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Summary.—A method is developed for the calculation of the pressure distribution and the aerodynamic forces and moments acting on a wing at incidence, a wing in (steady) roll and a wing in (steady) pitch. The calculation is based on the assumption of an inviscid potential flow and is restricted to small incidence and thickness ratio, so that quadratic terms in the perturbation velocities are neglected. The results are valid, if $|1 - M^2| . A^2$ is small compared to 1, *i.e.*, either for any Mach number M for wings of a small aspect ratio A, or for any aspect ratio for sonic speeds ($M \sim 1$). The aerodynamic coefficients and stability derivatives l_p and m_q for a wing family which is described by the parameters aspect ratio A, taper ratio λ, and sweep ratio a (Fig. 8), are given in the form of charts. The calculation indicates, that the plan-form of the wing is of similar importance as regards the pressure distribution at sonic speeds as the chordwise section of a wing at subsonic speeds for wings of larger aspect ratios.

Although the calculation is based on the assumption of an inviscid flow without shock-waves, the results are thought to be useful for showing the main trends of the behaviour of a wing near the speed of sound. Plan-forms, which show rapid variations of the aerodynamic properties near Mach number $M = 1$ according to the potential theory, will have to be abandoned in favour of other planforms with more favourable characteristics.

NOTATION

V		Undisturbed velocity
x, y, z		Cartesian co-ordinates
u, v, w		Perturbation velocities, caused by the wing
ρ		Air density
M	=	V/a_∞, Mach number
p		Static pressure
I	=	$(p - p_\infty)/\rho$, Enthalpy
α		Incidence
p, q		Angular velocities about the x and y-axis

* R.A.E. Report Aero. 2439, received 11th February, 1952.

A		Aspect ratio
λ		Taper ratio
S		Wing area
s	=	$s(x)(ds/dx \geqslant 0)$ Local half-span, equation of leading edge
s_m		Maximum half-span
$r(x), s(x)$		$(ds/dx \leqslant 0)$ Equation of trailing edge
c_r		Root chord
$\bar{c}$		Mean chord
$\Lambda_l\ (\Lambda_t)$		Sweep angle of leading (trailing) edge
a	=	$\dfrac{\tan \Lambda_t}{\tan \Lambda_l} = \dfrac{\mu}{1+\mu}$ (Fig. 8)
σ	=	$\dfrac{s - \bar{s}_0}{s_n - \bar{s}_0} = \dfrac{1}{\mu}\left(\dfrac{s}{\bar{s}_0} - 1\right)$; $1 + \mu\sigma = \dfrac{s}{\bar{s}_0}$ (Fig. 8)
$\dfrac{s_m}{\bar{s}_0}$	=	$\dfrac{1-\lambda}{1-a}$, $\dfrac{s_n}{\bar{s}_0} = \dfrac{1}{1-a}$ (Fig. 8)
K, E		Elliptic integrals of the first and second kind with the parameter $k = \sqrt{(1 - r^2/s^2)}$
Z	=	$y + iz$, Complex variable
χ	=	$\sin^{-1}\sqrt{\left(\dfrac{s^2 - Z^2}{s^2 - r^2}\right)}$, Auxiliary variable
l	=	$\dfrac{\Lambda p}{\frac{1}{2}\rho V^2}$, Non-dimensional load distribution
$h\,.\,\bar{c}$		Distance of the aerodynamic centre behind the apex

Aerodynamic coefficients (signs are indicated in Fig. 1b)

$$C_L = \frac{L}{\frac{1}{2}\rho V^2 S}$$

$$C_M = \frac{M}{\frac{1}{2}\rho V^2 S\bar{c}}$$

$$l_p = \frac{\partial L}{\partial(ps_m/V)}\,\frac{1}{\rho V^2 S\,.\,s_m}$$

$$z_q = \frac{\partial Z}{\partial(q\bar{c}/V)}\,\frac{1}{\rho V^2 S}$$

$$m_q = \frac{\partial M}{\partial(q\bar{c}/V)}\,\frac{1}{\rho V^2 S\bar{c}}$$

$$h = -\frac{C_M}{C_L} = -\frac{M}{\bar{c}L}$$

1. *Introduction.*—The theory of the lifting surface has been developed by various authors to such an extent, that we are now in a position to calculate the pressure and the lift distribution over a wing at either subsonic or supersonic speeds. The calculation is based on the assumption of an inviscid flow without any vorticity outside the wake. Furthermore it is assumed, that the disturbances, which are caused by the presence of the wing in a parallel flow, are small enough, so that it is sufficient to consider only linear terms in these perturbation velocities. Unfortunately the theory of the lifting surface, as it has been developed so far for subsonic speeds, breaks down, if the Mach number tends to 1. In a similar way many methods developed for supersonic flow, which are usually based on certain facts, which hold only in supersonic flow, lose their meaning near the speed of sound.

Thus it seems only natural to try another approach to the problem by starting from a wing flying at sonic speed and to extend the method later on to cover both subsonic and supersonic speeds. The present theory applies to wings of any aspect ratio moving with a speed very close to the speed of sound, and to wings of small enough aspect ratio at speeds remote from $M = 1$. This assumption permits the splitting up of our three-dimensional problem into two problems, namely to find a solution of the Laplace equation in two dimensions y and z (spanwise and normal to the wing span and the flight direction) and secondly to connect all these solutions by means of an integral equation in x. In other words the wing is divided into strips extending spanwise (not chordwise, as is usual in the theory of a wing of a not too small aspect ratio at subsonic speeds) and the solutions for each strip are combined afterwards to give the solution for the wing.

The justification for this method lies in the fact that in the three-dimensional Laplace equation for a steady flow the term containing the second derivative with respect to x, has the factor $(1 - M^2)$. If non-dimensional co-ordinates are introduced the assumption on which our theory is based may be put as

$$|1 - M^2|A^2 << 1$$

(A = aspect ratio). Thus all our results apply either for any wing at sonic speeds ($M \sim 1$) or for a wing of a small enough aspect ratio A at any speed.

After outlining the fundamentals of the theory in section 2, we shall consider the flat wing of an arbitrary plan-form at incidence in section 3, the wing in steady roll in section 4, and the wing in steady pitch in section 5. These results are applied in section 6 to the calculation of the lift and pitching moment due to incidence, the damping moment in roll (l_p) and the force (z_q) and pitching moment (m_q) due to pitch for a family of wings, which depends on three parameters (Fig. 8). We choose the aspect ratio A, the taper ratio λ and the ratio $a = \tan \Lambda_t / \tan \Lambda_l$ of the trailing and leading-edge sweep (Λ_t and Λ_l respectively), to describe the family, because with their help, the results for the forces and moments can be presented in a most convenient form (*see* Figs. 10 to 18).

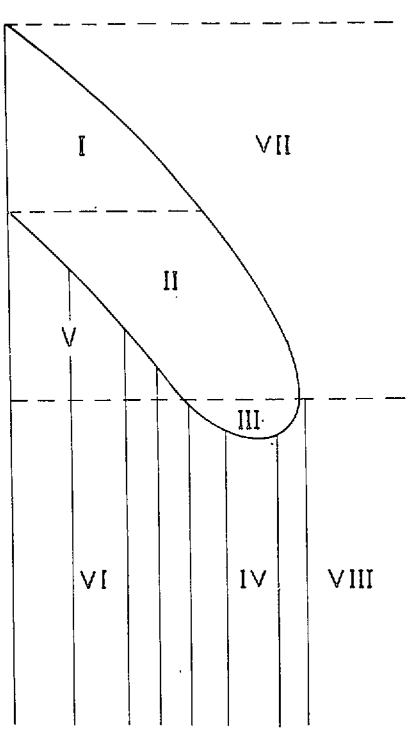
In order to show the form of the pressure distributions to be expected at sonic speeds, a pressure plotting is given for two particular wings (section 6.4, Figs. 19, 20 and 21). It shows the large effect of small changes in the wing plan form. As can be seen from the figures, a fairly slight rounding off of the leading edge results in a fairing of the pressure distribution.

It may be pointed out, that a theory for a wing of small aspect ratio was first developed by R. T. Jones[1], although on slightly different lines. It covered only the case of a Delta wing and similar plan forms, where a spanwise section consisted of only one part. It was applicable also to a plan form, where the local span reached its maximum in front of the foremost point of the trailing edge. Jones' theory was applied by Ribner[2] to determine most of the (steady) stability derivatives for such plan forms. Later on Heaslet and Lomax[4,5,6] extended the analysis to cover also the case of a spanwise section consisting of two separate parts, but they succeeded only in calculating the load over a particular wing plan form, which is shaped in such a way, that the load distribution over the middle part of the wing is constant, so that no wake is developed there, but only from the outer parts of the wing (compare section 3.4).

Robinson (revised version of Ref. 7) treated the case of a wing of an arbitrary plan form at incidence. Unfortunately his numerical results, which are given only for the fully tapered wing do not agree with ours, probably due to an inaccuracy in the numerical calculation*.

The main features of the pressure and velocity field around a flat wing at incidence α, moving at a Mach number $M = 1$, can be summarised as follows. The incoming flow is undisturbed in front of the Mach wave (Mach plane), which extends from the apex of the wing. The pressure field produces a load on the front part of the wing, which is bounded by the Mach wave (plane) extending from the wing tip (region I and II in the adjoining figure). There is no load on the rear part of the wing behind the tip Mach wave and the pressure is undisturbed everywhere behind this Mach wave. Between the two Mach waves the pressure varies, but is equal to the undisturbed pressure everywhere inside the plane $z = 0$ except for the wing area.

Incidence case. Plane $z = 0$



Region	Load	Down-wash (Depending on)	Side-wash (Depending on)	H(s)	
I	>0	const.	x, y	1	Wing
II	>0	const.	x, y	≠ 1, ≠ 0	
III	0	const.	y	0	
IV	0	const.	y	0	Wake
V	0	x, y	y	≠ 1, ≠ 0	
VI	0	y	y	0	
VII	0	x, y	0	≠ 1, ≠ 0	
VIII	0	y	0	0	

The pressure remains undisturbed except in regions I and II.

In front of the wing and beside the wing an upwash field is induced. On the wing itself and this part of the wake, which extends from the outer part of the wing, the downwash is constant ($w = V\alpha$), whereas in region V, the downwash depends on y and varies with x. In region VI, the downwash depends on the spanwise co-ordinate y and not on the downstream co-ordinate x. Similarly, the sidewash remains unchanged in region IV, V and VI of the wake; it depends only on y. The sidewash is discontinuous between the two faces of the wake, so that the wake contains a vortex field. The same applies for the wing area I + II + III. We have a sidewash between the two Mach waves, which extend from the apex and the wing tips, although the sidewash vanishes in the plane $z = 0$ outside the wing and wake. The perturbation velocities which have been created by the wing along the Mach wave, extending from the tip, remain unchanged in the downstream direction.

* Since the completion of this work as R.A.E. Report No. Aero 2439, a paper by H. Mirels[10] has appeared, which covers similar ground, although the numerical results cover only plan forms with constant chord. The author is indebted to Mr. Mirels for pointing out an error, which occurred in equation (72) in the original version. The error and the results in figures 14 and 15 have been put right.

2. *Principles of the Theory*.—2.1. *The Equations of Motion*.—We consider a thin wing at a small incidence in a steady parallel flow and apply the linearized potential theory, *i.e.*, we assume that all perturbation velocities, u, v, w, which are caused by the presence of the wing in the parallel flow V (which may be parallel to the x-direction in a cartesian co-ordinate system, *see* Fig. 1), are so small that only linear terms in u, v, w need be considered. Since the flow is irrotational outside the wake, a velocity-potential function exists so that

$$u = \frac{\partial \phi}{\partial x}, \qquad v = \frac{\partial \phi}{\partial y}, \qquad w = \frac{\partial \phi}{\partial z}. \quad \ldots \quad (1)$$

The Euler equations, which connect the velocities and the pressure p or the enthalpy I, can be linearized as:

$$\left.\begin{aligned} V\frac{\partial u}{\partial x} &= V\frac{\partial^2 \phi}{\partial x^2} = -\frac{\partial I}{\partial x} \\ V\frac{\partial v}{\partial x} &= V\frac{\partial^2 \phi}{\partial x\,\partial y} = -\frac{\partial I}{\partial y} \\ V\frac{\partial w}{\partial x} &= V\frac{\partial^2 \phi}{\partial x\,\partial z} = -\frac{\partial I}{\partial z} \end{aligned}\right\} \quad \ldots \quad (2)$$

where

$$I = I_\infty + \int_\infty^p \frac{dp}{\rho} = I_\infty + \frac{p - p_\infty}{\rho_\infty} \quad \ldots \quad (3)$$

since the air density ρ in the denominator may be replaced by its value ρ_∞ in the undisturbed flow within the accuracy of the linearized theory. In the same way we have for the speed of sound a:

$$a^2 = \frac{dp}{d\rho}$$

and thus

$$d\rho = \frac{dp}{a^2} \simeq \frac{dp}{a_\infty{}^2} = \frac{\rho_\infty}{a_\infty{}^2}\,dI\,.$$

The continuity equation can be linearized as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{M^2}{V}\frac{\partial I}{\partial x} = 0 \quad \ldots \quad (4)$$

where $M = V/a_\infty$ is the Mach number of the undisturbed flow. The integral of the Euler equations (2) is *Bernoulli's equation*

$$I + V\frac{\partial \phi}{\partial x} = I + Vu = I_\infty\,. \quad \ldots \quad (5)$$

We introduce equation (1) and the first equation (2) in equation (4) and obtain

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi - M^2\frac{\partial^2}{\partial x^2}\phi = 0 \quad \ldots \quad (6)$$

By differentiating this equation with respect to x and using (5) we have

$$(1 - M^2)\frac{\partial^2 I}{\partial x^2} + \frac{\partial^2 I}{\partial y^2} + \frac{\partial^2 I}{\partial z^2} = 0\,. \quad \ldots \quad (7)$$

Now, we introduce non-dimensional co-ordinates, referring x to the mean chord $\bar{c}$ and y and z to the span $b = S/\bar{c} = A\bar{c}$ (S = wing area, A = aspect ratio) and obtain (with $x' = x/\bar{c}$, $y' = y/(A\bar{c})$, $z' = z/(A\bar{c})$):

$$(1 - M^2)A^2\frac{\partial^2 I}{\partial x'^2} + \frac{\partial^2 I}{\partial y'^2} + \frac{\partial^2 I}{\partial z'^2} = 0 \quad \ldots \quad (8)$$

and the corresponding equation for the velocity potential ϕ. We shall consider only such problems, for which

$$|1 - M^2|A^2 << 1 \quad \ldots \quad (9)$$

is sufficiently small, so that the first term in the differential equation may be safely neglected compared with the other two*. Thus the motion of a wing of a small aspect ratio A at any Mach number or the motion of a wing of any aspect ratio at a Mach number M, which is very near to 1, is described by the differential equation:

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = 0 \quad \ldots \quad (10)$$

or

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)I = 0 . \quad \ldots \quad (11)$$

The variable x occurs in these equations only as a parameter.

2.2. *Boundary Conditions.*—The solutions of the differential equations (10) or (11) have to satisfy the following conditions:

(*a*) At a great distance from the wing ($y^2 + z^2 \to \infty$) the velocity potential ϕ and the ' acceleration potential ' $I - I_\infty$ must vanish.

(*b*) Along the surface of the wing, which may be given by the equation

$$S(x,y,z) = 0 \quad \ldots \quad (12)$$

(in case of a thick wing, both the upper and the lower surface have to be defined), the component v_n of the velocity normal to the surface S must either vanish, or is a function

$$v_n = V \, . \, g(x,y)$$

which is prescribed by the movement of the wing (*e.g.*, in roll or pitch):

$$[(V + u) \, . \, S_x + vS_y + wS_z] \frac{1}{\sqrt{(S_x^2 + S_y^2 + S_z^2)}} = V \, . \, g(x,y) \, . \quad \ldots \quad (13)$$

Here suffixes mean partial derivatives. Strictly speaking this condition (13) should be satisfied on the surface $S = 0$ of the wing, but it is well known that within the accuracy of a linearized theory, it is often sufficient, to satisfy this condition (13) inside the plan form of the wing in the plane $z = 0$ (since for points on the wing z is always small compared to $A\bar{c}$).

2.3. *Reduction to a Two-dimensional Problem* (*velocity potential*).—Since the differential equation (10) for the velocity potential ϕ depends only on the two variables y and z, we try to establish a two-dimensional problem with two-dimensional boundary conditions, which is equivalent to our three-dimensional problem.

We consider a cylinder with its axis parallel to the x-axis, the contour of which may be given by the equation

$$\bar{S}(y,z) = 0 . \quad \ldots \quad (14)$$

* In this connection the question arises, whether such an approach is justified, since it may be necessary to retain second-order terms in the potential equation if equation (9) holds. Experience shows that for a three-dimensional flow the linearized equation (10) has sensible solutions satisfying all boundary conditions (and the assumptions implied in neglecting the first term in (7)), provided the incoming flow is merely deflected and the pressure is an anti-symmetrical function of z. But there are no solutions (producing finite values for the pressure) as soon as thickness is involved. It is believed that the present calculations give a reasonable solution for the cases treated here, but that second-order terms may be required in order to allow for thickness effects near sonic speeds.

Along this contour the normal component may be a prescribed function $V \cdot \bar{g}(y)$:

$$\left\{ (v\bar{S}_y + w\bar{S}_z) \frac{1}{\sqrt{(\bar{S}_y^2 + \bar{S}_z^2)}} \right\}_{\bar{S}=0} = V \cdot \bar{g}(y)$$

or, if $\bar{\phi}$ denotes the potential of the two-dimensional flow:

$$\frac{1}{V}\left(\frac{\partial \bar{\phi}}{\partial y} \cdot \bar{S}_y + \frac{\partial \bar{\phi}}{\partial z} \cdot \bar{S}_z\right)_{\bar{S}=0} = \bar{g}(y) \left\{ \sqrt{(\bar{S}_y^2 + \bar{S}_z^2)} \right\}_{\bar{S}=0} . \quad .. \quad .. \quad .. \quad (15)$$

The boundary condition (13) for the three-dimensional problem may be written as

$$\frac{1}{V}\left(\frac{\partial \phi}{\partial z} S_z + \frac{\partial \phi}{\partial y} S_y\right)_{S=0} = \left\{ - S_x + g(x,y) \sqrt{(S_x^2 + S_y^2 + S_z^2)} \right\}_{S=0} \quad .. \quad .. \quad (16)$$

since u can be neglected compared with V.

By identifying (14) with (12) for any fixed value of x and by comparing (15) with (16) we can see, that our three-dimensional wing flow problem can be reduced to the following two-dimensional problem:

For any fixed value of x, we have to find a solution of the two-dimensional Laplace equation (10), which vanishes at a great distance from the surface of the wing ($y^2 + z^2 \to \infty$) and which has the following prescribed values $v_n = \partial\phi/\partial n = V \cdot \bar{g}$ along the contour $S = 0$ (or $z = 0$ respectively) with:

$$\bar{g} = \left\{ g(x,y) \frac{\sqrt{(S_x^2 + S_y^2 + S_z^2)}}{\sqrt{(S_y^2 + S_z^2)}} - \frac{S_x}{\sqrt{(S_y^2 + S_z^2)}} \right\}_{S=0} . \quad .. \quad .. \quad .. \quad (17)$$

Here $g(x,y)$ is the (non-dimensional) normal component of the velocity as it is prescribed along the wing surface $S = 0$ by the movement of the wing. For the wing at a constant incidence we have $g \equiv 0$.

This two-dimensional problem can be solved according to well-known methods (compare the theory of the analytic functions of a complex variable) for any fixed value x, which leads to the potential function ϕ for our three-dimensional problem. The pressure distribution over the wing can then be obtained by means of Bernoulli's equation (5).

It may be pointed out, that the solution of the above-mentioned two-dimensional problem is not always unique. In certain cases solutions exist for which the normal derivative along the contour is zero and any such function may be added to the solution of the problem. These additional solutions often contain such singularities on the surface of the wing which produce infinite forces and therefore must be excluded. In other cases the nature of the singularity, which is admissible (*e.g.*, at the sharp leading edge of a wing), is known, which helps to find the unique solution of the physical problem.

Another condition, which is useful in this connection, is the so-called *non-vorticity condition*: the integral over the tangential velocity v_t along any closed path of integration, which does not intersect the wake, must vanish:

$$\oint v_t \, ds = 0 . \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (18)$$

This means that the flow outside the wing and outside the wake does not contain any vorticity. It is obviously satisfied if ϕ is a unique function of y and z everywhere outside and on the surface of the wing.

Another condition, in order to obtain a unique solution, is the *Kutta-Joukowsky condition*. It has to be introduced if the plane $x =$ const where the two-dimensional problem is solved, contains points which belong to the trailing edge of the wing. It can best be formulated in the form: The pressure must remain finite (or bounded) along any sharp trailing edge of the wing. It will be seen that this is equivalent to the condition that there cannot be a pressure difference between both faces of the wake.

In certain cases, *e.g.*, for a unisectional wing (compare Fig. 6) no wake effects occur and the potential ϕ can very usefully be applied.

However the method runs into serious difficulties. It is directly applicable only to such (spanwise) sections of the wing which do not contain or touch a part of the wake (*see* Fig. 2, section AA). If a part of the wake occurs in the plane $x =$ const where the solution $\bar{\phi}$ must be determined, we have to admit discontinuities of $\bar{\phi}$ and the velocities along the wake (since the wake contains vortices) but these must be chosen in such a way that no pressure difference occurs between the lower and the upper surface of the wake.

Because of the serious restrictions in the applicability of this method, which is based on the use of the velocity potential ϕ, to the incidence case, we shall describe now another approach to this problem, which is based on the use of the enthalpy I—sometimes called the ' acceleration ' potential function (because of (2)).

2.4. *Reduction to a Two-dimensional Problem* (*acceleration potential*).—The enthalpy I is continuous everywhere outside the wing, and does not show directly the existence of a wake. We make use of this fact by identifying the potential $\bar{\phi}$ of the above mentioned two-dimensional problem with the acceleration potential $I \, . \, \bar{c}/V$ and the contour $S = 0$ of equation (14) with the contour $S = 0$ of equation (12) for any fixed value of x. By differentiating equation (13) with respect to x and using (2) we obtain the following boundary condition for I:

$$\left(\frac{\partial}{\partial z}\left(\frac{I}{V^2}\right) S_z + \frac{\partial}{\partial y}\left(\frac{I}{V^2}\right) S_y\right)_{S=0} =$$

$$(S_{xx})_{S=0} - \frac{\partial}{\partial x}\left\{g(x,y)\sqrt{(S_x^2 + S_y^2 + S_z^2}\right\}_{S=0} - \left(S_{zx}\frac{w}{V} + S_{yx}\frac{v}{V}\right)_{S=0} . \qquad (19)$$

Unfortunately this condition contains the velocities v and w, but fairly often, *e.g.*, in the important case of a flat wing (δ = dihedral)

$$S(x,y,z) \equiv z + \alpha(x - x_0) + \delta\,|y| \qquad (20)$$

the mixed derivatives S_{zx} and S_{yx} vanish and (19) becomes fairly simple. By comparing (19) and (15) our three-dimensional problem can be reduced to the following two-dimensional problem:

For any fixed value x we have to find a solution of the Laplace equation (11), which vanishes at a great distance from the contour $S = 0$ ($y^2 + z^2 \rightarrow \infty$) and which has the following prescribed values for the normal derivative

$$\frac{\partial}{\partial n}\left(\frac{I\bar{c}}{V}\right) = V \, . \, \bar{g}$$

along the contour $S = 0$ (or $z = 0$ respectively), where

$$\bar{g} = \left[\frac{\bar{c} \, . \, S_{xx}}{\sqrt{(S_y^2 + S_z^2)}} - \frac{\bar{c}}{\sqrt{(S_y^2 + S_z^2)}}\frac{\partial}{\partial x}\left\{g(x,y)\,\sqrt{(S_x^2 + S_y^2 + S_z^2)}\right\}\right.$$

$$\left. - \frac{\bar{c}}{\sqrt{(S_y^2 + S_z^2)}}\left\{\frac{w}{V}S_{zx} + \frac{v}{V}S_{yx}\right\}\right]_{S=0} . \qquad (21)$$

The solution I must be continuous and bounded everywhere outside and on the wing surface $S = 0$ (or $z = 0$ respectively), possibly with the exception of the points, which correspond to a sharp nose. Here only such singularities are admitted, which after the integration over the wing surface, produce finite forces and moments.

Since (21) represents only a condition for the x-derivative of the original boundary condition (13), we have to make sure by an integration over x, that this original condition (13) is also satisfied. For a flat wing (equation (20)) this leads by means of equation (2) to :

$$\left\{ S_z \int_{-\infty}^{x} \frac{\partial}{\partial z}\left(\frac{I}{V^2}\right) dx + S_y \int_{-\infty}^{x} \frac{\partial}{\partial y}\left(\frac{I}{V^2}\right) dx \right\}_{S=0} = \left\{ S_x - g(x,y)\sqrt{(S_x^2 + S_y^2 + S_z^2)} \right\}_{S=0}. \qquad (22)$$

Here the integration with respect to x must be performed for constant values of y and z (along a streamline), starting from a point $x = -\infty$ far ahead of the wing, where all perturbation velocities are zero, up to a point (x,y,z) on the surface $S = 0$ (or $z = 0$ respectively).

As will be seen later on, both conditions (21) and (22) can be satisfied, since the solution for (21) usually contains an arbitrary parameter (depending on x), which can be determined in such a way, that (22) is also satisfied.

In order to obtain a unique solution for the two-dimensional problem with the boundary condition (21) we have to introduce again, apart from conditions on the nature of the occurring singularities which were mentioned above, the *non-vorticity condition* and the *Kutta-Joukowsky condition.* The former was already explained in equation (18) and the latter will be applied in the form, that the pressure must remain bounded along the trailing edge of the wing, so that no flow round the sharp edge takes place and the flow can smoothly leave the trailing edge.

In the next section we shall apply this method to the problem of a flat wing of thickness ratio zero at incidence or in roll or in pitch.

