi}{W_1} \iint_C X F_1 dX dY = & \left[ \frac{2m_1}{(1-m_1^2)^{1/2}} Q_3 - \tau_1(m_1) (3S_{0,0} - S_{1,-1} - 4S_{2,-2} + 3m_1 X_0 S_{0,-1}) \right. \\
& + 10m_1 X_0 S_{1,-2} - 6m_1^2 X_0^2 S_{0,-2}) - \tau_2(m_1) (15S_{0,0} \\
& - 25S_{1,-1} - 136S_{2,-2} - 96S_{3,-3} + 45m_1 X_0 S_{0,-1} + 340m_1 X_0 S_{1,-2} \\
& \left. + 336m_1 X_0 S_{2,-3} - 240m_1^2 X_0^2 S_{0,-2} - 420m_1^2 X_0^2 S_{1,-3} + 180m_1^3 X_0^3 S_{0,-3}) \right] \\
& - iv \left[ \left( \beta + \frac{1}{\beta} \right) \left\{ \frac{2m_1}{(1-m_1^2)^{1/2}} Q_5 - \tau_1(m_1) (3S_{0,1} + 2S_{1,0} - 5S_{2,-1} \right. \right. \\
& - 4S_{3,-2} + 14m_1 X_0 S_{1,-1} + 14m_1 X_0 S_{2,-2} - 9m_1^2 X_0^2 S_{0,-1} - 16m_1^2 X_0^2 S_{1,-} \\
& + 6m_1^3 X_0^3 S_{0,-2}) - \tau_2(m_1) (15S_{0,1} - 10S_{1,0} - 161S_{2,-1} \\
& - 232S_{3,-2} - 96S_{4,-3} + 30m_1 X_0 S_{0,0} + 410m_1 X_0 S_{1,-1} + 812m_1 X_0 S_{2,-2} \\
& + 432m_1 X_0 S_{3,-3} - 285m_1^2 X_0^2 S_{0,-1} - 1000m_1^2 X_0^2 S_{1,-2} - 756m_1^2 X_0^2 S_{2,-3} \\
& \left. + 420m_1^3 X_0^3 S_{0,-2} + 600m_1^3 X_0^3 S_{1,-3} - 180m_1^4 X_0^4 S_{0,-3}) \right\} \\
& + \frac{2}{\beta} \left\{ \frac{(m_1 Q_5 + Q_4 + R_2)}{(1-m_1^2)^{1/2}} - \tau_1(m_1) (3S_{0,1} + 7S_{1,0} + 4S_{2,-1} \right. \\
& - 9m_1 X_0 S_{0,0} - 10m_1 X_0 S_{1,-1} + 6m_1^2 X_0^2 S_{0,-1}) - \tau_2(m_1) (15S_{0,1} \\
& + 55S_{1,0} + 72S_{2,-1} + 32S_{3,-2} - 75m_1 X_0 S_{0,0} - 180m_1 X_0 S_{1,-1} \\
& - 112m_1 X_0 S_{2,-2} + 120m_1^2 X_0^2 S_{0,-1} + 140m_1^2 X_0^2 S_{1,-2} \\
& \left. - 60m_1^3 X_0^3 S_{0,-2}) \right\} \left. \right]; \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (4.7)
\end{aligned}$$

$$\begin{aligned}
\frac{\pi}{q} \iint_C F_2 dX dY = & \left[ \frac{2}{(1-m_1^2)^{3/2}} \{ m_1 (2 - m_1^2) Q_3 + Q_2 + (1 - m_1 - m_1^2) R_1 \} \right. \\
& - \sigma_1(m_1) (45S_{0,0} + 40S_{1,-1} - 32S_{2,-2} - 60m_1 X_0 S_{0,-1} + 80m_1 X_0 S_{1,-2} \\
& - 60m_1^2 X_0^2 S_{0,-2}) - \sigma_2(m_1) (45S_{0,-1} - 40S_{1,-2} + 96S_{2,-3} \\
& \left. + 60m_1 X_0 S_{0,-2} - 240m_1 X_0 S_{1,-3} + 180m_1^2 X_0^2 S_{0,-3}) \right]; \quad \dots \quad (4.8)
\end{aligned}$$



$$\begin{aligned}
\frac{\pi}{q} \iint_C XF_2 dX dY = & \left[ \frac{2}{(1-m_1^2)^{3/2}} \{m_1(2-m_1^2)Q_5 + Q_4 + (1-m_1-m_1^2)R_2\} \right. \\
& - \sigma_1(m_1)(45S_{0,1} + 85S_{1,0} + 8S_{2,-1} - 32S_{3,-2} - 105m_1X_0S_{0,0}) \\
& - 20m_1X_0S_{1,-1} + 112m_1X_0S_{2,-2} - 140m_1^2X_0^2S_{1,-2} + 60m_1^3X_0^3S_{0,-2}) \\
& - \sigma_2(m_1)(45S_{0,0} + 5S_{1,-1} + 56S_{2,-2} + 96S_{3,-3} + 15m_1X_0S_{0,-1}) \\
& - 140m_1X_0S_{1,-2} - 336m_1X_0S_{2,-3} + 120m_1^2X_0^2S_{0,-2} + 420m_1^2X_0^2S_{1,-3} \\
& \left. - 180m_1^3X_0^3S_{0,-3} \right]; \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (4.9)
\end{aligned}$$

$$- \frac{\pi}{W_1} \iint_A F_1 dX dY = \frac{\pi}{E(k)} \left[ m_1^2 T_1 - iv \left\{ m_1^2 T_2 \left( \beta + \frac{1}{\beta} \right) + \frac{T_4}{\beta} \right\} \right]; \quad \dots \quad \dots \quad \dots \quad (4.10)$$

$$- \frac{\pi}{W_1} \iint_A XF_1 dX dY = \frac{\pi}{E(k)} \left[ m_1^2 T_2 - iv \left\{ m_1^2 T_3 \left( \beta + \frac{1}{\beta} \right) + \frac{T_5}{\beta} \right\} \right]; \quad \dots \quad \dots \quad \dots \quad (4.11)$$

$$\frac{\pi}{q} \iint_A F_2 dX dY = \frac{\pi}{e(m_1)} [m_1^2 T_2 + T_4]; \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (4.12)$$

$$\frac{\pi}{q} \iint_A XF_2 dX dY = \frac{\pi}{e(m_1)} [m_1^2 T_3 + T_5]. \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (4.13)$$

In these equations  $W_1$  and  $q$  are given by (2.7),  $e(m_1)$ ,  $\tau_1(m_1)$ ,  $\tau_2(m_1)$ ,  $\sigma_1(m_1)$ ,  $\sigma_2(m_1)$  are defined by (3.38), (3.33), (3.34), (3.53), (3.58) respectively and the  $P$ 's,  $Q$ 's,  $R$ 's,  $S$ 's and  $T$ 's are given in Appendices B, C, D, E and F respectively.

It is convenient to express the right-hand sides of equations (4.2), (4.3) . . . (4.13) in the forms

$$R_r(4.2) - ivR_i(4.2), R_r(4.3) - ivR_i(4.3), \dots R_r(4.13) - ivR_i(4.13),$$

respectively, so that the equations may be written in the form

$$\iint_{A+B} F_1 dX dY = - \frac{W_1}{\pi} \{R_r(4.2) - ivR_i(4.2)\}, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (4.14)$$

$$\iint_{A+B} XF_1 dX dY = - \frac{W_1}{\pi} \{R_r(4.3) - ivR_i(4.3)\}, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (4.15)$$

$$\iint_{A+B} F_2 dX dY = \frac{q}{\pi} R_r(4.4), \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (4.16)$$

$$\iint_{A+B} XF_2 dX dY = \frac{q}{\pi} R_r(4.5), \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (4.17)$$

$$\iint_C F_1 dX dY = - \frac{W_1}{\pi} \{R_r(4.6) - ivR_i(4.6)\}, \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (4.18)$$

$$\iint_C XF_1 dX dY = -\frac{W_1}{\pi} \{R_r(4.7) - ivR_i(4.7)\}, \quad \dots \quad (4.19)$$

$$\iint_C F_2 dX dY = \frac{q}{\pi} R_r(4.8), \quad \dots \quad (4.20)$$

$$\iint_C XF_2 dX dY = \frac{q}{\pi} R_r(4.9), \quad \dots \quad (4.21)$$

$$\iint_A F_1 dX dY = -\frac{W_1}{\pi} \{R_r(4.10) - ivR_i(4.10)\}, \quad \dots \quad (4.22)$$

$$\iint_A XF_1 dX dY = -\frac{W_1}{\pi} \{R_r(4.11) - ivR_i(4.11)\}, \quad \dots \quad (4.23)$$

$$\iint_A F_2 dX dY = \frac{q}{\pi} R_r(4.12), \quad \dots \quad (4.24)$$

$$\iint_A XF_2 dX dY = \frac{q}{\pi} R_r(4.13), \quad \dots \quad (4.25)$$

since to the first order in frequency  $F_2$  is real.

