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Summary.—(*a*) *Purpose and Range of Investigation.*—To present the general theory of diffusion of antisymmetrical concentrated end loads and edge loads* into parallel stiffened panels, including the theory of bending of a parallel stiffened panel under arbitrary transverse loads. By combining the results of this paper with the results on diffusion of symmetrical loads given in R. & M. 1969[5] and R. & M. 2038[6] or in Appendix I to this paper it is possible to analyse the diffusion in a parallel panel under any arbitrary load or edge stress distribution.

The methods developed in this paper permit a simplification and slight generalisation of the results obtained in R. & M. 1969[5] and 2038[6] for the symmetrical diffusion case in a parallel panel. The relevant formulae are given in Appendix I to this report.

An alternative approach to the diffusion problem in parallel panels with given boom areas is presented in Appendix II.

(*b*) *Conclusions.*—In general diffusion in parallel panels is determined by three parameters: the diffusion constant μl as defined by Cox (R. & M. 1860[1]), the ratio α of total area of edge members to total area of stringers plus effective sheet, and the ratio β of total area of stringers plus effective sheet to the product of length of panel and sheet thickness. In the particular case of a parallel panel with a *given* distribution of edge stress the direct stresses in the panel depend only on the parameter μl, and the shear stresses on μl and β.

It is shown that the effect of transverse loads on the direct stresses in a parallel panel is equivalent to that of antisymmetrical edge loads producing the same bending moment at each section. The shear stress distributions differ by a constant value across each section. This difference is the shear stress produced by the shear force of the transverse load system assumed uniformly distributed over each cross-section.

In all loading cases as μ increases the stress distribution in the panel approaches that indicated by the ordinary engineer's theory.

PART I

Introduction

1. *Nature of Problem.*—In R. & M. 1969[5] and 2038[6] the stress distribution in a stiffened panel was investigated: both the dimensions of the panel, including its taper if any, and the system of stresses or loads applied along its edges were assumed to be symmetrical about an axis parallel to the length of the panel. Such symmetrical loading might be realised in the top or bottom

* The term edge loads is used to describe loads which are applied to the edge members of the parallel panel and which act parallel to its axis of symmetry.

panel of a tapered or parallel box of rectangular cross-section bent about an axis parallel to the two panels. Thus in a wing this loading condition is approached in the box formed by the two spars and the wing cover, when the wing is bent by lifting forces about its chord.

Similarly the case of antisymmetrical edge-stress distribution which is the subject of the present report may be exemplified in the top or bottom panels of a parallel box under transverse (drag) loads parallel to these panels or under torsion. In the latter case if the end of the box is prevented from warping, direct stresses are induced in the panels, and the loading system applied to each panel can be represented by a combination of transverse loads in the plane of the panel and antisymmetrical edge loads. The effect of antisymmetrical edge loads is investigated in Part II of this report.

By combining the results of symmetrical and antisymmetrical edge loads and edge-stress distributions it is possible to analyse the general diffusion case in a parallel panel with any arbitrary edge load and edge-stress distribution.

It may be remarked that the effect of taper could easily be included by the method used for symmetrical edge-stress distributions in R. & M. 1969[5]; but it appeared preferable to restrict the present investigation to parallel panels since by so doing the main theme could be more fully expanded.

2. *Basic Assumptions.*—The problems of diffusion and bending analysed in this paper are based on the following assumptions (Cox, R. & M. 1860[1])

(*a*) that the panel has a finite number of stringers,

(*b*) that the effective sheet area is concentrated at the lines of attachment of the stringers,

(*c*) that the stringers are held apart by a closely spaced system of members, which are rigid against compression or extension but which offer no resistance to bending in the plane of the panel.

The method of analysis based on the above assumption is called the ' finite-stringer ' method. Applications of this method to symmetrical loading cases can be found in Cox, R. & M. 1860[1], Williams and others, R. & M. 2098[2], R. & M. 1969[5] and R. & M. 2038[6].

Because of assumptions (*b*) and (*c*) the shear stress at any cross-section is constant between two consecutive stringers. Furthermore assumption (*c*) implies that the deflection v of the panel is uniform across the width of each cross-section. Thus the shear stress is then given by,

$$q_r = G\{(u_{r-1} - u_r)/b + dv/dx\} \quad \ldots \quad (1)$$

where G is the effective shear modulus, u_r is the displacement of the rth stringer in the direction of the x-axis and at the section considered and b is the stringer spacing (*see* Figs. 1 and 2).

The method of analysis used in this paper is a generalisation and simplification of the method used in R. & M. 2038[6] and yields simple formulae for the direct stress and the shear stress for any number n of stringers at any point of the panel. The particular method of analysis when $n \to \infty$ is called the ' stringer-sheet ' method. In this the resistance of the panel to direct load is spread uniformly across the width of the panel. In the case of concentrated end loads the stringer-sheet method yields the anomalous result that the shear stress in the sheet adjacent to the edge member is infinite, whereas the finite-stringer method gives always a finite shear stress because b is finite. This anomaly of the stringer-sheet method can, however, be eliminated by calculating the u_r displacements by the stringer-sheet method and applying formula (1) for the computation of the shear stress. In all other applications the finite-stringer and stringer-sheet method may be assumed as being for all practical purposes identical, provided the number of stringers exceeds five. The agreement between the two methods is particularly good when the edge stress of the free end is zero.

A further question which arises is the influence of lateral strains. As stated above in both the finite-stringer and stringer-sheet method it is assumed that the stringers and edge members are held apart by a closely spaced system of ribs (cross-members) which are rigid against compression or extension but which offer no resistance to bending in the plane of the sheet. Actually these transverse members are at a finite distance and are not rigid. Approximately to represent their actual properties it is possible to extend the conception of the stringer-sheet method also to the lateral direction. The panel is then represented by an orthotropic plate. The general theory of diffusion in orthotropic plates will be necessarily very complicated. Only some very simple symmetrical diffusion problems have been investigated by this method. It has been shown that in the case of zero edge stress at the free end of the panel both the direct stresses and shear stresses calculated by the orthotropic plate and the finite-stringer or stringer-sheet method agree very closely. In the case of constant edge stress the agreement for the direct stresses is still very good. For the shear stresses the analysis shows that for a reasonably great number of stringers (say > 10) the maximum shear stresses calculated by the orthotropic plate treatment agree quite closely with those calculated by the finite-stringer method of this paper. For antisymmetrical diffusion problems comparison is possible with some results of the theory of bending by transverse loads of isotropic plates (Fine, R. & M. 2648[4]). Again there is excellent agreement in both the direct stresses and shear stresses with the corresponding results of the stringer-sheet method.

It is evident, however, that even the treatment by the orthotropic plate method cannot be termed exact. A full analysis ought to include the effect of the finite spacing of both stringers and cross-members; the effect of buckling of the sheet and the 'give' of the joints; and the effect of the bending stiffness of stringers and cross-members when deflected in the plane of the sheet. Even if such an extensive analysis were feasible, it would be very complicated and its applicability limited. Therefore, taking into account the excellent agreement between the finite-stringer method of this paper and the available results of the orthotropic plate method, it can be assumed that from the practical point of view the results of the present investigation are sufficiently accurate.

3. *Details of Present Investigation.*—In Part II a general analysis of a parallel stiffened panel with antisymmetrical edge-stress distribution is given. It is there assumed that the panel is under the action of concentrated end loads and/or edge loads only. Thus there is no resultant shear force at any cross-section; both constant and arbitrary antisymmetrical edge-stress distributions are investigated. As an application of the latter the parallel panel with constant area edge members is analysed. For this case the effect of both antisymmetrical concentrated end loads and arbitrary antisymmetrical edge loads is considered.

In Part III the effect of arbitrary transverse loads is investigated. It is shown that the analysis can be reduced to that of antisymmetrical edge loads treated in Part II.

In Appendix I a slight generalisation and simplification of the results of R. & M. 1969[5] and 2038[6] with respect to symmetrical edge-stress distributions in parallel panels is given. No derivation is included as the method is exactly the same as that of Part II of this report.

A new analysis for diffusion and shear lag in parallel panels with given boom areas is presented in Appendix II where it is preceded by a special introduction to which the reader is referred.

A number of diagrams at the end of the report show the variation of the moment carried by the panel and the shear stress at the edge for various numbers of stringers and values of the diffusion parameter μl. Two different edge conditions referring to the cases of constant edge stress and constant area edge members respectively are shown. In the latter case the stress distribution depends also on the ratio α of total area of edge members to total area of stringers plus effective sheet.

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4. NOTATION OF MAIN REPORT and APPENDIX I

x		Co-ordinate measured from free end of panel (*see* Figs. 1 and 2)
y		Co-ordinate measured from axis of symmetry of panel (*see* Figs. 1 and 2)
l		Length of panel
b		Stringer spacing
n		Number of stringers
w	=	$(n + 1)b$: width of panel
t		Thickness of sheet
A		Area of one stringer plus effective sheet
t_s		Stringer sheet thickness
B		Area of each edge member
α	=	$2B/nA$: ratio of total area of edge members to total area of stringers plus effective sheet
I_f		Moment of inertia of edge members about x-axis
I_p		Moment of inertia of stringers plus effective sheet about x-axis
f		Edge stress at section x
F_k, $\bar{F}_k$		Fourier coefficients of arbitrary edge stress distribution
P		Concentrated end load applied to edge member
f_0	=	P/B: edge stress at free end of panel
u_a, u_b u_0, u_{n+1}		Displacements of edge members parallel to x-axis
v		Deflection of panel at section x
f_s		Stress in sth stringer at section x
u_s		Displacement of sth stringer parallel to x-axis
q_s		Shear stress in sheet between sth and $(s + 1)$th stringers at section x
		Notation of stringers is in the s-system of reference (*see* below). In the rth system of reference substitute r for s.
f_a		Average stringer stress at section x
q		Shear stress in the sheet adjacent to edge member at section x
M		Moment carried by the panel at section x
e		Index, referring to moments, stresses and displacements given by ordinary engineer's theory
f_y		Direct stress in stringer-sheet at section x and ordinate y
q_y		Shear stress in stringer-sheet
$\bar{Q}$		Shear force of transverse load system
Q_k		Fourier coefficient of shear force diagram
$\bar{M}$		Moment of external forces at section x
S		Loads applied to edge members and acting parallel to x-axis

S_k — Fourier coefficient of edge-load distribution

$\bar{q}$ — Magnitude of shear stress system

$M' = \dfrac{n(n-1)}{6} bAf_b = \frac{1}{6} w^2 t_s f_b$

i — Positive integer varying from 1 to n

s — Ordinal of stringers, $y = sb$ being the distance of the stringers from the x-axis (*see* Figs. 1 and 2)

$s = 0, \pm 1, \pm 2 \ldots \pm (n-1)/2$, when n is odd

$s = \quad , \pm \frac{1}{2}, \pm \frac{3}{2} \ldots \pm (n-1)/2$, when n is even

r — Ordinal of stringers and positive integer varying from 1 to n

m — Positive integer varying from 1 to n

k — Odd integer varying from 1 to ∞

E — Young's modulus

G — Effective (secant) shear modulus of sheet

$\mu = (2/w)\sqrt{(Gbt/EA)}$

μl — Diffusion parameter of Cox, R. & M. 1860[1]

$\mu_i = \mu(n+1) \sin \dfrac{\pi i}{2(n+1)}$

λ_r — Coefficients

ω_i — Set of characteristic values

Stress functions of the anti-symmetrical diffusion case:

$$G_{ks} = \frac{\sinh \{2s\phi_k\}}{\sinh \{(n+1)\phi_k\}}$$

$$N_{ks} = \frac{1}{n \sinh \phi_k} \left[\frac{\cosh \{(2s+1)\phi_k\}}{\sinh \{(n+1)\phi_k\}} - \frac{1}{(n+1) \sinh \phi_k} \right]$$

$$C_k = (N_{ks})_{s=(n-1)/2} = \frac{1}{n} [\coth \{(n+1)\phi_k\} \coth \phi_k - 1] - \frac{1}{n(n+1)} \frac{1}{\sinh^2 \phi_k}$$

Stress functions of the symmetrical diffusion case:

$$H_{ks} = \frac{\cosh \{2s\phi_k\}}{\cosh \{(n+1)\phi_k\}}$$

$$R_{ks} = \frac{1}{n \sinh \phi_k} \frac{\sinh \{(2s+1)\phi_k\}}{\cosh \{(n+1)\phi_k\}}$$

$$T_k = (R_{ks})_{s=(n-1)/2} = \frac{1}{n} \left[\frac{\tanh \{(n+1)\phi_k\}}{\tanh \phi_k} - 1 \right]$$

$$\phi_k = \sinh^{-1} \left\{ \frac{k\pi}{2\mu l} \frac{1}{n+1} \right\}$$

In the limiting case if $n \to \infty$ (stringer-sheet) the stress functions are denoted by barred letters, e.g., $\bar{G}_{ky}$, $\bar{H}_{ky}$, etc. (*see* also formulae (62) and (146))

D, E — Constants

PART II

Diffusion of Antisymmetrical Concentrated End and Edge Loads into Stiffened Parallel Panels. Zero Transverse Load

1. *General Considerations.*—Consider a parallel panel stiffened by n stringers at uniform spacing b. The length of the panel is l and the width $w = (n + 1)b$ (*see* Fig. 1). It is assumed that there is a closely spaced system of cross-members infinitely stiff against compression or tension but offering no resistance to bending in the plane of the sheet. The edge of the panel at $x = l$ is held straight but the edge at $x = 0$ is entirely free to warp in its own plane. The displacements u in the direction of the x-axis are functions of both x and y, but the deflections v depend only on x. The edge stress at $y = + w/2$ is $- f(x)$ (compression) and at $y = - w/2$ is $+ f(x)$ (tension). This antisymmetrical edge-stress distribution may be produced by an antisymmetrical system of concentrated end loads and edge loads on the edge members (*see* Figs. 1 and 2), and it is assumed for the present that there are no transverse loads acting on the panel. The influence of transverse loads will be investigated in Part III of this report. It is assumed that the effective sheet is concentrated along the lines of attachment of the stringers, so that the shear stress is constant over that part of any particular cross-section which lies between two adjacent stringers. A is the area of stringer cross-section plus effective sheet. G is the effective secant shear modulus of the sheet and E the Young modulus of stringers and edge members. There is some difficulty in the evaluation of both A and G. The sheet next to the one edge member is under compression and shear, whereas the sheet next to the other edge member is under tension and shear. Thus in the latter case both the values of the effective sheet and secant shear modulus are higher than in the first. Furthermore, these values vary also with x. To make the analysis feasible it is necessary to assume uniform values of A and G over the whole panel. In the case of a constant antisymmetrical edge stress it is reasonable to make the following assumptions :

(*a*) An average value of G may be found on the assumption that the shear stress q in the sheet adjacent to the edge member is constant along the length of the panel and that at $x = l$ (built-in end) the engineering theory of bending applies. Let the moment in the panel at $x = l$ be M. It follows that

$$q = M/lwt$$

(*b*) An average value of A may be found by calculating the effective sheet on the assumption that the edge stress in each buckled plate (sheet) is $- f/2$.

For other edge-stress distributions the values of A and G may be estimated in a similar manner.

2. *The Differential Equations.*—On the basis of section 1 the analysis of antisymmetrical diffusion may be developed as follows. With the notation of Figs. 1 and 2 the shear stress q_r in the sheet between the rth and $(r - 1)$th stringers can be written:

$$q_r = G \left\{ \frac{u_{r-1} - u_r}{b} + \frac{dv}{dx} \right\} . \qquad (1)$$

The direct stress f_r in the rth stringer is,

$$f_r = E \frac{du_r}{dx} \qquad (2)$$

and the condition of equilibrium of the rth stringers is,

$$A \frac{df_r}{dx} = q_{r+1}t - q_r t . \qquad (3)$$

Substituting formula (1) it follows that

$$\frac{df_r}{dx} = \frac{Gt}{Ab}\left\{ - u_{r-1} + 2u_r - u_{r+1} \right\}$$

and by differentiation,

$$\frac{d^2f_r}{dx^2} = \frac{Gt}{EAb}\left\{ - f_{r-1} + 2f_r - f_{r+1} \right\} \qquad \ldots \quad (4)$$

for $r = 1$ to $r = n$ with the boundary conditions $f_0 = -f_{n+1} = -f(x)$.

From the condition of zero shear load at any cross-section it follows that,

$$\sum_{r=1}^{r=n} q_r = 0 . \qquad \ldots \quad (5)$$

By substitution of formula (1) one finds,

$$\frac{dv}{dx} = \frac{u_{n+1} - u_0}{(n+1)\,b} = \frac{u_{n+1} - u_0}{w} \qquad \ldots \quad (5a)$$

where u_0 and u_{n+1} are the displacements in the two edge members at the corresponding cross-section.

By differentiation of equation (5a)

$$\frac{d^2v}{dx^2} = \frac{2f}{Ew} \cdot \qquad \ldots \quad (5b)$$

It is worth noticing that the last formula is the same as that in the engineering theory of bending, when the edge-stress distribution is given and the transverse loads are zero. This result is a consequence of the assumption about the cross-members. It should, however, be borne in mind that for given edge loads, edge-stress distribution and structure of the panel the areas of the edge members when calculated by the engineering theory of bending differ from those calculated by the more accurate theory of this paper. For the influence of transverse loads see Part III.

Writing, $\mu = (2/w)\sqrt{(Gbt/EA)}$ it follows that

$$\frac{Gt}{EAb} = \mu^2\left(\frac{n+1}{2}\right)^2 . \qquad \ldots \quad (6)$$

Hence the differential equations (4) become,

$$\frac{d^2f_r}{dx^2} = \mu^2\left(\frac{n+1}{2}\right)^2\left\{ - f_{r-1} + 2f_r - f_{r+1} \right\} . \qquad \ldots \quad (7)$$

The boundary conditions for f_r and u_r in the x-direction are,

at $x = 0, f = 0$ and $du_r/dx = 0$ for $r = 1$ to n

at $x = l$, $df_r/dx = 0$ and $u_r = 0$ for $r = 1$ to n.

Furthermore there is $u_0 = u_{n+1} = 0$ at $x = l$. The condition $df_r/dx = 0$ at $x = l$ needs to be reconsidered, of course, when $\mu \to \infty$.

It follows from (5a) that at $x = l$

$$q_r = G \,.\, du/dx = 0 . \qquad \ldots \quad (8)$$

Thus in the absence of transverse loads the shear stress at the built-in end ($x = l$) is zero in the antisymmetrical loading case as it is also in the symmetrical loading case. The effect of transverse loads in modifying this conclusion is discussed in Part III.

3. *Constant Antisymmetrical Edge Stress.*—Consider now a parallel panel the edge members of which are so tapered that the edge stresses are constant ($-f$ in the upper edge member and $+f$ in the lower edge member, *see* also Fig. 1).

As in R. & M. 1969[5] and 2038[6] multiplying the rth equation by λ_r and summing with respect to r

$$\frac{d^2}{dx^2}\left(\sum_{r=1}^{r=n} \lambda_r f_r\right) = \mu^2\left(\frac{n+1}{2}\right)^2 \left\{\sum_{r=1}^{r=n} (-\lambda_{r-1} + 2\lambda_r - \lambda_{r+1}) f_r + f(\lambda_1 - \lambda_n)\right\}. \quad \ldots \quad (9)$$

For $r = 1$ to n with $\lambda_0 = \lambda_{n+1} = 0$.

Choosing the λ's and ω's so that

$$\lambda_{r-1} + (\omega^2 - 2)\lambda_r + \lambda_{r+1} = 0 \quad \ldots \quad (10)$$

and proceeding as in R. & M. 1969[5], there are n characteristic values $(\omega_i)^2$ and n corresponding sets of solution λ_r^i which satisfy equation (9).

They are,

$$\left.\begin{aligned} &(\omega_i)^2 = 4\sin^2\frac{\pi i}{2(n+1)} \text{ or } \omega_i = 2\sin\frac{\pi i}{2(n+1)} \\ &\text{and} \\ &\lambda_r^i = \sin\frac{\pi i r}{n+1} \end{aligned}\right\} \quad \ldots \quad (11)$$

where i takes all integral values 1 to n. The following relations of R. & M. 1969[5] section II.2 will be needed in subsequent analysis,

$$\lambda_r^i = -\lambda_{n+1-r}^i \text{ and } \lambda_1^i - \lambda_n^i = 2\sin\frac{\pi i}{n+1} \qquad \text{if } i \text{ is even}$$

and

$$\lambda_r^i = \lambda_{n+1-r}^i \text{ and } \lambda_1^i - \lambda_n^i = 0 \qquad \text{if } i \text{ is odd.}$$

Furthermore,

$$\sum_{r=1}^{r=n} \lambda_r^i = 0 \qquad \text{if } i \text{ is even}$$

and

$$\sum_{r=1}^{r=n} \lambda_r^i = \cot\frac{\pi i}{2(n+1)} \qquad \text{if } i \text{ is odd.}$$

4. *Solution of Differential Equations* (9).—For each value of i equation (9) can now be written,

$$\frac{d^2}{dx^2}\left(\sum_{r=1}^{r=n} \lambda_r^i f_r\right) = \mu^2\left(\frac{n+1}{2}\right)^2 \left\{\omega_i^2 \sum_{r=1}^{r=n} \lambda_r^i f_r + f(\lambda_1^i - \lambda_n^i)\right\} \quad \ldots \quad (9a)$$

The general solution of (9a) is found in the form

$$\sum_{r=1}^{r=n} \lambda_r^i f_r = -\frac{(\lambda_1^i - \lambda_n^i)}{\omega_i^2} f + L\sinh\mu_i x + M\cosh\mu_i x$$

where

$$\mu_i = \mu(n+1)\sin\frac{\pi i}{2(n+1)} \quad \ldots \quad (12)$$

and L and M are constants. Adjusting to the boundary conditions

$$\sum_{n=1}^{r=n} \lambda_r^i f_r = 0 \text{ at } x = 0 \text{ and } \frac{d}{dx}\left(\sum_{r=1}^{r=n} \lambda_r^i f_r\right) = 0 \text{ at } x = l,$$

the solution of (9a) becomes,

$$\left.\begin{aligned} &\text{for } i \text{ even} \sum_{r=1}^{r=n} \lambda_r{}^i f_r = -f \cot \frac{\pi i}{2(n+1)} \left\{1 - \frac{\cosh \mu_i(l-x)}{\cosh \mu_i l}\right\} \\ &\text{for } i \text{ odd} \sum_{r=1}^{r=n} \lambda_r{}^i f_r = 0\,. \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

Multiplying now each of the equations of order i by $\sum_{m=1}^{m=n} \lambda_m{}^i$ (which as shown above is zero for i even and $\cot \pi i/2(n+1)$ for i odd) and summing the resulting n equations we get an equation with the right-hand side zero.

