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The Deformation of a Long Swept Wing with Chordwise Variation of the thickness

By

E. C. Capey, B.Sc.

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ROYAL AIRCRAFT ESTABLISHMENT

The Deformation of a Long Swept Wing with Chordwise Variation of the Thickness

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SUMMARY

The stress distribution in a long, swept-back, solid, thin wing under a bending moment or a torsional moment is calculated using the inextensional theory for thin flat plates. Solutions are given for all sweep-back angles for strips whose cross-sections are rectangular, diamond shaped, parabolic and double wedge shaped.

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1 Notation

b	width of strip
t	thickness of strip at a point
t_o	thickness of strip at the centre
D	flexural rigidity at a point $= Et^3/12(1 - \nu^2)$
D_o	flexural rigidity at the centre
M_B	bending moment
M_T	torsional moment
Ox, Oy	axes in plane of strip, as shown in Fig.1
α	angle which generator makes with the x axis
α_o	angle at which strip is built in
η	distance measured along generator
η_1, η_2	distances of edges of plates measured along generator
M_α	moment acting perpendicular to generator
M_η	moment per unit length at a distance η along the generator
ε	$b \frac{d\alpha}{dx} \operatorname{cosec}^2 \alpha$
T	η/η_1
σ	stress at a point
σ'	$\frac{bt_o^2}{6M_B}\sigma$ or $\frac{bt_o^2}{3M_T}\sigma$
$f(\varepsilon)$	defined by equation (25)
$g(\varepsilon)$	defined by equation (32)
F	defined by equation (17)
α_1	defined by equation (7)
x^*	effective extra length, shown in Fig.3
γ	OQ/OP in Fig.3
K	a constant of integration
$\phi(\varepsilon)$	$\varepsilon\sqrt{\frac{df}{d\varepsilon}}$

2 Introduction

A long strip, whose thickness varies across its width, is built in at a sweep-back angle α_o, and a bending moment or a torsional moment is applied at its ends. The mode of deformation, stress distribution and stiffness are obtained using the inextensional theory for thin flat plates[1,2] in which it is assumed that the strip is bent without any middle surface strain, and consequently must take the form of a developable surface.

The assumption that there is no strain in the middle surface is generally valid when the deflection of the plate is large compared with its thickness, so it is only applicable to very thin wings.

General equations are obtained for a long strip with arbitrary cross-section subjected to a bending moment or a torsional moment. These equations are solved, for all sweep-back angles, for rectangular, diamond, parabolic and double wedge cross-sections.

3 Method of Solution

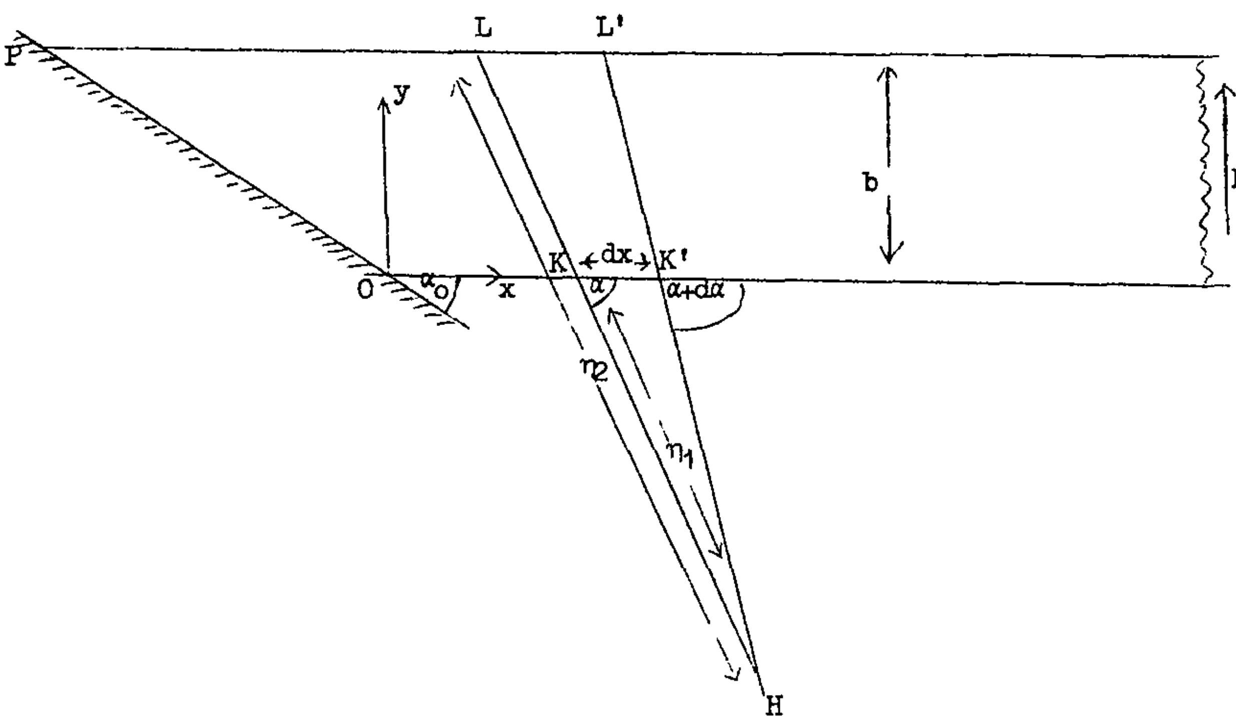

Fig.1

Fig.1 shows a strip to which a moment M_B is applied. HKL and HK'L' are two adjoining generators making angles α and $(\alpha + d\alpha)$ with the x axis. The strip KLL'K' forms part of a conical surface whose apex is at H.

The total moment acting perpendicular to the generator HKL is given by

$$M_\alpha = M_B \sin \alpha \qquad (1)$$

while if a torsional moment M_T is applied instead

$$M_\alpha = M_T \cos \alpha . \qquad (2)$$

Mansfield and Kleeman[1] and Mansfield[2] have shown that the strain energy in inextensional deformation is given by

$$U = \frac{1}{2} \int \left[\frac{M_\alpha^2}{\int_{\eta_1}^{\eta_2} \frac{D_\eta}{\eta} d\eta} \right] d\alpha , \qquad (3)$$

where D_η is the flexural rigidity at a distance η along the generator HKL from H. By maximisation of U a relationship can be obtained between x and α. It is shown in Appendix I that for a strip subjected to a bending moment this relationship is

$$\sin^2 \alpha = g(\varepsilon) , \tag{4}$$

while for a strip subjected to a torsional moment

$$\sin 2\alpha = g(\varepsilon) , \tag{5}$$

where

$$\varepsilon = b\frac{d\alpha}{dx} \operatorname{cosec}^2 \alpha \tag{6}$$

and g is a function of ε and of the shape of the cross section, though it is independent of the width and thickness of the cross-section. In Appendix IV expressions are obtained for g as a function of ε for each of the four cross-sections considered here.

As g is a complicated function of ε, equations (4) and (5) can be solved only by numerical methods. For each cross-section a number of values of ε were chosen, g was calculated for each of these, using the appropriate formula from Appendix IV, then equations (4) and (5) were used to calculate α for each value of ε. Integration of equation (6) by arithmetical methods then made it possible to plot x/b against α, and to draw the generators on a diagram of the strip. Figs.4 to 15 show the generators for the four types of strip considered in this paper.

It can be shown that, under inextensional deformation, the strain energy in a long strip subjected to a bending moment becomes a maximum when the generators at some distance from the ends are at 90° to Ox; while, if it is subjected to a torsional moment, the strain energy becomes a maximum when they are at 45° to Ox. From this it follows that, in the problem of a clamped strip, α must tend to 90° as x tends to infinity if a bending moment is applied, and to 45° if a torsional moment is applied.

When $\varepsilon = 0$, g is always equal to unity, giving an angle $\alpha = 90°$ in the bending case, and $\alpha = 45°$ in the torsional case. As ε increases, g always decreases; but g does not always decrease to zero, so that α does not always decrease to zero. In fact equation (4) has solutions only for values of α greater than α_1, where

$$\sin^2 \alpha_1 = g_{min} . \tag{7}$$

The generators for α less than α_1 all pass through the point O, as shown in Fig.2. As generators for varying angles all pass through the same point, it follows that

$$\frac{dx}{d\alpha} = 0$$

and consequently, on substituting this in equation (6), it is seen that ε has no finite value.

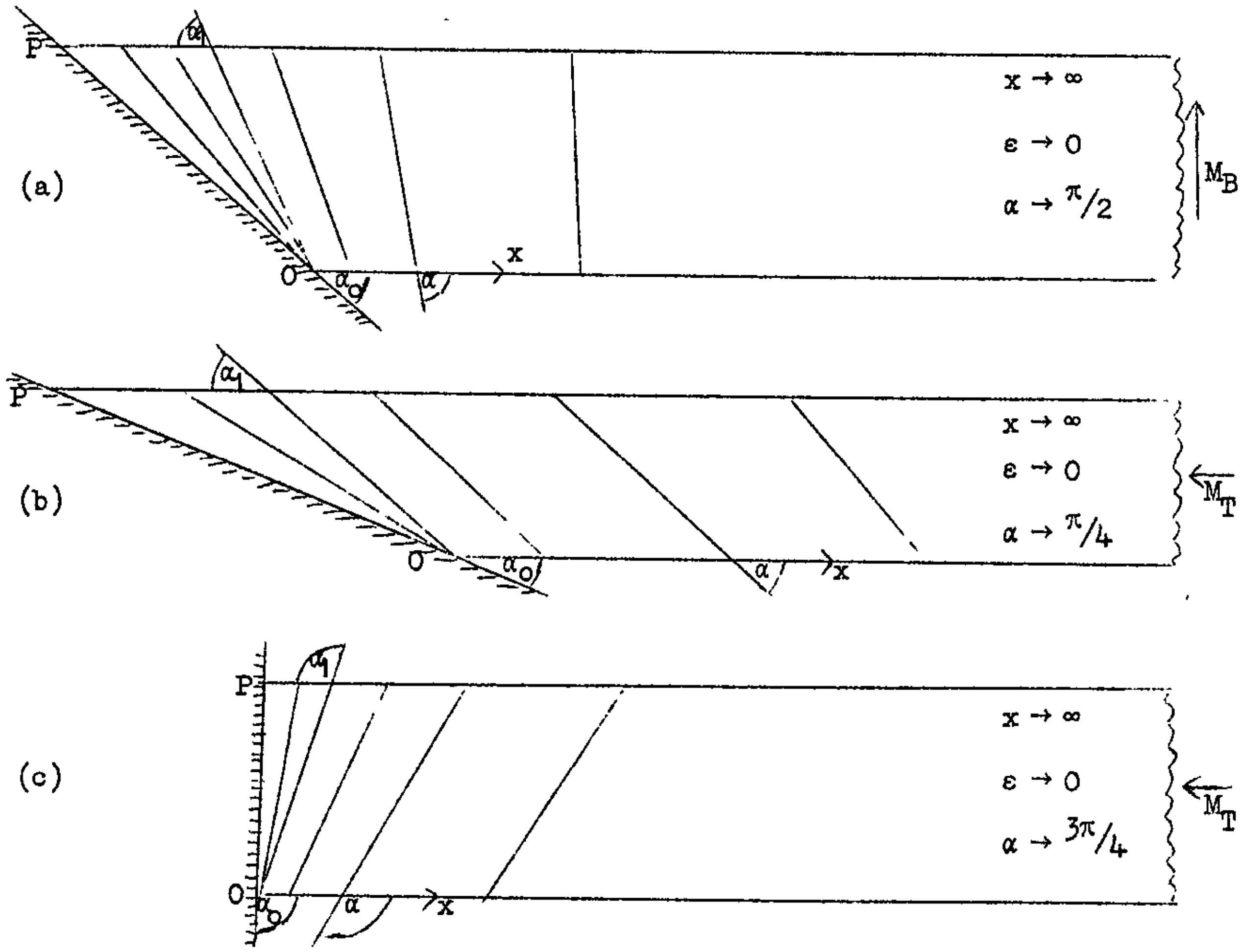

Fig.2

In the cases shown in Fig.2 (a) and (b), if the generators and stress distribution are known for $\alpha_o = 0$, then they are given for all other values of α_o, as the plate can be clamped along any of the generators of the $\alpha_o = 0$ system without altering the stresses and strains in the system. Similarly, for the case shown in Fig.2(c), it is sufficient to calculate the generators and stress distribution for $\alpha_o = 90^o$. The generators and stress distributions for rectangular, diamond shaped, parabolic and double wedge shaped cross-sections are shown in Figs.4 to 15.