Then from equations (2.11), (3.4) and (4.14) . . . (4.25), the complex lift coefficient becomes

$$\begin{aligned} C_L &= \frac{4 e^{ivT}}{U_S} \iint_{A+B+C} (F_1 + F_2) dX dY \\ &= \frac{4 e^{ivT}}{U_S} \left[ -\frac{W_1}{\pi} \left( \{R_r(4.2) + R_r(4.6)\} - iv\{R_i(4.2) + R_i(4.6)\} \right) \right. \\ &\quad \left. + \frac{q}{\pi} \{R_r(4.4) + R_r(4.8)\} \right]. \quad \dots \quad (4.26) \end{aligned}$$

Similarly the complex pitching-moment coefficient, (2.12), becomes

$$\begin{aligned} C_m &= -\frac{4\beta e^{ivT}}{U_S} \left[ -\frac{W_1}{\pi} \left( \{R_r(4.3) + R_r(4.7)\} - iv\{R_i(4.3) + R_i(4.7)\} \right) \right. \\ &\quad \left. + \frac{q}{\pi} \{R_r(4.5) + R_r(4.9)\} \right]. \quad \dots \quad (4.27) \end{aligned}$$

The complex hinge-moment coefficient, (2.13), becomes

$$\begin{aligned} C_H &= \left(\frac{\bar{c}}{c_f}\right)^2 \left[ C_m + h_0 C_L + \frac{4\beta e^{ivT}}{U_S} \left( -\frac{W_1}{\pi} \{R_r(4.11) - ivR_i(4.11)\} + \frac{q}{\pi} R_r(4.13) \right) \right. \\ &\quad \left. - \frac{h_0 4 e^{ivT}}{U_S} \left( -\frac{W_1}{\pi} \{R_r(4.10) - ivR_i(4.10)\} + \frac{q}{\pi} R_r(4.12) \right) \right]. \quad \dots \quad (4.28) \end{aligned}$$

After substituting for  $W_1$  and  $q$  from (2.7) and discarding terms in  $v^2$  we obtain the complex lift coefficient in the form

$$C_L = 4 \left( \frac{\bar{c}}{s\pi} \right) \left[ \alpha \left( R_r(4.2) + R_r(4.6) \right) - \left( \frac{\dot{\alpha}\bar{c}}{U} \right) \left( \{R_i(4.2) + R_i(4.6) + hR_r(4.2) + hR_r(4.6)\} - \left( 2\beta + \frac{1}{\beta} \right) \{R_r(4.4) + R_r(4.8)\} \right) \right], \quad \dots \dots \dots (4.29)$$

where  $\alpha_0 e^{ivT}$  has been replaced by  $\alpha$  and  $iv\alpha_0 e^{ivT}$  by  $(\dot{\alpha}\bar{c}/U)$ , where  $\dot{\alpha} = d\alpha/dt$ . Similarly the complex pitching-moment and hinge-moment coefficients become

$$C_m = -4\beta \left( \frac{\bar{c}}{s\pi} \right) \left[ \alpha \left( R_r(4.3) + R_r(4.7) \right) - \left( \frac{\dot{\alpha}\bar{c}}{U} \right) \left( \{R_i(4.3) + R_i(4.7) + hR_r(4.3) + hR_r(4.7)\} - \left( 2\beta + \frac{1}{\beta} \right) \{R_r(4.5) + R_r(4.9)\} \right) \right], \quad \dots \dots \dots (4.30)$$

$$C_H = \left( \frac{\bar{c}}{c_f} \right)^2 \left[ C_m + h_0 C_L + 4 \left( \frac{\bar{c}}{s\pi} \right) \left( \alpha \{ \beta R_r(4.11) - h_0 R_r(4.10) \} - \left( \frac{\dot{\alpha}\bar{c}}{U} \right) \{ \beta R_i(4.11) + \beta h R_r(4.11) - \beta \left( 2\beta + \frac{1}{\beta} \right) R_r(4.13) - h_0 R_i(4.10) - h_0 h R_r(4.10) + h_0 \left( 2\beta + \frac{1}{\beta} \right) R_r(4.12) \} \right) \right]. \quad \dots \dots \dots (4.31)$$

4.2. *Calculation of Aerodynamic Derivatives.*—Each aerodynamic coefficient may be split uniquely into two parts, viz.,

$$\left. \begin{aligned} C_L &= -2z_x \alpha - 2z_a \left( \frac{\dot{\alpha}\bar{c}}{U} \right) \\ (C_m)_h &= 2m_x \alpha + 2m_a \left( \frac{\dot{\alpha}\bar{c}}{U} \right) \\ C_H &= 2h_x \alpha + 2h_a \left( \frac{\dot{\alpha}\bar{c}}{U} \right) \end{aligned} \right\} \dots \dots \dots (4.32)$$

where  $\dot{\alpha} = d\alpha/dt$  and the real coefficients,  $z_x, z_a, m_x, m_a, h_x, h_a$  are the required derivatives.

The moment in the second equation of (4.32) will be taken about the axis of oscillation of the wing ( $x = h\bar{c}$ ) so that the moment coefficient  $(C_m)_h = C_m + hC_L$ , where  $C_m$  is taken about the pitching axis through the apex of the wing. Then equations (4.32) may be written

$$\left. \begin{aligned} C_L &= -2z_x \alpha - 2z_a \left( \frac{\dot{\alpha}\bar{c}}{U} \right) \\ C_m &= 2\{m_x + hz_x\} \alpha + 2\{m_a + hz_a\} \left( \frac{\dot{\alpha}\bar{c}}{U} \right) \\ C_H &= 2h_x \alpha + 2h_a \left( \frac{\dot{\alpha}\bar{c}}{U} \right) \end{aligned} \right\} \dots \dots \dots (4.33)$$

The derivatives may be evaluated by identifying (4.33) with the expressions for the aerodynamic coefficients given by (4.29), (4.30) and (4.31):

$$\left. \begin{aligned}
 -z_\alpha &= \frac{2\bar{c}}{s\pi} \{R_r(4.2) + R_r(4.6)\} \\
 -z_\alpha &= \frac{2\bar{c}}{s\pi} \left[ \left(2\beta + \frac{1}{\beta}\right) \{R_r(4.4) + R_r(4.8)\} - \{R_i(4.2) + R_i(4.6)\} \right] + h z_\alpha \\
 -m_\alpha &= \frac{2\beta\bar{c}}{s\pi} \{R_r(4.3) + R_r(4.7)\} + h z_\alpha \\
 -m_\alpha &= \frac{2\beta\bar{c}}{s\pi} \left[ \left(2\beta + \frac{1}{\beta}\right) \{R_r(4.5) + R_r(4.9)\} - \{R_i(4.3) + R_i(4.7)\} \right] \\
 &\quad + h(m_\alpha + h z_\alpha) + h z_\alpha \\
 -h_\alpha &= \left(\frac{\bar{c}}{c_f}\right)^2 \left[ \frac{2\bar{c}}{s\pi} \{h_0 R_r(4.10) - \beta R_r(4.11)\} - (m_\alpha + h z_\alpha) + h_0 z_\alpha \right] \\
 -h_\alpha &= \left(\frac{\bar{c}}{c_f}\right)^2 \left[ \frac{2\bar{c}}{s\pi} \left\{ \beta R_i(4.11) - h_0 R_i(4.10) - \beta \left(2\beta + \frac{1}{\beta}\right) R_r(4.13) \right. \right. \\
 &\quad \left. \left. + h_0 \left(2\beta + \frac{1}{\beta}\right) R_r(4.12) \right\} - \{(m_\alpha + h z_\alpha) + h(m_\alpha + h z_\alpha)\} \right. \\
 &\quad \left. + h_0(z_\alpha + h z_\alpha) \right] + h h_\alpha
 \end{aligned} \right\} \dots \quad (4.34)$$

On comparison of equations (4.14), (4.15), . . . (4.25) with equations (4.2), (4.3), . . . (4.13) the  $R_r$ 's and  $R_i$ 's are found as functions of the  $P, Q, R, S, T$  integrals which are given in Appendices B, C, D, E and F. It is easily seen that the  $R_r$ 's and  $R_i$ 's are functions of  $m_1 = \beta s / (c_0 - c_f)$ ,  $X_1 = c_0 / \beta \bar{c}$  and  $X_0 = (c_0 - c_f) / \beta \bar{c}$  only, so that for a given wing the derivatives are functions of Mach number and  $h$  only.

