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Summary.—The theories of supersonic flow past slender, smooth, pointed bodies of arbitrary cross-sectional shape, due to Ward[1], and of the flow past slender bodies of revolution with discontinuities in profile slope, due to Lighthill[2], are applied and extended to calculate first approximations for the aerodynamic forces on bodies of elliptic cross-section with discontinuities in profile slope. Open-nose bodies are included in this class, but only the external forces are considered. The investigation is restricted to bodies the major axes of whose cross-sections are co-planar and whose cross-sections have constant eccentricity.

General expressions are deduced for the wave drag, lift, induced drag, and pitching moments of such bodies. The drag formula bears a marked resemblance to that for the equivalent body of revolution (*i.e.*, the body of revolution with the same axial distribution of cross-sectional area), but the discontinuities introduce a slight difference in one term. The lift formula is identical with that already deduced by Ward for a particular case of the present problem.

The general theory is applied to elliptic cones, and a comparison is made with Squire's solution of this problem[5]. Numerical results[6] for the wave drag of bodies of revolution having straight and parabolic profiles are also extended to bodies of elliptic cross-section.

1. *Introduction*.—In Ref. 1 Ward has presented a method of obtaining approximate solutions of the linearized equation of supersonic flow for slender, pointed bodies of arbitrary cross-sectional shape. For Ward's method to be applicable, the bodies must be free from discontinuities in surface slope.

For the special case of bodies of revolution, Lighthill[2] has extended the slender-body theory to include bodies with discontinuities in surface slope. Open-nose bodies are included in this class†. It is apparent from Ref. 2 that discontinuities in slope introduce jumps in pressure which are wholly different in magnitude and form from the pressures encountered on smooth bodies (these pressure jumps are, in fact, two-dimensional expansions or compressions). However, the effects of the discontinuities decay very rapidly along the body, so that when the pressures are integrated over the body, the resulting expressions for drag and lift bear a marked resemblance to the results of the simpler theory for smooth bodies.

In the present paper we consider the supersonic flow past slender bodies of elliptic cross-section for which the major axes of the cross-sections are co-planar and whose cross-sections have constant eccentricity. A velocity potential is found of the general form derived by Ward for smooth bodies. This potential enables us to calculate not only the complete flow about smooth bodies of elliptic cross-section, but, with certain modifications, also the external aerodynamic forces on bodies having discontinuities in slope, provided that these discontinuities are spaced well apart.

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† It is assumed both in Lighthill's and in the present paper that the flow is undisturbed ahead of the nose, *i.e.*, that there is no 'spillage'.

The basis of the method for dealing with the discontinuities is that the general form of Ward's solution may be used to deduce asymptotic expressions for the disturbances from the discontinuities, and if these are handled with sufficient care they yield a first approximation to the aerodynamic forces. The method should be applicable to bodies other than elliptic ones, if the solution is known for smooth bodies of the same cross-sectional shape.

2. *An Outline of Ward's Theory.*—Let (r_1, θ_1, s_1) be a system of cylindrical polar co-ordinates such that an undisturbed stream at infinity, of velocity U, is flowing parallel to the s_1-axis in the direction of increasing s_1. Let ϕ be a pertubation potential such that the velocity components in the directions of increasing r_1, θ_1, s_1 are

$$U\frac{\partial \phi}{\partial r_1}, \quad \frac{U}{r_1}\frac{\partial \phi}{\partial \theta_1}, \quad U\left(1 + \frac{\partial \phi}{\partial s_1}\right).$$

Let p be the Heaviside operator with respect to s_1; we write $\underline{f}(p)$ for the operational form of a function $f(s_1)$. The linearized equation of supersonic flow is then

$$\frac{\partial^2 \underline{\phi}}{\partial r_1^2} + \frac{1}{r_1}\frac{\partial \underline{\phi}}{\partial r_1} + \frac{1}{r_1^2}\frac{\partial^2 \underline{\phi}}{\partial \theta_1^2} = \beta^2 p^2 \underline{\phi} \qquad \qquad (2.1)$$

where $\beta = \sqrt{(M^2 - 1)}$. The use of this equation as an approximate form of the exact equation of motion for the problem of flow past slender bodies has been fully investigated and justified by Lighthill[2,3] and Ward[1].

Consider a pointed body of length $O(1)$ and of maximum thickness t, where t is small compared with unity. The restrictions on the shape of the body are stated by Ward as follows. 'The angle which any tangent plane to the body boundary makes with the undisturbed stream direction must be small and $O(t)$, and the rate of change of this angle along the direction of the body must also be small and $O(t)$. One further restriction on the shape of the body is required in general: this concerns the radius of curvature of any section of the boundary of the body by a plane perpendicular to the stream direction. If d is the diameter of the section then the curvature must be (at most) $O(1/d)$ for all points where the section is convex outwards; there is no restriction at points where the section is concave outwards. This last condition is not always necessary for bodies at zero incidence, but is always required if we wish to calculate the flow at incidence within a known approximation.' The reasons for these restrictions will become apparent from what follows.

If ν_1 denotes the normal to a section of the body in a plane s_1 = constant (Fig. 1), the boundary condition may be written

$$\frac{\partial \phi}{\partial \nu_1} = \frac{d\nu_1}{ds_1}\left(1 + \frac{\partial \phi}{\partial s_1}\right) \simeq \frac{d\nu_1}{ds_1}. \qquad \qquad (2.2)$$

A general solution of (2.1), giving waves travelling outwards and backwards from the axis, is, assuming convergence of the series,

$$\underline{\phi} = \underline{A_0}K_0(\beta p r_1) + \sum_{n=1}^{\infty} [\underline{A_n}(p) \cos n\theta_1 + \underline{B_n}(p) \sin n\theta_1] \,.\, K_n(\beta p r_1), \qquad (2.3)$$

where the K_n are modified Bessel functions of the second kind, and the $\underline{A_n}$ and $\underline{B_n}$ are arbitrary functions of p which must be chosen to satisfy the boundary condition.

Near the body r_1 is small and $O(t)$; and by taking the largest two terms of K_0, and the largest term of each of the other K_n, Ward obtained an approximate perturbation potential ϕ_a, given by

$$\underline{\phi_a} = \mathscr{R}\left[\underline{a_0} \log z_1 + \underline{b_0} + \sum_{n=1}^{\infty} \underline{a_n} \,.\, z_1^{-n}\right] \qquad \qquad (2.4)$$

where

$$z_1 = r_1\, e^{i\theta_1}, \qquad \underline{a_0} = -\underline{A_0}(p),$$

$$\underline{b_0} = (\log \tfrac{1}{2}\beta p + \gamma)\underline{a_0}(p), \qquad \underline{a_n} = \tfrac{1}{2}(n-1)!\left(\frac{2}{\beta p}\right)^n (\underline{A_n} + i\underline{B_n}), \qquad (2.5)$$

and γ is Euler's constant. The error of ϕ_a as a solution of the exact equation of motion is $O(t^4 \log^2 t)$.

The value of ϕ_a as a solution lies not so much in its series form—which may, indeed, converge only outside the body, and will then have to be continued analytically elsewhere—but in the fact that we need now only seek a function $(\phi_a - b_0)$ which is a solution of Laplace's equation in (r_1, θ_1), and so all the methods of classical hydrodynamics are at our disposal.

The reason for Ward's restriction on the curvature of any section now follows from a knowledge of incompressible 'corner flows'. At a convex corner an incompressible stream attains an infinite velocity; at a concave corner it comes to rest. Thus his restriction ensures that the circumferential component of velocity $\partial\phi_a/\partial\tau_1$ is always $O(t)$.

By examining the boundary condition, Ward found that the coefficients a_n are $O(t^{n+2})$; and by integrating $\partial\phi_a/\partial\nu_1$ around the body contour he showed that

$$a_0(s_1) = \frac{S'(s_1)}{2\pi}, \qquad (2.6)$$

where $S(s_1)$ is the cross-sectional area of the body. Interpreting (2.5) by the product theorem, and using $a_0(0) = 0$ for a pointed body, Ward obtained

$$b_0(s_1) = \frac{1}{2\pi}\left[S'(s_1)\log \tfrac{1}{2}\beta - \int_0^{s_1} \log(s_1 - u)\, S''(u)\, du\right]. \qquad (2.7)$$

Now if the perturbation velocity $\partial\phi_a/\partial s_1$ is to be small everywhere clearly $S''(s_1)$ must be small everywhere, and this accounts for Ward's first two restrictions above on the shape of the body.

Since $\partial\phi/\partial s_1$ is $O(t^2 \log t)$ and $\partial\phi/\partial r_1$, $\partial\phi/r\partial\theta_1$ are $O(t)$, Ward took as his pressure coefficient

$$C_p = -2\frac{\partial\phi}{\partial s_1} - \left(\frac{\partial\phi}{\partial r_1}\right)^2 - \left(\frac{1}{r}\frac{\partial\phi}{\partial\theta_1}\right)^2 + O(t^4\log^2 t). \qquad (2.8)$$

He went on to develop general expressions for drag and lift: these forces are $O(t^4 \log t)$ and $O(t^3)$, respectively, and the errors due to the approximate form are $O(t^6 \log^2 t)$ and $O(t^5 \log^2 t)$.

3. *The Open-nose Body*.—A necessary condition for the use of the approximate form ϕ_a is that the body be pointed and that the streamwise surface slope dr_1/ds_1 be continuous along the body. This ensures that the source and multisource strengths $a_n(s_1)$ tend to zero as $s_1 \to 0$, and are continuous functions of s_1 (except at the base of the body). In this section we observe why this condition is necessary in the basic derivation of ϕ_a, and then examine the order of the potential and its derivatives when it is not satisfied. For convenience we place a discontinuity at the front of the body, writing $S(O) \neq 0$, $(dr_1/ds_1)_{s_1=0} \neq 0$, and assume that dr_1/ds_1 is continuous for $s_1 > 0$. This step involves no loss of generality, for if a discontinuity exists at some point further along the body we introduce a new disturbance potential at that point, and the observations below then apply to this potential.

For ϕ_a to be a valid approximation not only r_1 but also pr_1 must be small everywhere on the body. Now as $s_1 \to 0$ the operator p may be considered to tend to infinity, since

$$\lim_{s_1 \to 0+} f(s_1) = \lim_{p \to \infty} \underline{f}(p), \qquad (3.1)$$

provided that both limits exist. This is a standard rule in the operational calculus[4]. However, we also have the interpretation rule

$$f(pr_1) \doteqdot f\left(\frac{s_1}{r_1}\right), \quad \ldots \quad (3.2)$$

so that if s_1/r_1 is everywhere large and $O(t^{-1})$—and this is the case for a pointed body—we are justified in assuming pr_1 is everywhere small and $O(t)$. ϕ_a merely represents the first two terms of an asymptotic expansion of the true potential in the variable s_1/r_1, and is therefore a good approximation everywhere on a pointed body; for an open-nose body it is not valid over an initial region of the body of length $O(t)$, but thereafter it again becomes an increasingly good approximation.

We now establish two results, which we shall require later, regarding the magnitude of the true perturbation potential ϕ when s_1 is small. We assume that the velocity normal to the rim of the open-nose body is everywhere supersonic, and that the arbitrary constant in ϕ is chosen to make $\phi = 0$ upstream of the body.

(i) Immediately downstream of the rim we have $\phi = 0$ because the equations of continuity and momentum require that the potential be continuous.

Further, the perturbation velocities are those corresponding to two-dimensional flow. This may be shown by replacing the surface of the body by a distribution of 'supersonic sources'; the potential at any point on or outside the body is obtained by integrating the incremental source potential over the appropriate region (Fig. 2). As the point P approaches the disturbance from the rim of the body the area of integration becomes a small plane surface over which a constant normal velocity is required, and the flow is wholly analogous to that at the supersonic leading edge of a wing, all the perturbation velocities being $O(t)$. (In the limit the area of integration tends to zero, and since the incremental potential is integrable, the potential, unlike its derivatives, tends to zero at all points immediately behind the disturbance from the rim.)

(ii) When s_1 is $O(t)$, the potential is at most $O(t^2)$.

