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Neutral Holes in Plane Sheet: Reinforced Holes which are Elastically Equivalent to the Uncut Sheet

By

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Summary.—It is shown that, in a plane sheet under any particular loading system, certain reinforced holes may be made which do not alter the stress distribution in the main body of the sheet. These reinforced holes (hereafter called neutral holes) necessarily have exactly the same stiffness and at least the same strength as the portion of the sheet that has been cut out. The weight of the reinforcement is usually greater than the weight of the sheet that has been cut out, though there are cases where it is less.

The Airy stress function is used throughout because it admits of great generality and because the properties of a neutral hole can be expressed simply in terms of the function and its derivatives. Indeed, the stress function assumes a new and special significance in determining the shape of a neutral hole.

1. *Introduction.*—The use of stress-bearing sheet in structures, and especially aircraft structures, has given rise to a number of problems. The problem considered here is the general one of making and reinforcing a hole in a stress-bearing sheet so that the strength and stiffness are not altered. Such a reinforced hole which does not affect the stress distribution in the remainder of the sheet, and therefore gives the same strength and stiffness as the replaced sheet, is here described as a neutral hole.

Reissner and Morduchow[1] considered a neutral circular hole. They allowed themselves the choice of the tensile stiffness and bending stiffness of the reinforcement. Apart from a few particular loading systems their solutions were impracticable, mainly because the required bending stiffness was so large as to be practically incompatible with the required tensile stiffness. They concluded that for the types of reinforcement deemed technically important the bending stiffness may be ignored; in other words, the reinforcement around a neutral hole experiences primarily tensile or compressive loads rather than bending moments.

In this report it is shown that neutral holes are always possible if there is the choice of the shape of the hole as well as the tensile stiffness of the reinforcement. A number of cases representative of window holes and undercarriage cut-outs in aircraft has been considered and the hole shapes and tensile stiffnesses of the reinforcement are reasonable from an engineering point of view.

It is not possible to design a hole which is neutral for more than one type of loading system.

* R.A.E. Report Structures 90, received 25th January, 1951.

2. *Assumptions.*—The following assumptions are made regarding the structure:

(*a*) Stress-strain relations are linear.

(*b*) Buckling does not take place.

(*c*) Rivet flexibility is negligible.

(*d*) The bending stiffness of the reinforcing member is negligible compared with its tensile stiffness.

(*e*) If stringers (or ribs) are present their stiffening effect may be adequately represented by assuming them to be spread out into an elastic sheet with equivalent directional properties (R. & M. 2758[2]).

Assumptions (*a*) to (*c*) are standard practice; they are not essential in determining the *shape* of a neutral hole.

Ample justification for (*d*) is given in Appendix I where it is shown that the bending energy in the reinforcement is usually less than 1 per cent of the corresponding tensile energy. In addition it has been shown that a negligible bending stiffness is, in fact, a necessary quality of the reinforcement if the reinforcement is to have least weight.

Assumption (*e*) is not essential in that exact solutions taking account of discrete stringers may be obtained as in Appendix II.

3. *General Properties of a Neutral Hole.*—Consider first the sheet in the uncut state. The stresses are such that all elements of the sheet are in equilibrium which, for a typical element of the sheet, means that

$$\frac{\partial \bar{\sigma}_x}{\partial x} + \frac{\partial \bar{\tau}_{xy}}{\partial y} = 0$$

and

$$\frac{\partial \bar{\sigma}_y}{\partial y} + \frac{\partial \bar{\tau}_{xy}}{\partial x} = 0 \, .$$

Both these relations are automatically satisfied by introducing a stress function[4] ϕ such that the stresses are to be derived from it by the equation

$$\left.\begin{aligned} \bar{\sigma}_x &= \frac{\partial^2 \phi}{\partial y^2} \\ \bar{\sigma}_y &= \frac{\partial^2 \phi}{\partial x^2} \\ \bar{\tau}_{xy} &= -\frac{\partial^2 \phi}{\partial x \, \partial y} \end{aligned}\right\} \quad \ldots \quad (1)$$

The complete state of stress in a sheet can therefore be described by the stress function ϕ alone. This stress function is usually introduced as an aid to the determination of the stresses in a plate subjected to given boundary conditions. For example, for plain sheet it can be shown that ϕ satisfies a particular equation ($\nabla^4 \phi = 0$) and together with the boundary conditions, this is sufficient to determine ϕ and hence the stresses. Here, however, it is assumed that the complete stress distribution is already known, but ϕ will still be used because it admits of great generality and because the properties of a neutral hole can be expressed simply in terms of ϕ and its derivatives. The function ϕ itself assumes a new and special significance in determining the shape of a neutral hole.