The aerodynamic derivatives for a plunging motion of the wing are defined in terms of the corresponding aerodynamic coefficients by

$$\left. \begin{aligned}
 C_L &= -2z_z z - 2z_z \left(\frac{\dot{z}\bar{c}}{\bar{U}}\right) \\
 (C_m)_h &= 2m_z z + 2m_z \left(\frac{\dot{z}\bar{c}}{\bar{U}}\right) \\
 C_H &= 2h_z z + 2h_z \left(\frac{\dot{z}\bar{c}}{\bar{U}}\right)
 \end{aligned} \right\} \dots \quad (4.35)$$

where  $z$  denotes the complex downward displacement of the wing from its mean position and the moment in the second equation is taken about the pitching axis through  $x = h\bar{c}$ . On comparison with the coefficients in (4.29), (4.30) and (4.31), it is easily seen that

$$z_z = m_z = h_z = 0 \quad \dots \quad (4.36)$$

and

$$z_z = z_\alpha, m_z = m_\alpha, h_z = h_\alpha \quad \dots \quad (4.37)$$

It will be remembered that the derivatives (4.34) are given approximately only, because an approximation was made in calculating the loading over region  $C$  (section 4). This approximation was the replacement of the term  $(s - s_0)^{-1/2}$  in the integrands of (3.18) and (3.47) by the first two terms on the right-hand side of (3.19). However, (3.18) and (3.47) in general make only a small contribution to the loading in region  $C$  and consequently to the derivatives in (4.34). It can readily be shown from (3.33), (3.34), (3.53), (3.58) that  $\tau_1(m_1)$ ,  $\sigma_1(m_1)$  are of order  $k^2 = 1 - m_1^2$  and that  $\tau_2(m_1)$ ,  $\sigma_2(m_1)$  are of order  $k^4$  as  $m_1 \rightarrow 1$ . Thus we may consider the following three approximations:

- (a) A first approximation to the derivatives is obtained by omitting the contributions arising from (3.18) and (3.47) (this is equivalent to formally putting  $\tau_1(m_1)$ ,  $\tau_2(m_1)$ ,  $\sigma_1(m_1)$ ,  $\sigma_2(m_1)$  zero in (4.34) )
- (b) A second approximation is obtained by retaining only the first term in the right-hand side of (3.19) in place of the term  $(s - s_0)^{-1/2}$  (this is equivalent to formally putting  $\tau_2(m_1)$ , and  $\sigma_2(m_1)$  zero in (4.34) )
- (c) A third approximation is obtained by retaining only the first two terms in the right-hand side of (3.19) (this is equivalent to (4.34) as it stands).

Furthermore, the contribution arising from (3.18) and (3.47) is the contribution arising from region  $S_1$  (Fig. 4). Consequently, as the subsonic leading edges of the wing approach sonic leading edges, this contribution tends to zero, so that, as the Mach number increases, the above three approximations tend to each other and to the exact value.

For the particular wing of aspect ratio 1.8 and taper ratio 1/7, the derivatives  $-z_\alpha$ ,  $-z_{\alpha^2}$ , . . .  $-h_\alpha$  are tabulated in Table 1 for the special pitching axis  $h = 0$ , and the values of these derivatives for any given pitching axis then follow from (4.34).

5. *Discussion of Results.*—Of the three types of approximation defined at the end of section 4.2, the second approximations to the derivatives in (4.34) and (4.38) for the pitching axis at the apex of the wing ( $h = 0$ ) are tabulated against Mach number for a cropped delta wing (Fig. 1) of aspect ratio 1.8 and taper ratio 1/7. For most practical cropped delta plan-forms this second approximation is sufficient. In Table 1 the third approximations to the derivatives are given for the Mach number  $M = 1.1$ ; even at this unfavourably low Mach number, they are seen to differ little from the second approximation. The derivatives for pitching axes other than that through the apex can be found from (4.34) and (4.38).

The first and second approximations to the derivatives for the pitching axes given by  $h = \frac{1}{2}$  and  $h = 1$  are plotted for this wing against Mach number and shown in Figs. 5, 6, 7, 8. The third approximations for  $M = 1.1$ , are also shown. It is seen from these figures that the derivatives are given sufficiently accurately by their first approximations for a Mach-number range from about  $M = 1.3$  to  $M = 1.944$  at which the leading edges are sonic. From  $M = 1.1$  to 1.3 the derivatives are seen to be given sufficiently accurately by their second approximations. The quasi-steady values of  $-z_\alpha$  and  $-m_\alpha$  for  $M = 1$  from Mangler's<sup>5</sup> (1955) theory are shown in Figs. 5 and 6 for comparison.

6. *Acknowledgement.*—The computation, the results of which are given in the table and figures, was carried out by Miss S. W. Skan.

## LIST OF SYMBOLS

$a$	Speed of sound
$A, B, C$	Areas of integration (Fig. 3)
$c_0$	Root chord
$c_f$	Control chord
=	tip chord
$\bar{c}$	Mean chord
=	$\frac{1}{2}(c_0 + c_f)$
$C_H$	Complex hinge-moment coefficient
=	$H/\frac{1}{2}\rho_0 U^2 S_f c_f$
$C_L$	Complex lift coefficient
=	$L/\frac{1}{2}\rho_0 U^2 S$
$C_m$	Complex pitching-moment coefficient
=	$\mathcal{M}/\frac{1}{2}\rho_0 U^2 S \bar{c}$
$l(m_1)$	Defined by equation (3.38)
$E(k), K(k)$	Complete elliptic integrals of the first and second kinds with modulus $k$
$E(\phi, k), F(\phi, k)$	Corresponding incomplete elliptic integrals of argument
	$\phi = \sin^{-1} \frac{(X^2 - Y^2)^{1/2}}{kX}$
$F$	Complex non-dimensional loading (equation (2.9) )
$F_1, F_2$	Complex non-dimensional loading for constant incidence, pitch case (equations (3.2) and (3.3) )
$h\bar{c}$	Streamwise distance from apex to axis of oscillation
$h_0\bar{c}$	Streamwise distance from apex to hinge line ( $c_0 - c_f$ )
$h_\alpha, h_\alpha$	Stiffness, damping derivative of hinge moment due to pitching (equation (4.32) )
$h_z, h_z$	Stiffness, damping derivative of hinge moment due to plunging (equation (4.35) )
$H$	Complex hinge moment
$k$	= $(1 - m_1^2)^{1/2}$
$L$	Complex lift
$L(v)$	Function obtainable from equation (3.16)
$m_1$	Non-dimensional tangent of semi-apex angle ( <i>see</i> Fig. 3)
=	$\beta s/(c_0 - c_f)$
$m_\alpha, m_\alpha$	Direct stiffness, damping derivative of pitching moment (equation (4.32) )

