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Permissible Design Values and Variability Test Factors

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Summary.—For the design of structural elements it is postulated that :—

(*a*) not more than 10 per cent to any given design should have strength below the design value

(*b*) not more than 0·1 per cent should have strength below 90 per cent of the design value.

This rule forms a working basis for the interpretation of tests on statistical lines.

On the basis of a fixed probability, Part I deduces :—

(i) expressions for the derivation of permissible design values from a given number of test results

(ii) the number of test results required, on specimens chosen at random, so that the estimates of permissible design values can be regarded as sufficiently accurate

(iii) the factor which should be applied to the results of tests on any number of similar components designed to meet a specified requirement.

The effects of different probabilities and of different acceptable proportions of weak specimens are investigated in Part II.

PART I

1. *Introduction.*—There is an increasing interest being shown (Refs. 1, 2, 3, 4) in the application of statistical theory to the specification of test factors and to the derivation of design values. For the convenience of those engaged in work where statistical methods can usefully be employed, Part I presents a simple statement in graphical and tabular form, of the various factors which should be used when deriving design values from test results.

Proposals for acceptance standards are first put forward, and to maintain these standards the test factors which should be realised in any given number of tests are deduced; the design value calculated on the basis of these test figures is also shown.

2. *Statement of the Problem.*—Tests on nominally identical structures or materials show a variation in realised strength from specimen to specimen, which can be attributed to small variations in composition, in manufacturing processes, and in testing technique. From strength test results on a given material for instance, it is possible to plot a strength distribution diagram,

* R.A.E. Tech. Note Structures 15, received 3rd July, 1948.
R.A.E. Tech. Note Structures 61, received 1st February, 1951.

as shown in Fig. 1; if the number, n, of test results is very large, then the diagram approximates very closely to the true strength distribution diagram of the material, with a mean strength $\bar{X}$, and a standard deviation σ measuring the scatter; in general, however, the number of test results is relatively few, and the resultant distribution diagram has a mean strength $\bar{x}$ and a standard deviation s, which usually differ from the true values $\bar{X}$ and σ respectively. The problem then is to determine from n test results a strength value which is acceptable for design use, or conversely, given the design requirement, to establish mean values which should be realised in tests on n specimens.

3. *Acceptable Design Conditions.*—The first step is to postulate acceptable design values in terms of the true mean $\bar{X}$ and true standard deviation σ; for convenience the argument will be developed in terms of strength. The design value can be determined once the proportion of weak specimens which can be tolerated in service is decided. Suppose, now, that this decision has been made, and the two following conditions result:

(*a*) The design value shall be such that not more than 10 per cent material has values which fall below that value

(*b*) The design value shall be such that not more than $0 \cdot 1$ per cent of material has values less than $0 \cdot 9$ times the design value.

Putting these design conditions into statistical notation, and assuming that the strength frequency distribution approximates to the normal (*i.e.*, a Gaussian) curve,

$$f_{0\cdot 1} = \bar{X}(1 - 1\cdot 3v) \qquad (1)$$

and

$$f_{0\cdot 001} = 1\cdot 11\bar{X}(1 - 3v) \qquad (2)$$

where $v = \sigma/\bar{X}$ is the true coefficient of variation, and $f_{0\cdot 1}$ and $f_{0\cdot 001}$ are the design values fixed by the $0\cdot 1$ and $0\cdot 001$ proportions respectively of weak specimens. The actual design value will, of course, be whichever is the less; in practice this will be $f_{0\cdot 1}$ if $v < 0\cdot 055$ or $f_{0\cdot 001}$ if $v > 0\cdot 055$.

4. *Practical Interpretation of Design Conditions.*—4.1. Unfortunately, as mentioned in section 2, the number n of test results is usually relatively small, and consequently the mean value $\bar{x}$ and the standard deviation s differ in general from $\bar{X}$ and σ. Statistical theory shows that the discrepancy between the estimated and true standard deviation is liable to be greater than that for the mean values (*see* Tables in Ref. 5).

As it happens v for most aircraft materials and structures ranges from $0\cdot 03$ to $0\cdot 10$, and it is possible, on the bases of engineering judgment and experience, to assume a value of v which should be sufficiently accurate and yet on the conservative or high side; in other words, the standard deviation s should be calculated for the n results, and then modified if judged necessary.

If a number of samples, each of n specimens, were taken from a population with a mean $\bar{X}$ and a standard deviation σ the means of the samples, $\bar{x}$, will have a known distribution about the true mean $\bar{X}$. For design purposes, we are concerned lest the mean value of the particular sample of n specimens tested should be greater than the true mean, thus giving unconservative design on the basis of equations (1) and (2). Theory indicates that the probability that any particular mean value $\bar{x}$ lies above the limit $\bar{X} + 1\cdot 96\sigma/\sqrt{n}$ is $0\cdot 025$; similarly that $\bar{x}$ should be above $\bar{X} + 3\cdot 09\sigma/\sqrt{n}$ has a probability of $0\cdot 001$. The probability of $0\cdot 001$ may be regarded as fixing an upper limit, but for general application, a reasonable limit for $\bar{x}$ can be fixed by assuming a probability of $0\cdot 025$. On this somewhat pessimistic assumption, and taking round figures, in any particular instance

$$\bar{x} = \bar{X} + 2\sigma/\sqrt{n} \qquad (3)$$

Whence from equations (1) and (2) and putting $v = \sigma/\bar{X}$,

$$f_{0\cdot1} = \bar{x}\left[\frac{1 - 1\cdot3v}{1 + 2v/\sqrt{n}}\right]$$

and

$$f_{0\cdot001} = 1\cdot11\bar{x}\left[\frac{1 - 3v}{1 + 2v/\sqrt{n}}\right]$$

which equations can be rewritten as

$$f_{0\cdot1} = k_{0\cdot1}\bar{x} \qquad \ldots \qquad (4)$$

and

$$f_{0\cdot001} = k_{0\cdot001}\bar{x} \qquad \ldots \qquad (5)$$

where $k_{0\cdot1}$ and $k_{0\cdot001}$ represent the factors by which the mean $\bar{x}$ of n test results should be multiplied when estimating the design value. The values of k (the smaller of $k_{0\cdot1}$ and $k_{0\cdot001}$) are given in Table 1 and plotted in Fig. 2 for a range of v and n; in practice $k_{0\cdot1}$ is the smaller when $v < 0\cdot055$, and $k_{0\cdot001}$ when $v > 0\cdot055$.

4.2. There is still the possibility that in any particular case, the value of the mean $\bar{x}$ will lie above the limits implicit in equations (4) and (5). It is instructive then, to consider what might be the worst case, by assuming

$$\bar{x} = \bar{X}(1 + 3v/\sqrt{n}) \qquad \ldots \qquad (6)$$

If, by chance $\bar{x}$ did have the value given by equation (6), and if the design values were estimated from equations (4) and (5), then the actual values would be

$$f_{0\cdot1} = \bar{X}\left[\frac{1 + 3v/\sqrt{n}}{1 + 2v/\sqrt{n}}\right](1 - 1\cdot3v) \qquad \ldots \qquad (7)$$

and

$$f_{0\cdot001} = 1\cdot11\,\bar{X}\left[\frac{1 + 3v/\sqrt{n}}{1 + 2v/\sqrt{n}}\right](1 - 3v) \qquad \ldots \qquad (8)$$

Under these circumstances, the proportions $P_{0\cdot1}$ and $P_{0\cdot001}$ of the estimated design value below which respectively 10 per cent and $0\cdot1$ per cent. of specimens will fall, are given by

$$P_{0\cdot1} = \frac{1 + 2v/\sqrt{n}}{1 + 3v/\sqrt{n}} \qquad \ldots \qquad (9)$$

and

$$P_{0\cdot001} = \left[\frac{1 - 3v}{1 - 1\cdot3v}\right]\left[\frac{1 + 2v/\sqrt{n}}{1 + 3v/\sqrt{n}}\right] \qquad \ldots \qquad (10)$$

when $v < 0\cdot055$, and by

$$P_{0\cdot1} = 0\cdot9\left[\frac{1 - 1\cdot3v}{1 - 3v}\right]\left[\frac{1 + 2v/\sqrt{n}}{1 + 3v/\sqrt{n}}\right] \qquad \ldots \qquad (11)$$

and

$$P_{0\cdot001} = 0\cdot9\left[\frac{1 + 2v/\sqrt{n}}{1 + 3v/\sqrt{n}}\right] \qquad \ldots \qquad (12)$$

when $v > 0\cdot055$.

