R. & M. No. 2879
(14,814)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

Determination of the Stress Distribution in Reinforced Monocoque Structures

Part I. A Theory of Flat-sided Structures

By

L. S. D. MORLEY

LONDON: HER MAJESTY'S STATIONERY OFFICE
1955
PRICE 6*s* 6*d* NET

Determination of the Stress Distribution in Reinforced Monocoque Structures

Part I. A Theory of Flat-sided Structures

By

L. S. D. Morley

Communicated by the Principal Director of Scientific Research (Air), Ministry of Supply

*Reports and Memoranda No. 2879**
December, 1951

Summary.—This paper is concerned with the formation of basic differential equations for the determination of the stress distribution in reinforced monocoque flat-sided structures, such as rectangular or polygonal fuselages and wing boxes. The general scheme of the analysis is to develop the fundamental equations which govern the stresses, strains and displacements separately in the skin-stringer combination and rib flanges. Then, by identifying displacements along their intersections, the differential equations of compatibility are formed. The solution of these equations yields the stress distribution. It is intended that further papers will be devoted to the detailed solution and application of these equations to particular problems together with experimental verification for each type of problem.

A simple application of the theory is demonstrated in an appendix. A three-bay flat structure, containing a rectangular cut-out in the centre bay, under uniformly distributed tension loading is investigated. The calculated results for the longitudinal direct stress resultants in the skin-stringer combination compare favourably with those of experiment.

1. *Introduction.*—In recent years considerable attention has been given to the estimation of stress distributions in the reinforced monocoque construction encountered in modern aircraft. It was soon realised from the flexible nature of the construction that the stresses arising from discontinuities such as concentrated loads and abrupt changes in section were of primary importance and required quite comprehensive analytical and experimental investigations. Previous work has shown that the analytical estimation of the stress distribution in these reinforced monocoque structures can be broadly considered under the following three headings, *viz.*,

(*a*) Estimation of the overall stress distribution over regions far removed from a discontinuity (such as a concentrated load or abrupt change of section). Here the elementary theories of bending and torsion are usually applicable and no account is taken of the flexibility of the structure.

(*b*) Estimation of the stress distribution in the neighbourhood of a discontinuity (*i.e.*, within a range of the order of two or three chords or fuselage diameters). Here, account is taken of the flexibility of the structure.

(*c*) Estimation of the peak stresses *at* the discontuity (*i.e.*, within a range of two or three times a relevant stiffener cross-sectional dimension).

* R.A.E. Report Structures 120, received 21st April, 1952.

This paper is concerned with the estimation of the stress distribution in the neighbourhood of a discontinuity (*i.e.*, (*b*) above). Such estimations are usually made from the consideration of a 'shell model' that is an appropriate and convenient idealisation of the basic reinforced monocoque structure.

The most important shell models may be distinguished as follows, *viz.*,

(i) Basic reinforced monocoque structure.

(ii) Shell model with discrete stringers and ribs (frames). (Ebner and Köller[1,2], Cicala[3]).

(iii) Shell model with uniformly distributed stringers but discrete ribs (frames). (Goodey[4], Hoff[5]).

(iv) Shell model with uniformly distributed stringers and closely spaced rigid diaphragms. (Williams[6], Goodey[7], Hadji-Argyris and Dunne[8]).

In this paper, a theory is given which is based upon a shell model possessing uniformly distributed stringers but discrete ribs (frames). This shell model is used because it represents a close approximation to the truth, it is convenient for mathematical analysis and because account can be taken of the flexibility of the individual rib flanges (frames). Other authors[4,5] who have used this shell model have confined their attention to circular cylinders possessing complete cyclic symmetry. However, a theory is presented here in a form that will serve as a basis for the practical solution of a wide range of flat-sided structures such as rectangular or polygonal fuselages and wing boxes. It is intended that a corresponding theory of curved structures will be given in a further paper.

The general scheme of the analysis is to develop fundamental equations which govern the stresses, strains and displacements separately in the skin-stringer combination and the rib flanges or frames. Then, by identifying displacements along their intersections the differential equations of compatibility are formed. The solution of these compatibility equations yields the stress distribution. It is intended that further papers will be devoted to the detailed solution and application of these equations to particular problems together with individual experimental verification for each type of problem.

2. *Description of Structure.*—2.1. *Basic Reinforced Monocoque Structure.*—The basic structure is constructed from a thin flat metal sheet reinforced in the longitudinal direction by closely spaced stringers and in the transverse direction by a system of rib flanges (or frames).

2.2. *Derivation of the Shell Model.*—In the calculation of the stress distribution in the neighbourhood of a discontinuity, account is taken only of the most important effective work of the individual structural components. The resulting structure with its limited attributes is called a 'shell model.'

If the cross-section of the stiffeners is large in relation to the cross-section of the skin then the direct stresses are taken up almost exclusively by the stiffeners and a condition of almost pure shear exists in the individual sheet panels bounded by two stringers and two ribs. This notion is now applied to the case when the cross-sections of the skin and stiffeners are of the same order of value. Account is taken of the contributory direct stiffnesses of the skin by corresponding increases in the stiffener cross-sections, which are then usually called 'effective cross-sections.' As this yields the 'mean' of the actual stress conditions it affords a useful simplification for the calculation of the distribution of the stresses as a whole.

In most reinforced monocoque structures the stringers are so numerous that they may be considered as uniformly distributed over the surface of the skin. This skin-stringer combination will then have the nominal sheet thickness for resisting shear in its own plane and an effective thickness for resisting direct loads in the longitudinal direction.

2.3. *Assumptions.*—The following assumptions are made in the analysis, *viz.*,

(*a*) The stress-strain relationships are linear
(*b*) Buckling is excluded
(*c*) The stringers can resist only direct load
(*d*) The rib flanges (frames) can resist only direct load
(*e*) The thin sheet covering can resist only shear, account being taken of its contributory direct stiffnesses by corresponding increases in the stiffener cross-sections. (This means that the shear can only change at a rib or stringer.)
(*f*) The stringers are so closely spaced that they may be considered uniformly distributed over the surface of the skin
(*g*) The effects of the eccentricity of the neutral axes of the stringers and rib flanges (frames) from the skin median line are neglected.

3. *Determination of the Stress Distribution.*—3.1. *Fundamental Equations for the Skin-Stringer Combination.*—In preparation for forming the equations of compatibility for the shell model it is necessary to consider the detailed equations governing the individual behaviours of the skin-stringer combination and the rib flanges.

The structure and notation are shown in Fig. 1. In Fig. 2 are shown the stress resultants acting on an elemental portion of the skin-stringer combination taken from the jth bay of the shell model. Resolving in the longitudinal direction it is found for equilibrium of the element that

$$\frac{\partial T_j}{\partial x} + \frac{\partial S'_j}{\partial y} = 0. \qquad (1)$$

The longitudinal and shearing strains in the skin-stringer combination can be expressed in terms of the displacements and the stresses. They are respectively

$$e_{xx\,j} = \frac{\partial u_j}{\partial x} = \frac{T_j}{Et^*} \qquad (2)$$

and

$$e_{xy\,j} = \frac{\partial u_j}{\partial y} + \frac{\partial v_j}{\partial x} = \frac{S'_j}{\mu t}, \qquad (3)$$

where t^* is the effective thickness of the skin-stringer combination for resisting direct loads in the longitudinal direction, t is the nominal thickness of the skin, E is Young's modulus and μ is the shear modulus.

Now, in the shell model the shear stress resultant S'_j is independent of the longitudinal co-ordinate x, hence equation (1) may be written

$$T_j = -\frac{dS'_j}{dy}\int dx.$$

Observing that the longitudinal direct stress resultant T must be continuous from bay to bay, since the rib flanges can resist only direct load along their length, this expression integrates to

$$T_j = -x\frac{dS'_j}{dy} + \bar{T}_{j-1}, \qquad (4)$$

where a new origin for x is chosen on the left-hand side of each bay and $\bar{T}_{j-1}$ denotes the longitudinal direct stress resultant in the skin-stringer combination at the $j - 1$th rib.

The longitudinal displacement u_j is, from equation (2),

$$u_j = \frac{1}{Et^*} \int T_j \, dx$$

which, on using equation (4) and integrating, becomes

$$u_j = \frac{1}{Et^*}\left(- \frac{x^2}{2} \frac{dS'_j}{dy} + x\bar{T}_{j-1} \right) + \bar{u}_{j-1} \qquad \ldots \quad (5)$$

where $\bar{u}_{j-1}$ denotes the longitudinal displacement of the skin-stringer combination at the $j-1$th rib

From equation (3), the transverse displacement is

$$v_j = \int \left(\frac{S'_j}{\mu t} - \frac{\partial u_j}{\partial y} \right) dx.$$

Substituting from equation (5), and integrating, this becomes

$$v_j = \frac{x}{\mu t} S'_j - \frac{1}{Et^*}\left(- \frac{x^3}{6} \frac{d^2S'_j}{dy^2} + \frac{x^2}{2} \frac{d\bar{T}_{j-1}}{dy} \right) - x \frac{d\bar{u}_{j-1}}{dy} + \bar{v}_{j-1} \qquad \ldots \quad (6)$$

where $\bar{v}_{j-1}$ denotes the transverse displacement of the skin-stringer combination at the $j - 1$th rib. Putting $x = L$ into this equation yields

$$\bar{v}_j = \frac{L}{\mu t} S'_j - \frac{1}{Et^*}\left(- \frac{L^3}{6} \frac{d^2S'_j}{dy^2} + \frac{L^2}{2} \frac{d\bar{T}_{j-1}}{dy} \right) - L \frac{d\bar{u}_{j-1}}{dy} + \bar{v}_{j-1}, \qquad \ldots \quad (7)$$

which is an expression for the transverse displacement of the skin-stringer combination at the jth rib.

Equations (4), (5) and (7) which express the relationships between successive T_j, u_j and $\bar{v}_j$ form the fundamental equations for the skin-stringer combination.

3.2. *Fundamental Equations for the Rib Flanges (frames).*—The forces acting on an elemental portion of the jth rib flange are shown in Fig. 3, where the applied force is composed of the shearing forces applied by the skin-stringer combination plus the external force. Thus

$$\bar{S}_j = S'_{j+1} - S'_j + \mathscr{S}_j \qquad \ldots \quad (8).$$

where $\mathscr{S}_j$ is the external force.