LIST OF SYMBOLS—*continued*

$m_z, m_z$	Stiffness, damping derivative of pitching moment due to plunging (equation (4.35))
$M$	Mach number of free stream $= U/a$
$M(v)$	Function obtainable from equation (3.45)
$\mathcal{M}$	Complex pitching moment about pitching axis through apex (nose up)
$P$	Complex pressure difference across wing (equation (2.8))
$P_1, \dots, P_5$	Integrals evaluated in Appendix B
$q$	$= i\nu \alpha \bar{c}U\left(2\beta + \frac{1}{\beta}\right)$
$Q_1, \dots, Q_5$	Integrals evaluated in Appendix C
$(r, s)$	Axes parallel to Mach lines (equation (3.9)) (Fig. 4)
$(r_0, s_0)$	Variables of integration in $(r, s)$ -plane
$R_1, R_2$	Integrals evaluated in Appendix D
$R_r(4.2), \dots, R_r(4.13)$	Real parts of right-hand sides of equations (4.2), $\dots$ (4.13)
$\nu R_i(4.2), \dots, \nu R_i(4.13)$	Imaginary parts of right-hand sides of equations (4.2), $\dots$ (4.13)
$s$	Semi-span of wing
$S$	Surface area of wing $= 2s\bar{c}$
$S_f$	Surface area of full-span flap $= 2sc_f$
$S_{m,n}$	Integrals evaluated in Appendix E
$S_1, S_2$	Areas of integration shown in Fig. 4
$t$	Time
$T$	Non-dimensional time $= Ut/\bar{c}$
$T_1, \dots, T_5$	Integrals evaluated in Appendix F
$u$	Function defined by equation (3.12)
$U$	Speed of free stream
$v$	$= (s_0 - r_0)/(s_0 + r_0)$ (equation (3.14)) Variable of integration
$W$	Complex upward component of velocity $= (\partial\Phi/\partial Z)_{z=0}$
$W_1, W_2$	Complex upward component of velocity associated with constant incidence, pitch case

LIST OF SYMBOLS—*continued*

$x, y, z$	Rectangular Cartesian co-ordinates defined by Fig. 1.
$X, Y, Z$	Non-dimensional co-ordinates defined by equation (2.2)
$X_0, X_1$	Non-dimensional lengths shown in Fig. 3
	= $(c_0 - c_f)/\beta\bar{c}, c_0/\beta\bar{c}$
$z_\alpha, z_\dot{\alpha}$	Stiffness, damping derivative of lift due to pitching (equation (4.32))
$z_z, z_\dot{z}$	Stiffness, damping derivative of lift due to plunging (equation (4.35))
$\alpha$	Complex angle of incidence
	= $\alpha_0 e^{i\omega t}$
$\beta$	= $\sqrt{(M^2 - 1)}$
$\zeta$	Complex upward displacement ( <i>see</i> Fig. 2)
$\nu$	Frequency parameter based on mean chord
	= $\omega\bar{c}/U$
$\rho_0$	Density of free stream
$\sigma_1(m_1), \sigma_2(m_1)$	Functions defined by equations (3.53), (3.58)
$\tau_1(m_1), \tau_2(m_1)$	Functions defined by equations (3.33), (3.34)
$\phi$	Perturbation-velocity potential
$\Phi(X, Y)$	Time-independent complex perturbation-velocity potential on upper surface of wing
$\Phi_1(X, Y)$	Corresponding perturbation-velocity potential for uniform incidence
$\Phi_2(X, Y)$	Corresponding perturbation-velocity potential for uniform pitching
$\omega$	$2\pi$ (frequency of oscillation of wing)

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REFERENCES

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## APPENDIX A

### *Integrals required in section 4.1*

In order to calculate the aerodynamic coefficients the integrals (4.1) must be evaluated. The first four integrals in (4.1), namely, the integrals of  $F_1, F_2, XF_1, XF_2$  over the region  $A + B$ , where  $F_1, F_2$  are given by (3.7) and (3.40), reduce to a linear combination of the integrals  $\iint_{A+B} \frac{\chi dX dY}{(m_1^2 X^2 - Y^2)^{1/2}}$ , where  $\chi$  takes the forms  $X, X^2, X^3, (m_1^2 X^2 - Y^2)$  and  $X(m_1^2 X^2 - Y^2)$ . The values of these integrals are given in Appendix B as  $P_1, P_2, \dots, P_5$ . The next four integrals in (4.1), namely, the integrals of  $F_1, F_2, XF_1, XF_2$  over region  $C$ , where  $F_1, F_2$  are given by (3.36) and (3.62), reduce to a linear combination of three sets of integrals:

$$\begin{aligned} & \iint_C X^m Y^n \sinh^{-1} u^{1/2} dX dY, \text{ where } (m, n) \text{ take the values } (0, 0), (0, 1), (1, 0), (1, 1), (2, 0); \\ & R_1 = \iint_C (m_1 X + Y) u^{1/2} (1 + u)^{1/2} dX dY, R_2 = \iint_C X(m_1 X + Y) u^{1/2} (1 + u)^{1/2} dX dY; \\ & S_{m,n} = \iint_C (m_1 X_0 - Y)^{(2m+1)/2} (X + Y)^{(2n+1)/2} dX dY; \end{aligned}$$

where  $u$  is defined in equation (3.12). The first set of integrals may be found from Appendix C as  $Q_1, Q_2, \dots, Q_5$ , the integrals  $R_1$  and  $R_2$  are given in Appendix D and the integrals  $S_{m,n}$  in Appendix E. Finally, the last four integrals in (4.1) correspond to the first four integrals in (4.1) integrated over region  $A$  in place of region  $A + B$ ; these are given in Appendix F as  $T_1, T_2, \dots, T_5$ .

## APPENDIX B

### *Integrals $P_1, P_2, \dots, P_5$*

These are readily evaluated by writing the double integral over the region  $A + B$  in Fig. 3 as the sum of two repeated integrals

$$\iint_{A+B} dX dY = \int_0^{X_0(1+m_1)-X_1} dY \int_{Y/m_1}^{X_1} dX + \int_{X_0(1+m_1)-X_1}^{m_1 X_0} dY \int_{Y/m_1}^{X_0(1+m_1)-Y} dX.$$

Then, if

$$\xi = \frac{X_0(1+m_1) - X_1}{m_1 X_1}, \eta = \frac{1}{m_1} \left\{ 1 - \frac{X_1}{X_0} (1 - m_1) \right\}, \quad \dots \quad \dots \quad \dots \quad \dots \quad (B.1)$$

$$\begin{aligned} P_1 = \iint_{A+B} \frac{X dX dY}{(m_1^2 X^2 - Y^2)^{1/2}} &= \frac{1}{2} X_1^2 \{ \sin^{-1} \xi + \xi (1 - \xi^2)^{1/2} \} \\ &+ \frac{1}{2} \frac{(1 + m_1)^{1/2}}{(1 - m_1)^{3/2}} X_0^2 \{ \cos^{-1} \eta - \eta (1 - \eta^2)^{1/2} \}; \quad \dots \quad (B.2) \end{aligned}$$

$$\begin{aligned}
P_2 = \iint_{A+B} \frac{X^2 dX dY}{(m_1^2 X^2 - Y^2)^{1/2}} &= \frac{1}{3} X_1^3 \{ \sin^{-1} \xi + \frac{1}{2} \xi (1 - \xi^2)^{1/2} \} \\
&+ \frac{(1 + m_1)^{1/2} (2 + m_1^2)}{6(1 - m_1)^{5/2}} X_0^3 \cos^{-1} \eta \\
&- \frac{m_1(1 + m_1)^{1/2}}{6(1 - m_1)^{5/2}} X_0^3 (1 - \eta^2)^{1/2} \left[ 3 + \frac{1 - m_1}{m_1} \frac{X_1}{X_0} \eta \right]; \quad (B.3)
\end{aligned}$$

$$\begin{aligned}
P_3 = \iint_{A+B} \frac{X^3 dX dY}{(m_1^2 X^2 - Y^2)^{1/2}} &= \frac{1}{12} X_1^4 \{ 3 \sin^{-1} \xi + \xi (1 + 2\xi^2) (1 - \xi^2)^{1/2} \} \\
&+ \frac{(1 + m_1)^{1/2}}{24(1 - m_1)^{7/2}} X_0^4 \{ 3(2 + 3m_1^2) \cos^{-1} \eta - \{ 16m_1 + (2 + 7m_1^2)\eta \\
&- 16m_1\eta^2 + (4 + 2m_1^2)\eta^3 \} (1 - \eta^2)^{1/2} \}; \quad \dots \dots \dots (B.4)
\end{aligned}$$