3.1. *The Shape of a Neutral Hole.*—To fix ideas whilst considering the equilibrium of the reinforcing member and the adjacent sheet Figs. 1 and 2 are given below.

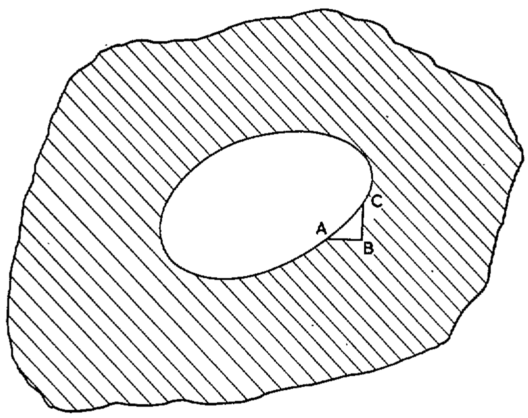

FIG. 1. Sheet with neutral hole.

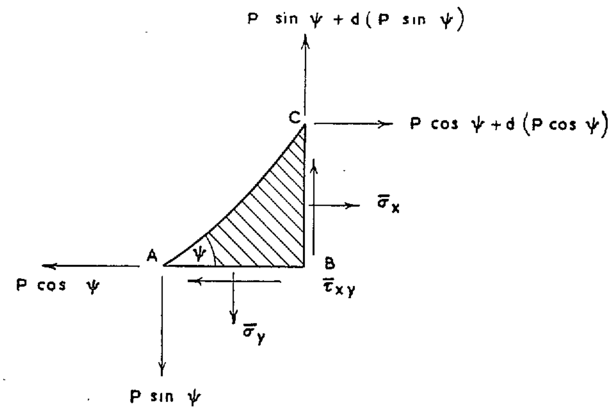

FIG. 2. Forces acting on the element ABC.

The bending stiffness of the reinforcing member is negligible compared with the tensile stiffness so that the reinforcing member has the properties of a chain in that the line of action of the load P in the member is directed along the length of the member. For the element shown in Fig. 2 the conditions of equilibrium are therefore

$$\left.\begin{aligned} d(P \sin \psi) &= t(\bar{\sigma}_y \, dx - \bar{\tau}_{xy} \, dy) \\ \text{and} \qquad d(P \cos \psi) &= t(\bar{\tau}_{xy} \, dx - \bar{\sigma}_x \, dy) \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Equations (1) and (2) can be combined to give

$$\left.\begin{aligned} (1/t) \, d(P \sin \psi) &= \frac{\partial^2 \phi}{\partial x^2} \, dx + \frac{\partial^2 \phi}{\partial x \, \partial y} \, dy \\ \text{and} \qquad (1/t) \, d(P \cos \psi) &= \frac{-\partial^2 \phi}{\partial x \, \partial y} \, dx - \frac{\partial^2 \phi}{\partial y^2} \, dy \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

These equations are in the form of total differentials and may therefore be integrated to give

$$\left.\begin{aligned} \frac{P \sin \psi}{t} &= \frac{\partial \phi}{\partial x} + a \\ \frac{P \cos \psi}{t} &= -\frac{\partial \phi}{\partial y} - b \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

where a and b are arbitrary constants.

P may be eliminated from equation (4) to give

$$\tan \psi = -\left[\frac{\partial \phi/\partial x + a}{\partial \phi/\partial y + b}\right] . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

Substituting dy/dx for $\tan \psi$, equation (5) may be integrated to give

$$\phi + ax + by + c = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (6)$$

as the equation for determining the shape of a neutral hole.

Terms of the type $(ax + by + c)$ can be added to ϕ without altering the stresses and so there will be no loss of generality by writing equation (6) as

$$\phi = 0 . \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (7)$$

Hereafter equation (7) will be used instead of equation (6) as this appreciably simplifies the presentation of results. The significance of the term $(ax + by + c)$ that may be included in ϕ is considered in the next section. *It is worth noting that equation (7) is derived purely from considerations of statics and is therefore independent of the elastic properties of the sheet.*

3.1.1. *Hole bounded by arcs.*—Since there are no restrictions on the constants a, b, c that may be included in ϕ it will be seen that there is a large variety of curves from which the hole shape may be chosen. Furthermore, the hole shape may be bounded by arcs of curves, each arc of which is determined by a different set of values of a, b, c; in this case it will be necessary to apply a balancing load at the junction point of adjacent arcs to ensure equilibrium of the loads in the reinforcing members. It will be shown later that such balancing loads can normally be produced by inserting a simple tension or compression member from one junction point to another, as in Fig. 3, or by utilising structural discontinuities already present in the main structure as in Fig. 4.