For equilibrium along the length of the rib flange it is necessary that

$$\bar{S}_j + \frac{d\bar{F}_j}{dy} = 0. \qquad \ldots \quad (9)$$

The strain along the rib flange can be expressed in terms of the stress and displacement It is

$$\bar{e}_{yy\,j} = \frac{d\bar{v}_j}{dy} = \frac{\bar{F}_j}{\bar{E}\bar{A}} \qquad \ldots \quad (10)$$

where $\bar{A}$ is the effective cross-sectional area of the rib flange and E is the Young's modulus.

From these last two equations it readily follows that

$$\frac{d^2\bar{v}_j}{dy^2} = - \frac{\bar{S}_j}{\bar{E}\bar{A}} \cdot \qquad \ldots \quad (11)$$

Equations (8) and (11), which relate the displacement $\bar{v}_j$ with the forces acting on the rib, form the fundamental equations for the rib flanges. The next step is to match the displacements $\bar{v}_j$ of the skin-stringer combination with those of the rib flanges and thus develop the compatibility equations for the structure.

3.3. *The Equations of Compatibility*.—For the stress and strain to be consistent throughout the structure it is now only necessary to match the transverse displacement $\bar{v}$ along each intersection of the skin-stringer combination and rib flange.

Proceeding thus, it is found from equations (7) and (11) for the transverse displacement to be identical that

$$\frac{1}{Et^*}\left(\frac{L^3}{6}\frac{d^4S'_j}{dy^4} - \frac{L^2}{2}\frac{d^3\bar{T}_{j-1}}{dy^3}\right) + \frac{L}{\mu t}\frac{d^2S'_j}{dy^2} - L\frac{d^3\bar{u}_{j-1}}{dy^3}$$

$$+ \frac{d^2\bar{v}_{j-1}}{dy^2} + \frac{1}{\bar{E}\bar{A}}\left(S'_{j+1} - S'_j\right) = -\frac{\mathscr{S}_j}{\bar{E}\bar{A}} \qquad \qquad (12)$$

where a substitution has been made from equation (8). The term on the right-hand side represents the influence of the external force acting on the rib flange. For successive values of the suffix j the above becomes the set of compatibility equations for the determination of the stress distribution in a flat structure undergoing a loading which is in the plane. When the terms containing the direct stress resultants $\bar{T}$, and the displacements $\bar{u}$ and $\bar{v}$ are eliminated by using the known relations

$$\bar{T}_j = -L\frac{dS'_j}{dy} + \bar{T}_{j-1}, \qquad \qquad (13)$$

$$\bar{u}_j = \frac{1}{Et^*}\left(-\frac{L^2}{2}\frac{dS'_j}{dy} + L\bar{T}_{j-1}\right) + \bar{u}_{j-1} \qquad \qquad (14)$$

by putting $x = L$ in equations (4) and (5)

$$\frac{d^2\bar{v}_j}{dy^2} = -\frac{1}{\bar{E}\bar{A}}\left(S'_{j+1} - S'_j + \mathscr{S}_j\right) \qquad \qquad (15)$$

which follows from the substitution of equation (8) into equation (11), and the boundary conditions at each end of the structure they form a set of simultaneous linear ordinary differential equations with constant coefficients. Each equation will be of fourth order involving only the even differentials and, in general, there will be one such equation for every bay.

In Appendix I of this paper it is demonstrated that the compatibility equation (12) and the displacement equations (14) and (15) are consistent with the strain energy being rendered a minimum.

It is intended that further papers will be devoted to the detailed solution and application of these equations to particular problems. However, in Appendix II a simple manipulation of the above equations is demonstrated by estimating the stress distribution in the three-bay reinforced monocoque flat structure shown in Fig. 4. It has a cut-out in the centre bay and is under a uniformly distributed tension loading. The calculated results for the longitudinal direct stress resultants in the skin-stringer combination compare favourably with those of experiment.

The compatibility and subsidiary equations may be conveniently combined and expressed in the form of a recurrence differential relationship. After manipulation this relationship is soon found to be

$$L^4\frac{d^4}{dy^4}\left(S'_{j+1} + 4S'_j + S'_{j-1}\right) + 6\frac{Et^*}{\mu t}L^2\frac{d^2}{dy^2}\left(S'_{+1} - 2S'_j + S'_{j-1}\right)$$

$$+ 6\frac{Et^*L}{\bar{E}\bar{A}}\left(S'_{j+2} - 4S'_{j+1} + 6S'_j - 4S'_{j-1} + S'_{j-2}\right)$$

$$= -6\frac{Et^*L}{\bar{E}\bar{A}}\left(\mathscr{S}_{j+1} - 3\mathscr{S}_j + 3\mathscr{S}_{j-1} - \mathscr{S}_{j-2}\right). \qquad \qquad (16)$$

In Appendix III it is shown how the compatibility equation appropriate to a shell model reinforced with closely spaced ribs may be derived from the one just above. It is obtained by considering the limit as L (the distance between the ribs) approaches zero with $\bar{A}/L$ tending towards an effective skin thickness $\bar{t}$ in the transverse direction.

Finally, in Appendix IV the equations of compatibility are derived for a more general shell model and with the inclusion of the effect of stringer bending and twisting. This shell model is more general because it is not necessary for the rib flanges (frames), skin-stringer combination and rib spacing to be constant throughout the shell model. The effect of stringer bending and twisting will usually be small except in such cases as a fuselage of a large aircraft under concentrated loading. In these fuselages the frames are often very flexible and it is then that the effect of stringer bending and twisting becomes important.

4. *Conclusions.*—In this paper a theory is given for the estimation of the stress distribution in the neighbourhood of a discontinuity (*i.e.*, within a range of the order of two or three chords or fuselage diameters) in reinforced monocoque flat-sided structures. The discontinuity may be considered as a concentrated load or an abrupt change of section. In Appendix IV equations are given in general terms so that they form a basis for the practical solution of a wide range of flat-sided structures such as rectangular or polygonal fuselages and wing boxes.

A numerical illustrative example is given in Appendix II. Here a three-bay flat structure, containing a rectangular cut-out in the centre bay, under a uniformly distributed tension loading is investigated. Comparison is favourable with the experimental results for the longitudinal direct stress-resultants in the skin-stringer combination.

It is intended that further papers will be devoted to the detailed solution and application of the equations to particular problems together with individual experimental verification for each type of problem.

LIST OF SYMBOLS

Structural properties

$\bar{A}_j$	Effective cross-sectional area of the jth rib flange
I_{Gj}	Effective moment of inertia per unit run of the skin-stringer combination for bending in the normal sense
I_{Kj}	Effective moment of inertia per unit run of the skin-stringer combination for bending in the transverse sense
$\bar{I}_j$	Effective moment of inertia of the jth rib flange
J_j	Effective polar moment of inertia per unit run of the skin-stringer combination for resisting twisting
L_j	Distance between the $j-1$th and jth ribs
t_j	Nominal thickness of the skin in the jth bay
t^*_j	Effective thickness of the skin-stringer combination for resisting load in the longitudinal direction
$\bar{t}$	Effective thickness of the skin-stringer combination for resisting load in the transverse direction.

Co-ordinate system

x	Longitudinal co-ordinate, a new origin being chosen on the left-hand side of each bay
y	Transverse co-ordinate

LIST OF SYMBOLS—*continued*

Loads and stresses

$\bar{F}_j$ Tension in the jth rib flange
G_j Stress couple in the jth bay of the skin-stringer combination, taken in the normal sense
$\bar{G}_j$ Stress couple in the skin-stringer combination at the jth rib, taken in the normal sense
G'_j Stress couple in the jth bay of the skin-stringer combination due to twisting
$\bar{G}'_j$ Bending moment per unit run acting on the jth rib flange
$\mathscr{G}'_j$ External bending moment per unit run applied to the jth rib flange
K_j Stress couple in the jth bay of the skin-stringer combination taken in the transverse sense
$\bar{M}_j$ Bending moment in the jth rib flange
N_j Normal shear stress resultant in the jth bay of the skin-stringer combination
$\bar{N}_j$ Normal load per unit run acting on the jth rib flange
$\mathscr{N}_j$ External normal load per unit run applied to the jth rib flange
$\bar{Q}_j$ Normal shear in the jth rib flange
S_j Transverse shear stress resultant in the jth bay of the skin-stringer combination (this includes the shear carried by the distributed stringers)
$\bar{S}_j$ Transverse load per unit run acting along the jth rib flange
S'_j Transverse shear stress resultant in the jth bay of the skin-stringer combination (this excludes the shear carried by the distributed stringers)
$\mathscr{S}_j$ External transverse load per unit run applied along the jth rib flange
T_j Longitudinal direct stress resultant in the jth bay of the skin-stringer combination
$\bar{T}_j$ Longitudinal direct stress resultant in the skin-stringer combination at the jth rib
T' Transverse direct stress resultant in the skin-stringer combination

Strains and displacements

$e_{xx\,j}$ Longitudinal strain in the jth bay of the skin-stringer combination
$e_{xy\,j}$ Shear strain in the jth bay of the skin-stringer combination
$\bar{e}_{yy\,j}$ Transverse strain in the jth rib flange along the skin median line
u_j Longitudinal displacement in the jth bay of the skin-stringer combination
$\bar{u}_j$ Longitudinal displacement of the skin-stringer combination at the jth rib.
v_j Transverse displacement in the jth bay of the skin-stringer combination
$\bar{v}_j$ Transverse displacement of the skin-stringer combination at the jth rib, this being identical to the displacement of the rib flange
w_j Normal displacement in the jth bay of the skin-stringer combination
$\bar{w}_j$ Normal displacement of the skin-stringer combination at the jth rib, this being identical to the displacement of the rib flange

Elastic constants

E_j Young's modulus of elasticity for the skin-stringer combination in the jth bay
$\bar{E}_j$ Young's modulus of elasticity for the jth rib flange
μ_j Shear modulus for the skin-stringer combination in the jth bay

A few additional symbols are introduced in Appendix II but these are defined as they are introduced.