$$\begin{aligned}
P_4 = \iint_{A+B} (m_1^2 X^2 - Y^2)^{1/2} dX dY &= \frac{1}{6} m_1^2 X_1^3 \{ \sin^{-1} \xi + 2\xi (1 - \xi^2)^{1/2} \} \\
&+ \frac{m_1^2(1 + m_1)^{3/2}}{6(1 - m_1)^{3/2}} X_0^3 \cos^{-1} \eta + \frac{m_1^3(1 + m_1)^{1/2}}{6(1 - m_1)^{5/2}} X_0^3 (1 - \eta^2)^{1/2} \\
&\times \left[ 1 - \frac{1}{m_1^2} \left\{ 1 - (1 - m_1)^2 \frac{X_1^2}{X_0^2} \right\} \right]; \quad \dots \dots \dots (B.5)
\end{aligned}$$

$$\begin{aligned}
P_5 = \iint_{A+B} X(m_1^2 X^2 - Y^2)^{1/2} dX dY &= \frac{1}{24} m_1^2 X_1^4 \{ 3 \sin^{-1} \xi + \xi (5 - 2\xi^2) (1 - \xi^2)^{1/2} \} \\
&+ \frac{m_1^2(1 + m_1)^{3/2}}{24(1 - m_1)^{5/2}} X_0^4 \{ 3 \cos^{-1} \eta - \eta (5 - 2\eta^2) (1 - \eta^2)^{1/2} \}. \quad \dots \dots \dots (B.6)
\end{aligned}$$

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## APPENDIX C

### *Integrals $Q_1, Q_2, \dots, Q_5$*

The integrals over the region C in Fig. 3 are evaluated by expressing the double integrals as repeated integrals

$$\iint_C dX dY = \int_{X_0(1+m_1)-X_1}^{m_1 X_0} dY \int_{X_0(1+m_1)-Y}^{X_1} dX$$

and then reducing them to a combination of subsidiary single integrals which are listed at the end of this Appendix. In terms of  $u = \{(1 - m_1)(m_1 X_0 - Y)\}/(m_1 X + Y)$ ,

$$Q_1 = \iint_C \sinh^{-1} u^{1/2} dX dY = 2m_1(1 - m_1)^2 (X_0 + X_1)^2 I_3^{(2)} - \frac{8m_1}{(1 - m_1)} X_0^2 J_3^{(2)}, \quad \dots \quad (C.1)$$

$$Q_2 = \iint_C Y \sinh^{-1} u^{1/2} dX dY = 2m_1^2(1 - m_1)^3(X_0 + X_1)^3 I_4^{(2)} \\ - 2m_1^2(1 - m_1)^2 X_1(X_0 + X_1)^2 I_3^{(2)} - \frac{16m_1^2}{(1 - m_1)^2} X_0^3 J_4^{(2)} + \frac{8m_1^2(1 + m_1)}{(1 - m_1)^2} X_0^3 J_3^{(2)}; \quad (C.2)$$

$$Q_3 = \iint_C X \sinh^{-1} u^{1/2} dX dY = -\frac{1}{m_1} Q_2 + \frac{1}{3} m_1 (1 - m_1)^3 (X_0 + X_1)^3 I_4^{(3)} \\ - \frac{8}{3} \frac{m_1}{(1 - m_1)} X_0^3 J_4^{(3)}; \quad \dots \quad \dots \quad \dots \quad \dots \quad (C.3)$$

$$Q_4 = \iint_C XY \sinh^{-1} u^{1/2} dX dY = -\frac{1}{m_1} \iint_C Y^2 \sinh^{-1} u^{1/2} dX dY + \\ + \frac{1}{3} m_1^2 (1 - m_1)^4 (X_0 + X_1)^4 I_5^{(3)} - \\ - \frac{1}{3} m_1^2 (1 - m_1)^3 X_1 (X_0 + X_1)^3 I_4^{(3)} - \frac{16}{3} \frac{m_1^2}{(1 - m_1)^2} X_0^4 J_5^{(3)} + \\ + \frac{8}{3} m_1^2 \frac{(1 + m_1)}{(1 - m_1)^2} X_0^4 J_4^{(3)}; \quad \dots \quad \dots \quad \dots \quad \dots \quad (C.4)$$

$$Q_5 = \iint_C X^2 \sinh^{-1} u^{1/2} dX dY = -\frac{1}{m_1^2} \iint_C Y^2 \sinh^{-1} u^{1/2} dX dY - \frac{2}{m_1} Q_4 \\ + \frac{2}{45} m_1 (1 - m_1)^4 (X_0 + X_1)^4 I_5^{(4)} - \frac{32}{45} \frac{m_1}{(1 - m_1)} X_0^4 J_5^{(4)}; \quad \dots \quad \dots \quad \dots \quad (C.5)$$

The integrals  $Q_4$  and  $Q_5$  involve the subsidiary double integral

$$\iint_C Y^2 \sinh^{-1} u^{1/2} dX dY = 2m_1^3(1 - m_1)^4 (X_0 + X_1)^4 I_5^{(2)} - 4m_1^3(1 - m_1)^3 X_1 (X_0 + X_1)^3 I_4^{(2)} \\ + 2m_1^3(1 - m_1)^2 X_1^2 (X_0 + X_1)^2 I_3^{(2)} - \frac{32m_1^3 X_0^4}{(1 - m_1)^3} J_5^{(2)} + \\ + \frac{32m_1^3(1 + m_1)}{(1 - m_1)^3} X_0^4 J_4^{(2)} - \frac{8m_1^3(1 + m_1)^2}{(1 - m_1)^3} X_0^4 J_3^{(2)}; \quad \dots \quad (C.6)$$

*Subsidiary single integrals.*—The quantities  $Q_1, Q_2, \dots, Q_5$  are evaluated with the aid of the  $I$ 's and  $J$ 's given below:

$$\text{Let } K = \left[ \frac{(1 - m_1)(X_1 - X_0)}{(X_0 - X_1) + m_1(X_0 + X_1)} \right]^{1/2} \dots \dots \dots \quad (C.7)$$

Since  $m_1 < 1$ ,  $X_0 \leq X_1$  and, by the condition (2.15),  $m_1 \geq (X_1 - X_0)/X_0$  then  $K$  is real.

$$\begin{aligned} \text{Let } I_n^{(1)} &= \int_0^K \frac{dv}{(1 - m_1 + v^2)^n (1 + v^2)^{1/2}} = \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial m_1^{n-1}} \int_0^K \frac{dv}{(1 - m_1 + v^2) (1 + v^2)^{1/2}} = \\ &= \frac{1}{(n-1)!} \frac{\partial^{n-1}}{\partial m_1^{n-1}} I_1^{(1)}, \quad (n \geq 1), \quad \dots \quad \dots \quad \text{(C.8)} \end{aligned}$$

where  $K$  is treated as a constant in the differentiation and where

$$I_1^{(1)} = m_1^{-1/2} (1 - m_1)^{-1/2} \tan^{-1} \left[ \left( \frac{m_1}{1 - m_1} \right)^{1/2} \frac{K}{(1 + K^2)^{1/2}} \right]. \quad \dots \quad \dots \quad \dots \quad \text{(C.9)}$$

$$\text{Let } J_n^{(1)} = \int_0^K \frac{dv}{(1 + v^2)^{(2n+1)/2}} = \frac{K}{(1 + K^2)^{1/2}} \sum_{r=0}^{n-1} \frac{(-)^r}{(2r+1)} \binom{n-1}{r} \frac{K^{2r}}{(1 + K^2)^r}, \quad (n \geq 1), \quad \dots \quad \text{(C.10)}$$

where  $\binom{n-1}{r}$  is a binomial coefficient.