Values of $P_{0\cdot1}$ and $P_{0\cdot001}$ are given in Tables 2 and 3 respectively, for a range of n and v, and are shown in Fig. 3.

5. *The Number of Tests Required.*—Fig. 3 illustrates the adage ' safety in numbers ', and leads inevitably to the question of the number of tests required to keep the risk within reasonable bounds. The answer must be largely a matter of opinion, but bearing in mind the pessimism of the

assumptions already made, coupled with the probability of the weakest specimens meeting the maximum design loads in service, it seems not unreasonable to choose some high proportion, say 98 per cent, of the original design conditions as the lowest acceptable limit. In general terms this high proportion can be called q, whence from equation (9) or equation (12)

$$\frac{\sqrt{n}}{v} = \frac{3q - 2}{1 - q} \quad \ldots \quad (13)$$

When $q = 0 \cdot 98$ approximately, equation (13) reduces to

$$n = 2500v^2 . \quad \ldots \quad (14)$$

Equation (14) is plotted in Fig. 4, illustrating the number n of test results needed on a material or structure of known coefficient of variation v, so that there is only a negligible chance that more than one in ten, or one in a thousand, of the specimens in service shall be weaker than 98 per cent or 88 per cent respectively of the desired strength.

6. *Test Factors.*—The converse problem is that of 'test factors', and is concerned with the mean values which should be realised in tests on a number n of specimens designed to achieve a specified value. These test factors can be derived directly from equations (4) and (5), being—

$$\begin{aligned} \text{V.T.F.}_{0\cdot 1} &= \bar{x}/f_{0\cdot 1} \\ &= 1/k_{0\cdot 1} \quad \ldots \quad (15) \end{aligned}$$

when $v < 0 \cdot 055$, and similarly

$$\text{V.T.F.}_{0\cdot 001} = 1/k_{0\cdot 001}$$

when $v > 0 \cdot 055$.

The values of the test factor are given in Table 4 for ranges of v and n, and are shown in Fig. 5.

7. *Discussion.*—The above arguments are based on the assumption of a 'normal' (*i.e.*, a Gaussian) frequency distribution, and their application would therefore appear to be limited. In the case of materials, however, the inspection procedure used to ensure an acceptable product, is one which aims at the rejection of all material having properties below certain specified minimum values; the net result is that the strength frequency distribution curve for example, tends to have a 'tail' at the high strength end; any such 'tail' in the material strength properties will tend to produce a similar 'tail' in the strength distribution curve for structures. Cox concludes[3] that if there is a positive skewness (*i.e.*, tail to the right or high side) estimates based on the normal Gaussian distribution will be severe; in such cases the suggestions put forward in this note would give a design value more conservative than indicated.

The test factors used at present[6] have been based on the minimum value realised in tests on a given number of specimens, whereas the test factors of section 6 are based on the mean value of a given number of test results; obviously with a specified acceptable proportion of weak specimens, either method will give the same factors to be applied to one test result. With more than one test specimen the mean value is the best parameter to use, because with random choice of specimens, the mean value is the parameter giving least scatter.

8. *Conclusions.*—On the basis of the proportion of weak specimens acceptable in service and a knowledge of the variability of the material or structure, Part I deduces :

(*a*) The smaller of the values given by the following formulae should be permissible in design

$$f_{0\cdot 1} = \bar{x}\left[\frac{1 - 1\cdot 3v}{1 + 2v/\sqrt{n}}\right]$$

$$f_{0\cdot 001} = 1\cdot 11\,\bar{x}\left[\frac{1 - 3v}{1 + 2v/\sqrt{n}}\right]$$

(*b*) The number of test results required, on specimens chosen at random, so that estimates of the permissible design values can be regarded as sufficiently accurate, is given by the formula

$$n = 2500v^2$$

(*c*) The higher of the test factors given by the following formulae should be applied to the mean result of tests on n similar components designed to meet a specified requirement

$$\text{V.T.F.}_{\cdot 0\cdot 1} = \frac{1 + 2v/\sqrt{n}}{1 - 1\cdot 3v}$$

$$\text{V.T.F.}_{\cdot 0\cdot 001} = 0\cdot 9\left[\frac{1 + 2v/\sqrt{n}}{1 - 3v}\right].$$

In the above formulae

n is the number of test results on similar specimens.

v ,, ,, true coefficient of variation of the specimens.

$\bar{x}$,, ,, mean of n test results.

PART II

1. *Introduction.*—In Part I methods were developed for obtaining from test results design values which satisfy given design conditions. The practical application of these methods is seldom as straightforward as might be expected. Part II, therefore, examines the implications of the accepted design conditions when applied to material with a normal (*i.e.*, a Gaussian) distribution of strength properties. The resulting tables and graphs are presented in one document in the belief that ease and quickness of reference will be of use when the designer has to decide whether the realised strength of a material or component is acceptable.

The first point of interest is the strength distribution of specimens which results from the application of the design conditions. With this knowledge we can assess the effects of slightly different design conditions.

Next we consider the Variability Test Factor, here defined as the ratio between the mean of a number of test results and the specified design strength, and associated with the variability of the material or structure. Probably the best known examples of these factors are the 'casting factors' of Chapter 406, A.P. 970. When the variability test factor is less than that called for we must be able to assess whether further tests are needed, or whether the increased risk is acceptable. In the latter case the increased risk can be looked at in two ways; first an increase in the probability that there will be more specimens weaker than the design condition, and second the proportion of specimens which might be expected to be weaker than the design condition.

Finally the note deals with the problem of the number of tests required to ensure a given accuracy. Clearly more tests are required on a material like glass than on a wrought metal extrusion, since the strength properties of glass are much more variable, but we still need some guide as to the minimum number of tests required in each case. We approach this problem by considering the worst that may happen on rare occasions when by chance the specimens chosen for test are very strong ones indeed.

2. *The Design Strength Distribution.*—If the design conditions of equations (1) and (2) (Part I) are *just* satisfied, then the strength distribution for any given coefficient of variation can be determined in terms of the number of weak specimens which might be expected to fall below p times the design value. We are interested in values of p between $0 \cdot 9$ and $1 \cdot 0$ since this denotes weakness, but equally we are interested in values of p greater than $1 \cdot 0$ since that denotes inefficient use of the material.

The following formulae derived in Appendix I can be used to estimate the proportion of weak specimens falling below p times the design value f.

$$a = \frac{p - 1}{v} - 1 \cdot 3p \qquad \ldots \quad (17)$$

when $v < 0 \cdot 055$, and

$$a = \frac{1 \cdot 11p - 1}{v} - 3 \cdot 33p \qquad \ldots \quad (18)$$

when $v > 0 \cdot 055$.