REFERENCES

No.	Author	Title, etc.
1	H. Ebner and H. Köller	Zur Berechnung des Kraftverlaufes in versteiften Zylinderschalen. *L.F.F.*, Vol. XIV, No. 12. 1937. Translated as A.R.C. 3470.
2	H. Ebner and H. Köller	Uber den Kraftverlauf in Langs- und querversteiften Scheiben. *L.F.F.*, Vol. XV, No. 10–11. 1938. Translated as A.R.C. 3896.
3	P. Cicala	Sul calcolo delle strutture a guscio. *L'Aerotecnica*, Vols. XXVI and XXVII. 1946 to 1947.
4	W. J. Goodey	The stresses in a circular fuselage. *J. R. Ae. Soc.*, Vol. 50. November, 1946.
5	N. J. Hoff	Thin-walled monocoques. Aeronautical Conference (London). September, 1947.
6	D. Williams, R. D. Starkey and A. H. Taylor.	Distribution of stress between spar flanges and stringers for a wing under distributed loading. R. & M. 2098. June, 1939.
7	W. J. Goodey	Stress diffusion problems. *Aircraft Engineering*, Vol. XVIII. June to November, 1946.
8	J. Hadji-Argyris and P. C. Dunne ..	The general theory of cylindrical and conical tubes under torsion and bending loads. *J. R. Ae. Soc.*, Vol. 51, February, September and November, 1947 ; Vol. 53, May and June, 1949.
9	L. S. D. Morley	Load distribution and relative stiffness parameters for a reinforced flat plate containing a rectangular cut-out under plane loading. N.L.L. Report S.347. 1949.
10	A. E. Green	Stress systems in isotropic and aeolotropic plates. V. *Proc. Roy. Soc.*, Vol. 184, Series A. 1945.

APPENDIX I

That the Equations of Compatibility are Consistent with the Strain Energy being Rendered a Minimum

It can soon be shown that the compatibility equations (12), together with the displacement equations (14) and (15), are consistent with the total strain energy stored in the shell model being rendered a minimum.

The total strain energy is given by

$$\text{S.E.} = \tfrac{1}{2} \sum_j \int \left\{ \frac{L}{\mu t} S'^2_j + \frac{1}{\bar{E}\bar{A}} \bar{F}_j^2 + \frac{1}{Et^*} \int_0^L T_j^2 \, dx \right\} dy$$

where, from equation (4)

$$T_j = - x \frac{dS'_j}{dy} + \bar{T}_{j-1} ,$$

and from equations (8) and (9)

$$\frac{d\bar{F}_j}{dy} = - (S'_{j+1} - S'_j + \mathscr{S}_j) .$$

The integral is taken over the width of the structure and the summation is taken over all the bays and ribs.

Now, the strain energy is to be minimized subject to the satisfaction of the two above conditions. The variational problem is therefore to minimize

$$\text{S.E.} + \sum_j \int \left\{ \bar{u}_j\left(L\frac{dS'_j}{dy} - \bar{T}_{j-1} + \bar{T}_j \right) + \bar{v}_j\left(\frac{d\bar{F}_j}{dy} + S'_{j+1} - S'_j + \mathscr{S}_j \right) \right\} dy$$

where $\bar{u}_j$ and $\bar{v}_j$ are undetermined functions which cannot as yet be associated with their interpretations as displacements. The integral and summation are taken over the same region as previously. Subjecting this expression to an arbitrary variation, which must have a null effect for a minimum, it is found on completing the integral with respect to x that

$$\sum_j \int \left\{ \frac{L}{\mu t} S'_j\,\delta S'_j + \frac{1}{\bar{E}\bar{A}}\bar{F}_j\,\delta\bar{F}_j + \frac{1}{Et^*}\left(\frac{L^3}{3}\frac{dS'_j}{dy}\frac{d\delta S'_j}{dy} + L\,\bar{T}_{j-1}\,\delta\bar{T}_{j-1} \right.\right.$$

$$\left. - \frac{L^2}{2}\frac{dS'_j}{dy}\,\delta\bar{T}_{j-1} - \frac{L^2}{2}\bar{T}_{j-1}\frac{d\delta S'_j}{dy} \right) + \bar{u}_j\left(L\frac{d\delta S'_j}{dy} - \delta\bar{T}_{j-1} + \delta\bar{T}_j \right)$$

$$\left. + \bar{v}_j\left(\frac{d\delta\bar{F}_j}{dy} + \delta S'_{j+1} - \delta S'_j \right) \right\} dy = 0.$$

This easily reduces to

$$\sum_j \int \left\{ \frac{L}{\mu t} S'_j\,\delta S'_j + \frac{1}{\bar{E}\bar{A}}\bar{F}_j\,\delta\bar{F}_j + \frac{1}{Et^*}\left(-\frac{L^3}{3}\frac{d^2S'_j}{dy^2}\,\delta S'_j + L\,\bar{T}_{j-1}\,\delta\bar{T}_{j-1} \right.\right.$$

$$\left. - \frac{L^2}{2}\frac{dS'_j}{dy}\,\delta\bar{T}_{j-1} + \frac{L^2}{2}\frac{d\bar{T}_{j-1}}{dy}\,\delta S'_j \right) - L\frac{d\bar{u}_j}{dy}\,\delta S'_j$$

$$\left. - \bar{u}_j\,\delta\bar{T}_{j-1} + \bar{u}_j\,\delta\bar{T}_j - \frac{d\bar{v}_j}{dy}\,\delta\bar{F}_j + \bar{v}_j\,\delta S'_{j+1} - \bar{v}_j\,\delta S'_j \right\} dy$$

$$+ \text{ terms involving only boundary conditions} = 0.$$

Therefore, by the usual arguments of the calculus of variations it is seen for the total strain energy to be rendered a minimum that

$$\frac{L}{\mu t} S'_j + \frac{1}{Et^*}\left(-\frac{L^3}{3}\frac{d^2S'_j}{dy^2} + \frac{L^2}{2}\frac{d\bar{T}_{j-1}}{dy} \right) - L\frac{d\bar{u}_j}{dy} + \bar{v}_{j-1} - \bar{v}_j = 0,$$

$$\frac{1}{\bar{E}\bar{A}}\bar{F}_j - \frac{d\bar{v}_j}{dy} = 0,$$

$$\frac{1}{Et^*}\left(-\frac{L^2}{2}\frac{dS'_j}{dy} + L\bar{T}_{j-1} \right) + \bar{u}_{j-1} - \bar{u}_j = 0.$$

It can now easily be verified that these equations are consistent with the compatibility equation (12) and the displacement equations (14) and (15).

APPENDIX II

Numerical Illustrative Example with Experimental Verification

A simple manipulation of the aforegoing equations will now be demonstrated by estimating the stress distribution in the three-bay reinforced monocoque flat structure shown in Fig. 4. It has a cut-out in the centre bay and is under a uniformly distributed tension loading.

This example, involving just the three bays, is not representative of the cut-out problems encountered in the wings or fuselage of an aircraft — where a greater number of bays will certainly be involved in the stress redistribution. However, the example was chosen for the following reasons, *viz.*,

(*a*) It demonstrates a simple manipulation of the equations

(*b*) It is very convenient for a laboratory test

(*c*) It affords a severe test of the theory.

This appendix therefore does not represent a proper investigation into the effects of cut-outs in aircraft structures, but, used with discretion, it will yield a first approximation to the longitudinal direct-stress resultants and the averaged shear-stress resultants between ribs in the skin-stringer combination of such structures.

There is a complete symmetry of the structure and elastic properties about the principal axes X and Y so the stress distribution will also be symmetrical about these axes. Furthermore, is is assumed that the rib flanges possess the same Young's modulus E as the skin-stringer combination. From considerations of symmetry it follows that the shear stress resultant $S'_1 = -S'_3$ and that $S'_2 = 0$. There is therefore only one such compatibility equation (12). Putting the suffix $j = 1$ in this equation yields

$$\frac{L^3}{6Et^*}\frac{d^4S'_1}{dy^4} + \frac{L}{\mu t}\frac{d^2S'_1}{dy^2} - L\frac{d^3\bar{u}_0}{dy^3} + \frac{d^2\bar{v}_0}{dy^2} - \frac{1}{E\bar{A}}S'_1 = 0, \quad \cdots \quad \cdots \quad (17)$$

since the longitudinal direct stress resultant $\bar{T}_0$ at any point in the skin-stringer combination at rib 0 is a constant and there are no externally applied distribution of forces such as $\mathscr{S}_1$.

The first step is to resolve equation (17) into terms of S'_1 only. Now, from equation (13), the longitudinal direct-stress resultant $\bar{T}_1$ in the skin-stringer combination at rib 1 is

$$\bar{T}_1 = -L\frac{dS'_1}{dy} + \bar{T}_0, \quad \cdots \quad (18)$$

while from equation (14) the axial displacements of the skin-stringer combination at ribs 1 and 2 are respectively

$$\bar{u}_1 = \frac{1}{Et^*}\left(-\frac{L^2}{2}\frac{dS'_1}{dy} + L\bar{T}_0\right) + \bar{u}_0$$

and

$$\bar{u}_2 = \frac{L}{Et^*}\bar{T}_1 + \bar{u}_1.$$

But, from symmetry it follows that $\bar{u}_1 = -\bar{u}_2$ and hence

$$\bar{u}_0 = \frac{L^2}{Et^*}\frac{dS'_1}{dy} - \frac{3L}{2Et^*}\bar{T}_0. \quad \cdots \quad (19)$$

Furthermore, from equation (15) it is seen that

$$\frac{d^2\bar{v}_0}{dy^2} = -\frac{1}{E\bar{A}}S'_1 \quad \cdots \quad (20)$$

since there are no externally applied distribution of forces such as $\mathscr{S}_0$.

Hence, on substituting equations (19) and (20) into equation (17) it becomes

$$\frac{5L^3}{6Et^*}\frac{d^4S'_1}{dy^4} - \frac{L}{\mu t}\frac{d^2S'_1}{dy^2} + \frac{2}{E\bar{A}}S'_1 = 0 \quad \ldots \qquad (21)$$

which is the differential equation of compatibility for the structure. A solution may be written

$$S'_1 = C_1 \sinh \lambda_1 (y - b)/L + C_2 \sinh \lambda_2 (y - b)/L$$
$$+ C_3 \cosh \lambda_1 (y - b)/L + C_4 \cosh \lambda_2 (y - b)/L \quad \ldots \qquad (22)$$

where C_1, C_2, C_3 and C_4 are arbitrary constants and

$$\lambda_1, \lambda_2 = 0{\cdot}7746 \left[\frac{Et^*}{\mu t} \pm \left\{\left(\frac{Et^*}{\mu t}\right)^2 - \frac{20}{3}\frac{t^*L}{\bar{A}}\right\}^{1/2}\right]^{1/2}.$$

Now, if the co-ordinate y has the origin as shown in Fig. 4 then equation (22) is valid only over the range $b < y < b + B$, because over the range $-b < y < b$ the shear stress resultant S'_1 follows immediately from equation (18). It is

$$S'_1 = \frac{\bar{T}_0}{L} y, \quad \ldots \qquad (23)$$

because the longitudinal direct stress resultant $\bar{T}_1$ is zero over this interval.