4 Stress Distribution

In Appendix II it is shown that, where $\alpha > \alpha_1$, the distribution of stress is given by the relationship

$$\sigma' = \frac{bt_o^2}{6M_B}\sigma = \left(\frac{t}{t_o}\right)\left(\frac{\phi}{f}\right)_{\varepsilon=0} \frac{1}{\phi\left(\frac{y}{b} + \frac{1}{\varepsilon}\right)} , \qquad (8)$$

or

$$\sigma' = \frac{bt_o^2}{3M_T}\sigma = \left(\frac{t}{t_o}\right)\left(\frac{\phi}{f}\right)_{\varepsilon=0} \frac{1}{\phi\left(\frac{y}{b} + \frac{1}{\varepsilon}\right)} ; \qquad (9)$$

where

$$\phi = \varepsilon\sqrt{\frac{df}{d\varepsilon}}$$

and f is a function of ε, for which a general expression is obtained in Appendix I. Special expressions for f are given in Appendix IV for the four strips with which we are dealing.

For each cross-section, graphs of equal stress were obtained by giving σ' selected values, then for any particular value of σ', y/b was calculated for each generator using equations (8) and (9), remembering that t/t_o is a function of y/b, and that the value of ε was known for each generator. Consequently, for each value of σ', the appropriate value of y/b was marked on each generator, and graphs of equal σ' were obtained. These are shown in Figs.4 to 15.

Equations (8) and (9) apply only where $\alpha > \alpha_1$. It is shown in Appendix II that, where $\alpha < \alpha_1$,

$$\sigma' = \frac{bt_o^2}{6M_B}\sigma = \left(\frac{t}{t_o}\right)\frac{\sin^2\alpha}{f(\infty)\,\frac{y}{b}}\ , \tag{10}$$

or

$$\sigma' = \frac{bt_o^2}{3M_T}\sigma = \left(\frac{t}{t_o}\right)\frac{\sin 2\alpha}{f(\infty)\,\frac{y}{b}}\ . \tag{11}$$

Graphs of equal stress were obtained in this region by the same method as was used for $\alpha > \alpha_1$.

5 Effective Lengths

If a long strip, shown in Fig.3(a), of length ℓ, built in at an angle α_o, rotates as much on application of M_B as a similar strip, of length $(\ell + x^*)$, built in at 90° then we can say that the effective length of the former strip is $(\ell + x^*)$. For large values of ℓ, x^* depends only on α_o. In Appendix III x^* is evaluated, and so is γ, which is defined by the equation

$$\gamma = OQ/OP\ . \tag{12}$$

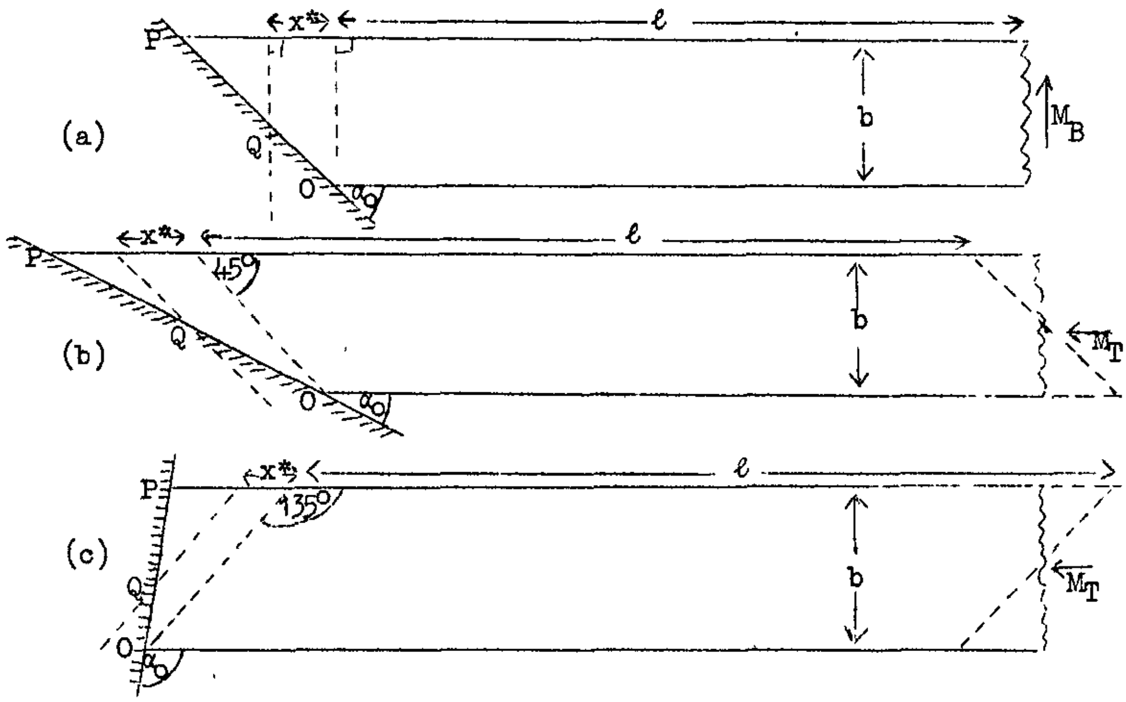

Fig.3

It is shown in Appendix III that in the bending case

$$\gamma \cot \alpha_o = x^*/b$$

$$= \left(\frac{f}{\varepsilon}\right)_o \frac{1}{f(\infty)}\left[\frac{\alpha_1}{2} - \frac{\sin 2\alpha_1}{4} - \frac{\alpha_o}{2} + \frac{\sin 2\alpha_o}{4}\right] + \int_{\alpha_1}^{\pi/2}\left[\left(\frac{f}{\varepsilon}\right)_o \frac{\sin^2\alpha}{f} - \frac{1}{\varepsilon \sin^2\alpha}\right] d\alpha . \qquad \text{..... (13)}$$

This equation was integrated numerically, and γ is plotted as a function of α_o in Fig.16 for the appropriate cross-sections.

If M_T is applied to a strip of length ℓ, and the strip rotates as much as a similar strip of length $(\ell + x^*)$ clamped at 45^o, then we can say that the effective length of the former strip is $(\ell + x^*)$. A strip clamped at 45^o has been chosen as a standard because such a strip bends in a simple manner, with all its generators parallel, on application of a torsional moment. It is shown in Appendix III that, for the case shown in Fig.3(b),

$$\gamma(\cot \alpha_o - 1) = x^*/b$$

$$= \left(\frac{f}{\varepsilon}\right)_o \frac{1}{f(\infty)} \left[2\alpha_1 + \sin \alpha_1 - 2\alpha_o - \sin \alpha_o\right] + \int_{\alpha_1}^{\pi/4}\left[\left(\frac{f}{\varepsilon}\right)_o \frac{4 \cos^2\alpha}{f} - \frac{1}{\varepsilon \sin^2\alpha}\right] d\alpha . \qquad \text{..... (14)}$$

while for the case shown in Fig.3(c),

$$\gamma(\cot \alpha_o + 1) = x^*/b \qquad (15)$$

where x^*/b is given by the same expression, except that the limit of integration in this instance is $3\pi/4$.

6 Discussion of Results

Figs.4 to 15 show the generators and stresses in strips with rectangular, diamond shaped, parabolic and double wedge shaped cross-sections when a bending moment or a torsional moment is applied on the strip. Fig.4 is taken from 'The Inextensional Theory for Thin Flat Plates' by Mansfield[2]. Figs.7, 8 and 9 refer to diamond shaped strips, and show that there is no extra stress concentration due to the clamping of the strip at a skew angle. Figs.10 to 15 refer to double wedge shaped strips and parabolic strips, and show highly localised stresses, with a stress concentration factor of up to 1.20 for the double wedge shaped strip, and of up to 1.52 for the parabolic strip. The rectangular strip (show in Figs.4, 5 and 6) has a very high stress concentration in a small region; but this must not be taken too seriously, as in this region the inextensional theory is inapplicable, even when it is accurate elsewhere.

Figs.16 and 17 show how the stiffnesses of the strips vary with the angle α_o at which the strip is to be clamped. Fig.16 refers to a strip subjected to a bending moment, and shows that as α_o approaches 90^o, γ approaches 0.5, so that the effective length of the strip is nearly equal to the average length; while for small values of α_o, γ becomes small, so that the effective length approaches the smallest length of the strip. Fig.17 refers to a strip subjected to a torsional moment. In this instance γ approaches 0.5 as α_o approaches 45^o or 135^o, and γ becomes small when α_o approaches 0^o or 90^o.

7 Conclusions

Methods of calculation are derived for a long strip, built in at any angle, subjected to a bending moment or a torsional moment. The method is applied to strips with rectangular, diamond shaped, parabolic and double wedge shaped cross-sections, and diagrams are presented showing generators and lines of equal stress.

It is shown that clamping the diamond shaped strip at a skew angle produces no extra stress concentration, that clamping the parabolic and double wedge shaped strips produces a small extra stress concentration, and that clamping the rectangular strip produces very high stresses in a small region.

The stiffness of the strips is discussed and expressions are given for the effective length of a strip under a bending moment or a torsional moment. From the graphs the effective length can be obtained for any of the four strips which have been considered, for any sweep-back angle.

REFERENCES

No.	Author	Title, etc.
1	E.H. Mansfield and P.W. Kleeman	A large deflection theory for thin plates. Aircraft Engineering, Vol.27, April, 1955, pp.102-108.
2	E.H. Mansfield	The inextensional theory for thin flat plates. Quarterly Journal of Mechanics and Applied Mathematics, Vol.8, September, 1955, pp.338-352.

APPENDIX I

Determination of the generators of a long strip built in at an angle, and subjected to a moment

Equation (3), for the strain energy of the strips, can be put in the form

$$U = \frac{1}{2} \int F \, d\alpha \tag{16}$$

where

$$F = \frac{M_\alpha^2}{\int_{\eta_1}^{\eta_2} \frac{D_\eta}{\eta} \, d\eta} . \tag{17}$$

It is difficult to deal with F in this form, and it is convenient to substitute other expressions for η, η_1 and η_2. Using the notation of Fig.1 it follows from geometrical considerations that

$$\eta_1 = \frac{dx}{d\alpha} \sin \alpha \tag{18}$$

and that

$$\eta_2 - \eta_1 = b \operatorname{cosec} \alpha . \tag{19}$$

If we also substitute

$$T = \eta/\eta_1 \tag{20}$$

and

$$\varepsilon = b \frac{d\alpha}{dx} \operatorname{cosec}^2 \alpha \tag{21}$$

$$= \frac{\eta_2 - \eta_1}{\eta_1} , \tag{22}$$

in equation (17), it becomes

$$F = \frac{M_\alpha^2}{D_o \int_1^{1+\varepsilon} \left(\frac{D}{D_o}\right) \frac{dT}{T}} \tag{23}$$

$$= \frac{M_\alpha^2}{D_o f} \tag{24}$$

where

$$f = \int_1^{1+\varepsilon} \left(\frac{D}{D_o}\right) \frac{dT}{T} . \tag{25}$$

As

$$\left(\frac{D}{D_o}\right) = \left(\frac{t}{t_o}\right)^3 ,$$

which is a function of y/b for a given shape of plate, and as

$$T = 1 + \varepsilon y/b \qquad (26)$$

it follows that f is a function of ε. Therefore, using equations (24) and (21), it follows that F is a function of α and of $dx/d\alpha$ for a given strip. Considering that

$$U = \tfrac{1}{2}\int F\left(\alpha, \frac{dx}{d\alpha}\right) d\alpha \qquad (27)$$

is to be maximised, it can be shown from the calculus of variations that

$$\frac{\partial F}{\partial\left(\frac{dx}{d\alpha}\right)} = K , \qquad (28)$$

where K is a constant. Using equations (24) and (21) this becomes

$$K = \frac{\partial F}{\partial f}\frac{df}{d\varepsilon}\frac{\partial \varepsilon}{\partial\left(\frac{dx}{d\alpha}\right)}$$

$$= \frac{M_\alpha^2}{D_o\, f^2}\frac{df}{d\varepsilon}\frac{\varepsilon^2 \sin^2\alpha}{b} , \qquad (29)$$

which can be put in the form

$$\frac{M_\alpha \sin\alpha}{(M_\alpha \sin\alpha)_\infty} = \frac{f}{\phi}\left(\frac{\phi}{f}\right)_\infty \qquad (30)$$

where

$$\phi = \varepsilon\sqrt{\frac{df}{d\varepsilon}} .$$

As x tends to infinity, α tends to $\pi/2$ in the bending case and $\pi/4$ or $3\pi/4$ in the torsion case. In either case $d\alpha/dx$ tends to zero, and consequently ε does the same. Substituting equations (1) and (2) for M_α equation (30) becomes

$$\left.\begin{aligned} \sin^2\alpha &= g(\varepsilon) \quad \text{(bending)} \\ \text{and} \qquad \sin 2\alpha &= g(\varepsilon) \quad \text{(torsion)} \end{aligned}\right\} \qquad (31)$$

where

$$g(\varepsilon) = \frac{f}{\phi}\left(\frac{\phi}{f}\right)_{\varepsilon=0} . \qquad (32)$$

As f can be calculated as a function of ε, these equations show the relationship between α and ε, from which a relationship can be obtained between x and α.