In these equations a represents the number of standard deviationswhich must be added to the mean strength $\bar{X}$ to give a strength value pf; the area under the distribution curve below the value pf represents the proportion of specimens with strength less than pf and can be found from the appropriate statistical tables[5] once the value of a has been found. Appendix I also shows that for $0 \cdot 9 < p < 1 \cdot 0$ the gloomiest strength picture is obtained when $v = 0 \cdot 055$, *i.e.*, when the material has a coefficient of variation such that meets exactly both the limiting conditions of Part I, section 3.

For various values of the coefficient of variation v, Table 5 and Fig. 6 show the proportion of weak specimens expected to fall below p times the design value.

3. *Variability Test Factors.*—3.1. When, in a particular series of tests, factors other than those specified by Part I are realised, they can be interpreted in two ways, either by estimation of the probability that the material will not meet the design conditions of Part I, or by estimation of the proportion of material which will fall below the specified design strength. For example consider a material whose coefficient of variation is $0 \cdot 05$. If in a test on one specimen the strength is shown to be 105 per cent of the specified design strength, then the realised variability test factor is $1 \cdot 05$. We see from Fig. 5 that the minimum factor required is $1 \cdot 18$, and it is therefore clear that the chance of there being more than 10 per cent material below the specified design strength is greater than $0 \cdot 025$. If, on the other hand, we were prepared to accept more than 10 per cent material below the specified design strength, then the realised variability test factor of $1 \cdot 05$ might be adequate. This sort of thing presents a very real practical problem, and we proceed to examine the two possible interpretations in the next sections.

3.2. The effect of varying the probability that the estimated design value will be too high can be examined by rewriting equations (15) and (16)

$$\text{V.T.F.} = \frac{1 + bv/\sqrt{n}}{1 - 1 \cdot 3v} \quad \ldots \quad (19)$$

when $v < 0 \cdot 055$ and

$$\text{V.T.F.} = \frac{1 + bv/\sqrt{n}}{1 \cdot 11(1 - 3v)} \quad \ldots \quad (20)$$

when $v > 0 \cdot 055$

where b is a constant depending on the probability chosen, e.g., $b = 2$ represents $0 \cdot 025$. Table 6 gives values of the variability test factor for combinations of v, n, and probability; Fig. 7 shows the same information for $v = 0 \cdot 20$, $0 \cdot 15$, $0 \cdot 10$ and $0 \cdot 03$.

3.3. Alternatively, if we keep the acceptable risk at $0 \cdot 025$, we can estimate the proportion of specimens which falls below the design value in all but 1 in 40 of the cases considered. This is given by the value of c in

$$\text{V.T.F.} = \frac{1 + 2v/\sqrt{n}}{1 - cv} . \quad \ldots \quad (21)$$

The proportion is equal to the integral of the normal distribution curve below $\bar{X}(1 - cv)$, so that if $c = 1 \cdot 3$ we can expect 10 per cent of the specimens to have a strength less than the design value. The variability test factors for a range of v, n, and the proportion of results below the design value are listed in Table 7. Figs. 8a, 8b and 8c show the same information graphically.

4. *The Number of Tests Required.*—Equations (4) and (5) which represent a practical interpretation of the design conditions of Part I, infer that in one case in 40 the design conditions will not be met. There is thus the possibility that in any particular case, the value of the mean ($\bar{x}$) will lie above the limit implicit in equations (4) and (5). We can consider what might be the worst case by assuming

$$\bar{x} = \bar{X}(1 + 3v/\sqrt{n}) . \quad \ldots \quad (6 \text{ bis})$$

With this assumption, as shown in Part I, it is possible to develop an equation which gives the number of test results required to ensure that the true design values are not less than q times the deduced values. This equation is

$$\frac{\sqrt{n}}{v} = \frac{3q - 2}{1 - q} . \quad \ldots \quad (13 \text{ bis})$$

The answer to the question of the number of tests required to keep risk within reasonable bounds must be largely a matter of opinion, but bearing in mind the pessimism of the assumptions already made, it seems not unreasonable to choose proportions of q not less than $0 \cdot 95$. Fig. 9 shows the number of tests needed to make certain that the true design values are not less than $0 \cdot 99$, $0 \cdot 98$, $0 \cdot 97$ and $0 \cdot 95$ times the values deduced from the test results.

It is also possible to deduce the number of tests required by setting a limit to the excess weight we can tolerate when the structure is too strong. In practice we do not know whether a particular sample is strong or weak, therefore, in the application of equations (4) and (5), which assume that the sample is relatively strong, we can expect that on the average we will be too severe. The degree of severity in the case of a sample of average strength is given by

$$R = 1 + 2v/\sqrt{n}$$

whence

$$\frac{\sqrt{n}}{v} = \frac{2}{R-1} \quad \cdots \quad (22)$$

where R is the ratio between the true design strength and the design strength deduced from equations (4) and (5).

From equations (13) and (22) we obtain the following relations between R and q.

q	R
0·99	1·02
0·98	1·04
0·97	1·07
0·95	1·12

It is clear then that the smaller the number of results the more we may encroach on safety and the greater, on the average, may be the excess weight we must tolerate.

5. *Discussion.*—The mere presentation of curves and tables gives insufficient emphasis to the meaning of the design conditions of Part I. The coefficient of variation, however, is a measure of the strength distribution of a particular type of structure or material, and any discussion is possibly more dramatic if illustrated in terms of a type of structure or material. For this purpose then let us choose :

(*a*) A built-up light-alloy structure[7] with a coefficient of variation of 0·03; this can be regarded as typical of a well-designed metal wing structure

(*b*) A wooden structure[8] with a coefficient of variation of 0·07; this represents a typical wooden wing structure

(*c*) Light-alloy castings with a coefficient of variation 0·10

(*d*) Glass in sheet form with a coefficient of variation of 0·20.

The coefficients of variation associated with the structures and materials mentioned above are of the right order of magnitude ; their main purpose here is to serve for illustration, and they should not be regarded as precise.

5.1. *Design Strength Distribution.*—The curves of Fig. 6 show that:

(*a*) The design of the metal structure ($v = 0·03$) is governed by the condition that not more than 10 per cent of specimens shall fall below the design value ; 0·1 per cent of specimens fall below 0·95 times design value; the mean strength of all specimens is 1·04 times design value; 10 per cent specimens are stronger than 1·08 times design value

(*b*) The wooden structure ($v = 0·07$) is governed by the condition that not more than 0·1 per cent specimens are weaker than 0·9 times design value ; 4 per cent fall below the design value; the mean strength is 1·14 times the design value; 10 per cent specimens are stronger than 1·24 times design value (*c*) $1 \cdot 3$ per cent light-alloy castings ($v = 0 \cdot 10$) are weaker than the design value; the mean strength is $1 \cdot 29$ times design value; 10 per cent are stronger than $1 \cdot 45$ times design value

(*d*) Of glass panels ($v = 0 \cdot 20$) owing to the very variable nature of the material, only some $0 \cdot 3$ per cent are weaker than the design value, whereas the mean strength of the panels is $2 \cdot 25$ times the design value; 10 per cent panels are stronger than $2 \cdot 84$ times design value.

The design conditions of Part I are determined by safety considerations which have nothing to do with the inherent properties of the material or structure. These safety criteria must be maintained but in the most economic way possible; this is obviously done by using the least variable material which meets all design requirements whether they be strength, stiffness, visibility, ease and quickness of production, etc.