The potential may be written

$$\phi(\nu_1, \tau_1, s_1) = \int_{PQ} \frac{\partial \phi}{\partial \nu_1}\, d\nu_1 , \quad \ldots \quad (3.3)$$

where the path of integration PQ is a line $\tau_1 =$ constant, $s_1 =$ constant between the disturbance from the rim and the point in question. Now it follows from the form of the general solution (2.3), and also from physical considerations of the flow, that the normal velocity $|\partial\phi/\partial\nu_1|$ cannot be of a greater order outside the body than on it, and is therefore $O(t)$. Since the length of the path PQ is also $O(t)$, ϕ must be $O(t^2)$.

4. *The Approximate Potentials for Bodies of Elliptic Cross-section.*—4.1. *The Elliptic Body.*—Let (x, y, s) be a system of rectangular Cartesian co-ordinates such that the s-axis is the axis of the body. We consider a body of the form

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = f^2(s) , \quad \ldots \quad (4.1)$$

and extending from $s = c_0$ to $s = c_n$. We assume that $f(s)$, $f'(s)$, $f''(s)$ are all $O(1)$ except at a finite number of points denoted by $s = c_i$ $(i = 0, 1, \ldots n)$ where $f'(s)$ is discontinuous; and that A, B are $O(t)$. In certain cases (*see* section 8.1) the minor-axis parameter B may be $O(t^2)$. We also assume that all the intervals $(c_i - c_{i-1})$ are $O(1)$. The cross-sectional area is

$$S(s) = \pi AB\, f^2(s),$$

and we write

$$S'(c_i+) - S'(c_i-) = 2\pi ABf(c_i)[f'(c_i+) - f'(c_i-)] = \Delta S_i',$$

with the convention that $f'(c_0-) = f'(c_n+) = 0$.

4.2. *The Potential at Zero Incidence.*—The boundary condition (2.2) may now be written for zero incidence,

$$\frac{\partial\phi}{\partial\nu} = \pm\frac{\partial y}{\partial s}\frac{1}{\sqrt{\left\{1+\left(\frac{\partial y}{\partial x}\right)^2\right\}}}, \quad (\pm \text{ for } y \gtrless 0), \quad \ldots \quad (4.2)$$

$$= \frac{ABf(s)f'(s)}{\sqrt{\left\{A^2f^2(s) - \left(1-\frac{B^2}{A^2}\right)x^2\right\}}}. \quad \ldots \quad (4.3)$$

We now make the following transformation :

$$\left.\begin{aligned} x &= \left[\lambda + \frac{(A^2-B^2)f^2(\sigma)}{4\lambda}\right]\cos\mu \\ y &= \left[\lambda - \frac{(A^2-B^2)f^2(\sigma)}{4\lambda}\right]\sin\mu \\ s &= \sigma \end{aligned}\right\} \quad \ldots \quad (4.4)$$

(We introduce σ because the operations $\partial/\partial s$ and $\partial/\partial\sigma$ will not be the same : the first will imply keeping x, y constant and the second keeping λ, μ constant.) (4.4) is of course the Joukowski transformation

$$z = \zeta + \frac{(A^2-B^2)f^2(\sigma)}{4\zeta}, \quad \ldots \quad (4.5)$$

where $z = x + iy$, and $\zeta = \lambda\, e^{i\mu}$.

The ellipse (4.1) becomes in the ζ-plane the circle

$$\lambda = \frac{A+B}{2}f(\sigma). \quad \ldots \quad (4.6)$$

Since the transformation is conformal in planes s = constant, an element of the normal at the contour in the z-plane will become an element of normal in the ζ-plane and be magnified by the factor $|d\zeta/dz|$. Hence if ν_2 is the normal in the ζ-plane, the boundary condition becomes

$$\frac{\partial\phi}{\partial\nu_2} = \frac{\partial\phi}{\partial\nu}\frac{d\nu}{d\nu_2}, \quad \ldots \quad (4.7a)$$

i.e.,

$$\frac{\partial\phi}{\partial\lambda} = \frac{ABf(\sigma)f'(\sigma)}{\sqrt{\left\{A^2f^2(\sigma) - \left(1-\frac{B^2}{A^2}\right)A^2f^2(\sigma)\cos^2\mu\right\}}}\left|\frac{dz}{d\zeta}\right|$$

$$= \frac{S'(\sigma)}{2\pi f(\sigma)\sqrt{(A^2\sin^2\mu + B^2\cos^2\mu)}}\left|\frac{dz}{d\zeta}\right|. \quad \ldots \quad (4.7b)$$

From (4.5) we find that on the body

$$\left|\frac{dz}{d\zeta}\right| = \frac{2}{A+B}\sqrt{(A^2\sin^2\mu + B^2\cos^2\mu)}, \quad \ldots \quad (4.8)$$

so that the boundary condition becomes

$$\frac{\partial \phi}{\partial \lambda} = \frac{S'(\sigma)}{\pi(A+B)f(\sigma)} . \qquad (4.9)$$

Therefore to obtain the potential ϕ_{a0} for a smooth pointed body at zero incidence we need merely place a source of suitable strength at $\zeta = 0$, and add the term b_0, and so we have

$$\phi_{a0} = \frac{S'(\sigma)}{2\pi} \log \lambda + \frac{1}{2\pi}\left[S'(\sigma) \log \frac{\beta}{2} - \int_0^\sigma S''(u) \log (\sigma - u)\, du\right] . \qquad (4.10)$$

Differentiation of (4.10) gives

$$\frac{\partial \phi_{a0}}{\partial \sigma} = \frac{1}{2\pi}\left[S''(\sigma) \log \frac{\beta\lambda}{2} - \int_{u=0-}^{\sigma} \log (\sigma - u)\, dS''(u)\right] , \qquad (4.11)$$

the last term being a Stieltjes integral*. Thus a discontinuity in profile curvature at a point $\sigma = g$, say, introduces a term

$$-\frac{1}{2\pi}[S''(g+) - S''(g-)] \log (\sigma - g) ,$$

which is singular at $\sigma = g$. However, as is shown in the Appendix, the error in pressure is very localized, and the resulting error in drag is only $O(t^6 \log^3 t)$, provided that the distance between successive discontinuities in curvature is $O(1)$.

For a body with discontinuities in $f'(\sigma)$ we follow the general approach of Ref. 2 and put

$$\phi_0 = \varphi_0 + \sum_{i=0}^{n} \varphi_{0i} ,$$

where φ_0 and its first derivatives are to be continuous along the body, and the φ_{0i} are to represent the disturbance effect of each discontinuity ; that is each φ_{0i} and its derivatives are to be zero for $\sigma < c_i$ and continuous for $\sigma > c_i$.

To obtain a suitable approximation for φ_0 we merely replace the $S'(\sigma)$ in (4.10) by the continuous function

$$\int_0^\sigma S''(u)\, du = S'(\sigma) - \sum_{j=0}^{i} \Delta S_j', \quad (c_i < \sigma < c_{i+1}) .$$

We now have †

$$\varphi_0 = \frac{1}{2\pi}\left\{\left[S'(\sigma) - \sum_{j=0}^{i} \Delta S_j'\right]\log \frac{\beta\lambda}{2} - \int_{c_0}^{\sigma} S''(u) \log (\sigma - u)\, du\right\} + O(t^4 \log^2 t) , \qquad (4.12)$$

$$\frac{\partial \varphi_0}{\partial \sigma} = \frac{1}{2\pi}\left\{S''(\sigma) \log \frac{\beta\lambda}{2} - \int_{u=c_0-}^{\sigma} \log (\sigma - u)\, dS''(u)\right\} , \qquad (4.13)$$

$$\frac{\partial \varphi_0}{\partial \lambda} = \frac{1}{2\pi\lambda}\left[S'(\sigma) - \sum_{j=0}^{i} \Delta S_j'\right] . \qquad (4.14)$$

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* When dealing with functions having discontinuities in an interval of integration we shall be careful to write Stieltjes integrals in the form $\int_{u=a-}^{b} f(u)\, dg(u)$. By expressions of the form $\int_a^b f(u)\, g'(u)\, du$ we shall mean the sum of Riemann integrals taken between the points of discontinuity.

† Although all the *expressions* for the potentials derived below are of the general form of ϕ_a as defined by equation (2.4), we shall omit the subscript a when dealing with bodies having discontinuities, because the approximate or asymptotic form of these expressions is in general indicated by the presence of an order term or by the symbol $\sim$.

It may be noted that in our approximation $\partial\varphi_0/\partial\sigma$ is not in fact continuous at $\sigma = c_i$, as was specified above. However, its singularities are the result of discontinuities in $S''(\sigma)$, and not of those in $S'(\sigma)$, and are of the same localized type as those already encountered on a smooth body.

Comparing (4.14) with the boundary condition (4.9) we deduce that the boundary condition for φ_{0i} is

$$\frac{\partial\varphi_{0i}}{\partial\lambda} = \frac{\Delta S_i'}{2\pi\lambda}\,, \quad \text{(on the body, for } \sigma > c_i) \,. \qquad (4.15)$$

Thus the asymptotic expression for φ_{0i} is

$$\varphi_{0i} \sim \frac{\Delta S_i'}{2\pi} \log \lambda + b_{0i}(\sigma)\,, \qquad (4.16)$$

where $b_{0i}(\sigma)$ must be evaluated from the original definition (2.5), since we now have $a_{0i}(c_i) \neq 0$. (4.16) then becomes

$$\varphi_{0i} \sim \frac{\Delta S_i'}{2\pi} \log \frac{\beta\lambda}{2(\sigma - c_i)}\,, \qquad (4.17)$$

and this expression becomes an increasingly good approximation for $\sigma - c_i > O(t)$, (section 3 and Appendix). Thus the asymptotic value of the complete potential is, for $C_i < \sigma < c_{i+1}$,

$$\phi_0 \sim \frac{1}{2\pi}\left\{S'(\sigma)\log\frac{\beta\lambda}{2} - \sum_{j=0}^{i} \Delta S_j' \log(\sigma - c_j) - \int_{c_0}^{\sigma} S''(u)\log(\sigma - u)\,du\right\}, \qquad (4.18)$$

where the $\sim$ here denotes that (4.18) becomes a valid approximation some distance behind each of the discontinuities on the body. Comparing this with the potential for a smooth body (4.10), we note that the two expressions are very similar, but that the b_{0i} have introduced a new term.

4.3. *The Incidence Potential.*—For the flow at incidence we consider incidences α_1 and α_2 applied in the x_1s_1- and y_1s_1-planes respectively (Fig. 3), and we assume that α_1 and α_2 are at most $O(t)$. The relation between the co-ordinates with respect to the wind axes (x_1, y_1, s_1), and the co-ordinates with respect to the body axes (x, y, s), is then given by

$$\begin{aligned} z &= z_1 + s_1\alpha e^{i\eta} + O(t^3) \\ s &= s_1 + O(t^2), \end{aligned} \qquad (4.19)$$

where $\alpha = \sqrt{(\alpha_1^2 + \alpha_2^2)}$ and $\eta = \tan^{-1}(\alpha_2/\alpha_1)$.