It only remains now to solve for the four arbitrary constants in equation (22). Along the sides of the skin-stringer combination, *i.e.*, at $y = b + B$, there is no external application of force and hence the shear stress resultant S'_1 in the skin stringer combination and the tension $\bar{F}_1$ in the rib flange must be zero. The tensions in the rib flanges are found from equations (8), (9), (22) and (23) and they are

$$\bar{F}_0 = -\bar{F}_1 = -\bar{F}_2 = \bar{F}_3 = -\frac{L}{\lambda_1}\{C_1 \cosh \lambda_1 (y - b)/L + C_3 \sinh \lambda_1 (y - b)/L\}$$
$$-\frac{L}{\lambda_2}\{C_2 \cosh \lambda_2 (y - b)/L + C_4 \sinh \lambda_2 (y - b)/L\} \quad \ldots \qquad (24)$$

over the range $b < y < b + B$, and

$$\bar{F}_0 = -\bar{F}_1 = -\bar{F}_2 = \bar{F}_3 = \frac{\bar{T}_0}{2L}(b^2 - y^2) - \frac{L}{\lambda_1}C_1 - \frac{L}{\lambda_2}C_2 \quad \ldots \qquad (25)$$

over the range $-b < y < b$. It is to be noted that the rib flange tensions are continuous at $y = b$. Finally, at $y = b$ it is known that the shear stress resultant S'_1 in the skin-stringer combination and the transverse displacement $\bar{v}_0$ in the rib must be continuous. This transverse displacement is found from equations (10), (24) and (25) which eventually yield

$$E\bar{A}\bar{v}_0 = -\frac{L^2}{\lambda_1^2}\{C_1 \sinh \lambda_1 (y - b)/L + C_3 \cosh \lambda_1 (y - b)/L\}$$
$$-\frac{L^2}{\lambda_2^2}\{C_2 \sinh \lambda_2 (y - b)/L + C_4 \cosh \lambda_2 (y - b)/L\} \quad \ldots \qquad (26)$$

over the range $b < y < b + B$, and

$$E\bar{A}\bar{v}_0 = \frac{\bar{T}_0}{2L}\left(b^2 y - \frac{y^3}{3}\right) - \frac{L}{\lambda_1}C_1 y - \frac{L}{\lambda_2}C_2 y \quad \ldots \qquad (27)$$

over the range $-b < y < b$.

There are therefore just four conditions for the determination of the arbitrary constants. By simple substitution these conditions are found to be the four simultaneous equations

$$\left.\begin{aligned}
&C_1 \sinh \lambda_1 B/L + C_2 \sinh \lambda_2 B/L + C_3 \cosh \lambda_1 B/L + C_4 \cosh \lambda_2 B/L = 0, \\
&\frac{C_1}{\lambda_1} \cosh \lambda_1 B/L + \frac{C_2}{\lambda_2} \cosh \lambda_2 B/L + \frac{C_3}{\lambda_1} \sinh \lambda_1 B/L + \frac{C_4}{\lambda_2} \sinh \lambda_2 B/L = 0, \\
&C_3 + C_4 = \frac{\bar{T}_0 b}{L}, \\
&-\frac{C_1}{\lambda_1}\frac{b}{L} - \frac{C_2}{\lambda_2}\frac{b}{L} + \frac{C_3}{\lambda_1^2} + \frac{C_4}{\lambda_2^2} = -\frac{\bar{T}_0}{3}\frac{b_3}{L_3}
\end{aligned}\right\} \quad (28)$$

For a numerical example the following values of the parameters have been chosen, *viz.*,

$$\frac{Et^*}{\mu t} = 4 \cdot 6, \quad \frac{t^* L}{\bar{A}} = 3 \cdot 15, \quad \frac{B}{L} = 0 \cdot 77, \quad \frac{b}{L} = 0 \cdot 46, \quad \bar{T}_0 = 1 \cdot 0.$$

Substituting these into the expression for λ_1 and λ_2 yields

$$\lambda_1 = 1 \cdot 587, \quad \lambda_2 = 1 \cdot 732,$$

whence the solution of the four simultaneous equations (28) gives

$$C_1 = 3 \cdot 377, \quad C_2 = -3 \cdot 888, \quad C_3 = -3 \cdot 779, \quad C_4 = 4 \cdot 239.$$

Substitution of the values of λ_1, λ_2 and C_1, C_2, C_3 and C_4 into the relevant expressions yields the stress distributions throughout the structure.

The calculated and experimental longitudinal direct-stress resultants in the skin-stringer combination are shown in Fig. 6. In this same figure are shown the calculated results using the theory of Ref. 9. Fig. 7 shows a panoramic view of the calculated stress resultants using the theory of the present paper.

The experimental results were obtained using the rig shown in Fig. 5. The longitudinal strains in the skin-stringer combination were measured at selected points and were then converted into direct stresses. It was not possible to measure these strains right at a rib line so readings were taken at various points along each stringer and the desired strains were estimated from these.

The calculated shear-stress resultants are shown in Fig. 8. It must be remembered that these are the 'averaged shears' along each bay of the structure and therefore do not indicate the peak concentrations that occur immediately at the corners of the cut-out.

The 'exact' solution involving the biharmonic equation encountered in the mathematical theory of elasticity yields infinite values of the stresses at the corners of the cut-out[10]. In practice, however, plastic deformation of the material will limit these stresses to some finite value.

APPENDIX III

Closely Spaced Ribs

If the basic structure is replaced by a shell model containing a closely spaced distribution of ribs there are some important simplifications of the compatibility equation.

The equation of compatibility has already been derived for discretely spaced ribs. It is, from equation 16,

$$L^4 \frac{d^4}{dy^4}(S'_{j+1} + 4S'_j + S'_{j-1}) + 6\frac{Et^*}{\mu t} L^2 \frac{d^2}{dy^2}(S'_{j+1} - 2S'_j + S'_{j-1})$$

$$+ 6\frac{Et^*L}{\bar{E}\bar{A}}(S'_{j+2} - 4S'_{j+1} + 6S'_j - 4S'_{j-1} + S'_{j-2})$$

$$= -6\frac{Et^*L}{\bar{E}\bar{A}}(\mathscr{S}_{j+1} - 3\mathscr{S}_j + 3\mathscr{S}_{j-1} - \mathscr{S}_{j-2}).$$

Now, using the notation of the ' Calculus of Finite Differences ', *viz.*,

$$\Delta^2 \!\to\! S'_j = \frac{1}{L^2}(S'_{j+1} - 2S'_j + S'_{j-1}),$$

this may be re-expressed

$$\left[\frac{d^4}{dy^4}(L^2\Delta^2 + 6) + 6\frac{Et^*}{\mu t}\frac{d^2}{dy^2}\Delta^2 + 6\frac{Et^*L}{\bar{E}\bar{A}}\Delta^4\right] \to S'_j = 0$$

when all the $\mathscr{S}_j$ are zero. In the limit as L, the distance between ribs, becomes zero it is known that

$$\lim_{L=0} \Delta^2 \to S'_j = \frac{d^2S'}{dx^2}$$

and so the equation of compatibility becomes

$$\frac{\partial^4 S'}{\partial y^4} + \frac{Et^*}{\mu t}\frac{\partial^4 S'}{\partial y^2 \partial x^2} + \frac{Et^*}{\bar{E}\bar{t}}\frac{\partial^4 S'}{\partial x^4} = 0, \quad \ldots \quad (29)$$

where it has been assumed for the rib flanges that

$$\bar{A}/L = \bar{t}$$

where $\bar{t}$ is an effective skin thickness in the transverse direction.

Equation (29) is the condition of compatibility for a shell model containing closely spaced ribs where the rib flanges possess a finite extensional rigidity. If it is assumed that they are rigid, i.e., $\bar{t} = \infty$, then equation (29) becomes

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 S'}{\partial y^2} + \frac{Et^*}{\mu t}\frac{\partial^2 S'}{\partial x^2}\right) = 0. \quad \ldots \quad (30)$$

This equation, in slightly modified form, occurs frequently in shear lag and diffusion problems such as those investigated by Williams[6], Goodey[7] and Hadji-Argyris and Dunne[8].

APPENDIX IV

Determination of the Stress Distribution in a More General Shell Model Including the Effect of Stringer Bending and Twisting

The reinforced monocoque structure considered in this appendix is shown in Fig. 9. The shell model is more general because it is not necessary for the rib flanges (frames), skin-stringer combination and rib spacing to be constant throughout the shell model. The effect of stringer bending and twisting will usually be small except in such cases as a fuselage of a large aircraft under concentrated loading. In these fuselages the frames are often very flexible and it is then that the effect of stringer bending and twisting becomes of importance.

1. *Fundamental Equations for the Skin-Stringer Combination.*—In preparation for forming the equations of compatibility for the shell model it is necessary to consider the detailed equations governing the individual behaviours of the skin-stringer combination and the rib flanges.

The stress resultants and couples acting on an elemental portion of the skin-stringer combination in the jth bay are shown in Fig. 10 (the position suffixes j have been omitted in the figure). Resolving forces in the longitudinal direction it is found for equilibrium of the element that

$$\frac{\partial T_j}{\partial x} + \frac{\partial S'_j}{\partial y} = 0, \qquad (31)$$

while in the transverse direction, neglecting T',

$$\frac{\partial S_j}{\partial x} = 0 \qquad (32)$$

and in the normal direction

$$\frac{\partial N_j}{\partial x} = 0. \qquad (33)$$

Taking moments about an axial line, it follows that

$$\frac{\partial G'_j}{\partial x} = 0, \qquad (34)$$

about a transverse line that

$$\frac{\partial G_j}{\partial x} - N_j = 0 \qquad (35)$$

and, finally, taking moments about a normal

$$\frac{\partial K_j}{\partial x} - S'_j + S_j = 0. \qquad (36)$$

There is one more equation belonging to this category and this refers to an assumption that the shear S' is independent of the axial co-ordinate, *viz.*,

$$\frac{\partial S'_j}{\partial x} = 0. \qquad (37)$$

The longitudinal and shear strains in the skin-stringer combination can be expressed in terms of the displacements and the stresses. They are respectively

$$e_{xx\,j} = \frac{\partial u_j}{\partial x} = \frac{T_j}{E_j t^*_j} \qquad (38)$$

neglecting T', and

$$e_{xy\,j} = \frac{\partial u_j}{\partial y} + \frac{\partial v_j}{\partial x} = \frac{S'_j}{\mu_j t_j}. \qquad (39)$$

Similarly, the bending couples may be expressed in terms of the displacements. They are

$$\frac{\partial^2 v_j}{\partial x^2} = \frac{K_j}{E_j I_{K\,j}} \qquad (40)$$

and

$$\frac{\partial^2 w_j}{\partial x^2} = -\frac{G_j}{E_j I_{G\,j}}, \qquad (41)$$

from elementary bending theory.