In terms of these results, by integration by parts we obtain

$$\begin{aligned} I_n^{(2)} &= \int_0^K \frac{v \{ \sinh^{-1} v + v(1 + v^2)^{1/2} \} dv}{(1 - m_1 + v^2)^n} = - \frac{ \{ \sinh^{-1} K + K(1 + K^2)^{1/2} \} }{2(n-1)(1 - m_1 + K^2)^{n-1}} + \\ &\quad + \frac{1}{(n-1)} \left[ I_{n-2}^{(1)} + m_1 I_{n-1}^{(1)} \right], \quad (n \geq 3); \quad \dots \quad \text{(C.11)} \end{aligned}$$

$$\begin{aligned} J_n^{(2)} &= \int_0^K \frac{v \{ \sinh^{-1} v + v(1 + v^2)^{1/2} \} dv}{(1 + v^2)^n} = - \frac{ \{ \sinh^{-1} K + K(1 + K^2)^{1/2} \} }{2(n-1)(1 + K^2)^{n-1}} + \\ &\quad + \frac{J_{n-2}^{(1)}}{(n-1)}, \quad (n \geq 3); \quad \dots \quad \dots \quad \dots \quad \text{(C.12)} \end{aligned}$$

$$\begin{aligned} I_n^{(3)} &= \int_0^K \frac{v \{ 3 \sinh^{-1} v + v(1 + v^2)^{3/2} - 3v^3(1 + v^2)^{1/2} \} dv}{(1 - m_1 + v^2)^n} \\ &= - \frac{ \{ 3 \sinh^{-1} K + K(1 + K^2)^{3/2} - 3K^3(1 + K^2)^{1/2} \} }{2(n-1)(1 - m_1 + K^2)^{n-1}} + \frac{2}{(n-1)} \{ -2I_{n-3}^{(1)} + \\ &\quad + (3 - 4m_1)I_{n-2}^{(1)} + m_1(3 - 2m_1)I_{n-1}^{(1)} \}, \quad (n \geq 4); \quad \dots \quad \dots \quad \dots \quad \dots \quad \text{(C.13)} \end{aligned}$$

$$\begin{aligned} J_n^{(3)} &= \int_0^K \frac{v \{ 3 \sinh^{-1} v + v(1 + v^2)^{3/2} - 3v^3(1 + v^2)^{1/2} \} dv}{(1 + v^2)^n} \\ &= - \frac{ \{ 3 \sinh^{-1} K + K(1 + K^2)^{3/2} - 3K^3(1 + K^2)^{1/2} \} }{2(n-1)(1 + K^2)^{n-1}} + \frac{2}{(n-1)} \{ -2J_{n-3}^{(1)} + \\ &\quad + 3J_{n-2}^{(1)} \}, \quad (n \geq 4); \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \text{(C.14)} \end{aligned}$$

$$\begin{aligned}
I_n^{(4)} &= \int_0^K \frac{v\{15 \sinh^{-1} v + v(8v^4 - 4v^2 + 3)(1 + v^2)^{1/2}\} dv}{(1 - m_1 + v^2)^n} \\
&= -\frac{\{15 \sinh^{-1} K + K(8K^4 - 4K^2 + 3)(1 + K^2)^{1/2}\}}{2(n-1)(1 - m_1 + K^2)^{n-1}} + \frac{3}{(n-1)} \{m_1(15 - 20m_1 + \\
&\quad + 8m_1^2) I_{n-1}^{(1)} + (15 - 40m_1 + 24m_1^2) I_{n-2}^{(1)} - 4(5 - 6m_1) I_{n-3}^{(1)} + 8I_{n-4}^{(1)}\}, \quad (n \geq 5); \quad (C.15)
\end{aligned}$$

$$\begin{aligned}
J_n^{(4)} &= \int_0^K \frac{v\{15 \sinh^{-1} v + v(8v^4 - 4v^2 + 3)(1 + v^2)^{1/2}\} dv}{(1 + v^2)^n} \\
&= -\frac{\{15 \sinh^{-1} K + K(8K^4 - 4K^2 + 3)(1 + K^2)^{1/2}\}}{2(n-1)(1 + K^2)^{n-1}} + \frac{3}{(n-1)} \{15J_{n-2}^{(1)} - 20J_{n-3}^{(1)} + \\
&\quad + 8J_{n-4}^{(1)}\}, \quad (n \geq 5). \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (C.16)
\end{aligned}$$

## APPENDIX D

### Integrals $R_1, R_2$

These integrals are evaluated in the same way as the integrals  $Q_1, \dots, Q_5$  of Appendix C. The double integral over region C is expressed as the same repeated integral. In terms of  $u = \{(1 - m_1)(m_1 X_0 - Y)\}/(m_1 X + Y)$

$$\begin{aligned}
R_1 &= \iint_C (m_1 X + Y) u^{1/2} (1 + u)^{1/2} dX dY = \frac{2}{3} m_1^2 (1 - m_1)^3 (X_0 + X_1)^3 G_1 - \\
&\quad - \frac{16}{3} \frac{m_1^2}{(1 - m_1)} X_0^3 H_1, \quad \dots \quad \dots \quad (D.1)
\end{aligned}$$

$$\begin{aligned}
R_2 &= \iint_C X(m_1 X + Y) u^{1/2} (1 + u)^{1/2} dX dY = \frac{2}{3} m_1^2 (1 - m_1)^3 X_1 (X_0 + X_1)^3 G_1 - \\
&\quad - \frac{4}{15} m_1^2 (1 - m_1)^4 (X_0 + X_1)^4 G_2 - \\
&\quad - \frac{16}{15} \frac{m_1^2 X_0^4}{(1 - m_1)^2} \{10m_1 H_2 + (1 - m_1) H_1\}, \quad \dots \quad (D.2)
\end{aligned}$$

where

$$H_1 = \frac{2}{3} \frac{K^3}{(1 + K^2)^{3/2}}, \quad \dots \quad \dots \quad \dots \quad (D.3)$$

$$H_2 = \frac{2}{5} \frac{K^5}{(1 + K^2)^{5/2}}, \quad \dots \quad \dots \quad \dots \quad (D.4)$$

$$G_1 = \frac{1}{24} m_1^{-3/2} (1 - m_1)^{-5/2} \{3\theta - \sin \theta \cos \theta (3 - 14 \sin^2 \theta + 8 \sin^4 \theta)\}, \quad \dots \quad (D.5)$$

$$G_2 = \frac{1}{192} m_1^{-3/2} (1 - m_1)^{-7/2} \{15\theta - \sin \theta \cos \theta (15 - 118 \sin^2 \theta + 136 \sin^4 \theta - 48 \sin^6 \theta)\}, \quad (D.6)$$

$$\theta = \tan^{-1} \left[ \left( \frac{m_1}{1 - m_1} \right)^{1/2} \frac{K}{(1 + K^2)^{1/2}} \right], \quad \dots \quad (D.7)$$

where  $K$  is given by (C.7) of Appendix C.

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## APPENDIX E

### *Integrals $S_{m,n}$*

Again the double integral over region  $C$  is expressed as the repeated integral in Appendix C and

$$S_{m,n} = \iint_C (m_1 X_0 - Y)^{m+1/2} (X + Y)^{n+1/2} dX dY \quad \dots \quad (E.1)$$

is found to reduce to

$$S_{m,n} = \frac{4}{(2n+3)} (m_1 X_0 + X_1)^{m+n+3} \int_0^\phi \sin^{2m+2} \theta \cos^{2n+4} \theta d\theta - \frac{4(1+m_1)^{n+3/2} X_0^{n+3/2} (X_1 - X_0)^{m+3/2}}{(2m+3)(2n+3)}, \quad \dots \quad (E.2)$$

where

$$\phi = \sin^{-1} \left( \frac{X_1 - X_0}{m_1 X_0 + X_1} \right)^{1/2} \quad \dots \quad (E.3)$$

and the integral on the right-hand side of (E.2) may be evaluated by means of the reduction formulae

$$\left. \begin{aligned} I_{n,m} &= \frac{\sin^{n+1} \theta \cos^{m-1} \theta}{(n+m)} + \left( \frac{m-1}{n+m} \right) I_{n,m-2} \\ I_{n,m} &= - \frac{\sin^{n-1} \theta \cos^{m+1} \theta}{(n+m)} + \left( \frac{n-1}{n+m} \right) I_{n-2,m} \end{aligned} \right\}, \quad \dots \quad (E.4)$$

where

$$I_{n,m} = \int \sin^n \theta \cos^m \theta d\theta. \quad \dots \quad \therefore \quad \dots \quad (E.5)$$