On the left-hand side we have,

$$\sum_{i=1}^{i=n} \sum_{m=1}^{m=n} \lambda_m{}^i \sum_{r=1}^{r=n} \lambda{}^i f_r = \sum_{r=1}^{r=n} \left\{ \sum_{m=1}^{m=n} \sum_{i=1}^{i=n} \lambda_m{}^i \lambda_r{}^i \right\} f_r$$

as in R. & M. 1969[5] we have,

$$\left.\begin{aligned} &\sum_{i=1}^{i=n} \lambda_r{}^i \lambda_m{}^i = \sum_{i=1}^{i=n} \lambda_i{}^r \lambda_i{}^m = 0 && \text{if } m \neq r \\ &\sum_{i=1}^{i=n} (\lambda_r{}^i)^2 = \sum_{i=1}^{i=n} \sin^2 \frac{\pi r i}{m+1} = \frac{n+1}{2} && \text{if } m = r \end{aligned}\right\}. \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

Thus the left-hand side is $\frac{n+1}{2} \sum_{r=1}^{r=n} f_r$ and the equation is finally

$$\sum_{r=1}^{r=n} f_r = 0\,.$$

This is, of course, an obvious result, in view of the antisymmetrical loading.

To find now the stress in the rth stringer each of the equations of system (13) is multiplied by $\lambda_r{}^i = \sin \pi i r/n + 1$. By summation of the resulting n equations we get an equation with the left-hand side $f_r(n+1)/2$. This follows immediately from the relations (14). On the right-hand side we have

$$-f \sum_{i\,\text{even}} \cot \frac{\pi i}{2(n+1)} \sin \frac{\pi i r}{n+1} \left\{1 - \frac{\cosh \mu_i(l-x)}{\cosh \mu_i l}\right\}.$$

Thus the stress in the rth stringer is

$$f_r/f = -\frac{2}{n+1} \sum_{i\,\text{even}} \cot \frac{\pi i}{2(n+1)} \sin \frac{\pi i r}{n+1} \left\{1 - \frac{\cosh \mu_i(l-x)}{\cosh \mu_i l}\right\}.$$

It can be shown that,

$$\frac{2}{n+1} \sum_{i\,\text{even}} \cot \frac{\pi i}{2(n+1)} \sin \frac{\pi i r}{n+1} = \frac{n+1-2r}{n+1} \text{ for } n \text{ odd or even.}$$

Hence,

$$f_r/f = -\frac{n+1-2r}{n+1} + \frac{2}{n+1} \sum_{i\,\text{even}} \cot \frac{\pi i}{2(n+1)} \sin \frac{\pi i r}{n+1} \frac{\cosh \mu_i(l-x)}{\cosh \mu_i l}\,. \quad \ldots \quad (15)$$

The first term on the right-hand side corresponds to the stress distribution in the panel according to the engineering theory of bending. It is preferable for reasons of presentation to replace the ordinal r by the ordinal s where

$$s = \frac{n + 1 - 2r}{2} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

and to renumber the stringers positively and negatively from the centre-line of the panel.

Note that $2s/(n + 1) = 2y/w$ and

$$\left.\begin{array}{ll} s = 0, \pm 1, \pm 2, \ldots, & \pm \dfrac{n-1}{2} \text{ for } n \text{ odd} \\ s = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots, & \pm \dfrac{n-1}{2} \text{ for } n \text{ even} \end{array}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17)$$

$\pm \dfrac{n-1}{2}$ indicates the outer stringers adjacent to the edge members.

Continuing,

$$\sin \frac{\pi i r}{n+1} = \sin \left[\frac{\pi i}{n+1}\left\{\frac{n+1}{2} - s\right\}\right] = -\cos \frac{\pi i}{2} \sin \frac{\pi i s}{n+1}$$

$$= (-1)^{1+i/2} \sin \frac{\pi i s}{n+1} \text{ for } i \text{ even.}$$

Writing $f_{se} = -f\dfrac{2s}{n+1}$, where f_{se} is the stress in the sth stringer given by the engineering theory of bending, we have,

$$f_s/f_{se} = 1 - \frac{1}{s} \sum_{i \text{ even}} (-1)^{1+i/2} \cot \frac{\pi i}{2(n+1)} \sin \frac{\pi i s}{n+1} \frac{\cosh \mu_i (l - x)}{\cosh \mu_1 l} \,. \qquad \ldots \quad \ldots \quad (18)$$

With increasing values of μ (diffusion constant) f_s approaches f_{se}.

Substituting $f_s = E \,.\, du_s/dx$, integrating and adjusting to the boundary conditions $u_s = 0$ at $x = l$, one finds,

$$Eu_s = \frac{2sf(l - x)}{n + 1} - \frac{2fl}{n + 1} \sum_{i \text{ even}} (-1)^{1+i/2} \cot \frac{\pi i}{2(n+1)} \sin \frac{\pi i s}{n+1} \frac{\sinh \mu_i (l - x)}{\mu_i l \cosh \mu_i l} \,. \qquad \ldots \quad (19)$$

The engineering theory of bending indicates displacements u_{se}

$$Eu_{se} = \frac{2sf(l - x)}{n + 1} \,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

Hence,

$$\frac{u_s}{u_{se}} = 1 - \frac{l}{s(l - x)} \sum_{i \text{ even}} (-1)^{1+i/2} \cot \frac{\pi i}{2(n+1)} \sin \frac{\pi i s}{n+1} \frac{\sinh \mu_i (l - x)}{\mu_i l \cosh \mu_i l} \,. \qquad \ldots \quad \ldots \quad (21)$$

The symbols for the displacements u of the edge members in the s-system are $u_{+(n+1)/2}$ and $u_{-(n+1)/2}$. For brevity they will be denoted from now on by u_a and u_b respectively. They are given by,

$$Eu_a = f(l - x) \text{ and } Eu_b = -f(l - x) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

The shear stress q_s in the sheet between the sth and the $(s + 1)$th stringer is (*see* equation (4))

$$q_s = G\left\{\frac{u_{s+1} - u_s}{b} + \frac{dv}{dx}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

where

$$\frac{dv}{dx} = \frac{u_b - u_a}{(n + 1)b} \,.$$

Substituting equations (19) and (22) one finds,

$$(q_s/f)\sqrt{(Ebt/GA)} = \frac{2}{n+1} \sum_{i\,\text{even}} (-1)^{i/2} \cot \frac{\pi i}{2(n+1)} \cos \frac{\pi i(2s+1)}{2(n+1)} \frac{\sinh \mu_i(l-x)}{\cosh \mu_i l} \quad \ldots \quad (24)$$

The maximum shear stresses occur at $x = 0$ (free end) and the ratio of the hyperbolic functions in (24) becomes $\tanh \mu_i l$. It is usually permissible to put $\tanh \mu_i l = 1$. This yields a simple approximate formula for the shear stresses at $x = 0$.

$$(q_s/f)\sqrt{(Ebt/GA)} = \frac{2}{n+1} \sum_{i\,\text{even}} (-1)^{i/2} \cot \frac{\pi i}{2(n+1)} \cos \frac{\pi i(2s+1)}{2(n+1)} \quad \ldots \quad (25)$$

The shear stress distribution across any cross-section is symmetrical about the x-axis and has its maximum value in the plates adjacent to the edge members.

Substituting $s = (n-1)/2$ in (24) one finds for the shear stress q in the sheet connected to the edge members,

$$(q/f)\sqrt{(Ebt/GA)} = \frac{2}{n+1} \sum_{i\,\text{even}} \cot \frac{\pi i}{2(n+1)} \cos \frac{\pi i}{2(n+1)} \frac{\sinh \mu_i(l-x)}{\cosh \mu_i l} \quad \ldots \quad (26)$$

This formula should be compared with formula (34) of R. & M. 1969[5] which gives the shear stress at the edges for the symmetrical constant edge stress. In the latter case the sum has to be taken over i odd, the formula being otherwise identical.

If n is odd $s = 0$ defines the sheet adjacent to the middle stringer and the shear stress q_0 in this sheet becomes,

$$(q_0/f)\sqrt{(Ebt/GA)} = \frac{2}{n+1} \sum_{i\,\text{even}} (-1)^{i/2} \cot \frac{\pi i}{2(n+1)} \cos \frac{\pi i}{2(n+1)} \frac{\sinh \mu_i(l-x)}{\cosh \mu_i l} \quad \ldots \quad (27)$$

It follows immediately that $|q_0| < q$, because all the separate factors in the sum (26) are positive. For $n = 9$ and $\mu l = 2$ Fig. 3 shows a typical distribution of shear stress across the sections at $x/l = 0$ and $0 \cdot 1$.

In Figs. 4, 5 and 6, $(q/f)\sqrt{(Ebt/GA)}$ is plotted against x/l for $n = 5$, 10 and 30 and various values of μl. These diagrams should be compared with Figs. 5, 6 and 7 in R. & M. 1969[5] corresponding to the symmetrical constant edge stress. It can be seen that the values of the maximum shear stresses in the antisymmetrical case are approximately half those found in the symmetrical case for the same absolute value of the edge stress.

An alternative form of the left-hand side of equation (24), etc., is

$$(q/f)\sqrt{(Ebt/GA)} = \{(q/f)/(nA/tl)\} \frac{2n}{n+1} \frac{1}{\mu l} \quad \ldots \quad (28)$$

Using this relation it is found that the maximum shear stress at $x = 0$ does not vary rapidly with the number of stringers, provided that the total area of section of stringers plus effective sheet is maintained constant and n is in the practical range (say 10 to 20).

A further point of interest is the moment M which the panel carries at any particular cross-section x.

By definition,

$$M = -\sum_{s=-(n-1)/2}^{s=+(n-1)/2} sbAf_s \quad \ldots \quad (29)$$

Substituting formula (18) and taking into account that,

$$\sum_{-(n-1)/2}^{+(n-1)/2} s^2 = \frac{1}{12} n(n-1)(n+1)$$

and that,

$$\sum_{-(n-1)/2}^{+(n-1)/2} s \sin \frac{\pi i s}{n+1} = (-1)^{1+i/2} \frac{n+1}{2} \cot \frac{\pi i}{2(n+1)} \text{ for all values of } n, \text{ one finds that,}$$

$$M = \tfrac{1}{6} n(n-1) bfA - bfA \sum_{i \text{ even}} \cot^2 \frac{\pi i}{2(n+1)} \frac{\cosh \mu_i (l-x)}{\cosh \mu_i l} . \qquad \ldots \quad \ldots \quad (30)$$

The first term in equation (30) corresponds to the moment M_e indicated by the engineering theory of bending.

Hence formula (30) can be written,

$$\left.\begin{aligned} M/M_e &= 1 - \frac{6}{n(n-1)} \sum_{i \text{ even}} \cot^2 \frac{\pi i}{2(n+1)} \frac{\cosh \mu_i (l-x)}{\cosh \mu_i l} \\ \text{where} \quad M_e &= \frac{n(n-1)}{6} bfA \end{aligned}\right\} . \qquad \ldots \quad \ldots \quad (31)$$

Equation (31) satisfies the boundary conditions $M = 0$ at $x = 0$ and $dM/dx = 0$ at $x = l$ (note that $\sum_{i \text{ even}} \cot^2 \frac{\pi i}{2(n+1)} = \frac{1}{6} n(n-1)$). With increasing values of μ, M approaches M_e. In Fig. 7, M/M_e is plotted against x/l for $n = 10$ and various values of μl*. The ratio M/M_e varies only slightly with n when $n > 7$.

To find the areas $B(x)$ of constant-stress edge members under a given system of end loads and/or edge loads consider the equilibrium condition of moments at the section x. Let the moment of the external loads be $\bar{M}$, which may vary with x, then

$$B(x) = (\bar{M} - M)/wf . \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

The deflection v of the panel can be found from equations (5a) and (22), and the resulting formula

$$Ev = f(l-x)^2/w \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

shows that the deflection is unaffected by diffusion except in so far as the edge stress f itself is affected.

4.1. *Special Case : Stringer-sheet when number of stringers is infinite.*—Assuming the convergence of the infinite series when $n \to \infty$, we have:

$$\left.\begin{aligned} &\lim_{n \to \infty} \frac{s}{n+1} = y/w \\ &\lim_{n \to \infty} \mu_i = \lim_{n \to \infty} \mu(n+1) \sin \frac{\pi i}{2(n+1)} = \pi i \mu/2 \\ &\lim_{n \to \infty} \frac{6}{n(n-1)} \cot^2 \frac{\pi i}{2(n+1)} = 24/\pi^2 i^2 \end{aligned}\right\} . \qquad \ldots \quad \ldots \quad (34)$$

* Values of the diffusion constant μl for typical aircraft structures are between 1 and 4.

Formula (18) becomes,

$$f_y/f = -\frac{2y}{w} + \frac{4}{\pi}\sum_{i\text{ even}}^{\infty} (-1)^{1+i/2}\frac{1}{i}\sin\frac{\pi i y}{w}\frac{\cosh\{\pi i\mu(l-x)/2\}}{\cosh\{\pi i\mu l/2)\}} \quad \ldots \quad \ldots \quad (35)$$

or

$$f_y/f_{ye} = 1 - \frac{2}{\pi}\frac{w}{y}\sum_{i\text{ even}}^{\infty} (-1)^{1+i/2}\frac{1}{i}\sin\frac{\pi i y}{w}\frac{\cosh\{\pi i\mu(l-x)/2\}}{\cosh\{\pi i\mu l/2\}} \quad \ldots \quad \ldots \quad (35a)$$

where $f_{ye} = -f(2y/w)$ is the direct stress indicated by the engineering theory of bending.

Formula (24) becomes,

$$(q_y/f)\sqrt{(EtG/t_s)} = \frac{4}{\pi}\sum_{i\text{ even}}^{\infty} (-1)^{i/2}\frac{1}{i}\cos\frac{\pi i y}{w}\frac{\sinh\{\pi i\mu(l-x)/2\}}{\cosh\{\pi i\mu l/2\}} \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

and the shear stress q at the edge,

$$(q/f)\sqrt{(Et/Gt_s)} = \frac{4}{\pi}\sum_{i\text{ even}}^{\infty}\frac{1}{i}\frac{\sinh\{\pi i\mu(l-x)/2\}}{\cosh\{\pi i\mu l/2\}}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

Note that in this case the value of $q \to \infty$ as $x \to 0$. This anomaly results from making $b \to 0$. The moment M carried by the panel is given by,

$$M/M_e = 1 - \frac{24}{\pi^2}\sum_{i\text{ even}}^{\infty}\frac{1}{i^2}\frac{\cosh\{\pi i\mu(l-x)/2\}}{\cosh\{\pi i\mu l/2\}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

where $M_e = w^2t_sf/6$ is the moment carried by the ordinary engineer's theory.

It is interesting to note that for $l \to \infty$ the series in formulae (35) and (36) can be summed. One finds

$$f_y/f = -\frac{2}{\pi}\tan^{-1}\left\{\tan\left(\frac{\pi y}{w}\right)\tan\left(\frac{\pi}{2}\mu x\right)\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (35b)$$

and

$$(q_y/f)\sqrt{(Et/Gt_s)} = \frac{1}{\pi}\left\{-\ln\left[4\left(\cos^2\frac{\pi y}{w} + \sinh^2\frac{\pi}{2}\mu x\right)\right] + \pi\mu x\right\}. \quad \ldots \quad \ldots \quad \ldots \quad (36a)$$

The shear stress q at the edge becomes,

$$(q/f)\sqrt{(Et/Gt_s)} = \frac{2}{\pi}\left\{-\ln\left[2\sinh\left(\frac{\pi}{2}\mu x\right)\right] + \frac{\pi}{2}\mu x\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37a)$$

and $q \to \infty$ as $x \to 0$.

The shear stress q_0 along the x-axis,

$$(q_0/f)\sqrt{(Et/Gt_s)} = \frac{2}{\pi}\left\{-\ln\left[2\cosh\left(\frac{\pi}{2}\mu x\right)\right] + \frac{\pi}{2}\mu x\right\}. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

The right-hand side of equation (39) reduces to $-\frac{2}{\pi}\ln 2$ when $x = 0$.

Formula (35b) has been found very useful for quickly estimating the stress distribution in long panels ($l > 3w$ say).

5. *Presentation of the Results of Section* 4 *in Fourier Series*.—For subsequent analysis in Section 6 and other applications it is necessary to expand formulae (15) or (18) and (24) in Fourier series. One method is to expand first the hyperbolic functions

$$\left(1 - \frac{\cosh\mu_i(l-x)}{\cosh\mu_i l}\right) \quad \text{and} \quad \frac{\sinh\mu_i(l-x)}{\cosh\mu_i l} \quad \text{in Fourier series.}$$

The formulae for the direct stresses and shear stresses are then transformed into double series, one finite over i and one Fourier infinite. It is now possible to sum in each case over i. This method has been applied in R. & M. 1969[5] but the derivation of the corresponding formulae by the same means in the present problem would be rather lengthy and cumbersome. For this reason a more direct method is applied, which, moreover, has the advantage that the λ-coefficients are avoided.

A constant edge stress f can be represented by the Fourier series,

$$f = \frac{4}{\pi} f \sum_{k \text{ odd}}^{\infty} \frac{1}{k} \sin \frac{k\pi x}{2l}, \qquad 0 < x \leqslant l. \quad \ldots \quad (40)$$

This Fourier series does not converge to the value f at $x = 0$, but this is of no importance to the developments of this section.

Throughout the subsequent analysis it is only necessary to consider a typical term $f \frac{1}{k} \sin \frac{k\pi x}{2l}$ and the summation of the resulting expressions over k may be deferred until the final stage; a short discussion of the convergence of the series will then be given.

Consider now the general relations (7) which in the present case can be written:

$$\frac{d^2 f_r}{dx^2} = \mu^2 \left(\frac{n+1}{2}\right)^2 \{-f_{r-1} + 2f_r - f_{r+1}\} \text{ for } r = 1 \text{ to } n. \quad \ldots \quad (7)$$

and the boundary conditions $f = 0$ at $x = 0$, $(df_r/dx) = 0$ at $x = l$ and $f_0 = -f \frac{1}{k} \sin \frac{k\pi x}{2l}$ and $f_{n+1} = +f \frac{1}{k} \sin \frac{k\pi x}{2l}$.

The form of these differential equations and boundary conditions suggests the following solution,

$$f_r = G_r' f \frac{1}{k} \sin \frac{k\pi x}{2l} \quad \ldots \quad (41)$$

where G_r' is a function of r solely. Substituting the solution (41) into (7)

$$-\left(\frac{k\pi}{2l}\right)^2 G_r' = \mu^2 \left(\frac{n+1}{2}\right)^2 \{-G_{r-1}' + 2G_r' - G_{r+1}'\} \quad \ldots \quad (42)$$

with the boundary conditions $G_0' = -1$ and $G_{n+1}' = +1$. Substituting

$$\sinh \phi_k = \frac{k\pi}{2\mu l} \frac{1}{n+1} {}^{*} \quad \ldots \quad (43)$$

into (42) one obtains after some elementary transformations the finite difference equations

$$-G_{r-1}' + 2 \cosh 2\phi_k \, G_r' - G_{r+1}' = 0 \quad \ldots \quad (44)$$

for $r = 1$ to n and the same boundary conditions. The general solution of (44) can be written in the form,

$$G_r' = D \sinh \alpha r + E \cosh \alpha r \quad \ldots \quad (45)$$

where D and E are constants and α is a characteristic value. The substitution of either of the two particular solutions of (45) into (44) conditions the unknown characteristic value α by,

$$\cosh \alpha = \cosh 2\phi_k \quad \ldots \quad (46)$$

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* This substitution was also used in R. & M. 2038[6].

The only real solution of this equation is,

$$\alpha = 2\phi_k \,. \qquad (47)$$

Hence (45) becomes,

$$G_r' = D \sinh 2\phi_k r + E \cosh 2\phi_k r \,. \qquad (45a)$$

Adjusting to the boundary conditions $G_0' = -1$ and $G_{n+1}' = +1$ one obtains,

$$G_r' = -\frac{\sinh\{(n+1-2r)\phi_k\}}{\sinh\{(n+1)\phi_k\}} \,. \qquad (48)$$

The antisymmetrical character of G_r' appears more clearly by referring the stringers again to the axis of symmetry of the panel, so that

$$G_s' = -\frac{\sinh\{2s\phi_k\}}{\sinh\{(n+1)\phi_k\}} = -G_{k,s} \qquad (49)$$

where $s = \dfrac{n+1}{2} - r$

and $s = 0, \pm 1, \pm 2, \ldots, \pm\left(\dfrac{n-1}{2}\right)$ when n is odd

$s = \pm \tfrac{1}{2}, \pm \tfrac{3}{2}, \ldots, \pm\left(\dfrac{n-1}{2}\right)$ when n is even.

The index k in the function G_{ks} denotes the dependence on the corresponding Fourier term k.

Hence the stress in the sth stringer, when the antisymmetrical edge-stress distribution is

$$\pm f \frac{1}{k} \sin \frac{k\pi x}{2l}\,, \text{ becomes}$$

$$f_s = -\frac{f}{k}\frac{\sinh\{2s\phi_k\}}{\sinh\{(n+1)\phi_k\}} \sin\frac{k\pi x}{2l} = -\frac{f}{k} G_{ks} \sin\frac{k\pi x}{2l} \,. \qquad (50)$$

In the case of a constant edge-stress distribution (40), f_s becomes by summation of the k-series,

$$f_s = -\frac{4}{\pi} f \sum_{k\ \mathrm{odd}}^{\infty} \frac{1}{k} \frac{\sinh\{2s\phi_k\}}{\sinh\{(n+1)\phi_k\}} \sin\frac{k\pi x}{2l} \,. \qquad (51)$$

With increasing μl

$$\frac{\sinh\{2s\phi_k\}}{\sinh\{(n+1)\phi_k\}} \to \frac{2s}{n+1} = \frac{2y}{w}$$

and formula (51) reduces to the stress distribution indicated by the engineering theory of bending.