APPENDIX II

Determination of the Stress Distribution in the Strip

The equation for the principal stress on the surface of a plate subjected to a bending moment in one dimension is

$$\sigma = 6M_\eta / t^2 . \tag{33}$$

Using the equations given by Mansfield[1,2] and Kleeman[1], this becomes

$$\sigma = \frac{6D_\eta}{t^2\eta} \frac{M_\alpha}{\int_{\eta_1}^{\eta_2} \frac{D_\eta}{\eta} d\eta} \tag{34}$$

for any plate under inextensional deformation. Substituting equations (17) and (24) we have

$$\sigma = \frac{6D_\eta . M_\alpha}{D_o \, t^2\eta . f(\varepsilon)} . \tag{35}$$

For the bending case, equations (1), (4) and (32) are substituted to give

$$\sigma = \frac{6M_B}{t^2} \frac{D_\eta}{D} \left(\frac{\phi}{f}\right)_o \frac{1}{\eta \phi \sin \alpha} \tag{36}$$

which is valid only when $\alpha > \alpha_1$. From geometrical considerations it can be shown that

$$\eta \sin \alpha = \eta_1 \sin \alpha + y , \tag{37}$$

and substitution of this and equation (18) into equation (36) gives

$$\sigma' = \frac{bt_o^2}{6M_B} \sigma = \left(\frac{t}{t_o}\right) \left(\frac{\phi}{f}\right)_o \frac{1}{\phi \left(\frac{1}{\varepsilon} + \frac{y}{b}\right)} \quad (\alpha > \alpha_1) . \tag{38}$$

A similar treatment of the torsional case produces the equation

$$\sigma' = \frac{bt_o^2}{3M_T} \sigma = \left(\frac{t}{t_o}\right) \left(\frac{\phi}{f}\right)_o \frac{1}{\phi \left(\frac{1}{\varepsilon} + \frac{y}{b}\right)} \quad (\alpha > \alpha_1) . \tag{39}$$

When $\alpha < \alpha_1$, ε is infinite. So for bending equation (35) becomes

$$\sigma = \frac{6D_\eta}{D_o \, t^2} \frac{M_B \sin \alpha}{\eta f(\infty)_o} . \tag{40}$$

As $\eta_1 = 0$ for this region, equation (37) is simplified, and on substitution in equation (40) gives

$$\sigma' = \frac{bt_o^2}{6M_B}\sigma = \left(\frac{t}{t_o}\right)\frac{\sin^2\alpha}{f(\infty)\ y/b} \qquad (\alpha < \alpha_1)\ , \tag{41}$$

while a similar treatment of the torsional case gives

$$\sigma' = \frac{bt_o^2}{3M_T}\sigma = \left(\frac{t}{t_o}\right)\frac{\sin 2\alpha}{f(\infty)\ y/b} \qquad (\alpha < \alpha_1)\ . \tag{42}$$

APPENDIX III

Determination of the Stiffness of the Strip

The flexibility of a strip subjected to a particular type of loading is equal to the strain energy produced by that loading divided by half the square of the magnitude of the load. The flexibility of a long strip built in at an angle α_o under a moment M_B is therefore equal to

$$\frac{U}{\frac{1}{2}M_B^2} ,$$

$$= \frac{1}{M_B^2}\int_{\alpha=\alpha_o}^{x=\ell}\frac{M_\alpha^2}{D_o f}\,d\alpha \qquad (43)$$

substituting equations (16) and (24) for U . The flexibility of the equivalent strip of length $(\ell + x^*)$, shown in Fig.3, is

$$\frac{\ell + x^*}{\int_o^\omega D\,dy} . \qquad (44)$$

This integral can be rewritten by replacing

$$\left(\frac{f}{\varepsilon}\right)_{\varepsilon=0} = \frac{1}{\varepsilon}\int_{\eta_1}^{\eta_2}\frac{D_\eta}{D_o}\frac{d\eta}{\eta}$$

o

$$= \frac{1}{b}\frac{dx}{d\alpha}\int_o^\omega \frac{D}{D_o}\frac{dy}{\eta} .$$

As $\eta/\eta_1 \to 1$ as $\varepsilon \to 0$, equation (18) can be substituted, giving

$$\left(\frac{f}{\varepsilon}\right)_o = \frac{1}{bD_o}\int_o^\omega D\,dy \qquad (45)$$

so that the flexibility of the equivalent strip becomes

$$\frac{\ell + x^*}{bD_o\,(f/\varepsilon)_o} . \qquad (46)$$

Equating expressions (43) and (46), the equivalent length is determined by the equation

$$\int_{\alpha_o}^{\alpha_1} \frac{\sin^2\alpha}{D_o f(\infty)}\, d\alpha + \int_o^{\ell} \frac{\sin^2\alpha}{D_o f} \frac{d\alpha}{dx}\, dx = \frac{x^*}{bD_o (f/\varepsilon)_o} + \int_o^{\ell} \frac{1}{bD_o (f/\varepsilon)_o}\, dx \qquad \ldots\ldots \quad (47)$$

so that

$$\frac{x^*}{b} = \left(\frac{f}{\varepsilon}\right)_o \frac{1}{f(\infty)} \int_{\alpha_o}^{\alpha_1} \sin^2\alpha\, d\alpha + \int_o^{\ell} \left[\left(\frac{f}{\varepsilon}\right)_o \frac{\sin^2\alpha}{f} \frac{d\alpha}{dx} - \frac{1}{b}\right] dx \; . \qquad \ldots\ldots \quad (48)$$

When ℓ is large the integral from 0 to ℓ can be considered as an integral from $\alpha = \alpha_1$ to $\alpha = \pi/2$. When, in addition, equation (21) is used to substitute for $dx/d\alpha$, and the first integral is evaluated, equation (48) becomes

$$\frac{x^*}{b} = \left(\frac{f}{\varepsilon}\right)_o \frac{1}{f(\infty)} \left[\frac{\alpha_1}{2} - \frac{\sin 2\alpha_1}{4} - \frac{\alpha_o}{2} + \frac{\sin 2\alpha_o}{4}\right] + \int_{\alpha_1}^{\pi/2} \left[\left(\frac{f}{\varepsilon}\right)_o \frac{\sin^2\alpha}{f} - \frac{1}{\varepsilon \sin^2\alpha}\right] d\alpha \, . \qquad \ldots\ldots \quad (49)$$

A similar treatment of the torsional case gives the equation

$$\frac{x^*}{b} = \left(\frac{f}{\varepsilon}\right)_o \frac{1}{f(\infty)} \left[2\alpha_1 + \sin 2\alpha_1 - 2\alpha_o - \sin 2\alpha_o\right] + \int_{\alpha_1}^{\pi/4 \text{ or } 3\pi/4} \left[\left(\frac{f}{\varepsilon}\right)_o \frac{4\cos^2\alpha}{f} - \frac{1}{\varepsilon \sin^2\alpha}\right] d\alpha \qquad \ldots\ldots \quad (50)$$

where the upper limit of the integral is $\pi/4$ for the situation shown in Fig.3(b), and $3\pi/4$ for the situation shown in Fig.3(c).

APPENDIX IV

Evaluation of $f(\varepsilon)$ and $g(\varepsilon)$ for particular cross-sections

It is desired to evaluate f and g for each cross-section, where f is given by equation (25), that is

$$f = \int_1^{1+\varepsilon} \frac{D}{D_o} \frac{dT}{T} ,$$

while, from equation (32),

$$g = \frac{f}{\phi}\left(\frac{\phi}{f}\right)_{\varepsilon=0} ,$$

where

$$\phi = \varepsilon\sqrt{\frac{df}{d\varepsilon}} .$$

1 Rectangular Strip

As $t = t_o$, f is determined by the equation

$$f = \int_1^{1+\varepsilon} \frac{dT}{T}$$

$$= \ell n(1+\varepsilon) . \qquad (51)$$

When $\varepsilon = 0$

$$\left.\begin{aligned} \frac{f}{\varepsilon} &= \frac{df}{d\varepsilon} \\ &= 1 \end{aligned}\right\} \qquad (52)$$

and therefore

$$g = \sqrt{1+\varepsilon}\ \ell n(1+\varepsilon)/\varepsilon . \qquad (53)$$

2 Diamond shaped strip

The shape is defined by the equations

$$\left.\begin{aligned} \frac{t}{t_o} &= 2\frac{y}{b} \quad \text{for } \frac{y}{b} < \tfrac{1}{2} \\ \text{and} \quad \frac{t}{t_o} &= 2\left(1-\frac{y}{b}\right) \quad \text{for } \frac{y}{b} > \tfrac{1}{2} \end{aligned}\right\} \qquad (54)$$

Remembering that (D/D_o) is the cube of (t/t_o), and substituting equation (26) for y/b we have

$$\left.\begin{aligned} \frac{D}{D_o} &= 8\left(\frac{T-1}{\varepsilon}\right)^3 \quad \text{for} \quad 1 < T < 1 + {}^{\varepsilon}/2 \\ &\text{and} \\ \frac{D}{D_o} &= 8\left(1 - \frac{T-1}{\varepsilon}\right)^3 \quad \text{for} \quad 1 + {}^{\varepsilon}/2 < T < 1 + \varepsilon \,. \end{aligned}\right\} \qquad (55)$$

Therefore

$$f = 8\int_1^{1+\varepsilon/2} \left(\frac{T-1}{\varepsilon}\right)^3 \frac{dT}{T} + 8\int_{1+\varepsilon/2}^{1+\varepsilon} \left(\frac{\varepsilon+1-T}{\varepsilon}\right)^3 \frac{dT}{T} \,. \qquad (56)$$

On integration this expression becomes

$$f = 8\left(\frac{1}{\varepsilon}+1\right)^3 \ell n \left(\frac{1+\varepsilon}{1+\varepsilon/2}\right) - \frac{8}{\varepsilon^3}\,\ell n\left(1 + {}^{\varepsilon}/2\right) - 5\left(1 + {}^{2}/\varepsilon\right) \qquad (57)$$

which tends to ${}^{\varepsilon}/4$ as $\varepsilon \to 0$, so that

$$\left(\frac{f}{\phi}\right)_o = \tfrac{1}{2} \,. \qquad (58)$$

Differentiation of equation (57) gives

$$\frac{df}{d\varepsilon} = -\frac{24}{\varepsilon^2}\left(1+\frac{1}{\varepsilon}\right) \ell n\left(\frac{1+\varepsilon}{1+\varepsilon/2}\right) + \frac{24}{\varepsilon^4}\,\ell n\left(1+\frac{\varepsilon}{2}\right) + \frac{18}{\varepsilon^2} \qquad (59)$$

from which it follows that

$$g = \frac{8\left(\frac{1}{\varepsilon}+1\right)^3 \ell n\left(\frac{1+\varepsilon}{1+\varepsilon/2}\right) - \frac{8}{\varepsilon^3}\,\ell n\left(1+\frac{\varepsilon}{2}\right) - 5\left(1+\frac{2}{\varepsilon}\right)}{\frac{1}{2}\sqrt{\left\{18 + \frac{24}{\varepsilon^2}\,\ell n\left(1+\frac{\varepsilon}{2}\right) - 24\left(1+\frac{1}{\varepsilon}\right)^2 \ell n\left(\frac{1+\varepsilon}{1+\varepsilon/2}\right)\right\}}} \,. \qquad (60)$$