5.2. *Variability Test Factors*.—The variability test factors given by Fig. 5 ensure that the probability is $0 \cdot 025$ that a material that does not meet the design conditions of Part I will be accepted; Fig. 7 illustrates how the factors are affected by variation of the probability of exceeding the design conditions; Fig. 8 demonstrates the effect of variation in the design conditions when the probability is $0 \cdot 025$ that these will be exceeded. These figures should be considered together, and the example of one test result will be taken, thus

(i) For a metal structure ($v = 0 \cdot 03$) Fig. 5 indicates a desirable variability test factor of $1 \cdot 10$. If the structure realises only $1 \cdot 05$, *i.e.*, $0 \cdot 95 \times 1 \cdot 10$, the probability that there will be more than $0 \cdot 1$ specimens below the specified design value rises to about $0 \cdot 4$, *i.e.*, we should expect that the proportion of specimens weaker than the specified value will be greater than $0 \cdot 1$ in 4 cases out of 10. The probability is $0 \cdot 025$ that there will be more than $0 \cdot 64$ specimens below the specified value, so that in one case in 40 we can expect that the proportion of specimens below the design value will be greater than $0 \cdot 64$. These concepts seem startling from the safety point of view until we consider that this same variability test factor of $1 \cdot 05$ means that the probability is $0 \cdot 025$ that more than $0 \cdot 001$ specimens will be below $0 \cdot 9$ times the design value.

(ii) A light-alloy casting ($v = 0 \cdot 10$) according to Fig. 5 should realise a factor of $1 \cdot 54$. If a factor of $1 \cdot 46$ is realised ($0 \cdot 95 \times 1 \cdot 54$), the probability rises to $0 \cdot 09$ that an accepted casting will not meet the design conditions, or the probability is $0 \cdot 025$ that there will be more than $0 \cdot 04$ specimens below the design value and more than $0 \cdot 001$ specimens below $0 \cdot 855$ times the design value.

(iii) A glass panel ($v = 0 \cdot 20$) should realise a variability test factor of $3 \cdot 15$. If a factor of $3 \cdot 0$ is realised ($0 \cdot 95 \times 3 \cdot 15$) the probability is $0 \cdot 05$ that the design condition will be exceeded; or the probability is $0 \cdot 025$ that there will be more than $0 \cdot 004$ specimens below the design value and more than $0 \cdot 001$ specimens below $0 \cdot 855$ times the design value.

5.3. *The Number of Tests Required*.—It should be emphasised that the curves of Fig. 9 represent a lower limit. In rare cases application of the factors of Fig. 2 to the mean of n test results may give optimistic or unconservative design values. Fig. 9 shows the worst that may happen for:

(*a*) A metal structure ($v = 0 \cdot 03$). With one test result the true design value should be not less than 97 per cent and, on the average, not more than 107 per cent of the deduced design value.

(*b*) A wooden structure ($v = 0 \cdot 07$). The true design value should be not less than some 94 per cent. and, on the average, not more than 115 per cent of the value deduced from one test result.

(*c*) Light-alloy castings ($v = 0 \cdot 10$). Three tests are needed to get the true design value within 95 per cent and, on the average, not more than 112 per cent of the deduced design value.

(*d*) Glass panels ($v = 0 \cdot 20$). Twelve tests are needed to make sure that the true design value is not less than $0 \cdot 95$ and, on the average, not more than $1 \cdot 12$ times the deduced design value.

When planning a test programme to obtain design data it is suggested that the number of tests should be not less than that given by the curve $q = 0 \cdot 98$ in Fig. 9. For approval of a particular component, that is something which will not have such wide application as basic design data, then perhaps less test specimens are needed, but it is suggested that the number should be not less than that given by $q = 0 \cdot 95$.

LIST OF SYMBOLS

f	Permissible design value
$f_{0 \cdot 1}$	Design value fixed by the condition that not more than $0 \cdot 1$ specimens shall fall below it
$f_{0 \cdot 001}$	Design value fixed by the condition that not more than $0 \cdot 001$ specimens shall fall below $0 \cdot 9 f_{0 \cdot 001}$
k	The factor by which the mean $\bar{x}$ of n test results should be multiplied to give the design value f
$k_{0 \cdot 1}$	The factor by which the mean $\bar{x}$ of n test results should be multiplied to give the design value $f_{0 \cdot 1}$
$k_{0 \cdot 001}$	The factor by which the mean $\bar{x}$ of n test results should be multiplied to give the design value $f_{0 \cdot 001}$
n	Number of test results from which design values are deduced
$\bar{x}$	Mean value of a sample of n test results
$\bar{X}$	True mean value
s	Standard deviation of sample of n test results
σ	True standard deviation
$v = \frac{\sigma}{\bar{X}}$	True coefficient of variation
$P_{0 \cdot 1}$	Proportion of estimated design value below which $0 \cdot 1$ specimens will fall
$P_{0 \cdot 001}$	Proportion of estimated design value below which $0 \cdot 001$ specimens will fall
V.T.F.	Variability test factor
$\text{V.T.F.}_{0 \cdot 1}$	Test factor appropriate to the condition of not more than $0 \cdot 1$ specimens below the design value
$\text{V.T.F.}_{0 \cdot 001}$	Test factor appropriate to the condition of not more than $0 \cdot 001$ specimens below 90 per cent design value

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APPENDIX I

The Critical Value of the Coefficient of Variation

The permissible design strength is given by

$$f_{0\cdot1} = \bar{X}(1 - 1\cdot3v) \quad \ldots \quad (1)$$

when $v < 0\cdot055$, and by

$$f_{0\cdot001} = 1\cdot11\bar{X}(1 - 3v) \quad \ldots \quad (2)$$

when $v > 0\cdot055$.

The proportion which falls below p times the design strength can be determined from

$$pf_{0\cdot1} = \bar{X}(1 + av)$$

which by substitution from equation (1) reduces to

$$a = \frac{p - 1}{v} - 3\cdot3p \quad \ldots \quad (3)$$

when $v < 0\cdot055$.

Similarly from equation (2)

$$a = \frac{1\cdot11p - 1}{v} - 3\cdot3p \quad \ldots \quad (4)$$

when $v > 0\cdot055$.

In equation (3) and (4) a represents the number of standard deviations which must be added to the mean strength $\bar{X}$ to give a strength value which is $pf_{0\cdot1}$ or $pf_{0\cdot001}$, as the case may be. The proportion of material whose strength is below p times the permissible design strength can then easily be determined from the appropriate statistical tables.

When $0\cdot9 < p < 1\cdot0$, examination of equation (3) shows that the first term is negative and therefore the greater the value of v the less the numerical value of a. Therefore since the specified design conditions must also be satisfied a will have its least numerical value when $v = 0\cdot055$. Similarly in the case of equation (4), the first term is positive and a has its least numerical value when $v = 0\cdot055$, *i.e.*, the smallest value of v which will satisfy the specified design condition. Thus the greatest proportion of weak specimens falling below p times the design strength is obtained when $v = 0\cdot055$.