3. *The Flat Wing of Thickness Ratio Zero at Incidence.*—We consider a flat wing with the thickness ratio zero at a small incidence α. The surface of this wing is defined by

$$S \equiv z + \alpha(x - x_0) = 0 \qquad (23)$$

(*see* Fig. 2). Since we intend to consider any plan form of this wing, including a swallow-tail wing, where the wake interferes with the pressure distribution on the wing, we prefer the use of the enthalpy I to the use of ϕ. Thus we have to find a solution of the Laplace equation

$$I_{yy} + I_{zz} = 0 \qquad (11)$$

which vanishes for $y^2 + z^2 \rightarrow \infty$ and satisfies the boundary condition inside the wing plan form for $z = 0$:

$$\bar{c}\left\{\frac{\partial}{\partial z}\left(\frac{I}{V^2}\right)\right\}_{z=0} = \bar{g} = -\bar{c}\frac{\partial}{\partial x}\left\{g(x,y)\sqrt{(1 + \alpha^2)}\right\}. \qquad (24)$$

For the wing at *incidence* α, the normal component along the wing surface is zero

$$g = 0\,, \qquad \bar{g} = 0\,. \qquad (25a)$$

For the wing in *steady roll* (starboard wing going down if $p > 0$) we would have

$$g = -\frac{py}{V}\,, \quad \bar{g} = 0 \qquad (25b)$$

and for the wing in *steady pitch* (nose going up for $q > 0$) :

$$g = -\frac{qx}{V}\,, \quad \bar{g} = -\frac{q\bar{c}}{V}\,. \qquad (25c)$$

Equation (22) becomes now

$$\left(\int_{-\infty}^{x} \frac{\partial}{\partial z}\left(\frac{I}{V^2}\right) dx\right)_{z=0} = \alpha - g(x,y)\sqrt{(1 + \alpha^2)} = \alpha - g(x,y) \qquad (26)$$

for points $(x, y, 0)$ inside the wing plan form.

3.1. *Pressure Distribution for the Front Part of the Wing.*—First we consider the front part of the wing (section A in Fig. 2) where every section of the wing consists only of one part $-s \leqslant y \leqslant s$. We consider the incidence case ($\bar{g} = 0$) and have to find a solution I of the Laplace equation (11), for which the normal derivative $\partial I/\partial z$ vanishes along the section $y^2 < s^2$ of the y-axis $z = 0$, which is the cross-section of the wing (Fig. 3).

As is well known, any analytical function $f_1 + if_2$ of the complex variable

$$Z = y + iz \qquad (27)$$

is a solution of the Laplace equation (11). Thus the real part (abbreviation $\mathscr{R}$) of the function

$$f_1 + if_2 = \frac{isZ}{(s^2 - Z^2)^{3/2}} = \frac{d}{dZ}\frac{is}{\sqrt{(s^2 - Z^2)}} \qquad (28)$$

and of its integral $is/\sqrt{(s^2 - Z^2)}$ are solutions of (11). We put

$$\frac{I}{V^2} = -\,h(x)\,\mathscr{R}\left\{\frac{s}{\sqrt{(s^2 - Z^2)}}\right\} \qquad (29)$$

where $h(x)$ is a real function of the 'parameter' x. This function I vanishes for $|Z|^2 = y^2 + z^2 \rightarrow \infty$ and has the appropriate singularities at the leading edge $y = \pm s$, $z = 0$*. Since

$$\frac{d}{dZ} = \frac{\partial}{\partial y} = \frac{1}{i}\frac{\partial}{\partial z} \qquad (30)$$

we have

$$\frac{\partial}{\partial z}\left(\frac{I}{V^2}\right) = h(x)\,\mathscr{R}\left\{\frac{d}{dZ}\frac{-\,is}{\sqrt{(s^2 - Z^2)}}\right\} \qquad (31)$$

and this function vanishes for any point $Z = y$ of the section $y^2 < s^2$ as can be seen from (28). Thus the solution (29) satisfies all conditions, and it can be shown (*see* Appendix I) that this solution is unique, provided that $h(x)$ can be determined in a unique way.

This is done by means of equation (26). For this purpose we have to remember that both the 'half-span' s and the factor h are functions of x. Without any loss of generality we may introduce

$$h(x) = \alpha\,.\,H(s)\,\frac{ds(x)}{dx} \qquad (32)$$

where the function $H(s)$ is to be determined by means of equation (26).

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* The sign of the square root in (28) and (29) shall be fixed in the following way. We put

$$Z - s = |Z - s| \exp i\phi_1,\ Z + s = |Z + s| \exp i\phi_2$$

where ϕ_1 and ϕ_2 vary between $-\pi$ and π. Then we have

$$\sqrt{(Z^2 - s^2)} = |Z^2 - s^2|^{1/2} \exp i\,\frac{\phi_1 + \phi_2}{2}$$

$$\sqrt{(s^2 - Z^2)} = .\,|Z^2 - s^2|^{1/2} \exp i\left(\frac{\phi_1 + \phi_2}{2} - \frac{\pi}{2}\right)$$

which means that the sign of the square root along the y-axis $z = 0$ for $y^2 < s^2$ ($\phi_1 = \pm\pi$, $\phi_2 = 0$) is positive on the upper side and negative on the lower side of the slit.

Since there is no pressure discontinuity in all planes $x = \text{const}$ ahead of the apex ($x = 0$) of the wing, we have $I = I_\infty = 0$ for $x < 0$ and we may take $x = 0$ as the lower limit of the integral in (26). In order to avoid the singularity at the leading edge, we integrate first along a streamline $y = \text{const}$, $z = \text{const} > 0$ and go to the limit $z \to 0$ after the integration. Thus we obtain from (26)

$$-\frac{w(x,y,z)}{V} = \alpha \int_0^x H(s)\, \mathcal{R} \left\{ \frac{d}{dZ} \frac{-is}{\sqrt{(s^2 - Z^2)}} \right\} \frac{ds}{dx}\, dx$$

$$= \alpha \int_0^x H(s)\, \mathcal{R} \left\{ \frac{d}{ds} \frac{iZ}{\sqrt{(s^2 - Z^2)}} \right\} \frac{ds}{dx}\, dx\,. \quad \ldots \quad (33)$$

Integration by parts leads to

$$-\frac{w(x,y,z)}{V} = \alpha H(s)\, \mathcal{R} \left\{ \frac{iZ}{\sqrt{(s^2 - Z^2)}} \right\} - \alpha H(0)\, \mathcal{R} \left\{ \frac{iZ}{-i\sqrt{(Z^2 - s^2)}} \right\}_{s=0}$$

$$- \alpha \int_0^x \frac{dH(s)}{ds}\, \mathcal{R} \left\{ \frac{iZ}{\sqrt{(s^2 - Z^2)}} \right\} \frac{ds}{dx}\, dx.$$

Along the first part of the path of integration, which is in front of the wing (*see* Fig. 2), we have $s^2 \leqslant y^2$ and $\sqrt{(s^2 - Z^2)}/i = -\sqrt{(Z^2 - s^2)}$ tends to a real value for $z \to 0$. Along the second part of the path, which is inside the wing area, we have $y^2 \leqslant s^2$ and $\sqrt{(s^2 - Z^2)}/i$ tends to an imaginary value, so that in the limit $z \to 0$ only the first part of the integral from $s = 0$ to the leading edge $s = y$ contributes to the integral, and we have the condition

$$\alpha = -\frac{w(x,y,0)}{V} = \alpha H(0) + \alpha \int_0^y \frac{dH(s)}{ds} \frac{y}{\sqrt{(y^2 - s^2)}}\, ds\,. \quad \ldots \quad (34)$$

One solution of this integral equation for $H(s)$ is given by

$$H(s) = H(0) = 1\,. \quad \ldots \quad (35)$$

This solution is unique, since any other solution must be of the form $1 + H_1(s)$, where $H_1(0) = 0$ (the second term in (34) vanishes for small values of y) and $H_1(s)$ satisfies the condition

$$\int_0^y \frac{dH_1(s)}{ds} \frac{y}{\sqrt{(y^2 - s^2)}}\, dy = 0$$

for all values of y inside the plan form ($y^2 < s_m^2$). Thus we have $H_1(s) = \text{const} = H_1(0) = 0$.

Therefore the solution for I is

$$\frac{I}{V^2} = -\alpha H(s) \frac{ds}{dx}\, \mathcal{R} \left\{ \frac{s}{\sqrt{(s^2 - Z^2)}} \right\}, \qquad (z \geqslant 0) \quad \ldots \quad (36)$$

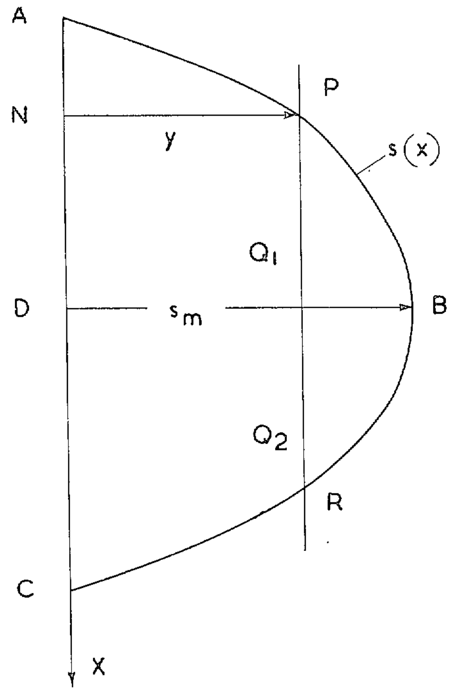
$$H(s) = 1\,.$$

The sign in (36) applies in the upper half-plane only. Along the wing section $z = 0$, $y^2 < s^2$, the function I is discontinuous, but it is continuous ($I = 0$) along the parts $y^2 > s^2$ of the axis $z = 0$. The load distribution over the wing is

$$l = \frac{\Delta p}{\frac{1}{2}\rho V^2} = -\frac{4I}{V^2} = \frac{4\alpha s}{\sqrt{(s^2 - y^2)}} H(s) \frac{ds}{dx}\,, \quad \ldots \quad (37)$$

$$H(s) = 1$$

which agrees with the results obtained by R. T. Jones[1] in a different way.

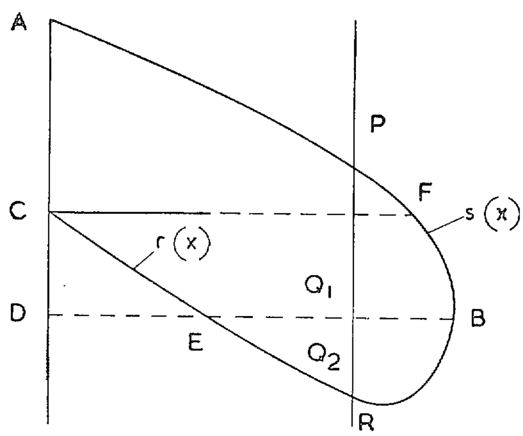


The function I has been determined in such a way that the downwash condition at any point Q of the wing section PR is satisfied, if $H(s)$ at P is properly determined (equation (34)). Thus we satisfy the boundary condition over the whole wing by determining H along the leading edge AB, where $ds/dx \geqslant 0$. The load is then known at each spanwise section PN up to BD, where it tends to zero since $ds/dx = 0$.

This process tells us nothing about $H(s)$ for spanwise sections behind BD. To get this, we use the Kutta-Joukowsky condition, that the load must be zero along the trailing edge BC, where $ds/dx \leqslant 0$. But equation (37), which is the unique solution of our two-dimensional problem (*see* Appendix I), violates that condition as y tends to s unless to the rear of BD the function $H(s)$ vanishes.

There is no load beyond the maximum span:

$$\left.\begin{aligned} l = -4I = \frac{4\alpha s}{\sqrt{(s^2 - y^2)}}\, H(s)\, \frac{ds}{dx} \\ H(s) = 1,\ \text{if}\ \frac{ds}{dx} \geqslant 0; \qquad H(s) = 0,\ \text{if}\ \frac{ds}{dx} \leqslant 0 \end{aligned}\right\} \quad \ldots \quad (37)$$



In the general case, where there is a cut-out CDE in front of the maximum span BD, the form of I will be chosen (compare section 3.2), so that the same argument applies with the additional condition, that $I = 0$ along the trailing edge CE. The boundary condition at any point Q of a section PR is satisfied, if $H(s)$ at P is properly determined. It will be found, that H is 1 as before between A and F, but is different from 1 between F and B. As before we have $H = 0$ to the rear of BD.

3.2. *Pressure Distribution for the Rear Part of the Wing.*—In the rear part of the wing a spanwise section usually consists of two separate parts $-s < y < -r$ and $r < y < s$ (compare section BB in Fig. 2). In this case we have to find a function I, which is the real part of a complex analytic function of $Z = y + iz$, so that the derivative $\partial I/\partial z$ vanishes for $z = 0$ along the two sections $r^2 < y^2 < s^2$. For $y = +s$, i.e., at the leading edge, the function must behave in the same way as the solution I, given in equation (29). Along the trailing edge $y = \pm r$, the Kutta-Joukowsky condition must be satisfied, *i.e.*, the pressure must remain bounded there. We shall satisfy this condition by making I equal to the undisturbed conditions ($I_\infty = 0$) along the trailing edge, since this is the only possibility of making the pressure equal on both faces of the wake which extends for $z = 0$ between $-r$ and r.

In order to find such an analytical function, we have to use some results of the theory of the complex functions (*see* Appendix I). Here we shall give only the result and show that this solution satisfies all the boundary conditions. The uniqueness of the solution is proved in Appendix I.

We start with the derivative $\partial I/\partial z$, which is given by

$$\frac{\partial}{\partial z}\frac{I}{V^2} = - h(x)\,\mathscr{R}\left\{\frac{is\sqrt{(Z^2 - r^2)}}{(s^2 - Z^2)^{3/2}} + \varkappa(s)\,\frac{is}{\sqrt{(s^2 - Z^2)}\sqrt{(Z^2 - r^2)}}\right\}. \quad .. \quad (38)$$

The parameter $\varkappa$ is real. It depends on the ratio r/s. We introduce

$$k = \sqrt{(1 - r^2/s^2)}\,, \qquad r/s = \sqrt{(1 - k^2)} \quad .. \quad (39)$$

and choose

$$\varkappa(s) = \frac{E(k)}{K(k)} \quad .. \quad (40)$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first and second kind respectively:

$$\left.\begin{aligned} K(k) &= \int_0^{\pi/2} \frac{d\chi}{\sqrt{(1 - k^2\sin^2\chi)}} \\ E(k) &= \int_0^{\pi/2} \sqrt{(1 - k^2\sin^2\chi)}\,d\chi \end{aligned}\right\} \quad .. \quad (41)$$

Since $\varkappa$ tends to zero for $k \to 1$ or $r \to 0$ equation (38) reduces to equation (28) or (31) for $r \to 0$, so that $\partial I/\partial z$ has the required singularity at the leading edge $y = \pm s$.

Since the bracket in (38) is purely imaginary along the sections $z = 0$, $r^2 < y^2 < s^2$, the derivative $\partial I/\partial z$ vanishes there, as is required by the boundary conditions.

In order to state the sign of the square roots in (38) it is sufficient to consider the function $\sqrt{\{(s^2 - Z^2)/(Z^2 - r^2)\}}$. We put

$$Z - s = |Z - s|\exp i\phi_1, \quad Z + s = |Z + s|\exp i\phi_2$$
$$Z - r = |Z - r|\exp i\phi_3, \quad Z + r = |Z + r|\exp i\phi_4$$

where the angles ϕ_j vary between $-\pi$ and π and define

$$\sqrt{\left\{\frac{s^2 - Z^2}{Z^2 - r^2}\right\}} = \sqrt{\left\{\left|\frac{Z^2 - s^2}{Z^2 - r^2}\right|\right\}}\exp \tfrac{1}{2}i(\phi_1 + \phi_2 - \phi_3 - \phi_4 - \pi)\,. \quad .. \quad (42)$$

The enthalpy I itself is given by

$$\frac{I}{V^2} = - h(x)\,\mathscr{R}\left\{\frac{Z}{s}\sqrt{\left(\frac{Z^2 - r^2}{s^2 - Z^2}\right)} - \int_s^Z \frac{(Z^2 - \varkappa\,.\,s^2)}{\sqrt{(s^2 - Z^2)}\sqrt{(Z^2 - r^2)}}\,\frac{dZ}{s}\right\} \quad .. \quad (43)$$

as can be seen by differentiating this equation with respect to z ($\partial/\partial z = i\,d/dZ$), which leads back to equation (38). The bracket in (43) vanishes for $|Z| \to \infty$, as can be proved by expanding both terms for large values of $|Z|$.

For $Z = s$ the second term in the bracket vanishes and the first produces the required singularity at the leading edge. For $Z = r$ the bracket vanishes as can be seen, when the integral is evaluated by means of the following transformation. We introduce

$$\sin\chi = \sqrt{\left\{\frac{s^2 - Z^2}{s^2 - r^2}\right\}}\,, \qquad k^2 = 1 - \frac{r^2}{s^2} \quad .. \quad (44)$$

(χ is real for $z = 0$, $r^2 < y^2 < s^2$) and have

$$\frac{Z}{s} = \sqrt{(1 - k^2 \sin^2 \chi)}\ , \qquad \frac{s\, dZ}{\sqrt{(s^2 - Z^2)}\sqrt{(Z^2 - r^2)}} = \frac{-\, d\chi}{\sqrt{(1 - k^2 \sin^2 \chi)}}$$

and thus

$$\frac{I}{V^2} = -\, h(x)\, \mathscr{R} \left\{ \cot \chi \,.\, \sqrt{(1 - k^2 \sin^2 \chi)} + \int_0^{\chi} \sqrt{(1 - k^2 \sin^2 \chi)}\, d\chi \right.$$
$$\left. -\, \varkappa \int_0^{\chi} \frac{d\chi}{\sqrt{(1 - k^2 \sin^2 \chi)}} \right\} . \qquad (45)$$

This function is plotted in Fig. 4 against χ for various values of $\sin^{-1} k$. Along the trailing edge $y = r$, $\chi = \pi/2$, the function I tends to zero, as follows from (45) and (40). The Kutta-Joukowsky condition is satisfied. The enthalpy I becomes zero not only at the trailing edge $r = s$, but also along the entire surface of the wake ($z = 0$, $y^2 < r^2$), since an integration from $Z = r$ along the axis $z = 0$, where the integrand is purely imaginary, towards $Z = 0$ does not change the value of I. Any other choice for $\varkappa$, different from the value given in equation (40), would have resulted in a value I different from zero for $y = r$ on the upper surface of the wing and the same value with the opposite sign on the lower surface of the wing, and there would have been a pressure difference between both faces of the wake. This is the reason why the second term had to be included in equation (38) when equation (28) was generalized, although the term with $\varkappa$ vanishes for $r \to 0$.

It may be pointed out that $\partial I/\partial z$ in (38) changes its sign, if Z is replaced by $(-\bar{Z})$ (which means according to (42) that the sign of the square-root must be altered at the same time), and the function I remains unaltered, if Z is replaced by $(-\bar{Z})$. The pressure distribution is symmetrical.

Finally the function

$$h(x) = \alpha \,.\, H(s) \frac{ds}{dx} \qquad (32)$$

is determined from condition (26). The integration with respect to x is performed along a line $y = \text{const}$, $z = \text{const} > 0$, as in equation (33), which leads to the downwash:

$$-\, \frac{w(x,y,z)}{V} = \int_0^x \frac{\partial}{\partial z} \frac{I}{V^2}\, dx = -\, \alpha \int_0^x H(s)\, \mathscr{R} \left\{ \frac{is\sqrt{(Z^2 - r^2)}}{(s^2 - Z^2)^{3/2}} \right.$$
$$\left. + \frac{\varkappa \,.\, is}{\sqrt{(s^2 - Z^2)}\sqrt{(Z^2 - r^2)}} \right\} \frac{ds}{dx}\, dx \,. \qquad (46)$$

Here both r and s are functions of x. Since r and s are prescribed by the plan form, r may also be expressed as a function of s

$$r = r(s) \qquad (47)$$

and we have

$$\frac{s\sqrt{(Z^2 - r^2)}}{(s^2 - Z^2)^{3/2}} = -\, \frac{d}{ds} \sqrt{\left\{ \frac{Z^2 - r^2}{s^2 - Z^2} \right\}} - \frac{r\, dr/ds}{\sqrt{(s^2 - Z^2)}\sqrt{(Z^2 - r^2)}}$$

Thus w/V can be written as

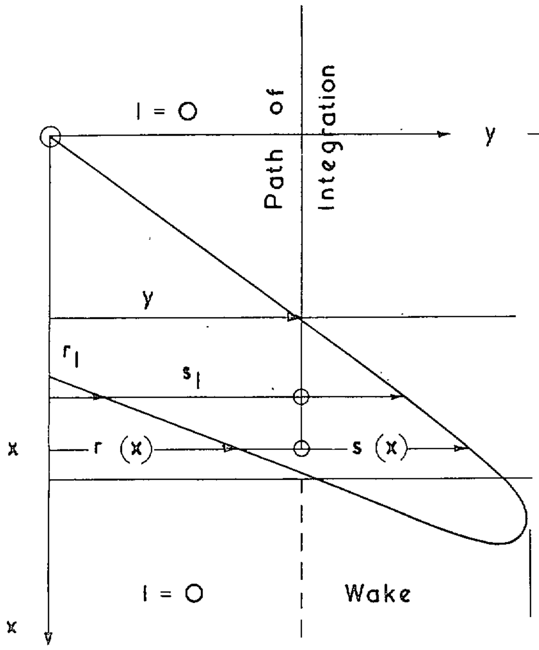
$$-\, \frac{w}{V} = \alpha \int_0^{s(x)} H(s_1)\, \mathscr{R} \left\{ i \frac{d}{ds_1} \sqrt{\left\{ \frac{Z^2 - r_1^2}{s_1^2 - Z^2} \right\}} + i \frac{r_1\, dr_1/ds_1 - \varkappa_1 \,.\, s_1}{\sqrt{(s_1^2 - Z^2)}\sqrt{(Z^2 - r_1^2)}} \right\} ds_1$$

where the integration variable s_1 occurs also in the functions $r_1 = r(s_1)$ and $\varkappa_1 = \varkappa(s_1)$. This equation is now treated in the same way as the corresponding equation (33). When integrating the first term by parts we obtain:

$$-\frac{w}{V}(x,y,z) = \alpha H(0) + \alpha H(s)\,\mathcal{R}\left\{ i\sqrt{\left(\frac{Z^2 - r^2}{s^2 - Z^2}\right)}\right\}$$

$$-\alpha\int_0^s H'(s_1)\,\mathcal{R}\left\{ i\sqrt{\left(\frac{Z^2 - r_1^2}{s_1^2 - Z^2}\right)}\right\} ds_1$$

$$+\alpha\int_0^s H(s_1)\,\mathcal{R}\left\{ i\,\frac{r_1\,dr_1/ds_1 - \varkappa_1 \,.\, s_1}{\sqrt{(s_1^2 - Z^2)}\sqrt{(Z^2 - r_1^2)}}\right\} ds_1\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (48)$$



Region	$\mathcal{R}\left\{ i\sqrt{\frac{y^2-r_1^2}{s_1^2-y^2}}\right\}$	$\mathcal{R}\left\{ \sqrt{\frac{y^2-r_1^2}{s_1^2-y^2}}\right\}$
I	>0	0
II	0	$\lesseqgtr 0$
III	<0 ($z = \pm 0$)	0

In the limit $z \to 0$ the second term on the right and the integrands vanish for points $(x, y, 0)$ inside the plan form and thus it is sufficient to extend the integration between $s_1 = 0$ and $s_1 = y > 0$ (*see* Fig.). When allowing for the proper signs of the square roots ($z \geqslant 0$), we arrive finally at the condition ($z \to +0$), which is the generalized equation (34):

$$1 = H(0) + \int_0^y H'(s_1)\sqrt{\left\{\frac{y^2 - r_1^2}{y^2 - s_1^2}\right\}}\,ds_1 - \int_0^y H(s_1)\,\frac{r_1\,dr_1/ds_1 - \varkappa_1 \,.\, s_1}{\sqrt{(y^2 - s_1^2)}\sqrt{(y^2 - r_1^2)}}\,ds_1\,. \qquad (49)$$

It must be satisfied for all points $y = s(x)$ of the leading edge ($ds/dx \geqslant 0$). For the rear part of the wing (where $ds/dx \leqslant 0$), the boundary conditions are satisfied without introducing a pressure difference and a load, as was explained in section 3.1. Since $H(0) = 1$ and $H' = 0$, $r = 0$, $\varkappa = 0$ along the front part of the wing, we may write instead of (49)

$$0 = \int_{s_0}^y H'(s_1)\sqrt{\left\{\frac{y^2 - r_1^2}{y^2 - s_1^2}\right\}}\,ds_1 - \int_{s_0}^y H(s_1)\,\frac{r_1\,dr_1/ds_1 - \varkappa_1 \,.\, s_1}{\sqrt{(y^2 - s_1^2)}\sqrt{(y^2 - r_1^2)}}\,ds_1 \quad \ldots \quad (50)$$

where the integration must be extended between the half-span $s_0 = s(c_r)$ and the point y in question (*see* Fig. 8).

Since this equation for the function $H(s)$ is fairly complicated, we shall determine the solution by a numerical method (compare Appendix II).

The load distribution on the wing is then obtained from (45) and (32) as

$$l = \frac{\Delta p}{\frac{1}{2}\rho V^2} = 4\alpha H(s)\frac{ds}{dx}\left\{\cot \chi\sqrt{(1 - k^2\sin^2\chi)} + \int_0^\chi \sqrt{(1 - k^2\sin^2\chi)}\,d\chi - \varkappa\int_0^\chi \frac{d\chi}{\sqrt{(1 - k^2\sin^2\chi)}}\right\} \quad \ldots \quad (51)$$

where

$$\sin\chi = \sqrt{\left\{\frac{s^2 - y^2}{s^2 - r^2}\right\}}, \qquad r^2 \leqslant y^2 \leqslant s^2 .$$

If r tends to zero (i.e., $k \to 1$, $\varkappa \to 0$, $H(s) \to 1$), we obtain from (51)

$$l = 4\alpha\frac{ds}{dx}\frac{1}{\sin\chi}, \qquad \sin\chi = \sqrt{\left(1 - \frac{y^2}{s^2}\right)} \quad \ldots \quad (52)$$

which agrees with equation (37).

3.3. *Forces and Moments for a Flat Wing at Incidence.*—When determining the forces and moments acting on a wing at incidence, we have to integrate the load distributions equation (51) and (52) over the area of the wing. We notice, that for a flat wing at incidence only those parts of the wing, for which ds/dx is positive, produce a pressure difference. The rear parts of the wing (section CC in Fig. 2) cannot influence the flow and therefore produce no lift. Since we have $l = 0$ everywhere on this part of the wing, the Kutta-Joukowsky condition is automatically satisfied.