Finally, for the twisting couple,

$$\frac{\partial^2 w_j}{\partial y \partial x} = \frac{G'_j}{\mu_j J_j} \dagger \quad \ldots \quad (42)$$

Now, it has been assumed in equation (37) that the shear S'_j is a function only of the transverse co-ordinate y, hence equation (31) may be written

$$T_j = -\frac{dS'_j}{dy}\int dx$$

and, on observing that the longitudinal direct-stress resultant must be continuous from bay to bay since the ribs are assumed to have no rigidity for bending in the plane of the sheet this integrates to

$$T_j = -x\frac{dS'_j}{dy} + \bar{T}_{j-1}, \quad \ldots \quad (43)$$

where a new origin for x is chosen on the left-hand side of each bay and $\bar{T}_{j-1}$ denotes the longitudinal direct-stress resultant in the skin-stringer combination at the $j-1$th rib.

The axial displacement u_j is, from equation (38),

$$u_j = \frac{1}{E_j t^*_j}\int T_j\, dx$$

which on using equation (43), and integrating, becomes

$$u_j = \frac{1}{E_j t^*_j}\left(-\frac{x^2}{2}\frac{dS'_j}{dy} + x\bar{T}_{j-1}\right) + \bar{u}_{j-1}, \quad \ldots \quad (44)$$

where $\bar{u}_{j-1}$ denotes the longitudinal displacement of the skin-stringer combination at the $j-1$th rib.

From equation (39), the transverse displacement is

$$v_j = \int\left(\frac{S'_j}{\mu_j t_j} - \frac{\partial u_j}{\partial y}\right) dx.$$

Substituting from equation (44), and integrating, this becomes

$$v_j = \frac{x}{\mu_j t_j} S'_j - \frac{1}{E_j t^*_j}\left(-\frac{x^3}{6}\frac{d^2 S'_j}{dy^2} + \frac{x^2}{2}\frac{d\bar{T}_{j-1}}{dy}\right) - x\frac{d\bar{u}_{j-1}}{dy} + \bar{v}_{j-1}, \quad \ldots \quad (45)$$

where $\bar{v}_{j-1}$ denotes the transverse displacement of the skin-stringer combination at the $j-1$th rib. Putting $x = L_j$ into this equation yields

$$\bar{v}_j = \frac{L_j}{\mu_j t_j} S'_j - \frac{1}{E_j t^*_j}\left(-\frac{L^3_j}{6}\frac{d^2 S'_j}{dy^2} + \frac{L^2_j}{2}\frac{d\bar{T}_{j-1}}{dy}\right) - L_j\frac{d\bar{u}_{j-1}}{dy} + \bar{v}_{j-1} \quad \ldots \quad (46)$$

which is an expression for the transverse displacement of the skin-stringer combination at the jth rib.

† Due to the construction of the skin-stringer combination, *i.e.*, a very thin skin reinforced by uniformly distributed but *distinct* longitudinal elements, derivatives of the normal deflection w with respect to y possess no real significance unless they refer to the displacement of a rib flange. However, the notation $\partial w/\partial y$ is most convenient to express the rotation of the axial elements between the ribs, provided it is remembered that it is only a notation. Thus the anomaly presented by equations (41) and (42), *viz.*,

$$\frac{\partial^3 w_j}{\partial y\, \partial x^2} = -\frac{1}{E_j I_{aj}}\frac{\partial G_j}{\partial y} = \frac{1}{\mu_j J_j}\frac{\partial G'_j}{\partial x} = 0,$$

from equation (34), simply does not arise.

From equations (40) and (45) it is found that the stress couple K_j in the transverse sense is

$$K_j = -\frac{I_{Kj}}{t^*_j}\left(-x\frac{d^2S'_j}{dy^2} + \frac{d\bar{T}_{j-1}}{dy}\right), \quad \ldots \quad (47)$$

and then from equation (36) the shear stress resultant S_j is

$$S_j = S'_j - \frac{I_{Kj}}{t^*_j}\frac{d^2S'_j}{dy^2}. \quad \ldots \quad (48)$$

In equation (34) it is seen that the twisting couple G'_j is independent of the longitudinal co-ordinate x and hence equation (42) may be integrated with respect to x to give

$$\frac{\partial w_j}{\partial y} = \frac{x}{\mu_j J_j} G'_j + \frac{\partial \bar{w}_{j-1}}{\partial y} \quad \ldots \quad (49)$$

where $\bar{w}_{j-1}$ denotes the normal displacement of the skin-stringer combination at the $j-1$th rib. Putting $x = L_j$ into this equation yields

$$\frac{\partial \bar{w}_j}{\partial y} = \frac{L_j}{\mu_j J_j} G'_j + \frac{\partial \bar{w}_{j-1}}{\partial y}, \quad \ldots \quad (50)$$

which is an expression for the rotation of the skin-stringer combination at the jth rib.

From equations (33) and (35) it follows that the bending couple G_j may be expressed

$$G_j = xN_j + \bar{G}_{j-1} \quad \ldots \quad (51)$$

where $\bar{G}_{j-1}$ denotes the bending couple in the skin-stringer combination at the $j-1$th rib. The ribs are assumed to possess no bending rigidity in this sense and hence the bending couple in the skin-stringer combination will be continuous from bay to bay.

Substituting equation (51) into equation (41) and integrating, yields

$$\frac{\partial w_j}{\partial x} = -\frac{1}{E_j I_{Gj}}\left(\frac{x^2}{2}N_j + x\bar{G}_{j-1}\right) + \frac{\partial \bar{w}_{j-1}}{\partial x} \quad \ldots \quad (52)$$

and

$$w_j = -\frac{1}{E_j I_{Gj}}\left(\frac{x^3}{6}N_j + \frac{x^2}{2}\bar{G}_{j-1}\right) + x\frac{\partial \bar{w}_{j-1}}{\partial x} + \bar{w}_{j-1}. \quad \ldots \quad (53)$$

Putting $x = L_j$ into this last equation gives

$$\bar{w}_j = -\frac{1}{E_j I_{Gj}}\left(\frac{L_j^3}{6}N_j + \frac{L_j^2}{2}\bar{G}_{j-1}\right) + L_j\frac{\partial \bar{w}_{j-1}}{\partial x} + \bar{w}_{j-1}, \quad \ldots \quad (54)$$

which is an expression for the normal displacement of the skin-stringer combination at the jth rib. Substituting this into equation (50), the twisting couple is given by

$$G'_j = -\frac{\mu_j J_j}{E_j I_{Gj}}\left(\frac{L_j^2}{6}\frac{dN_j}{dy} + \frac{L_j}{2}\frac{d\bar{G}_{j-1}}{dy}\right) + \mu_j J_j \frac{\partial^2 \bar{w}_{j-1}}{\partial y\, \partial x}. \quad \ldots \quad (55)$$

This completes the formulation of the fundamental equations for the skin-stringer combination.

2. *Fundamental Equations for the Rib Flanges (frames)*.—The forces acting on an elemental portion of the jth rib flange are shown in Fig. 11, where the applied forces are composed of those applied by the skin-stringer combination plus the external forces. Thus

$$\left.\begin{aligned} \bar{N}_j &= N_{j+1} - N_j + \mathscr{N}_j \\ \bar{S}_j &= S_{j+1} - S_j + \mathscr{S}_j \\ \bar{G}_j &= G'_{j+1} - G'_j + \mathscr{G}_j \end{aligned}\right\}, \quad \ldots \quad (56)$$

where $\mathscr{N}_j$, $\mathscr{S}_j$ and $\mathscr{G}_j$ are the external forces.

The equations of equilibrium of the elemental portion of rib flange are derived in the usual way. Resolving forces in the normal direction

$$\bar{N}_j + \frac{d\bar{Q}_j}{dy} = 0 \qquad (57)$$

and then in the transverse direction

$$\bar{S}_j + \frac{d\bar{F}_j}{dy} = 0 \qquad (58)$$

and finally by taking moments

$$\bar{G}'_j - \frac{d\bar{M}_j}{dy} + \bar{Q}_j = 0. \qquad (59)$$

The strain in the rib flange along the skin median line can be expressed in terms of the displacements and stresses. It is

$$\bar{e}_{yy\,j} = \frac{d\bar{v}_j}{dy} = \frac{\bar{F}_j}{\bar{E}_j\bar{A}_j}. \qquad (60)$$

For the bending of the rib flange it follows that

$$\frac{\bar{M}_j}{\bar{E}_j\bar{I}_j} = -\frac{d^2\bar{w}_j}{dy^2} \qquad (61)$$

where the shear deflection is neglected.

Using the above equations it is possible to express the forces and displacements in the rib flange in terms of the applied forces. They are

$$\frac{d\bar{Q}_j}{dy} = -\bar{N}_j, \qquad (62)$$

$$\frac{d\bar{F}_j}{dy} = -\bar{S}_j, \qquad (63)$$

$$\frac{d^2\bar{M}_j}{dy^2} = \frac{d\bar{G}'_j}{dy} - \bar{N}_j, \qquad (64)$$

$$\frac{d^2\bar{v}_j}{dy^2} = -\frac{\bar{S}_j}{\bar{E}_j\bar{A}_j}. \qquad (65)$$

and finally

$$\frac{d^4\bar{w}_j}{dy^4} = -\frac{1}{\bar{E}_j\bar{I}_j}\left(\frac{d\bar{G}'_j}{dy} - \bar{N}_j\right). \qquad (66)$$

This completes the formulation of the fundamental equations and the next step is to match the displacements of the skin-stringer combination and rib flanges along their various intersections and thus develop the compatibility equations for the structure.

3. *The Equations of Compatibility*.—For the stress and strain to be consistent throughout the structure it is now only necessary to match the transverse and normal displacements v and w along each intersection of the skin-stringer combination and rib flanges.