3 Parabolic Strip

For a parabolic strip

$$\frac{t}{t_o} = 4\,\frac{y}{b}\left(1-\frac{y}{b}\right) ; \qquad (61)$$

so that, on substituting equation (26), we have

$$\frac{D}{D_o} = 64\left(\frac{T-1}{\varepsilon}\right)^3 \left(1 - \frac{T-1}{\varepsilon}\right)^3 , \qquad (62)$$

and f is given by the equation

$$f = -\frac{64}{\varepsilon^6} \int_1^{1+\varepsilon} (T-1)^3 (T-\varepsilon-1)^3 \frac{dT}{T} . \tag{63}$$

On integration this becomes

$$f = 64 \left\{ \frac{1}{60} - \frac{1}{20\varepsilon} + \frac{1}{4\varepsilon^2} + \frac{11}{6\varepsilon^3} + \frac{5}{2\varepsilon^4} + \frac{1}{\varepsilon^5} - \frac{(1+\varepsilon)^3}{\varepsilon^6} \ln (1+\varepsilon) \right\} , \qquad \ldots\ldots \tag{64}$$

and its differential coefficient with respect to ε is given by

$$\frac{df}{d\varepsilon} = 64 \left\{ \frac{1}{20\varepsilon^2} - \frac{1}{2\varepsilon^3} - \frac{13}{2\varepsilon^4} - \frac{12}{\varepsilon^5} - \frac{6}{\varepsilon^6} + 3\,(2+\varepsilon) \frac{(1+\varepsilon)^3}{\varepsilon^6} \ln (1+\varepsilon) \right\} . \qquad \ldots\ldots \tag{65}$$

When $\varepsilon = 0$,

$$\frac{f}{\varepsilon} = \frac{64}{140} , \tag{66}$$

so that

$$g = \frac{\sqrt{140} \left\{ \frac{1}{60\varepsilon} - \frac{1}{20\varepsilon^2} + \frac{1}{4\varepsilon^3} + \frac{11}{6\varepsilon^4} + \frac{5}{2\varepsilon^5} + \frac{1}{\varepsilon^6} - \frac{(1+\varepsilon)^3}{\varepsilon^7} \ln (1+\varepsilon) \right\}}{\sqrt{\left\{ \frac{1}{20\varepsilon^2} - \frac{1}{2\varepsilon^3} - \frac{13}{2\varepsilon^4} - \frac{12}{\varepsilon^5} - \frac{6}{\varepsilon^6} + 3\,(2+\varepsilon) \frac{(1+\varepsilon)^3}{6} \ln (1+\varepsilon) \right\}}} . \qquad \ldots\ldots \tag{67}$$

4 Double wedge shaped strip

The shape is defined by the equations

$$\left.\begin{aligned} \frac{t}{t_o} &= 3\,\frac{y}{b} && \text{for } 0 < \frac{y}{b} < \frac{1}{3} , \\ \frac{t}{t_o} &= 1 && \text{for } \frac{1}{3} < \frac{y}{b} < \frac{2}{3} , \\ \text{and} \quad \frac{t}{t_o} &= 3\left(1-\frac{y}{b}\right) && \text{for } \frac{2}{3} < \frac{y}{b} < 1 . \end{aligned}\right\} \tag{68}$$

Substituting for $\frac{y}{b}$ as before, the expression for f is

$$f = \int_1^{1+\varepsilon/3} 27 \left(\frac{T-1}{\varepsilon}\right)^3 \frac{dT}{T} + \int_{1+\varepsilon/3}^{1+2\varepsilon/3} \frac{dT}{T} + \int_{1+2\varepsilon/3}^{1+\varepsilon} 27 \left(1 - \frac{T-1}{\varepsilon}\right)^3 \frac{dT}{T} , \qquad \ldots\ldots \tag{69}$$

which becomes on integration

$$f = \ln\left(\frac{1+2\varepsilon/3}{1+\varepsilon/3}\right) - \frac{27}{\varepsilon^3}\ln(1+\varepsilon/3) + \frac{27}{\varepsilon^3}(1+\varepsilon)^3 \ln\left(\frac{1+\varepsilon}{1+2\varepsilon/3}\right) - 21\left(\frac{1}{\varepsilon}+\frac{1}{2}\right). \qquad \ldots\ldots \quad (70)$$

On differentiating and reorganising the terms we have

$$\frac{df}{d\varepsilon} = \frac{81}{\varepsilon^4}\ln(1+\varepsilon/3) - \frac{81}{\varepsilon^2}\left(1+\frac{1}{\varepsilon}\right)^2 \ln\left(\frac{1+\varepsilon}{1+2\varepsilon/3}\right) + \frac{36}{\varepsilon^2}. \qquad (71)$$

When $\varepsilon = 0$

$$\frac{f}{\varepsilon} = \frac{1}{2}, \qquad (72)$$

and substitution of (70), (71) and (72) into the equation for g gives

$$g = \frac{\sqrt{2}\left\{\ln\left(\frac{1+2\varepsilon/3}{1+\varepsilon/3}\right) - \frac{27}{\varepsilon^3}\ln(1+\varepsilon/3) + \frac{27}{\varepsilon^3}(1+\varepsilon)^3\ln\left(\frac{1+\varepsilon}{1+2\varepsilon/3}\right) - 21\left(\frac{1}{\varepsilon}+\frac{1}{2}\right)\right\}}{\sqrt{\left\{\frac{81}{\varepsilon^2}\ln(1+\varepsilon/3) - 81\left(1+\frac{1}{\varepsilon}\right)^2\ln\left(\frac{1+\varepsilon}{1+2\varepsilon/3}\right) + 36\right\}}}.$$

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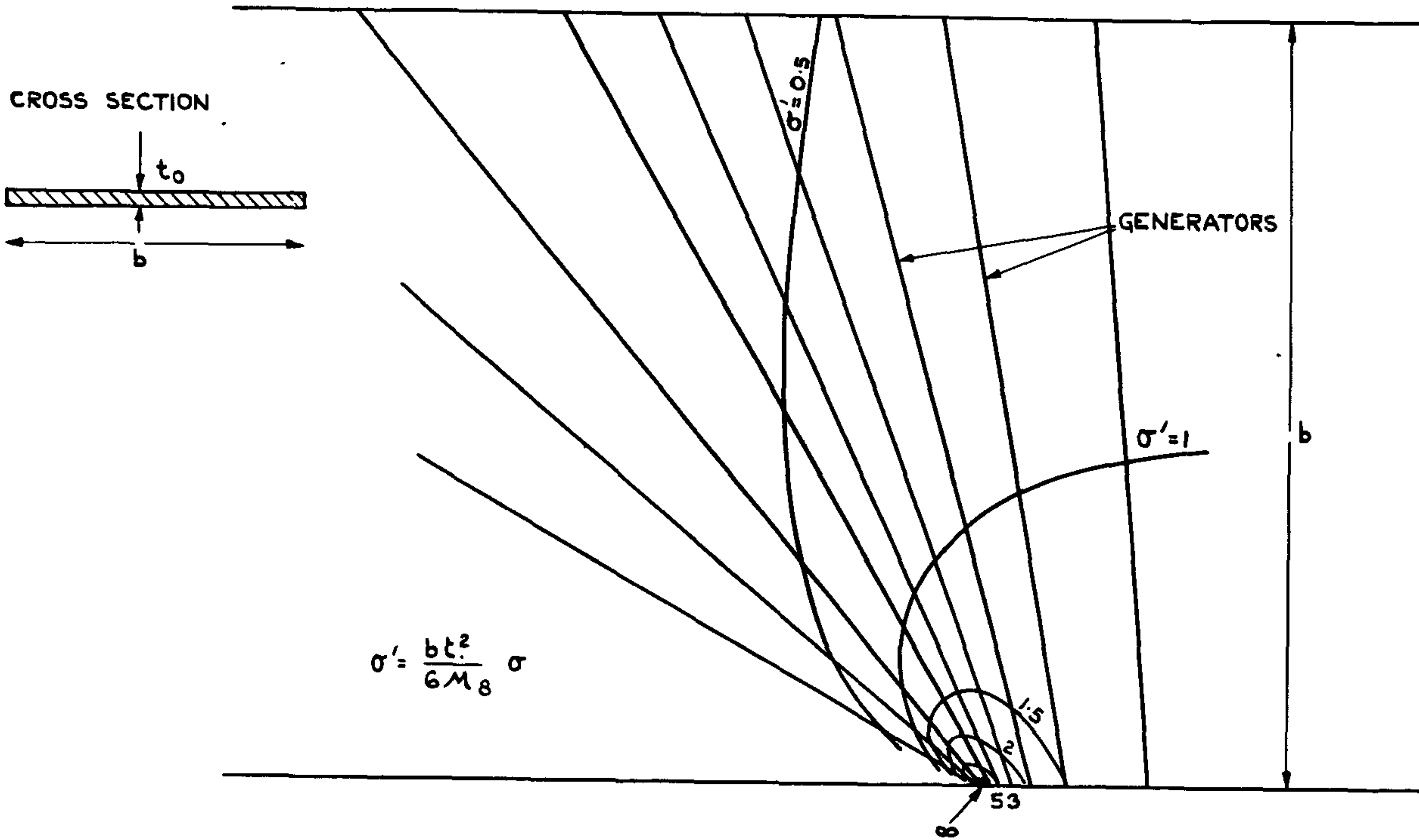

FIG. 4. STRESSES AND GENERATORS IN A RECTANGULAR STRIP SUBJECTED TO A BENDING MOMENT WITH CLAMPING ANGLE $\alpha_0 = 0$.

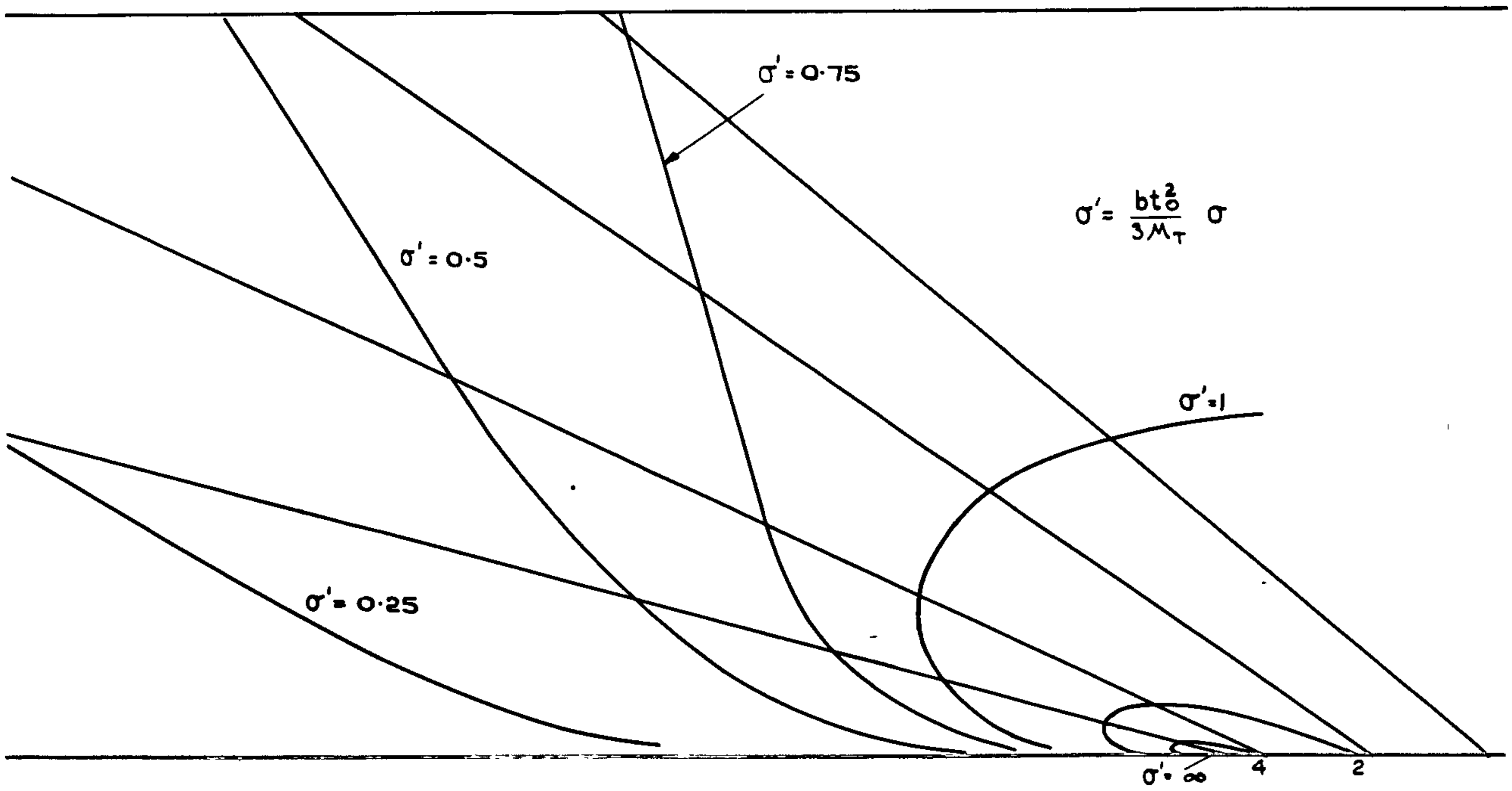

FIG. 5. RECTANGULAR STRIP SUBJECTED TO A TORSIONAL MOMENT WITH CLAMPING ANGLE $\alpha_0 = 0$.