Thus it is sufficient to extend our integration only over those parts of the wing, for which $ds/dx > 0$. We integrate first in the spanwise direction and have, using equation (51) and (44),

$$\int_{-s}^{s} l\,dy = 2\int_r^s l\,dy = 8\alpha H(s)\frac{ds}{dx}s\int_0^{\pi/2}\left\{\cot\chi\sqrt{(1 - k^2\sin^2\chi)} + \int_0^\chi \sqrt{(1 - k^2\sin^2\chi)}\,d\chi - \varkappa\int_0^\chi\frac{d\chi}{\sqrt{(1 - k^2\sin^2\chi)}}\right\}\frac{k^2\sin\chi\cos\chi\,d\chi}{\sqrt{(1 - k^2\sin^2\chi)}} .$$

These integrals can be evaluated after an integration by parts and we have finally

$$\int_{-s}^{s} l\,dy = 8\alpha H(s)\frac{ds}{dx}s\left(1 - \frac{E}{K}\right)\frac{\pi}{2} = 4\pi\alpha H(s)s\frac{ds}{dx}[1 - \varkappa(s)] . \quad \ldots \quad (53)$$

A second integration along the chord gives the lift coefficient, if $A = 4s_m^2/S$ denotes the aspect ratio of the wing ($2s_m$ = maximum value of the local span $2s$)

$$C_L = \frac{1}{S}\int_S\int l\,dy\,dx = \frac{4\pi\alpha}{S}\int_0^{s_m} H(s)[1 - \varkappa(s)]s\,ds$$

$$= \pi A\alpha\int_0^{s_m} H(s)[1 - \varkappa(s)]\frac{s\,ds}{s_m^2} \quad \ldots \quad (54)$$

and the moment coefficient, referred to the apex $x = 0$ (C_M positive, if nose is turning upwards):

$$-C_M = \frac{1}{S\bar{c}}\int\left(\int l\,dy\right)x\,dx = \frac{4\pi\alpha}{S\bar{c}}\int_0^{s_m} H(s)[1 - \varkappa(s)]s \,.\, x(s)\,ds . \quad \ldots \quad (55)$$

Here $\bar{c}$ denotes the standard mean chord, so that $-C_M/C_L$ gives the distance of the aerodynamic centre behind the apex in terms of the mean chord $\bar{c}$. For a *straight leading edge* we have

$$s = x\cot\Lambda_l , \qquad \bar{s}_0 = c_r\cot\Lambda_l \quad \ldots \quad (56)$$

(Λ_l = sweep angle of the leading edge) and we may write

$$-C_M = \frac{\pi A \alpha c_r}{\bar{c}} \int_0^{s_m} H(s)[1 - \varkappa(s)] \frac{s^2 \, ds}{s_m^2 \bar{s}_0} \cdot \quad \ldots \quad (57)$$

If we prefer, to refer C_M to the root chord c_r, we have to omit the factor $c_r/\bar{c}$ in (57) and then $-C_M/C_L$ would give the aerodynamic-centre position in terms of the root chord.

3.4. *The Induced Velocities Behind the Wing.*—After the function $H(s)$ has been determined according to the integral equation (49), we can use equation (48) for the evaluation of the downwash at any point of the space, if we bear in mind that $H(s) = 0$ for all points behind the plane x = const which extends from the wing tip (tip Mach wave). Therefore, there is no disturbance in front of the plane $x = 0$, which corresponds to the Mach wave, extending from the apex. We obtain an upwash in front and sideways of the leading edge and a constant downwash over the wing. For points in the wake, which are in front of the tip Mach wave (region V of the Fig. in section 1), the downwash depends on x and y, but after the tip Mach wave has been reached, the velocities, which are induced there, remain unchanged for all points downstream, since $H(s)$ vanishes in this region. Thus the velocity field behind the tip Mach wave depends only on y and z and is independent of x. Here the upper limits in the integral (48) must be replaced by s_m. For the outer part of the wake (region IV), the downwash remains constant ($V . \alpha$). For delta-shaped plan forms (which are unisectional for every x), the downwash remains constant ($V . \alpha$) over the entire wake, since then the regions V and VI disappear.

The spanwise component v of the induced velocity, can be obtained from (2) and (43) in a similar way to w. After an integration by parts, we have

$$-\frac{v(x,y,z)}{V} = \alpha H(s) \, \mathscr{R}\left\{\sqrt{\left(\frac{Z^2 - r^2}{s^2 - Z^2}\right)}\right\} + \alpha \int_0^{s(x)} H(s_1) \, \mathscr{R}\left\{\frac{r_1 \, dr_1/ds_1 - s_1 . \varkappa_1}{\sqrt{(s_1^2 - Z^2)}\sqrt{(Z^2 - r_1^2)}}\right\} ds_1$$

$$- \alpha \int_0^{s(x)} H'(s_1) \, \mathscr{R}\left\{\sqrt{\left(\frac{Z^2 - r_1^2}{s_1^2 - Z^2}\right)}\right\} ds_1 \, . \quad \ldots \quad (58)$$

Again we can see, that no disturbance occurs in front of the apex Mach wave, and that the conditions at the tip Mach wave remain unchanged for all points downstream. When evaluating the values for the plane $z = 0$ we can use the figure and table in section 3.2, which shows that the lower limit 0 may be replaced in this case by y (the integral vanishes between 0 and y). Along the trailing edge of the wing, we have $y = r(x)$, and for region V (*see* figure in section 1), where $0 \leqslant y \leqslant r(s_m)$:

$$-\frac{v(x, r(x), +0)}{V} = \alpha \int_{r(x)}^{s(x)} H(s_1) \frac{r_1 \, dr_1/ds_1 - s_1 . \varkappa_1}{\sqrt{(s_1^2 - r^2)}\sqrt{(r^2 - r_1^2)}} \, ds_1$$

$$- \alpha \int_{r(x)}^{s(x)} H'(s_1) \sqrt{\left(\frac{r^2 - r_1^2}{s_1^2 - r^2}\right)} \, ds_1 \qquad (z = +0) \, . \quad \ldots \quad (59)$$

The same expression is valid for the entire wake in regions V and VI. For region IV ($r(s_m) < y < s_m$) we have to use (58) for $z = 0$ with s replaced by s_m and r replaced by $r(s_m)$. The vorticity in the wake is easily obtained from the sidewash v.

It may be mentioned that the velocity field behind a Delta-shaped wing, where the wake does not interfere with the wing, is given by the first term in (58). It corresponds to the velocity field induced by a plate (replacing the wake), which moves in the direction of $-z$. For the general case no such simple comparison exists because of the influence of the vortex distribution in the wake, which in turn is determined by the shape of the wing.

Now we define a particular wing plan form by demanding (compare Heaslet and Lomax[4]), that

$$r\,dr/ds = s\varkappa(s) = s\,\frac{E}{K} \qquad \text{(60)}$$

which can be integrated (*see* Appendix III) as

$$s_0/s = E(k) - (1 - k^2)K(k)\,, \qquad k = \sqrt{(1 - r^2/s^2)}\,. \qquad \text{(61)}$$

For this particular plan form, we have from (50) the solution

$$H(s) \equiv H(0) = 1$$

and thus from (59)

$$\frac{v}{V}\,(x, r(x), \pm 0) = 0\,.$$

If the middle part of the wing (or the entire wing plan form) were defined by equation (61), it would not produce a wake, since there would be no difference between the v-velocities on either surface of the wake.

4. *The Wing in Steady Roll.*—4.1. *Pressure Distribution for the Forward Part of the Wing.*—We have to find a solution I of the Laplace equation, which vanishes at a great distance from the wing, and satisfies the conditions (24) and (26), where

$$g = -\,py/V\,, \qquad \bar{g} = 0\,. \qquad \text{(25b)}$$

We consider first the front part of the wing (*see* section AA of Fig. 2), and have to find an anti-symmetrical function I, for which the normal derivative $\partial I/\partial z$ vanishes for $z = 0$, $y^2 < s^2$. We choose

$$\frac{I}{V^2} = -\,h_p(x)\,\mathscr{R}\left\{\frac{Z}{\sqrt{(s^2 - Z^2)}} + \frac{1}{i}\right\} \qquad \text{(62)}$$

which vanishes for $|Z| \to \infty$. The parameter $h_p(x)$ is to be determined later. I contains the required singularities at the leading edges $Z = \pm s$. We obtain for the derivative:

$$\frac{\partial}{\partial z}\left(\frac{I}{V^2}\right) = -\,h_p(x)\,\mathscr{R}\left\{\frac{is^2}{(s^2 - Z^2)^{3/2}}\right\} = h_p(x)\,\mathscr{R}\left\{\frac{i}{s}\frac{d}{ds}\frac{2Z^2 - s^2}{\sqrt{(s^2 - Z^2)}}\right\}. \qquad \text{(63)}$$

This function vanishes for any point $z = 0$, $y^2 < s^2$. The signs of the square root are defined in the same way as in equation (29).

Now we use equation (26) for determining the function $h_p(x)$. We introduce instead of h_p

$$h_p(x) = \frac{ps}{2V}\frac{ds}{dx}\,H_p(s) \qquad \text{(64)}$$

(the 2 is introduced in the denominator, to make $H_p = 1$ near the apex) and obtain for any point $(x, y, 0)$ inside the plan form:

$$\frac{py}{V} = \int_0^x \frac{\partial}{\partial z}\frac{I}{V^2}\,dx = \frac{p}{2V}\int_0^x H_p(s)\,\mathscr{R}\left\{\frac{d}{ds}\frac{i(2Z^2 - s^2)}{\sqrt{(s^2 - Z^2)}}\right\}_{z=0}\frac{ds}{dx}\,dx$$

where the integration is extended over the same path as in section 3.1 (Fig. 2).

Since $\mathscr{R}\{(2y^2 - s^2)/\sqrt{(s^2 - y^2)}\} = 0$ for points inside the wing and $\sqrt{(s^2 - Z^2)}$ tends to $-\,i\sqrt{(y^2 - s^2)}$ for $z \to 0$ and points outside the wing ($y^2 > s^2$) according to our rule for the signs of the square root, the last equation is, by means of an integration by parts, equivalent to

$$\frac{py}{V} = \frac{p}{V}\left\{y\,.\,H_p(0) + \int_0^y H_p{}'(s)\,\frac{2y^2 - s^2}{2\sqrt{(y^2 - s^2)}}\,ds\right\}. \qquad \text{(65)}$$

One solution of this equation is

$$H_p(s) \equiv H_p(0) = 1 \qquad \ldots \quad (66)$$

and this is the only solution, as can be shown in the same way as for equation (34).

Thus the load distribution for the forward part of a wing in roll is given by

$$l = \frac{\Delta p}{\frac{1}{2}\rho V^2} = -4\,\frac{I}{V^2} = 4\,\frac{ps}{2V}\,\frac{ds}{dx}\,\frac{y}{\sqrt{(s^2 - y^2)}}\,. \qquad \ldots \quad (67)$$

4.2. *Pressure Distribution for the Rear Part of the Wing.*—In order to find the solution for the wing in roll for the rear part of the wing, where a spanwise section consists of two parts ($z = 0$, $r^2 < y^2 < s^2$), we generalize equation (62) in the following way

$$\frac{I}{V^2} = -\,h_p(x)\,\mathcal{R}\left\{\frac{\sqrt{(Z^2 - r^2)}}{\sqrt{(s^2 - Z^2)}} + \frac{1}{i}\right\}. \qquad \ldots \quad (68)$$

By means of the transformation (44) we have $I/V^2 = -\,h_p(x)\,\mathcal{R}\{\cot \chi + 1/i\}$. This function vanishes for $|Z| \to \infty$ and has the required singularity at the leading edge $Z = \pm s$. Along the trailing edge $Z = \pm r$ the enthalpy vanishes, so that the Kutta-Joukowsky condition is satisfied. I is a unique function of y and z, if the sign of the square root is defined in the same way as in equation (42). The boundary condition (24) is also satisfied, as can be seen from

$$\frac{\partial}{\partial z}\left(\frac{I}{V^2}\right) = -\,h_p(x)\,\mathcal{R}\left\{\frac{iZ(s^2 - r^2)}{(s^2 - Z^2)^{3/2}(Z^2 - r^2)^{1/2}}\right\} \qquad \ldots \quad (69)$$

which vanishes for $z = 0$, $r^2 < y^2 < s^2$.

Finally, we introduce $H_p(s)$ instead of $h_p(x)$ by means of equation (64) and determine $H_p(s)$, using the boundary condition (26). This leads to

$$\frac{py}{V} = -\,\frac{p}{2V}\int_0^x H_p(s)\,.\,s\,.\,\mathcal{R}\left\{\frac{iZ(s^2 - r^2)}{(s^2 - Z^2)^{3/2}(Z^2 - r^2)^{1/2}}\right\}_{z=0}\frac{ds}{dx}\,dx \qquad \ldots \quad (70)$$

where the integration is again performed along the path, indicated in Fig. 2. Now we may write

$$-\,\frac{iZ(s^2 - r^2)}{(s^2 - Z^2)^{3/2}(Z^2 - r^2)^{1/2}} = \frac{1}{s}\,\frac{d}{ds}\left(\frac{iZ\sqrt{(Z^2 - r^2)}}{\sqrt{(s^2 - Z^2)}}\right) + \frac{iZ(r\,dr/ds - s)}{s\sqrt{(s^2 - Z^2)}\sqrt{(Z^2 - r^2)}}\,.$$

We introduce this expression in equation (70), and obtain, integrating the first term by parts, for any point $(x, y, 0)$ inside the plan form:

$$\frac{py}{V} = \frac{p}{2V}\left\{H_p(s)\,\mathcal{R}\left\{iy\sqrt{\left(\frac{y^2 - r^2}{s^2 - y^2}\right)}\right\} + H_p(0)\,.\,y + \int_0^y \frac{d}{ds}\,(H_p(s))\,\frac{y}{s}\sqrt{\left(\frac{y^2 - r^2}{y^2 - s^2}\right)}\,ds\right.$$
$$\left. -\int_0^y H_p(s)\left[\frac{y(r\,dr/ds - s)}{\sqrt{(y^2 - s^2)}\sqrt{(y^2 - r^2)}}\right]ds\right\}.$$

Here is is sufficient, to extend the integration with respect to s between $s = 0$ and $s = y$, as indicated, since $\mathcal{R}\{i\,.\,Z/s\,.\,\sqrt{[(Z^2 - r^2)/(s^2 - Z^2)]}\} = 0$ for $z = 0$ and values of s between y and $s(x) > y$. (Compare section 3.2 and the figure in section 3.2.) For the same reason the first term in this equation vanishes and we have the following condition for $H_p(s)$, which must be satisfied for all points of the leading edge $y = s(x)$ with $ds/dx \geqslant 0$:

$$\frac{py}{V} = \frac{py}{2V}\,H_p(0) + \frac{py}{2V}\int_0^y H_p{}'(s)\sqrt{\left(\frac{y^2 - r^2}{y^2 - s^2}\right)}\,ds$$
$$-\,\frac{py}{2V}\int_0^y H_p(s)\,\frac{(r\,dr/ds - s)}{\sqrt{(y^2 - s^2)}\sqrt{(y^2 - r^2)}}\,ds\,. \qquad \ldots \quad (71)$$

For the same reasons, as were explained in section 3.1, the rear part of the wing, where $ds/dx < 0$, does not contribute any load.

In the forward part of the wing ($s \leqslant s_0$, $x \leqslant c_r$) we have $H_p(s) \equiv 1$, $r = 0$, as shown above (equation (66)).

Using this result we obtain from (71) for $y \geqslant s_0$:

$$\sqrt{\left(1 - \frac{s_0^2}{y^2}\right)} = \int_{s_0}^{y} H_p'(s) \sqrt{\left(\frac{y^2 - r^2}{y^2 - s^2}\right)} ds - \int_{s_0}^{y} H_p(s) \frac{(r\, dr/ds - s)}{\sqrt{(y^2 - s^2)}\sqrt{(y^2 - r^2)}} ds \quad \ldots \quad (72)$$

This integral equation is very similar to equation (50) and can thus be treated by the same method (compare Appendix II).

The load distribution for the wing is thus given by

$$l = -\frac{4I}{V^2} = 4\frac{ds}{dx} H_p(s) \frac{ps}{2V} \cdot \sqrt{\left(\frac{y^2 - r^2}{s^2 - y^2}\right)} \quad \ldots \quad (73)$$

which for $r = 0$ agrees with equation (67).

4.3. *Rolling Moment Due to Roll.*—In order to determine the rolling moment due to roll, we have to multiply the load l by the arm y and integrate over the area of the wing. At first we integrate spanwise:

$$\int_{-s(x)}^{s(x)} l \,.\, y\, dy = 2\int_r^s l \,.\, y\, dy = 4\frac{ps}{V}\frac{ds}{dx} H_p(s) \int_r^s \sqrt{\left(\frac{y^2 - r^2}{s^2 - y^2}\right)} y\, dy \,.$$

Using the transformation (44) for $Z = y$(χ real), we have

$$\int_r^s \sqrt{\left(\frac{y^2 - r^2}{y^2 - s^2}\right)} y\, dy = s^2k^2 \int_0^{\pi/2} \cot\chi \sin\chi \cos\chi\, d\chi$$

$$= s^2k^2 \int_0^{\pi/2} \cos^2\chi\, d\chi = \tfrac{1}{4}\pi(s^2 - r^2)$$

and thus for the rolling moment L due to roll:

$$L = -\tfrac{1}{2}\rho V^2 \int_S\int ly\, dy\, dx = -\tfrac{1}{2}\rho V^2\pi \,.\, \frac{p}{V}\int_0^{s_m} H_p(s)(s^2 - r^2)s\, ds \quad \ldots \quad (74)$$

where $2s_m$ is the maximum value of the local span $2s$. For the same reasons as above, in the incidence case, the integration over s (which takes the place of the integration over x) must be extended between $s = 0$ and $s = s_m$. We introduce the roll-damping coefficient

$$l_p = \frac{\partial L}{\partial(ps_m/V)} \frac{1}{\rho V^2 S s_m} \quad \ldots \quad (75)$$

and obtain ($A = 4s_m^2/S$ = aspect ratio):

$$l_p = -\frac{1}{8}\pi A \int_0^{s_m} H_p(s) \frac{(s^2 - r^2)s\, ds}{s_m^4} \,. \quad \ldots \quad (76)$$

5. *The Flat Wing in Steady Pitch.*—5.1. *The Pressure Distribution.*—The pressure distribution for a wing in steady pitch which satisfies the boundary conditions (24) and (26) with

$$g = -\frac{qx}{V}\,, \quad \bar{g} = -\bar{c}\frac{\partial}{\partial x} g = \frac{q\bar{c}}{V} \quad \ldots \quad (77)$$

is very closely related to the pressure distribution for a wing at a constant incidence. We can expect to use similar functions for I in both cases and shall therefore deal immediately with the general case that the spanwise section of the wing consists of two parts $-s < y < -r$ and $r < y < s$. For the forward part of the wing we may then put $r = 0$.

For the derivative $\partial I/\partial z$ we use now the function:

$$\frac{\partial}{\partial z}\frac{I}{V^2} = -\frac{qx}{V}H_q(s)\frac{ds}{dx}\mathscr{R}\left\{\frac{is\sqrt{(Z^2-r^2)}}{(s^2-Z^2)^{3/2}} + \frac{is\varkappa(s)}{\sqrt{(s^2-Z^2)}\sqrt{(Z^2-r^2)}}\right\}$$

$$+\frac{q}{V}\mathscr{R}\left\{1+i\frac{Z^2-s^2\,.\,\varkappa(s)}{\sqrt{(s^2-Z^2)}\sqrt{(Z^2-r^2)}}\right\}. \quad \ldots \quad (78)$$

The first term agrees with equation (38) apart from the parameter $H_q(s)$. The second term has been introduced in order to satisfy the boundary condition

$$\frac{\partial}{\partial z}\frac{I}{V^2} = -\frac{\partial}{\partial x}g(x,y) = \frac{q}{V} \quad \ldots \quad (79)$$

inside the plan form of the wing ($z = 0$, $r^2 < y^2 < s^2$). The sign of the square root shall be defined in the same way as before in equation (43). Then both brackets in (78) vanish for $|Z| \to \infty$.

It can be verified by differentiation ($\partial/\partial z = i\, d/dZ$) that (78) is the derivative of

$$\frac{I}{V^2} = -\frac{qx}{V}H_q(s)\frac{ds}{dx}\mathscr{R}\left\{\frac{Z}{s}\sqrt{\left(\frac{Z^2-r^2}{s^2-Z^2}\right)} - \int_s^Z \frac{Z^2-s^2\,.\,\varkappa}{\sqrt{(s^2-Z^2)}\sqrt{(Z^2-r^2)}}\frac{dZ}{s}\right\}$$

$$+\frac{q}{V}\mathscr{R}\left\{-iZ + \int_s^Z \frac{(Z^2-s^2\,.\,\varkappa)\,dZ}{\sqrt{(s^2-Z^2)}\sqrt{(Z^2-r^2)}}\right\}. \quad \ldots \quad (80)$$

Here the first term agrees with equation (43), if $\varkappa$ is again determined by

$$\varkappa(s) = \frac{E(k)}{K(k)}, \qquad k = \sqrt{\left(1-\frac{r^2}{s^2}\right)}. \quad \ldots \quad (81)$$

The first term produces the required singularity at the leading edge $y = \pm s$ and vanishes at the trailing edge $y = \pm r$ and for $|Z| \to \infty$. The second term vanishes both for $y = \pm s$ and $y = \pm r$ and for $|Z| \to \infty$, but it produces also a pressure difference between both faces of the wing.

Using the transformation (44) we may write equation (80) as

$$\frac{I}{V^2} = -\frac{qx}{V}H_q(s)\frac{ds}{dx}\mathscr{R}\left\{\cot\chi\sqrt{(1-k^2\sin^2\chi)}\right.$$

$$\left.+\int_0^\chi \sqrt{(1-k^2\sin^2\chi)}\,d\chi - \frac{E}{K}\int_0^\chi \frac{d\chi}{\sqrt{(1-k^2\sin^2\chi)}}\right\}$$

$$-\frac{qs}{V}\mathscr{R}\left\{i\sqrt{(1-k^2\sin^2\chi)} + \int_0^\chi \sqrt{(1-k^2\sin^2\chi)}\,d\chi - \frac{E}{K}\int_0^\chi \frac{d\chi}{\sqrt{(1-k^2\sin^2\chi)}}\right\} \quad (82)$$

where

$$\sin\chi = \sqrt{\left(\frac{s^2-Z^2}{s^2-r^2}\right)} \qquad (Z = y + iz)$$

becomes real on the surface of the wing $z = 0$, $r^2 < y^2 < s^2$. The first bracket is plotted against χ in Fig. 4 (incidence case), the second bracket in Fig. 5.

Finally, we have to determine the 'parameter' $H_q(s)$ in such a way that the condition (26) is also satisfied for any point inside the plan form:

$$\frac{qx}{V} = \int_0^x \frac{\partial}{\partial z}\frac{I}{V^2}\,dx = \frac{q}{V}\int_0^x x\,.\,H_q(s)\,\mathscr{R}\left\{\frac{d}{ds}\,i\sqrt{\left(\frac{Z^2-r^2}{s^2-Z^2}\right)}\right.$$

$$\left.+\,i\frac{r\,dr/ds - s\,.\,\varkappa(s)}{\sqrt{(s^2-Z^2)}\sqrt{(Z^2-r^2)}}\right\}_{z=0}\frac{ds}{dx}dx + \frac{q}{V}\int_0^x \mathscr{R}\left\{1+i\frac{Z^2-s^2\,.\,\varkappa(s)}{\sqrt{(s^2-Z^2)}\sqrt{(Z^2-r^2)}}\right\}_{z=0}dx\,.$$

Here the integration must be performed along the same path as in section 3.1 (Fig. 2).

The first bracket has been written in a similar way as in the corresponding equation, preceding equation (48). We integrate the first term by parts and carry out the integration in the first term of the second bracket. This leads to:

$$\frac{qx}{V} = \frac{q}{V} x(s) \, . \, H_q(s) \, \mathscr{R} \left\{ i \sqrt{\left(\frac{y^2 - r^2}{s^2 - y^2}\right)} \right\}$$

$$+ \frac{q}{V} \int_0^{s(x)} \frac{d}{ds} (x(s) \, . \, H_q(s)) \, \mathscr{R} \left\{ \sqrt{\left(\frac{y^2 - r^2}{y^2 - s^2}\right)} \right\} ds$$

$$- \frac{q}{V} \int_0^{s(x)} x(s) \, H_q(s) \, \mathscr{R} \left\{ \frac{r \, dr/ds - s \, . \, \varkappa(s)}{\sqrt{(y^2 - s^2)} \sqrt{(y^2 - r^2)}} \right\} ds$$

$$+ \frac{qx}{V} - \frac{q}{V} \int_0^x \mathscr{R} \left\{ \frac{y^2 - s^2 \, . \, \varkappa(s)}{\sqrt{(y^2 - s^2)} \sqrt{(y^2 - r^2)}} \right\} dx \, . \quad \ldots \qquad (83)$$

The first term on the right is zero for all points inside the plan form $r^2 < y^2 < s^2$. (Compare figure and table in section 3.2.) For the same reason it is sufficient to integrate in the second, third and last term over the part of the path of integration, which is in front of the wing ($0 \leqslant s(x) \leqslant y$). In other words: it is sufficient to satisfy equation (83) for points ($x, y = s(x), 0$) of the leading edge only, where $ds/dx \geqslant 0$. Since all the real parts $\mathscr{R}\{\sqrt{[(y^2 - r^2)/(y^2 - s^2)]}\}$ vanish for points inside the wing, equation (83) is then automatically satisfied in any point of the wing. The Kutta-Joukowsky condition excludes the occurrence of a load distribution of the type used in the incidence case (equation (43)), in the rear part of the wing, where $ds/dx < 0$, since it would introduce an infinite pressure along the trailing edge.