The reasoning leading to equation (51) is purely formal. A verification of the solution is therefore necessary. This will not be given in full, the only difficult step being the proof that series (51) can be twice differentiated for $-(n-1)/2 < s < +(n-1)/2$. Note particularly that these differentiations are not implied and necessary for the case $s = \pm(n+1)/2$ when indeed they are not possible. In outline one may proceed as follows:

differentiating formally

$$\frac{df_s}{dx} = -\frac{2f}{l} \sum_{k\ \mathrm{odd}}^{\infty} \frac{\sinh\{2s\phi_k\}}{\sinh\{(n+1)\phi_k\}} \cos\frac{k\pi x}{2l}$$

and

$$\frac{d^2 f_s}{dx^2} = \frac{f\pi}{l^2} \sum_{k\ \mathrm{odd}}^{\infty} k \frac{\sinh\{2s\phi_k\}}{\sinh\{(n+1)\phi_k\}} \sin\frac{k\pi x}{2l} \,.$$

Now $\phi_k = \sinh^{-1}\left\{\frac{k\pi}{2\mu l}\frac{1}{n+1}\right\}$ and hence for large values of k, $e^{\phi_k} \to \frac{k\pi}{\mu l}\frac{1}{n+1}$. It follows that for large k, i.e., large ϕ_k, the typical terms in the above series tend to

$$\frac{\sinh\{2s\phi_k\}}{\sinh\{(n+1)\phi_k\}} \to \pm \exp\{2|s| - (n+1)\}\phi_k \to \pm \left(\frac{k\pi}{\mu l}\frac{1}{n+1}\right)^{\{2|s|-(n+1)\}}$$

$$k\frac{\sinh\{2s\phi_k\}}{\sinh\{(n+1)\phi_k\}} \to \pm k \exp\{2|s| - (n+1)\}\phi_k \to \pm k\left(\frac{k\pi}{\mu l}\frac{1}{n+1}\right)^{\{2|s|-(n+1)\}}$$

The least convergent case is when $|s| = (n-1)/2$ in which case the power is -2. Thus the convergence of the series can be made to depend on that of

$$\sum_{k \text{ odd}}^{\infty} \frac{1}{k^2}\cos\frac{k\pi x}{2l} \quad \text{and} \quad \sum_{k \text{ odd}}^{\infty} \frac{1}{k}\sin\frac{k\pi x}{2l}$$

which are known to be convergent. It is easy to see by a similar procedure that (51) is uniformly convergent over the complete range.

To find the shear stresses it is necessary to calculate the displacements u_s.

By integration of (51)

$$Eu_s = \frac{8l}{\pi^2} f \sum_{k \text{ odd}}^{\infty} \frac{1}{k^2}\frac{\sinh\{2s\phi_k\}}{\sinh\{(n+1)\phi_k\}}\cos\frac{k\pi x}{2l}\,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (52)$$

This expression satisfies the boundary conditions $Edu_s/dx = f_s = 0$ at $x = 0$ and $u_s = 0$ at $x = 0$. For $s = \pm(n+1)/2$, i.e., for the edge members, the above formula reduces to

$$Eu_a = -Eu_b = \frac{8l}{\pi^2} f \sum_{k \text{ odd}}^{\infty} \frac{1}{k^2}\cos\frac{k\pi x}{2l} = f(l-x) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (53)$$

which follows also directly from (40).

Substituting expressions (53) and (52) into (23) one finds

$$(q_s/f)/(nA/tl) = \sum_{k \text{ odd}}^{\infty} \frac{1}{n\sinh\phi_k}\left[\frac{\cosh\{(2s+1)\phi_k\}}{\sinh\{(n+1)\phi_k\}} - \frac{1}{(n+1)\sinh\phi_k}\right]\cos\frac{k\pi x}{2l}\,. \qquad \ldots \quad \ldots \quad (54)$$

For some applications it may be preferable to separate again the term $(u_b - u_a)/(n+1)$. Formula (54) then becomes,

$$(q_s/f)/(nA/tl) = -\left(\frac{n+1}{2n}\right)(\mu l)^2(1 - x/l) + \sum_{k \text{ odd}}^{\infty} \frac{1}{n\sinh\phi_k}\frac{\cosh\{2s+1)\phi_k\}}{\sinh\{(n+1)\phi_k\}}\cos\frac{k\pi x}{2l}\,. \qquad \ldots \quad (54a)$$

For the further developments it is useful to introduce the function N_{ks} defined by,

$$N_{ks} = \frac{1}{n\sinh\phi_k}\left[\frac{\cosh\{(2s+1)\phi_k\}}{\sinh\{(n+1)\phi_k\}} - \frac{1}{(n+1)\sinh\phi_k}\right]. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (55)$$

Hence

$$(q_s/f)/(nA/tl) = \sum_{k \text{ odd}}^{\infty} N_{ks}\cos\frac{k\pi x}{2l}\,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (54b)$$

Of particular interest are the shear stresses in the plates adjacent to the edge members and along the x-axis. The former can be found by substituting $s = (n-1)/2$ into equation (54).

The particular function N_{ks} for $s = (n - 1)/2$ is important and for brevity this function will be denoted by C_k where

$$C_k = \frac{1}{n}[\coth\{(n+1)\phi_k\}\coth\phi_k - 1] - \frac{1}{n(n+1)\sinh^2\phi_k} \cdot \quad \ldots \quad \ldots \quad \ldots \quad (56)$$

Then the shear stress q at the edge given by,

$$(q/f)/(nA/tl) = \sum_{k \text{ odd}}^{\infty} C_k \cos\frac{k\pi x}{2l} \quad \ldots \quad \ldots \quad \ldots \quad (57)$$

or

$$(q/f)/(nA/tl) = \sum_{k \text{ odd}}^{\infty} \frac{1}{n}[\coth\{(n+1)\phi_k\}\coth\phi_k - 1]\cos\frac{k\pi x}{2l} - \left(\frac{n+1}{2n}\right)(\mu l)^2(1 - x/l) \,. \quad (57a)$$

When n is even, $s = -1/2$ represents the sheet in the middle of the panel.

It can readily be seen that,

$$(q_{-1/2}/f)/(nA/tl) = \sum_{k \text{ odd}}^{\infty} \left\{\frac{1}{n\sinh\phi_k}\left[\frac{1}{\sinh\{(n+1)\phi_k\}} - \frac{1}{(n+1)\sinh\phi_k}\right]\right\}\cos\frac{k\pi x}{2l} \,. \quad \ldots \quad (58)$$

The moment M which the panel carries at a cross-section x is by definition,

$$M = bA \sum_{-(n-1)/2}^{+(n-1)/2} sf_s \,. \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

Substituting equation (51) into (29) and taking into account that

$$\sum_{-(n-1)/2}^{+(n-1)/2} s\frac{\sinh\{2s\phi_k\}}{\sinh\{(n+1)\phi_k\}} = \frac{n(n+1)}{2}C_k \quad \ldots \quad \ldots \quad \ldots \quad (59)$$

where C_k is defined by (56), one finds,

$$M = \frac{2}{\pi}n(n+1)\,bAf\sum_{k \text{ odd}}^{\infty}\frac{1}{k}C_k\sin\frac{k\pi x}{2l} \quad \ldots \quad \ldots \quad \ldots \quad (60)$$

or

$$M/M_e = \frac{12}{\pi}\frac{n+1}{n-1}\sum_{k \text{ odd}}^{\infty}\frac{1}{k}C_k\sin\frac{k\pi x}{2l} \quad \ldots \quad \ldots \quad \ldots \quad (60a)$$

where $M_e = \{n(n-1)/6\}bAf$ is the moment carried by the panel according to the engineering theory of bending. With increasing values of μ, M approaches M_e. This follows directly from equation (60a) because

$$\lim_{\mu l \to \infty} C_k = \frac{1}{3}\frac{n-1}{n+1}$$

and $\displaystyle \frac{4}{\pi}\sum_{k \text{ odd}}^{\infty}\frac{1}{k}\sin\frac{k\pi x}{2l} = 1 \,.$

It is worth noticing that the shear stress in the sheet adjacent to the edge members can also be found from the relation,

$$q = \frac{1}{wt}\frac{dM}{dx} \,. \quad \ldots \quad \ldots \quad \ldots \quad (61)$$

The deflections v can best be calculated from the equation (33).

5.1. *Special case : stringer-sheet, when number of stringers is infinite.*—The limiting values of the functions G_{ks}, N_{ks}, and C_k for $n \to \infty$ are:

$$\left.\begin{aligned}
\lim_{n\to\infty} G_{ks} &= \lim_{n\to\infty} \frac{\sinh\{2s\,\phi_k\}}{\sinh\{(n+1)\phi_k\}} = \frac{\sinh\{(2y/w)(k\pi/2\mu l)\}}{\sinh\{k\pi/2\mu l\}} = \bar{G}_{ky} \\
\lim_{n\to\infty} N_{ks} &= \lim_{n\to\infty} \left\{\frac{1}{n\sinh\phi_k}\left[\frac{\cosh\{(2s+1)\phi_k\}}{\sinh\{(n+1)\phi_k\}} - \frac{1}{(n+1)\sinh\phi_k}\right]\right\} \\
&= \frac{2\mu l}{k\pi}\left[\frac{\cosh\{(2y/w)(k\pi/2\mu l)\}}{\sinh\{k\pi/2\mu l\}} - \frac{2\mu l}{k\pi}\right] = \bar{N}_{ky} \\
\lim_{n\to\infty} C_k &= \frac{2\mu l}{k\pi}\left[\coth\left\{\frac{k\pi}{2\mu l}\right\} - \frac{2\mu l}{k\pi}\right] = \bar{C}_k
\end{aligned}\right\} \quad \ldots \quad (62)$$

Hence one finds for; the direct stress f_y,

$$\left.\begin{aligned}
f_y/f &= -\frac{4}{\pi}\sum_{k\text{ odd}}^{\infty}\frac{1}{k}\frac{\sinh\{(2y/w)(k\pi/2\mu l)\}}{\sinh\{k\pi/2\mu l\}}\sin\frac{k\pi x}{2l} \\
&= -\frac{4}{\pi}\sum_{k\text{ odd}}^{\infty}\frac{1}{k}\bar{G}_{ky}\sin\frac{k\pi x}{2l}
\end{aligned}\right\} \quad \ldots \quad (63)$$

the shear stress q_y,

$$\left.\begin{aligned}
(q_y/f)/(wt_s/tl) &= \sum_{k\text{ odd}}^{\infty}\bar{N}_{ky}\cos\frac{k\pi x}{2l} \\
\text{or} \quad &= \frac{2\mu l}{\pi}\sum_{k\text{ odd}}^{\infty}\frac{1}{k}\frac{\cosh\{(2y/w)(k\pi/2\mu l)\}}{\sinh\{k\pi/2\mu l\}}\cos\frac{k\pi x}{2l} \\
&\quad - \tfrac{1}{2}(\mu l)^2(1 - x/l)
\end{aligned}\right\} \quad \ldots \quad (64)$$

The particular formulae for the shear stress at the edges and along the middle axis of the panel are straightforward.

The moment M is given by

$$M/M_e = \frac{12}{\pi}\sum_{k\text{ odd}}^{\infty}\frac{1}{k}\bar{C}_k\sin\frac{k\pi x}{2l} \quad \ldots \quad (65)$$

where $M_e = w^2t_sf/6$ is the moment carried by the panel according to the engineering theory of bending. For $\mu l \to \infty$, $M \to M_e$ because $\lim_{\mu l\to\infty}\bar{C}_k = \frac{1}{3}$.

6. *Arbitrary Edge-Stress Distribution.*—An arbitrary antisymmetrical edge-stress distribution $\pm f$ may be represented by

$$f = f_b + \frac{4}{\pi}\sum_{k\text{ odd}}^{\infty}\frac{1}{k}F_k\sin\frac{k\pi x}{2l} \quad \ldots \quad (66a)$$

or

$$f = \frac{4}{\pi}\sum_{k\text{ odd}}^{\infty}\frac{1}{k}\bar{F}_k\sin\frac{k\pi x}{2l} = \frac{4}{\pi}\sum\frac{1}{k}(f_b + F_k)\sin\frac{k\pi x}{2l} \quad \ldots \quad (66b)$$

where f_b is the edge stress at the free end of the edge member. When no end loads are applied to the edge member, $f_b = 0$.

Owing to the fact that the Fourier expansion of a constant (in this case the expansion of f_b) cannot be differentiated term by term, the representation (66b) will be inappropriate whenever df/dx occurs and $f_b \neq 0$. In such cases form (66a) has to be used (the series in this expression will be assumed to be differentiable). The application of this remark will appear in section 7.

To find the stress distribution in the panel the method of section **5** can be applied immediately. One has only to substitute the Fourier coefficient $(1/k)(f_b + F_k) = (1/k)(\bar{F}_k)$ into derivation of section **5** for the Fourier coefficient $(1/k)(f_b)$.

One obtains for, the stress f_s in the sth stringer,

$$f_s = -\frac{4}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} (f_b + F_k) G_{ks} \sin \frac{k\pi x}{2l} \qquad \text{(67)}$$

the shear stress q_s in the sheet between the $(s + 1)$th and sth stringers,

$$q_s/(nA/tl) = \sum_{k \text{ odd}}^{\infty} (f_b + F_k) N_{ks} \cos \frac{k\pi x}{2l} \qquad \text{(68)}$$

the shear stress q in the sheet adjacent to the edge member,

$$q/(nA/tl) = \sum_{k \text{ odd}}^{\infty} (f_b + F_k) C_k \cos \frac{k\pi x}{2l} \qquad \text{(69)}$$

the moment M carried by the panel,

$$M = \frac{2}{\pi} n(n + 1) bA \sum_{k \text{ odd}}^{\infty} \frac{1}{k} (f_b + F_k) C_k \sin \frac{k\pi x}{2l} \qquad \text{(70)}$$

where G_{ks}, N_{ks} and C_k are defined by equations (49), (55) and (56).

The deflections v are given by

$$\left.\begin{aligned} Ev &= \frac{32l^2}{\pi^3 w} \sum_{k \text{ odd}}^{\infty} \frac{1}{k^3} (f_b + F_k) \left((-1)^{(k-1)/2} - \sin \frac{k\pi x}{2l} \right) \\ &= f_b \frac{(l - x)^2}{w} + \frac{32l^2}{\pi^3 w} \sum_{k \text{ odd}}^{\infty} \frac{1}{k^3} F_k \left((-1)^{(k-1)/2} - \sin \frac{k\pi x}{2l} \right) \end{aligned}\right\} \qquad \text{(71)}$$

The areas $B(x)$ of the edge members for a given edge-stress distribution $\pm f$ may be found from equation (32) noting that the condition $(\bar{M} - M)/f \geqslant 0$ must be satisfied throughout.

6.1. *Special case : Stringer-sheet, when number of stringers is infinite.*—In the limiting case of stringer-sheet one has only to substitute $\bar{G}_{ky}$, $\bar{N}_{ky}$ and $\bar{C}_k$ into formulae (67) to (71) for G_{ks}, N_{ks} and C_k. The functions $\bar{G}_{ky}$, $\bar{N}_{ky}$ and $\bar{C}_k$ are defined by equations (62).

7. *Parallel Panel with Constant-Area Edge Members under Concentrated Antisymmetrical End Loads.*—Consider a parallel panel stiffened with n stringers. Each edge member has a constant area B and the area of one stringer plus effective sheet is A. Two antisymmetrical concentrated end loads P are applied to the edge members at the free end of the panel (*see* Fig. 2).

The unknown antisymmetrical edge-stress distribution $\pm f$ can be represented by formulae (66a) and (66b) of section **6**.

$$f = f_b + \frac{4}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} F_k \sin \frac{k\pi x}{2l} \qquad \text{(66a)}$$

or

$$f = \frac{4}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} \bar{F}_k \sin \frac{k\pi x}{2l} \qquad \text{(66b)}$$

where $\bar{F}_k = f_b + F_k$ and $f_b = P/B$.

The equilibrium condition can be written in two alternative forms, namely

$$B\frac{df}{dx} + qt = 0 \qquad \text{(72a)}$$

$$M + Bwf \equiv Bwf_b = Pw\,. \qquad \text{(72b)}$$

The first equation refers to the equilibrium condition of an element dx of the edge member. The second equation indicates the equilibrium condition of the moments at a cross-section x. The equivalence of the two forms appears from the fact that (a) can be obtained by differentiating (b). Remembering the remark in section 6 one sees that while both (66a) and (66b) can be substituted into (72b) only (66a) is appropriate for (72a).

Substituting equation (69) into (72a) one obtains

$$B\frac{2}{l}\sum_{k\text{ odd}}^{\infty} F_k \cos\frac{k\pi x}{2l} + \frac{nA}{l}\sum_{k\text{ odd}}^{\infty}(f_b + F_k)C_k \cos\frac{k\pi x}{2l} = 0\,.$$

This equation can only be satisfied if

$$F_k = -f_b\frac{C_k}{\alpha + C_k} \qquad \text{(73)}$$

where

$$\alpha = \frac{2B}{nA} = \frac{\text{total area of edge members}}{\text{total area of stringers plus effective sheet}} \qquad \text{(74)}$$

and C_k is defined by equation (56). Formula (72b), of course, leads to the same result. A similar formula was derived in R. & M. 1969[5] for the symmetrical loading case of a parallel panel with constant-area edge members.

The substitution of formula (73) into the appropriate formulae of section 6 yields for: the edge stress f,

$$\frac{f}{f_b} = 1 - \frac{4}{\pi}\sum_{k\text{ odd}}^{\infty}\frac{1}{k}\frac{C_k}{\alpha + C_k}\sin\frac{k\pi x}{2l} \qquad \text{(75)}$$

the stress f_s in the sth stringer,

$$\frac{f_s}{f_b} = -\frac{4}{\pi}\alpha\sum_{k\text{ odd}}^{\infty}\frac{1}{k}\frac{G_{ks}}{\alpha + C_k}\sin\frac{k\pi x}{2l} \qquad \text{(76)}$$

the shear stress q_s in the sheet between the sth and $(s + 1)$th stringers,

$$\frac{q_s/f_b}{nA/tl} = \alpha\sum_{k\text{ odd}}^{\infty}\frac{N_{ks}}{\alpha + C_k}\cos\frac{k\pi x}{2l} \qquad \text{(77)}$$

the shear stress q in the sheet adjacent to the edge member,

$$\frac{q/f_b}{nA/tl} = \alpha\sum_{k\text{ odd}}^{\infty}\frac{C_k}{\alpha + C_k}\cos\frac{k\pi x}{2l} \qquad \text{(78)}$$

the moment M in the panel,

$$M/M' = \frac{12}{\pi}\frac{n+1}{n-1}\alpha\sum_{k\text{ odd}}^{\infty}\frac{1}{k}\frac{C_k}{\alpha + C_k}\sin\frac{k\pi x}{2l} \qquad \text{(79)}$$

where $$M' = \frac{n(n-1)}{6}BAf_b$$

the deflections v,

$$Ev = \frac{32l^2}{\pi^3 w}f_b\alpha\sum_{k\text{ odd}}^{\infty}\frac{1}{k^3}\frac{1}{\alpha + C_k}\left((-1)^{(k-1)/2} - \sin\frac{k\pi x}{2l}\right). \qquad \text{(80)}$$

Formulae (75) and (79) are quickly convergent, but formulae (76), (77) and (78) are less satisfactory in this respect. For computational reasons it is therefore preferable to separate once again the contribution of the constant edge stress f_b as given by the equations of section 4.

One obtains for: the stress f_s,

$$\frac{f_s}{f_b} = -\frac{2s}{n+1} + \frac{2}{n+1} \sum_{i \text{ even}} (-1)^{1+i/2} \cot \frac{\pi i}{2(n+1)} \sin \frac{\pi i s}{n+1} \frac{\cosh \mu_i(l-x)}{\cosh \mu_i l}$$
$$+ \frac{4}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} \frac{C_k G_{ks}}{\alpha + C_k} \sin \frac{k\pi x}{2l} \quad \ldots \quad (81)$$

the shear stress q_s

$$\frac{q_s/f_b}{nA/tl} = \frac{\mu l}{n} \sum_{i \text{ even}} (-1)^{i/2} \cos \frac{\pi i(2s+1)}{2(n+1)} \cot \frac{\pi i}{2(n+1)} \frac{\sinh \mu_i(l-x)}{\cosh \mu_i l}$$
$$- \sum_{k \text{ odd}}^{\infty} \frac{C_k}{\alpha + C_k} N_{ks} \cos \frac{k\pi x}{2l} \quad \ldots \quad (82)$$

the shear stress q at the edge,

$$\frac{q/f_b}{nA/tl} = \frac{\mu l}{n} \sum_{i \text{ even}} \cot \frac{\pi i}{2(n+1)} \cos \frac{\pi i}{2(n+1)} \frac{\sinh \mu_i(l-x)}{\cosh \mu_i l}$$
$$- \sum_{k \text{ odd}}^{\infty} \frac{C_k^2}{\alpha + C_k} \cos \frac{k\pi x}{2l} \quad \ldots \quad (83)$$

the moment M carried by the panel,

$$\frac{M}{M'} = 1 - \frac{6}{n(n-1)} \sum_{i \text{ even}} \cot^2 \frac{\pi i}{2(n+1)} \frac{\cosh \mu_i(l-x)}{\cosh \mu_i l}$$
$$- \frac{12}{\pi} \frac{n+1}{n-1} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} \frac{C_k^2}{\alpha + C_k} \sin \frac{k\pi x}{2l} \quad \ldots \quad (84)$$

and the deflections v,

$$Ev = f_b \frac{(l-x)^2}{w} - \frac{32 l^2}{\pi^3 w} f_b \sum_{k \text{ odd}}^{\infty} \frac{1}{k^3} \frac{C_k}{\alpha + C_k} \left((-1)^{(k-1)/2} - \sin \frac{k\pi x}{2l} \right). \quad \ldots \quad (85)$$

The infinite series in (81) to (83) converge rapidly, quicker than $1/k^2$. Series (84) is very rapidly convergent, quicker than $1/k^3$.

For a constant total stringer plus effective sheet area and constant values of μl and α the maximum shear stress at $x = 0$ increases with the number of stringers. This increase is small in the practical range of n (say 10 to 20). The evaluation of (83) is not too laborious since the finite series has already been evaluated for $n = 5$, 10 and 30 and various values of μl (Figs. 4, 5 and 6, *see* also section 7).

For a constant values of μl, α and $2s/(n+1) = 2y/w$ the value of f_s/f_b varies only slightly with n provided that $n > 7$. The same applies also to the ratio M/M'.

7.1. *Alternative formulae.*—For certain applications it is preferable to use the ratio f_s/f_{se} instead of f_s/f_b.

One finds readily that

$$f_{se} = -f_b \frac{\alpha}{\alpha + \{(n-1)/3(n+1)\}} \frac{2s}{n+1}.$$

Hence

$$\frac{f_s}{f_{se}} = -\left\{\frac{\alpha + \{(n-1)/3(n-1)\}}{\alpha}\right\} \frac{n+1}{2s} \frac{f_s}{f_b} \quad \ldots \quad \ldots \quad \ldots \quad (86)$$

where f_s/f_b is given by equation (76).