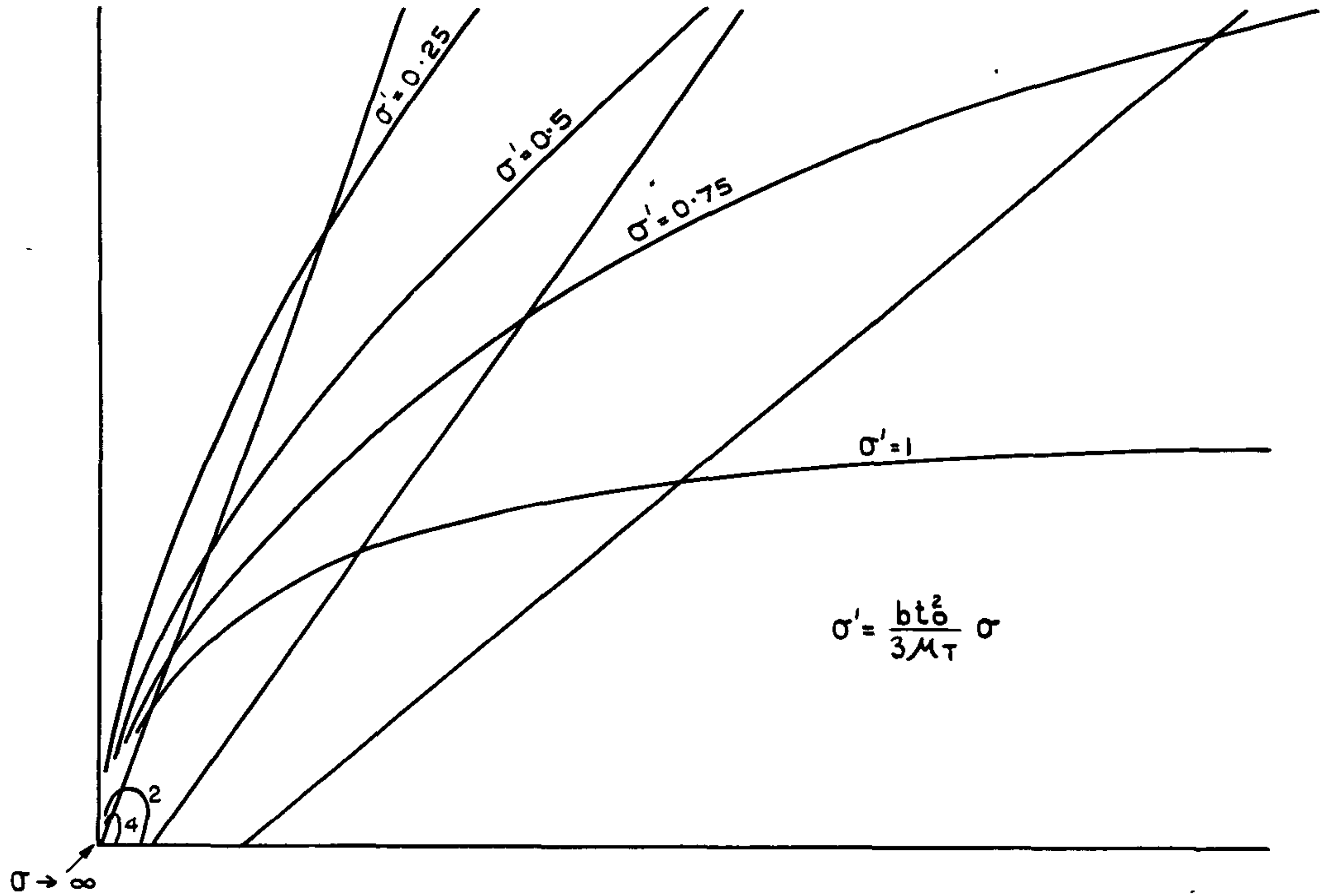

FIG. 6. RECTANGULAR STRIP SUBJECTED TO A TORSIONAL MOMENT WITH CLAMPING ANGLE $\alpha_0 = 90$.

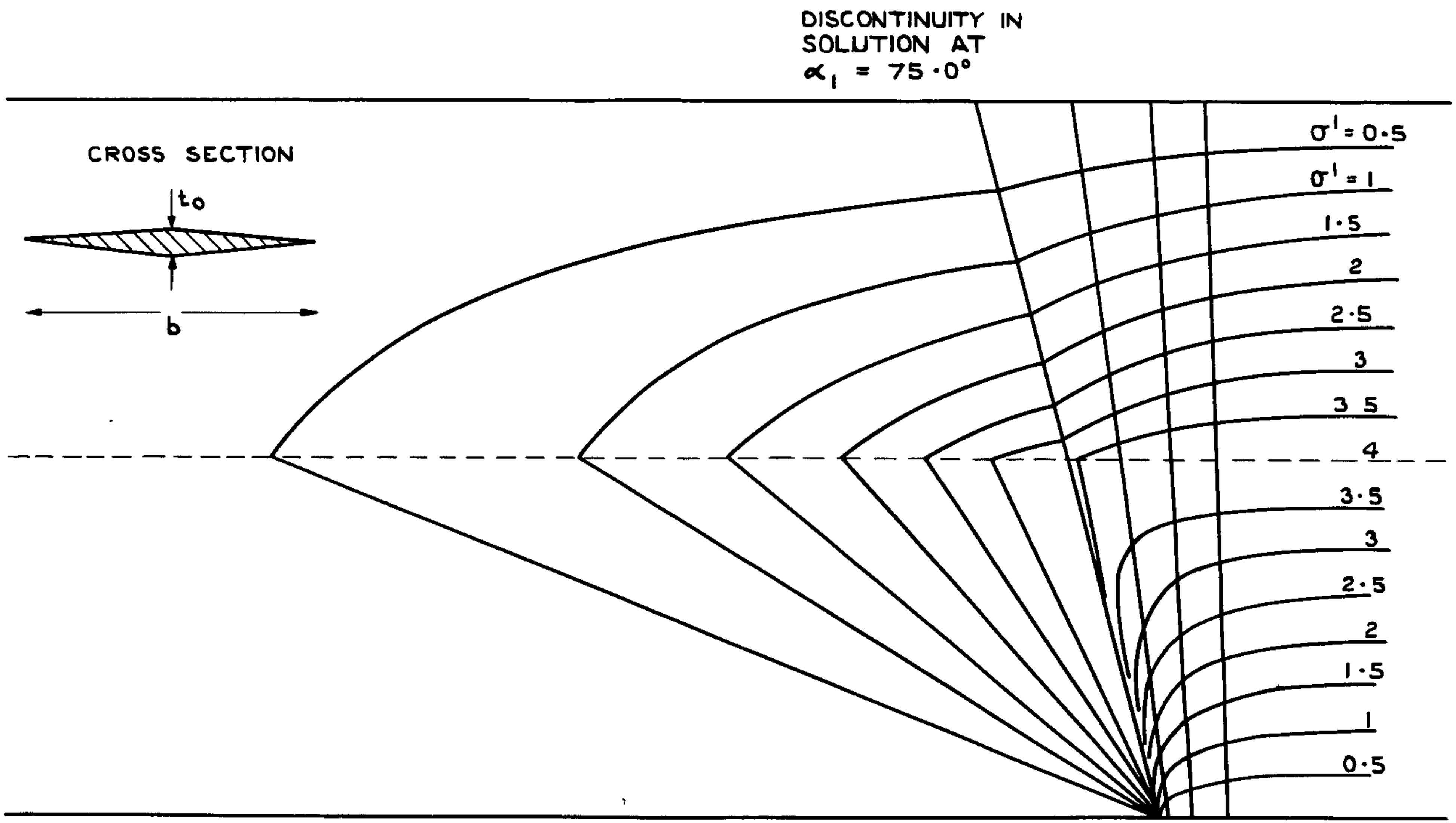

FIG. 7. DIAMOND SHAPED STRIP SUBJECTED TO A BENDING MOMENT WITH CLAMPING ANGLE $\alpha_0 = 0$.