Thus we have for all points $y = s(x)$ of the leading edge ($ds/dx \geqslant 0$) the condition ($y \geqslant 0$):

$$0 = \frac{q}{V} \int_0^y \frac{d}{ds} [x(s) \, . \, H_q(s)] \sqrt{\left(\frac{y^2 - r^2}{y^2 - s^2}\right)} ds$$

$$- \frac{q}{V} \int_0^y x(s) \, . \, H_q(s) \frac{r \, dr/ds - s \, . \, \varkappa(s)}{\sqrt{(y^2 - s^2)} \sqrt{(y^2 - r^2)}} ds - \frac{q}{V} \int_0^x \frac{y^2 - s^2 \, . \, \varkappa(s)}{\sqrt{(y^2 - s^2)} \sqrt{(y^2 - r^2)}} dx \, . \qquad (84)$$

For the forward part of the wing we have $r = 0$, $\varkappa(s) = 0$ and from (84):

$$\int_0^y \frac{d}{ds} [x(s)(H_q(s) - 1)] \frac{y}{\sqrt{(y^2 - s^2)}} ds = 0$$

or

$$x(s)(H_q(s) - 1) = c$$

is constant. The enthalpy I on the forward part of the wing is then according to (82):

$$\frac{I}{V^2} = - \frac{q(x + c)}{V} \frac{ds}{dx} \frac{1}{\sin \chi} - \frac{qs}{V} \sin \chi \, ,$$

$$\sin \chi = \sqrt{(1 - y^2/s^2)}$$

which should vanish at the apex $x = 0$ of the wing where the 'local incidence' qx/V vanishes. Thus we have $c = 0$ and

$$H_q(s) \equiv 1 \qquad (s \leqslant s_0) \, . \quad \ldots \qquad (85)$$

On introducing this into equation (84) we arrive for $s \geqslant s_0$ or $y \geqslant s_0$ at the condition

$$0 = \int_{s_0}^y \frac{d}{ds} [x(s) \, H_q(s)] \sqrt{\left(\frac{y^2 - r^2}{y^2 - s^2}\right)} ds$$

$$- \int_{s_0}^y [x(s) \, . \, H_q(s)] \frac{(r \, dr/ds - s \, . \, \varkappa(s))}{\sqrt{(y^2 - s^2)} \sqrt{(y^2 - r^2)}} ds$$

$$- \int_{c_r}^{x(y)} \frac{[y^2 - s^2 \, . \, \varkappa(s)]}{\sqrt{(y^2 - s^2)} \sqrt{(y^2 - r^2)}} dx \, . \quad \ldots \qquad (86)$$

For a wing with a *straight leading edge* we have the simpler condition:

$$\int_{s_0}^{y} \frac{d}{ds}\,[s\,.\,H_q(s)] \sqrt{\left(\frac{y^2 - r^2}{y^2 - s^2}\right)}\, ds - \int_{s_0}^{y} s\,.\,H_q(s)\, \frac{(r\, dr/ds - s\,.\,\varkappa(s))}{\sqrt{(y^2 - s^2)}\sqrt{(y^2 - r^2)}}\, ds$$

$$= \int_{s_0}^{y} \frac{y^2 - s^2\,.\,\varkappa(s)}{\sqrt{(y^2 - s^2)}\sqrt{(y^2 - r^2)}}\, ds \qquad (y \geqslant s_0)\,. \quad \ldots \quad \ldots \quad \ldots \quad (87)$$

Apart from the term on the right side, equation (87) for the function $s\,.\,H_q(s)$ agrees with equation (50) for the function $H(s)$, and can therefore be solved in a similar way (*see* Appendix II). In particular, we have for the tail part of the plan form, where $s = s_m =$ const, the solution $H_q \equiv 0$, so that only the front part of the plan form $s \leqslant s_m$ contributes to the first term in (82), whereas the second term depends on the entire plan form.

5.2. *Forces and Moments on a Wing in Pitch.*—In order to obtain the force $(-Z)$ and the pitching moment M with respect to the apex, we have to integrate the load

$$l = -4I/V^2 \ldots \quad \ldots \quad (88)$$

as given by equation (82), first with respect to the span. Since all the integrals involved occurred already in the incidence case (compare equation (53)), we can use these results and obtain now

$$\int_{-s}^{s} l\, dy = 2\int_{r}^{s} l\, dy = \frac{8qx}{V} H_q(s) s \frac{ds}{dx} \frac{\pi}{2} [1 - \varkappa(s)]$$

$$+ \frac{8qs}{V}\,.\,s\,.\left\{[1 - \varkappa(s)]\frac{\pi}{2} - \left(1 - \frac{r^2}{s^2}\right)\frac{\pi}{4}\right\} \quad \ldots \quad (89)$$

From this equation we obtain for the stability derivatives ($\bar{c}$ = mean chord $= 2s_m/A$)

$$z_q = \frac{\partial Z}{\partial(q\bar{c}/V)} \frac{1}{\rho V^2 S} \quad \ldots \quad (90)$$

and

$$m_q = \frac{\partial M}{\partial(q\bar{c}/V)} \frac{1}{\rho V^2 S\bar{c}} \quad \ldots \quad (91)$$

the results

$$-2z_q \frac{q\bar{c}}{V} = \frac{1}{S}\iint l\, dy\, dx = \frac{q\bar{c}}{V} A\pi \left\{\int_0^{s_m} \frac{x(s)}{\bar{c}} H_q(s)[1 - \varkappa(s)] \frac{s\, ds}{s_m^2}\right.$$

$$\left. + \int_0^{c_m} \left[\frac{1}{2}\left(1 + \frac{r^2}{s^2}\right) - \varkappa(s(x))\right] \frac{s^2\, dx}{s_m^2 \bar{c}}\right\} \quad \ldots \quad (92)$$

and

$$-2m_q \frac{q\bar{c}}{V} = \frac{1}{S\bar{c}} \int_0^{c_m} x \int_{-s}^{s} l\, dy\, dx$$

$$= \frac{q\bar{c}}{V} A\pi \left\{\int_0^{s_m} \frac{x^2(s)}{\bar{c}^2} H_q(s)\Big(1 - \varkappa(s)\Big) \frac{s\, ds}{s_m^2}\right.$$

$$\left. + \int_0^{c_m} \left[\frac{1}{2}\left(1 + \frac{r^2}{s^2}\right) - \varkappa(s)\right] \frac{s^2\,.\,x\, dx}{s_m^2 \bar{c}^2}\right\}. \quad \ldots \quad (93)$$

In both expressions the first integral has to be extended over s between $s = 0$ and $s = s_m$ (= maximum value of the local half-span), and the second integral, where $r(s)$ and $\varkappa(s)$ are by means of $s = s(x)$ functions of x, has to be extended over x between $x = 0$ and $x = c_m$ (= maximum value of x for any point inside the wing plan form). The first integral in either expression depends only on the forward part of the wing, the second integral on the entire plan form.

5.3. *Pitching Derivatives for an Arbitrary Axis Position.*—In order to obtain the pitching derivatives for an arbitrary axis position $x = x_0$, we have to satisfy the boundary condition

$$w(x, y, 0) = - q(x - x_0) \quad \ldots \quad (94)$$

inside the plan form of the wing. The resultant load consists of two parts, one corresponding to the boundary condition $w = - qx$, which may be denoted by l_0 and leads to derivatives z_{q0} and m_{q0}, which are given in section 5.2, and a second load l_{inc}, which was obtained above (3.3) for the constant incidence $\alpha = - qx_0/V$. Thus we obtain for the stability derivatives z_q and m_q for the axis position x_0

$$- 2z_q \frac{q\bar{c}}{V} = \frac{1}{S} \iint l \, dy \, dx = \frac{1}{S} \iint l_0 \, dy \, dx - \frac{qx_0}{V\alpha} \frac{1}{S} \iint l_{inc} \, dy \, dx$$

or

$$- z_q = - z_{q0} - \frac{1}{2} \frac{x_0}{\bar{c}} \frac{\partial C_L}{\partial \alpha} \quad \ldots \quad (95)$$

and

$$- 2m_q \frac{q\bar{c}}{V} = \frac{1}{S\bar{c}} \int (x - x_0) \int l \, dy \, dx$$

$$= \frac{1}{S\bar{c}} \iint \left(l_0 - \frac{qx_0}{V\alpha} l_{inc} \right) (x - x_0) \, dy \, dx$$

or

$$- m_q = - m_{q0} + \frac{x_0}{\bar{c}} \left(\frac{1}{2} \frac{\partial C_M}{\partial \alpha} + z_{q0} \right) + \frac{1}{2} \frac{\partial C_L}{\partial \alpha} \frac{x_0^2}{\bar{c}^2} \,. \quad \ldots \quad (96)$$

6. *Results for Some Particular Wings.*—In the following section the results of the preceding sections shall be applied to some particular plan forms namely the Delta wing, a plan form with straight edges and unswept trailing edge, and finally a family of wings, which includes the two preceding cases and consists of wings the plan forms of which (Fig. 8) have straight edges and can be described by three parameters, *e.g.*, the aspect ratio A, the taper ratio λ, and the sweep ratio a. In all these cases we shall give a list of the stability derivatives, as far as they have been determined above.

6.1. *The Delta Wing.*—In the case of a Delta wing, the function $r(s)$ is always zero and therefore $\varkappa(s) = 0$, $H(s) = 1$, $H_p(s) = 1$, $H_q(s) = 1$. Since $s = \cot \Lambda_l$ (Λ_l = leading-edge sweep), all the integrals can easily be worked out. We obtain

$$\left. \begin{aligned} - z_w &= \frac{1}{2} \frac{\partial C_L}{\partial \alpha} = \frac{1}{4} \pi A && \text{(compare (54))} \\ - m_w &= - \frac{1}{2} \frac{\partial C_M}{\partial \alpha} = \frac{1}{3} \pi A && \text{(compare (57))} \\ - l_p &= \frac{1}{32} \pi A && \text{(compare (76))} \\ - z_q &= \frac{1}{2} \pi A \left(1 - \frac{x_0}{2\bar{c}} \right) && \text{(compare (92) and (95))} \\ - m_q &= \frac{1}{2} \pi A \left\{ \frac{3}{2} - \frac{5}{3} \frac{x_0}{\bar{c}} + \frac{1}{2} \frac{x_0^2}{\bar{c}^2} \right\} && \text{(compare (93) and (96))} \end{aligned} \right\} \quad \ldots \quad (97)$$

The results have been obtained before by Ribner[2]. These results also agree with the stability coefficients for a Delta wing at supersonic speeds (*e.g.*, Ref. 9), if we take there the limiting case of a wing of small aspect ratio.

6.2. *The Cropped Delta Wing.*—Now we consider the family of 'cropped' Delta wings (*see* Fig. 6), where the leading edge is swept at an angle Λ_l and the trailing edge is unswept. The family can be described by two geometrical parameters, the aspect ratio A and the taper ratio λ (tip chord divided by root chord). Here we have no direct interference between the wake and the wing ($r = 0$), and thus $\varkappa(s) = 0$, $H(s) = 1$, $H_p(s) = 1$, $H_q(s) = 1$. All the integrals for C_L, C_M, and l_p can easily be worked out between $s = 0$ and $s = s_m$ according to equations (54), (57) and (76). This leads to the same results for z_w and l_p as in the case of the Delta wing (equation (97)), *i.e.*, these two derivatives are independent of the taper ratio λ. But for m_w we obtain a different answer:

$$-z_w = \tfrac{1}{2}\frac{\partial C_L}{\partial \alpha} = \tfrac{1}{4}\pi A \, , \qquad -l_p = \tfrac{1}{32}\pi A \, ,$$

$$-m_w = -\tfrac{1}{2}\frac{\partial C_M}{\partial \alpha} = \tfrac{1}{3}\pi A \frac{1-\lambda}{1+\lambda} \, . \qquad \ldots \quad (99)$$

Here only the forward part of the wing ($x \leqslant s_m \tan \Lambda_l$) contributes to the forces. For the derivatives z_q and m_q also the rear part of the wing becomes important.

Since for the front part of the wing

$$x = s \tan \Lambda_l \, , \qquad c_r = s_0 \tan \Lambda_l \qquad \ldots \quad (100)$$

and for the rear part of the wing ($s_m \tan \Lambda_l \leqslant x \leqslant c_r$):

$$s = s_m \, , \qquad \ldots \quad (101)$$

we obtain for an axis position in the apex $x = 0$ according to (92) and (93) the following integrals:

$$-z_q = A\frac{\pi}{2}\frac{c_r}{\bar{c}}\left[\tfrac{3}{2}\int_0^{s_m}\frac{s^2\,ds}{s_m^{\,2}s_0} + \tfrac{1}{2}\int_{(1-\lambda)c_r}^{c_r}\frac{dx}{c_r}\right] ,$$

$$-m_q = A\frac{\pi}{2}\frac{c_r^{\,2}}{\bar{c}^2}\left[\tfrac{3}{2}\int_0^{s_m}\frac{s^3\,ds}{s_m^{\,2}s_0^{\,2}} + \tfrac{1}{2}\int_{(1-\lambda)c_r}^{c_r}\frac{x\,dx}{c_r^{\,2}}\right] .$$

We introduce

$$\frac{s_m}{s_0} = 1 - \lambda \, , \qquad \bar{c} = \frac{1+\lambda}{2}\,c_r \qquad \ldots \quad (102)$$

and have (axis at the apex $x = 0$):

$$\left.\begin{aligned} -z_q &= A\frac{\pi}{2}\frac{1}{1+\lambda} \\ -m_q &= A\frac{\pi}{2}\frac{1}{(1+\lambda)^2}\left[1 + \tfrac{1}{2}(1-\lambda)^2\right] \end{aligned}\right\} \qquad \ldots \quad (103)$$

For an arbitrary axis position at $x = x_0$ we obtain from (95) and (96):

$$-z_q = \frac{\pi A}{4}\left[\frac{2}{1+\lambda} - \frac{x_0}{\bar{c}}\right] \qquad \ldots \quad (104)$$

$$-m_q = \frac{\pi A}{2(1+\lambda)^2}\left[1 + \tfrac{1}{2}(1-\lambda)^2 - \frac{x_0}{\bar{c}}\left(\tfrac{5}{3} + \lambda - \tfrac{2}{3}\lambda^2\right) + \tfrac{1}{2}\frac{x_0^{\,2}}{\bar{c}^2}(1+\lambda)^2\right] .$$

In order to show the effect of cropping the wing tips on the derivatives, we consider a wing family with a constant leading-edge sweep Λ_l. Thus we express A by means of Λ_l:

$$A = \frac{4s_m}{(1+\lambda)c_r} = 4\,\frac{1-\lambda}{1+\lambda}\cot\Lambda_l \quad \ldots \quad (105)$$

and introduce the root chord c_r as reference length, in order to illustrate more clearly the effect of cropping the tips. This leads to*

$$\left.\begin{aligned}
-z_w &= \pi\,\frac{1-\lambda}{1+\lambda}\cot\Lambda_l \\
-m_w &= \frac{4\pi}{3}\left(\frac{1-\lambda}{1+\lambda}\right)^2\cot\Lambda_l\,.\,\frac{\bar{c}}{c_r} = \frac{2\pi}{3}\,\frac{(1-\lambda)^2}{1+\lambda}\cot\Lambda_l \\
-l_p &= \frac{\pi}{8}\,\frac{1-\lambda}{1+\lambda}\cot\Lambda_l \\
-z_q &= 2\pi\,\frac{1-\lambda}{(1+\lambda)^2}\cot\Lambda_l\left(1-\frac{x_0}{c_r}\right) \\
-m_q &= 2\pi\,\frac{1-\lambda}{(1+\lambda)^3}\cot\Lambda_l\left[\left(1+\tfrac{1}{2}(1-\lambda)^2\right)-2\,\frac{x_0}{c_r}\left(1+\tfrac{2}{3}(1-\lambda)\right)+2\,\frac{x_0^2}{c_r^2}\right]
\end{aligned}\right\} \quad . \quad (106)$$

In Fig. 7 the function $-m_q\tan\Lambda_l$ has been plotted against the taper ratio λ for the axis positions $x_0 = \frac{1}{2}c_r$, $x_0 = \frac{2}{3}c_r$ and for the axis position in the aerodynamic centre $x_0 = \frac{2}{3}(1-\lambda)c_r$.

Near $\lambda = 0$ the derivative m_q is fairly sensitive to a small alteration of λ, in particular for an axis position between the apex and the middle of the root chord. The variation of m_q with λ is nearly linear, if the axis is kept always in the aerodynamic centre.

6.3. *A Wing Family with Three Geometric Parameters.*—Finally we consider a wing plan form with a straight leading and trailing edge, the sweep of which is Λ_l and Λ_t respectively. These two edges (or their extension beyond the wing tip) may intersect at a distance $y = s_n$ from the line of symmetry ($y = 0$) and the wing tip may be cut along a line $y = s_m$ parallel to the x-axis so that the wing has the taper ratio λ (*see* Fig. 8). We define

$$\bar{s}_0 = c_r\cot\Lambda_l \quad \ldots \quad (107)$$

and have $s_0 = s_m$ in Fig. 8a ($s_m \leqslant s_0$) and $s_0 = \bar{s}_0$ in Fig. 8b ($s_m \geqslant s_0$). In order to describe the plan form we shall use the aspect ratio A, the taper ratio λ and the ratio (sweep ratio)

$$a = \frac{\tan\Lambda_t}{\tan\Lambda_l}\,. \quad \ldots \quad (108)$$

These three parameters are most suitable since all derivatives are then proportional to the aspect ratio A and a function of the two remaining parameters λ and a, as we shall see later on. The relations between these parameters and the quantities, occurring in the analysis, are

$$\frac{s_m}{s_n} = 1-\lambda\,,\quad \frac{\bar{s}_0}{s_n} = 1-a\,,\quad \frac{s_m}{\bar{s}_0} = \frac{1-\lambda}{1-a}\,. \quad \ldots \quad (109)$$

In the limit $a \to 0$ we obtain the family of cropped Delta wings treated in section 6.2.

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* If we prefer the use of the standard mean chord $\bar{c}$, the values of m_w and m_q have to be multiplied by $c_r/\bar{c} = 2/(1+\lambda)$.

In order to show the influence of the three geometric parameters, three families, each with one varying and two fixed values for the three parameters A, λ, and a are plotted in Fig. 9.

The contours of the plan form are given by the equations

$$\left.\begin{array}{lll} \text{Leading edge:} & x = s \tan \Lambda_l & (0 \leqslant s \leqslant s_m) \\ \text{Tip:} & s = s_m & \\ \text{Trailing edge:} & r = (x - c_r) \cot \Lambda_t = \dfrac{x - c_r}{a} \cot \Lambda_l & \\ & (c_r \leqslant x \leqslant s_m \tan \Lambda_l + \lambda c_r) & \end{array}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (110)$$

where

$$\cot \Lambda_l = \frac{1 - a}{4} \frac{1 + \lambda}{1 - \lambda} A . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (111)$$

If $s_m > \bar{s}_0 = s_0$, we shall need a relation between r and s:

$$\frac{r}{s_0} = \frac{r}{\bar{s}_0} = \frac{s_n}{\bar{s}_0} \cdot \frac{s - \bar{s}_0}{s_n - \bar{s}_0} = \frac{1}{a}\left(\frac{s}{s_0} - 1\right) \qquad (s_0 \leqslant s \leqslant s_m) . \ldots \quad \ldots \quad (112)$$

For the rear part of the wing ($s = s_m > s_0$), we have to use

$$\frac{r}{s_0} = \frac{1}{a}\left(\frac{x}{c_r} - 1\right) \qquad (c_r \leqslant x) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (113)$$

which follows from (110).

6.3.1. *Lift and pitching moment due to incidence.*—The lift and the pitching moment of a flat wing of the plan form described in 6.3 at the incidence α can be obtained from equations (54) and (57).

For plan-forms, where $s_m \leqslant \bar{s}_0$, i.e., $\lambda \geqslant a$, we have no interaction between the trailing edge $r(x)$ and the leading edge $s(x)$. Thus $H(s) \equiv 1$ and

$$C_L = \frac{\pi}{2} A \alpha \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (114)$$

$$(\lambda \geqslant a)$$

$$- C_M = \frac{2\pi}{3} A \frac{1 - \lambda}{1 + \lambda} \frac{1}{1 - a} \alpha \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (115)$$

(C_M is referred to the standard mean chord $\bar{c}$ and the apex $x = 0$). For $s_m \geqslant \bar{s}_0 = s_0$ we have to determine the function $H(s)$ from equation (50).

First, we consider the case that $\mu = a/(1 - a)$ is very small. As has been shown in Appendix IV, the solution $H(s)$ of equation (50), when replotted against the variable

$$\sigma = \frac{s - s_0}{s_n - s_0} = \frac{1 - a}{a}\left(\frac{s}{s_0} - 1\right) = \frac{1}{\mu}\left(\frac{s}{s_0} - 1\right) , \quad \ldots \quad \ldots \quad \ldots \quad (116)$$

is in the limit $\mu \rightarrow 0$ given by

$$\lim_{s_n \bar{s}_0 \rightarrow 1} H(s) = \frac{1}{\sqrt{(1 - \sigma^2)}} = \frac{s_n - \bar{s}_0}{\sqrt{(s_n - s)} \sqrt{(s_n + s - 2\bar{s}_0)}} . \quad \ldots \quad \ldots \quad \ldots \quad (117)$$

For other values of $s_n/s_0 = 1 + \mu$ a numerical method was used to calculate the solution $H(s)$ of equation (50). The method is described in detail in Appendix II. The results are given in Tables 1 and 2 and are plotted in Fig. 10 against $s/\bar{s}_0 = s/s_0$. When replotted against $\sigma = (s/s_0 - 1)/\mu$ all the curves nearly coincide with the limiting case $\mu \to 0$. The biggest difference occurs for $\mu \to \infty$. Except for the neighbourhood of the wing tip, where the load decreases anyhow and a slight inaccuracy does not affect very much the overall results, the difference between the solution for $\mu \to 0$ and $\mu = 1$ is of the order 0·01. The solution for the constant chord wing ($\mu \to \infty$) decreases first, has a minimum of the order 0·942 and must go back to $H = 1$ for $s \to \infty$. The curve for $s_n/s_0 = 1 \cdot 5$, which is plotted in Fig. 10, was calculated from equation (117), since it was felt that this approximation gives sufficient accuracy in this case.

Using these values for $H(s)$ and the function $\varkappa(s)$ according to equation (40), we obtain the coefficients C_L and C_M by a numerical integration according to equations (54) and (57). By extending the integration up to a certain value $s = s_m$ a family of wings with varying taper parameter λ and fixed values of A and a is obtained from each of the curves $H(s)$ in Fig. 10. Because of the labour involved in determining each of these H-functions, an approximate method was applied for values of a between 0 and $\frac{1}{2}$.

Here it was thought permissible to use the approximation (117) for $H(s)$ and to replace the function $\varkappa(s)$, which is defined by (40), by

$$\varkappa(s) = 1 - \varkappa_0\left(1 - \frac{r}{s}\right) \quad \ldots \quad (118)$$

where $\varkappa_0 = 0 \cdot 85$ yields a reasonable approximation of the function (*see* Fig. 13). Then the integration in (54) and (57) can be carried out in general terms. We obtain

$$C_L = \alpha \frac{\pi}{2} A \left[\frac{s_0^2}{s_m^2} + 2\int_{s_0}^{s_m} \frac{(s_n - s_0)\varkappa_0}{\sqrt{(s_n - s)}\sqrt{(s_n + s - 2s_0)}} \left(1 - \frac{s_n}{s}\frac{s - s_0}{s_n - s_0}\right)\frac{s\,ds}{s_m^2}\right]$$

or, when substituting equation (116):

$$C_L = \alpha \frac{\pi}{2} A \frac{s_0^2}{s_m^2}\left[1 + 2\int_0^{\sigma_m} \frac{\varkappa_0\mu}{\sqrt{(1 - \sigma^2)}}(1 - \sigma)\,d\sigma\right]$$

with

$$\sigma_m = \frac{1}{\mu}\left(\frac{s_m}{s_0} - 1\right), \qquad \mu = \frac{s_n}{s_0} - 1 = \frac{a}{1 - a} \quad \ldots \quad (119)$$

and in a similar way:

$$-C_M = \alpha \frac{\pi}{3} A \frac{2}{1 + \lambda}\frac{s_0^2}{s_m^2}\left[1 + 3\int_0^{\sigma_m} \frac{\varkappa_0\mu}{\sqrt{(1 - \sigma^2)}}(1 - \sigma)(1 + \mu\sigma)\,d\sigma\right].$$

The evaluation of these two integrals yields the result ($\lambda \leqslant a \leqslant \frac{1}{2}$):

$$C_L = \alpha \frac{\pi}{2} A \frac{s_0^2}{s_m^2}\left\{1 + 2\varkappa_0\mu(\sqrt{(1 - \sigma_m^2)} - 1 + \sin^{-1}\sigma_m)\right\} \quad \ldots \quad (120)$$

$$-C_M = \alpha \frac{2\pi}{3(1 + \lambda)} A \frac{s_0^2}{s_m^2}\left\{1 + 3\varkappa_0\mu\left[\left(1 - \frac{\mu}{2}\right)\sin^{-1}\sigma_m + \frac{\sigma_m\mu}{2}\sqrt{(1 - \sigma_m^2)} - (1 - \mu)\left(1 - \sqrt{(1 - \sigma_m^2)}\right)\right]\right\} \quad \ldots \quad (121)$$

where $$\frac{s_m}{s_0} = \frac{1 - \lambda}{1 - a}.$$

Since the aspect ratio A occurs only as a factor and since μ and σ_m are according to (119) directly connected with λ and a, the functions C_L/A and $-C_M/C_L$ can be plotted as functions of the taper ratio λ with a as a parameter.

In Fig. 11 and 12 equations (120) and (121) have been used for values of a between 0 and $\frac{1}{2}$. For $a = \frac{1}{2}$ the difference between the approximation (120) and (121) and the rigorous solution according to (54) and (57), using the exact values for $H(s)$ and $\varkappa(s)$, was not noticeable in the pictures. For the constant chord wing ($\lambda = 1$) also a rigorous solution without any approximations was determined by the method in Appendix II.1 and a numerical integration. The gap for $\frac{1}{2} < a < 1$ was filled by interpolation, whereby the curves with the parameter $(1 - \lambda)/(1 - a) = s_m/s_0$ were employed and the results (120) and (121) were used as a first approximation.

As can be seen from Fig. 11, C_L is, for a given aspect ratio A and a given sweep parameter a, independent of the taper ratio λ if $\lambda \geqslant a$, and then C_L decreases rapidly with decreasing λ. For a constant chord wing ($\lambda = a = 1$) the plan form is better described by the parameter

$$\frac{s_m}{s_0} = \frac{1 - \lambda}{1 - a} = \frac{1 + \lambda}{4} A \tan \Lambda_t \quad \ldots \quad (122)$$

which is also indicated in the figures.