We take as our complete velocity potential

$$\begin{aligned} \Phi &= [U\cos\alpha \,.\, s + U\,\phi_0] + [U\sin\alpha(x\cos\eta + y\sin\eta) + U\,\phi_1^{(1)}] \\ &= U\cos\alpha\,.\,s + U\,\phi_0 + U\,\phi_1^{(2)}\,, \end{aligned} \qquad (4.20)$$

where $\phi_1^{(2)}$ is more than a perturbation potential, for it includes the uniform cross-flow at infinity. The complete boundary condition is then

$$\frac{\partial}{\partial\nu}(\phi_0 + \phi_1^{(2)}) = \frac{d\nu}{ds}\left[\cos\alpha + \frac{\partial}{\partial s}(\phi_0 + \phi_1^{(2)})\right], \qquad (4.21)$$

so that the boundary condition for $\phi_1^{(2)}$ (we shall hereafter drop the superscript (2)) is

$$\frac{\partial\phi_1}{\partial\nu} = O + O(t^3 \log t)\,, \qquad (4.22)$$

the order term being $O(t^2)$ immediately behind a discontinuity in slope. Hence for smooth, pointed bodies a suitable potential is given by

$$\phi_{a1} = \mathscr{R}\alpha\left[\zeta\, e^{-i\eta} + \frac{(A+B)^2 f^2(\sigma)}{4\zeta}\, e^{i\eta}\right] \quad \ldots \quad (4.23)$$

and

$$\phi_a = \mathscr{R}\left\{\frac{S'(\sigma)}{2\pi}\log\zeta + b_0(\sigma) + \alpha\left[\zeta\, e^{-i\eta} + \frac{(A+B)^2 f^2(\sigma)}{4\zeta}\, e^{i\eta}\right]\right\}. \quad \ldots \quad (4.24)$$

Although it is written in terms of the co-ordinates with respect to the body axes, (4.24) is of the required form (2.4) for ϕ_a, because by (4.19) the expression inside the curly brackets is an analytic function of z_1 as well as of ζ or z, except for negligible terms of $O(t^4)$; and also

$$b_0(\sigma) = b_0(s_1) + O(t^4)\ .$$

For a body with discontinuities we could again divide the potential ϕ_1 into continuous and discontinuous parts, but since the term b_0 occurs only in conjunction with the zero-incidence potential, it is immediately apparent that the asymptotic form of ϕ_1 will in this case be an expression identical to (4.23).

5. *The Pressure Coefficient.*—We have, writing q for total velocity,

$$C_p = 1 - \frac{q^2}{U^2} + O\left[\left(1 - \frac{q^2}{U^2}\right)^2\right],$$

and by (4.20) this becomes

$$C_p = \alpha^2 - 2\left(\frac{\partial\phi_0}{\partial s} + \frac{\partial\phi_1}{\partial s}\right) - \left(\frac{\partial\phi_0}{\partial\nu} + \frac{\partial\phi_1}{\partial\nu}\right)^2 - \left(\frac{\partial\phi_0}{\partial\tau} + \frac{\partial\phi_1}{\partial\tau}\right)^2 + O\left[\left(\frac{\partial\phi}{\partial s}\right)^2\right]. \quad \ldots \quad (5.1)$$

For smooth, pointed bodies we may use the approximate forms ϕ_{a0}, ϕ_{a1}, and this expression is sufficiently accurate for all points on and near to the body.

For bodies with discontinuities in slope (5.1) may be used on the body to predict the aerodynamic forces. It is true that behind points of discontinuity the O-term is only $O(t^2)$, but as these regions are only of length $O(t)$ the effect on lift and drag is $O(t^4)$ and $O(t^5)$, respectively, and therefore negligible. Similarly in the term $(\partial\phi/\partial\tau)^2$ we may use the asymptotic forms of the previous section, although these are not good approximations immediately behind points of discontinuity*. $\partial\phi/\partial\nu$ is of course known exactly on the body from the boundary condition.

It remains to consider the term $-2\partial\phi/\partial s$, which is $O(t)$ immediately behind a discontinuity and will have to be handled with some care. We cannot simply substitute the asymptotic value of $\partial\phi/\partial s$, for two reasons : (i) the lift and drag contributions of the true $\partial\phi/\partial s$ in the regions where the asymptotic form is invalid are $O(t^3)$ and $O(t^4)$ respectively, *i.e.*, of the same order as the total lift and drag ; and (ii) the singularities of the asymptotic form at $s = c_i$ are non-integrable (*see* (4.18)).

We now transform (5.1) into a more convenient form. From (4.18) and (4.22) it follows that on the body

$$\frac{\partial\phi_0}{\partial\mu} \sim 0, \quad \text{i.e.,}\ \frac{\partial\phi_0}{\partial\tau} \sim 0\ ; \qquad \frac{\partial\phi_1}{\partial\nu} = 0, \quad \text{i.e.,}\ \frac{\partial\phi_1}{\partial\lambda} = 0\ ,$$

so that on the body (5.1) becomes in terms of the variables (λ, μ, σ)

$$C_p = C_{p\,0} + C_{p\,1},$$

where

$$C_{p\,0} = -2\left(\frac{\partial\phi_0}{\partial\sigma} + \frac{\partial\phi_0}{\partial\lambda}\frac{\partial\lambda}{\partial s}\right) - \left(\frac{\partial\phi_0}{\partial\lambda}\right)^2\left|\frac{d\zeta}{dz}\right|^2, \quad \ldots \quad (5.2)$$

$$C_{p\,1} = \alpha^2 - 2\left(\frac{\partial\phi_1}{\partial\sigma} + \frac{\partial\phi_1}{\partial\mu}\frac{\partial\mu}{\partial s}\right) - \left(\frac{1}{\lambda}\frac{\partial\phi_1}{\partial\mu}\right)^2\left|\frac{d\zeta}{dz}\right|^2. \quad \ldots \quad (5.3)$$

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* These arguments can also be developed more rigorously by the method of the Appendix.

Differentiating the first two equations of the transformation (4.4) partially with respect to s, substituting the value of λ on the body, and solving for $\partial\lambda/\partial s$, $\partial\mu/\partial s$, we obtain

$$\frac{\partial\lambda}{\partial s} = f'(\sigma)\,\frac{A+B}{2}\left(1 - \frac{AB}{A^2\sin^2\mu + B^2\cos^2\mu}\right), \qquad (5.4)$$

$$\frac{\partial\mu}{\partial s} = \frac{f'(\sigma)}{f(\sigma)}\cot\mu\left(1 - \frac{B^2}{A^2\sin^2\mu + B^2\cos^2\mu}\right). \qquad (5.5)$$

These expressions are $O(t)$ and $O(1)$, respectively, so that in (5.2) and (5.3) only $\partial\phi_0/\partial\sigma$ and $\partial\phi_1/\partial\sigma$ are ever $O(t)$, the terms $(\partial\phi_0/\partial\lambda)(\partial\lambda/\partial s)$ and $(\partial\phi_1/\partial\mu)(\partial\mu/\partial s)$ being always $O(t^2)$. Accordingly we may use the asymptotic forms for these latter terms ; (in the case of $\partial\phi_0/\partial\lambda$ the exact and asymptotic forms are of course identical on the body by virtue of the boundary condition).

It is convenient to combine the last two terms of (5.2) at this stage. From (4.8), (4.9) and (5.4), we have on the body

$$-2\frac{\partial\phi_0}{\partial\lambda}\frac{\partial\lambda}{\partial s} = -\frac{2S'(\sigma)}{\pi(A+B)f(\sigma)}f'(\sigma)\,\frac{A+B}{2}\left(1 - \frac{AB}{A^2\sin^2\mu + B^2\cos^2\mu}\right),$$

$$-\left(\frac{\partial\phi_0}{\partial\lambda}\right)^2\left|\frac{d\zeta}{dz}\right|^2 = -\frac{S'^2(\sigma)}{\pi^2(A+B)^2f^2(\sigma)}\,\frac{(A+B)^2}{4(A^2\sin^2\mu + B^2\cos^2\mu)}.$$

Hence

$$C_{p\,0} = -\frac{2\,\partial\phi_0}{\partial\sigma} + ABf'^2(\sigma)\left(\frac{AB}{A^2\sin^2\mu + B^2\cos^2\mu} - 2\right). \qquad (5.6)$$

It may be of interest to note that while there is no circumferential velocity on a smooth pointed body of elliptic cross-section at zero incidence, or at some distance behind a discontinuity on any elliptic body [*see* equations (4.10), (4.18)], a circumferential pressure gradient does exist. The effect of this gradient is of course to curve the streamlines on the surface of the body.

6. *The Drag at Zero Incidence.*—The drag integral may be written

$$\frac{D}{\frac{1}{2}\rho U^2} = \int_{c_0}^{c_n} ds \oint C_{p0}\frac{d\nu}{ds}\,d\tau \qquad (6.1)$$

where

$$\frac{d\nu}{ds} = \frac{\partial y}{\partial s}\,\frac{1}{\sqrt{\left\{1+\left(\frac{\partial y}{\partial x}\right)^2\right\}}} = \frac{S'(s)}{2\pi\sqrt{\{A^2f^2(s) - x^2\}}}\,\frac{1}{\sqrt{\left\{1+\left(\frac{\partial y}{\partial x}\right)^2\right\}}}, \qquad (6.2)$$

and

$$d\tau = dx\sqrt{\left\{1+\left(\frac{\partial y}{\partial x}\right)^2\right\}}. \qquad (6.3)$$

By substituting these expressions into (6.1) and writing $Af(\sigma)\cos\mu$ for x on the body, we obtain

$$\frac{D}{\frac{1}{2}\rho U^2} = \frac{1}{2\pi}\int_{c_0}^{c_n} S'(\sigma)\,d\sigma\int_0^{2\pi} C_{p\,0}\,d\mu\,. \qquad (6.4)$$

We now consider the contributions of the various terms in C_{p0}, (5.6). From $-2\partial\varphi_0/\partial\sigma$ we have, by (4.13),

$$-\frac{1}{\pi}\int_{c_0}^{c_n} S'(\sigma)\,d\sigma\left[S''(\sigma)\log\frac{\beta(A+B)f(\sigma)}{4} - \int_{u=c_0-}^{\sigma}\log(\sigma - u)\,dS''(u)\right]. \qquad (6.5)$$

From each $-2\partial\varphi_{0i}/\partial\sigma$ we have a term

$$-\frac{1}{\pi}\int_0^{2\pi} d\mu \int_{c_i}^{c_n} S'(\sigma)\,\frac{\partial\varphi_{0i}}{\partial\sigma}\,d\sigma\,. \qquad (6.6)$$

We cannot substitute our asymptotic value for $\partial\varphi_{0i}/\partial\sigma$ here, for the reasons already noted; however, we may proceed as follows. Let

$$\int_{c_i}^{\sigma} \frac{\partial\varphi_{0i}}{\partial\sigma}\,d\sigma = F_i(\sigma, \mu)\,, \qquad (6.7)$$

where $\partial\varphi_{0i}/\partial\sigma$ represents the true value, and the integral is taken along a line μ = constant on the body. Then integrating (6.6) by parts, and noting that $F_i(c_i, \mu) = 0$ by (6.7), we obtain

$$-\frac{1}{\pi}\int_0^{2\pi} d\mu\left[-\int_{\sigma=c_i+}^{c_n+} F_i(\sigma, \mu)\,dS'(\sigma)\right]. \qquad (6.8)$$

Now

$$\varphi_{0i}(\sigma, \mu) = \varphi_{0i}(c_i, \mu) + \int_{c_i}^{\sigma} \frac{\partial\varphi_{0i}}{\partial\sigma}\,d\sigma + \int_{c_i}^{\sigma} \frac{\partial\varphi_{0i}}{\partial\lambda}\,\frac{d\lambda}{d\sigma}\,d\sigma\,,$$

where the true (as opposed to the asymptotic) value of $\varphi_{0i}(c_i, \mu)$ is zero (section 3), and on the body, by (4.15) and (4.6),

$$\frac{\partial\varphi_{0i}}{\partial\lambda} = \frac{\Delta S_i'}{\pi(A + B)f(\sigma)}\,,$$

$$\frac{d\lambda}{d\sigma} = \frac{A + B}{2} f'(\sigma)\,.$$

Hence

$$F_i(\sigma, \mu) = \int_{c_i}^{\sigma} \frac{\partial\varphi_{0i}}{\partial\sigma}\,d\sigma = \varphi_{0i}(\sigma, \mu) - \int_{c_i}^{\sigma} \frac{\Delta S_i'}{2\pi}\,\frac{f'(\sigma)}{f(\sigma)}\,d\sigma$$

by (4.17)

$$\sim \frac{\Delta S_i'}{2\pi}\left[\log\frac{\beta(A + B)f(\sigma)}{4(\sigma - c_i)} - \int_{c_i}^{\sigma} d\log f(\sigma)\right]$$

$$= \frac{\Delta S_i'}{2\pi}\log\frac{\beta(A + B)f(c_i)}{4(\sigma - c_i)}\,. \qquad (6.9)$$