TABLE 1

Values of k

n	v							
	0·02	0·04	0·055	0·06	0·08	0·10	0·15	0·20
1	0·937	0·878	0·836	0·813	0·727	0·648	0·4696	0·317
4	0·955	0·912	0·880	0·858	0·781	0·707	0·531	0·370
16	0·964	0·929	0·903	0·884	0·811	0·741	0·568	0·404
64	0·969	0·939	0·915	0·897	0·827	0·758	0·588	0·423

TABLE 2

Estimated minimum likely proportion ($P_{0\cdot1}$) of the design value below which not more than 10 per cent of specimens should fall, when the design value is based on Equations (4) and (5)

n	v					
	0·02	0·04	0·055	0·06	0·08	1·10
1	0·981	0·964	0·952	0·961	0·993	1·033
4	0·990	0·981	0·975	0·984	1·023	1·070
16	0·995	0·990	0·987	0·997	1·041	1·092
64	0·998	0·995	0·993	1·005	1·051	1·105

TABLE 3

Estimated minimum likely proportion ($P_{0\cdot001}$) of the design value below which not more than 0·1 per cent of specimens should fall, when the design value is based on Equations (4) and (5)

n	v					
	0·02	0·04	0·055	0·06	0·08	0·10
1	0·947	0·895	0·856	0·854	0·842	0·831
4	0·956	0·911	0·876	0·875	0·868	0·861
16	0·960	0·919	0·887	0·887	0·883	0·879
64	0·963	0·924	0·893	0·893	0·891	0·889

TABLE 4

Variability Test Factors Required

n	v						
	0·02	0·04	0·055	0·08	0·10	0·15	0·20
1	1·07	1·14	1·20	1·37	1·54	2·13	3·15
4	1·05	1·10	1·14	1·28	1·41	1·88	2·70
16	1·04	1·08	1·11	1·23	1·35	1·76	2·48
64	1·03	1·07	1·09	1·21	1·32	1·70	2·36

TABLE 5

Proportion of Specimens Below p Times the Design Value

	v						
p	0·03	0·055	0·07	0·10	0·12	0·15	0·20
0·900		0·00138	0·00124	0·00127	0·0013	0·0013	0·0013
0·925		0·00517	0·00353	0·00251	0·0022	0·0019	0·0014
0·950	0·00187	0·0159	0·00843	0·00442	0·0034	0·0026	0·00195
0·975	0·0179	0·0423	0·0185	0·00807	0·0052	0·0035	0·00234
1·000	0·097	0·0967	0·0396	0·0128	0·0079	0·0047	0·00277
1·050	0·618	0·288	0·118	0·0299	0·0156	0·0077	0·00361
1·100	0·9712	0·649	0·307	0·0728	0·0342	0·0142	0·00529
1·150		0·891	0·548	0·143	0·0640	0·0236	0·0069
1·200		0·981	0·773	0·247	0·1095	0·0374	0·00955

TABLE 6

The Effect of Variation in Probability on Variability Test Factors (Design Conditions of Part I)

v	n	Probability					
		0·001	0·025	0·070	0·160	0·300	0·500
0·030	1	1·1342	1·1103	1·0874	1·0718	1·0561	1·0406
	4	1·0874	1·0718	1·0640	1·0561	1·0484	1·0406
	16	1·0640	1·0561	1·0523	1·0484	1·0445	1·0406
0·100	1	1·6731	1·5444	1·4801	1·4157	1·3514	1·2870
	4	1·4801	1·4157	1·3835	1·3514	1·3192	1·2870
	16	1·3835	1·3514	1·3353	1·3192	1·3031	1·2870
0·150	1	2·3727	2·1273	2·0045	1·8818	1·7591	1·6364
	4	2·0045	1·8818	1·8205	1·7591	1·6977	1·6364
	16	1·8205	1·7591	1·7284	1·6977	1·6670	1·6364
0·200	1	3·6036	3·1532	2·9279	2·7027	2·4775	2·2522
	4	2·9279	2·7027	2·5901	2·4775	2·3649	2·2522
	16	2·5901	2·4775	2·4212	2·3649	2·3086	2·2522

TABLE 7

Variability Test Factors as Affected by the Proportion of Results Acceptable Below the Design Value

v	n	Proportion of results below design value					
		0·500	0·300	0·160	0·070	0·025	0·001
0·030	1	1·0600	1·0761	1·0927	1·1099	1·1276	1·1648
	4	1·0300	1·0457	1·0618	1·0785	1·0957	1·1319
	16	1·0150	1·0305	1·0464	1·0628	1·0797	1·1154
0·055	1	1·1100	1·1413	1·1746	1·2098	1·2471	1·3293
	4	1·0550	1·0848	1·1164	1·1499	1·1854	1·2635
	16	1·0275	1·0566	1·0873	1·1198	1·1544	1·2305
0·100	1	1·2000	1·2631	1·3333	1·4117	1·5000	1·7143
	4	1·1000	1·1579	1·2222	1·2941	1·3750	1·5714
	16	1·0500	1·1053	1·1667	1·2353	1·3125	1·5000
0·150	1	1·3000	1·4054	1·5294	1·6774	1·8571	2·3636
	4	1·1500	1·2432	1·3529	1·4839	1·6429	2·0909
	16	1·0750	1·1622	1·2647	1·3871	1·5357	1·9545
0·200	1	1·4000	1·5556	1·7500	2·0000	2·3333	3·5000
	4	1·2000	1·3333	1·5000	1·7142	2·0000	3·0000
	16	1·1000	1·2222	1·3750	1·5714	1·8333	2·7500

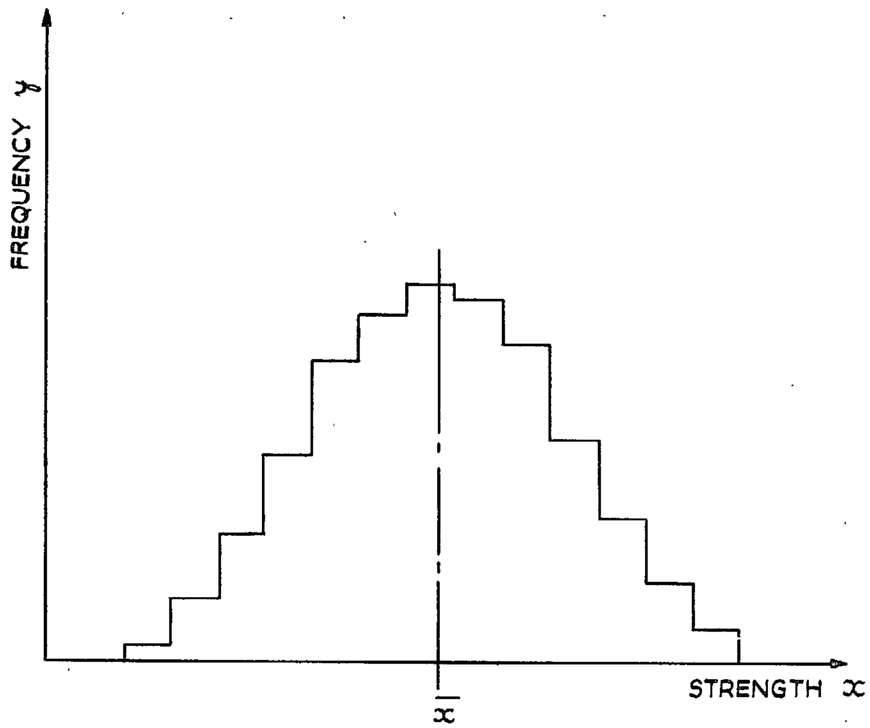

FIG. 1. Strength distribution diagram. $\int_{x_1}^{x_2} y\,dx$ gives the proportion of specimen whose strength lies between x_1 and x_2.