The engineering theory of bending indicates a moment M_e in the panel given by

$$M_e = \frac{\alpha}{\alpha + \{(n-1)/3(n+1)\}} \frac{n(n-1)}{6} bAf_b \, .$$

Hence

$$\frac{M}{M_e} = \left\{\frac{\alpha + \{(n-1)/3(n+1)\}}{\alpha}\right\} \frac{M}{M'} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (87)$$

where M/M' is given by equation (79).

For $\mu \to \infty$, $C_k \to (n-1)/3(n+1)$ and $G_{ks} \to 2s/(n+1)$ and therefore $f_s \to f_{se}$ and $M \to M_e$.

The physical interpretation of the stress and moment equations of this section is facilitated by noting that

$$\frac{\alpha}{\alpha + \{(n-1)/3(n+1)\}} = \frac{\frac{1}{2}B(n+1)^2b^2}{\{\frac{1}{2}B(n+1)^2b^2\} + \{\frac{1}{12}An(n-1)(n+1)b^2\}} = \frac{I_f}{I_f + I_p} \quad \ldots \quad (88)$$

where I_f is moment of inertia of edge members (flanges) about the x-axis.

and I_p is moment of inertia of panel about x-axis.

Furthermore it follows that

$$\lim_{\mu \to \infty} \frac{\alpha}{\alpha + C_k} = \frac{\alpha}{\alpha + \{(n-1)/3(n+1)\}} = \frac{I_f}{I_f + I_p} \, . \quad \ldots \quad \ldots \quad \ldots \quad (89)$$

For graphical representations of q it is usually preferable to apply formula (28),

$$(q/f_b)\sqrt{(Ebt/GA)} = \frac{q/f_b}{nA/tl} \frac{2n}{n+1} \frac{1}{\mu l} \, . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

In Figs. 8, 9 and 10 $(q/f_b)\sqrt{(Ebt/GA)}$ is plotted against x/l for $n = 5$, 10 and 30 and various values of μl and α. These diagrams should be compared with Figs. 8, 9 and 10 of R. & M. 2038[6] corresponding to the symmetrical loading case.

Fig. 11 shows the variation of

$$M/Bwf_b = [(n-1)/\{3(n+1)\alpha\}](M/M') \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (79)$$

with x/l for various values of μl and α and $n = 10$. At $x = l$ the ratio M/Bwf_b varies only slightly with μl if $\mu l > 1$. It follows that at the built-in end the moment carried by the panel is nearly equal to that indicated by the engineering theory of bending.

7.2. *Special case: stringer-sheet, when number of stringers is infinite.*—In the limiting case of stringer-sheet one has only to substitute $\bar{G}_{ky}$, $\bar{N}_{ky}$, and $\bar{C}_k$ into formulae (75) to (80) for G_{ks}, N_{ks}, and C_k. The functions $\bar{G}_{ky}$, $\bar{N}_{ky}$ and $\bar{C}_k$ are defined by equations (62). It is again easy to separate the contribution of f_b by taking into account the formulae section 4.1.

8. *Parallel Panel with Constant-Area Edge Members under Arbitrary Antisymmetrical Edge Loads.*—Consider a parallel panel stiffened with n stringers and with constant-area edge members. Arbitrary antisymmetrical edge loads $\pm S$ are applied to the edge members (*see* Fig. 2). Taking

into account that the edge stress at the free end must be zero the unknown antisymmetrical edge-stress distribution $\pm f$ can be represented by the Fourier sine series

$$f = \frac{4}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} F_k \sin \frac{k\pi x}{2l} \,. \qquad \text{(90)}$$

The given edge-load distribution $\pm S(x)$ can be expanded in a Fourier cosine series (*see* also R. & M. 2038[6])

$$S = \bar{q}t \sum_{k \text{ odd}}^{\infty} S_k \cos \frac{k\pi x}{2l} \qquad \text{(91)}$$

where $\bar{q}$ is a parameter expressing the magnitude of the edge-load system.

The equilibrium condition can be written in two alternative forms, namely

$$B \frac{df}{dx} + qt = S \qquad \text{(92a)}$$

$$M + Bwf = w \int_0^x S \, dx \,. \qquad \text{(92b)}$$

By substitution of formula (90) for f, (91) for S and the appropriate part of (70) for M into (92b) one obtains

$$\frac{2}{\pi} n(n+1)bA \sum_{k \text{ odd}}^{\infty} \frac{1}{k} F_k C_k \sin \frac{k\pi x}{2l} + \frac{4}{\pi}(n+1)bB \sum_{k \text{ odd}}^{\infty} \frac{1}{k} F_k \sin \frac{k\pi x}{2l} = \frac{2}{\pi} \bar{q}(n+1)btl \sum_{k \text{ odd}}^{\infty} \frac{1}{k} S_k \sin \frac{k\pi x}{2l} \,.$$

Hence

$$F_k \frac{nA}{tl} = \bar{q} \frac{S_k}{\alpha + C_k} \qquad \text{(93)}$$

where α is given by equation (74).

Substituting equation (93) into the appropriate parts of (90) and (67) to (70) it follows that: the edge stress $\pm f$,

$$\frac{f}{\bar{q}} \frac{nA}{tl} = \frac{4}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} \frac{S_k}{\alpha + C_k} \sin \frac{k\pi x}{2l} \qquad \text{(94)}$$

the stress f_s in the sth stringer,

$$\frac{f_s}{\bar{q}} \frac{nA}{tl} = -\frac{4}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} \frac{S_k}{\alpha + C_k} G_{ks} \sin \frac{k\pi x}{2l} \qquad \text{(95)}$$

the shear stress q_s,

$$\frac{q_s}{\bar{q}} = \sum_{k \text{ odd}}^{\infty} \frac{S_k}{\alpha + C_k} N_{ks} \cos \frac{k\pi x}{2l} \qquad \text{(96)}$$

the shear stress q at the edge,

$$\frac{q}{\bar{q}} = \sum_{k \text{ odd}}^{\infty} \frac{S_k}{\alpha + C_k} C_k \cos \frac{k\pi x}{2l} \qquad \text{(97)}$$

the moment M in the panel,

$$\left.\begin{aligned} \frac{M}{M''} &= \frac{2}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} \frac{S_k}{\alpha + C_k} C_k \sin \frac{k\pi x}{2l} \\ \text{where} \quad M'' &= (n+1)btl\bar{q} \end{aligned}\right\} \qquad \text{(98)}$$

Equation (95) fulfills the boundary conditions $df_s/dx = 0$ at $x = l$. For $\mu \to \infty$ the stress distribution across the panel approaches the stress distribution given by the engineering theory of bending. The proof is exactly the same as in section 7.1. If edge loads are applied to the edge members up to the built-in end then $(df/dx)_{x=l} \neq 0$. If, however, the edge loads are discontinued before the built-in end, $(df/dx)_{x=l} = 0$. With decreasing α, q approaches $\bar{q}$.

The deflections v are given by

$$Ev(nA/tl) = \frac{32l^2}{\pi^3 w} \bar{q} \sum_{k \text{ odd}}^{\infty} \frac{1}{k^3} \frac{S_k}{\alpha + C_k} \left((-1)^{(k-1)/2} - \sin \frac{k\pi x}{2l} \right). \qquad (99)$$

8.1. *Example.*—The physical background of the equations of the last section appears more clearly when dealing with a particular example. As such the case of a constant antisymmetrical edge load $\pm S$ is chosen. It can easily be shown that

$$\left.\begin{aligned} S_k &= \frac{4}{\pi k} (-1)^{(k-1)/2},\ k \text{ odd} \\ \text{and} \qquad S &= \bar{q} t \end{aligned}\right\} \qquad (100)$$

The engineering theory of bending indicates a moment M_e in the panel,

$$M_e(x) = \frac{Sxb(n-1)/3}{\alpha + \{(n-1)/3(n+1)\}}. \qquad (101)$$

When $\alpha \to 0$ formula (101) converges to the obvious result $M_e = Sxb(n+1)$.

The direct stress $f_{se}(x)$ in the sth stringer according to the engineering theory is

$$f_{se}(x) = \frac{-2Sx/nA}{\alpha + \{(n-1)/3(n+1)\}} \frac{2s}{n+1}. \qquad (102)$$

Substituting equations (100) and (102) into (95) it can readily be found that

$$f_s(x)/f_{se}(l) = \frac{n+1}{2s} [\alpha + \{(n-1)/3(n+1)\}] \frac{8}{\pi^2} \sum_{k \text{ odd}}^{\infty} \frac{1}{k^2} \frac{(-1)^{(k-1)/2}}{\alpha + C_k} G_{ks} \sin \frac{k\pi x}{2l} \qquad (103)$$

and

$$f_s(x)/f_{se}(x) = \{f_s/f_{se}(l)\} \frac{l}{x} \qquad (104)$$

where $f_{se}(l)$ is the stress in the sth stringer at the built-in end as indicated by the engineering theory of bending.

The moment $M(x)$ in the panel is

$$M(x)/M_e(l) = \frac{3(n+1)}{n-1} [\alpha + \{(n-1)/3(n+1)\}] \frac{8}{\pi^2} \sum_{k \text{ odd}}^{\infty} \frac{1}{k^2} \frac{(-1)^{(k-1)/2}}{\alpha + C_k} C_k \sin \frac{k\pi x}{2l} \qquad (105)$$

and

$$M(x)/M_e(l) = \{M(x)/M_e(l)\} \frac{l}{x} \qquad (106)$$

8.2. *Special case : stringer-sheet, when number of stringers is infinite.*—One obtains easily that

$$\frac{f}{\bar{q}}\frac{wt_s}{tl} = \frac{4}{\pi}\sum_{k \text{ odd}}^{\infty}\frac{1}{k}\frac{S_k}{\alpha + \bar{C}_k}\sin\frac{k\pi x}{2l} \quad \text{.. .. ..} \quad (107)$$

$$\frac{f_s}{\bar{q}}\frac{wt_s}{tl} = -\frac{4}{\pi}\sum_{k \text{ odd}}^{\infty}\frac{1}{k}\frac{S_k}{\alpha + \bar{C}_k}\bar{G}_{ky}\sin\frac{k\pi x}{2l} \quad \text{.. .. .. ..} \quad (108)$$

$$\frac{q_s}{\bar{q}} = \sum_{k \text{ odd}}^{\infty}\frac{S_k}{\alpha + \bar{C}_k}\bar{N}_{ky}\cos\frac{k\pi x}{2l} \quad \text{.. .. ..} \quad (109)$$

$$\frac{M}{M''} = \frac{2}{\pi}\sum_{k \text{ odd}}^{\infty}\frac{1}{k}\frac{S_k}{\alpha + \bar{C}_k}\bar{C}_k\sin\frac{k\pi x}{2l} \quad \text{.. .. .. ..} \quad (110)$$

where $\quad M'' = wtl\bar{q}$.

The direct stresses computed by either the stringer-sheet or finite-stringer method agree very closely. There is also good agreement in the values of the shear stresses at the edge for a reasonably large number of stringers (say > 10). It may be preferable to use the stringer-sheet method in all cases where edge loads only are applied to the edge members as the corresponding formulae for f_s and q_s are somewhat simpler.

PART III

Analysis of Parallel Panels under Arbitrary Transverse Loads

1. *General Considerations.*—Consider a parallel panel under any arbitrary transverse load system (*see* Fig. 12). The physical assumptions underlying the analysis of this part are the same as in Part II. The basic equations (1), (2) and (3) of Part II, section 2, are, therefore, still valid. Hence the same applies to the differential equations (4) or (7) of the direct-stress distribution. Furthermore the boundary conditions for f_s are also identical to those in Part II, namely, $f_s = 0$ at $x = 0$ and $(df_s/dx) = 0$ at $x = l$. It follows that for a *given* antisymmetrical edge-stress distribution and any transverse loads the direct stresses in the panel must be the same as those calculated in Part II for zero transverse load. But if the direct stresses are the same, then so also are the shear stresses except for an added term, which is constant across the width of the panel. This may be demonstrated in detail as follows.

The shear stresses are defined by equation (1) of Part II

$$q_s = G\left\{\frac{u_{s+1} - u_s}{b} + \frac{dv}{dx}\right\} . \quad \text{.. .. ..} \quad (1)$$

The only difference between the analysis of this part and that given in Part II arises from the consideration of the equilibrium of the shear stresses at a cross-section x of the panel.

Instead of equation (5) one obtains

$$\sum_{-\frac{1}{2}(n-1)}^{+\frac{1}{2}(n-1)} q_s bt = Gt\left\{(u_a - u_b) + (n+1)b\frac{dv}{dx}\right\} = -\bar{Q}(x) \quad \text{.. ..} \quad (111)$$

where $\bar{Q}(x)$ is the resultant shear force of the transverse loads at the section x.

Hence

$$\frac{dv}{dx} = -\frac{\bar{Q}(x)}{Gwt} + \frac{u_b - u_a}{w} \quad \ldots \quad (112)$$

and

$$q_s = \frac{G}{b}\left\{(u_{s+1} - u_s) + \frac{(u_b - u_a)}{n+1}\right\} - \frac{\bar{Q}(x)}{wt}. \quad \ldots \quad (113)$$

It can immediately be seen that the term in brackets of equation (113) is exactly the same as that indicated by equations (1) and (5a) of section 2, Part II, for a panel without transverse load. The additional term $-\bar{Q}(x)/wt$ is the shear stress due to the transverse loads and is constant over each cross-section and may vary only with x. It follows that in case of a *given* antisymmetrical edge-stress distribution the shear stresses in the panel can be found by adding the shear $-\bar{Q}(x)/wt$ to the appropriate formulae of Part II.

At $x = l$, the displacements u must be zero. It follows that

$$\left(\frac{dv}{dx}\right)_{x=l} = -\frac{\bar{Q}(l)}{Gwt} \quad \ldots \quad (114)$$

and

$$(q_s)_{x=l} = -\frac{\bar{Q}(l)}{wt}. \quad \ldots \quad (115)$$

This expression is independent of s. Hence the shear stresses at the built-in end of a panel under transverse loads are constant over the cross-section. This result is entailed by the assumption of a closely spaced system of transverse members infinitely stiff against compression or tension.

Integrating equation (112) one obtains for the deflection $v(x)$

$$v(x) = v_0(x) + \left\{\int_x^l \bar{Q}(\xi)\, d\xi\right\} \Big/ Gwt. \quad \ldots \quad (116)$$

where $v_0(x)$ is the deflection of the panel for the given edge-stress distribution disregarding the transverse loads and is found by the methods of Part II.; the term

$$\left\{\int_x^l \bar{Q}(\xi)\, d\xi\right\} \Big/ Gwt$$

represents the ' deflection due to shear '.

Let $\bar{M}$ be the moment of the transverse loads at a section x and M the moment carried by the panel at x for an antisymmetrical edge stress $\pm f$. The latter moment, of course, is the one found in Part II. The areas $B(x)$ of the edge members are then given by

$$B(x) = (\bar{M} - M)/wf. \quad \ldots \quad (117)$$

Formula (117) is identical with (72b). It follows that the areas of the edge members for a given panel with a given antisymmetrical edge stress depend only on the *value* of the external moment $\bar{M}$ irrespectively whether it be produced by transverse, edge loads or a combination of the two.

It is now obvious from equations (72b) or (117) and the preceding arguments that exactly the same results can be deduced if the areas of the edge members are given and the edge-stress distribution is initially unknown.

An arbitrary shear force diagram $\bar{Q}$ can be represented by the Fourier series

$$\bar{Q}(x) = \bar{q}tw \sum_{k \text{ odd}}^{\infty} Q_k \cos\frac{k\pi x}{2l} \quad \ldots \quad (118)$$

where $\bar{q}$ is a parameter expressing the magnitude of the transverse-load system. The corresponding moment $\bar{M}(x)$ follows by integration,

$$\bar{M}(x) = \frac{2}{\pi} \bar{q}tlw \sum_{k \text{ odd}}^{\infty} \frac{1}{k} Q_k \sin \frac{k\pi x}{2l} \quad \ldots \quad (119)$$

An arbitrary edge-load distribution was represented in section 8, Part II

$$S = \bar{q}t \sum_{k \text{ odd}}^{\infty} S_k \cos \frac{k\pi x}{2l} \quad \ldots \quad (91)$$

and the moment which these edge loads apply to the panel is (*see* also equation (92b))

$$\bar{M} = w \int_0^x S \, dx = \frac{2}{\pi} \bar{q}tlw \sum_{k \text{ odd}}^{\infty} \frac{1}{k} S_k \sin \frac{k\pi x}{2l} \quad \ldots \quad (120)$$

The load systems (118) and (91), apart from the additional term $-\bar{Q}(x)/wt$ in the shear-stress distribution of the former case, are identical in their effect upon the stress distribution in the panel and edge members if $S_k = Q_k$. Thus a single transverse load $P = \bar{q}tw$ corresponds to a uniform edge load $S = \bar{q}t$ and a uniformly distributed transverse load $p = \bar{q}tw/l$ corresponds to a linearly increasing edge load of a magnitude $S_x = \bar{q}tx/l$, etc.

2. *Parallel Panel with Constant-Area Edge Members under Arbitrary Transverse Loads.*— Consider a parallel panel stiffened with n stringers and with constant-area edge members. Arbitrary transverse loads are applied to the panel. The corresponding shear force diagram $\bar{Q}(x)$ can be represented by the Fourier cosine series (118).

To find the stress distribution in the panel, in accordance with the developments of section 1, one has only to substitute the Fourier coefficients Q_k for S_k into the appropriate formulae of section 8, Part II, and to take into account the additional terms for the shear-stress distribution and deflection as indicated by equations (113) and (116).

One obtains for: the edge stress $\pm f$,

$$\frac{f}{\bar{q}} \frac{nA}{tl} = + \frac{4}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} \frac{Q_k}{\alpha + C_k} \sin \frac{k\pi x}{2l} \quad \ldots \quad (121)$$

the stress f_s in the sth stringer,

$$\frac{f_s}{\bar{q}} \frac{nA}{tl} = - \frac{4}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} \frac{Q_k}{\alpha + C_k} G_{ks} \sin \frac{k\pi x}{2l} \quad \ldots \quad (122)$$

the shear stress q_s,

$$\begin{aligned} \frac{q_s}{\bar{q}} &= \sum_{k \text{ odd}}^{\infty} \frac{Q_k}{\alpha + C_k} N_{ks} \cos \frac{k\pi x}{2l} - \frac{\bar{Q}}{\bar{q}tw} \\ &= \sum_{k \text{ odd}}^{\infty} \frac{Q_k}{\alpha + C_k} (N_{ks} - C_k - \alpha) \cos \frac{k\pi x}{2l} \quad \ldots \quad (123) \end{aligned}$$

the shear stress q at the edge,

$$\frac{q}{\bar{q}} = - \alpha \sum_{k \text{ odd}}^{\infty} \frac{Q_k}{\alpha + C_k} \cos \frac{k\pi x}{2l} \quad \ldots \quad (124)$$

the moment M carried by the panel,

$$\left.\begin{aligned} \frac{M}{M''} &= \frac{2}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} \frac{Q_k}{\alpha + C_k} C_k \sin \frac{k\pi x}{2l} \\ \text{where} \quad M'' &= wtl\bar{q} \end{aligned}\right\} \quad \ldots \quad (125)$$

the deflections $v(x)$,

$$E v \frac{nA}{tl} = \frac{32 l^2}{\pi^3 w} \bar{q} \sum_{k \text{ odd}}^{\infty} \frac{1}{k^3} \frac{Q_k}{\alpha + C_k} \left\{ (-1)^{(k-1)/2} - \sin \frac{k\pi x}{2l} \right\}$$

$$+ \frac{2}{\pi} \frac{E}{G} \frac{nA}{tl} \bar{q} l \sum_{k \text{ odd}}^{\infty} \frac{1}{k} Q_k \left\{ (-1)^{(k-1)/2} - \sin \frac{k\pi x}{2l} \right\} . \quad .. \quad (126)$$

It is now interesting to follow in detail the limiting case of $\mu \to \infty$.

We have

$$\lim_{\mu l \to \infty} C_k = \{(n-1)/3(n+1)\} . \quad .. \quad .. \quad .. \quad .. \quad .. \quad (56a)$$

Hence the limit of the edge stress

$$\lim_{\mu \to \infty} \frac{f}{\bar{q}} \frac{nA}{tl} = \frac{4}{\pi} \frac{1}{\alpha + \{(n-1)/3(n+1)\}} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} Q_k \sin \frac{k\pi x}{2l}$$

$$= \frac{2}{\bar{q} t l w} \frac{1}{\alpha + \{(n-1)/3(n+1)\}} \bar{M}(x)$$

and

$$\lim_{\mu \to \infty} f = \frac{1}{nAw} \frac{1}{\alpha + \{(n-1)/3(n+1)\}} \bar{M}(x)$$

$$= \frac{(n+1)b/2}{\{B(n+1)^2 b^2/2\} + \{n(n-1)(n+1)b^2 A/12\}} \bar{M}(x)$$

$$= \frac{w/2}{I_f + I_p} \bar{M}(x) = f_e$$

where f_e is the edge stress given by the engineering theory of bending.

Similarly

$$\lim_{\mu \to \infty} f_s = f_e \frac{2s}{n+1} = f_{se} .$$

Furthermore,

$$\lim_{\mu \to \infty} M = \bar{M} \frac{I_p}{I_f + I_p} ,$$

which is the result predicted by the engineering theory of bending. For shear stresses in the limiting case we note that

$$\lim_{\mu l \to \infty} (N_{ks} - C_k) = -\frac{2}{n}(n+1) - \frac{2}{n(n+1)} \left\{ \left(\frac{n-1}{2} \right) - s \right\} \left\{ \left(\frac{n-1}{2} \right) + (s+1) \right\} .$$

Hence

$$\lim_{\mu l \to \infty} \frac{N_{ks} - C_k - \alpha}{\alpha + C_k} = \frac{-\frac{2}{n}(n+1) - \frac{2}{n(n+1)} \left\{ \left(\frac{n-1}{2} \right) - s \right\} \left\{ \left(\frac{n-1}{2} \right) + (s+1) \right\} - \alpha}{\alpha + \{(n-1)/3(n+1)\}} .$$

Multiplying numerator and denominator by $\frac{1}{4} n A b^2 (n+1)^2$ one obtains for the limit

$$\lim_{\mu l \to \infty} \frac{N_{ks} - C_k - \alpha}{\alpha + C_k} = -w \frac{\frac{Ab}{2} \left\{ \left(\frac{n-1}{2} \right) - s \right\} \left\{ \left(\frac{n-1}{2} \right) + (s+1) \right\} + Bb \left(\frac{n+1}{2} \right)}{I_f + I_p} .$$

It can readily be seen that the numerator of the ratio is the static moment $\bar{S}_s$ about the x axis of the edge member and the stringers $(s+1)$, $(s+2) \ldots \{(n-1)/2\}$.