It is shown in the Appendix that if we substitute this asymptotic expression into (6.8) the error is $O(t^5)$. We have in fact overcome the difficulty by working with the asymptotic form of φ_{0i}, instead of that for $\partial\varphi_{0i}/\partial\sigma$, and by applying the true value of φ_{0i} at $\sigma = c_i$. We now have for the drag contribution of the $-2\partial\varphi_{0i}/\partial\sigma$ terms

$$-\frac{1}{\pi}\sum_{i=0}^{n-1} \Delta S_i'\left[S'(c_i+)\log\frac{\beta(A + B)f(c_i)}{4} + \int_{\sigma=c_i+}^{c_n+} \log(\sigma - c_i)\,dS'(\sigma)\right]. \qquad (6.10)$$

From the last term in C_{p0}, (5.6), we have

$$\frac{1}{2\pi}\int_{c_0}^{c_n} S'(\sigma)\, d\sigma \int_0^{2\pi} ABf'^2(\sigma)\left[\frac{AB}{A^2\sin^2\mu + B^2\cos^2\mu} - 2\right] d\mu$$

$$= -\frac{1}{2\pi}\int_{c_0}^{c_n} S'(\sigma)\,.\, ABf'^2(\sigma)\,.\, 2\pi\, d\sigma$$

$$= -\frac{1}{2\pi}\int_{c_0}^{c_n} S'^2(\sigma)\frac{d}{d\sigma}\log f(\sigma)\, d\sigma$$

$$= -\frac{1}{2\pi}\sum_{i=0}^{n-1}\left[S'^2(c_{i+1}-)\log f(c_{i+1}) - S'^2(c_i+)\log f(c_i)\right]$$

$$+\frac{1}{\pi}\int_{c_0}^{c_n} S'(\sigma)S''(\sigma)\log f(\sigma)\, d\sigma\,. \qquad \ldots\ldots (6.11)$$

The drag is the sum of (6.5), (6.10) and (6.11). (6.5) may be written

$$-\frac{1}{\pi}\left\{\int_{c_0}^{c_n} S'(\sigma)\, S''(\sigma)\log f(\sigma)\, d\sigma + \log\frac{\beta(A+B)}{4}\sum_{i=0}^{n-1}\tfrac{1}{2}[S'^2(c_{i+1}-) - S'^2(c_i+)]\right.$$

$$\left. - \int_{c_0}^{c_n} S'(\sigma)\, d\sigma\int_{u=c_0-}^{\sigma}\log(\sigma-u)\, dS''(u)\right\}. \qquad \ldots\ldots (6.12)$$

The first terms of (6.10) and (6.11), and the second term of (6.12) combine to give

$$-\frac{1}{2\pi}\sum_{i=0}^{n}(\Delta S_i')^2\log\frac{\beta(A+B)f(c_i)}{4}, \qquad \ldots\ldots (6.13)$$

and the remainder of the drag is

$$-\frac{1}{\pi}\sum_{i=0}^{n-1}\Delta S_i'\int_{\sigma=c_i+}^{c_n+}\log(\sigma-c_i)\, dS'(\sigma) + \frac{1}{\pi}\int_{c_0}^{c_n} S'(\sigma)\, d\sigma\int_{u=c_0-}^{\sigma}\log(\sigma-u)\, dS''(u)\,. \qquad \ldots\ldots (6.14)$$

Reversing the order of integration in the last term, and then integrating twice by parts gives

$$\frac{1}{\pi}\int_{u=c_0-}^{c_n+} dS''(u)\int_u^{c_n}\log(\sigma-u)\, S'(\sigma)\, d\sigma$$

$$= -\frac{1}{\pi}\int_{c_0}^{c_n} S''(u)\, du\int_{\sigma=u}^{c_n+}\log(\sigma-u)\, dS'(\sigma)$$

$$= -\frac{1}{\pi}\left\{\int_{c_0}^{c_n} S''(u)\, du\int_u^{c_n}\log(\sigma-u)\, S''(\sigma)\, d\sigma\right.$$

$$\left. + \sum_{i=1}^{n}\Delta S_i'\int_{c_0}^{c_i} S''(u)\log(c_i-u)\, du\right\}. \qquad \ldots\ldots (6.15)$$

The last term in (6.15) combines with part of the first term in (6.14) to give

$$-\frac{1}{\pi}\sum_{i=0}^{n} \Delta S_i' \int_{c_0}^{c_n} S''(\sigma) \log |\sigma - c_i| \, d\sigma \, , \quad \ldots \quad (6.16)$$

and the remainder of the first term in (6.14) is

$$-\frac{1}{\pi}\sum_{i=0}^{n-1} \Delta S_i' \sum_{j=i+1}^{n} \Delta S_j' \log (c_j - c_i) \, . \quad \ldots \quad (6.17)$$

Hence, collecting terms, we finally have

$$\frac{D}{\frac{1}{2}\rho U^2} = \frac{1}{2\pi}\int_{c_0}^{c_n}\int_{c_0}^{c_n} S''(\sigma)\, S''(u) \log \frac{1}{|\sigma - u|} \, d\sigma \, du + \frac{1}{\pi}\sum_{i=0}^{n} \Delta S_i' \int_{c_0}^{c_n} S''(\sigma) \log \frac{1}{|\sigma - c_i|} \, d\sigma$$

$$+ \frac{1}{\pi}\sum_{i=0}^{n-1} \Delta S_i' \sum_{j=i+1}^{n} \Delta S_j' \log \frac{1}{c_j - c_i} + \frac{1}{2\pi}\sum_{i=0}^{n} (\Delta S_i')^2 \log \frac{4}{\beta(A+B)f(c_i)} + O(t^5) \, , \quad \ldots \quad (6.18a)$$

$$= \frac{1}{2\pi} {}^{*}\!\!\int_{\sigma=c_0-}^{c_n+} \int_{u=c_0-}^{c_n+} \log \frac{1}{|\sigma - u|} \, dS'(\sigma) \, dS'(u)$$

$$+ \frac{1}{2\pi}\sum_{i=0}^{n} (\Delta S_i')^2 \log \frac{4}{\beta(A+B)f(c_i)} + O(t^5) \, , \quad \ldots \quad (6.18b)$$

where the asterisk denotes the 'finite part of' the double Stieltjes integral (*cf.* Ref. 2, equation (37)). (6.18) is extremely similar to the expression for the drag of the equivalent body of revolution (*i.e.*, the body of revolution which has the same area distribution $S(s)$), the only difference being in the last term. The drag of the body of revolution is greater by an amount

$$\frac{1}{2\pi}\sum_{i=0}^{n} (\Delta S_i')^2 \log \frac{A+B}{2\sqrt{(AB)}} \, , \quad \ldots \quad (6.19)$$

which is always positive and independent of Mach number. However, if A and B are approximately equal, so that

$$\frac{A}{B} = 1 + O(t^{1/2}) \, ,$$

then

$$\log \frac{A+B}{2\sqrt{(AB)}} = O(t) \, ,$$

so that the difference in drag of the two bodies in this case is only $O(t^5)$, which is negligible to the order of accuracy of the theory.

It is apparent from the symmetrical form of (6.18) that the 'reverse' of any elliptic body has the same drag as the original body.

7. *The Forces and Moments at Incidence.*—7.1. *The Lateral Forces.*—The incidence problem is somewhat simpler than the drag problem for two reasons. Firstly, the cross-flow potential (4.23) gives, without further modifications, an asymptotic expression for the true potential when discontinuities are present; and secondly, in the integrals below for the lateral forces and pitching moments the continuous function $f(\sigma)$ appears where previously we had $S'(\sigma)$.

We have for the lateral forces in the directions of increasing x and y,

$$\frac{X}{\frac{1}{2}\rho U^2} = \int_{c_0}^{c_n} ds \left[\int_{-Bf(s)}^{Bf(s)} (-C_{p1}|_{x>0})\, dy + \int_{-Bf(s)}^{Bf(s)} (C_{p1}|_{x<0})\, dy \right]$$

$$= -B \int_{c_0}^{c_n} f(\sigma)\, d\sigma \int_0^{2\pi} C_{p1} \cos \mu \, d\mu \,, \qquad (7.1)$$

$$\frac{Y}{\frac{1}{2}\rho U^2} = -A \int_{c_0}^{c_n} f(\sigma)\, d\sigma \int_0^{2\pi} C_{p1} \sin \mu \, d\mu \,. \qquad (7.2)$$

Our potential is, from (4.23),

$$\phi_1 \sim \alpha \left[\lambda + \frac{(A+B)^2 f^2(\sigma)}{4\lambda} \right] \cos(\mu - \eta) \,, \qquad (7.3)$$

where the $\sim$ here denotes that (7.3) becomes a valid approximation some distance behind each of the discontinuities on the body. We have shown that this is in adequate form for all the terms in C_{p1}, (5.3), except $-2\,\partial\phi_1/\partial\sigma$.

The term α^2, and the term $-(\partial\phi_1/\lambda\partial\mu)^2\,|d\zeta/dz|^2$, which occurs in the drag of an infinite elliptic cylinder in incompressible flow, will not contribute anything to X and Y.

The contribution to X of $-2\,\partial\phi_1/\partial\sigma$ may be written

$$2B \int_0^{2\pi} \cos \mu \, d\mu \int_{c_0}^{c_n} \frac{\partial \phi_1}{\partial \sigma} f(\sigma)\, d\sigma \,,$$

and if

$$\int_{c_0}^{\sigma} \frac{\partial\phi_1}{\partial\sigma}\, d\sigma = G(\sigma, \mu) \,,$$

(the integral being taken along a line μ = constant on the body), this may be written

$$2B \int_0^{2\pi} \cos \mu \, d\mu \left[f(c_n)\, G(c_n, \mu) - \int_{c_0}^{c_n} G(\sigma, \mu) f'(\sigma)\, d\sigma \right] . \qquad (7.4)$$

Now since $\partial\phi_1/\partial\lambda = 0$ on the body

$$G(\sigma, \mu) = \phi_1(\sigma, \mu) - \phi_1(c_0, \mu) \,. \qquad (7.5)$$

We have shown (section 3) that a perturbation potential tends to zero as $\sigma \to c_0$, but our ϕ_1 includes the uniform cross-flow; hence at $\sigma = c_0$ it must represent a uniform cross-flow in the entire z-plane:

$$\phi_1(c_0, \mu) = \mathscr{R}(\alpha z e^{-i\eta})$$

$$= \mathscr{R}\left\{ \alpha \left[\zeta + \frac{(A^2 - B^2) f^2(\sigma)}{4\zeta} \right] e^{-i\eta} \right\}$$

$$= \alpha f(c_0)[A \cos\mu \cos\eta + B \sin\mu \sin\eta] \qquad \text{on the body.}$$

Hence $$G(\sigma, \mu) \sim \alpha[(A+B) f(\sigma) \cos(\mu - \eta) - f(c_0)(A \cos\mu \cos\eta + B \sin\mu \sin\eta)] , \qquad (7.6)$$

and as before we may substitute this into (7.4); the error will be $O(t^4)$. (7.4) now becomes

$$2B \int_0^{2\pi} \cos\mu \,.\, \alpha \left[\frac{A+B}{2} (f^2(c_n) + f^2(c_0)) \cos(\mu - \eta) - f^2(c_0)(A \cos\mu \cos\eta + B \sin\mu \sin\eta) \right] d\mu$$

$$= \pi\alpha \cos\eta \,.\, B[(A+B) f^2(c_n) - (A-B) f^2(c_0)] \,. \qquad (7.7)$$

The term $-2(\partial\phi_1/\partial\mu)(\partial\mu/\partial s)$ in C_{p1} contributes to X, by (5.5) and (7.3),

$$-2B\int_{c_0}^{c_n} f(\sigma)\,d\sigma\int_0^{2\pi}\alpha(A+B)\sin(\mu-\eta)\,.\,f'(\sigma)\cot\mu\left[1-\frac{B^2}{A^2\sin^2\mu+B^2\cos^2\mu}\right].\cos\mu\,d\mu$$

$$=-2\pi\,\alpha\cos\eta\,.\,B(A-B)\int_{c_0}^{c_n} f(\sigma)f'(\sigma)\,d\sigma$$

$$=-\pi\,\alpha\cos\eta\,.\,B(A-B)\,[f^2(c_n)-f^2(c_0)]\,. \qquad (7.8)$$

Adding (7.7) and (7.8) we have

$$\frac{X}{\frac{1}{2}\rho U^2}=2\pi B^2\,\alpha\cos\eta\,.\,f^2(c_n)+O(t^4)$$

$$C_X=\frac{X}{\frac{1}{2}\rho U^2 S(c_n)}=2\frac{B}{A}\alpha\cos\eta=2\frac{B}{A}\alpha_1+O(t^2)\,. \qquad (7.9)$$

Similarly

$$C_Y=2\frac{A}{B}\alpha\sin\eta=2\frac{A}{B}\alpha_2+O(t^2)\,. \qquad (7.10)$$

This result is the same as that obtained by Ward[1] for the particular case of a smooth, pointed body of elliptic cross-section ending in a cylinder. X and Y have been derived as forces normal to the body axis, but they are equally ' lift ' forces to the order of accuracy shown.