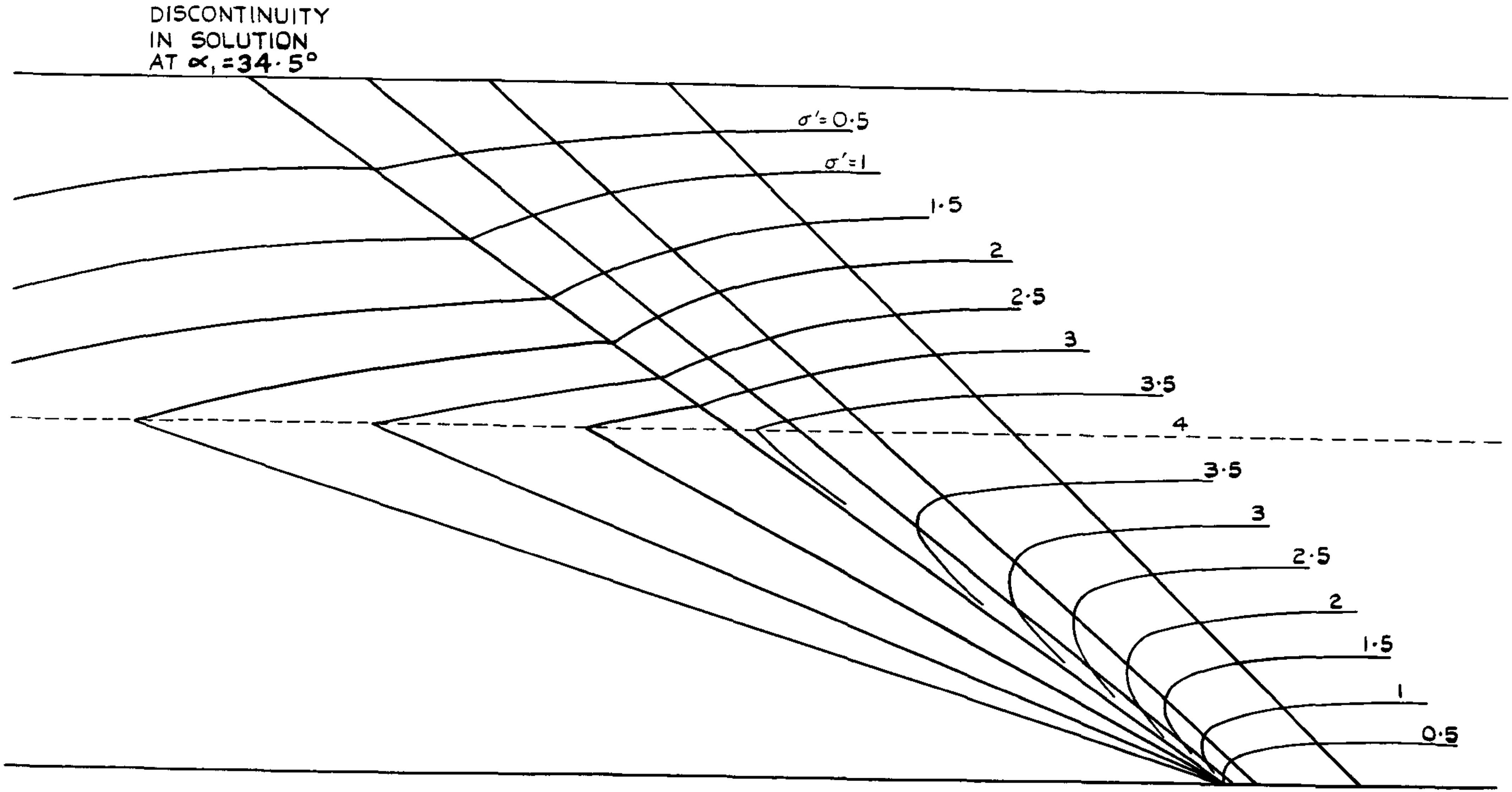

FIG. 8. DIAMOND SHAPED STRIP SUBJECTED TO A TORSIONAL MOMENT, WITH CLAMPING ANGLE $\alpha_0 = 0$

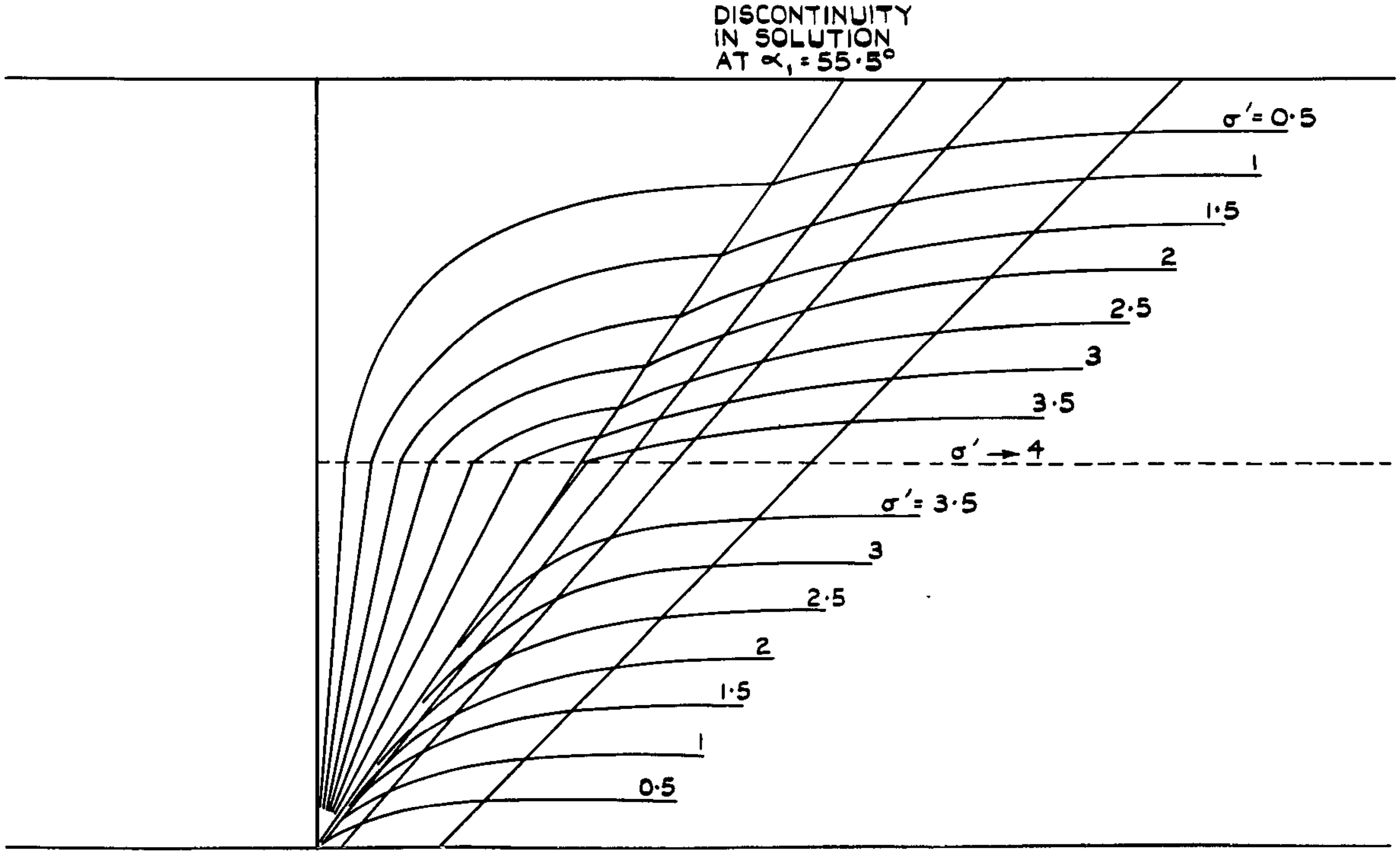

FIG. 9. DIAMOND SHAPED STRIP SUBJECTED TO A TORSIONAL MOMENT, WITH CLAMPING ANGLE $\alpha_0 = 90^\circ$

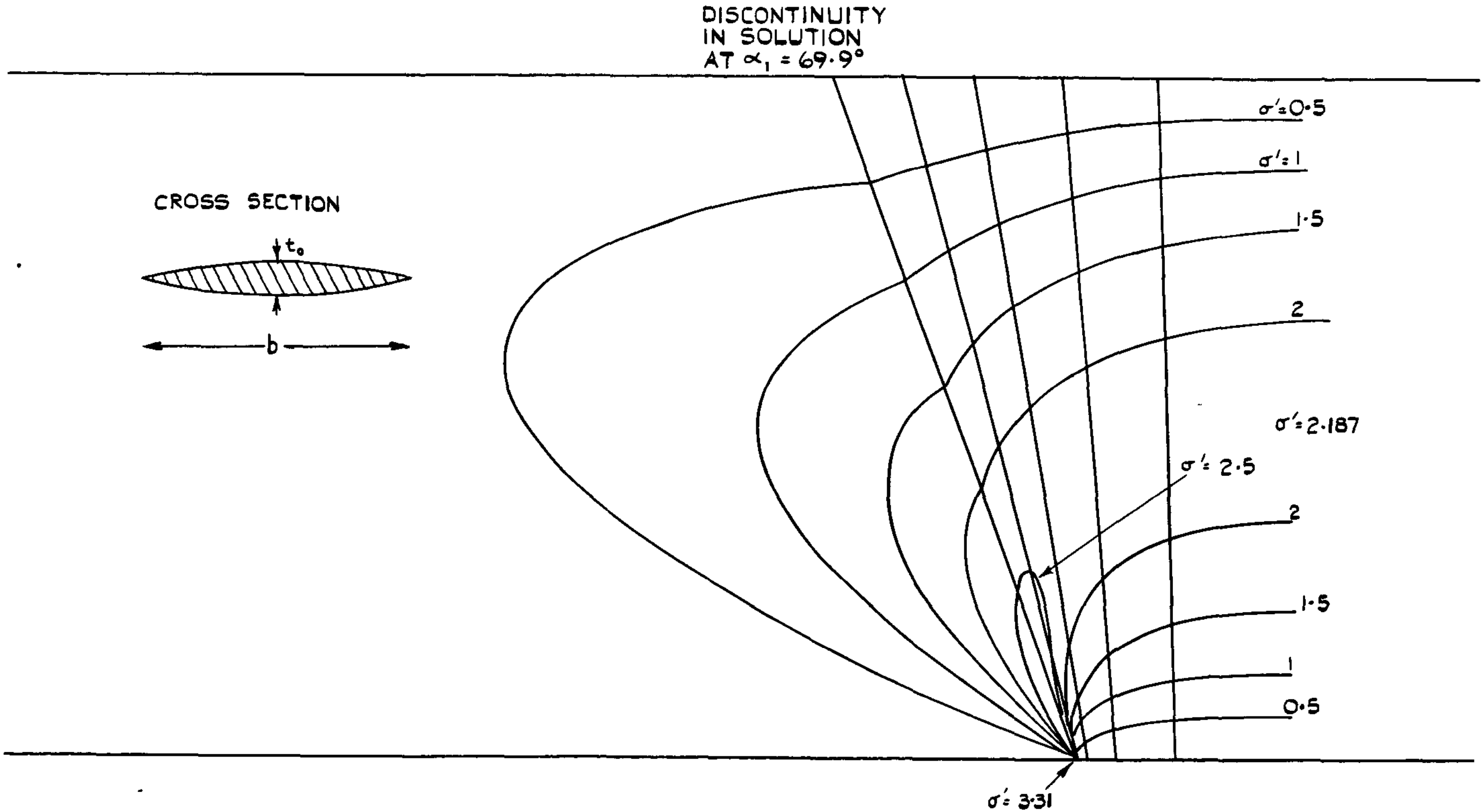

FIG. 10. PARABOLIC STRIP SUBJECTED TO A BENDING MOMENT, WITH CLAMPING ANGLE $\alpha_0 = 0$

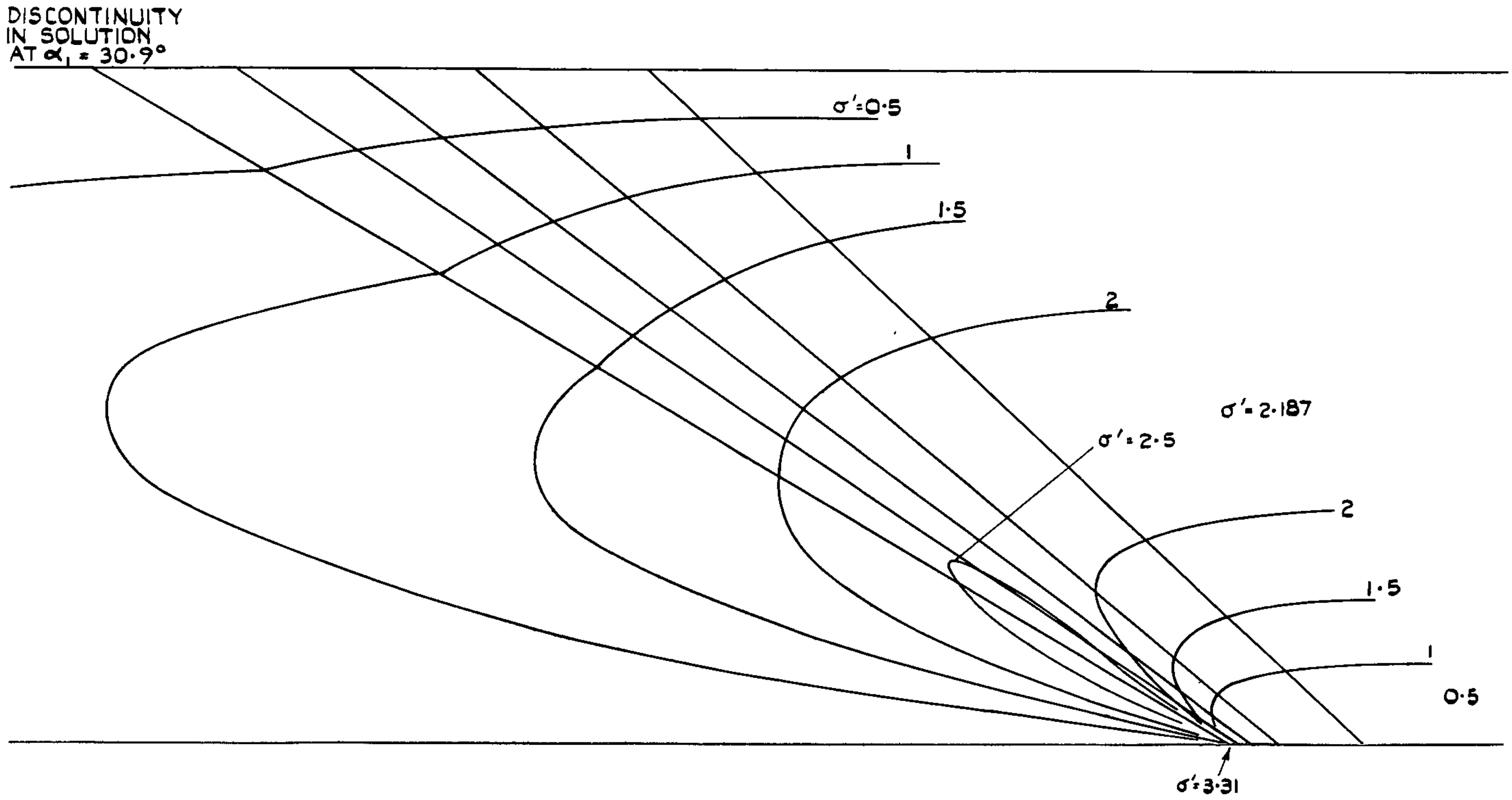