Proceeding thus, it is found from equations (46) and (65) for the transverse displacement $\bar{v}$ to be identical in the plate and rib flange along an intersection that

$$\frac{1}{E_j t^*_j}\left(\frac{L_j^3}{6}\frac{d^4S'_j}{dy^4} - \frac{L_j^2}{2}\frac{d^3\bar{T}_{j-1}}{dy^3}\right) + \frac{L_j}{\mu_j t_j}\frac{d^2S'_j}{dy^2} - L_j\frac{d^3\bar{u}_{j-1}}{dy^3} + \frac{d^2\bar{v}_{j-1}}{dy^2}$$
$$+ \frac{1}{\bar{E}_j\bar{A}_j}\left(S'_{j+1} - \frac{I_{K\,j+1}}{t^*_{j+1}}\frac{d^2S'_{j+1}}{dy^2} - S'_j + \frac{I_{K\,j}}{t^*_j}\frac{d^2S'_j}{dy^2}\right) = -\frac{\mathscr{S}_j}{\bar{E}_j\bar{A}_j}, \quad \ldots \quad (67)$$

where substitutions have been made from equations (48) and (56). The term on the right-hand side represents the influence of the external forces acting on the rib flange. For successive values of the suffix j the above becomes the set of compatibility equations for the determination of the stress distribution in a flat structure undergoing a loading which is in the plane of the structure. When the terms containing the direct stress resultants $\bar{T}$, and the displacements $\bar{u}$ and $\bar{v}$ are eliminated by using the known relations

$$\bar{T}_j = -L_j\frac{dS'_j}{dy} + \bar{T}_{j-1}, \quad \ldots \quad (68)$$

$$\bar{u}_j = \frac{1}{E_j t^*_j}\left(-\frac{L_j^2}{2}\frac{dS'_j}{dy} + L_j\bar{T}_{j-1}\right) + \bar{u}_{j-1} \quad \ldots \quad (69)$$

by putting $x = L_j$ in equations (43) and (44),

$$\frac{d^2\bar{v}_j}{dy^2} = -\frac{1}{\bar{E}_j\bar{A}_j}\left(S'_{j+1} - \frac{I_{K\,j+1}}{t^*_{j+1}}\frac{d^2S'_{j+1}}{dy^2} - S'_j + \frac{I_{K\,j}}{t^*_j}\frac{d^2S'_j}{dy^2} + \mathscr{S}_j\right), \quad \ldots \quad (70)$$

which follows from the substitution of equations (48) and (56) into equation (65), and the boundary conditions at each end of the structure they form a set of simultaneous linear ordinary differential equations with constant coefficients. Each equation will be of fourth order involving only the even differentials and there will be one such equation for every bay.

For the normal displacement $\bar{w}$ to be identical in the skin-stringer combination and rib flange along an intersection, it is found by substituting equations (54) to (56) into equation (66) that

$$\frac{1}{E_j I_{Gj}}\left(\frac{L_j^3}{6}\frac{d^4N_j}{dy^4} + \frac{L_j^2}{2}\frac{d^4\bar{G}_{j-1}}{dy^4}\right) - L_j\frac{\partial^5\bar{w}_{j-1}}{\partial y^4\,\partial x} - \frac{\partial^4\bar{w}_{j-1}}{\partial y^4}$$
$$+ \frac{1}{\bar{E}_j\bar{I}_j}\left\{\frac{\mu_{j+1}\;\;J_{j+1}}{E_{j+1}\,I_{G\,j+1}}\left(\frac{L_{j+1}{}^2}{6}\frac{d^2N_{j+1}}{dy^2} + \frac{L_{j+1}}{2}\frac{d^2\bar{G}_j}{dy_2}\right) - \mu_{j+1}J_{j+1}\frac{\partial^3\bar{w}_j}{\partial y^2\,\partial x}\right.$$
$$\left. - \frac{\mu_j J_j}{E_j I_{G\,j}}\left(\frac{L_j^2}{6}\frac{d^2N_j}{dy^2} + \frac{L_j}{2}\frac{d^2\bar{G}_{j-1}}{dy^2}\right) + \mu_j J_j\frac{\partial^3\bar{w}_{j-1}}{\partial y^2\,\partial x} + N_{j+1} - N_j\right\}$$
$$= \frac{1}{\bar{E}_j\bar{I}_j}\left(\frac{d\mathscr{G}_j}{dy} - \mathscr{N}_j\right). \quad \ldots \quad (71)$$

The term on the right-hand side represents the influence of the external forces acting on the rib flange. For successive values of the suffix j the above becomes the set of compatibility equations for the determination of the stress distribution in a flat structure undergoing a loading which is normal to the plane of the structure. When the terms containing the stress couples $\bar{G}$ and the displacements $\bar{w}$ are eliminated by using the known relations

$$\bar{G}_j = L_j N_j + \bar{G}_{j-1}, \quad \ldots \quad (72)$$

$$\frac{\partial\bar{w}_j}{\partial x} = -\frac{1}{E_j I_{G\,j}}\left(\frac{L_j^2}{2}N_j + L_j\bar{G}_{j-1}\right) + \frac{\partial\bar{w}_{j-1}}{\partial x} \quad \ldots \quad (73)$$

by putting $x = L_j$ in equations (51) and (52),

$$\frac{\partial^4 \bar{w}_j}{\partial y^4} = \frac{1}{\bar{E}_j \bar{I}_j}\left\{\frac{\mu_{j+1}\ J_{j+1}}{E_{j+1}\ I_{G\,j+1}}\left(\frac{L_{j+1}{}^2}{6}\frac{d^2 N_{j+1}}{dy^2} + \frac{L_{j+1}}{2}\frac{d^2 \bar{G}_j}{dy^2}\right)\right.$$
$$- \mu_{j+1}\, J_{j+1}\frac{\partial^3 \bar{w}_j}{\partial y^2 \partial x} - \frac{\mu_j J_j}{E_j I_{G\,j}}\left(\frac{L_j^2}{6}\frac{d^2 N_j}{dy^2} + \frac{L_j}{2}\frac{d^2 \bar{G}_{j-1}}{dy^2}\right)$$
$$\left. + \mu_j\, J_j \frac{\partial^3 \bar{w}_{j-1}}{\partial y^2\, \partial x} - \frac{d\mathcal{G}}{dy} + N_{j+1} - N_j + \mathcal{N}_j\right\} \quad \ldots \quad (74)$$

by substituting equations (55) and (56) into equation (66), and the boundary conditions at each end of the structure they form a set of simultaneous linear ordinary differential equations similar in nature to equation (67).

In the special case when there are only twisting couples, or the effects of all the N_j are neglected, the compatibility equation becomes

$$\frac{L_j}{\mu_j J_j}\frac{d^3 G'_j}{dy^3} + \frac{d^4 \bar{w}_{j-1}}{dy^4} + \frac{1}{\bar{E}_j \bar{I}_j}\left(\frac{dG'_{j+1}}{dy} - \frac{dG'_j}{dy}\right) = -\frac{1}{\bar{E}_j \bar{I}_j}\left(\frac{d\mathcal{G}_j}{dy} - \mathcal{N}_j\right) \quad (75)$$

which is obtained by substituting equations (50) and (56) into equation (66). When the terms containing the displacements $\bar{w}$ are eliminated by using

$$\frac{d^4 \bar{w}_j}{dy^4} = -\frac{1}{\bar{E}_j \bar{I}_j}\left(\frac{dG'_{j+1}}{dy} - \frac{dG'_j}{dy} + \frac{d\mathcal{G}_j}{dy} - \mathcal{N}_j\right) \quad \ldots \quad (76)$$

obtained by substituting equation (56) into (66) and the boundary conditions, the resulting set of differential equations is of third order involving only the odd differentials.

Solutions to the differential equations of compatibility are reserved for later papers in which specific problems will be investigated.

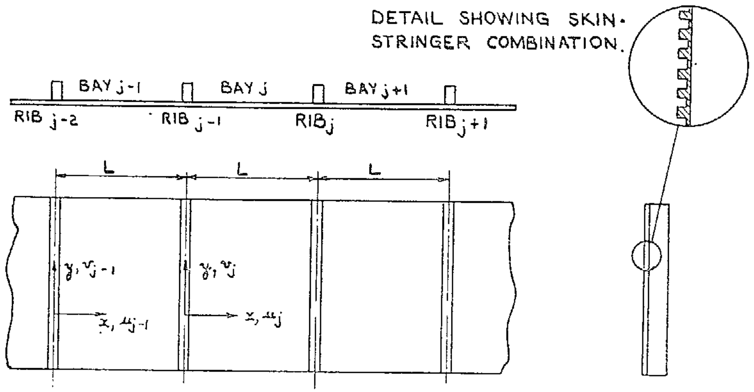

FIG. 1. The reinforced monocoque flat structure.

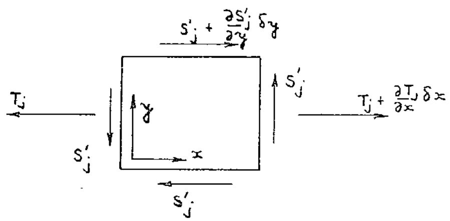

FIG. 2. Stress resultants acting on an elemental portion of the skin-stringer combination in the shell model.

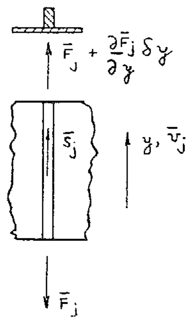

FIG. 3. Forces acting on an elemental portion of a rib flange in the shell model.

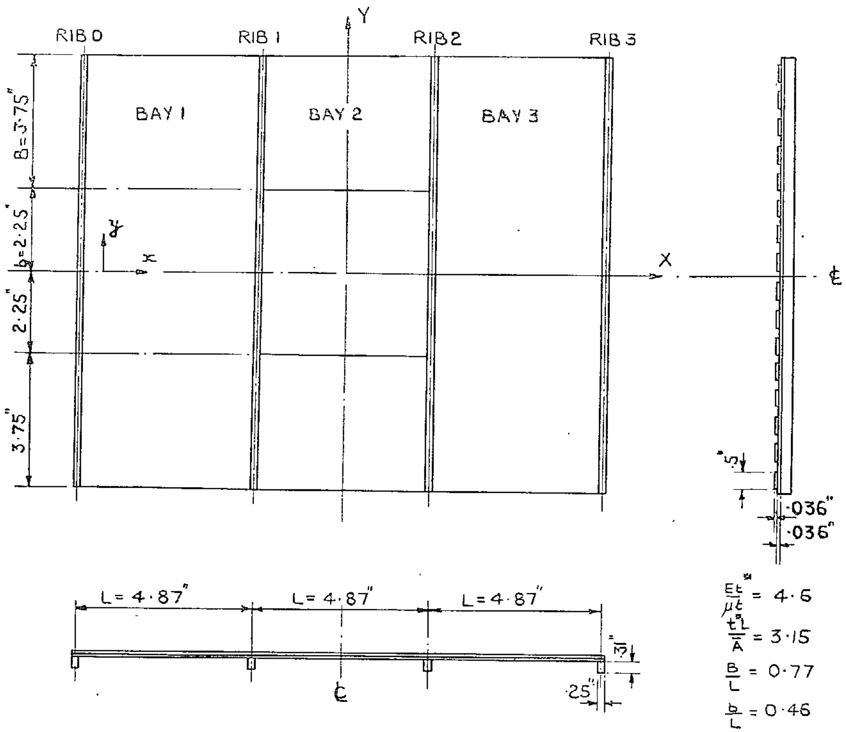

FIG. 4. Three-bay reinforced monocoque flat structure with a rectangular cut-out in the centre bay.