The distance of the a.c. behind the apex in terms of the root chord is plotted in Fig. 12. It depends only on the taper ratio λ and the sweep parameter a. For a given value of a, the a.c. which is in the leading edge for an unswept constant-chord wing, moves backwards with decreasing taper ratio λ. In Fig. 9 the position of the a.c. is indicated for various plan forms. The a.c. position in terms of the standard mean chord is obtained from this picture by a multiplication by the conversion factor

$$\frac{c_r}{\bar{c}} = \frac{2}{1 + \lambda} .$$

6.3.2. *Rolling moment due to roll.*—The rolling moment due to roll (roll damping) of a flat wing of the plan form given in Fig. 8 can be calculated from equation (76). If $s_m \leqslant \bar{s}_0$, we have $H_p \equiv 1$ and (compare equation (99)):

$$l_p = -\frac{\pi}{32} A \qquad (a \leqslant \lambda) \ldots \quad (123)$$

For $\bar{s}_m \geqslant s_0$, the function $H_p(s)$ was determined from equation (72) with $H_p(s_0) = 1$ according to the numerical method, described in Appendix II. The parameter values used in the calculation were $a = 0, \frac{1}{2}, \frac{2}{3}, 1$ or $\mu = 0, 1, 2, \infty$. The results are given in Tables 1 and 2 and plotted in Fig. 14 as functions of s/s_0. The function $H_p\sqrt{(1 - \sigma^2)}$ was replotted against σ, in order to include very small values of μ, for which $H\sqrt{(1 - \sigma^2)}$ tends to 1. Both pictures can be used to interpolate additional curves for $H_p(s)$. These were checked by the method described in Appendix II.2.

The results for l_p are plotted in Fig. 15. Since l_p is proportional to the aspect ratio A, the function $-l_p(32/\pi A)$ is given as a function of the taper ratio λ with the sweep ratio a as a parameter.

For a given value of A and of the sweep parameter a the roll damping coefficient $-l_p$ is independent of λ, if $\lambda \geqslant a$, and decreases rapidly with decreasing λ, if $\lambda \leqslant a$. Curves for constant values of $(1 - \lambda)/(1 - a) = \frac{1}{4}(1 + \lambda)A \tan \Lambda_t$ are also included in the picture.

6.3.3. *Force and pitching moment due to pitch.*—The force and the pitching moment due to pitch for the same wing plan form follow from equations (92) and (93) after the function $H_q(s)$ is determined from the integral equation (87) with $H_q(s) \equiv 1$ for $0 \leqslant s \leqslant s_0$. In Fig. 16 the function $H_q(s)$ is plotted against s/s_0 for several values of s_n/s_0 ($s_n/s_0 = 2, 3, \infty$ are computed according to Appendix II, the remainder of the curves are obtained by interpolation and checking by the method described in Appendix II.2). The function $H_q(s)$ is also plotted against σ.

Using these curves we are in a position to carry out the integrations in (92) and (93). We obtain for $s_m \leqslant \bar{s}_0$

$$- z_q = A \frac{\pi}{2} \frac{c_r}{\bar{c}} \left\{ \frac{1}{2} \frac{s_m}{s_0} + \int_{c_1}^{c_m} \left[\frac{1}{2} \left(1 + \frac{r^2}{s_m^2} \right) - \varkappa(r) \right] \frac{dx}{c_r} \right\} \quad \ldots \quad (124)$$

$$- m_q = A \frac{\pi}{2} \frac{c_r^2}{\bar{c}^2} \left\{ \frac{3}{8} \frac{s_m^2}{s_0^2} + \int_{c_1}^{c_m} \left[\frac{1}{2} \left(1 + \frac{r^2}{s_m^2} \right) - \varkappa(r) \right] \frac{x\, dx}{c_r^2} \right\} \quad \ldots \quad (125)$$

with

$$\left. \begin{array}{ll} c_1 = c_r \dfrac{1 - \lambda}{1 - a}, & c_m = c_r \dfrac{1 - \lambda a}{1 - a} \\ \dfrac{r}{s_m} = \dfrac{1 - a}{a(1 - \lambda)} \left(\dfrac{x}{c_r} - 1 \right) & \lambda \geqslant a \\ \dfrac{s_m}{\bar{s}_0} = \dfrac{1 - \lambda}{1 - a} & \end{array} \right\} \quad \ldots \quad (126)$$

For $s_m \geqslant \bar{s}_0 = s_0$ we have three contributions to the integrals, namely one for $0 \leqslant s \leqslant s_0$, the second for $s_0 \leqslant s \leqslant s_m$, where r may be considered as a function of s:

$$\frac{r}{s} = \frac{1}{a} \left(\frac{s}{s_0} - 1 \right) \quad \ldots \quad (127)$$

and the third for the tail end of the wing, where x is used as a variable:

$$c_1 = c_r(1 - \lambda)/(1 - a) \leqslant x \leqslant c_r(1 - \lambda a)/(1 - a) = c_m$$

and r is given by equation (114). We obtain

$$\begin{aligned} - z_q = \frac{\pi}{2} A \frac{c_r}{\bar{c}} \Bigg\{ & \frac{1}{2} \frac{s_0^2}{s_m^2} + \int_{s_0}^{s_m} H_q(s) \left(1 - \varkappa(s) \right) \frac{s^2\, ds}{s_m^2 s_0} \\ & + \int_{s_0}^{s_m} \left[\frac{1}{2} \left(1 + \frac{r^2}{s^2} \right) - \varkappa(s) \right] \frac{s^2\, ds}{s_m^2 s_0} + \int_{c_1}^{c_m} \left[\frac{1}{2} \left(1 + \frac{r^2}{s_m^2} \right) - \varkappa(r) \right] \frac{dx}{c_r} \Bigg\} \quad \ldots \quad (128) \end{aligned}$$

$$\begin{aligned} - m_q = \frac{\pi}{2} A \frac{c_r^2}{\bar{c}^2} \Bigg\{ & \frac{3}{8} \frac{s_0^2}{s_m^2} + \int_{s_0}^{s_m} H_q(s) \left(1 - \varkappa(s) \right) \frac{s^3\, ds}{s_m^2 s_0^2} \\ & + \int_{s_0}^{s_m} \left[\frac{1}{2} \left(1 + \frac{r^2}{s^2} \right) - \varkappa(s) \right] \frac{s^3\, ds}{s_m^2 s_0^2} + \int_{c_1}^{c_m} \left[\frac{1}{2} \left(1 + \frac{r^2}{s_m^2} \right) - \varkappa(r) \right] \frac{x\, dx}{c_r^2} \Bigg\} . \quad (129) \end{aligned}$$

Here all the integrals were determined numerically or graphically. The results are given in Figs. 17 and 18. For a given value of the taper ratio λ and a constant aspect ratio A the derivative $- z_q$ increases first with increasing sweep parameter a and then decreases, whereas the derivative $- m_q$ increases all the time with increasing sweep parameter a, except for very small values of λ. The curves $(1 - \lambda)/(1 - a) = s_m/s_0 =$ const are also indicated for the region $\lambda \leqslant a$. They can usefully be employed for interpolation between the calculated curves.

6.4. *Pressure Distributions for Two Particular Plan Forms.*—In order to show the form of the pressure distribution over a swept wing at sonic speed, the pressure has been calculated for two flat wings at incidence α. The plan form of both wings is given in Fig. 19. The first wing (cropped wing) belongs to the family considered in 6.3, the aspect ratio is $A = 2 \cdot 835$, the taper ratio $\lambda = 0 \cdot 4$, and the sweep of the quarter-chord line is $\Lambda_{1/4} = 45$ deg. The pressure distribution at a number of chordwise sections ($y/c_r = 0,\ 0 \cdot 2,\ 0 \cdot 4,\ 0 \cdot 6,\ 0 \cdot 8,\ 0 \cdot 9$) is given in Fig. 20. Then the plan form was faired according to Fig. 19 in order to avoid the sharp kink at the leading edge of the tip. (A mathematical expression was not used for the faired leading edge, but both the shape $y = s(x)$ and the derivative $s'(x)$ must be defined numerically or graphically.) The pressure distribution of the 'faired' wing is plotted in Fig. 21. In both cases the approximation (117) for the function $H(s)$ has been used, which should be good enough, since $a = 0 \cdot 4747$ is not too big.

A comparison of Figs. 20 and 21 shows, that the fairing has a considerable effect on the pressure distribution. Instead of the two kinks in the pressure curves, which correspond to the two Mach waves (at $M = 1$ a Mach cone degenerates into a plane normal to the undisturbed flow) in Fig. 20, arising from the tip and the trailing edge of the centre-section, only the last one appears in Fig. 21, whereas the first one has been smoothed out.

The local aerodynamic centre for the chordwise sections is plotted in Fig. 19 for both plan forms. Apart from the region of the wing tips, the fairing has not much influence on the a.c. position. The overall a.c. and the overall C_L are practically the same for both wings as can be seen from Table 3.

Thus the fairing of a plan form can be used in order to 'fair' the pressure curves. Whereas the load of the cropped wing is concentrated on the front part of the wing, the faired wing shows a more evenly distributed load. In order to avoid the sharp pressure rise along the Mach line $x = c_r$, extending from the kink at the trailing edge, a similar fairing of the trailing edge near the middle of the wing may be useful (compare Appendix II.1). It would spread the pressure rise more evenly over a greater area.

7. *Calculation of the Drag.*—The induced drag and the drag coefficient C_D (neglecting the drag C_{D0} due to friction) can easily be obtained, after the lift coefficient C_L has been calculated. As was shown by Ward[8], the following relation holds:

$$C_D = C_{D0} + \frac{1}{2}\left(\frac{\partial C_L}{\partial \alpha}\right)_{\alpha=0} \cdot \alpha^2 . \qquad \qquad (130)$$

Thus the drag coefficients can be determined using the chart of Fig. 11, provided one can obtain a reasonable estimate for the drag coefficient C_{D0} due to friction. The latter probably depends very much on the wing sections and on the wing plan form, since this is responsible for the formation of shock-waves, which may affect the drag to a great extent.

8. *Conclusions.*—Theoretical values, based on the assumption of an inviscid potential flow (linearized with respect to the magnitude of the disturbance caused by the wing), are given for the lift slope $dC_L/d\alpha$, the pitching moment C_M for a wing at incidence, and for the stability derivatives l_p, z_q, and m_q. The results are valid, if $A^2(1 - M^2)$ is small compared to 1. All these quantities are proportional to the aspect ratio A. For a wing family depending on three parameters (*see* Fig. 8) the results are given in the form of charts (Figs. 11, 12, 15, 17 and 18). These charts can be used to estimate the magnitude of the lift slope, pitching moment, roll damping l_p and damping in pitch m_q for a wide range of plan forms. Minor alterations of the plan form, *e.g.*, rounding off of corners or tips, usually do not affect very much these overall values, as can be seen in the example treated in section 6.4. But these minor alterations may have a big effect on the shape of the pressure distribution. A plan form with curved edges without corners has a much smoother pressure distribution than a plan form consisting of straight lines linked by sharp corners (compare Figs. 19, 20 and 21). Each corner gives rise to a Mach cone (which is here degenerated into a plane, normal to the flight direction). In a real flow this Mach cone may develop into a shock-wave with the associated sudden pressure rise and drag, whereas it is likely that a rounded leading edge produces a smoother pressure gradient possibly

without a great increase in drag. Thus the choice of the plan form of the wing becomes equally important in sonic flight as the choice of the chordwise section in subsonic flight for a wing of a larger aspect ratio.

As regards the applicability of these results, which are based on the assumption of a non-viscid flow without interference of shock-waves, to actual flow problems at transonic speeds it may be pointed out that this theory will at least show the main trends of the behaviour of a sonic flow at a small incidence. All plan forms, for which the linearized theory yields pressure distributions with steep pressure rises, are likely to produce shock-waves and the resultant drag increase and rapid variation of the aerodynamic characteristics with Mach number near the speed of sound. Thus such plan forms, where even the potential theory shows rapid variations of the stability derivatives near $M = 1$, will have to be abandoned in favour of other plan forms with more favourable characteristics. These will have to be tested as to their behaviour in a real flow. Thus the linearized theory, given in this report, will provide a useful tool in the design of transonic aircraft and the investigation of transonic flow problems. The range of validity is usually restricted by an incidence of about 4 deg, since, *e.g.*, for a Delta wing at this incidence the flow tends to break away from the surface near the apex and non-linear effects have to be accounted for.

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APPENDIX I

The Uniqueness of the Solution for the Two-dimensional Problem

In order to show that the solutions of the Laplace equation (11), which we have given in sections 3, 4 and 5, are unique, we employ a conformal transformation of the Z-plane ($Z = y + iz$) of Fig. 3 into an auxiliary U-plane.

At first we introduce the T-plane by

$$T = \sqrt{\left(\frac{s^2 - Z^2}{s^2 - r^2}\right)} \qquad \text{(I.1)}$$

(*see* Fig. 22) and go from there into the U-plane by means of

$$\frac{Z}{s} = \text{dn}\, U = \sqrt{(1 - k^2 \,\text{sn}^2\, U)} = \sqrt{(1 - k^2 T^2)}$$

$$k^2 = 1 - r^2/s^2 = 1 - k'^2 . \qquad \text{(I.2)}$$

Here $\text{sn}\, U$, $\text{cn}\, U = \sqrt{(1 - \text{sn}^2\, U)}$, and $\text{dn}\, U = \sqrt{(1 - k^2\, \text{sn}^2\, U)}$ denote the elliptic functions, introduced by Jacobi, with the periods ($4K$, $2iK'$), ($4K$, $4iK'$) and ($2K$, $4iK'$) respectively, where

$$K = K(k) = \int_0^{\pi/2} \frac{d\chi}{\sqrt{(1 - k^2 \sin^2 \chi)}}$$

and

$$K' = K'(k) \equiv K(k')$$

are complete elliptic integrals of the first kind.

In Fig. 22 points, which correspond to each other in the Z-, T- and U-plane, are denoted by the same letter. The entire Z-plane is transformed into the interior of the rectangle B″ BB′ B‴, of the U-plane, the point D ($U = iK'$) corresponds to $Z = \infty$, the points A ($U = 0$) and A′ ($U = 2iK'$) to the leading edges $Z = -s$ and $Z = +s$, the points B, B″ ($U = K$, $U = K + 2iK'$) and B′, B‴ ($U = -K$, $U = -K + 2iK'$) to the trailing edges $Z = -r$, $Z = +r$.

Since the pressure $I = \text{const}\ \mathscr{R}\{F\}$ must be a continuous function everywhere in the Z-plane (which is slit along AB and A′B′), with the only exception of the neighbourhood of the leading edges A and A′ (where the pressure may tend to infinity), the complex pressure function $F(U)$ must be bounded and continuous everywhere in the U-plane except near A, A′ and their equivalent points $U = 2mK + 2niK'$ (m, n = integers). $F(U)$ must have the period $2K$ along the real axis in order to ensure continuity of the pressure between both faces B C B′ of the wake. Since the line $\mathscr{I}(U) = 0$ corresponds to the left-hand section AB of the wing, the line $\mathscr{I}(U) = 2iK'$ to the right-hand section B′A′ and so on, the pressure function $F(U)$ or at least its derivative dF/dU must have the period $4iK'$ along the imaginary axis. F itself may take on an imaginary additive constant if U is replaced by $U + 4iK'$.

Near $U = 0$ the function $F(U)$ must behave as U^{-1} and dF/dU as U^{-2}, since only this singularity leads to the well-known form of a singularity near the leading edge $Z = \pm S$.

Our solution I for the wing at incidence (compare equation (43), (45)) may be written as

$$\frac{I}{V^2} = -H(s) \frac{ds}{dx} \alpha \,.\, \mathscr{R}\{F\}$$

with

$$F(U) = \frac{\text{cn } U \text{ dn } U}{\text{sn } U} + \int_0^U \left[\text{dn}^2 U - \frac{E(k)}{K(k)}\right] dU \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (\text{I.3})$$

$$\frac{dF}{dU} = 1 - \frac{E(k)}{K(k)} - \frac{1}{\text{sn}^2 U} .$$

For the wing in roll (equation (68)) we have the solution

$$\frac{I}{V^2} = - h_p(x) \, \mathscr{R}\{F_p\}$$

with

$$\left.\begin{aligned} F_p(U) &= \frac{\text{cn } U}{\text{sn } U} + \frac{1}{i} = i \, (\text{dn } (U - iK') - 1) \\ \frac{dF_p}{dU} &= - \frac{\text{dn } U}{\text{sn}^2 U} = - ik^2 \text{ sn } (U - iK') \text{ cn } (U - iK') \end{aligned}\right\} \quad \ldots \quad \ldots \quad (\text{I.4})$$

Finally for the wing in pitch we have (compare equation (80))

$$\frac{I}{V^2} = - \frac{qx}{V} H_q(S) \frac{ds}{dx} \mathscr{R}\{F\} - \frac{qs}{V} \mathscr{R}\{F_q\}$$

where $F(U)$ is given in (I.3) and

$$\left.\begin{aligned} F_q(U) &= - i \text{ dn } U + \int_0^U \left[\text{dn}^2 U - \frac{E(k)}{K(k)}\right] dU \\ \frac{dF_q}{dU} &= + ik^2 \text{ sn } U \text{ cn } U + \text{dn}^2 U - \frac{E(k)}{K(k)} \\ &= 1 - \frac{E(k)}{K(k)} - k^2 \text{ sn } U \, (\text{sn } U - i \text{ cn } U) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (\text{I.5})$$

The function dF/dU has the periods $2K$ and $2iK'$ and a pole of the order 2 at $U = 2mK + 2niK'$. Its integral $F(U)$ vanishes for $U = iK'$ and the constant $\varkappa = E/K$ has been chosen as to make $F(U + 2K) = F(U)$. Furthermore we have $F(U + 2iK') = F(U) - \pi i/(2K)$ so that $\mathscr{R}\{F\}$ has the period $2iK'$ as required. Since an elliptic function is (apart from an additive constant) determined in a unique way by the principal part of the expansions at the poles, the only arbitrary constant is the coefficient of U^{-2} of the expansion at $U = 0$ which is determined afterwards by an integral equation. Thus our solution is unique.

The function $F_p(U)$ has the periods $2K$ and $4iK'$ and poles at $U = 2mK + 2inK'$. The integral $F(U)$ vanishes at $U = iK'$ and we have

$$F_p(U + 2K) = F_p(U) \, , \qquad F_p(U + 2iK') = - F_p(U) \, .$$

Thus the pressure $I \sim \mathscr{R}\{F_p\}$ has a different sign along the right and the left wing section, as required. Again the solution is determined in a unique way by the principal part of dF_p/dU at the pole.

For the wing in pitch we have to use two functions, one $F(U)$ for the forward part of the wing with $ds/dx \geqslant 0$ and another function $F_q(U)$ for the whole wing. The coefficient of U^{-2} near $U = 0$ of the first function $F(U)$ follows in a similar way as for the incidence case from an integral equation. The second function $F_q(U)$ is introduced, to satisfy the boundary condition (79) $\partial(I/V^2)/\partial z = q/V$ along the wing. Since this contribution to the pressure function must remain finite everywhere in the Z-plane (for $Z = \pm s$ may now correspond to a side edge or even a trailing edge of the wing), we have to take care that no singularities occur inside the rectangle B″ B B′ B‴, which means that the singularity must occur in the adjoining rectangle. Thus the function dn^2U has been introduced in dF_q/dU, so that dF_q/dU has the periods $2K$ and $4iK'$ and a singularity at $U = 2mK + (4n - 1)iK'$ of the order $- 2$. Its integral $F_q(U)$ vanishes for $U = iK'$ and we have

$$F_q(U + 2K) = F_q(U)$$

$$F_q(U + 2iK') = F_q(U) - \frac{\pi i}{2K} + 2i \,\mathrm{dn}\, U$$

$$F_q(U + 4iK') = F_q(U) - \frac{2\pi i}{2K}$$

so that the real part of F_q has the period $4K'$, and is symmetrical with respect to the line $\mathcal{I}(U) = iK'$.

APPENDIX II

Solution of the Integral Equation (50)

In this Appendix we shall describe a numerical method for the solution of the three integral equations (50), (72) and (87) for the functions $H(s)$, $H_p(s)$ and $H_q(s)$. The equations may be written in the form ($s_0 \leqslant s$)

$$\int_{s_0}^{s} H'(s_1)\sqrt{\left(\frac{s^2 - r_1^2}{s^2 - s_1^2}\right)}\,ds_1 - \int_{s_0}^{s} H(s_1)\,\frac{(r_1\,dr_1/ds_1 - \varkappa_1 \,.\, s_1)}{\sqrt{(s^2 - s_1^2)}\sqrt{(s^2 - r_1^2)}}\,ds_1 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (50)$$

$$\int_{s_0}^{s} H_p'(S_1)\sqrt{\left(\frac{s^2 - r_1^2}{s^2 - s_1^2}\right)}\,ds_1 - \int_{s_0}^{s} H_p(s_1)\,\frac{(r_1\,dr_1/ds_1 - s_1)}{\sqrt{(s^2 - s_1^2)}\sqrt{(s^2 - r_1^2)}}\,ds_1 = \sqrt{\left(1 - \frac{s_0^2}{s^2}\right)} \quad \ldots \quad (72)$$

and

$$\int_{s_0}^{s} \frac{d}{ds_1}\left(s_1 \,.\, H_q(s_1)\right)\sqrt{\left(\frac{s^2 - r_1^2}{s^2 - r_1^2}\right)}\,ds_1 - \int_{s_0}^{s}\left(s_1 \,.\, H_q(s_1)\right)\frac{(r_1\,dr_1/ds_1 - \varkappa_1 \,.\, s_1)\,ds_1}{\sqrt{(s^2 - s_1^2)}\sqrt{(s^2 - r_1^2)}}$$

$$= \int_{s_0}^{s} \frac{(s^2 - s_1^2 \,.\, \varkappa_1)\,ds_1}{\sqrt{(s^2 - s_1^2)}\sqrt{(s^2 - r_1^2)}} \,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (87)$$

Here $r_1 = r(s_1)$ and $\varkappa_1 = \varkappa(s_1)$ are for any plan form known functions of s_1.

1. *Solution Near* $s = s_0$.—We consider first the neighbourhood of $s = s_0$, where the influence of the trailing edge comes in first. We may assume there that the leading edge is straight and the trailing edge may be either curved or straight ($m = 1$) and have thus for the right half-wing ($y > 0$):

$$\left.\begin{aligned} s_1 &= s_0 + (x - c_r)B_1 \\ r_1 &= s_0 B_2 \,.\left|\frac{x - c_r}{c_r}\right|^m \qquad (0 < m \leqslant 1) \end{aligned}\right\} \quad \ldots \quad \ldots \quad (\mathrm{II}.1)$$

so that (B_1, B_2, B_3 are suitable constants)

$$r_1 = s_0 B_2 \left(\frac{s_1 - s_0}{c_r B_1}\right)^m = s_0 B_3 \left(\frac{s_1 - s_0}{s_0}\right)^m$$

$$k^2 = 1 - \frac{r_1^{\,2}}{s_1^{\,2}} \simeq 1 - B_3^{\,2}\left(\frac{s_1}{s_0} - 1\right)^{2m}$$

and

$$\varkappa_1 = \frac{E(k)}{K(k)} \simeq \frac{1}{\log\,(4s_1/r_1)} \simeq \frac{1}{\log\,\{4s_0^{\,m}/B_3(s_1 - s_0)^m\}} \qquad \ldots \qquad \ldots \text{(II.2)}$$

$$r_1 \frac{dr_1}{ds_1} \simeq m s_0 B_3^{\,2} \left(\frac{s_1 - s_0}{s_0}\right)^{2m-1} .$$

Furthermore we may write

$$\sqrt{(s^2 - r_1^{\,2})} \simeq s \simeq s_0$$

$$\sqrt{(s^2 - s_1^{\,2})} \simeq \sqrt{(s - s_1)}\sqrt{(2s_0)}$$

so that finally we obtain instead of (50), (72), and (87) the simplified equations:

$$\int_{s_0}^{s} H'(s_1) \frac{ds_1}{\sqrt{(s - s_1)}} - \int_{s_0}^{s} H(s_1) \frac{\left(mB_3^{\,2}\left(\frac{s_1 - s_0}{s_0}\right)^{2m-1} - \varkappa_1\right) ds_1}{s_0\sqrt{(s - s_1)}} = 0$$

$$\int_{s_0}^{s} H_p'(s_1) \frac{ds_1}{\sqrt{(s - s_1)}} - \int_{s_0}^{s} H_p(s_1) \frac{\left(mB_3^{\,2}\left(\frac{s_1 - s_0}{s_0}\right)^{2m-1} - 1\right) ds_1}{s_0\sqrt{(s - s_1)}} = \sqrt{\frac{2}{s_0}}\sqrt{\left(1 - \frac{s_0^{\,2}}{s^2}\right)}$$

$$\int_{s_0}^{s} \frac{d}{ds_1}\left(s_1 H_q(s_1)\right) \frac{ds_1}{\sqrt{(s - s_1)}} - \int_{s_0}^{s} H_q(s_1) \frac{\left(mB_3^{\,2}\left(\frac{s_1 - s_0}{s_0}\right)^{2m-1} - \varkappa_1\right) ds_1}{\sqrt{(s - s_1)}}$$

$$= \int_{s_0}^{s} \frac{(1 - \varkappa_1)\, ds_1}{\sqrt{(s - s_1)}} .$$

In the second term the functions H, H_p, H_q may be replaced by 1 and we obtain the solution in the approximate form:

$$\left.\begin{aligned} s_0 H'(s_1) &\simeq -\varkappa_1 + mB_3^{\,2}\left(\frac{s_1 - s_0}{s_0}\right)^{2m-1} \\ s_0 H_p'(s_1) &\simeq \qquad\quad mB_3^{\,2}\left(\frac{s_1 - s_0}{s_0}\right)^{2m-1} \\ \frac{d}{ds_1}\left(s_1 H_q(s_1)\right) &\simeq 1 + s_0 \frac{dH_q}{ds_1} \simeq 1 - 2\varkappa_1 + mB_3^{\,2}\left(\frac{s_1 - s_0}{s_0}\right)^{2m-1} \end{aligned}\right| \qquad \ldots \text{(II.3)}$$

or

$$s_0 H_q'(s_1) \simeq -2\varkappa_1 + mB_3^{\,2}\left(\frac{s_1 - s_0}{s_0}\right)^{2m-1} .$$

and finally

$$\left.\begin{aligned}H(s_1) &\simeq 1 - \int_{s_0}^{s_1} \frac{\varkappa_1\, ds_1}{s_0} + \tfrac{1}{2} B_3{}^2 \left(\frac{s_1 - s_0}{s_0}\right)^{2m} \\ H_p(s_1) &\simeq 1 \qquad\qquad\qquad + \tfrac{1}{2} B_3{}^2 \left(\frac{s_1 - s_0}{s_0}\right)^{2m} \\ H_q(s_1) &\simeq 1 - 2 \int_{s_0}^{s_1} \frac{\varkappa_1\, ds_1}{s_0} + \tfrac{1}{2} B_3{}^2 \left(\frac{s_1 - s_0}{s_0}\right)^{2m}\end{aligned}\right\} \quad \ldots \ldots \ldots \text{(II.4)}$$

Thus the influence of the second term, which produces a fairly quick decrease in H and thus an adverse pressure gradient in the chordwise direction, may be cancelled by the third term, which is of opposite sign, if m is chosen in a proper way. For a straight trailing edge ($m = 1$) the influence of this term may be noticed only for bigger values of $s - s_0$, but for a parabolic trailing edge ($m = \frac{1}{2}$) its influence appears very soon and may even ($m < \frac{1}{2}$) produce a favourable pressure gradient in the chordwise direction.