7.2. *The Induced Drag.*—The induced drag is given by

$$\frac{D_i}{\frac{1}{2}\rho U^2}=\frac{\sin\alpha}{\frac{1}{2}\rho U^2}(X\cos\eta+Y\sin\eta)+\frac{\cos\alpha}{2\pi}\int_{c_0}^{c_n}S'(\sigma)\,d\sigma\int_0^{2\pi}C_{p1}\,d\mu\,.$$

To our order of accuracy this may be written

$$\frac{D_i}{\frac{1}{2}\rho U^2}=\frac{X}{\frac{1}{2}\rho U^2}\alpha_1+\frac{Y}{\frac{1}{2}\rho U^2}\alpha_2+\frac{1}{2\pi}\int_{c_0}^{c_n}S'(\sigma)\,d\sigma\int_0^{2\pi}C_{p1}\,d\mu\,. \qquad (7.11)$$

The α^2 term in C_{p1}, (5.3), contributes to the integral in (7.11)

$$\alpha^2\,[S(c_n)-S(c_0)]. \qquad (7.12)$$

Considerations of symmetry show that there is no contribution from $-2(\partial\phi_1/\partial s)$, (*i.e.*, from the whole second term in (5.3)), and from the last term in C_{p1} we have, by (4.8) and (7.3),

$$-\frac{1}{2\pi}\int_{c_0}^{c_n}S'(\sigma)\,d\sigma\int_0^{2\pi}4\,\alpha^2\sin^2(\mu-\eta)\,.\,\frac{(A+B)^2}{4(A^2\sin^2\mu+B^2\cos^2\mu)}\,d\mu$$

$$=-\frac{1}{2\pi}\int_{c_0}^{c_n}S'(\sigma)\,.\,2\pi\,\alpha^2\left(1+\frac{B}{A}\cos^2\eta+\frac{A}{B}\sin^2\eta\right)d\sigma$$

$$=-[S(c_n)-S(c_0)]\left[\alpha^2+\frac{B}{A}\alpha_1^2+\frac{A}{B}\alpha_2^2\right]. \qquad (7.13)$$

Adding (7.12), (7.13) and the contributions of the lateral forces, we have

$$\frac{D_i}{\frac{1}{2}\rho U^2}=[S(c_n)+S(c_0)]\left[\frac{B}{A}\alpha_1^2+\frac{A}{B}\alpha_2^2\right]+O(t^5)\,. \qquad (7.14)$$

7.3. *The Pitching Moments.*—For convenience we now take the x- and y-axes in the plane of the nose, writing $c_0 = 0$. The moments about these axes in the sense tending to decrease the applied incidence are then, respectively,

$$\frac{M_x}{\frac{1}{2}\rho U^2} = -A\int_0^{c_n} \sigma f(\sigma)\, d\sigma \int_0^{2\pi} C_{p1} \sin\mu\, d\mu\,, \qquad \ldots \quad (7.15)$$

$$\frac{M_y}{\frac{1}{2}\rho U^2} = -B\int_0^{c_n} \sigma f(\sigma)\, d\sigma \int_0^{2\pi} C_{p1} \cos\mu\, d\mu\,. \qquad \ldots \quad (7.16)$$

The integration is of course very similar to that for the lateral forces. From $-2\,\partial\phi_1/\partial\sigma$ we have the contribution to M_y (*cf.* (7.4))

$$2B\int_0^{2\pi} \cos\mu\, d\mu \left\{ c_n f(c_n) G(c_n, \mu) - \int_0^{c_n} G(\sigma, \mu)[f(\sigma) + \sigma f'(\sigma)]\, d\sigma \right\},$$

and upon substitution for G from (7.6) this becomes

$$2B\int_0^{2\pi} \cos\mu \,.\, \alpha \frac{A+B}{2} \cos(\mu - \eta) \left[c_n f^2(c_n) - \int_0^{c_n} f^2(\sigma)\, d\sigma \right] .\, d\mu$$

$$= \pi\alpha\cos\eta \,.\, B(A+B)\left[c_n f^2(c_n) - \int_0^{c_n} f^2(\sigma)\, d\sigma \right]. \qquad \ldots \quad (7.17)$$

From $-2(\partial\phi_1/\partial\mu)(\partial\mu/\partial s)$ we have [*cf.* (7.8)]

$$-2\pi\alpha\cos\eta\, B(A-B)\int_0^{c_n} \sigma f(\sigma) f'(\sigma)\, d\sigma$$

$$= -\pi\alpha\cos\eta\, B(A-B)\left[c_n f^2(c_n) - \int_0^{c_n} f^2(\sigma)\, d\sigma \right]. \qquad \ldots \quad (7.18)$$

Hence, adding (7.17) and (7.18), we obtain

$$\frac{M_y}{\frac{1}{2}\rho U^2} = 2\pi\alpha\cos\eta \,.\, B^2\left[c_n f^2(c_n) - \int_0^{c_n} f^2(\sigma)\, d\sigma \right]. \qquad \ldots \quad (7.19)$$

Hence

$$\frac{M_y}{\frac{1}{2}\rho U^2} = 2\frac{B}{A}\alpha_1\left[(c_n - c_0) S(c_n) - \int_{c_0}^{c_n} S(\sigma)\, d\sigma \right] + O(t^4)\,, \qquad \ldots \quad (7.20)$$

$$\frac{M_x}{\frac{1}{2}\rho U^2} = 2\frac{A}{B}\alpha_2\left[(c_n - c_0) S(c_n) - \int_{c_0}^{c_n} S(\sigma)\, d\sigma \right] + O(t^4)\,. \qquad \ldots \quad (7.21)$$

Thus the X and Y forces act through the same point. The distance from the nose of the centre of pressure (or, in this case, of the aerodynamic centre), expressed as a fraction of body length, is given by

$$h = 1 - \frac{1}{(c_n - c_0) S(c_n)} \int_{c_0}^{c_n} S(\sigma)\, d\sigma + O(t) \qquad \ldots \quad (7.22)$$

$$= 1 - \frac{\text{mean cross-section area}}{\text{base area}}\,.$$

8. *Applications.*—8.1. *The Restriction on Section Curvature.*—In order to satisfy Ward's general restriction on the curvature of the cross-section we originally specified that both the major- and minor-axis parameters, A and B, be $O(t)$. In practice it may sometimes be desirable to consider bodies whose elliptic cross-section is very flat, that is, to let B be $O(t^2)$.

This is wholly justifiable for smooth bodies at zero incidence, since the circumferential velocity is zero in this case. We might also expect it to be permissible for bodies with discontinuities in slope at zero incidence, since at the actual points of maximum curvature symmetry requires zero circumferential velocity. However, in this case no rigorous justification is possible by the methods of the present paper, because we have no detailed knowledge of the flow immediately behind points of discontinuity.

The flat elliptic body at incidence has already been discussed by Ward[1] for the case of smooth bodies. Ward suggested that although the flow could no longer be calculated within a known approximation, the solutions obtained by his method gave a first approximation for sufficiently small angles of attack, since the forces and moments tend to a finite limit as $B \to 0$. This argument may be extended to the more general class of bodies considered here.

8.2. *The Elliptic Cone at Zero Incidence.*—For the particular problem of an elliptic cone we may write

$$f(s) = s\,, \qquad S(s) = \pi AB\, s^2\,, \qquad S''(s) = 2\pi AB\,. \qquad \text{(8.1)}$$

From equations (5.6) and (4.11), we then have for the pressure coefficient on the cone at zero incidence

$$C_{p0} = -\frac{1}{\pi}\left[S''(\sigma)\log\frac{\beta\lambda}{2} - \int_{u=0-}^{\sigma}\log(\sigma - u)\,dS''(u)\right] + ABf'^2(\sigma)\left[\frac{AB}{A^2\sin^2\mu + B^2\cos^2\mu} - 2\right]$$

$$= AB\left[2\log\frac{4}{\beta(A+B)} + \frac{AB}{A^2\sin^2\mu + B^2\cos^2\mu} - 2\right]. \qquad \text{(8.2)}$$

This pressure coefficient is plotted for a range of ratios B/A in Fig. 4. It may be noted that as the eccentricity increases the pressure gradient over most of the ellipse first increases and then decreases, and that a pressure peak appears at the end of the major axis or 'leading edge'.

The drag coefficient is, by (6.18),

$$C_D = AB\left[2\log\frac{4}{\beta(A+B)} - 1\right]. \qquad \text{(8.3)}$$

In Ref. 5, Squire has obtained an exact solution of the linearized equation for flat elliptic cones, the major axis of his ellipse being finite and the minor axis infinitesimal. Squire's boundary condition is that of a wing problem, and may be written

$$\frac{\partial\phi}{\partial y} = \frac{\partial y}{\partial s}\,,$$

where the corresponding form of our boundary condition is

$$\frac{\partial\phi}{\partial y} - \frac{\partial\phi}{\partial x}\frac{\partial y}{\partial x} = \frac{\partial y}{\partial s}\,.$$

Nevertheless we would expect the two solutions to be similar for the common problem of a cone having A of $O(t)$ and B of $O(t^2)$, and this is in fact the case.

Squire's corrected solution consists of two parts as follows :

(i) A constant pressure over the elliptic cone, which in our notation is given by

$$C_p = 2AB\frac{K\sqrt{(1 - A^2\beta^2)} - E\sqrt{(1 - A^2\beta^2)}}{1 - A^2\beta^2}, \quad \ldots \quad (8.4)$$

where K and E are complete elliptic integrals of the first and second kinds.

(ii) A drag contribution from the singularity at the leading edge. Expressed as force per unit length normal to the leading edge, this is

$$\frac{F_n}{\frac{1}{2}\rho U^2} = \pi r_c \frac{A^2}{(A^2 + 1)\sqrt{\left\{1 - (\beta^2 + 1)\frac{A^2}{A^2 + 1}\right\}}}, \quad \ldots \quad (8.5)$$

where r_c is the radius of curvature of the leading edge.

In (i), when A is $O(t)$ we may expand the elliptic integrals in series, writing

$$K\sqrt{(1 - A^2\beta^2)} = \log\frac{4}{A\beta} + O(t^2 \log t) ,$$

$$E\sqrt{(1 - A^2\beta^2)} = 1 + O(t^2 \log t),$$

and so we obtain, for B of $O(t^2)$,

$$C_p = AB\left(2\log\frac{4}{A\beta} - 2\right) + O(t^5 \log t) . \quad \ldots \quad (8.6$$

In Fig. 5 this pressure coefficient is compared with that predicted by the theory of this paper (equation (8.2)). It is interesting to note that the two values agree well over most of the surface, and that, instead of Squire's singularity, the theory of this paper predicts a finite pressure peak at the leading edge. This peak is in fact $O(t^2)$, whereas the lower pressures are $O(t^3 \log t)$.