FIG. 11. PARABOLIC STRIP SUBJECTED TO A TORSIONAL MOMENT, WITH CLAMPING ANGLE $\alpha_0 = 0$

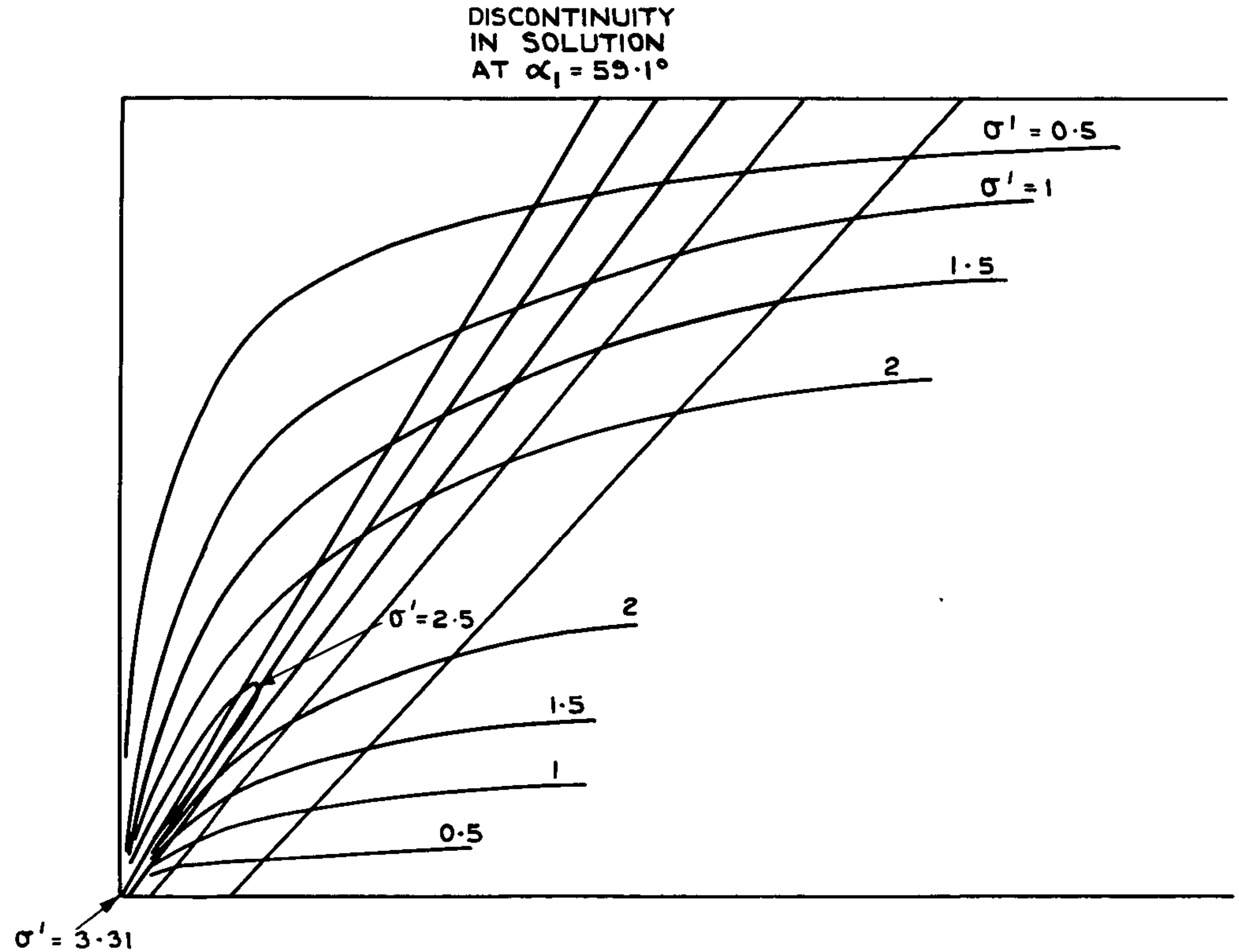

FIG. 12. PARABOLIC STRIP SUBJECTED TO A TORSIONAL MOMENT WITH CLAMPING ANGLE $\alpha_0 = 90°$

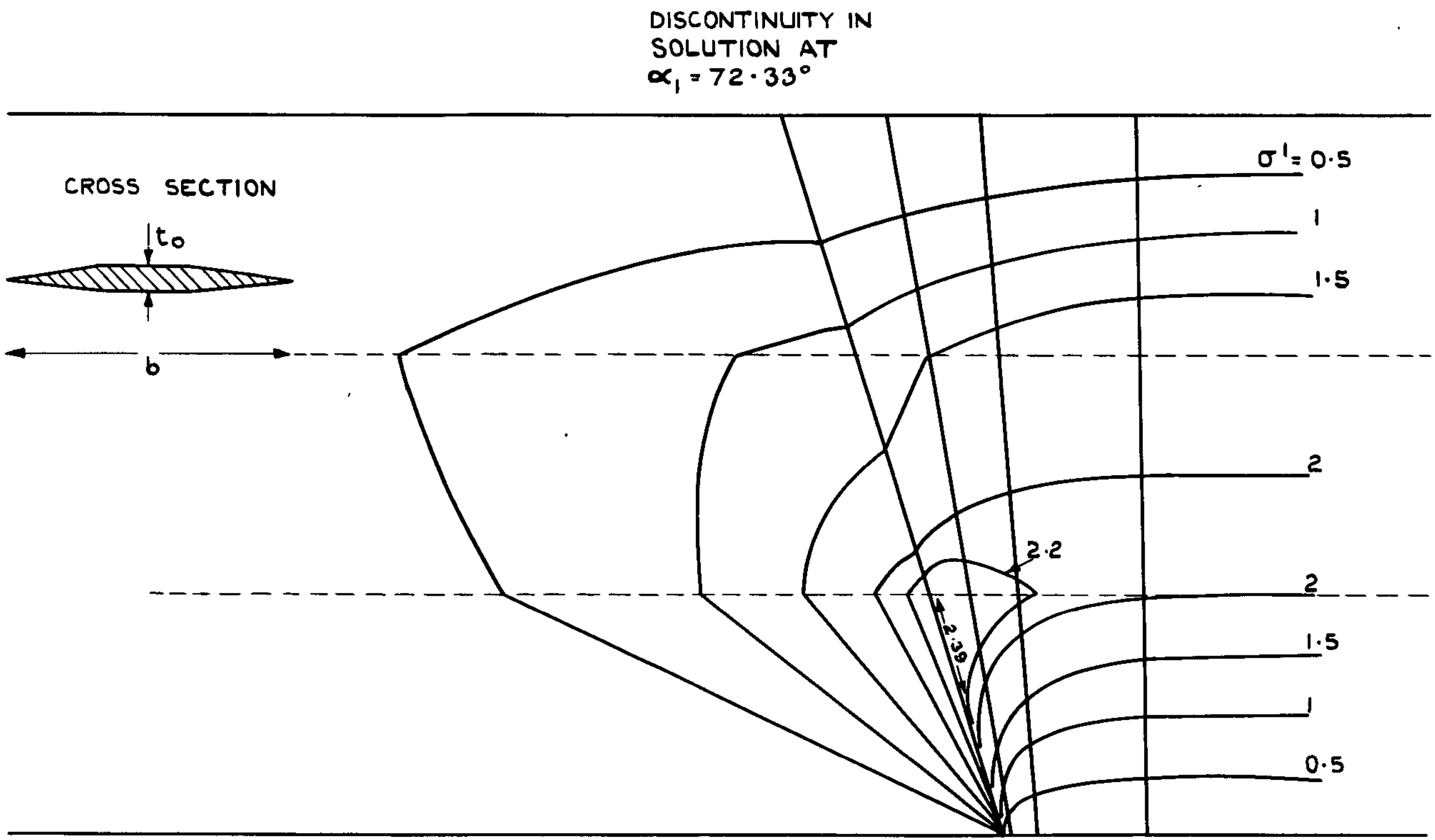

FIG. 13. DOUBLE WEDGE SHAPED STRIP SUBJECTED TO A BENDING MOMENT WITH CLAMPING ANGLE $\alpha_0 = 0$

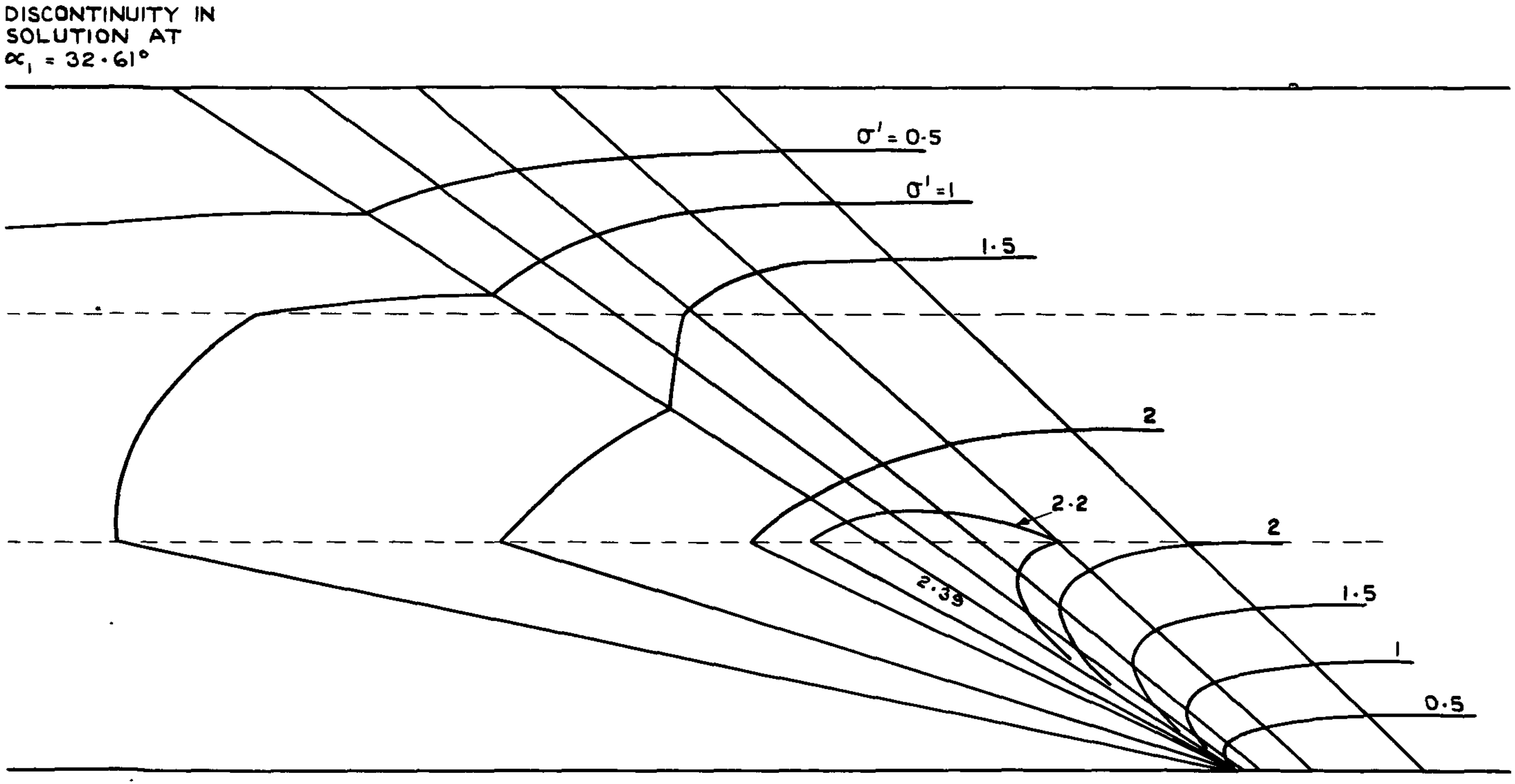

FIG. 14. DOUBLE WEDGE SHAPED STRIP SUBJECTED TO A TORSIONAL MOMENT WITH CLAMPING ANGLE $\alpha_0 = 0$.