2. *An Alternative Form for the Integral Equation.*—In order to eliminate the derivative H' in the equation (50) we integrate by parts and obtain

$$-\int_{s_0}^{s} \frac{d}{ds_1} [H(s) - H(s_1)] \sqrt{\left(\frac{s^2 - r_1{}^2}{s^2 - s_1{}^2}\right)}\, ds_1$$

$$= [H(s) - H(s_0)] \frac{s}{\sqrt{(s^2 - s_0{}^2)}} + \int_{s_0}^{s} \frac{\{H(s) - H(s_1)\} [s_1(s^2 - r_1{}^2) - r_1\, dr_1/ds_1 \,.\, (s^2 - s_1{}^2)]\, ds_1}{\sqrt{(s^2 - r_1{}^2)}(s^2 - s_1{}^2)^{3/2}}$$

$$= \int_{s_0}^{s} H(s_1) \frac{(r_1\, dr_1/ds_1 - \varkappa_1 s_1)\, ds_1}{\sqrt{(s^2 - r_1{}^2)}\sqrt{(s^2 - s_1{}^2)}}$$

or

$$\frac{s}{\sqrt{(s^2 - s_0{}^2)}} [H(s) - H(s_0)] = \int_{s_0}^{s} \frac{H(s) r_1\, dr_1/ds_1 - H(s_1) s_1 \,.\, \varkappa_1}{\sqrt{(s^2 - r_1{}^2)}\sqrt{(s^2 - s_1{}^2)}}\, ds_1$$

$$- \int_{s_0}^{s} \frac{[H(s) - H(s_1)]}{(s^2 - s_1{}^2)^{3/2}} s_1 \sqrt{(s^2 - r_1{}^2)}\, ds_1 \,. \quad \ldots \ldots \ldots \text{(II.5)}$$

The corresponding equation for H_p is obtained from (II.5) by replacing $\varkappa_1$ by 1 and adding the term on the left-hand side of equation (72) on the right. In the equation for $s_1 H_q(s_1)$ we have to add on the right of (II.5) the term on the right of equation (87).

Equation (II.5) suggests, to try a solution by means of an iteration process where an approximation for $H(s)$ (obtained by guessing or another calculation) is introduced in the right-hand side and the improved function $H(s)$ is obtained from the left. Unfortunately this method does not work very well, unless the approximation introduced in the right is already fairly near to the actual solution. Thus this procedure was only used to check solutions calculated by another method (*see* (II.3)) or to check solutions which had been obtained by interpolation between functions $H(s)$ calculated according to (II.3).

For the accuracy, required to compile the graphs in this report, it was sufficient to determine the integrals on the right of (II.5) numerically or by plotting the integrands and counting out the areas.

3. *A Direct Method of Solution.*—Since the iteration method, explained in section 2, did not yield satisfactory results a more elaborate method was developed, to solve equation (50) or (II.5) for the function $H(s)$. It was used to calculate a few curves for $H(\bar{s})$, $H_p(s)$, and $H_q(s)$.

Then a number of additional curves for different geometrical parameters of the plan form were interpolated and the entire picture was checked and corrected if necessary, by the method in section 2.

The new method is based on the assumption that our solution $H(s)$ may be approximated by a straight line for any interval $\bar{s}_j \leqslant s \leqslant \bar{s}_{j+1}$, which is not too big. Since the singularity at the beginning $s = s_0$ is such (compare (II.4)) that this assumption holds even there with sufficient accuracy, we may apply our method straight from the beginning $s = s_0$.

Thus we are entitled to approximate also the function

$$\frac{H(s)\, r_1\, dr_1/ds_1 - H(s_1)\,.\, s_1\,.\,\varkappa_1}{\sqrt{(s^2 - r_1^2)}} \equiv f(s, s_1) \quad \ldots \quad \text{(II.6)}$$

by a straight line for the interval $\bar{s}_j \leqslant s_1 \leqslant \bar{s}_{j+1}$ $(j = 0, 1, 2 \ldots)$:

$$f(s, s_1) = \frac{(\bar{s}_{j+1} - s_1) f(s, \bar{s}_j) + (s_1 - \bar{s}_j) f(s, \bar{s}_{j+1})}{\bar{s}_{j+1} - \bar{s}_j}.$$

Then the first integral in equation (II.5) can be evaluated as a sum of integrals, each taken over a small interval $\bar{s}_j \leqslant s_1 \leqslant \bar{s}_{j+1}$ $(s \equiv \bar{s}_N)$:

$$\int_{\bar{s}_j}^{\bar{s}_{j+1}} \frac{f(\bar{s}_N, s_1)\, ds_1}{\sqrt{(\bar{s}_N^2 - s_1^2)}} = \frac{\bar{s}_{j+1} f(\bar{s}_N, \bar{s}_j) - \bar{s}_j f(\bar{s}_N, \bar{s}_{j+1})}{\bar{s}_{j+1} - \bar{s}_j} \,.\, I^N_{1,j}$$

$$+ \frac{f(\bar{s}_N, \bar{s}_{j+1}) - f(\bar{s}_N, \bar{s}_j)}{\bar{s}_{j+1} - \bar{s}_j} \,.\, I^N_{2,j} \quad \ldots \quad \text{(II.8)}$$

where

$$\left.\begin{aligned} I^N_{1,j} &= \int_{\bar{s}_j}^{\bar{s}_{j+1}} \frac{ds_1}{\sqrt{(\bar{s}_N^2 - s_1^2)}} = \sin^{-1}\frac{\bar{s}_{j+1}}{\bar{s}_N} - \sin^{-1}\frac{\bar{s}_j}{\bar{s}_N} \\ I^N_{2,j} &= \int_{\bar{s}_j}^{\bar{s}_{j+1}} \frac{s_1\, ds_1}{\sqrt{(\bar{s}_N^2 - s_1^2)}} = -\sqrt{(\bar{s}_N^2 - \bar{s}_{j+1}^2)} + \sqrt{(\bar{s}_N^2 - \bar{s}_j^2)} \end{aligned}\right\} \quad \ldots \quad \text{(II.9)}$$

In the second integral in (II.5) we may replace

$$[H(s) - H(s_1)]\, s_1 \sqrt{(s^2 - r_1^2)} \equiv g(s, s_1) \quad \ldots \quad \text{(II.10)}$$

by a straight line:

$$g(s, s_1) = \frac{(\bar{s}_{j+1} - s_1) g(s, \bar{s}_j) + (s_1 - \bar{s}_j) g(s, \bar{s}_{j+1})}{\bar{s}_{j+1} - \bar{s}_j}. \quad \ldots \quad \text{(II.11)}$$

Then the integral over a small interval $\bar{s}_j \leqslant s_1 \leqslant \bar{s}_{j+1}$ becomes $(s \equiv \bar{s}_N)$:

$$\int_{\bar{s}}^{\bar{s}_{j+1}} \frac{g(\bar{s}_N, s_1)}{(\bar{s}_N^2 - s_1^2)^{3/2}}\, ds_1 = \frac{\bar{s}_{j+1}\, g(\bar{s}_N, \bar{s}_j) - \bar{s}_j g(\bar{s}_N, \bar{s}_{j+1})}{\bar{s}_{j+1} - \bar{s}_j} \,.\, I^N_{3,j}$$

$$+ \frac{g(\bar{s}_N, \bar{s}_{j+1}) - g(\bar{s}_N, \bar{s}_j)}{\bar{s}_{j+1} - \bar{s}_j} \,.\, I^N_{4,j} \quad \ldots \quad \text{(II.12)}$$

where

$$\left.\begin{aligned} I_{3,j}^N &= \int_{\bar{s}_j}^{\bar{s}_{j+1}} \frac{ds_1}{(\bar{s}_N^{\,2} - s_1^{\,2})^{3/2}} = \frac{1}{\bar{s}_N^{\,2}} \left[\frac{\bar{s}_{j+1}}{\sqrt{(\bar{s}_N^{\,2} - \bar{s}_{j+1}^{\,2})}} - \frac{\bar{s}_j}{\sqrt{(\bar{s}_N^{\,2} - \bar{s}_j^{\,2})}} \right] \\ I_{4,j}^N &= \int_{\bar{s}_j}^{\bar{s}_{j+1}} \frac{s_1\, ds_1}{(\bar{s}_N^{\,2} - s_1^{\,2})^{3/2}} = \frac{1}{\sqrt{(\bar{s}_N^{\,2} - \bar{s}_{j+1}^{\,2})}} - \frac{1}{\sqrt{(\bar{s}_N^{\,2} - \bar{s}_j^{\,2})}} \end{aligned}\right\} \quad \ldots \text{(II.13)}$$

For the last interval $\bar{s}_{N-1} \leqslant s_1 \leqslant \bar{s}_N \equiv s$ we have instead $(j = N - 1)$:

$$g(s, s_1) = \frac{\bar{s}_N - \bar{s}_1}{\bar{s}_N - \bar{s}_{N-1}}\, g(s, \bar{s}_{N-1}) \qquad \ldots \text{(II.14)}$$

and

$$\int_{\bar{s}_{N-1}}^{\bar{s}_N} \frac{g(\bar{s}_N, s_1)\, ds_1}{(s^2 - s_1^{\,2})^{3/2}} = \frac{\bar{s}_N - \bar{s}_{N-1}}{g(\bar{s}_N, \bar{s}_{N-1})} \cdot I_5^N \qquad \ldots \text{(II.15)}$$

where

$$\begin{aligned} I_5^N &= \int_{\bar{s}_{N-1}}^{\bar{s}_N} \frac{(\bar{s}_N - s_1)\, ds_1}{(\bar{s}_N^{\,2} - s_1^{\,2})^{3/2}} = \int_{\bar{s}_{N-1}}^{\bar{s}_N} \frac{ds_1}{\sqrt{(\bar{s}_N - s_1)}(\bar{s}_N + s_1)^{3/2}} \\ &= (\bar{s}_N I_{3,j}^N - I_{4,j}^N)_{j=N-1} = \frac{1}{\bar{s}_N} \sqrt{\left(\frac{\bar{s}_N - \bar{s}_{N-1}}{\bar{s}_N + \bar{s}_{N-1}}\right)}. \end{aligned} \qquad \ldots \text{(II.16)}$$

Using only one interval $\bar{s}_0 \leqslant s_1 \leqslant \bar{s}_1 \equiv s$ $(N = 1)$ we obtain from equation (II.5) the following equation:

$$\begin{aligned} \frac{s_1}{\sqrt{(s_1^{\,2} - s_0^{\,2})}} [H(s_1) - H(s_0)] &= \frac{s_1 f(s_1, s_0) - s_0 f(s_1, s_1)}{s_1 - s_0} \cdot I_{1,0}^1 \\ &+ \frac{f(s_1, s_1) - f(s_1, s_0)}{s_1 - s_0} \cdot I_{2,0}^1 - \frac{g(s_1, s_0)}{s_1 - s_0} I_5^1 \end{aligned}$$

or $(H(s_0) = 1)$:

$$\begin{aligned} \frac{s_1}{\sqrt{(s_1^{\,2} - s_0^{\,2})}} [H(s_1) - 1] &= - \frac{H(s_0) s_0 \varkappa(s_0)}{s_1} \cdot \frac{s_1\, I_{1,0}^1 - I_{2,0}^1}{s_1 - s_0} \\ &- \frac{H(s_1)(r_1\, dr_1/ds_1) - s_1 \varkappa(s_1))}{\sqrt{(s_1^{\,2} - r_1^{\,2})}} \, \frac{s_0\, I_{1,0}^1 - I_{2,0}^1}{s_1 - s_0} \\ &- \frac{s_0 \sqrt{(s_1^{\,2} - r_0^{\,2})}}{s_1 - s_0} I_5^1 \, [H(s_1) - 1] \end{aligned}$$

which is a linear equation for $H(s_1)$. It holds only if the interval $\bar{s}_0 \leqslant s_1 \leqslant \bar{s}_1 \equiv s$ is not too big, so that the approximation of certain functions by straight lines, as indicated above, is permissible. After $H(s_1)$ has been determined, we may calculate $H(s_2)$ at a point $s_2 > s_1$ using the values $H(s_0)$ and $H(s_1)$ already determined. The linear equation for $H(s_2)$ is more complicated and more tedious to establish and it shall not be given here explicitly. In the same way we calculate the remainder of the function $H(s)$, replacing equation (II.5) by a sum of integrals over sufficiently small intervals in the way explained in equation (II.8), (II.12) and (II.15), which leads to a linear equation for the value $H(\bar{s}_n)$. Since the greatest contribution to the coefficients of this equation arises always from the last part-interval, it is essential to keep the last intervals (preceding $\bar{s}_n$) small enough, whereas the first intervals (near s_0) may be chosen much bigger, in order to reduce the time required for the work.

This method becomes a little unreliable as one approaches the wing tip of a fully tapered wing since there $(s^2 - r_1^2)^{1/2}$ tends to zero. The steps $\bar{s}_{j+1} - \bar{s}_j$ must be chosen fairly small there. All results were checked by the method of II.2. In general the contribution of the wing tips in such a case is fairly small (since the wing sections $|s - r|$ are small there), so that a certain inaccuracy of the results near the tips was considered to be fairly harmless.

Apparently equation (72) can be treated in exactly the same way, the only difference being that $\varkappa$ has to be replaced by 1. The same applies for equation (87), where only the additional term on the right has to be introduced, which is a given function of s for every plan form. Thus it seems unnecessary to give more details of the calculation here.

APPENDIX III

A Particular Wing with a Spanwise Constant Load Distribution (*Ref.* 4)

As was shown in section 3.4, equation (60), a wing plan form, which is defined by

$$r\frac{dr}{ds} = s\,.\,\varkappa(s) = s\,\frac{E(k)}{K(k)}$$

$$k^2 = 1 - r^2/s^2 \qquad \ldots \qquad (60)$$

does not develop a wake between the ' tails ' of the wing $-r < y < r$, and has thus a constant sectional lift over the middle part of the span (compare Ref. 4).

The plan form of this wing is obtained by integrating the differential equation (60). We obtain

$$-\frac{s}{2}K\frac{dk^2}{ds} = K\left(\frac{r^2}{s^2} + \frac{r}{s}\frac{dr}{ds}\right) = E - (1 - k^2)K\,. \qquad \ldots \qquad (III.1)$$

Since we have the following relation for the complete elliptic integrals K and E

$$\frac{d}{dk^2}[E - (1 - k^2)K] = \tfrac{1}{2}K \qquad \ldots \qquad (III.2)$$

we may write instead of equation (III.1):

$$-s\frac{d}{ds}[E - (1 - k^2)K] = E - (1 - k^2)K$$

which may be integrated as

$$\frac{s_0}{s} = E - (1 - k^2)K \qquad \ldots \qquad (III.3)$$

since for $s = s_0$ we have $r = 0$, $k^2 = 1$, and $E = 1$, $K = \infty$.

For small values of $r^2/s^2 = 1 - k^2$, we may write

$$\frac{s_0}{s} = 1 - \frac{1}{2}\frac{r^2}{s^2}\left(\Lambda + \tfrac{1}{2}\right) + \ldots$$

where

$$\Lambda = \log\frac{4s}{r} \simeq \log\frac{4s_0}{r}\,.$$

If the leading edge is assumed to be straight the shape of the trailing edge is approximately parabolic.

If the leading edge is cut off at a value $s = s_m$ and then joined to the trailing edge by a straight line $s = s_m$ parallel to the x-axis, this part of the wing sheds vortices and a wake between $y = r(s_m)$ and $y = s_m$. The load drops down to zero immediately after the span $s(x)$ has reached its maximum, due to the Mach cone, extending from the wing tip.

If we consider a wing of a great span, the asymptotic value for the trailing edge is obtained for big values of s and r and small values of $1 - r^2/s^2 = k^2$ as

$$\frac{2s_0}{\pi} = (s - r)\left[1 - \tfrac{3}{8}\frac{s - r}{s} + \ldots\right] \quad \ldots \quad \text{(III.4)}$$

as described in more detail in Ref. 4.

APPENDIX IV

Solution of the Integral Equations for H_1, H_p, and H_q for Small Values of μ

We consider the integral equation (50) for the function $H(s)$ for the flat wing at incidence α in the case of a wing, which is not very different from a Delta wing. We introduce

$$\mu = \frac{a}{1 - a}\,, \qquad \sigma_1 = \frac{1}{\mu}\left(\frac{s_1}{s_0} - 1\right), \qquad \sigma = \frac{1}{\mu}\left(\frac{s}{s_0} - 1\right) \quad \ldots \quad (116)$$

and obtain from (50) $(H(s_1) \equiv H^*(\sigma_1)\,;\ \varkappa_1(s_1) \equiv \varkappa_1^*(\sigma_1))$:

$$0 = \int_0^\sigma H^{*\prime}(\sigma_1)\sqrt{\left[\frac{(1+\mu\sigma)^2 - (1+\mu)^2\sigma_1^2}{(1+\mu\sigma)^2 - (1+\mu\sigma_1)^2}\right]}\,d\sigma_1$$

$$- \int_0^\sigma H^*(\sigma_1)\frac{(1+\mu)^2\sigma_1 - \mu(1+\mu\sigma_1)\varkappa_1^*(\sigma_1)}{\sqrt{[(1+\mu\sigma)^2 - (1+\mu)^2\sigma_1^2]}\sqrt{[(1+\mu\sigma)^2 - (1+\mu\sigma_1)^2]}}\,d\sigma_1 \quad \ldots \quad \text{(IV.1)}$$

since

$$\left.\begin{aligned}\frac{r_1}{s_0} &= \frac{1}{a}\left(\frac{s_1}{s_0} - 1\right) = (1+\mu)\sigma_1\\ \frac{dr_1}{ds_1} &= \frac{1}{a} = \frac{1+\mu}{\mu}\,, \qquad ds_1 = \mu\,d\sigma_1\end{aligned}\right\} \quad \ldots \quad \text{(IV.2)}$$

We multiply equation (IV.1) by $\sqrt{(2\mu)}$ and go to the limit $\mu \to 0$:

$$0 = \int_0^\sigma \frac{dH^*(\sigma_1)}{d\sigma_1}\frac{\sqrt{(1-\sigma_1^2)}}{\sqrt{(\sigma-\sigma_1)}}\,d\sigma_1 - \int_0^\sigma \frac{H^*(\sigma_1)\sigma_1\,d\sigma_1}{\sqrt{(1-\sigma_1^2)}\sqrt{(\sigma-\sigma_1)}}$$

which is equivalent to

$$0 = \int_0^\sigma \frac{d}{d\sigma_1}\left[H^*(\sigma_1)\sqrt{(1-\sigma_1^2)}\right]\frac{d\sigma_1}{\sqrt{(\sigma-\sigma_1)}}\,. \quad \ldots \quad \text{(IV.3)}$$

This is Abel's integral equation for the function $H^*(\sigma_1)\sqrt{(1-\sigma_1^2)}$. The (unique) solution is

$$H(\sigma_1)\sqrt{(1-\sigma_1^2)} = \frac{1}{\pi}\int_0^{\sigma_1} \frac{d\sigma}{\sqrt{\sigma}\sqrt{(\sigma_1-\sigma)}} = 1 ,$$

since $H^*(0) = 1$. Thus we have for small values of $\mu = \dfrac{a}{1-a}$:

$$H(s) = \frac{s_n - s_0}{\sqrt{(s_n - s)}\sqrt{(s_n + s - 2s_0)}} \, . \qquad \text{(IV.4)}$$

By similar considerations it can be proved that for small values of μ, $H_p(s) = H(s)$ and $H_q(s)$ are also given by equation (IV.4). Whereas the approximation (IV.4) for $H(s)$ can be used over quite a considerable range of μ in the incidence case, as has been shown in section 6.3, the applicability of this approximation for the functions $H_p(s)$ and $H_q(s)$ is much more restricted.

TABLE 1

The H-functions as Functions of s/s_0

	$H(s)$		$H_p(s)$			$H_q(s)$		
$\frac{s}{s_0}$	$\frac{s_n}{s_0} = \infty$	$\frac{s_n}{s_0} = 2$	$\frac{s_n}{s_0} = \infty$	$\frac{s_n}{s_0} = 3$	$\frac{s_n}{s_0} = 2$	$\frac{s_n}{s_0} = \infty$	$\frac{s_n}{s_0} = 3$	$\frac{s_n}{s_0} = 2$
1·0	1·0	1·0	1·0	1·0	1·0	1·0	1·0	1·0
1·05	0·992	0·995	1·001	1·003	1·004	0·980	0·982	0·984
1·1	0·984	0·997	1·005	1·012	1·015	0·967	0·968	0·975
1·2	0·972	1·011	1·015	1·034	1·053	0·938	0·947	0·963
1·4	0·957	1·095	1·041	1·102	1·190	0·898	0·926	0·987
1·6	0·948	1·256	1·070	1·186	1·425	0·870	0·926	1·079
1·8	0·944	1·622	1·099	1·285	1·905	0·850	0·942	1·314
1·9		2·077						1·621
2·0	0·942	—	1·127	1·405	—	0·836	0·972	—
2·2	0·942		1·154	1·547		0·826	1·023	
2·4	0·942		1·179	1·731		0·819	1·099	
2·6							1·220	
2·8							1·451	
2·9							1·709	
3·0							—	

$$\mu = \frac{s_n}{s_0} - 1, \qquad \sigma = \frac{1}{\mu}\left(\frac{s}{s_0} - 1\right)$$

TABLE 2

The H-functions as Functions of $\sigma = \frac{1}{\mu}\left(\frac{s}{s_0} - 1\right)$

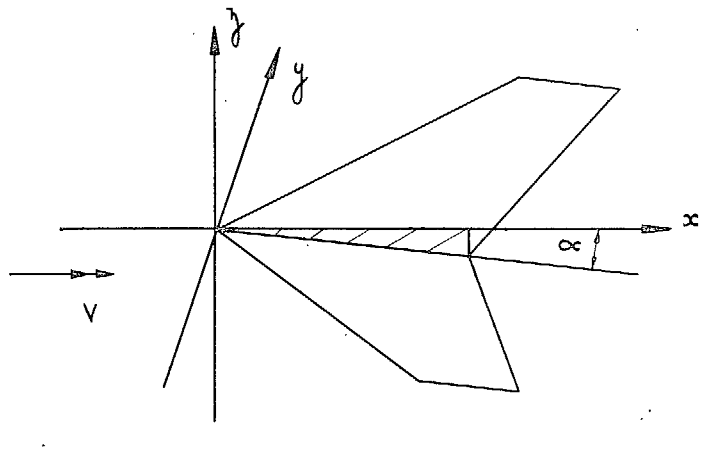
	$H(s)\sqrt{(1-\sigma^2)}$	$H_p(s)\sqrt{(1-\sigma^2)}$		$H_q(s)\sqrt{(1-\sigma^2)}$	
σ	$\mu = 1$	$\mu = 2$	$\mu = 1$	$\mu = 2$	$\mu = 1$
0	1·0	1·0	1·0	1·0	1·0
0·05	0·994	1·010	1·002	0·967	0·983
0·1	0·992	1·029	1·010	0·942	0·970
0·2	0·991	1·080	1·032	0·907	0·944
0·3		1·131		0·883	0·922
0·4	1·004	1·178	1·091	0·863	0·905
0·5		1·216		0·842	0·885
0·6	1·005	1·237	1·140	0·818	0·863
0·7		1·236		0·785	0·835
0·8	0·973		1·143	0·732	0·788
0·9	0·905			0·632	0·707

$$\lim_{\mu \to 0} H(s)\sqrt{(1-\sigma^2)} = 1, \quad \lim_{\mu \to 0} H_p(s)\sqrt{(1-\sigma^2)} = 1, \quad \lim_{\mu \to 0} H_q(s)\sqrt{(1-\sigma^2)} = 1$$

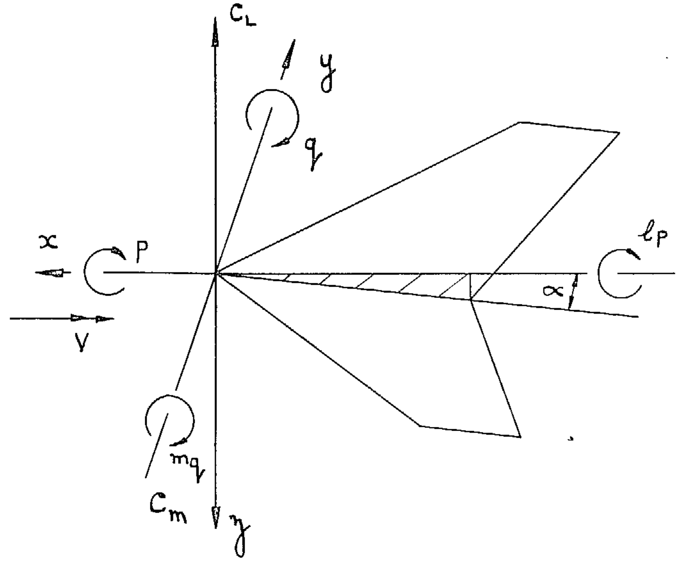
TABLE 3

Aerodynamic Characteristics for the Wings in Fig. 19

	Cropped wing	Faired wing
$\frac{dC_L}{d\alpha}$	4·11	4·11
$\frac{dC_M}{d\alpha}$	3·01	3·06
a.c. position $\frac{x}{c_r}$ behind apex	0·73	0·74



(a). COORDINATES USED IN THE THEORY.



(b). STABILITY AXES (STABILITY DERIVATIVES).

FIGS. 1a and 1b. Notations.