In (ii) above we have

$$r_c = \frac{B^2}{A}s + O(t^5) . \quad \ldots \quad (8.7)$$

Hence we deduce

$$\Delta C_D = AB + O(t^5),$$

and

$$C_D = AB\left(2\log\frac{4}{A\beta} - 1\right) + O(t^5 \log t) . \quad \ldots \quad (8.8)$$

The two expressions (8.3) and (8.8) for the drag coefficient are equal to the lowest order, for B of $O(t^2)$, the difference being $O(t^4 \log t)$.

Similar agreement may be obtained for the lift of the elliptic cone when incidence is applied in the ys-plane. It is already well known that the expression for the lift of a delta wing reduces to that given by (7.10) when the aspect ratio becomes small.

8.3. *Conical and Parabolic Forebodies and Afterbodies.*—Ref. 6 presented curves of the drag predicted by slender body theory for certain families of bodies of revolution. These included curves of the drag of forebodies and afterbodies of straight and parabolic profile ; the afterbodies were assumed to be situated behind an infinitely long parallel portion, so that by the reversibility property mentioned in section 6, forebodies and afterbodies which are the ' reverse ' of one another have equal drag.

It is now a simple matter to extend these results to slender bodies of elliptic cross-section with the same area distribution $S(s)$ as the bodies of revolution in question.

Let R_0 and R_1 denote the radii at the initial and final sections of a forebody of circular cross-section, and let l be the length. Then we have forebodies of straight profile

$$\Delta S_0' = 2\pi R_0 \frac{R_1 - R_0}{l},$$

$$\Delta S_1' = -2\pi R_1 \frac{R_1 - R_0}{l}; \quad \ldots \quad (8.9)$$

and for bodies of parabolic profile,

$$\Delta S_0' = 4\pi R_0 \frac{R_1 - R_0}{l}. \quad \ldots \quad (8.10)$$

It follows from equation (6.19) that the drag of the equivalent bodies of elliptic cross-section is that of the bodies of revolution plus the following term :

for bodies of straight profile,

$$\Delta C_D \left(\frac{l}{R_1}\right)^2 = \Delta C_D \left(\frac{\pi l^2}{S_1}\right) = -2\left(1 + \frac{S_0}{S_1}\right)\left(1 - \sqrt{\frac{S_0}{S_1}}\right)^2 \log \frac{1 + B/A}{2\sqrt{(B/A)}}, \quad \ldots \quad (8.11)$$

and for bodies of parabolic profile,

$$\Delta C_D \left(\frac{\pi l^2}{S_1}\right) = -8\frac{S_0}{S_1}\left(1 - \sqrt{\frac{S_0}{S_1}}\right)^2 \log \frac{1 + B/A}{2\sqrt{(B/A)}}. \quad \ldots \quad (8.12)$$

These correction terms are plotted in Fig. 6 : comparison with the values of Ref. 6 shows that in general they are a very small proportion of the drag of the body.

8.4. *The Interference Effect of a Forebody on an Afterbody.*—It was demonstrated in Ref. 6 that when one considers the drag of a body consisting of a forebody, a parallel mid-portion and an afterbody, it is often convenient to calculate the drag as the sum of the following three components :

(i) The forebody drag.

(ii) The 'principal afterbody drag' (denoted by D_{A1}), which is the drag that the afterbody would have if it were situated behind an infinitely long parallel portion.

(iii) The interference drag (denoted by D_{A2}) due to the effect of the forebody on the afterbody.

Curves of interference drag for a number of conical and parabolic bodies of revolution were given in Ref. 6.

For all cases where the distance between successive discontinuities in slope is large with respect to the thickness of the body, the interference drag is unaffected by the change from a circular to an elliptic cross-section. For such bodies the interference drag may be deduced from the drag formula (6.18a) by subtracting the forebody and principal afterbody drags from the drag of the complete body. We denote the ends of the parallel portion by $s = c_k$ and c_{k+1}, and the result is :

$$\frac{D_{A2}}{\frac{1}{2}\rho U^2} = \frac{1}{\pi}\int_{c_0}^{c_k} S''(\sigma)\, d\sigma \int_{c_{k+1}}^{c_n} S''(u) \log \frac{1}{u - \sigma}\, du$$

$$+ \frac{1}{\pi}\sum_{i=0}^{k} \Delta S_i' \int_{c_{k+1}}^{c_n} S''(\sigma) \log \frac{1}{\sigma - c_i}\, d\sigma + \frac{1}{\pi}\sum_{i=k+1}^{n} \Delta S_i' \int_{c_0}^{c_k} S''(\sigma) \log \frac{1}{c_i - \sigma}\, d\sigma$$

$$+ \frac{1}{\pi}\sum_{i=0}^{k} \Delta S_i' \sum_{j=k+1}^{n} \Delta S_j' \log \frac{1}{c_j - c_i}. \quad \ldots \quad (8.13)$$

This is identical with equation (27) of Ref. 6, and is independent of Mach number.

When discontinuities in slope occur at both ends of the parallel portion, and the length of that portion is only of the order of the thickness of the body, we have no sufficiently accurate expression for the interference drag of the elliptic body, for we have no function corresponding to the exact form of $U_1(x)$ in Ref. 6. However, if this case should prove of real interest, the difficulty could be partially overcome as follows.

The value of the interference drag when the length of the parallel portion is zero may again be deduced from (6.18*a*). (The appropriate expression is given below, and in this case the interference drag is a function both of Mach number and of the eccentricity of the ellipse.) It should now be possible to guess fairly accurately the variation of interference drag with the length of the parallel portion, using as a guide the known initial values (8.14), the unique curve given by (8.13) for long parallel portions, and the known variation for the equivalent body of revolution (*cf.* Figs. 8 and 11, Ref. 6).

To evaluate the interference drag when the length of the parallel portion is zero, we let $\Delta S_{k-}'$ denote the final discontinuity in $S'(s)$ of the forebody alone (according to the convention of section 4.1), and $\Delta S_{k+}'$ the initial discontinuity of the afterbody alone. Then when the two are joined there is a discontinuity in $S'(s)$ at $s = c_k$ given by

$$\Delta S_k' = \Delta S_{k-}' + \Delta S_{k+}' \ ,$$

and the interference drag is found to be

$$\begin{aligned}\frac{D_{A2}}{\frac{1}{2}\rho U^2} = {} & \frac{1}{\pi}\int_{c_0}^{c_k} S''(\sigma)\,d\sigma \int_{c_k}^{c_n} S''(u)\log\frac{1}{u-\sigma}\,du \\ & + \frac{1}{\pi}\sum_{i=0}^{k-} \Delta S_i' \int_{c_k}^{c_n} S''(\sigma)\log\frac{1}{\sigma - c_i}\,d\sigma + \frac{1}{\pi}\sum_{i=k+}^{n} \Delta S_i' \int_{c_0}^{c_k} S''(\sigma)\log\frac{1}{c_i-\sigma}\,d\sigma \\ & + \frac{1}{\pi}\sum_{i=0}^{k-1} \Delta S_i' \sum_{j=k+1}^{n} \Delta S_j' \log\frac{1}{c_j - c_i} \\ & + \frac{1}{\pi}\Delta S_{k-}' \sum_{i=k+1}^{n} \Delta S_j' \log\frac{1}{c_i - c_k} + \frac{1}{\pi}\Delta S_{k+}' \sum_{i=0}^{k-1} \Delta S_i' \log\frac{1}{c_k - c_i} \\ & + \frac{1}{\pi}(\Delta S_{k-}')(\Delta S_{k+}')\log\frac{4}{\beta(A+B)f(c_k)} \,. \qquad \ldots \quad (8.14)\end{aligned}$$

Because the last term of (6.18*a*) has contributed the last term of this expression, the interference drag is now a function both of Mach number and of the eccentricity of the ellipse.

NOTATION

a_n	Coefficients in Ward's approximate potential ϕ_a, equation (2.4)
A	Parameter defining the semi-major-axis of the elliptic cross-section
b_0	Function of s_1 in Ward's approximate potential ϕ_a, equation (2.4)
B	Parameter defining the semi-minor-axis of the elliptic cross-section
c_i	Values of s at which discontinuities occur
C_p	Pressure coefficient $(p - p_0)/\frac{1}{2}\rho U^2$
D	Drag

$f(s)$		Function defining the variation with s of the major and minor axes of the ellipse
F_i		*See* equation (6.7)
G		*See* equation (7.5)
M		Mach number of the free stream
M_x, M_y		Pitching moments about the x- and y-axes, respectively, in the plane of the nose
p		Heaviside operator with respect to s_1
(r_1, θ_1, s_1)		Cylindrical polar co-ordinates with respect to the ' wind axes '
$\mathcal{R}$		' Real part of '
S		Cross-section area
t		Maximum thickness of body
(x, y, s)		Cartesian co-ordinates with respect to the body axes
(x_1, y_1, s_1)		Cartesian co-ordinates with respect to the ' wind axes '
X, Y		Lateral forces in the direction of increasing x and y
z, (z_1)	=	$x + iy$, $(x_1 + iy_1)$
α		Total incidence
α_1, α_2		Incidences in the x_1s_1- and y_1s_1-planes, respectively
β		$\sqrt{(M^2 - 1)}$
γ		Euler's constant
ζ		Complex variable $\lambda\, e^{i\eta}$ of transformed (circle) plane
η	=	$\tan^{-1} \alpha_2/\alpha_1$
(λ, μ, σ)		Elliptic-type co-ordinates with respect to the body axes, *see* equation (4.4)
ν, (ν_1)		Normal to the body contour in a plane s = constant (s_1 = constant)
ρ		Density of the free stream
τ, (τ_1)		Circumferential variable in a plane s = constant (s_1 = constant)
ϕ		Exact perturbation potential
ϕ_a		Ward's approximate form of perturbation potential, equation (2.4)
ϕ_b, ϕ_c		Errors of ϕ_a, *see* Appendix
ϕ_0, ϕ_{a0}		Zero-incidence (drag) potential } *see* footnote to equation (4.12)
ϕ_1, ϕ_{a1}		Cross-flow (lift) potential } *see* footnote to equation (4.12)
φ_0, φ_{0i}		Components of ϕ_0, *see* section 4.2

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APPENDIX

The Effect of Discontinuities in the Source and Multi-source Strengths and their Derivatives.

A.1. *The Error of the Asymptotic Solutions.*—In this section we examine the accuracy of the slender body approximation as a solution of the linearized problem when there are discontinuities in the source and multi-source strengths $a_n(s_1)$. We let ϕ_a denote the asymptotic slender-body solution ; we let $(\phi_a + \phi_b)$ denote the exact solution of the linearized equation with the source and multi-source strengths of the slender-body solution (i.e., $(\phi_a + \phi_b)$ is defined by (2.3), with the A_n and B_n derived from (2.5) and from the solution of the harmonic problem) ; and we let $(\phi_a + \phi_b + \phi_c)$ denote the exact solution of the linearized equation which satisfies the boundary condition (2.2) on the surface of the body. We consider values at a point for which the series (2.3) and (2.4) converge, and assume that the resultant orders of magnitude are valid everywhere on and near the body.