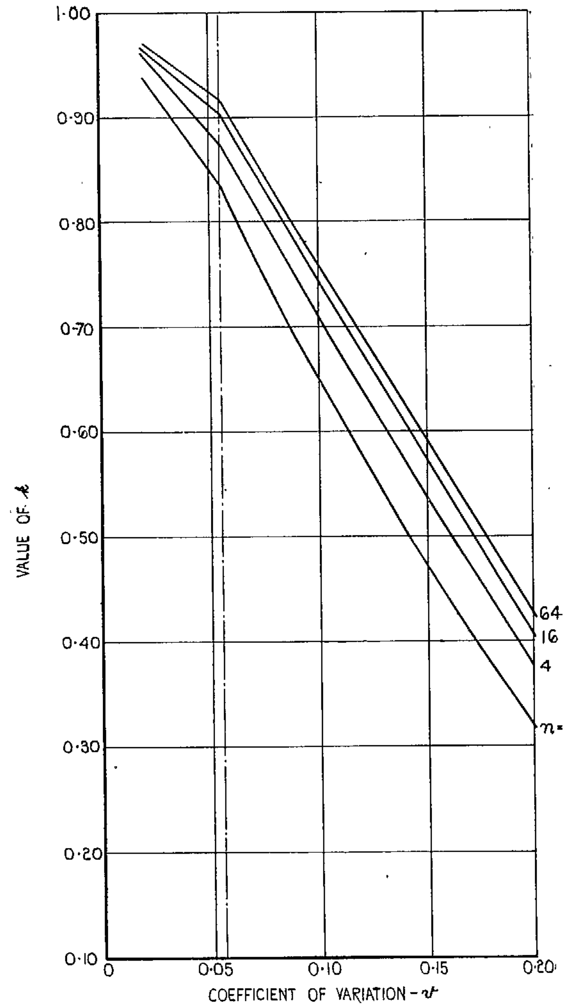

FIG. 2. Values of k.

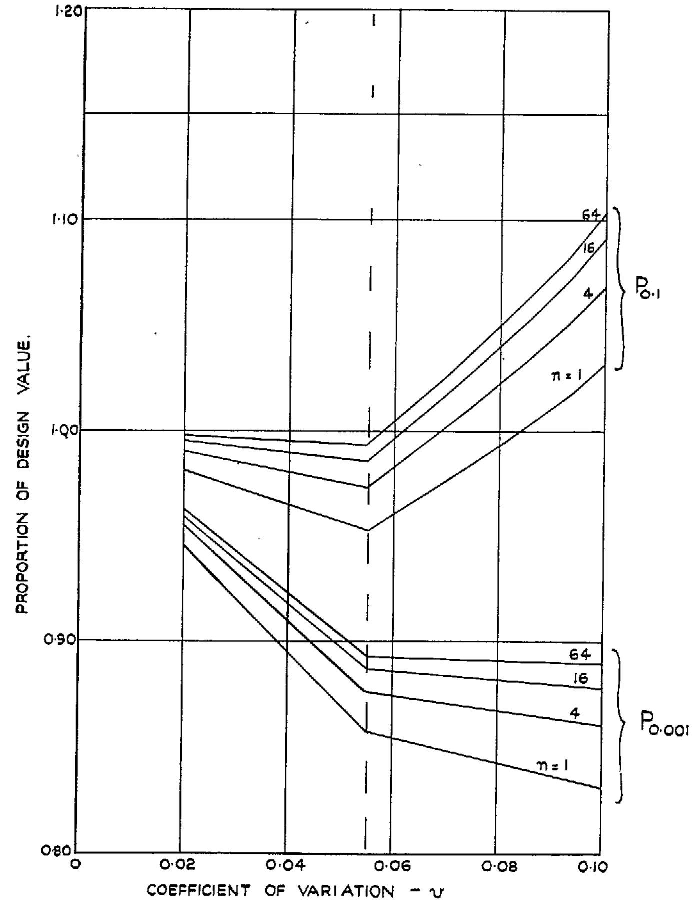

FIG. 3. Estimated minimum likely proportions of the design value below which not more than 0·1 and 0·001 specimens should fall, when the design value is based on equations (4) and (5).

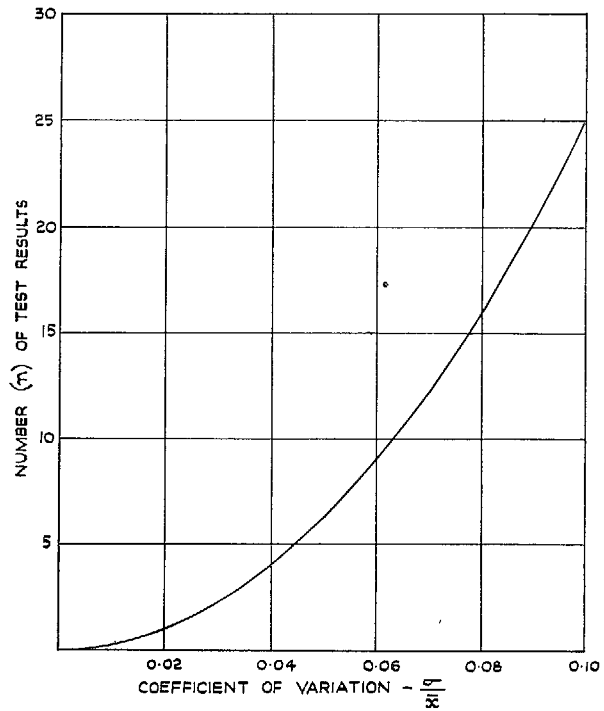

FIG. 4. The number of test results required to ensure not more than 10 per cent specimens are weaker than 98 per cent and not more than 0·1 per cent are weaker than 88 per cent of the deduced design value.

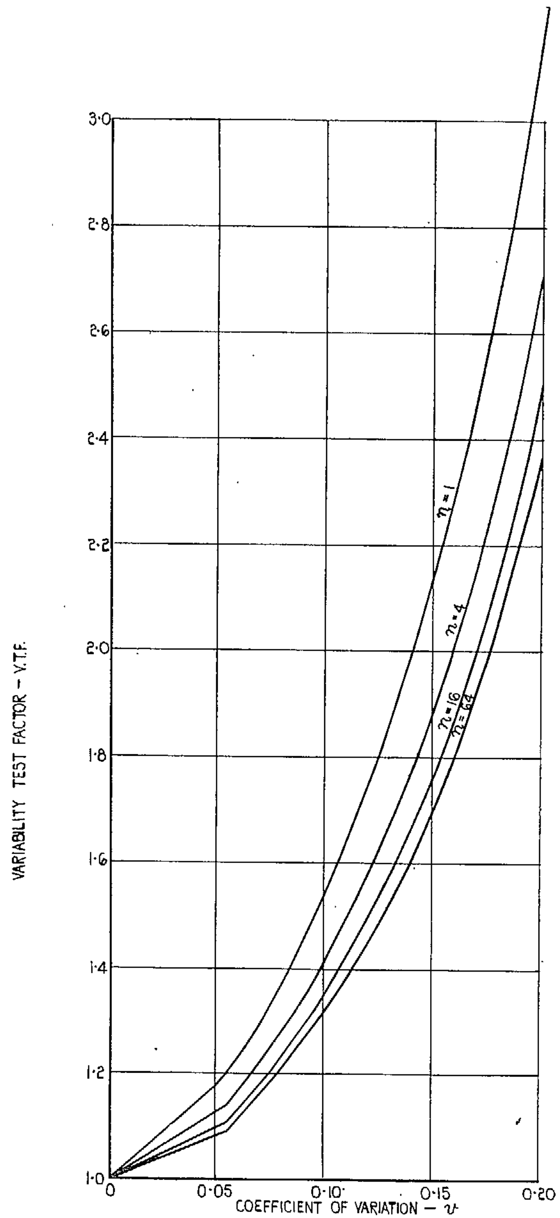

FIG. 5. Variability test factors required on the mean $\bar{x}$ of n test results.