## APPENDIX F

### *Integrals $T_1, T_2, \dots, T_5$*

These are evaluated by expressing the double integral over region  $A$  in Fig. 3 as a repeated integral:

$$\iint_A dX dY = \int_0^{X_0} dX \int_0^{m_1 X} dY.$$

Then it is easily found that

$$T_1 = \iint_A \frac{X}{(m_1^2 X^2 - Y^2)^{1/2}} dX dY = \frac{\pi}{4} X_0^2; \dots \dots \dots \quad (\text{F.1})$$

$$T_2 = \iint_A \frac{X^2}{(m_1^2 X^2 - Y^2)^{1/2}} dX dY = \frac{\pi}{6} X_0^3; \dots \dots \dots \quad (\text{F.2})$$

$$T_3 = \iint_A \frac{X^3}{(m_1^2 X^2 - Y^2)^{1/2}} dX dY = \frac{\pi}{8} X_0^4; \dots \dots \dots \quad (\text{F.3})$$

$$T_4 = \iint_A (m_1^2 X^2 - Y^2)^{1/2} dX dY = \frac{\pi}{12} m_1^2 X_0^3; \dots \dots \dots \quad (\text{F.4})$$

$$T_5 = \iint_A X(m_1^2 X^2 - Y^2)^{1/2} dX dY = \frac{\pi}{16} m_1^2 X_0^4. \dots \dots \dots \quad (\text{F.5})$$

TABLE 1

*Second Approximation to Derivatives for the Delta Wing  
of Aspect Ratio 1.8 and Taper Ratio 1/7  
with Subsonic Leading Edges*

$M$	$-z_\alpha = -z_{\dot{\alpha}}$	$-z_{\dot{\alpha}}$	$-m_\alpha = -m_{\dot{\alpha}}$	$-m_{\dot{\alpha}}$	$-h_\alpha = -h_{\dot{\alpha}}$	$-h_{\dot{\alpha}}$
1.1*	1.536	1.774	1.676	2.305	0.3880	1.2724
1.1	1.537	1.762	1.678	2.285	0.3905	1.2382
1.15	1.508	1.763	1.656	2.282	0.4208	1.2142
1.2	1.478	1.734	1.629	2.243	0.4358	1.1871
1.3	1.417	1.644	1.570	2.128	0.4469	1.1264
1.4	1.361	1.546	1.513	2.001	0.4482	1.0655
1.5	1.309	1.453	1.457	1.882	0.4428	1.0030
1.6	1.260	1.367	1.405	1.771	0.4350	0.9487
1.944†	1.116	1.125	1.249	1.460	0.4027	0.7923

\* Indicates that the third approximations to the derivatives are given.

† Indicates the Mach number at which the wing has sonic leading edges.

The pitching axis is taken at the apex of the wing.



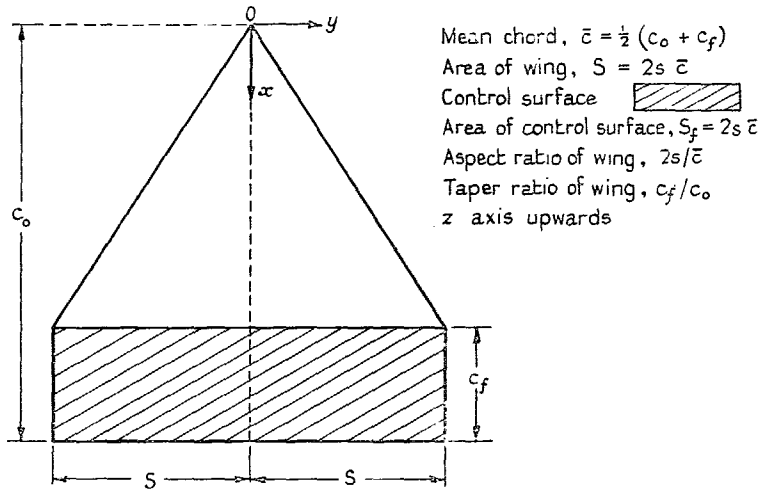


FIG. 1. Plan of wing and control surface.

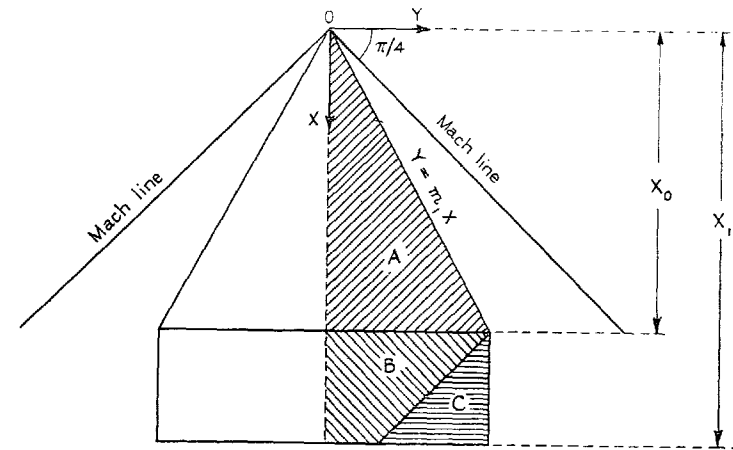


FIG. 3. 'Transform' of wing.

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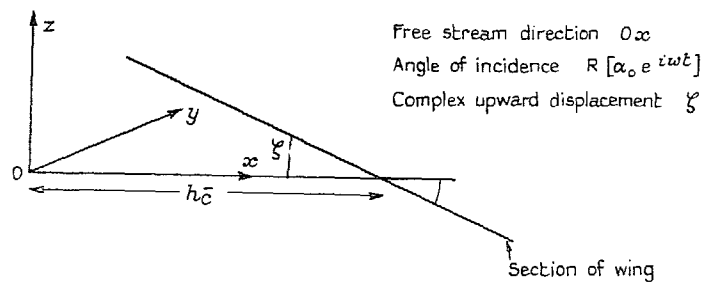


FIG. 2. Motion of wing surface.

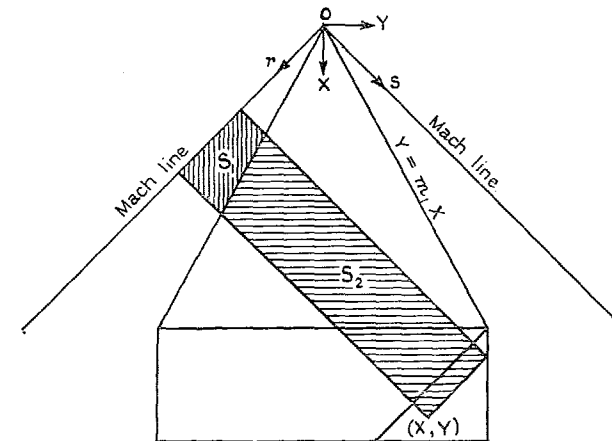


FIG. 4. Areas of integration to obtain velocity potentials.