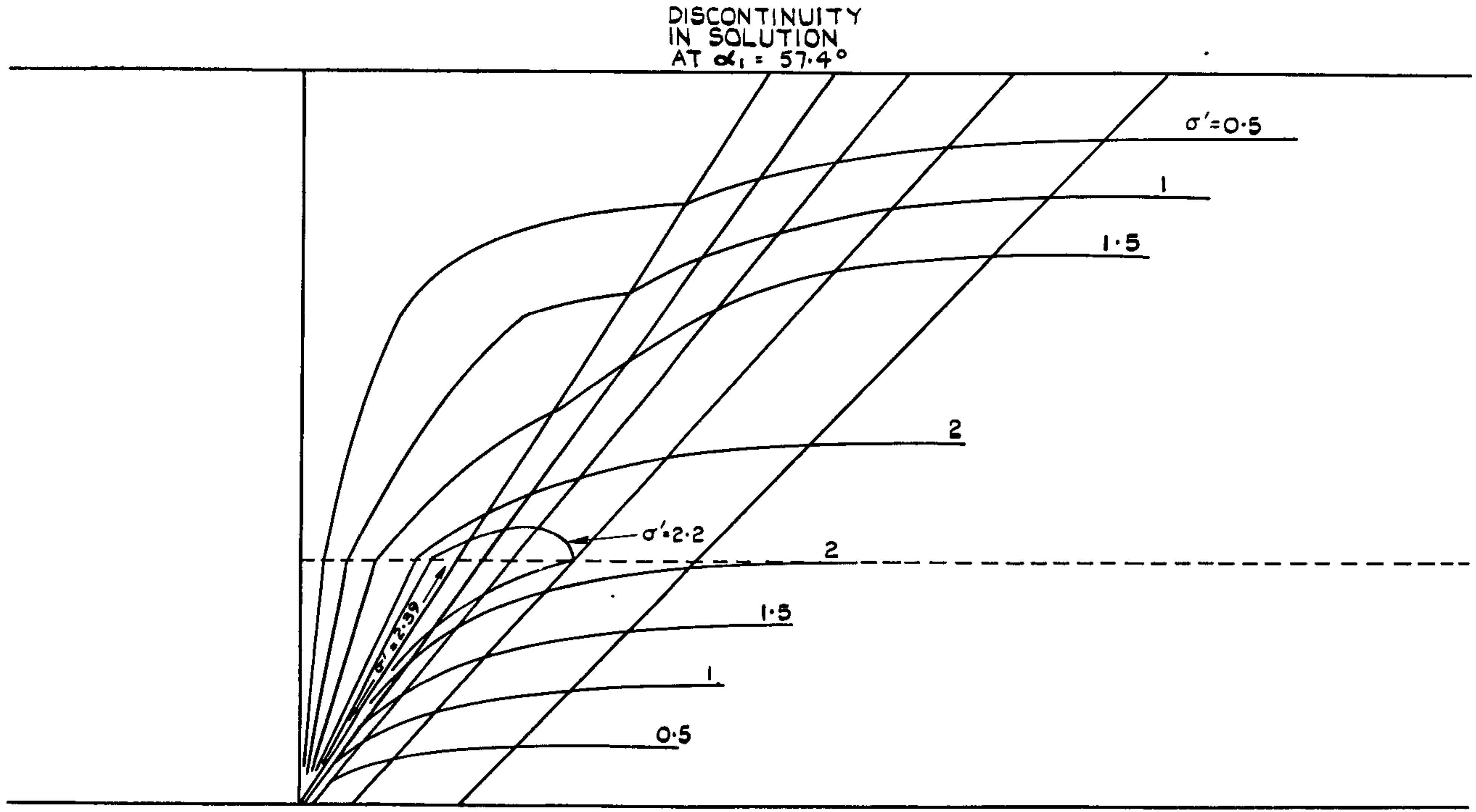

FIG. 15. DOUBLE WEDGE SHAPED STRIP SUBJECTED TO A TORSIONAL MOMENT, WITH CLAMPING ANGLE $\alpha_0 = 90°$

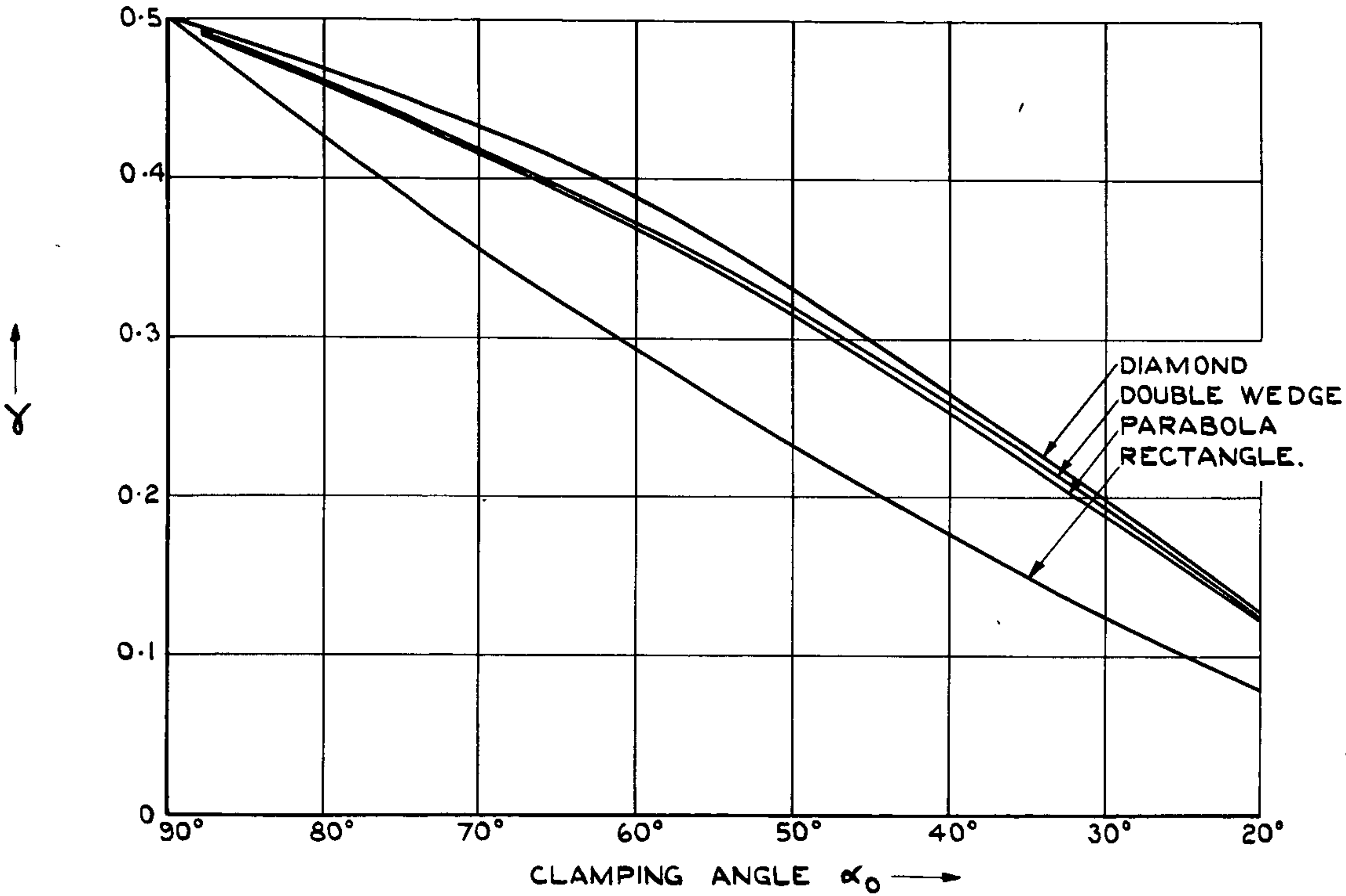

FIG. 16. γ FOR A STRIP SUBJECTED TO A BENDING MOMENT.

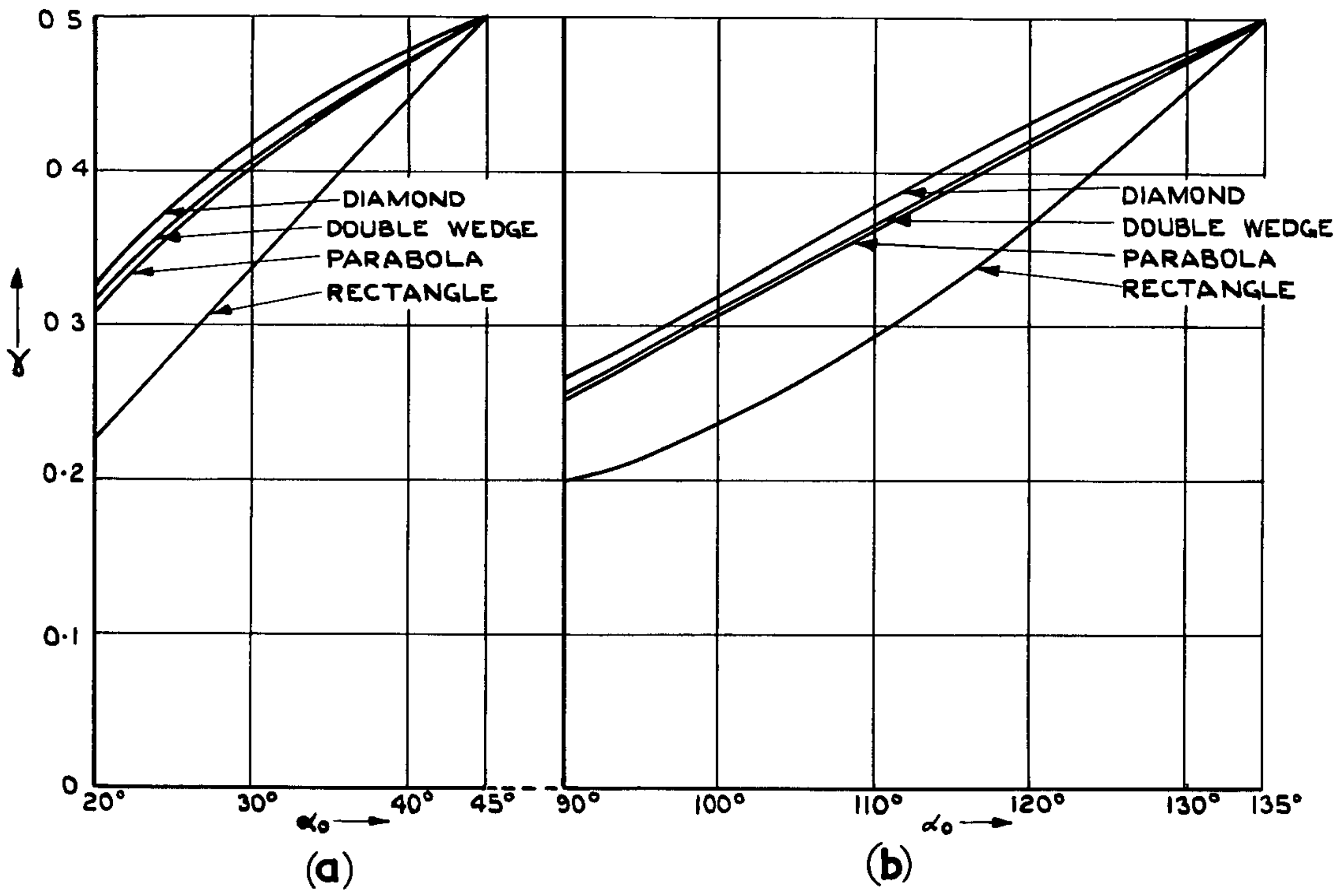

FIG. 17. (a & b) γ FOR A STRIP SUBJECTED TO A TORSIONAL MOMENT.

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