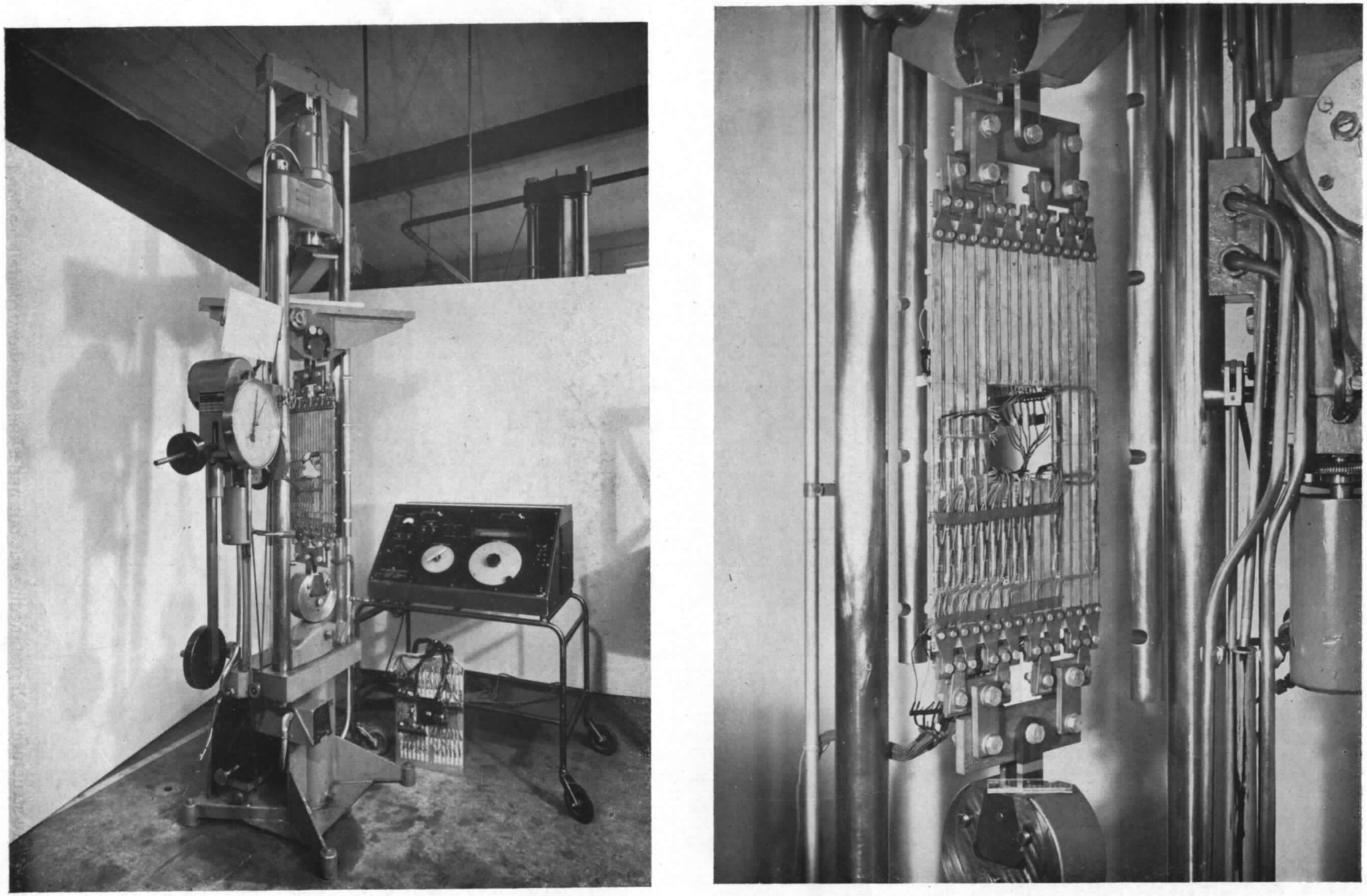

FIG. 5. Test rig for experimental investigation of a three-bay reinforced monocoque flat structure with a rectangular cut-out in the centre bay under uniformly distributed tension loading.

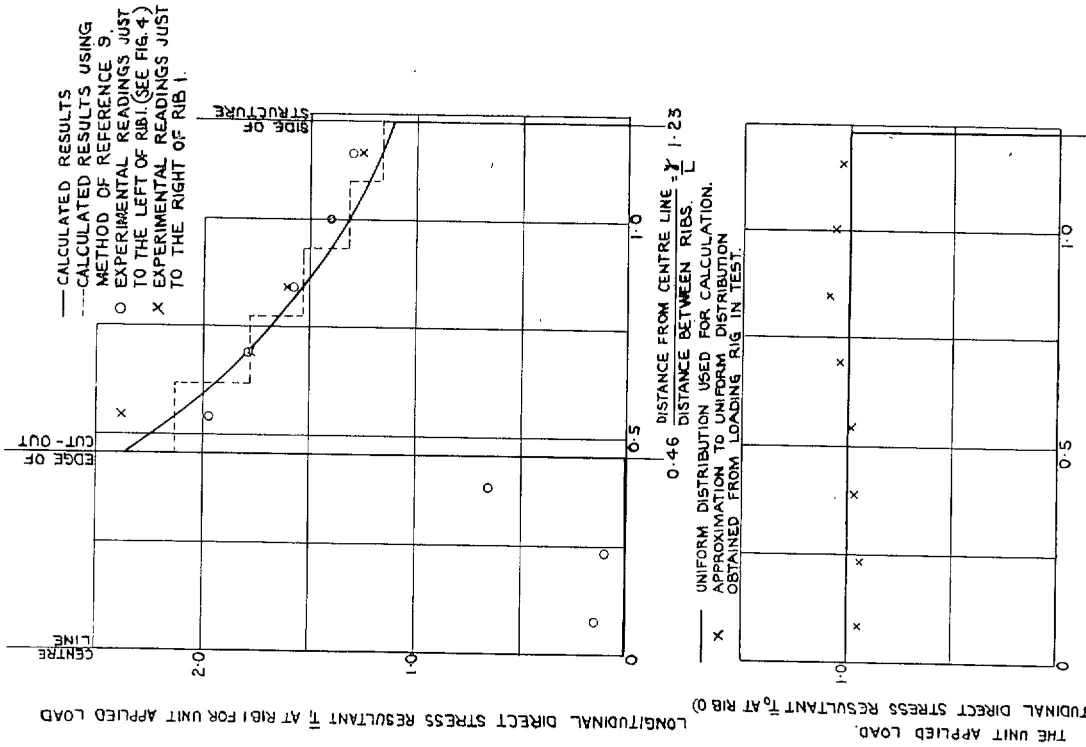

FIG. 6. Longitudinal direct-stress resultants in the skin-stringer combination of a three-bay structure with rectangular cut-out under uniformly distributed tension loading.

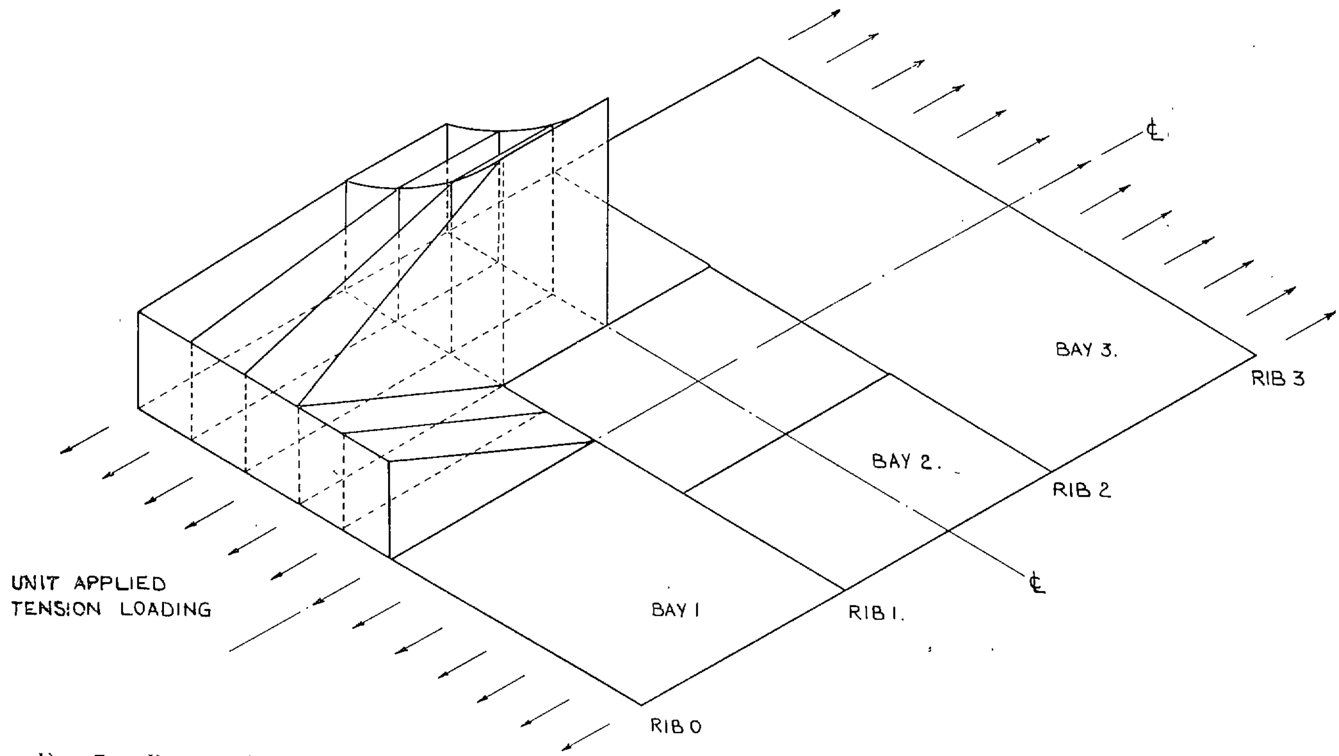

FIG. 7. Panoramic view of the calculated longitudinal direct-stress resultants in the skin-stringer combination of a three-bay structure with rectangular cut-out under uniformly distributed tension loading.