It follows that

$$\lim_{\mu \to \infty} q_s = - \bar{q}w \frac{\bar{S}_s}{I_f + I_p} \sum_{k \text{ odd}}^{\infty} Q_k \cos \frac{k\pi x}{2l} = - \frac{\bar{Q}(x)\, \bar{S}_s}{(I_f + I_p)t} = q_{se}$$

where q_{se} is the shear stress predicted by the engineering theory of bending.

2.1. *Special case: stringer-sheet, when number of stringers is infinite.*—In the limiting case of stringer-sheet one has to substitute the limiting values $\bar{G}_{ky}$, $\bar{N}_{ky}$ and $\bar{C}_k$ defined by (62) into formulae (121) to (126).

Provided that $n > 7$ it is usually preferable to use for transverse loading cases the stringer-sheet method. The application of the formulae given in this section is straightforward and no particular example need be given.

APPENDIX I

Parallel Stiffened Panel with Symmetrical Edge-Stress Distribution

In this section a collection of formulae will be given for the stress distribution in parallel stiffened panels with *symmetrical* edge-stress distribution. The equations marked with an asterisk have already been derived in R. & M. 1969[5] and 2038[6], the others are new. The derivations of the latter formulae are not given, as they can readily be obtained by the methods indicated in this paper. By combining the results of this section with those given in Parts II and III the stress distribution in a panel with any asymmetric edge-stress can easily be found.

1. *Constant Symmetrical Edge-Stress* $- f$.—Stress f_r in the rth stringer,

$$\frac{f_r}{f} = - \frac{2}{n+1} \sum_{i \text{ odd}} \cot \frac{\pi i}{2(n+1)} \sin \frac{\pi i r}{n+1} \left(1 - \frac{\cosh \mu_i(l-x)}{\cosh \mu_i l}\right) \quad \ldots \quad \ldots \quad \ldots (127)^*$$

for $r = 1$ to n (*see* Fig. 1 for r-system of notation for the stringers).

It can be shown that

$$\frac{2}{n+1} \sum_{i \text{ odd}} \cot \frac{\pi i}{2(n+1)} \sin \frac{\pi i r}{n+1} = 1 \,.$$

Hence

$$\frac{f_r}{f} = - 1 + \frac{2}{n+1} \sum_{i \text{ odd}} \cot \frac{\pi i}{2(n+1)} \sin \frac{\pi i r}{n+1} \frac{\cosh \mu_i(l-x)}{\cosh \mu_i l} \,. \quad \ldots \quad \ldots \quad \ldots (128)$$

The corresponding formula in the s-system of notation is

$$\frac{f_s}{f} = - 1 + \frac{2}{n+1} \sum_{i \text{ odd}} (-1)^{(i-1)/2} \cot \frac{\pi i}{2(n+1)} \cos \frac{\pi i s}{n+1} \frac{\cosh \mu_i(l-x)}{\cosh \mu_i l} \quad \ldots \quad \ldots (129)$$

for $-(n-1)/2 \leqslant s \leqslant +(n-1)/2$.

Average stringer stress f_a:

$$\frac{f_a}{f} = - 1 + \frac{2}{n(n+1)} \sum_{i \text{ odd}} \cot^2 \frac{\pi i}{2(n+1)} \frac{\cosh \mu_i(l-x)}{\cosh \mu_i l} \quad \ldots \quad \ldots \quad \ldots \quad \ldots (130)^*$$

Shear stress q_s in the sheet between the $(s+1)$th and sth stringers:

$$(q_s/f)\sqrt{(Ebt/GA)} = \frac{2}{n+1}\sum_{i\ \mathrm{odd}} (-1)^{(i-1)/2} \cot\frac{\pi i}{2(n+1)} \sin\frac{\pi i(2s+1)}{2(n+1)} \frac{\sinh \mu_i(l-x)}{\cosh \mu_i l} \quad \ldots \ (131)$$

Shear stress q in the sheet adjacent to the edge members:

$$(|q|/f)\sqrt{(Ebt/GA)} = \frac{2}{n+1}\sum_{i\ \mathrm{odd}} \cot\frac{\pi i}{2(n+1)} \cos\frac{\pi i}{2(n+1)} \frac{\sinh \mu_i(l-x)}{\cosh \mu_i l}. \quad \ldots \ (132)^*$$

The shear-stress distribution at a cross-section is antisymmetric about the x-axis.

1.1. *Stringer-sheet when number of stringers is infinite.*—When $n \to \infty$, formulae (129) to (132) become,

$$\frac{f_y}{f} = -1 + \frac{4}{\pi}\sum_{i\ \mathrm{odd}}^{\infty} (-1)^{(i-1)/2} \frac{1}{i} \cos\frac{\pi i y}{w} \frac{\cosh\{\pi i\mu(l-x)/2\}}{\cosh\{\pi i\mu l/2\}} \quad \ldots \ (133)^*$$

$$\frac{f_a}{f} = -1 + \frac{8}{\pi^2}\sum_{i\ \mathrm{odd}}^{\infty} \frac{1}{i^2} \frac{\cosh\{\pi i\mu(l-x)/2\}}{\cosh\{\pi i\mu l/2\}} \quad \ldots \ (134)^*$$

$$(q_y/f(\sqrt{(Ebt/GA)} = \frac{4}{\pi}\sum_{i\ \mathrm{odd}}^{\infty} (-1)^{(i-1)/2} \frac{1}{i} \sin\frac{\pi i y}{w} \frac{\sinh\{\pi i\mu(l-x)/2\}}{\cosh\{\pi i\mu l/2\}} \quad \ldots \ (135)$$

$$(|q|/f)\sqrt{(Ebt/GA)} = \frac{4}{\pi}\sum_{i\ \mathrm{odd}}^{\infty} \frac{1}{i} \frac{\sinh\{\pi i\mu(l-x)/2\}}{\cosh\{\pi i\mu l/2\}}. \quad \ldots \ (136)^*$$

For $l \to \infty$ series (133) and (135) can be summed. One obtains

$$\frac{f_y}{f} = -\frac{2}{\pi}\tan^{-1}[\sinh\{\pi\mu x/2\}/\cos\{\pi y/w\}] \quad \ldots \ (137)$$

$$(q_y/f)\sqrt{(Et/Gt_s)} = \frac{1}{\pi}\ln\left[\frac{\cosh\{\pi\mu x/2\} + \sin\{\pi y/w\}}{\cosh\{\pi\mu x/2\} - \sin\{\pi y/w\}}\right] \quad \ldots \ (138)$$

$$(|q|/f)\sqrt{(Et/Gt_s)} = \frac{2}{\pi}\ln[\coth\{\pi\mu x/4\}] \quad \ldots \ (139)$$

$|q| \to \infty$ as $x \to 0$. Formula (137) has been found very useful for the quick computation of the direct-stress distribution in long panels ($l > 3w$ say).

2. *Arbitrary Edge-Stress Distribution.*—An arbitrary symmetrical edge-stress distribution $-f$ may be represented by

$$f = \frac{4}{\pi}\sum_{k\ \mathrm{odd}}^{\infty} \frac{1}{k} \bar{F}_k \sin\frac{k\pi x}{2l} = \frac{4}{\pi}\sum_{k\ \mathrm{odd}}^{\infty} \frac{1}{k} (f_b + F_k)\sin\frac{k\pi x}{2l} \quad \ldots \ (140)$$

where $-f_b$ is the edge stress at the free end.

For the stress distribution in the panel one obtains

$$f_s = -\frac{4}{\pi}\sum_{k\ \mathrm{odd}}^{\infty} \frac{1}{k}(f_b + F_k)H_{ks}\sin\frac{k\pi x}{2l} \quad \ldots \ (141)$$

$$f_a = -\frac{4}{\pi}\sum_{k\ \mathrm{odd}}^{\infty} \frac{1}{k}(f_b + F_k)T_k\sin\frac{k\pi x}{2l} \quad \ldots \ (142)^*$$

$$\frac{q_s}{nA/tl} = \sum_{k\ \mathrm{odd}}^{\infty} (f_b + F_k)R_{ks}\cos\frac{k\pi x}{2l} \quad \ldots \ (143)$$

$$\frac{|q|}{nA/tl} = \sum_{k\ \mathrm{odd}}^{\infty} (f_b + F_k)T_k\cos\frac{k\pi x}{2l}. \quad \ldots \ (144)^*$$

The functions H_{ks}, T_k and R_{ks} are defined by:

$$\left.\begin{aligned} H_{ks} &= \frac{\cosh\{2\phi_k s\}}{\cosh\{(n+1\phi_k\}}, \quad R_{ks} = \frac{1}{n\sinh\phi_k}\frac{\sinh\{(2s+1)\phi_k\}}{\cosh\{(n+1)\phi_k\}} \\ T_k &= \{R_{ks}\}_{s=(n-1)/2} = \frac{1}{n}\left[\frac{\tanh\{(n+1)\phi_k\}}{\tanh\phi_k} - 1\right] \\ \text{and} \qquad \phi_k &= \sinh^{-1}\left\{\frac{k\pi}{2\mu l}\frac{1}{n+1}\right\} \end{aligned}\right\} \quad \ldots \quad \ldots \quad (145)$$

If the edge stress is constant, $F_k \equiv 0$ and the resulting equations represent the Fourier series expansion of formulae (120) to (132).

2.1. *Stringer-sheet, when number of stringers is infinite.*— When $n \to \infty$ one has only to substitute the limiting values $\bar{H}_{ky}$, $\bar{R}_{ky}$, and $\bar{T}_k$ into equations (141) to (144), where

$$\left.\begin{aligned} \bar{H}_{ky} &= \frac{\cosh\{(2y/w)(k\pi/2\mu l)\}}{\cosh\{k\pi/2\mu l\}} \\ \bar{R}_{ky} &= \frac{2\mu l}{k\pi}\frac{\sinh\{(2y/w)(k\pi/2\mu l)\}}{\cosh\{k\pi/2\mu l\}} \\ \bar{T}_k &= \frac{2\mu l}{k\pi}\tanh\left\{\frac{k\pi}{2\mu l}\right\} \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (146)$$

3. *Parallel Panel with Constant-Area Edge Members under Concentrated Symmetrical End Loads.*—For a parallel panel with constant-area members under concentrated symmetrical end loads P one obtains

$$\frac{f}{f_b} = -1 + \frac{4}{\pi}\sum_{k\,\mathrm{odd}}^{\infty}\frac{1}{k}\frac{T_k}{\alpha + T_k}\sin\frac{k\pi x}{2l} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (147)^*$$

$$\frac{f_s}{f_b} = -\frac{4}{\pi}\alpha\sum_{k\,\mathrm{odd}}^{\infty}\frac{1}{k}\frac{H_{ks}}{\alpha + T_k}\sin\frac{k\pi x}{2l} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (148)$$

$$\frac{f_a}{f_b} = -\frac{4}{\pi}\alpha\sum_{k\,\mathrm{odd}}^{\infty}\frac{1}{k}\frac{T_k}{\alpha + T_k}\sin\frac{k\pi x}{2l} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (149)^*$$

$$\frac{q_s/f_b}{nA/tl} = \alpha\sum_{k\,\mathrm{odd}}^{\infty}\frac{R_{ks}}{\alpha + T_k}\cos\frac{k\pi x}{2l} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (150)$$

$$\frac{|q|/f_b}{nA/tl} = \alpha\sum_{k\,\mathrm{odd}}^{\infty}\frac{T_k}{\alpha + T_k}\cos\frac{k\pi x}{2l} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (151)^*$$

where $-f_b = -P/B$ is the edge stress at the free end and H_{ks}, R_{ks}, and T_k are defined by equations (145).

For computational reasons it is preferable to separate the influence of a constant edge-stress $-f_b$ by taking into account formulae (129) to (132). One obtains

$$\frac{f_s}{f_b} = -1 + \frac{2}{(n+1)} \sum_{i \text{ odd}} (-1)^{(i-1)/2} \cot \frac{\pi i}{2(n+1)} \cos \frac{\pi i s}{(n+1)} \frac{\cosh \mu_i(l-x)}{\cosh \mu_i l}$$
$$+ \frac{4}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} \frac{T_k}{\alpha + T_k} H_{ks} \sin \frac{k\pi x}{2l} \quad \ldots \quad (152)$$

$$\frac{f_a}{f_b} = -1 + \frac{2}{n(n+1)} \sum_{i \text{ odd}} \cot^2 \frac{\pi i}{2(n+1)} \frac{\cosh \mu_i(l-x)}{\cosh \mu_i l}$$
$$+ \frac{4}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} \frac{T_k^2}{\alpha + T_k} \sin \frac{k\pi x}{2l} \quad \ldots \quad (153)^*$$

$$\frac{q_s/f_b}{nA/tl} = \frac{\mu l}{n} \sum_{i \text{ odd}} (-1)^{(i-1)/2} \cot \frac{\pi i}{2(n+1)} \sin \frac{\pi i(2s+1)}{2(n+1)} \frac{\sinh \mu_i(l-x)}{\cosh \mu_i l}$$
$$- \sum_{k \text{ odd}}^{\infty} \frac{T_k}{\alpha + T_k} R_{ks} \cos \frac{k\pi x}{2l} \quad \ldots \quad (154)$$

$$\frac{|q|/f_b}{nA/tl} = \frac{\mu l}{n} \sum_{i \text{ odd}} \cot \frac{\pi i}{2(n+1)} \cos \frac{\pi i}{2(n+1)} \frac{\sinh \mu_i(l-x)}{\cosh \mu_i l}$$
$$- \sum_{k \text{ odd}}^{\infty} \frac{T_k^2}{\alpha + T_k} \cos \frac{k\pi x}{2l} . \quad \ldots \quad (155)^*$$

For the limiting case of a stringer-sheet, formulae (146) have to be substituted.

4. *Parallel Panel with Constant-Area Edge Members under Arbitrary Symmetrical Edge Loads.*— An arbitrary symmetrical edge-load distribution $S(x)$ may be represented by

$$S = \bar{q}t \sum_{k \text{ odd}}^{\infty} S_k \cos \frac{k\pi x}{2l} . \quad \ldots \quad (91)^*$$

The corresponding stresses in edge members and panel are

$$\frac{f}{\bar{q}} \frac{nA}{tl} = -\frac{4}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} \frac{S_k}{\alpha + T_k} \sin \frac{k\pi x}{2l} \quad \ldots \quad (156)^*$$

$$\frac{f_s}{\bar{q}} \frac{nA}{tl} = -\frac{4}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} \frac{S_k}{\alpha + T_k} H_{ks} \sin \frac{k\pi x}{2l} \quad \ldots \quad (157)$$

$$\frac{f_a}{\bar{q}} \frac{nA}{tl} = -\frac{4}{\pi} \sum_{k \text{ odd}}^{\infty} \frac{1}{k} \frac{S_k}{\alpha + T_k} T_k \sin \frac{k\pi x}{2l} \quad \ldots \quad (158)^*$$

$$\frac{q_s}{\bar{q}} = \sum_{k \text{ odd}}^{\infty} \frac{S_k}{\alpha + T_k} R_{ks} \cos \frac{k\pi x}{2l} \quad \ldots \quad (159)$$

$$\frac{|q|}{\bar{q}} = \sum_{k \text{ odd}}^{\infty} \frac{S_k}{\alpha + T_k} T_k \cos \frac{k\pi x}{2l} . \quad \ldots \quad (160)^*$$

APPENDIX II

An Alternative Approach to Diffusion and Shear Lag in Parallel Panels

Introduction.—The series of reports R. & M. 1969[5] and 2038[6] and the present paper give a fairly complete analysis of diffusion and shear lag in flat parallel panels under very general loading conditions. It is hence not inappropriate to attempt to consider now, critically, the methods used in the solution of the problems.

The first interesting conclusion one would probably draw is that the formulae for diffusion in panels under constant or linearly varying edge stress are diametrically opposed to those for panels with constant boom areas. Thus, whilst in the first case the cross-sectional distribution of stresses is described by trigonometrical functions and the longitudinal variation by hyperbolic functions, exactly the opposite applies to the second type of problem. A further revealing difference is that in the formulae corresponding to the first group it is always possible, and in fact appears as a natural step, to separate from the series for the direct stresses a constant stress (for symmetrical cases) or a linearly varying stress (for antisymmetrical cases) but that this is not so readily achieved in the second group.

These observations should be sufficient to show that the analysis for panels with a given edge stress is more logical and attractive than for the panels with constant boom areas. It is, after all, a well-known characteristic property of a diffusion phenomenon in a semi-infinite region or strip that the dying-out process is expressed by exponental functions of the type $e^{-\mu x}$ which indicates that the longitudinal variation of the stresses in a uniform panel should be expressed by hyperbolic functions. Also the splitting-up of the formulae in a so-called engineers' theory term and an additional series expansion will appeal to the physical instinct of most structural analysts.

The above arguments convinced the author that an alternative analysis of parallel panels with given boom areas should be sought. The results of this attempt are given in this Appendix. The method consists in all loading cases of finding first the simple engineers' theory stress system which, at every station, is in equilibrium with the applied external load. Then the difference between the true stresses and these engineers' theory stresses must be obviously self-equilibrating or self-balancing and the main task of the analysis is to find the expression for this stress-system. Note that the *raison d'être* of these self-equilibrating stress systems is not only to satisfy the boundary conditions but also to contribute, in general, to the elastic compatibility of the total stresses. The latter may be necessary if the engineers' theory stresses are not by themselves elastically compatible. In the author's opinion the new analysis is preferable to the old one and the series expansions are very quickly convergent. One drawback, but probably the only one, of the new method is that it involves the solution of transcendental equations. A great advantage of the present approach is that it allows one to derive, without undue complications, the differential equations when the thicknesses and boom areas vary similarly lengthwise. Note that the direct and shear-stress-carrying thicknesses may vary independently lengthwise but the variation of the boom areas must be the same as that of the direct-stress-carrying thickness. The investigation has been restricted here to stringer-sheet panels but the extension to a finite stringer-panel would present no difficulties.

It is believed that the new analysis has considerable potentialities for the solution of diffusion and shear-lag problems in tubular cylindrical or conical structures. It should be pointed out that the mathematics of this Appendix could have been made more rigorous and concise by the use of the Sturm-Liouville Theorem for eigen-values and eigen-functions. But it was thought preferable to give here a discussion mainly in physical terms and to avoid mathematical complications and terminologies.

The notation and signs of this Appendix are in some respects different from those of the main report. Figs. 13 and 14 should be sufficient in explaining the differences. For simplicity of printing the symbol sn ω is used to denote

$$\text{sn}\ \omega = \sin \omega / \omega$$

There should be no danger of confusing it with a Jacobian sine. The suffixes $+$ and $-$ are used to denote values of a function at the two edges $y = + w/2$ and $y = - w/2$ respectively. The numbering of the equations starts at (1) again.

1. *Basic Equations.*—Consider a flat stringer-sheet combination symmetrical about the Ox axis with effective thicknesses t_s', t' in direct and shear stress respectively. t_s' and t' may vary both with x and y. If one denotes the direct and shear flows* by N_x and N_{xy} the following equilibrium conditions in the x direction may immediately be written down from a perusal of Fig. 13a.

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \qquad (1)$$

where,

$$N_x = f t_s', \qquad N_{xy} = q t' . \qquad (2)$$

Let the panel be bounded by booms (flange) of area $B_{(x)}'$ parallel to the x-axis at $y = \pm w/2$. Then the equilibrium of an element dx of the boom at $y = + w/2$ yields

$$\frac{dP_+}{dx} - (N_{xy})_{y=w/2} - S = 0 \qquad (3)$$

where P_+ is the boom load at $y = w/2$ and S the edge load per unit length (*see* Fig. 13 for positive signs, etc.). A similar equation may be found for the boom at $y = - w/2$.

The usual stress-strain relations are

$$f = E\frac{\partial u}{\partial x}, \qquad q = G\left\{\frac{\partial u}{\partial y} + \frac{dv}{dx}\right\} \qquad (4)$$

Eliminating u from equations (4) and using equations (2) one obtains the compatibility equation

$$\frac{1}{E}\frac{\partial}{\partial y}\left(\frac{N_x}{t_s'}\right) - \frac{1}{G}\frac{\partial}{\partial x}\left(\frac{N_{xy}}{t'}\right) = -\frac{d^2v}{dx^2} \qquad (5)$$

or

$$\frac{1}{E}\frac{\partial^2}{\partial y^2}\left(\frac{N_x}{t_s'}\right) - \frac{1}{G}\frac{\partial^2}{\partial x\,\partial y}\left(\frac{N_{xy}}{t'}\right) = 0 . \qquad (5a)$$

It will now be assumed that t_s' and t' vary only in the x-direction. Thus

$$t_s' = t_s\phi_s$$
$$t' = t\phi \qquad (6)$$

where t_s and t are constant and ϕ_s and ϕ are non-dimensional functions of x. For a panel with booms of cross-section $B_{(x)}'$ it will also be assumed that

$$B' = B\phi_s. \qquad (6a)$$

Laws (6) and (6a) will be taken to apply throughout this Appendix. For convenience t_s and t are taken as the actual thicknesses at $x = 0$. Thus, the boundary conditions for ϕ_s and ϕ are

$$\phi_s(0) = \phi(0) = 1 . \qquad (6b)$$

Using equations (1) and (6) in equation (5a) one derives the ultimate form of the compatibility relation as it will be used here,

$$\frac{1}{E\phi_s t_s}\frac{\partial^2 N_x}{\partial y^2} + \frac{1}{Gt}\frac{\partial}{\partial x}\left(\frac{1}{\phi}\frac{\partial N_x}{\partial x}\right) = 0 \qquad (7)$$

which is a partial differential equation in the sole unknown N_x. Note that it is valid in the region

$$l \geqslant x \geqslant 0 \text{ and } w/2 > y > - w/2 . \qquad (8)$$

* The direct flows N_y are ignored in this presentation.

If the panel is unloaded at $x = 0$

$$N_x = 0 \quad \ldots \quad (9)$$

and if the panel is built-in at $x = l$ one finds from (1) using (2), (4) and (6)

$$\left(\frac{\partial N_x}{\partial x}\right)_{x=l} = 0 \,. \quad \ldots \quad (9a)$$

At the edges $y = \pm w/2$ the strains in panel and booms must obviously be equal. Thus, if the boom and the direct-stress-carrying material of the panel are of the same material one obtains

$$P_{\pm} = \frac{B}{t_s} N_{x\pm} \,. \quad \ldots \quad (10)$$

Having derived N_x from (7) and the appropriate boundary conditions, N_{xy} is found from (see equation (1))

$$N_{xy} = -\int_0^y \frac{\partial N_x}{\partial x}\, dy + N_{xy0} \quad \ldots \quad (11)$$

where N_{xy0} is the shear flow at $y = 0$ and can be determined from the equilibrium in the y-direction

$$\int_{-w/2}^{+w/2} N_{xy}\, dy = \bar{Q} \quad \ldots \quad (12)$$

(*see* Fig. 13 for signs).