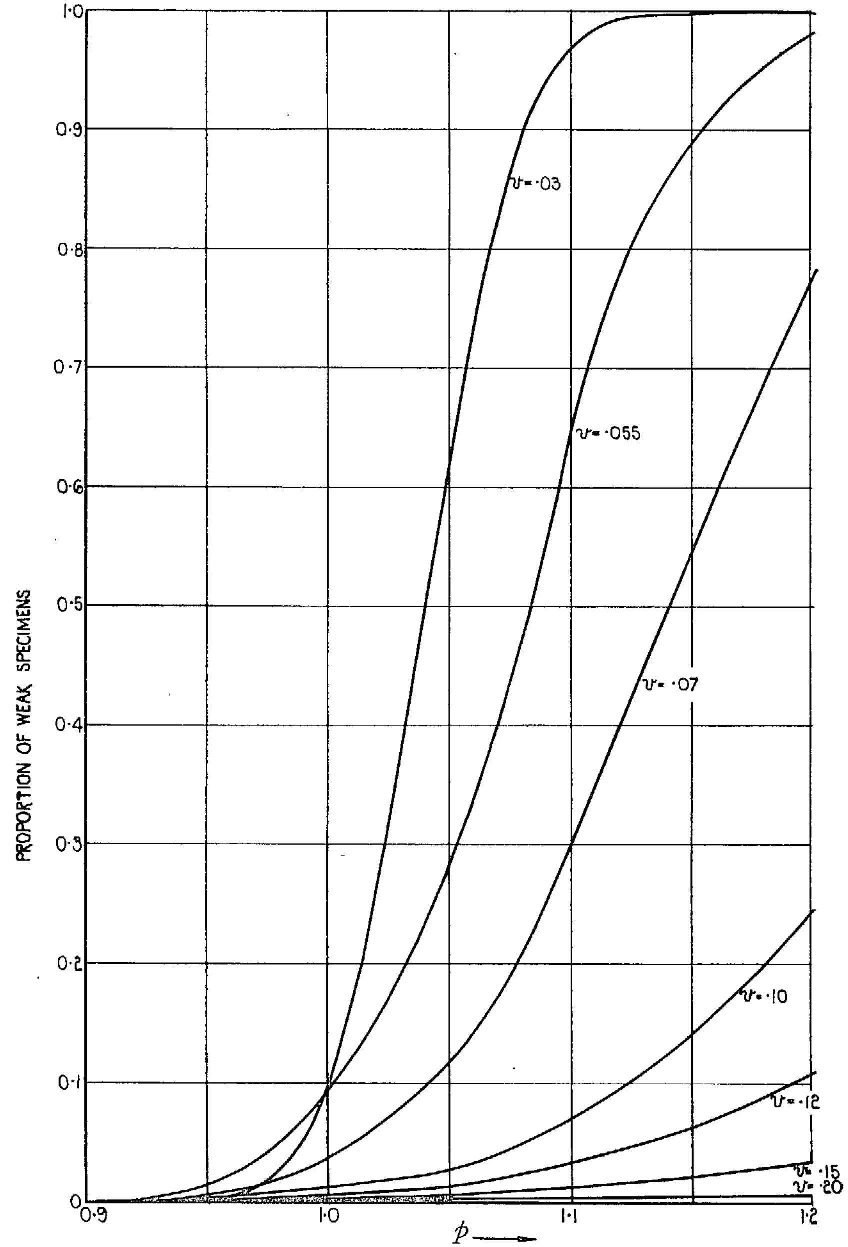

FIG. 6. Proportion of specimens below p times the design value.

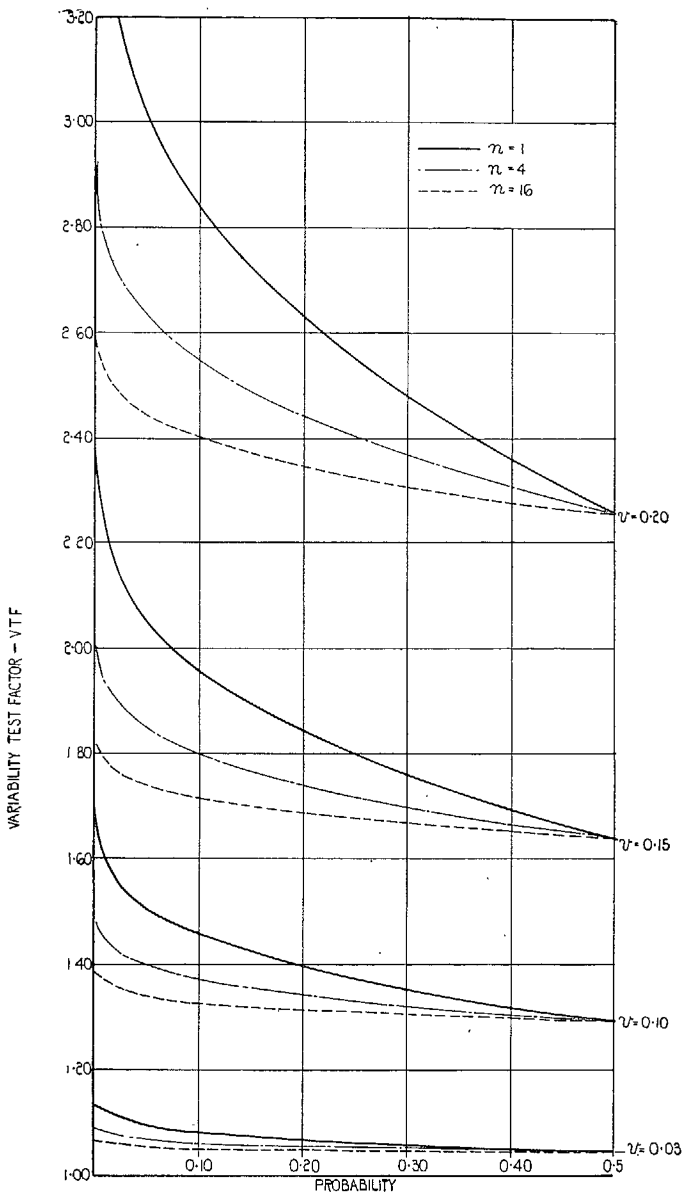

FIG. 7. Variability test factors for a range of probabilities that the design conditions will be exceeded.

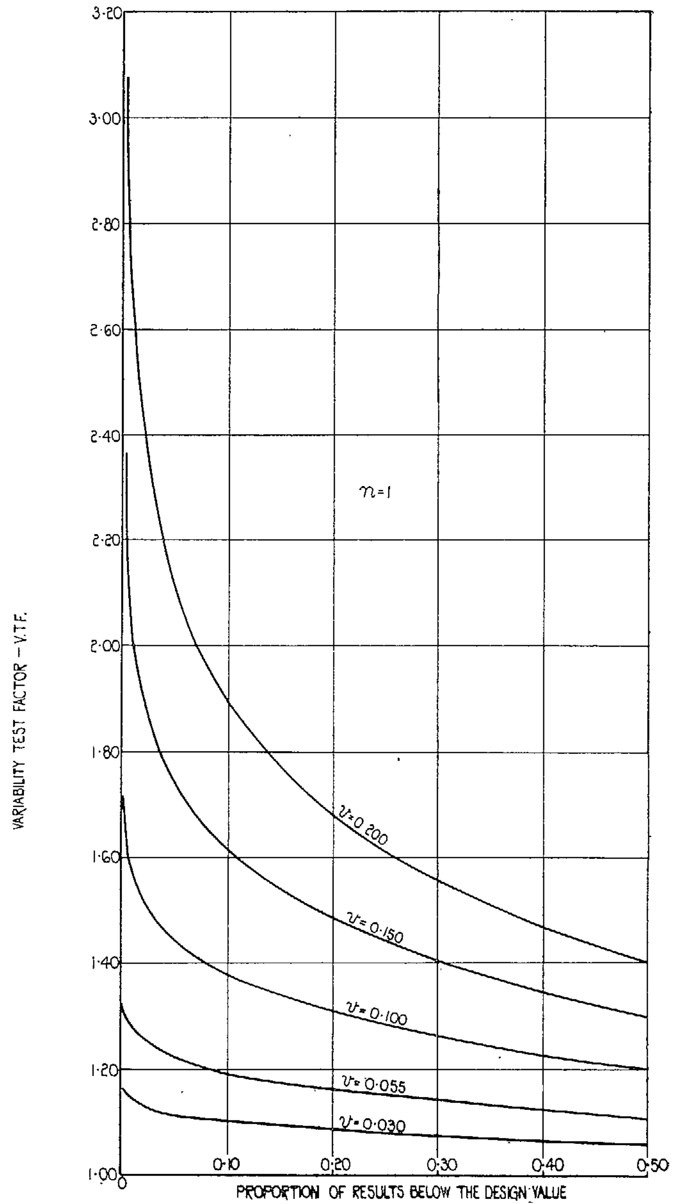

FIG. 8a. Variability test factors as affected by the proportion of results acceptable below the design value.