J4310 Wt.17/680 K9 5/55 D&Co. 34/263

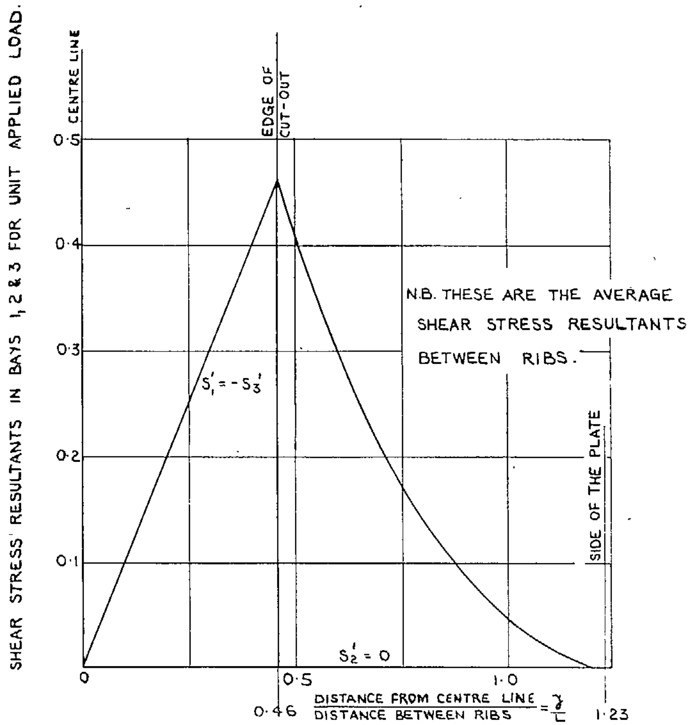

FIG. 8. Calculated shear-stress resultants in the skin of a three-bay structure with rectangular cut-out under uniformly distributed tension loading.

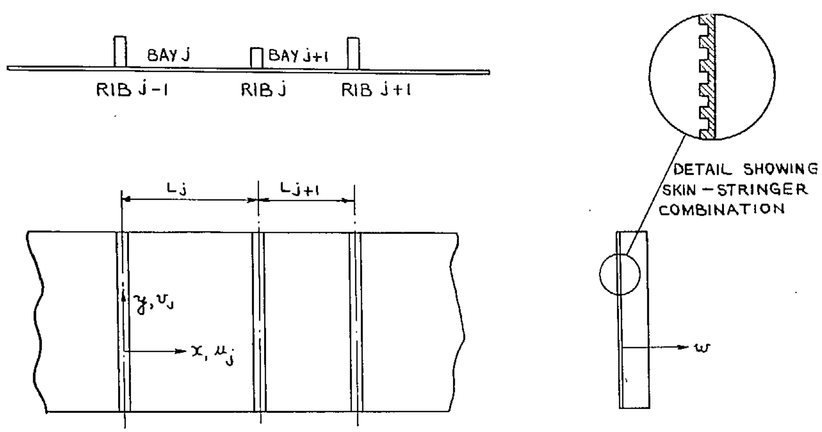

FIG. 9. The reinforced monocoque flat structure in general.

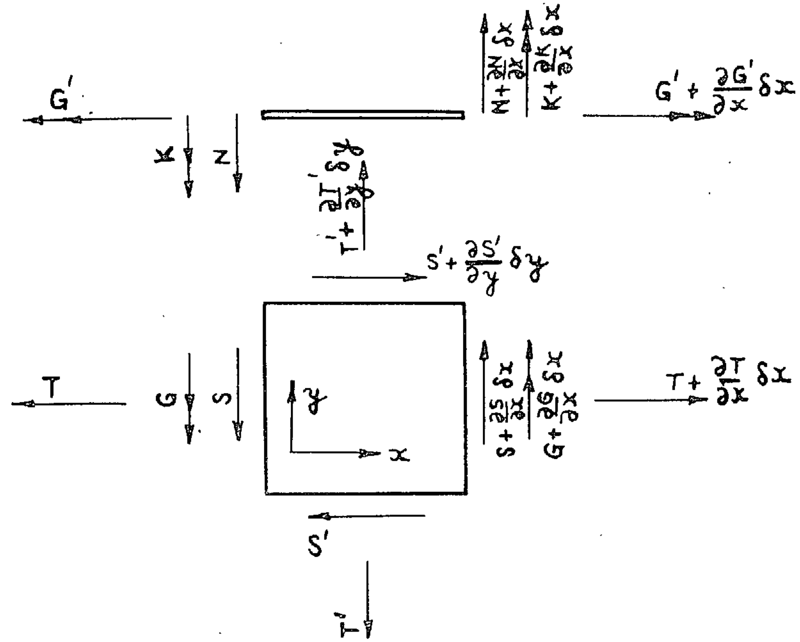

FIG. 10. Stress resultants and couples acting on an elemental portion of the skin-stringer combination in the general shell model.

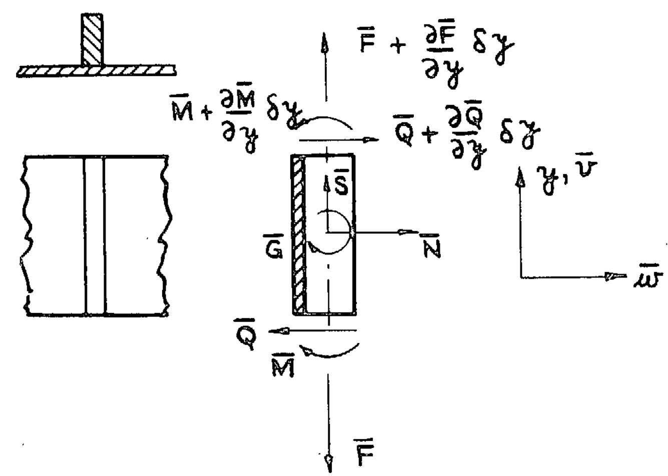

FIG. 11. Forces acting on an elemental portion of a rib flange in the general shell model.

PRINTED IN GREAT BRITAIN

Publications of the Aeronautical Research Council

ANNUAL TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL (BOUND VOLUMES)

1936 Vol. I. Aerodynamics General, Performance, Airscrews, Flutter and Spinning. 40s. (41s. 1d).
Vol. II. Stability and Control, Structures, Seaplanes, Engines, etc. 50s. (51s. 1d.)
1937 Vol. I. Aerodynamics General, Performance, Airscrews, Flutter and Spinning. 40s. (41s. 1d.)
Vol. II. Stability and Control, Structures, Seaplanes, Engines, etc. 60s. (61s. 1d.)
1938 Vol. I. Aerodynamics General, Performance, Airscrews. 50s. (51s. 1d.)
Vol. II. Stability and Control, Flutter, Structures, Seaplanes, Wind Tunnels, Materials. 30s. (31s. 1d.)
1939 Vol. I. Aerodynamics General, Performance, Airscrews, Engines. 50s. (51s. 1d.)
Vol. II. Stability and Control, Flutter and Vibration, Instruments, Structures, Seaplanes, etc. 63s. (64s. 2d.)
1940 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Icing, Stability and Control, Structures, and a miscellaneous section. 50s. (51s. 1d.)
1941 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Stability and Control, Structures. 63s. (64s. 2d.)
1942 Vol. I. Aero and Hydrodynamics, Aerofoils, Airscrews, Engines. 75s. (76s. 3d.)
Vol. II. Noise, Parachutes, Stability and Control, Structures, Vibration, Wind Tunnels. 47s. 6d. (48s. 7d.)
1943 Vol. I. Aerodynamics, Aerofoils, Airscrews, 80s. (81s. 4d.)
Vol. II. Engines, Flutter, Materials, Parachutes, Performance, Stability and Control, Structures. 90s. (91s. 6d.)
1944 Vol. I. Aero and Hydrodynamics, Aerofoils, Aircraft, Airscrews, Controls. 84s. (85s. 8d.)
Vol. II. Flutter and Vibration, Materials, Miscellaneous, Navigation, Parachutes, Performance, Plates, and Panels, Stability, Structures, Test Equipment, Wind Tunnels. 84s. (85s. 8d.)

ANNUAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL—

1933–34	1s. 6d. (1s. 8d.)	1937	2s. (2s. 2d.)
1934–35	1s. 6d. (1s. 8d.)	1938	1s. 6d. (1s. 8d.)
April 1, 1935 to Dec. 31, 1936.	4s. (4s. 4d.)	1939–48	3s. (3s. 2d.)

INDEX TO ALL REPORTS AND MEMORANDA PUBLISHED IN THE ANNUAL TECHNICAL REPORTS, AND SEPARATELY—

April, 1950 - - - - R. & M. No. 2600. 2s. 6d. (2s. 7½d.)

AUTHOR INDEX TO ALL REPORTS AND MEMORANDA OF THE AERONAUTICAL RESEARCH COUNCIL—

1909–1949 - - - - - R. & M. No. 2570. 15s. (15s. 3d.)

INDEXES TO THE TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL—

December 1, 1936 — June 30, 1939.	R. & M. No. 1850.	1s. 3d. (1s. 4½d.)
July 1, 1939 — June 30, 1945.	R. & M. No. 1950.	1s. (1s. 1½d.)
July 1, 1945 — June 30, 1946.	R. & M. No. 2050.	1s. (1s. 1½d.)
July 1, 1946 — December 31, 1946.	R. & M. No. 2150.	1s. 3d. (1s. 4½d.)
January 1, 1947 — June 30, 1947.	R. & M. No. 2250.	1s. 3d. (1s. 4½d.)
July, 1951 - - - -	R. & M. No. 2350.	1s. 9d. (1s. 10½d.)

Prices in brackets include postage.

Obtainable from

HER MAJESTY'S STATIONERY OFFICE

York House, Kingsway, London W.C.2 ; 423 Oxford Street, London W.1 (Post Orders : P.O. Box No. 569, London S.E.1) ; 13A Castle Street, Edinburgh 2 ; 39 King Street, Manchester 2 ; 2 Edmund Street, Birmingham 3 ; 109 St. Mary Street, Cardiff ; Tower Lane, Bristol 1 ; 80 Chichester Street, Belfast OR THROUGH ANY BOOKSELLER

S.O. Code No. 23-2879

R. & M. No. 2879