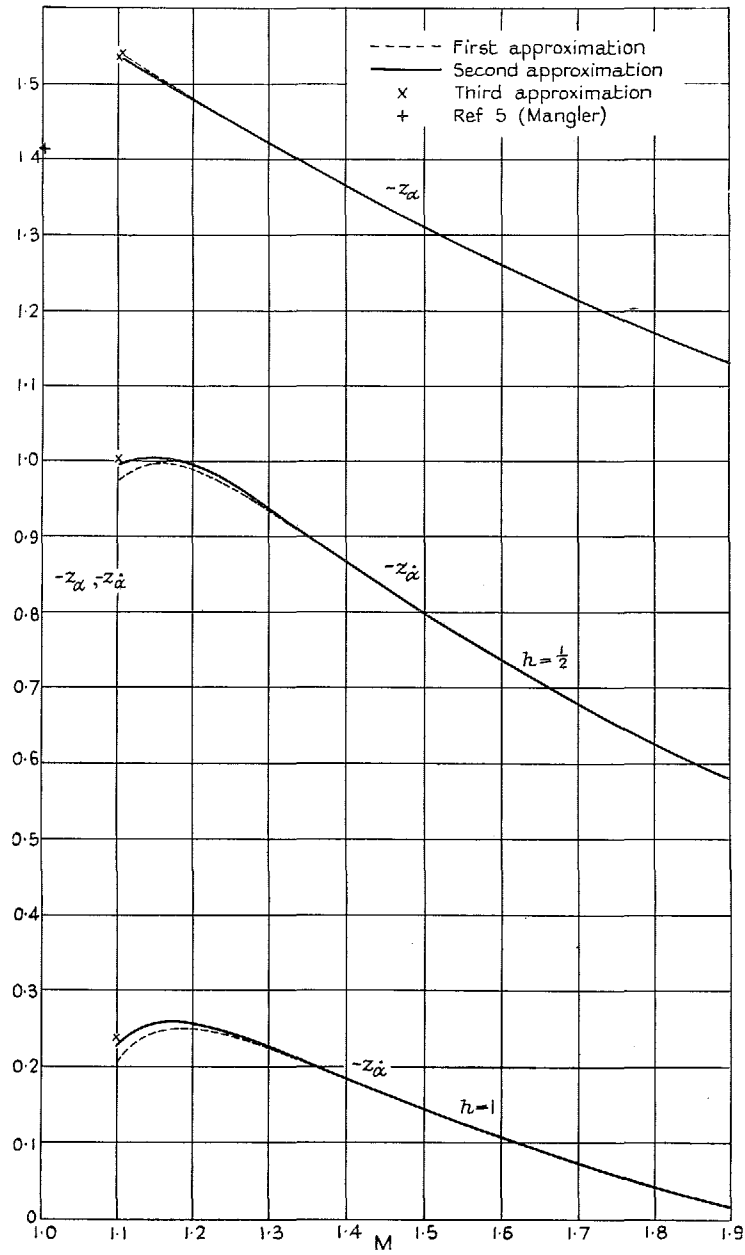


FIG. 5. Variation of lift derivatives with Mach number.

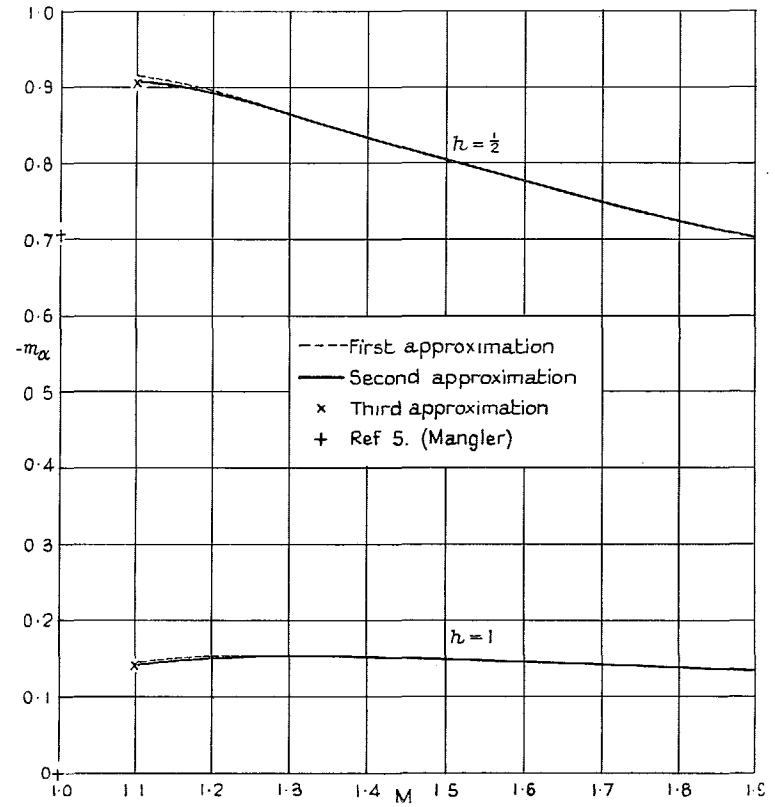


FIG. 6. Variation of pitching-moment stiffness derivative with Mach number.

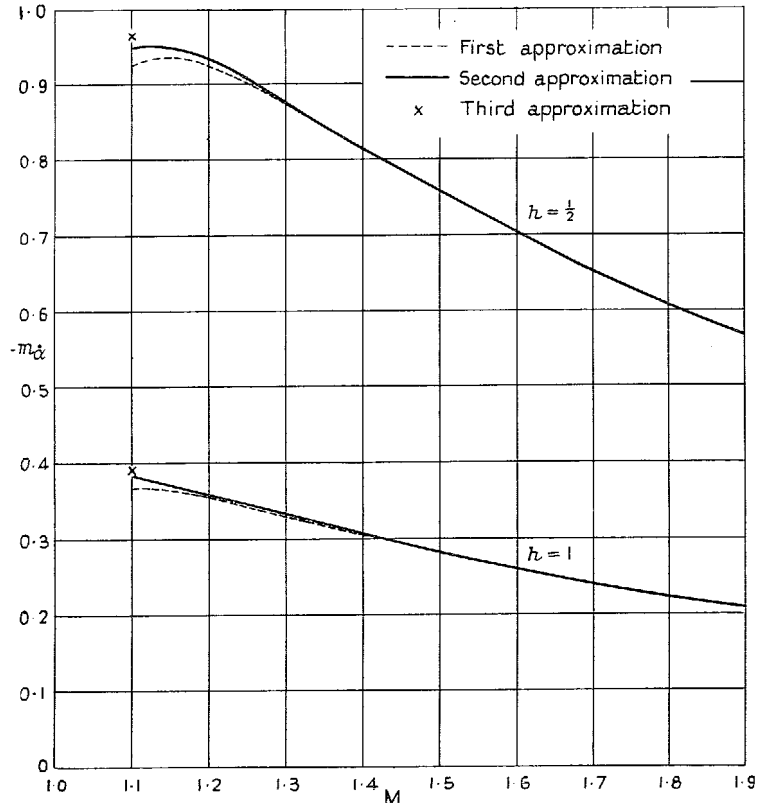


FIG. 7. Variation of pitching-moment damping derivative with Mach number.

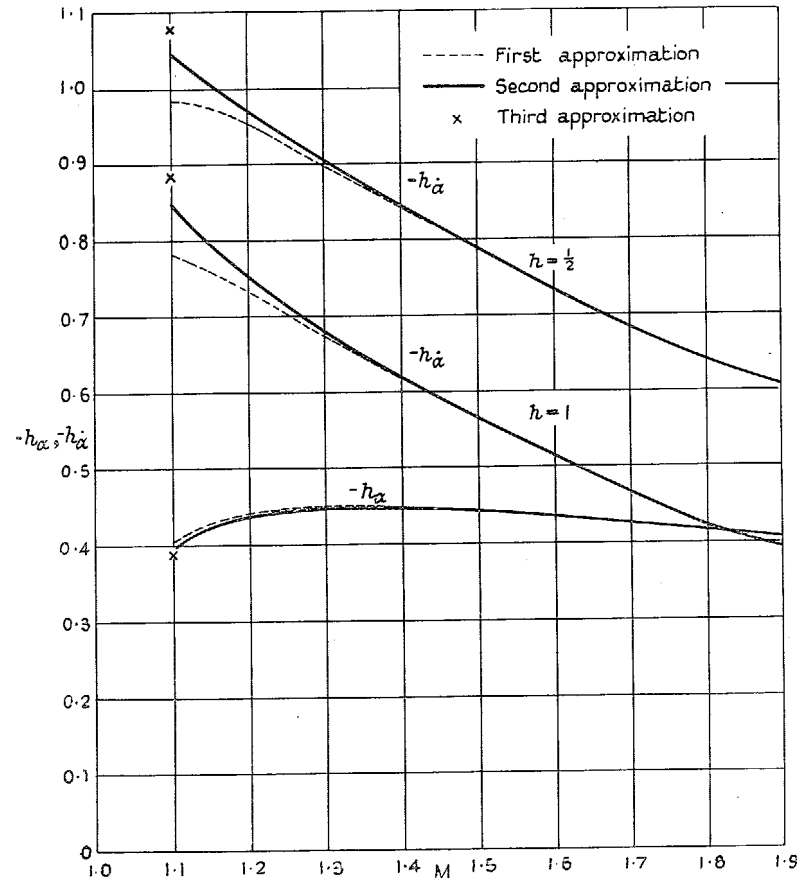


FIG. 8. Variation of hinge-moment derivatives with Mach number.

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