Writing $a_n(s_1) = a_{nR}(s_1) + i a_{nI}(s_1)$ we have

$$\underline{\phi}_a + \underline{\phi}_b = - \underline{a}_0(p) K_0(\beta p r_1) + \sum_{n=1}^{\infty} \frac{2}{(n-1)\,!} \left(\frac{\beta p}{2}\right)^n [\underline{a}_{nR}(p) \cos n\,\theta_1 + \underline{a}_{nI}(p) \sin n\,\theta_1]\, K_n(\beta p r_1)\,, \quad \text{(A.1)}$$

and

$$\underline{\phi}_a = \underline{a}_0(p)\left(\gamma + \log \frac{\beta p r_1}{2}\right) + \sum_{n=1}^{\infty} [\underline{a}_{nR}(p) \cos n\,\theta_1 + \underline{a}_{nI}(p) \sin n\theta_1] r_1^{-n}\,. \quad \text{.. .. .. (A.2)}$$

We interpret the first (source) terms in these expressions by standard methods, using the product theorem in the form

$$\underline{f}(p)\,\underline{g}(p) \doteqdot \int_{u=0-}^{s_1} f(s_1 - u)\, dg(u)\,.$$

The second (doublet) terms are interpreted by differentiating the source terms and their interpretations with respect to x or y and replacing a_0 by a_{1R} or a_{1I}. Subsequent terms may be found similarly. There result

$$\phi_a + \phi_b = -\int_{u=0-}^{s_1-\beta r_1} \cosh^{-1}\frac{s_1-u}{\beta r_1}\, da_0(u) + \frac{\cos\theta_1}{r_1}\int_{u=0-}^{s_1-\beta r_1} \frac{s_1-u}{\sqrt{\{(s_1-u)^2-\beta^2 r_1^2\}}}\, da_{IR}(u) + \ldots , \quad \text{(A.3)}$$

and

$$\phi_a = -\int_{u=0-}^{s_1} \log\frac{2(s_1-u)}{\beta r_1}\, da_0(u) + \frac{\cos\theta_1}{r_1}\int_{u=0-}^{s_1} da_{IR}(u) + \ldots \quad \text{(A.4)}$$

Hence from a discontinuity in the $a_n(s_1)$ at $s_1 = c_i$ we get

$$[\Delta(\phi_a+\phi_b)]_i = -\cosh^{-1}\frac{s_1-c_i}{\beta r_1}\,.\,(\Delta a_0)_i + \frac{\cos\theta_1}{r_1}\frac{s_1-c_i}{\sqrt{\{(s_1-c_i)^2-\beta^2 r_1^2\}}}\,.\,(\Delta a_{IR})_i + \ldots , \quad \text{(A.5)}$$

and

$$(\Delta\phi_a)_i = -\log\frac{2(s_1-c_i)}{\beta r_1}(\Delta a_0)_i + \frac{\cos\theta_1}{r_1}(\Delta a_{IR})_i + \ldots \quad \text{(A.6)}$$

Subtracting, we see that there exists some positive finite number C_1 such that

$$(\Delta\phi_b)_i \text{ is } O\left[\frac{t^4}{(s_1-c_i)^2}\right] \quad \text{for } s_1 - c_i \geqslant C_1 t\,. \quad \text{(A.7)}$$

ϕ_c is closely related to ϕ_b. By definition ϕ_c is an exact solution of the linearized equation, tending to zero as $r_1 \to \infty$, such that

$$\frac{\partial\phi_c}{\partial r_1} = -\frac{\partial\phi_b}{\partial r_1} \quad \text{on the body}\,. \quad \text{(A.8)}$$

Accordingly we assume that

$$(\Delta\phi_c)_i \text{ is } O\left[\frac{t^4}{(s_1-c_i)^2}\log\frac{t}{s_1-c_i}\right] \quad \text{for } s_1 - c_i \geqslant C_1 t\,, \quad \text{(A.9)}$$

where the log term has been included because with it $\partial\phi_c/\partial r_1$ may still be $O[t^3/(s_1-c_i)^2]$, as is required by (A.8). This assumption is confirmed for the particular case of a body of revolution by Ward's quasi-cylinder solution[7].

In the region $s_1 - c_i \leqslant C_1 t$, $(\Delta\phi_a)_i$ is $O\{t^2\log[t/(s_1-c_i)]\}$, and by the arguments of section 2 $[\Delta(\phi_a+\phi_b+\phi_c)]_i$ is $O(t^2)$, so that we may write

$$[\Delta(\phi_b+\phi_c)]_i \text{ is } O\left[t^2\log\frac{t}{s_1-c_i}\right] \quad \text{for } s_1 - c_i \leqslant C_1 t \ldots \quad \text{(A.10)}$$

We now examine the case where the $a_n(s_1)$ themselves are continuous, but the derivatives $a_n'(s_1)$ are discontinuous. The integrals in (A.3) and (A.4) are then ordinary Riemann integrals, and integrating these by parts we have

$$\phi_a + \phi_b = -\int_{u=0-}^{s_1-\beta r_1}\left[(s_1-u)\cosh^{-1}\frac{s_1-u}{\beta r_1} - \sqrt{\{(s_1-u)^2-\beta^2 r_1^2\}}\right] da_0'(u)$$

$$+ \frac{\cos\theta_1}{r_1}\int_{u=0-}^{s_1-\beta r_1}\sqrt{\{(s_1-u)^2-\beta^2 r_1^2\}}\, da_{iR}'(u) + \ldots , \quad \text{(A.11)}$$

and $$\phi_a = -\int_{u=0-}^{s_1} \left[(s_1 - u) \log \frac{2(s_1 - u)}{\beta r_1} - (s_1 - u) \right] da_0'(u)$$

$$+ \frac{\cos \theta_1}{r_1} \int_{u=0-}^{s_1} (s_1 - u) \, da_{IR}'(u) + \ldots \qquad \ldots \text{(A.12)}$$

Proceeding as before we find that

$$(\Delta\phi_b)_i \text{ is } O\left[\frac{t^4}{s_1 - c_i}\right] \quad \text{for } s_1 - c_i \geqslant C_2 t , \qquad \ldots \text{(A.13)}$$

and assume that

$$(\Delta\phi_c)_i \text{ is } O\left[\frac{t^4}{s_1 - c_i} \log \frac{t}{s_1 - c_i}\right] \quad \text{for } s_1 - c_i \geqslant C_2 t , \qquad \ldots \text{(A.14)}$$

and find that

$$[\Delta(\phi_b + \phi_c)]_i \text{ is } O\left[t^2(s_1 - c_i) \log \frac{t}{s_1 - c_i}\right], \text{ i.e., is } O(t^3), \text{ for } s_1 - c_i \leqslant C_2 t . \qquad \ldots \text{(A.15)}$$

So far we have only considered the accuracy of our asymptotic forms in the light of the linearized equation. However it is known that when no discontinuities are present the error of the slender body approximation, considered as a solution of the linearized equation, is $O(t^4 \log t)$, while the error of this approximation, considered as a solution of the exact equation, is $O(t^4 \log^2 t)$. Accordingly we multiply the order terms in (A.7), (A.9), (A.13) and (A.14) by $\log \{t/(s - c_i)\}$ and assume that the errors are then those of our approximation as a solution of the exact equation.

A.2. *The Effect of a Discontinuity in $S''(\sigma)$ on the Drag of a Smooth Pointed Body.*—A discontinuity in profile curvature and $S''(\sigma)$, (profile slope and $S'(\sigma)$ being continuous), is a particular case of a discontinuity in the $a_n'(s_1)$, (the $a_n(s_1)$ being continuous). For convenience let us take our origin at the discontinuity in question, ($c_i = 0$). We wish to examine the effect of neglecting ($\phi_b + \phi_c$) in the integral, taken along a line μ = constant on the body,

$$I(\mu) = \int_0^{l_1} \frac{\partial}{\partial \sigma} (\phi_a + \phi_b + \phi_c) S'(\sigma) \, d\sigma , \qquad \ldots \text{(A.16)}$$

where l_1 is the length of the body downstream of the discontinuity, and is taken as $O(1)$, which clearly gives the maximum possible error. If we introduce

$$F(\sigma, \mu) = \int_0^{\sigma} \frac{\partial}{\partial \sigma} (\phi_a + \phi_b + \phi_c) \, d\sigma$$

$$= \phi_a(\sigma, \mu) + \phi_b(\sigma, \mu) + \phi_c(\sigma, \mu) - \phi_a(O, \mu) - \phi_b(O, \mu) - \phi_c(O, \mu)$$

$$- \int_0^{\sigma} \frac{\partial}{\partial \lambda} (\phi_a + \phi_b + \phi_c) \frac{d\lambda}{d\sigma} \, d\sigma , \qquad \ldots \text{(A.17)}$$

(A.16) may be written

$$I(\mu) = F(l_1, \mu) S'(l_1) - \int_0^{l_1} F(\sigma, \mu) \, S''(\sigma) \, d\sigma . \qquad \ldots \text{(A.18)}$$

It may be noted that ϕ_a is continuous in this case, so that the forms (A.16) and (A.18) are wholly equivalent even when $(\phi_b + \phi_c)$ is neglected. Now since the distance between successive discontinuities is $O(1)$, and l_1 is $O(1)$, $(\phi_b + \phi_c)$ is $O(t^4 \log^2 t)$ at $\sigma = l_1$ and 0.

Also $\partial\phi/\partial\lambda$ is known from the boundary condition. Hence the dominant term in the error due to neglecting $(\phi_b + \phi_c)$ in (A.18) is

$$\int_0^{l_1} [\phi_b(\sigma, \mu) + \phi_c(\sigma, \mu)] S''(\sigma)\, d\sigma = O\left\{\int_0^{C_2 t} t^3 |S''(\sigma)|\, d\sigma + \int_{C_2 t}^{l_1} \frac{t^4}{\sigma} \log^2 \frac{t}{\sigma} |S''(\sigma)|\, d\sigma\right\}$$

$$= O(t^6 \log^3 t) \quad \ldots \quad (A.19)$$

A.3. *The Error Due to the Asymptotic Expressions for F_i and G.*—Finally we justify the substitution of the asymptotic form of F_i into the equation (6.8) of the main text. The $a_n(s_1)$ themselves are now discontinuous. Essentially we have to justify the neglect of $(\phi_b + \phi_c)$ in the integral

$$\int_{\sigma = 0+}^{l_1 +} (\phi_a + \phi_b + \phi_c)\, dS'(\sigma) \, .$$

The error due to this step is

$$O\left\{\int_{\sigma = 0+}^{C_1 t} t^2 \left|\log \frac{t}{\sigma}\right| |dS'(\sigma)| + \int_{C_1 t}^{l_1 +} \frac{t^4}{\sigma^2} \log^2 \frac{t}{\sigma} |dS'(\sigma)|\right\} . \quad \ldots \quad (A.20)$$

If $(c_{i+1} - c_i)$ is always $O(1)$, as we have specified, the discontinuities in $S'(\sigma)$ will contribute only to the second integral of this expression, and their contributions will by $O(t^6 \log^2 t)$. The remainder of (A.20) is

$$O\left\{t^4\left[\int_0^t \log \frac{t}{\sigma}\, d\sigma + \int_t^{C_1 t} \log \frac{\sigma}{t}\, d\sigma\right] + t^6 \int_{C_1 t}^{l_1} \frac{1}{\sigma^2} \log^2 \frac{t}{\sigma}\, d\sigma\right\} = O(t^5) \, . \quad \ldots \quad (A.21)$$

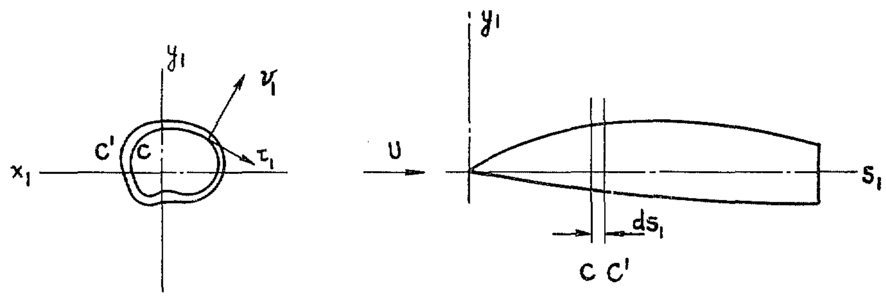

FIG. 1. Notation.

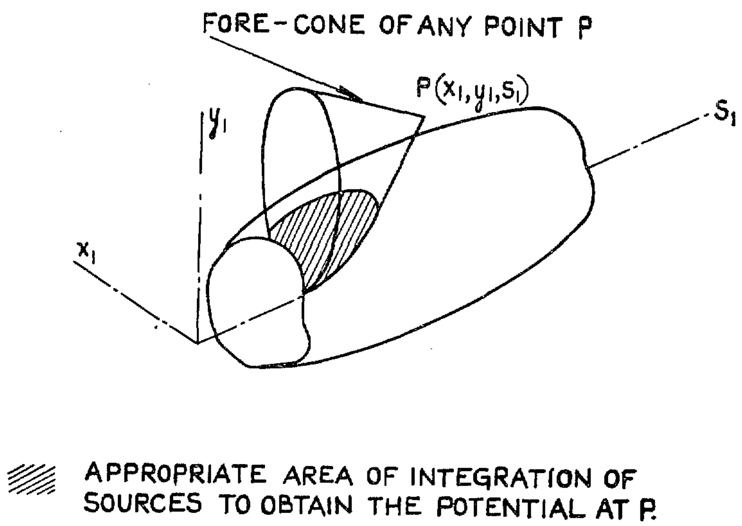

FIG. 2. Area of integration of sources.