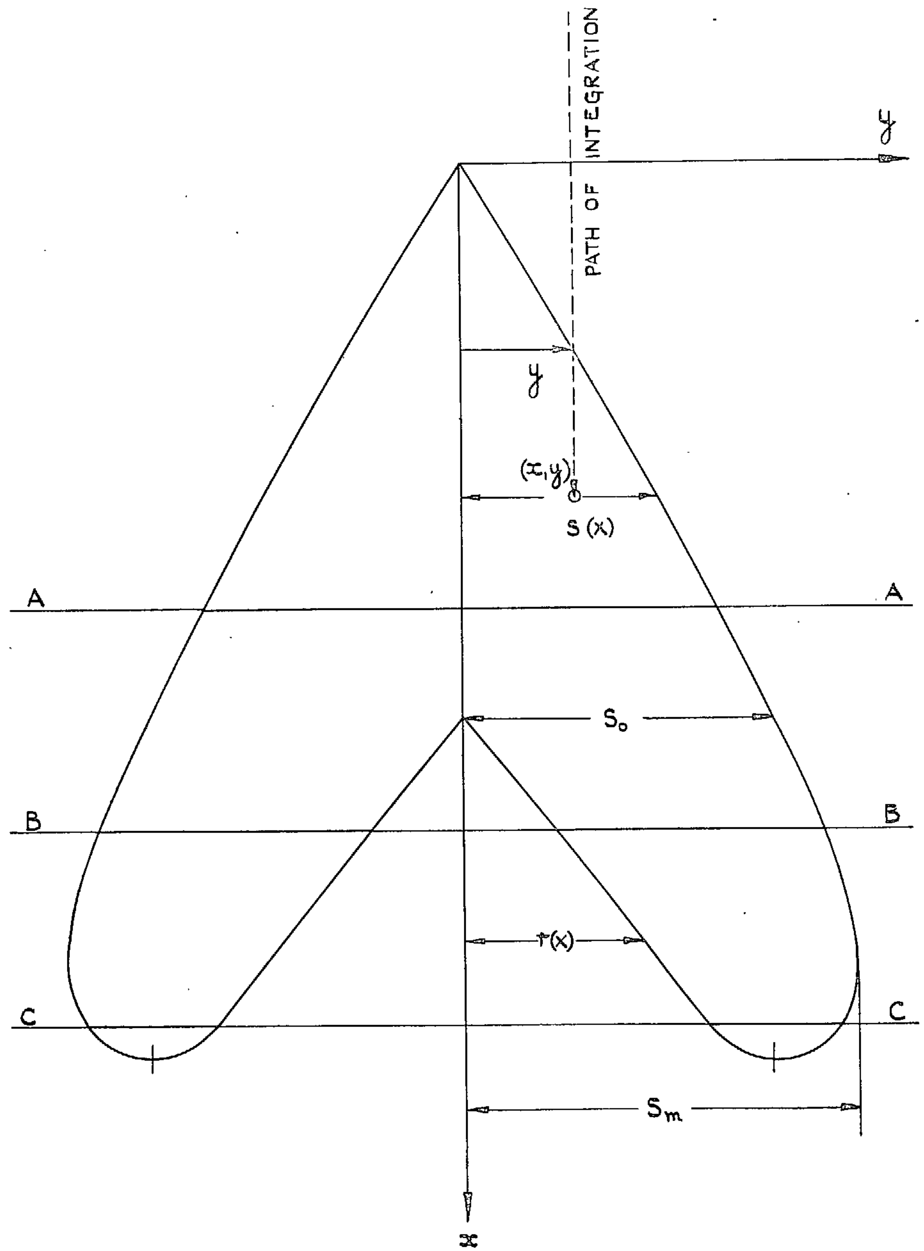


FIG. 2. The plan form of the wing.

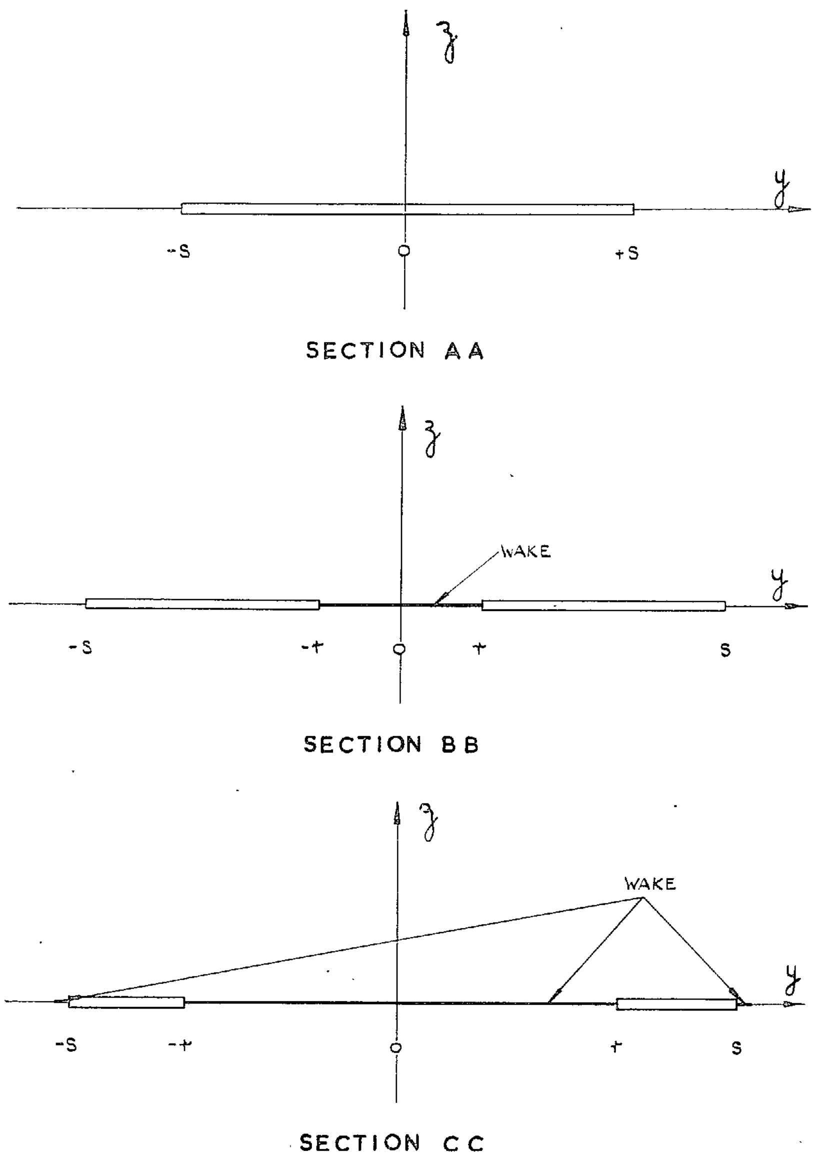


FIG. 3. The (y, z)-plane (sections AA, BB and CC).

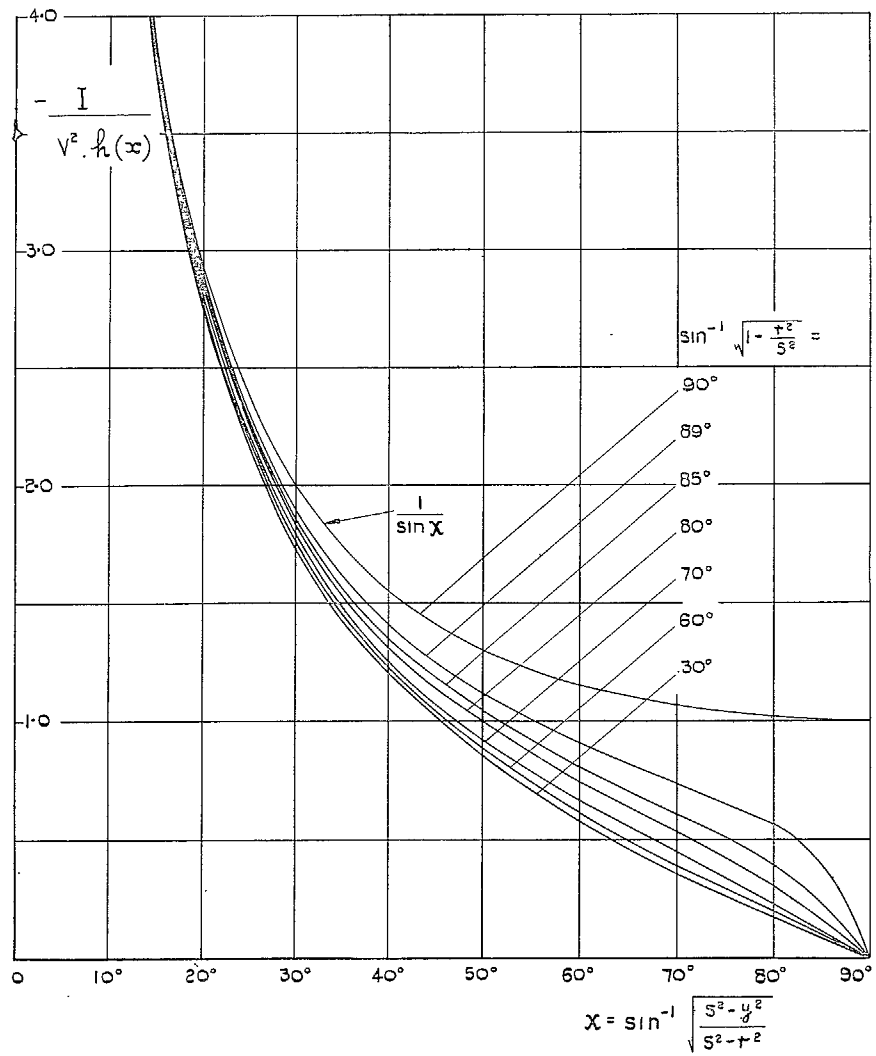


FIG. 4. Pressure curves for the incidence case.

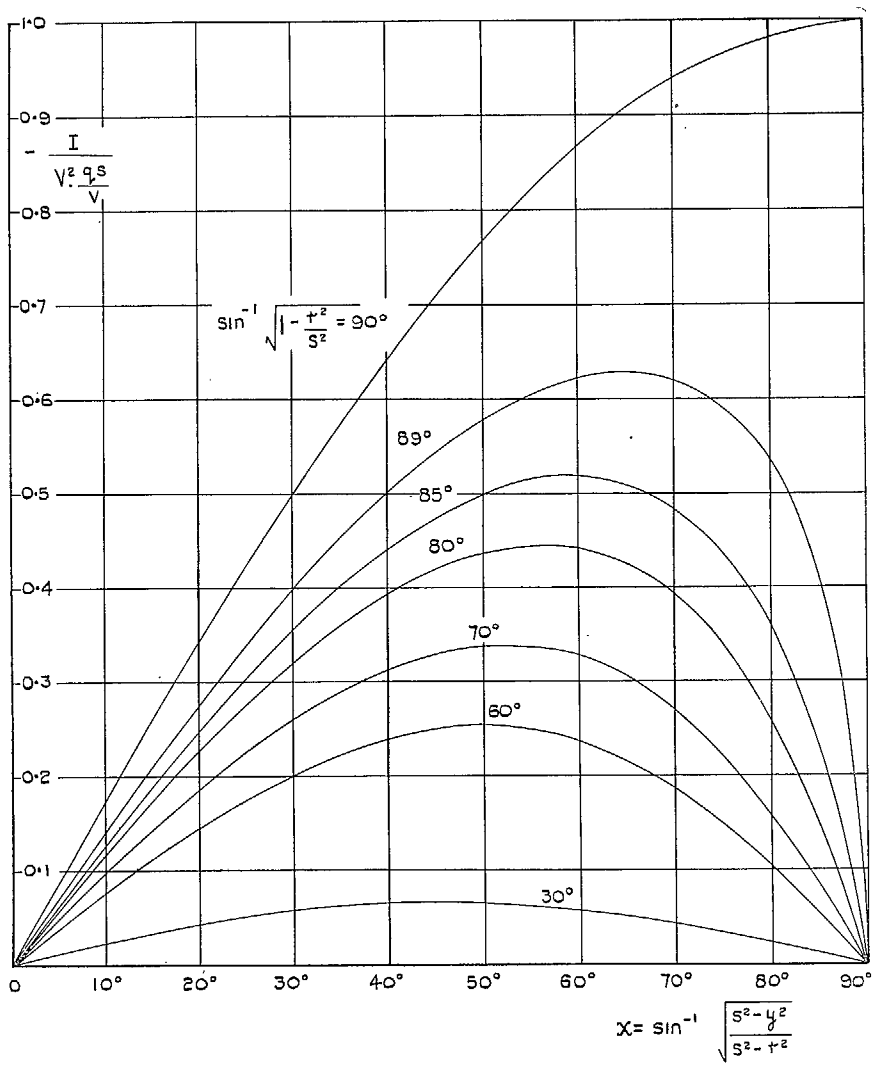


FIG. 5. Pressure curves for the wing in pitch (second contribution).

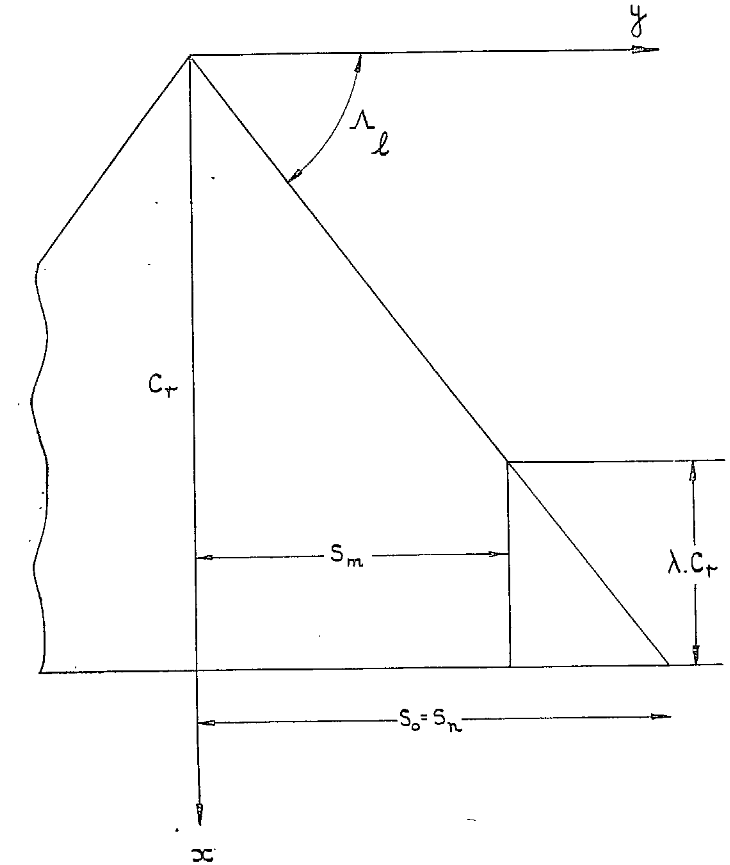


FIG. 6. The cropped Delta wing.

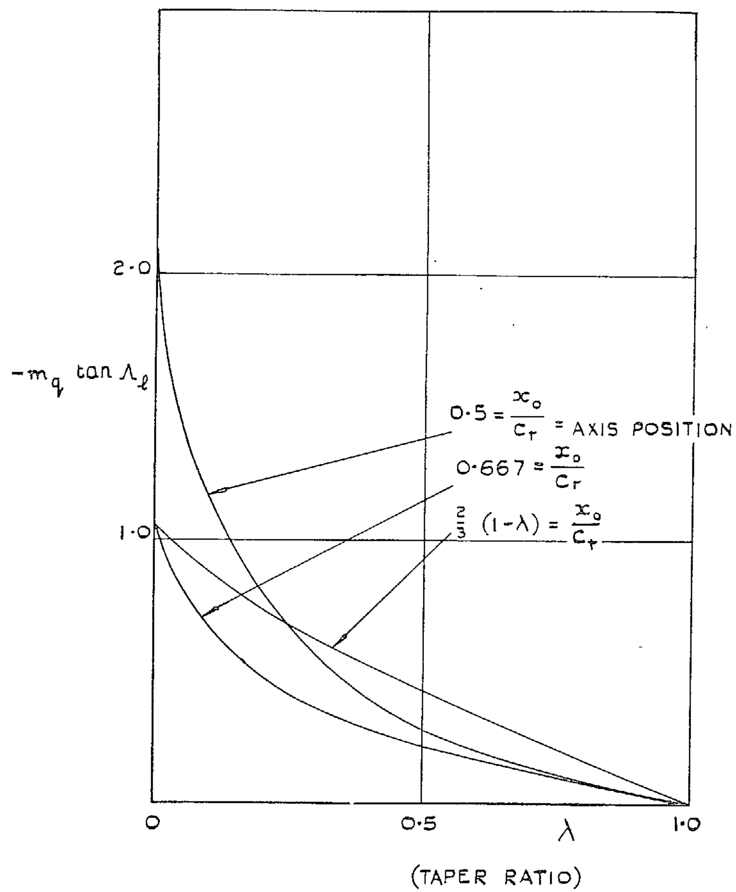


FIG. 7. The derivative $-m_q$ for a cropped Delta wing and three axis positions ($x_0 = \frac{1}{2}c_r$, $x_0 = \frac{2}{3}c_r$ and $x_0 = \frac{2}{3}(1 - \lambda)c_r$ = position of the centre of pressure).

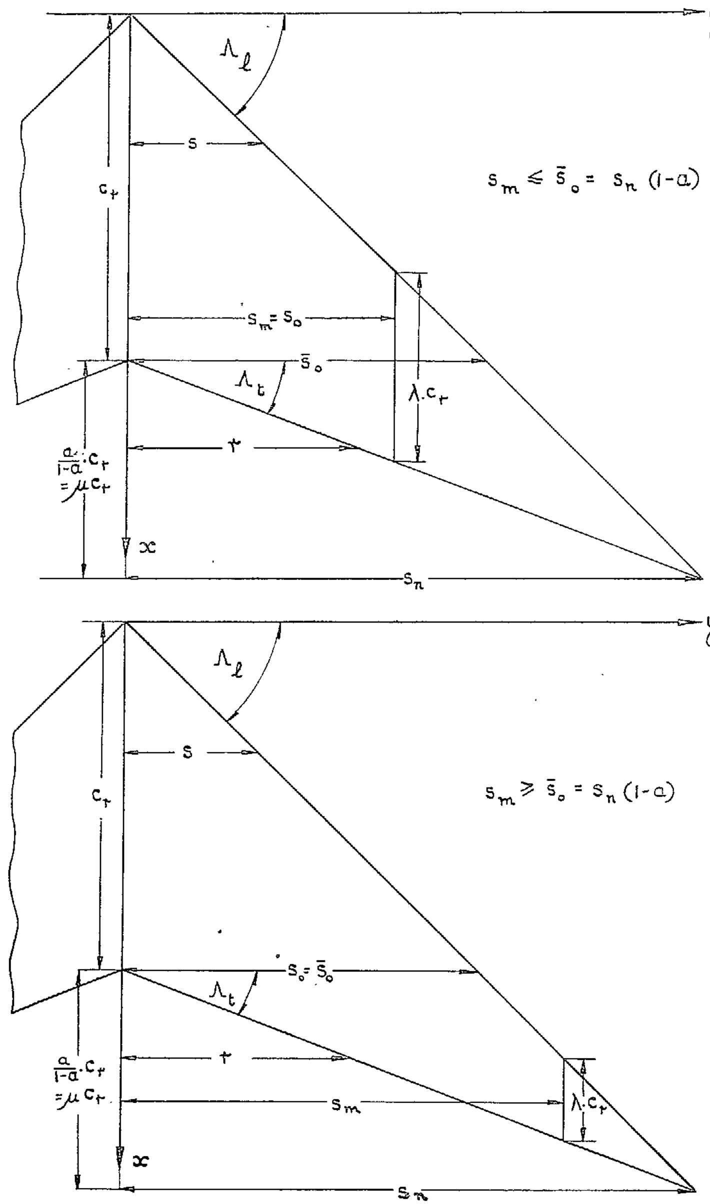


FIG. 8. A plan form with three parameters

($a = (\tan \Lambda_t)/(\tan \Lambda_l)$; $s_m/s_n = 1 - \lambda$; $\bar{c}/c_r = \frac{1}{2}(1 + \lambda)$; $A = [4/(1 - a)][(1 - \lambda)/(1 + \lambda)] \cot \Lambda_l$; sweep of $\varkappa$-line: $\tan \Lambda_\varkappa = \tan \Lambda_l\{1 - \varkappa(1 - a)\}$, $(0 \leqslant \varkappa \leqslant 1)$.)

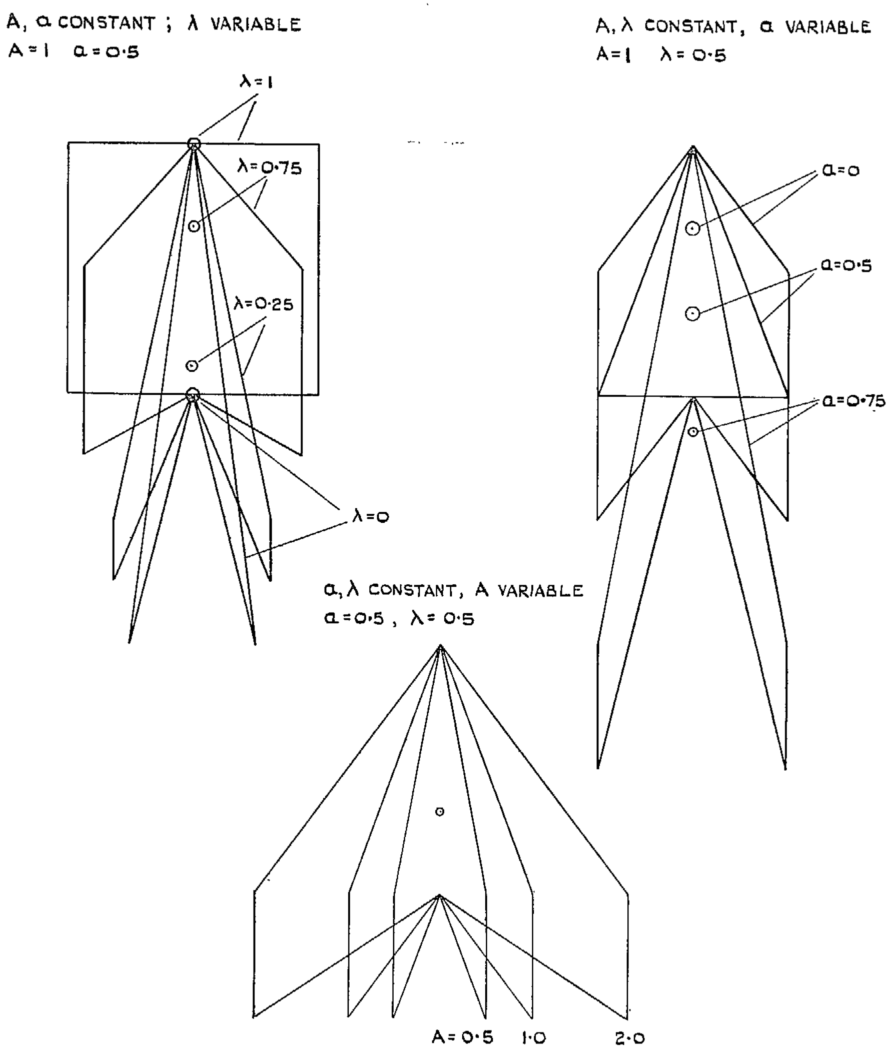


FIG. 9. Three families of wing plan forms (the position of the aerodynamic centre is indicated by ⊙).

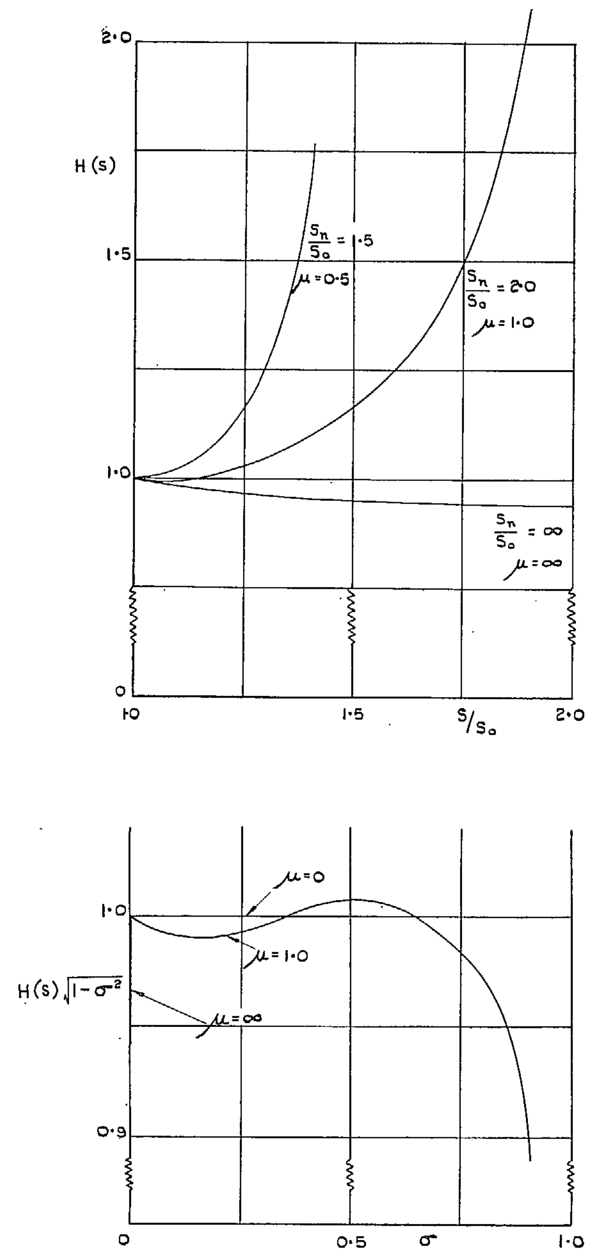


FIG. 10. The functions $H(s)$. (Incidence case.)