The shear stress (and shear flow) at a built-in end, say $x = l$, must be constant over the width. This follows immediately from (4) and $\left(\frac{\partial u}{\partial y}\right)_{x=l} = 0$. Thus,

$$(N_{xy})_{x=l} = \frac{\bar{Q}_{x=l}}{w} \,. \quad \ldots \quad (12a)$$

Finally the deflection u may be found from

$$\frac{d^2 v}{dx^2} = -\frac{f_{+1} - f_{-1}}{Ew} + \frac{1}{Gt}\frac{d}{dx}\left(\frac{\bar{Q}}{\phi}\right) . \quad \ldots \quad (13)$$

2. *Engineers' Theory Stress Systems.*—The end and edge loads on a parallel panel may always be analysed in a symmetrical system and an antisymmetrical system (*see* Figs. 13b and 13c).

Assume first that the loading consists solely of end loads P_0 and that the panel is very long ($l/w \geqslant 1$). It is obvious that for symmetrical end loads, P_0 the stress distribution for large x, must approach a constant value both with respect to x and y. Similarly for an antisymmetrical couple $\pm P_0$ the stress distribution at a cross-section approaches asymptotically the linearly varying stress of the simple engineers' theory of bending (Euler-Bernouilli assumption). Thus, the asymptotic flow distributions and boom loads are :

(*a*) Symmetrical case

$$\left.\begin{aligned} N_{x\mathrm{E}} &= \frac{2P_0}{wt_s' + 2B'}\, t_s' = \frac{2P_0}{w(1+\alpha)}\,; \quad P_{\mathrm{E}} = \frac{2P_0}{wt_s' + 2B'}\, B' = \frac{P_0 \alpha}{1+\alpha} \\ N_{xy\,\mathrm{E}} &= 0 \end{aligned}\right\} \quad \ldots \quad (14)$$

(*b*) Antisymmetrical case

$$\left.\begin{aligned} N_{x\mathrm{E}} &= \frac{6P_0}{wt_s' + 6B'}\, t_s' \frac{2y}{w} = \frac{6P_0}{w(1+3\alpha)} \frac{2y}{w}\,; \quad P_{\mathrm{E}\,\pm} = \pm \frac{3P_0\alpha}{1+3\alpha} \\ N_{xy\,\mathrm{E}} &= 0 \end{aligned}\right\} \quad \ldots \quad (15)$$

where $\alpha = 2B'/wt_s' = 2B/wt_s \,. \quad \ldots \quad (16)$

The suffix $_E$ will be used throughout this Appendix to indicate the usual simple engineers' theory stress systems, either in direct load or bending. Note that the distributions (14) and (15) do satisfy the compatibility condition (7).

As next let the panel be submitted also to edge loads S (*see* Figs. 14b and 14c). Then it is still possible to equilibrate the applied loads with the direct stress flows and boom loads of (14) or (15) if one substitutes $\bar{P}$ for P_0, where

$$\bar{P} = P_0 + \int_0^x S\,dx. \qquad \qquad (17)$$

But these stress-systems will, in general, violate the compatibility condition (7) since the variation of $\bar{P}$ with x will entail also shear flows. Thus one finds from equations (11) and (12), with $\bar{Q} = 0$, for the

(i) Symmetrical case

$$N_{xy\,E} = -\frac{2S}{wt_s' + 2B'}\,t_s'y = -\frac{S}{1+\alpha}\left(\frac{2y}{w}\right) \qquad \qquad (18)$$

(ii) Antisymmetrical case

$$N_{xy\,E} = \frac{S}{2(1+3\alpha)}\left\{1 - 3\left(\frac{2y}{w}\right)^2\right\} \quad \text{(parabolic !)} . \qquad \qquad (19)$$

When the panel is subjected to transverse loads $\bar{w}(z)$ (*see* Figs. 13d and 13e), it is again possible to equilibrate the applied moment $\bar{M}$ with $N_{x\,E}$ and P_E of (15) if one replaces $P_0 w$ by $\bar{M}(x)$.

In fact,

$$N_{x\,E} = \frac{6\bar{M}}{wt_s' + 6B'}\,\frac{t_s'}{w}\,\frac{2y}{w} = \frac{\bar{M}}{\dfrac{w^2t_s'}{12} + \dfrac{B'w^2}{2}}\,t_s'y = \frac{M}{I'}\,t_s'y \qquad \qquad (20)$$

which is the standard engineers' theory result. For the shear flow distribution one obtains from (11), (12) and (20),

$$\left.\begin{aligned} N_{xy\,E} &= \frac{\bar{Q}}{I'}\left\{\frac{B'w}{2} + \frac{t_s'}{2}\left(\frac{w^2}{4} - y^2\right)\right\} \\ &= \frac{\bar{Q}}{I}\left\{\frac{Bw}{2} + \frac{t_s}{2}\left(w^2 - y^2\right)\right\} \end{aligned}\right\} \qquad \qquad (21)$$

the well-known formula for the shear flow in the web of an I-beam.

3. *Condition of Compatibility of the Engineers' Theory Stress Systems of Section* 2.—To obviate continual reference in full to 'engineers' theory stress systems', and to 'self-equilibrating stress systems', the abbreviations E.T.S.S. and S.E.S.S. will be used.

The elementary E.T.S.S. of section 2 satisfy the compatibility condition of equation (7) only in some simple cases. Thus, as already mentioned, this is obviously the case with the distributions (14) and (15) corresponding to end loads P_0.

When edge loads S are applied to the panel substitution of (14) and (15) with (17) in (7) yields the following condition for compatibility:

$$\frac{d}{dx}\left(\frac{S}{\phi}\right) = 0 \qquad \qquad (22)$$

or

$$S \propto \phi . \qquad \qquad (22a)$$

For example, in a uniform panel the stress distributions (14) or (15) and (18) or (19) corresponding to constant S are elastically compatible. In fact, in an infinitely long panel these solutions will satisfy also the boundary conditions and thus be the true stresses. They are in the antisymmetrical case

$$N_{x\,\mathrm{E}} = \frac{6Sx}{w(1+3\alpha)}\,\frac{2y}{w} \quad , \qquad P_{\mathrm{E}\,\pm} = \pm\,\frac{3Sx\alpha}{1+3\alpha} \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

and $N_{xy\,\mathrm{E}}$ given by equation (19).

For a panel under transverse loads the E.T.S.S. are, but for a constant in the shear flow, the same as for a panel under antisymmetrical end and edge loads subject to the following relation between the two loading systems,

$$\bar{M} = \bar{P}w \text{ or } \bar{Q} = Sw \;. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (24)$$

Hence, the condition of compatibility or the E.T.S.S. in a beam under transverse loads is

$$\bar{Q} \propto \phi \;. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22b)$$

Thus, in a uniform cantilever under a constant shear force, which corresponds to the case of constant antisymmetrical edge loads $\pm\,S$, the simple theory does satisfy equation (7). But it will not represent the true solution if the end $x = l$ is fully built-in. It is easy to give many more examples where the E.T.S.S., although internally elastically compatible, is not true due to the boundary conditions.

However, in all cases it is possible to represent the stress distribution as a combination of—

(*a*) an E.T.S.S. which is in equilibrium with the applied loads, and

(*b*) a self-equilibrating stress system conditioned by the requirement that the total stresses satisfy the compatibility and boundary conditions.

Consider, for example, an infinite panel under end loads P_0. The E.T.S.S. is given by (14) or (15) which satisfies (7). But in order to achieve the correct boundary condition at $x = 0$ one must superimpose on the E.T.S.S., an S.E.S.S. at $x = 0$ which is equal to the difference between the P_0 system and the E.T.S.S. This is shown in Fig. 14 both for symmetrical and antisymmetrical loadings. The next step is now obviously the study of self-equilibrating stress systems.

4. *Self-equilibrating Stress Systems.*—It is natural to inquire into the possibility of S.E.S.S. which take the form

$$N_x = h(y)\,.\,g(x) \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

with shear flows N_{xy} in accordance with formulae (12) and (12a) for $\bar{Q} = 0$.

Since the direct flows are self-equilibrating they must satisfy the conditions of zero total end load and moment, *i.e.*,

$$\int_{-w/2}^{+w/2} h\,dy + \frac{B}{t_s}\,(h_+ + h_-) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26)$$

$$\int_{-w/2}^{+w/2} hy\,dy + \frac{B}{t_s}\,\frac{w}{2}\,(h_+ - h_-) = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

where h_+ and h_- denote the values of h and $y = +\,w/2$ and $-\,w/2$ respectively.

Substitution of (25) in (7) yields an equation which may be written as follows:

$$\frac{E}{G}\,\frac{\dfrac{d}{dx}\left(\dfrac{1}{\phi}\,\dfrac{dg}{dx}\right)}{\dfrac{1}{\phi_s}\,g} = -\,\frac{\dfrac{1}{t_s}\,\dfrac{d^2h}{dy^2}}{\dfrac{h}{t}} \;. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

By definition the first ratio can only be a function of x, or a constant, and the second a function of y or a constant. Thus, the common value of these ratios must be a constant, say λ^2. Equation (28) can now be split into two ordinary differential equations,

$$\frac{1}{t_s}\frac{d^2h}{dy^2} + \lambda^2\frac{h}{t} = 0 \qquad (29)$$

$$\frac{d}{dx}\left(\frac{1}{\phi}\frac{dg}{dx}\right) - \frac{G}{E}\lambda^2\frac{g}{\phi_s} = 0 \qquad (30)$$

It is interesting to pause at this stage and to see what solutions correspond to $\lambda^2 = 0$.

They are

$$h = C_1 + C_{2y} \qquad (31)$$

and

$$\frac{d}{dx}\left(\frac{1}{\phi}\frac{dg}{dx}\right) = 0$$

or

$$\frac{dg}{dx} \propto \phi . \qquad (32)$$

But equation (31) does obviously not represent a S.E.S.S. and in fact is the most general form of a E.T.S.S. Equation (32) is identical in substance with (22a) and indicates the condition under which the E.T.S.S. do satisfy the internal elastic compatibility. Thus the solutions for $\lambda^2 = 0$ merely restate the results of sections 2 and 3 and are hence of no interest any more.

The general solution of (29) for $\lambda^2 \neq 0$ is

$$h = C_1 \cos \bar{\lambda}y + C_2 \sin \bar{\lambda}y \qquad (33)$$

where

$$\bar{\lambda} = \lambda\sqrt{(t_s/t)} \qquad (34)$$

The terms $C_1 \cos \bar{\lambda}y$ and $C_2 \sin \bar{\lambda}y$ correspond to symmetrical and antisymmetrical N_x-distributions respectively and are best considered separately. Since the C-constants can be absorbed in the g-functions they will be taken here as unity.

(*a*) *The Symmetrical h-functions.*—The solution

$$h = \cos \bar{\lambda}y \qquad (35)$$

does automatically satisfy the zero moment condition (27).

Substitution of (35) into the zero direct-load condition (26) yields the transcendental equation for $\bar{\lambda}$

$$\tan(\bar{\lambda}w/2) + \alpha(\bar{\lambda}w/2) = 0 \qquad (36)$$

where α is given by (16).

It appears now that λ cannot take an arbitrary value but must be one of the infinite roots

$$\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_i \ldots \qquad (37)$$

of (36). Thus the λ_i's are the eigen-values and h_i's the eigen-functions of equation (29). To each of the roots there corresponds a different and independent S.E.S.S.* As the order i increases the roots approach asymptotically the values

$$\bar{\lambda}_i w/2 \rightarrow (2i - 1)\pi/2 \qquad (38)$$

and the h_i-functions approach the form $\cos\left\{(2i-1)\frac{\pi}{2}\frac{2y}{w}\right\}$. Thus, it follows that for large i's the boom loads tend to zero and self-equilibrium is achieved practically by the N_x-distribution in the stringer-sheet alone.

* Note that a different g_i-function corresponds to each value λ_i.

Of interest are also the extreme values of λ for $\alpha \to 0$ and $\alpha \to \infty$, i.e., $B \to 0$ and $B \to \infty$.

In the first case

$$\left.\begin{aligned} &\bar{\lambda}_i w/2 \to i\pi \\ &\bar{\lambda}_i w/2 \to (2i-1)\pi/2 \end{aligned}\right\} \quad \ldots \quad (39)$$

and in the second

The stress distribution in the panel for each value of $\bar{\lambda}_i$ may be determined from

$$N_x = \cos \bar{\lambda}_i y \,.\, g_i \,, \qquad P = \frac{B}{t_s} \cos (\bar{\lambda}_i w/2) \,.\, g_i = -\frac{\sin (\bar{\lambda}_i w/2)}{\bar{\lambda}_i} g_i$$

$$N_{xy} = -\frac{\sin \bar{\lambda}_i y}{\bar{\lambda}_i} \frac{dg_i}{dx} \,. \quad \ldots \quad (40)$$

In the derivation of the second form of the boom-load equation (36) was used.

(*b*) *The Antisymmetrical h-functions.*—The solution

$$h = \sin \bar{\lambda} y \quad \ldots \quad (41)$$

does satisfy automatically the zero and load condition (26).

Substitution of (41) into the zero moment condition (27) yields the transcendental equation

$$\tan (\bar{\lambda} w/2) - \frac{(\bar{\lambda} w/2)}{1 + \alpha(\bar{\lambda} w/2)^2} = 0 \quad \ldots \quad (42)$$

the infinite set of roots of which give the appropriate $\bar{\lambda}_i$-values for the antisymmetrical case. A similar discussion to that of the symmetrical case applies here too. Thus, for large i's the roots approach asymptotically the values

$$\bar{\lambda}_i w/2 \to i\pi \,. \quad \ldots \quad (43)$$

The stress distribution in the panel for each value of $\bar{\lambda}_i$ may be determined from,

$$N_x = \sin \bar{\lambda}_i y \,.\, g_i \,, \qquad P_{\pm} = \pm \frac{B}{t_s} \sin (\bar{\lambda}_i w/2) \,.\, g_i$$

$$N_{xy} = \frac{\cos \bar{\lambda}_i y - \mathrm{sn}\, (\bar{\lambda}_i w/2)}{\bar{\lambda}_i} \frac{dg_i}{dx} \,. \quad \ldots \quad (44)$$

The S.E.S.S. (43) and (44) will be denoted collectively as the eigenloads of the structure.

An important relation, which corresponds to the usual orthogonality conditions of Fourier series, holds for the h-functions of either kind. Thus one can easily prove that

$$\int_{-w/2}^{+w/2} t_s h_i h_j \, dy + B[h_{i+} h_{+} + h_{i-} h_{j-}] = 0 \qquad \text{when } i \neq j \quad \ldots \quad (45)$$

the following relations apply:

(i) Symmetrical case, $h_i = \cos \bar{\lambda}_i y$

$$\left.\begin{aligned} \int_{-w/2}^{+w/2} t_s h_i^2 \, dy + 2B(h_{i+})^2 &= \frac{t_s w}{2} \left[1 + \alpha \cos^2 (\bar{\lambda}_i w/2)\right] \\ &= \frac{t_s w}{2} \left[1 - \mathrm{sn}\, (\bar{\lambda}_i w)\right] \end{aligned}\right\} \quad \ldots \quad (46)$$

(ii) Antisymmetrical case, $h_i = \sin \bar{\lambda}_i y$

$$\left.\begin{aligned} \int_{-w/2}^{+w/2} t_s h_i^2 \, dy + B[h_{i+}^2 + h_{i-}^2] &= \frac{t_s w}{2} \left[1 + \alpha \sin^2 (\bar{\lambda}_i w/2) - \mathrm{sn}^2 (\bar{\lambda}_i w/2)\right] \\ &= \frac{t_s w}{2} \left[1 + \mathrm{sn}\, (\bar{\lambda}_i w) - 2\, \mathrm{sn}^2 (\bar{\lambda}_i w/2)\right] \end{aligned}\right\} . \quad (47)$$

The next step is to investigate the lengthwise variation of the self-equilibrating stress systems. For this it will be necessary to find the general solution of equation (30).

$$g_i(x) = D_{i1}\Psi_{i1}(x) + D_{i2}\Psi_{i2}(x) \quad \ldots \qquad (48)$$

where Ψ_{i1} and Ψ_{i2} are the complementary functions and the D's are constants depending on the end conditions.

At a free end $N_x = 0$ and hence

$$g_i = 0 \, . \quad \ldots \qquad (49)$$

At a built-in end $N_{xy} = 0$ and hence

$$dg_i/dx = 0 \, . \quad \ldots \qquad (50)$$

In the case of a uniform panel ($\phi = \phi_s = 1$)

$$g_i = D_{i1} \cosh \mu_i x + D_{i2} \sinh \mu_i x \quad \ldots \qquad (51)$$

where

$$\mu_i = \lambda_i \sqrt{(G/E)} \, . \quad \ldots \qquad (52)$$

From equations (51) and (50) one can derive the following expression for the g_i-function in a panel built-in at $x = l$.

$$g_i = D_i \frac{\cosh \mu_i(l - x)}{\cosh \mu_i l} \, . \quad \ldots \qquad (53)$$

Equation (53) shows that the larger i and hence the greater the quicker the self-equilibrating system dies out.

It is often useful to have an expression for the warping, *i.e.*, the out-of-plane displacements in a cross-section. In the estimation of the warping, which will be denoted by u^*, it is immaterial if any translational or rotational rigid body movements are superimposed. The following formula for u^* can be obtained from the shear-shear strain relation (4) using equation (29)

$$u_i^* = -\frac{1}{Gt_s} \frac{h_i}{\lambda_i^2} \frac{dg_i}{dx} \quad \ldots \qquad (54)$$

(Built-in condition $dg_i/dx = 0$!). Note that the warping is proportional to the direct stress. This is a characteristic property of the self-equilibrating stress systems described in equations (35) and (41).

Assume now that an arbitrary S.E.S.S. is applied at the free end $x = 0$ of a panel built-in at $x = l$. The stress distribution is obtained if one succeeds in expressing the given S.E.S.S. in terms of the eigen-loads (43) and (44). Again symmetrical and antisymmetrical loading groups will be considered separately.

(1) *Symmetrical Arbitrary S.E.S.S.*—Let, at $x = 0$, the direct stress flow in the stringer-sheet be N_{xs} (which may vary with y) and the boom loads P_s. It is required to express this system in the form of an infinite series in the h_i's.

$$\left.\begin{aligned} N_{xs} &= \sum_{i=1}^{\infty} (g_i)_0 h_i \\ P_{s+} &= \frac{B}{t_s} \sum_{i=1}^{\infty} (g_i)_0 h_i + \end{aligned}\right| \quad \ldots \qquad (55)$$

where $h_i = \cos \bar{\lambda}_i y$ and $(g_i)_0$ is found from equation (53) for a uniform panel. For panels with lengthwise variation of thickness one must obtain the appropriate solution (48) and adjust it to the boundary condition (50). In all cases it is possible to write equations (55) in the form

$$\begin{aligned} N_{xs} &= \sum_{i=1}^{\infty} D_i h_i \\ P_s &= \frac{B}{t_s} \sum_{i=1}^{\infty} D_i h_{i+} \, . \end{aligned} \quad \ldots \qquad (55a)$$

Thus, the problem consists in the determination of the constants D_i. Multiplying now the first of (55a) by $h_j = \cos \bar{\lambda}_j y$ and integrating between 0 and $w/2$ and adding to the second multiplied by $\cos (\bar{\lambda}_j w/2)$ one obtains, using equations (45) and (46), an explicit expression for each constant D_j.

$$D_j(\mathrm{w}/4)\,[1 - \mathrm{sn}\,(\bar{\lambda}_j w)] = \int_0^{w/2} N_{xs} \cos \bar{\lambda}_j y \, dy + P_s \cos (\bar{\lambda}_j w/2)\,. \qquad \text{(56)}$$

It is hence possible to derive for any self-equilibrating stress system a unique expansion of the type (55a).

(2) *Antisymmetrical Arbitrary S.E.S.S.*—Let at $x = 0$ the direct stress flow be N_{xa} and the boom loads $\pm P_a$. An expansion similar to equation (55a) is sought with the only difference that now $h_i = \sin \lambda_i y$. Proceeding as in the previous case one obtains

$$D_j(w/4)\,[1 + \mathrm{sn}\,(\lambda_j w) - 2\,\mathrm{sn}^2\,(\lambda_j w/2)] = \int_0^{w/2} N_{xa} \sin \lambda_j y \, dy + P_{a+} \sin (\lambda_j w/2) \qquad \text{(57)}$$

which solves the antisymmetrical problem uniquely.

Having the D_j coefficients for a symmetrical or antisymmetrical loading one determines the stress distribution in the stringer-sheet and the booms of a uniform panel from the following formulae:

$$\left.\begin{aligned}
N_x &= \sum_{i=1}^{\infty} D_i \frac{\cosh \mu_i (l - x)}{\cosh \mu_i l}\, h_i \\
P_+ &= \frac{B}{t_s} \sum_{i-1}^{\infty} D_i \frac{\cosh \mu_i (l - x)}{\cosh \mu_i l}\, h_{i+} \\
N_{xy} &= \sqrt{\left(\frac{Gt}{Et_s}\right)} \sum_{i=1}^{\infty} D_i \frac{\sinh \mu_i (l - x)}{\cosh \mu_i l} \sin \bar{\lambda}_i y \quad \text{for a symmetrical case} \\
N_{xy} &= -\sqrt{\left(\frac{Gt}{Et_s}\right)} \sum_{i=1}^{\infty} D_i \frac{\sinh \mu_i (l - x)}{\cosh \mu_i l} (\cos \bar{\lambda}_i y - \mathrm{sn}\,(\bar{\lambda}_i w/2)) \quad \text{for an antisymmetrical case}
\end{aligned}\right\} \qquad \text{(58)}$$

The series are so quickly convergent that, in general, only a few terms are required to obtain a good approximation.

5. *The Panel Under End Loads P_0.*—The results of the previous section will be applied to the problem of a panel under symmetrical or antisymmetrical end loads P_0.

(*a*) *Symmetrical End Loads.*—The symmetrical loads P_0 on the booms at $x = 0$ can be regarded as the super position of the uniform loading indicated by $N_{x\mathrm{E}}$ and P_E of equations (14) and a S.E.S.S. N_{xs} and P_s defined by

$$\begin{aligned} N_{xs} &= -N_{x\mathrm{E}} \\ P_s &= P_0 - P_\mathrm{E} \end{aligned} \qquad \text{(59)}$$

(*see* Fig. 14a).