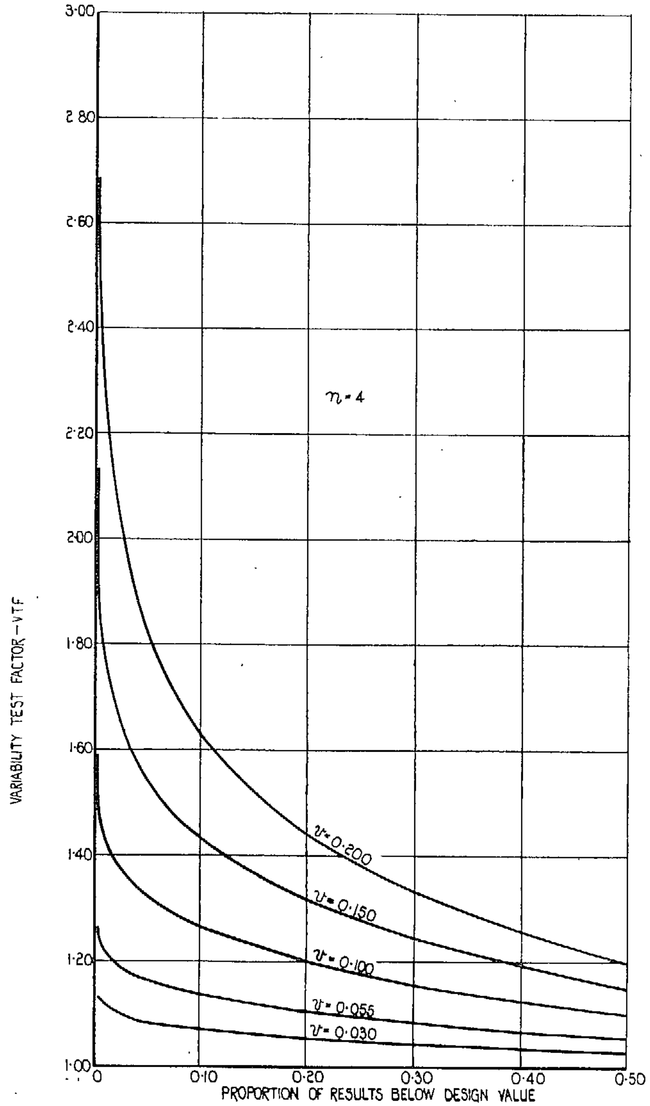

FIG. 8b. Variability test factors as affected by the proportion of results acceptable below the design value.

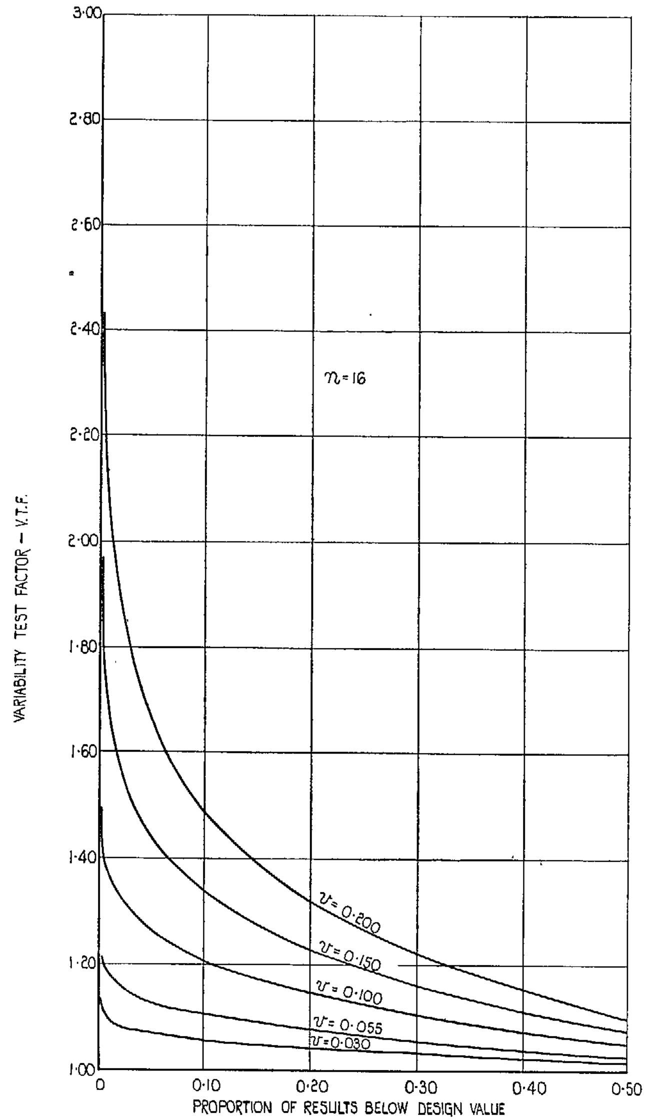

FIG. 8c. Variability test factors as affected by the proportion of results acceptable below the design value.

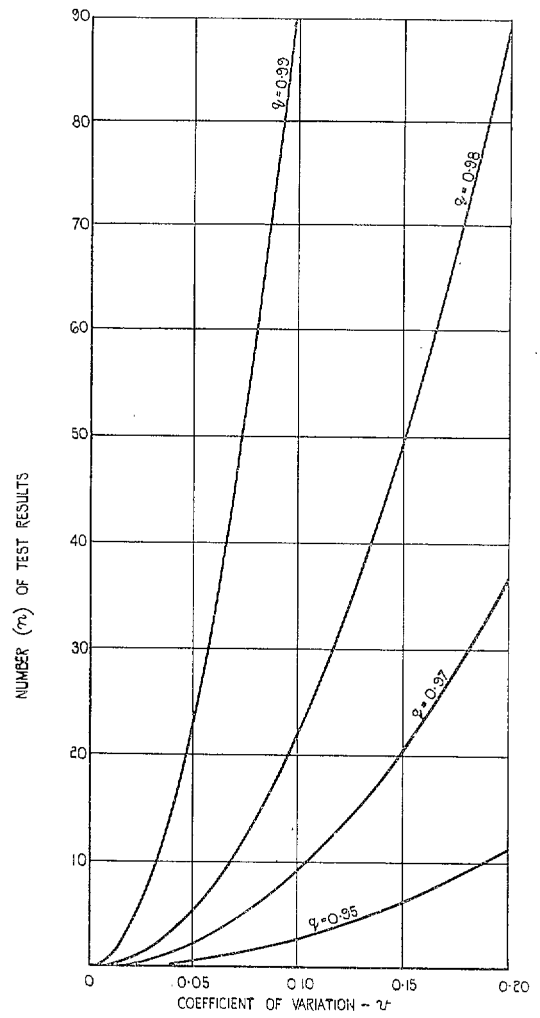

FIG. 9. The number of test results required to ensure that the true design values are not less than q times the deduced design value.

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