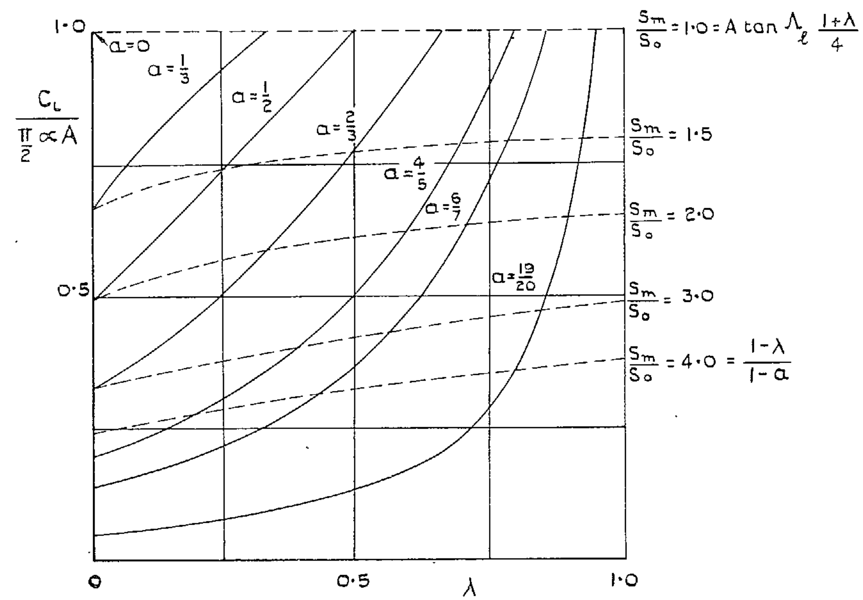


FIG. 11. Lift coefficient for the wing plan form of Fig. 8.

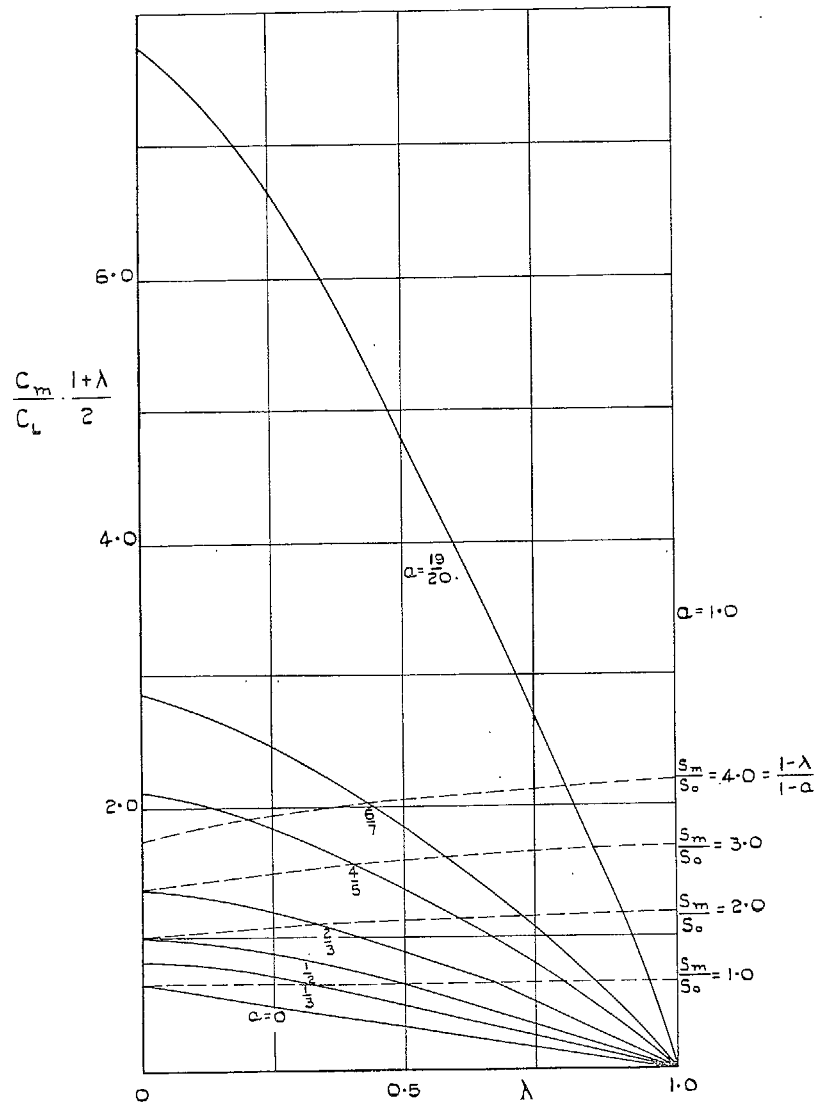


FIG. 12. Position of the aerodynamic centre for the family of Fig. 8. (Measured from apex in terms of root chord.)

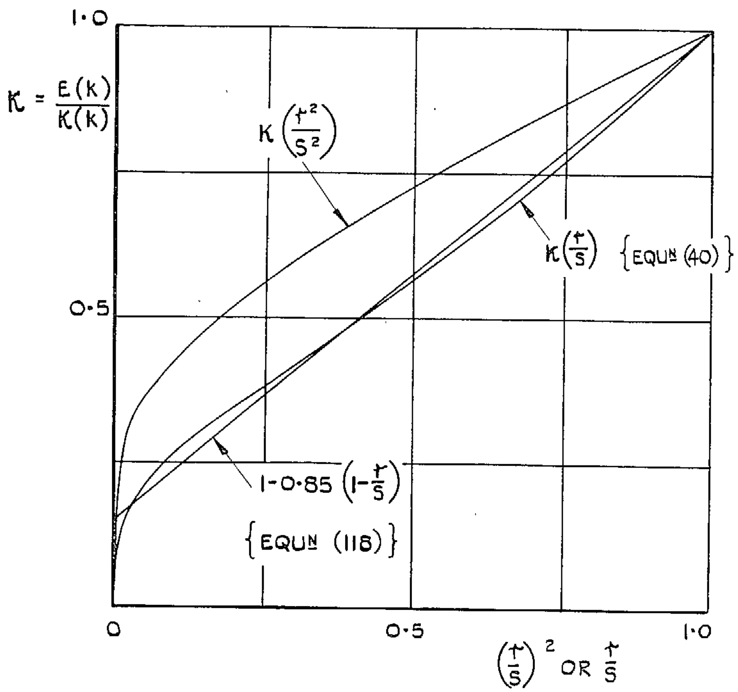


FIG. 13. Approximation to the function $\varkappa = E(k)/K(k)$ (equation (40)).

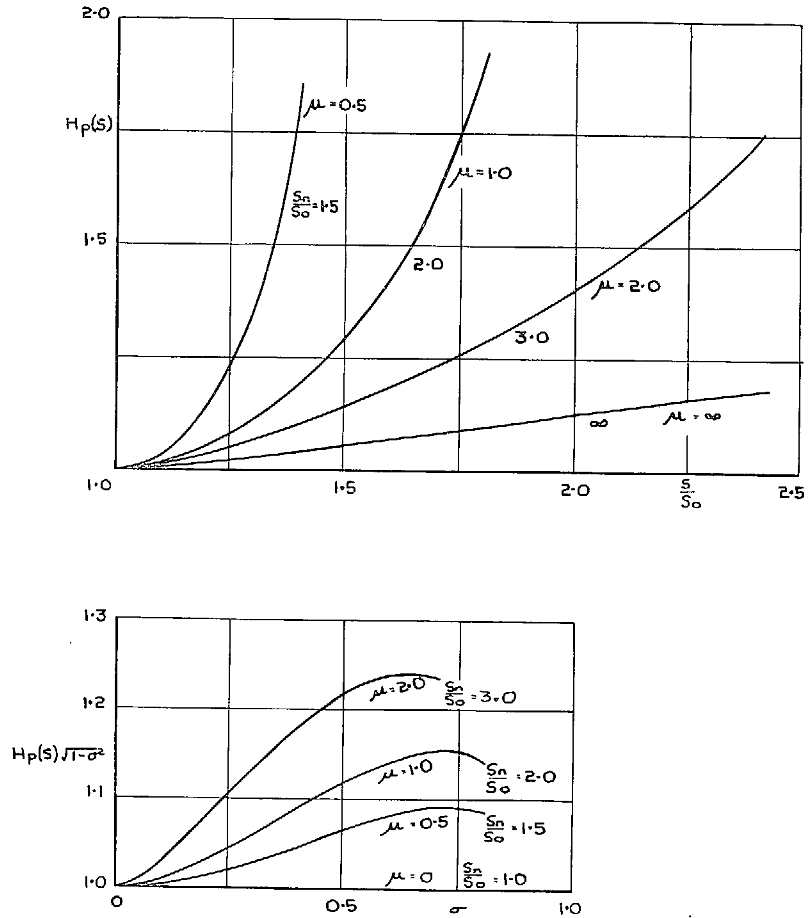


FIG. 14. The function $H_p(s)$. (Wing in roll.)

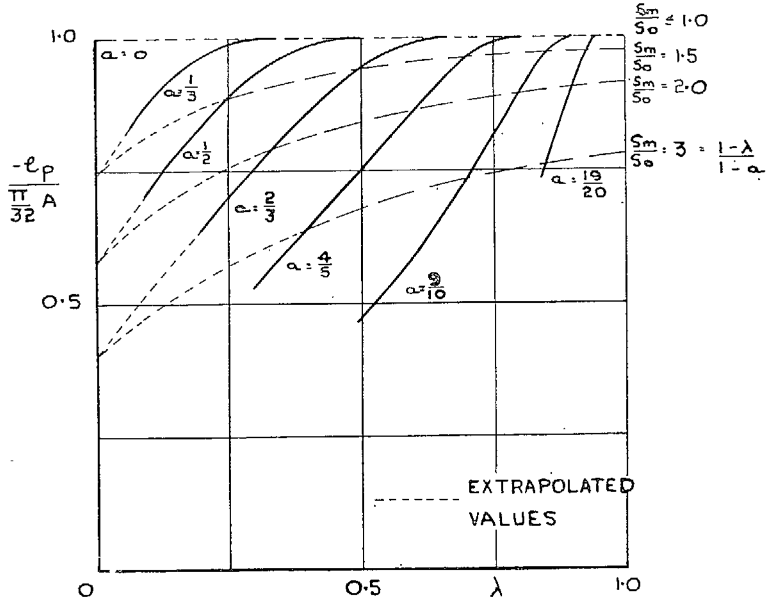


FIG. 15. The derivative l_p for the wing family of Fig. 8.

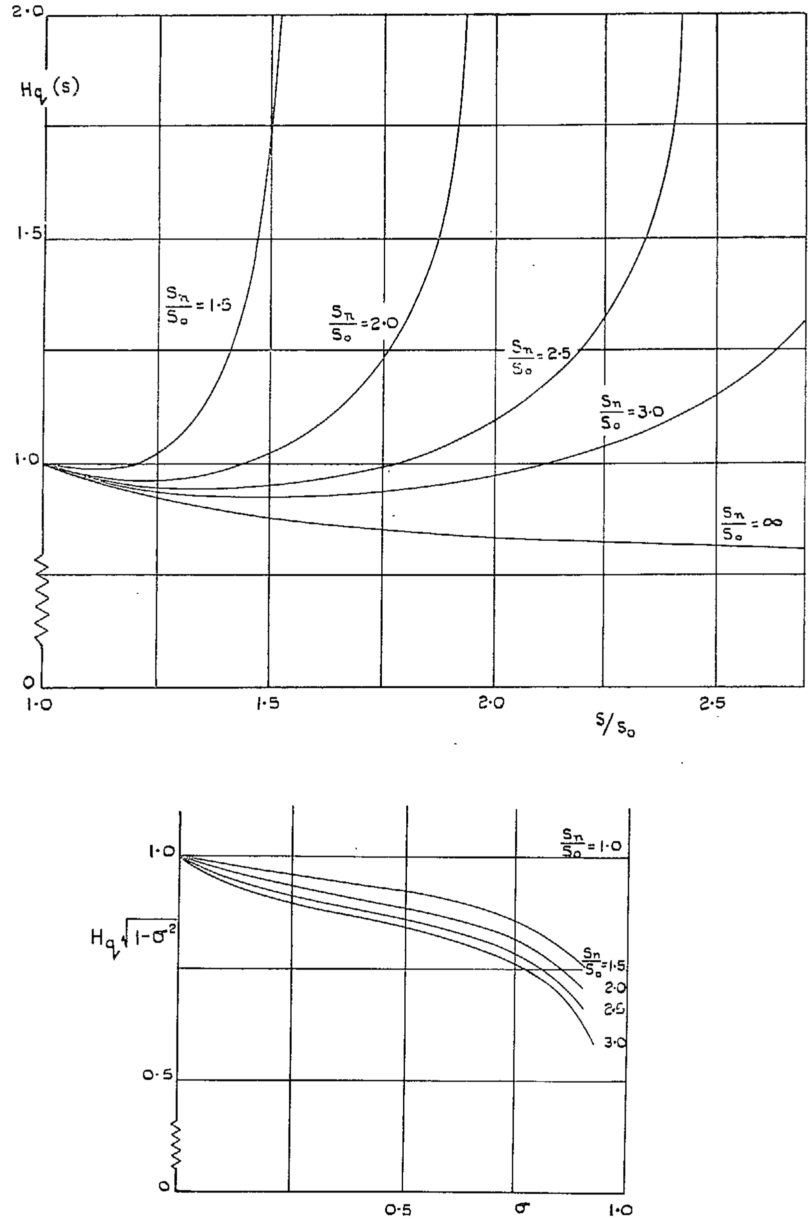


FIG. 16. The function $H_q(s)$. (Wing in pitch.)

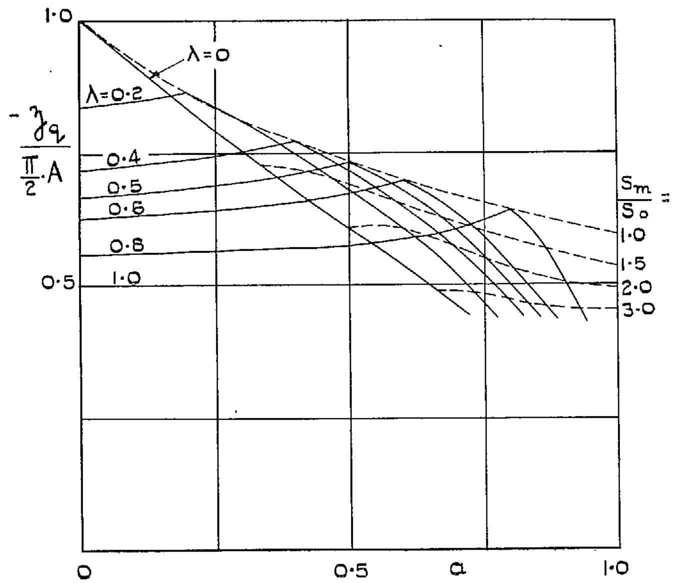


FIG. 17. The derivative z_q for the wing family of Fig. 8.

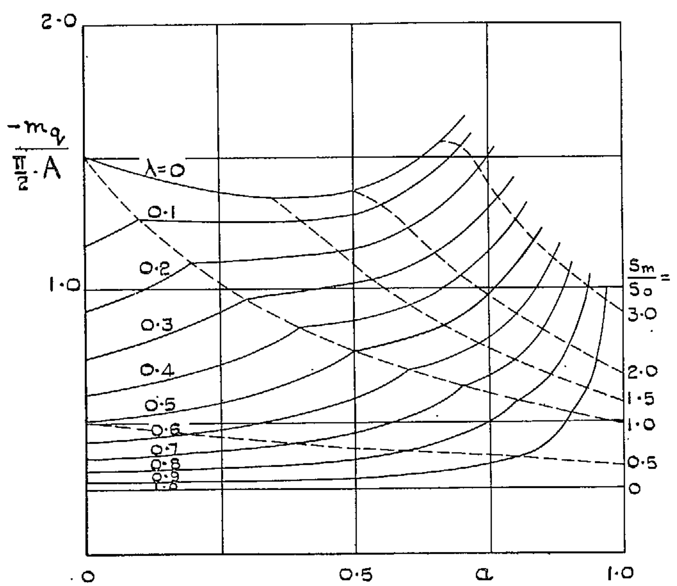


FIG. 18. The derivative m_q for the wing family of Fig. 8.

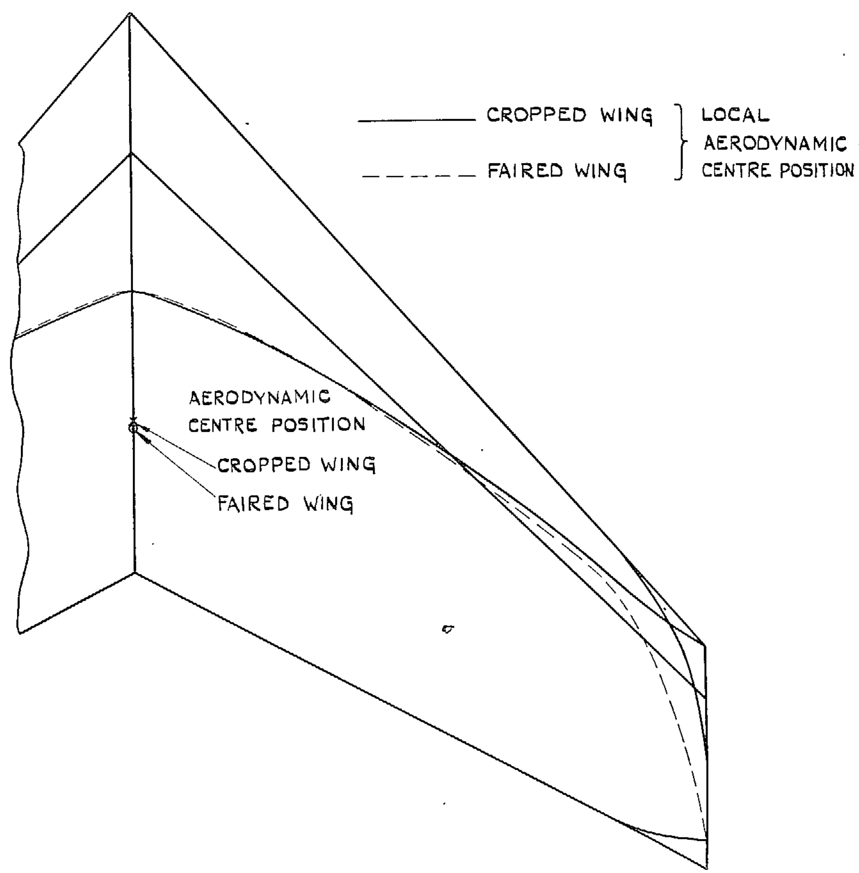


FIG. 19. Two particular wings, one with straight edges (cropped wing) and one with faired tip.

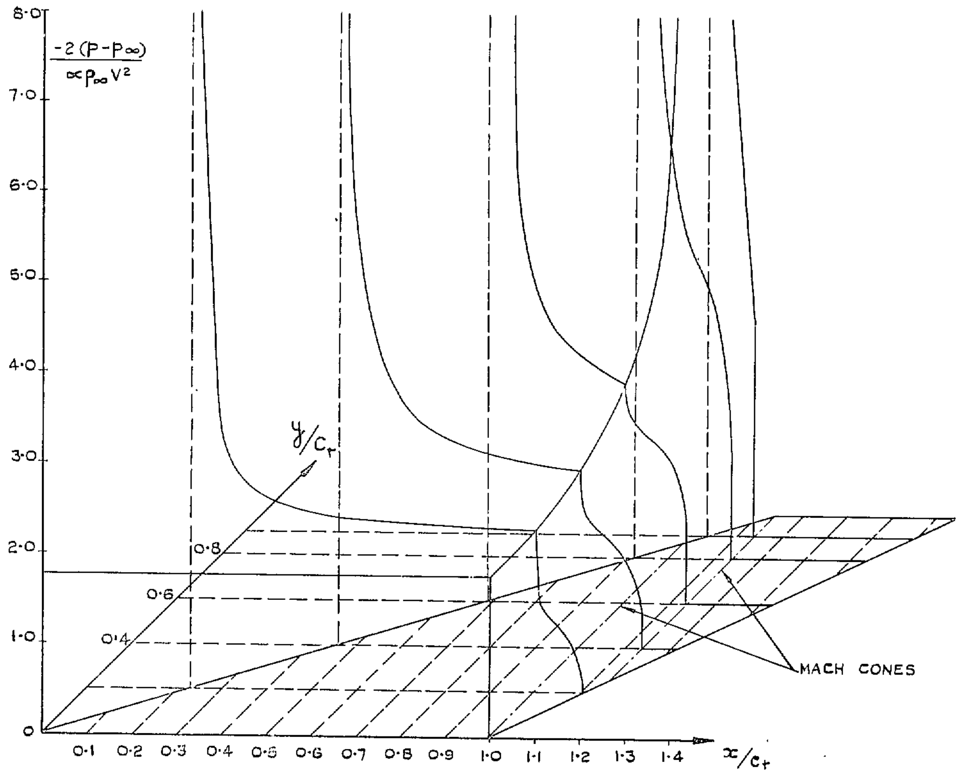


FIG. 20. Pressure distribution for the cropped wing of Fig. 19.

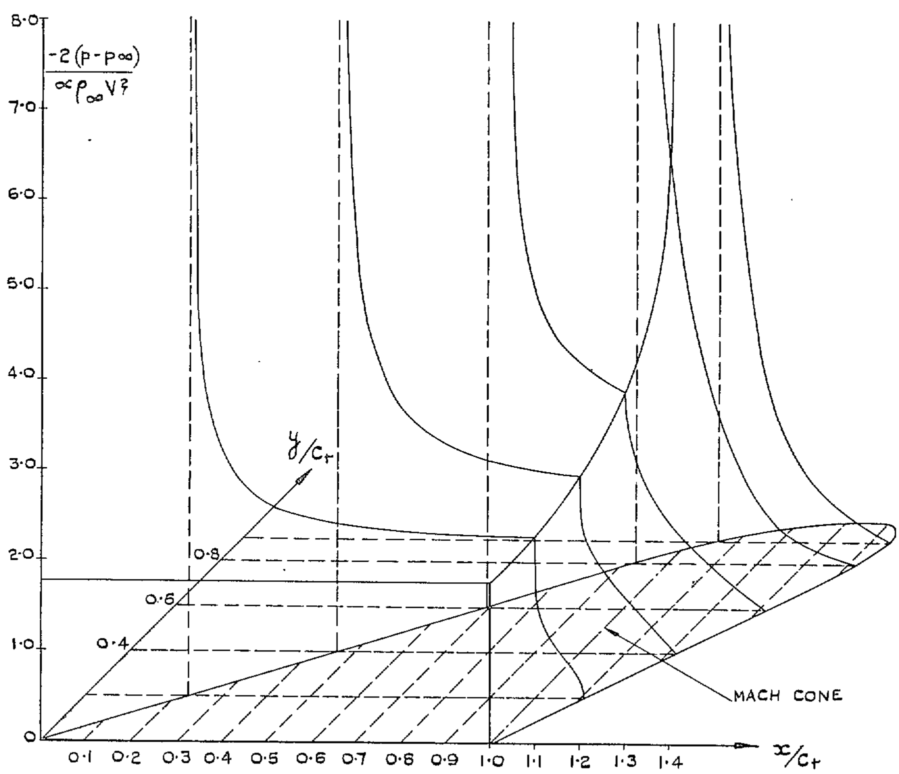


FIG. 21. Pressure distribution for the faired wing of Fig. 19.

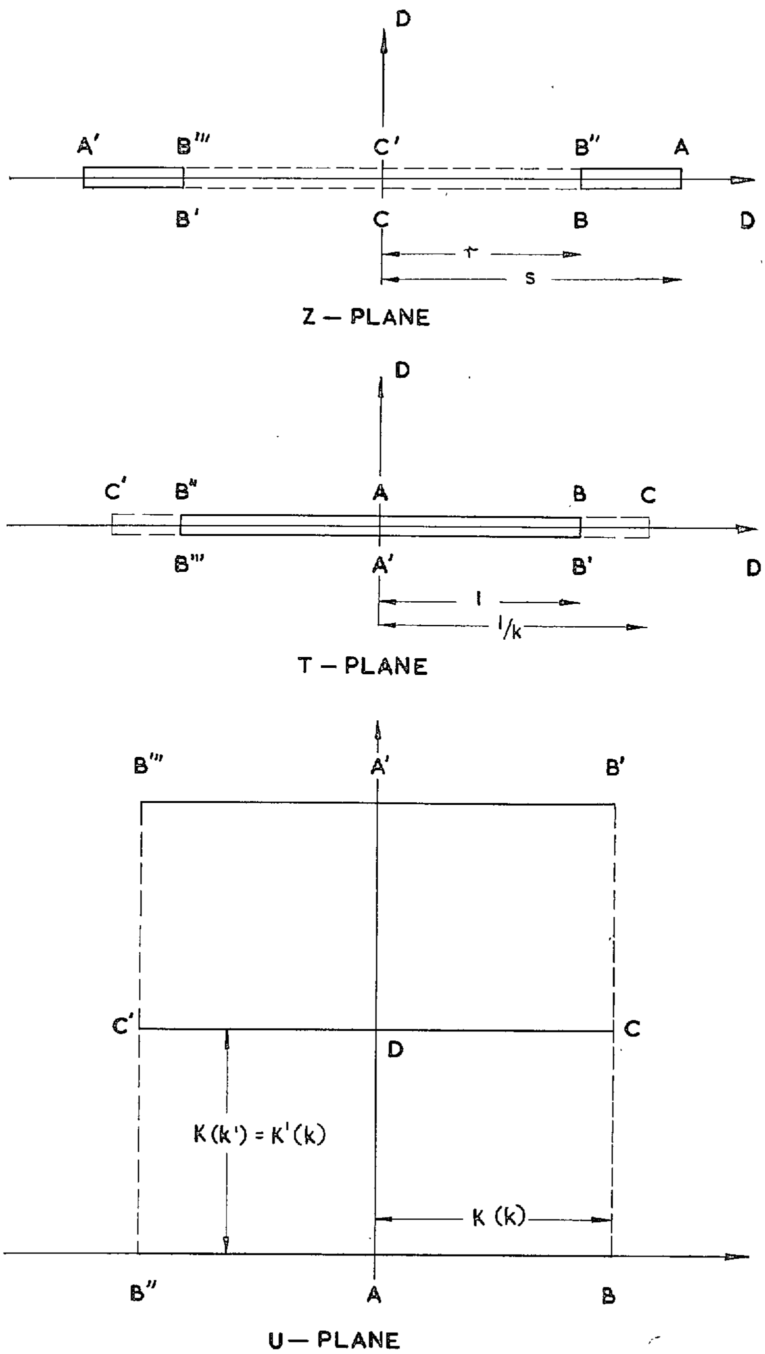


FIG. 22. A conformal transformation.

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