Substituting equations (59) on the right-hand side of equation (56) and remembering that the contribution of the uniform E-system must be zero, one obtains

$$D_j = \frac{4P_0}{w} \frac{\cos (\lambda_j w/2)}{1 - \mathrm{sn}\,(\lambda_j w)} = K_j P_0 \qquad \text{(60)}$$

where

$$K_j = \frac{4}{w} \frac{\cos (\lambda_j w/2)}{1 - \mathrm{sn}\,(\lambda_j w)}\,. \qquad \text{(60a)}$$

Thus, the total stress distribution is given by

$$\left.\begin{aligned} N_x &= \frac{2P_0}{w}\left\{\frac{1}{1+\alpha} + 2\sum_{j=1}^{\infty}\left[\frac{\cos(\bar{\lambda}_j w/2)}{1-\operatorname{sn}(\bar{\lambda}_j w)}\,\frac{\cosh\mu_j(l-x)}{\cosh\mu_j l}\cos\bar{\lambda}_j y\right]\right\} \\ P &= P_0\alpha\left\{\frac{1}{1+\alpha} + 2\sum_{j=1}^{\infty}\left[\frac{\cos^2(\bar{\lambda}_j w/2)}{1-\operatorname{sn}(\bar{\lambda}_j w)}\,\frac{\cosh\mu_j(l-x)}{\cosh\mu_j l}\right]\right\} \\ N_{xy} &= \sqrt{\left(\frac{Gt}{Et_s}\right)}\frac{4P_0}{w}\sum_{j=1}^{\infty}\left[\frac{\cos(\bar{\lambda}_j w/2)}{1-\operatorname{sn}(\bar{\lambda}_j w)}\,\frac{\sinh\mu_j(l-x)}{\cosh\mu_j l}\sin\bar{\lambda}_j y\right] \end{aligned}\right\} \quad \ldots \quad (61)$$

(*b*) *Antisymmetrical End Loads* $\pm P_0$.—The method is similar to that under (*a*). Thus, if one notes that the linearly distributed $\bar{E}$-system of equation (15) cannot contribute to the right-hand side of (57) one obtains

$$D_j = \frac{4P_0}{w}\,\frac{\sin(\bar{\lambda}_j w/2)}{1+\operatorname{sn}(\bar{\lambda}_j w) - 2\operatorname{sn}^2(\bar{\lambda}_j w/2)} = K_j P_0 \quad \ldots \quad (62)$$

where

$$K_j = \frac{4}{w}\,\frac{\sin(\bar{\lambda}_j w/2)}{1+\operatorname{sn}(\bar{\lambda}_j w) - 2\operatorname{sn}^2(\bar{\lambda}_j w/2)} \quad \ldots \quad (62a)$$

$$\left.\begin{aligned} N_x &= P_0\left\{\frac{6}{w(1+3\alpha)}\,\frac{2y}{w} + \sum_{j=1}^{\infty}\left[K_j\,\frac{\cosh\mu_j(l-x)}{\cosh\mu_j l}\sin\bar{\lambda}_j y\right]\right\} \\ P_{\pm} &= \pm P_0\alpha\left\{\frac{3\alpha}{1+3\alpha} + \frac{w}{2}\sum_{j=1}^{\infty}\left[K_j\sin(\bar{\lambda}_j w/2)\,\frac{\cosh\mu_j(l-x)}{\cosh\mu_j l}\right]\right\} \\ N_{xy} &= -\sqrt{\left(\frac{Gt}{Et_s}\right)}P_0\sum_{j=1}^{\infty}K_j\,\frac{\sinh\mu_j(l-x)}{\cosh\mu_j l}\left(\cos\bar{\lambda}_j y - \operatorname{sn}(\bar{\lambda}_j w/2)\right)\Big]. \end{aligned}\right\} \quad \ldots \quad (63)$$

6. *Panel Under Arbitrary Edge Loads S*.—The next loading cases to be considered are those of a panel under arbitrary edge loads S; as in the previous sections symmetrical and anti-symmetrical loads will be investigated separately.

Whatever the nature of the edge loads S it is always possible to equilibrate them with stress systems of the type (14), (18) or (15), (19) subject to the substitution $\bar{P}$ for P_0. The analytical character of the additional self-equilibrating stress systems, however, will in general be different from those discussed in section 4, since the present E.T.S.S.'s may violate the compatibility condition (7). Thus the purpose of the S.E.S.S. will not only be to satisfy the boundary conditions but also *in combination* with the E.T.S.S. the compatibility equation (7). This will obviously only affect the g_i-function which in the present case will be denoted by $\bar{g}_i$.

Following this preamble one can express the direct-stress distribution in the panel and booms as follows :

$$\left.\begin{aligned} N_x &= N_{x\mathrm{E}} + \sum_{i=1}^{\infty}\bar{g}_i h_i \\ P_{\pm} &= P_{\mathrm{E}\pm} + \frac{B}{t_s}\sum_{i=1}^{\infty}\bar{g}_i h_i \end{aligned}\right\} \quad \ldots \quad (64)$$

Substituting the first of these equations into (7) and noting that

$$\frac{\partial^2 N_{x\mathrm{E}}}{\partial y^2} = 0$$

for all E.T.S.S. one obtains

$$-\frac{1}{Gt}\frac{\partial}{\partial x}\left(\frac{1}{\phi}\frac{\partial N_{x\mathrm{E}}}{\partial x}\right) = +\sum_{i=1}^{\infty}\left\{\frac{1}{E\phi_s t_s}\bar{g}_i\frac{d^2 h_i}{dy^2} + \frac{h_i}{Gt}\frac{d}{dx}\left(\frac{1}{\phi}\frac{d\bar{g}_i}{dx}\right)\right\} \quad \ldots \quad (65)$$

or, using equation (29),

$$-\frac{\partial}{\partial x}\left(\frac{1}{\phi}\frac{\partial N_{xE}}{\partial x}\right) = +\sum_{i=1}^{\infty} h_i \left\{\frac{d}{dx}\left(\frac{1}{\phi}\frac{d\bar{g}_i}{dx}\right) - \mu_i^2 \frac{\bar{g}_i}{\phi_s}\right\} \quad \ldots \quad (66)$$

A corresponding equation for the booms can easily be derived from the second of (64) using (3) and (30). Thus, for the boom at $y = +w/2$,

$$\frac{d}{dx}\left(\frac{1}{\phi}\frac{d\bar{P}}{dx}\right) - \frac{d}{dx}\left(\frac{1}{\phi}\frac{dP_{E+}}{dx}\right) = \frac{B}{t_s}\sum_{i=1}^{\infty} h_{i+} \left\{\frac{d}{dx}\left(\frac{1}{\phi}\frac{d\bar{g}_i}{dx}\right) - \mu_i^2 \frac{\bar{g}_i}{\phi_s}\right\} \quad \ldots \quad (67)$$

where $\bar{P}$ is given by equation (17). Equation (67) can also be obtained by a simple physical argument. The right-hand side of (66) represents essentially the expansion in an h_i-series of a self-equilibrating stress system with a direct-flow distribution in the stringer sheet equal to

$$-\frac{\partial}{\partial x}\left(\frac{1}{\phi}\frac{\partial N_{xE}}{\partial x}\right).$$

The corresponding boom load of this S.E.S.S. is at $y = +w/2$,

$$\frac{d}{dx}\left(\frac{1}{\phi}\frac{d(\bar{P} - P_{E+})}{dx}\right) = \frac{d}{dx}\left(\frac{1}{\phi}\frac{d\bar{P}}{dx}\right) - \frac{d}{dx}\left(\frac{1}{\phi}\frac{dP_{E+}}{dx}\right)$$

and can obviously be expressed by the right-hand side series of (67), (*see* also the discussion of sections 4 and 5).

The subsequent analysis is very similar to that given in section 5 for end loads P_0. Thus, multiplying (66) by h_i and integrating between 0 and $w/2$ and adding (67) after multiplying it by h_{i+} one obtains, using (17), (45), (46) or (47) the required differential equation in $\bar{g}_i$,

$$\frac{d}{dx}\left(\frac{1}{\phi}\frac{d\bar{g}_i}{dx}\right) - \mu_i^2 \frac{\bar{g}_i}{\phi_s} = K_i \frac{d}{dx}\left(\frac{S}{\phi}\right) \quad \ldots \quad (68)$$

where K_i is for symmetrical loads given by (60a) and for antisymmetrical loads by (62a).

If $S \propto \phi$ the right-hand side of (68) is zero and $\bar{g}_i = g_i$. This confirms also condition (22a) for elastic compatibility of the E.T.S.S.

As next one must define the boundary conditions. At a free end $N_x = N_{xE} = 0$ and hence

$$\bar{g}_i = 0 . \quad \ldots \quad (69)$$

At a built-in end $u = 0$ or $\partial N_x/\partial x = 0$ and hence, by a method similar to that applied for the derivation of (68), one finds

$$d\bar{g}_i/dx = K_i S . \quad \ldots \quad (70)$$

Examples

(1) Uniform panel, free at $x = 0$, built-in at $x = l$; loading:

symmetrical and anti-symmetrical S = constant.

The differential equation for $\bar{g}_i$ reduces to that of $\bar{g}_l$ which in the present case is

$$\frac{d^2 g_i}{dx^2} - \mu_i^2 g_i = 0 . \quad \ldots \quad (71)$$

The solution of (71) adjusted to the boundary conditions (69) and (70) is

$$g_i = \frac{K_i S}{\mu_i \cosh \mu_i l} \sinh \mu_i x . \quad \ldots \quad (72)$$

Thus, the stress distribution in the panel is as follows:

(*a*) Symmetrical case

$$\left.\begin{aligned} N_x &= \frac{2Sx}{w(1+\alpha)} + S\sum_{i=1}^{\infty}\left[\frac{K_i}{\mu_i}\frac{\sinh\mu_i x}{\cosh\mu_i l}\cos\bar{\lambda}_i y\right] \\ P &= \frac{Sx\alpha}{1+\alpha} + \frac{Sl\alpha}{2}\sum_{i=1}^{\infty}\left[\frac{K_i w}{\mu_i l}\frac{\sinh\mu_i x}{\cosh\mu_i l}\cos(\bar{\lambda}_i w/2)\right] \\ N_{xy} &= -\frac{S}{1+\alpha}\frac{2y}{w} - S\sum_{i=1}^{\infty}\left[\frac{K_i}{\bar{\lambda}_i}\frac{\cosh\mu_i x}{\cosh\mu_i l}\sin(\bar{\lambda}_i y)\right] \end{aligned}\right\} \quad \ldots \quad (73)$$

(*b*) Anti-symmetrical case

$$\left.\begin{aligned} N_x &= \frac{6Sx}{w(1+3\alpha)}\frac{2y}{w} + S\sum_{i=1}^{\infty}\left[\frac{K_i}{\mu_i}\frac{\sinh\mu_i x}{\cosh\mu_i l}\sin\bar{\lambda}_i y\right] \\ P_{\pm} &= \pm\frac{3Sx\alpha}{w(1+3\alpha)} \pm \frac{Sl\alpha}{2}\sum_{i=1}^{\infty}\left[\frac{K_i w}{\mu_i l}\frac{\sinh\mu_i x}{\cosh\mu_i l}\sin(\bar{\lambda}_i w/2)\right] \\ N_{xy} &= \frac{S}{2(1+3\alpha)}\left\{1 - 3\left(\frac{2y}{w}\right)^2\right\} + S\sum_{i=1}^{\infty}\left[\frac{K_i}{\bar{\lambda}_i}\frac{\cosh\mu_i x}{\cosh\mu_i l}(\cos\bar{\lambda}_i y - \mathrm{sn}\,(\bar{\lambda}_i w/2))\right] \end{aligned}\right\} \quad . \quad (74)$$

(2) Uniform panel, free at $x = 0$, built-in at $x = l$; Loading: symmetrical and antisymmetrical linearly increasing edge-loads $S = S_0(x/l)$.

The differential equation (68) reduces in the present case to

$$\frac{d^2\bar{g}_i}{dx^2} - \mu_i^2\bar{g}_i = K_i\frac{S_0}{l}, \quad \ldots \quad (75)$$

the solution of which adjusted to the boundary conditions (69) and (70) is

$$\bar{g}_i = \frac{K_i S_0}{\mu_i^2 l}\left\{\frac{\cosh\mu_i(l-x) + \mu_i l\sinh\mu_i x}{\cosh\mu_i l} - 1\right\}. \quad \ldots \quad (76)$$

The final formulae for the stresses will not be given here, but they may be found very simply from equations (64) and (76).

In all cases only a few terms of the series need be taken to obtain a very good accuracy in the stresses.

7. *The Panel Under Transverse Loads.*—It was stated repeatedly in the main report and also in this appendix that the stress distribution in a panel under transverse loads is, but for a constant $\bar{Q}/w$ in the shear flow, the same as in the panel under antisymmetrical loads as long as the following reciprocal relation holds between the two loadings,

$$\bar{M} = \bar{P}w \text{ or } \bar{Q} = Sw\,. \quad \ldots \quad (24)$$

Hence, the differential equation (68) for the $\bar{g}_i$-function takes the following form in a panel under transverse loads:

$$\frac{d}{dx}\left(\frac{1}{\phi}\frac{d\bar{g}_i}{dx}\right) - \mu_i^2\frac{\bar{g}_i}{\phi_s} = \frac{K_i}{w}\frac{d}{dx}\left(\frac{\bar{Q}}{\phi}\right). \quad \ldots \quad (77)$$

Similarly, the boundary condition (70) may be written as

$$\frac{d\bar{g}_i}{dx} = \bar{K}_i \frac{\bar{Q}}{w} . \quad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (78)$$

The two examples investigated in the previous section find an immediate interesting application here. Thus, the case of a uniform antisymmetrical edge load S corresponds to that of a constant shear force $\bar{Q}$ and equations (74) give the stress distribution subject to the substitution (74) and the superposition of a constant shear flow $\bar{Q}/w$. Furthermore, the results of example (2) may similarly be used for a panel under uniform transverse load.

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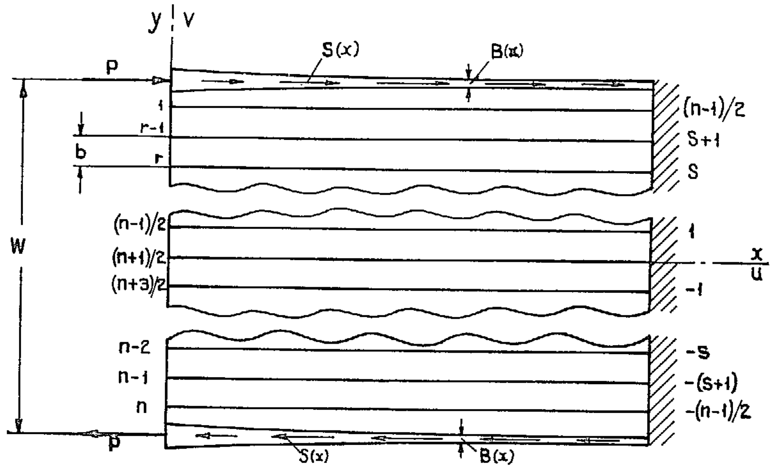

FIG. 1. Parallel stiffened panel with constant-stress edge members, antisymmetrical edge loading, zero shear load.

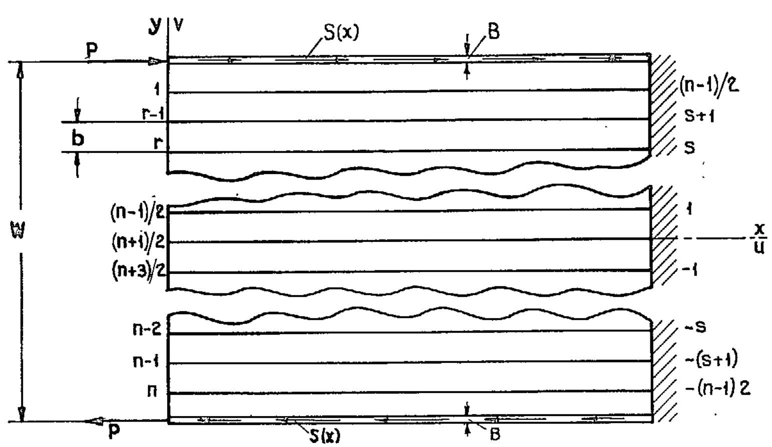

FIG. 2. Parallel stiffened panel with constant-area edge members, antisymmetrical edge loading, zero shear load.

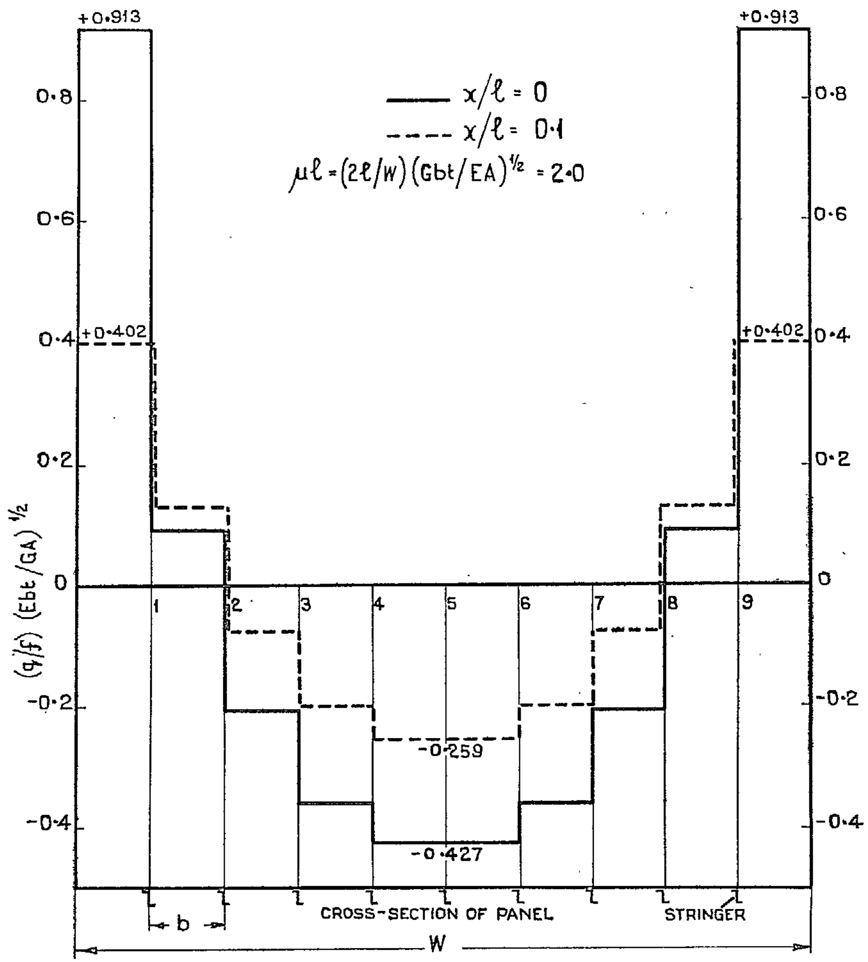

FIG. 3. Shear distribution across a 9-stringer panel with constant antisymmetrical edge stress at $x/l = 0$ and $x/l = 0 \cdot 1$. Diffusion parameter $\mu l = 2$. Zero shear load.

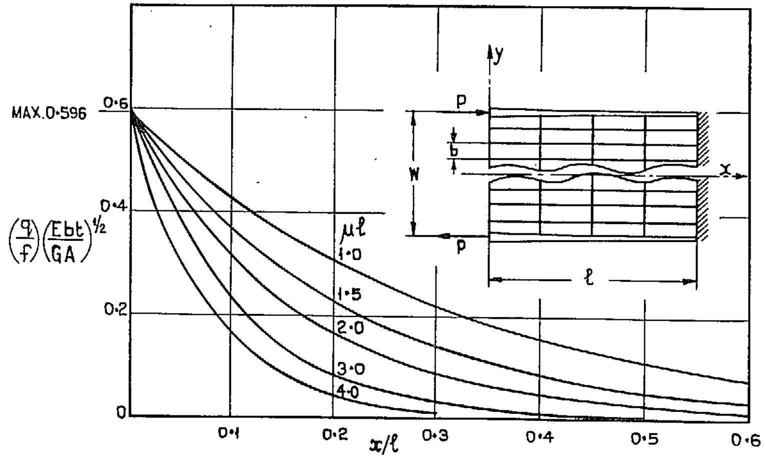

FIG. 4. Edge shear stresses in 5-stringer panel with constant antisymmetrical edge strain. Zero shear load.

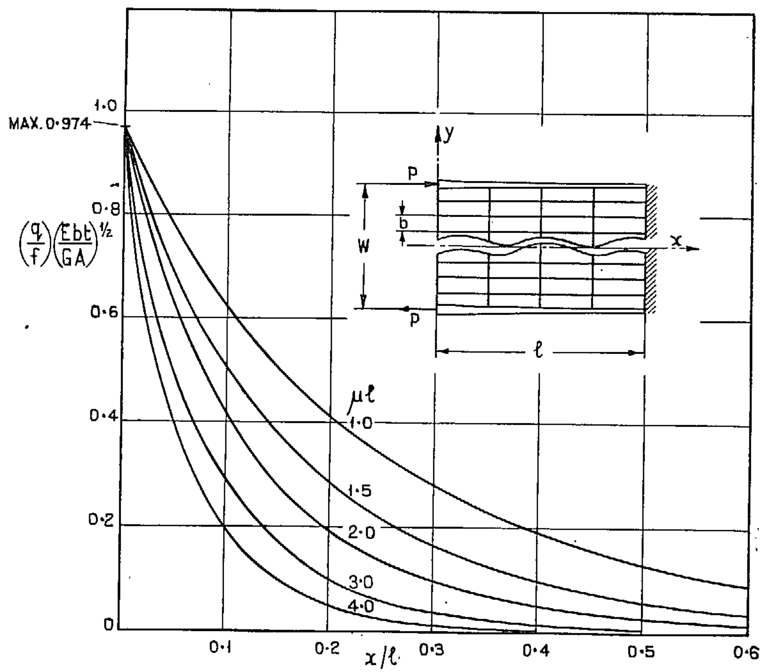

FIG. 5. Edge shear stresses in 10-stringer panel with constant antisymmetrical edge strain. Zero shear load.