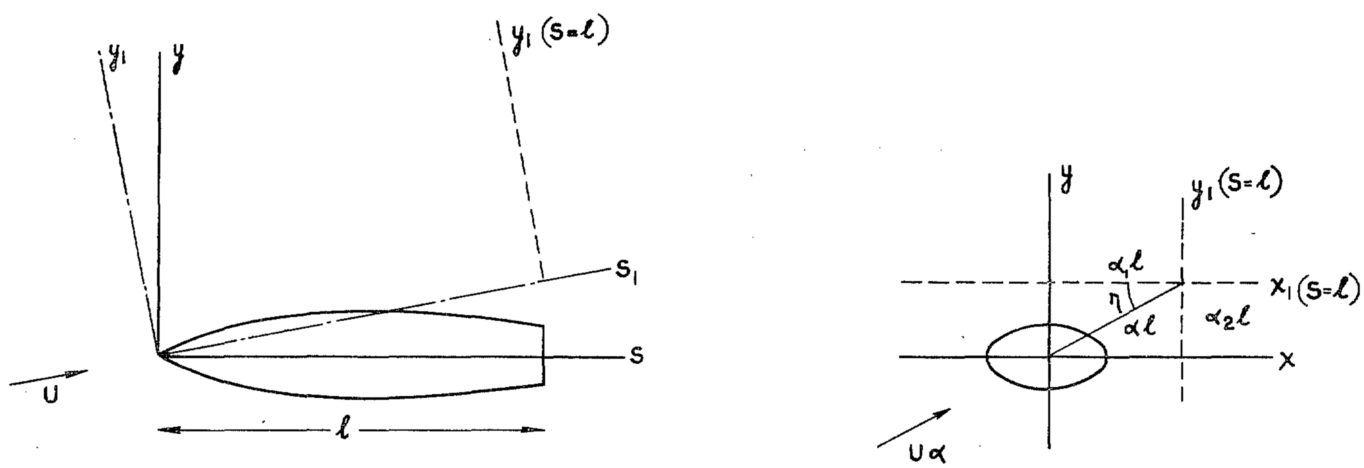

FIG. 3. Body and wind axes.

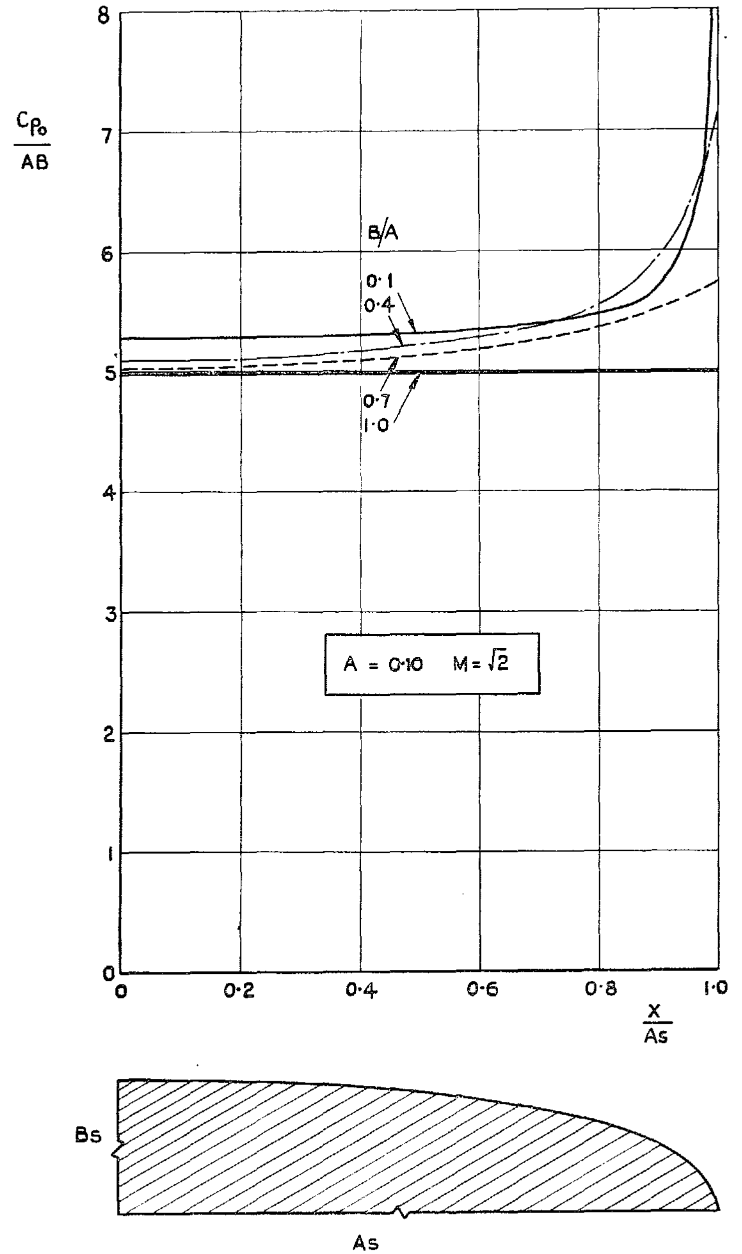

FIG. 4. Examples of the pressure distribution on elliptic cones.

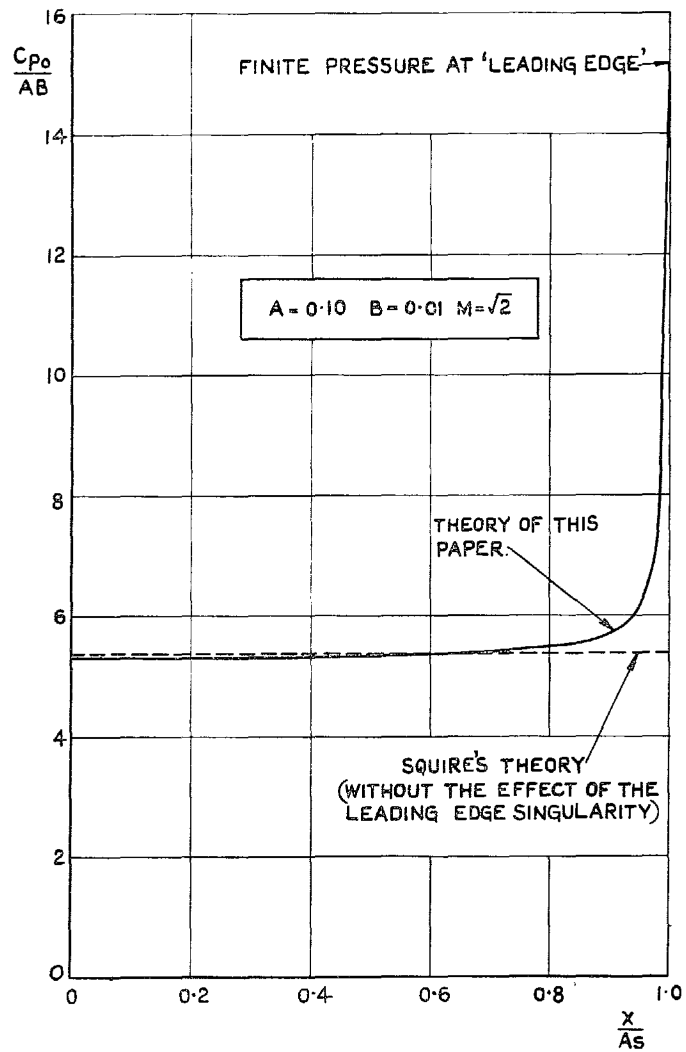

FIG. 5. Comparison with Squire's theory of the pressure on a flat elliptic cone.

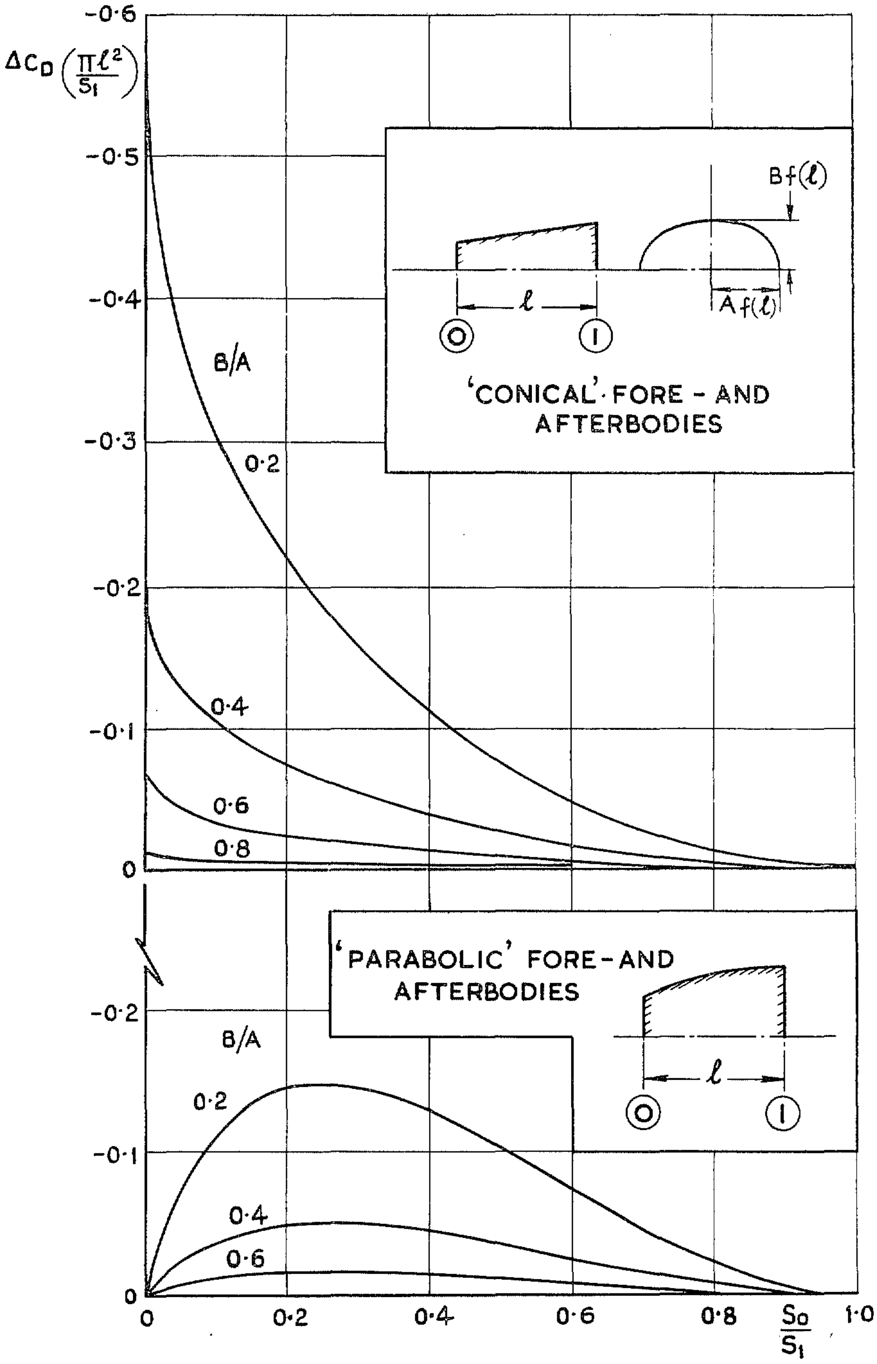

FIG. 6. Differences between the drags of bodies of elliptic and circular cross-section.

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