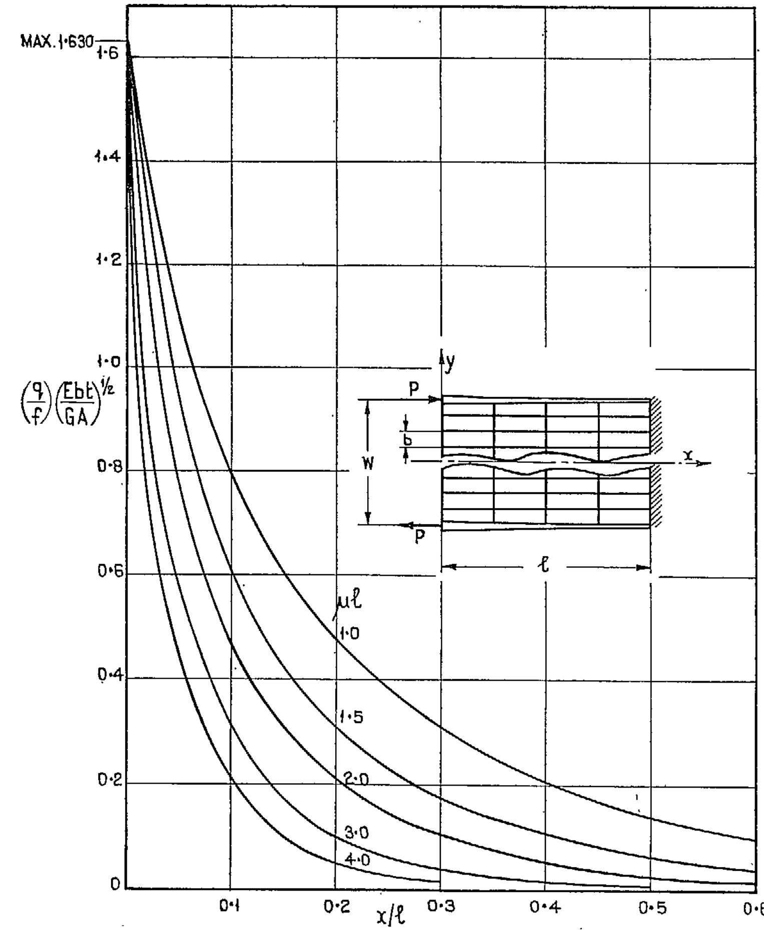

FIG. 6. Edge shear stresses in 30-stringer panel with constant antisymmetrical edge strain. Zero shear load.

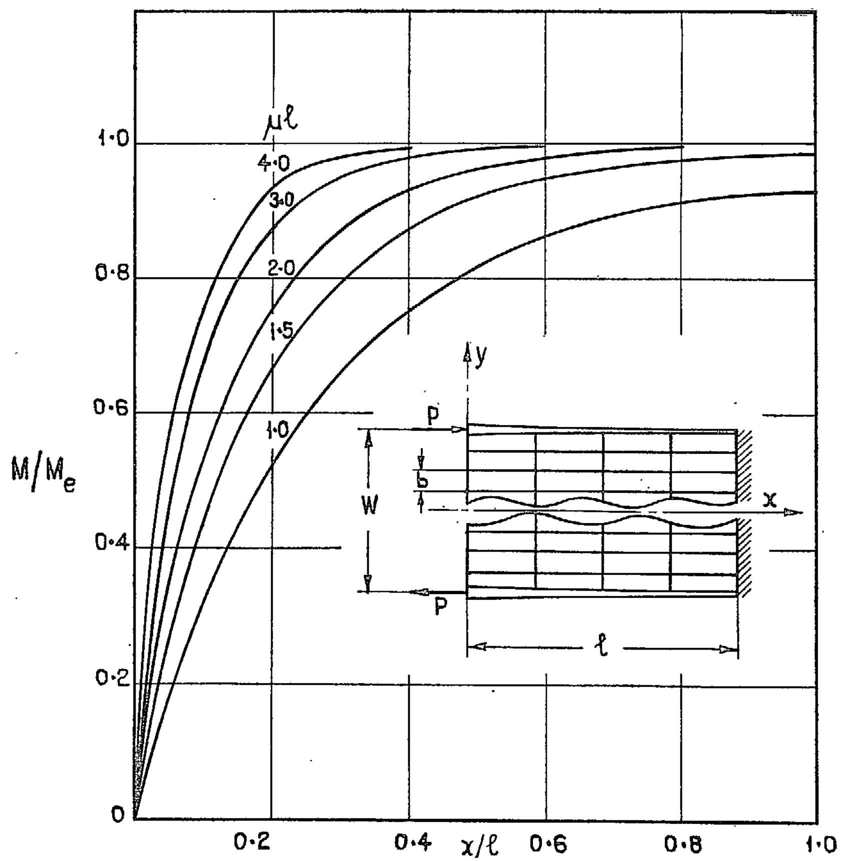

FIG. 7. Moment carried by the panel for constant antisymmetrical edge strain.

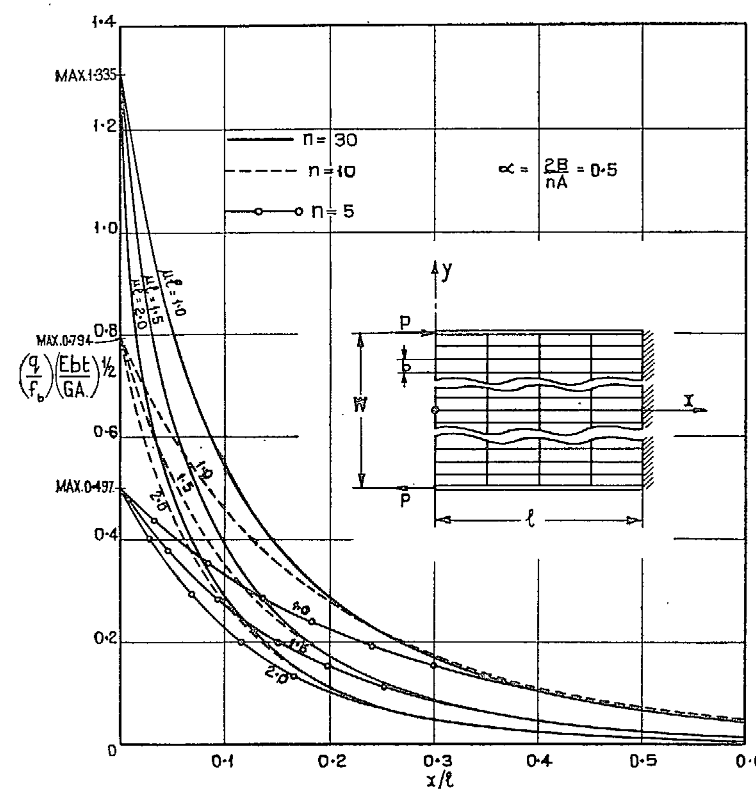

FIG. 8. Edge shear stresses in a parallel panel under concentrated antisymmetrical end loads with uniform edge members. $\alpha = 2B/nA = 0{\cdot}5$.

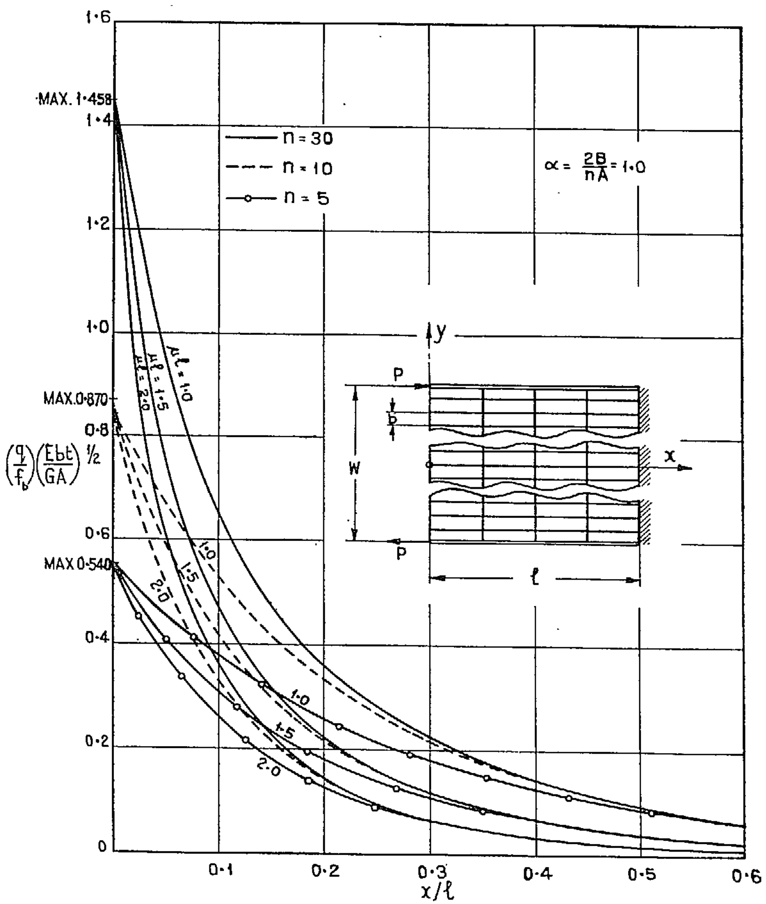

FIG. 9. Edge shear stresses in a parallel panel under concentrated antisymmetrical end loads with uniform edge members. $\alpha = 2B/nA = 1\cdot0$.

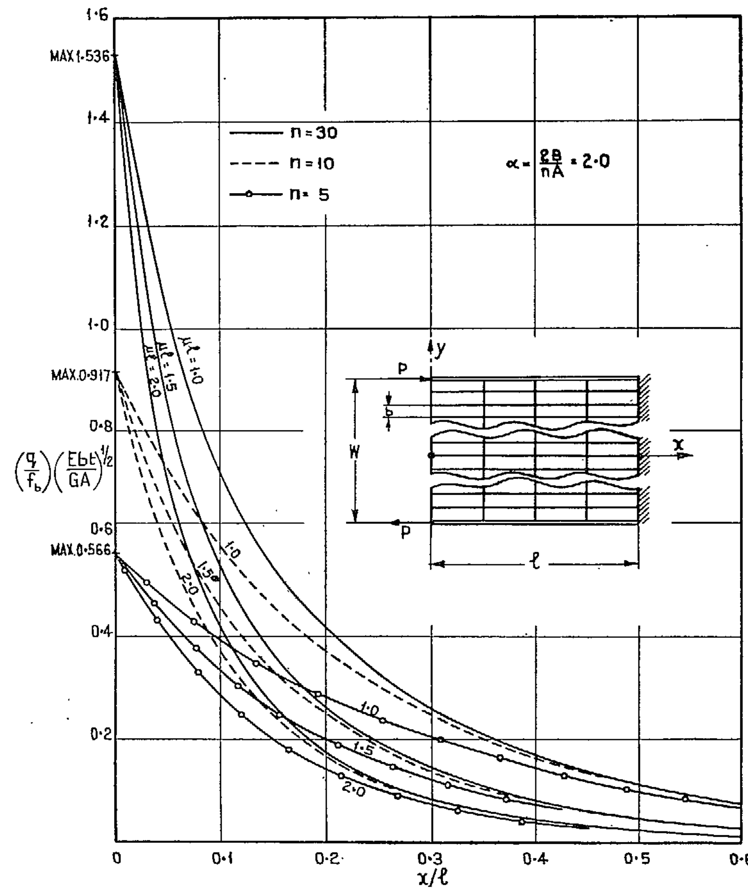

FIG. 10. Edge shear stresses in a parallel panel under concentrated antisymmetrical end loads with uniform edge members. $\alpha = 2B/nA = 2\cdot0$.

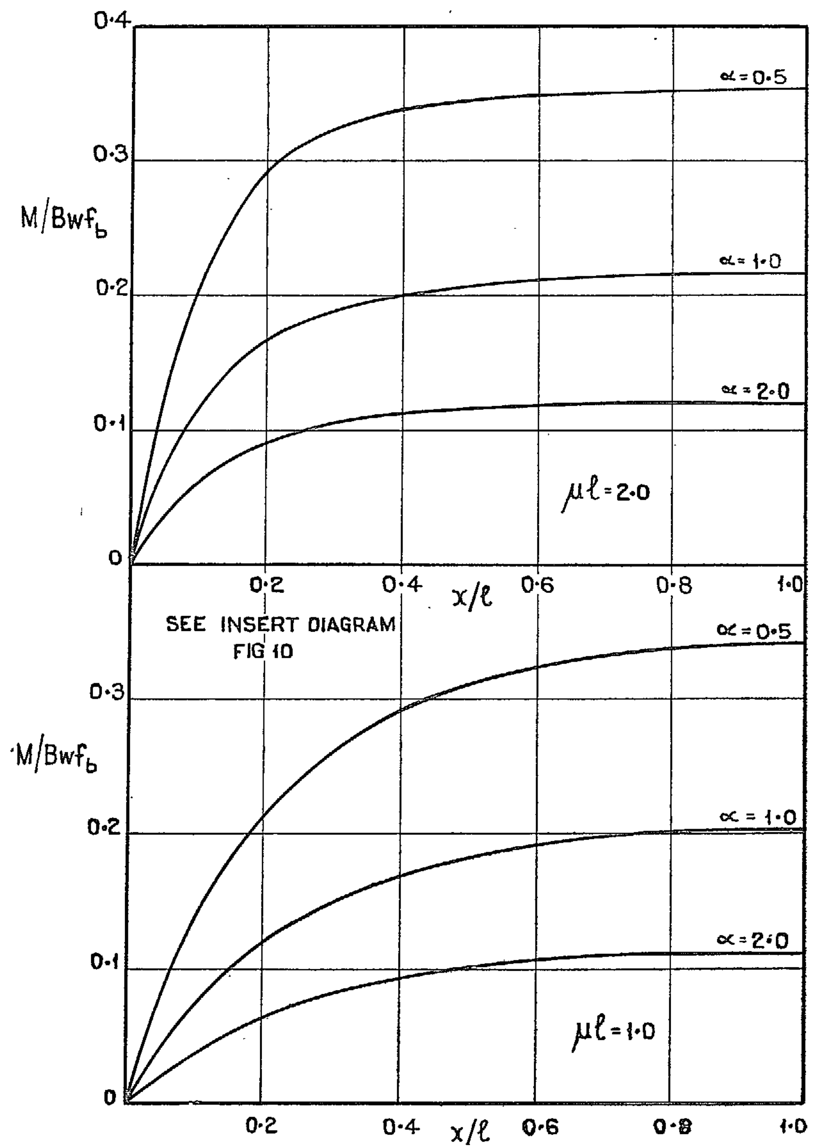

FIG. 11. Moment carried by a parallel panel under concentrated antisymmetrical end loads with uniform edge members.

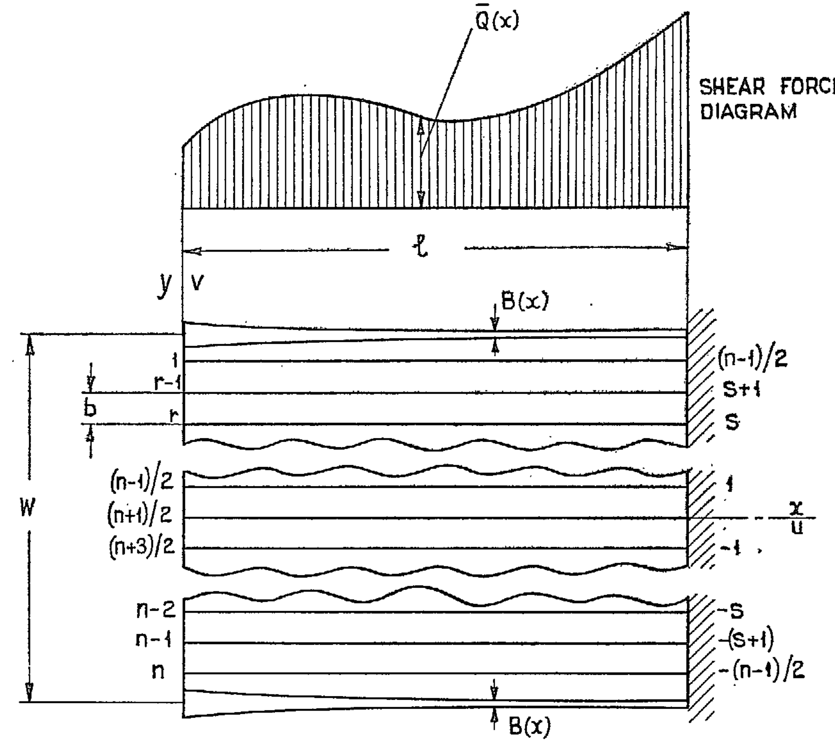

FIG. 12. Parallel stiffened panel under arbitrary transverse loads.

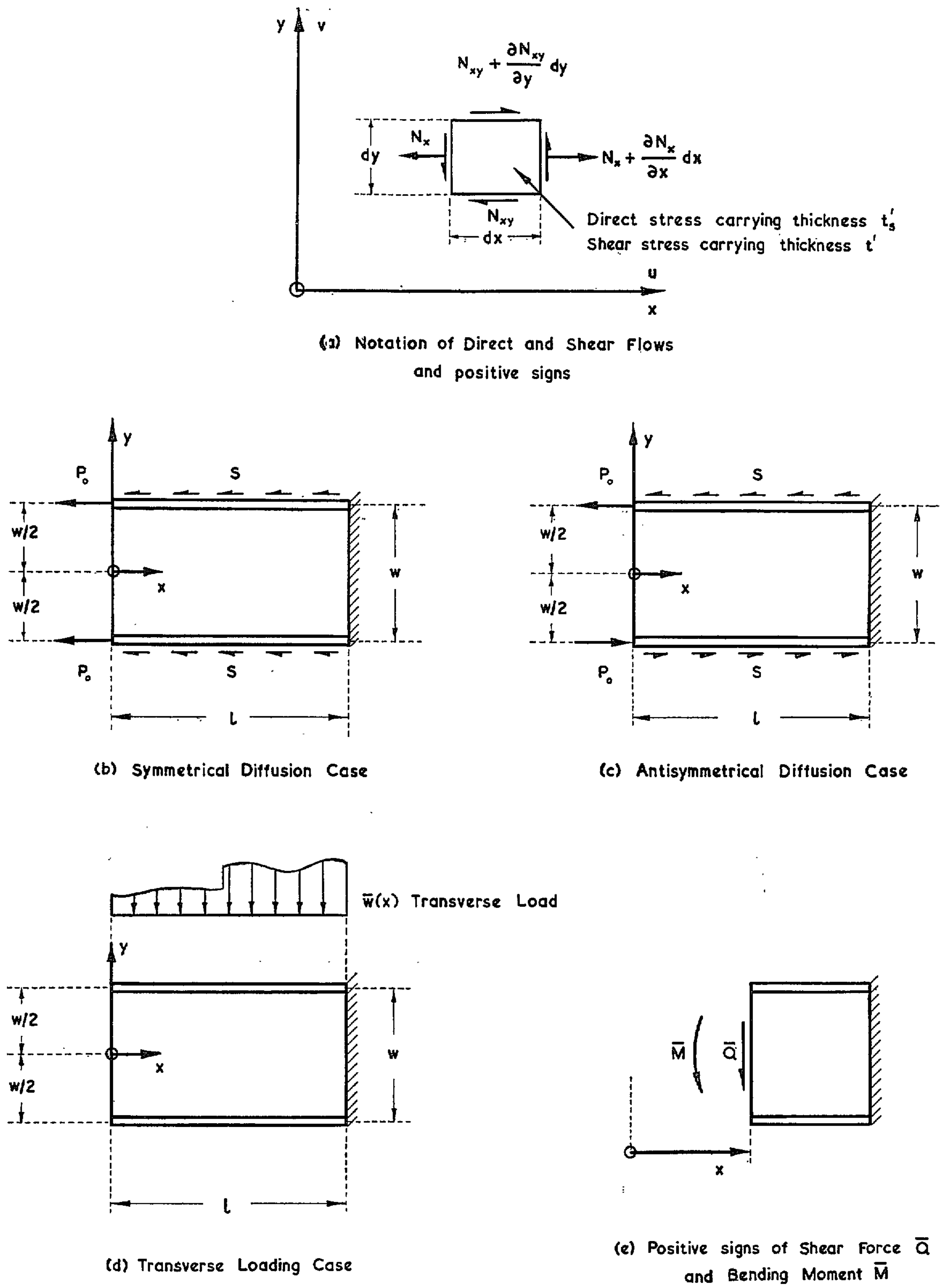

FIG. 13. General notation for Appendix II.

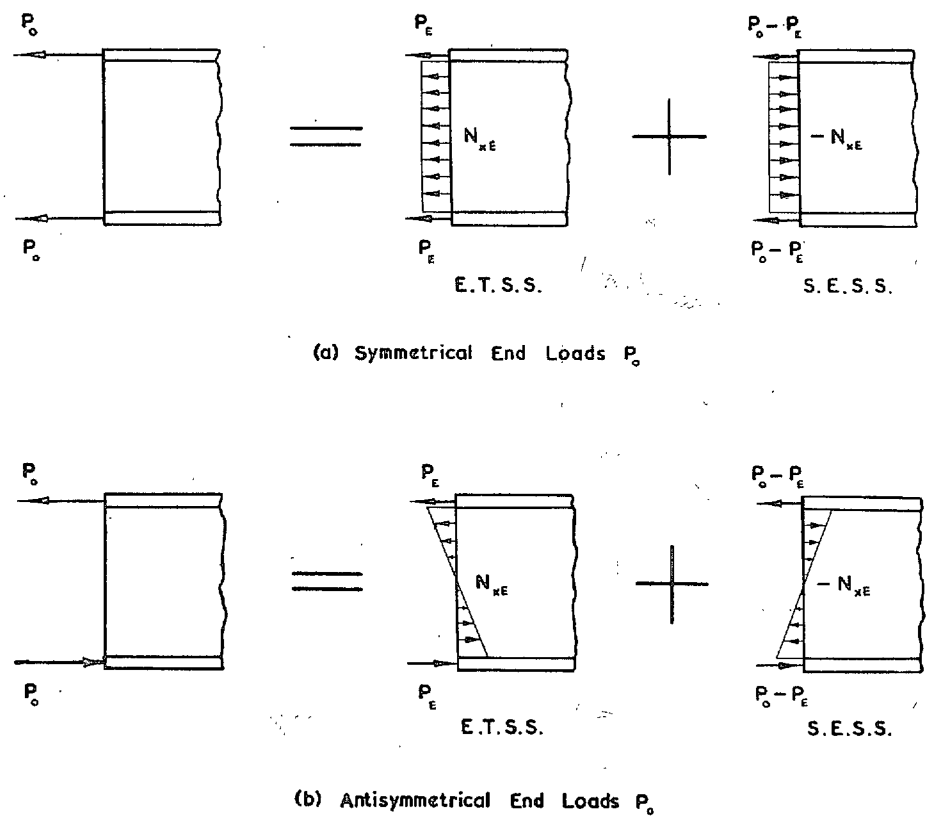

FIG. 14. Analysis of a symmetrical or antisymmetrical end-load system P_0 into an engineers'-theory stress system (E.T.S.S.) and a self-equilibrating stress system (S.E.S.S.).

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