The calculation of such balancing loads is straightforward. Suppose arc_1 is given by

$$\phi_1 = \phi' + a_1 x + b_1 y + c_1 = 0$$

and arc_2 by

$$\phi_2 = \phi' + a_2 x + b_2 y + c_2 = 0 ;$$

then from equation (4) the vertical (or y) component of the load in arc_1 is $t\left(\frac{\partial \phi'}{\partial x} + a_1\right)$ and in arc_2 is $t\left(\frac{\partial \phi'}{\partial x} + a_2\right)$, and there are similar expressions for the x-components. It follows that the vertical and horizontal components of the balancing load are respectively

and

$$\left.\begin{array}{l} t(a_1 - a_2) \\ t(b_2 - b_1) \end{array}\right\} . \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (8)$$

3.2. *Section Area of the Reinforcing Member.*—The section area of the reinforcing member can be determined from a knowledge of the load P and the strain ε_m in the reinforcing member by the relation

$$A_m = P/E\varepsilon_m .$$

P is determined from equation (4) by eliminating ψ:

$$P = t\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right]^{1/2} . \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (9)$$

The strain ε_m is related to the sheet stresses as follows[4]:

$$E\varepsilon_m = \cos^2\psi(\sigma_x - \nu\sigma_y) + \sin^2\psi(\sigma_y - \nu\sigma_x) + 2\sin\psi\cos\psi(1+\nu)\tau_{xy} \,. \quad .. \qquad .. \qquad (10)$$

For the simple case in which the sheet is not reinforced by stringers (i.e., $\sigma_x = \bar{\sigma}_x$, etc., equations (9) and (10) can be combined with (1) and (7) to give

$$A_m = t\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]^{3/2}\left[\frac{\partial^2\phi}{\partial x^2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{\partial^2\phi}{\partial y^2}\left(\frac{\partial\phi}{\partial y}\right)^2 + \frac{2\partial^2\phi}{\partial x\,\partial y}\cdot\frac{\partial\phi}{\partial x}\cdot\frac{\partial\phi}{\partial y}\right.$$

$$\left. - \nu\left\{\frac{\partial^2\phi}{\partial x^2}\left(\frac{\partial\phi}{\partial y}\right)^2 + \frac{\partial^2\phi}{\partial y^2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \frac{2\partial^2\phi}{\partial x\,\partial y}\cdot\frac{\partial\phi}{\partial x}\cdot\frac{\partial\phi}{\partial y}\right\}\right]^{-1}. \quad .. \qquad .. \qquad .. \qquad (11)$$

3.3. *Rib and Stringer-reinforced Sheet.*—A stress function has been developed in R. & M. 2758[2] from which the sheet stresses in a sheet reinforced by stringers and skew ribs can be derived. The stress function satisfies equation (1) and so the analysis up to and including equation (7) is still valid.

Equation (11) is of a different form and may be obtained from Appendix III.

4. *Particular Stress Distributions.*—In this section four examples are given to demonstrate the application of the results of section 3 and to show that these neutral holes have not unreasonable shapes and reinforcing members from an engineering point of view. The examples are considered in order of simplicity rather than practical importance. They are:

(*a*) A circular hole for equal stress in *x*- and *y*-directions.

(*b*) A $\sqrt{2} : 1$ elliptical hole for stress in *y*-direction twice that in *x*-direction.

(*c*) A parabolic cut-out for stress in *y*-direction alone.

(*d*) A $\sqrt{3} : 1$ elliptical hole for equal bending stresses in two directions.

In these examples, the sheet is not reinforced by stringers or ribs. Further examples, dealt with in less detail, are given in Appendix IV. An example with discrete stringers is given in Appendix II.

4.1. *Equal Stress in x- and y-directions.*—Such a stress distribution occurs, for example, in a thin-walled sphere subjected to hydrostatic pressure.

The most general form for ϕ is

$$\phi = \frac{f}{2}(x^2 + y^2) + ax + by + c\,,$$

so that the shape of the neutral hole ($\phi = 0$) is circular. The fact that the constants *a*, *b*, *c* are at present arbitrary means that the circle can be chosen to have any radius and to be situated anywhere. If we take the centre of the circle at the origin and let it have a radius *r* we can substitute

$$\phi = \frac{f}{2}(x^2 + y^2 - r^2) = 0$$

in equations (11) and (9) to find

$$\left.\begin{aligned} A_m &= \frac{rt}{1-\nu} \\ P &= frt \end{aligned}\right\} \quad \text{and} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (12)$$

These properties of a neutral circular hole could, of course, have been deduced simply from first principles, though it is interesting to note that the circular hole is the only possible hole; furthermore the circular hole, with this value for A_m, will not be neutral for any other stress distribution.

There is no need to limit the number of such holes to one, in fact there can be two or more overlapping, as mentioned in section 3.1.1, provided there is inserted a tension or compression member to ensure continuity of load P and compatibility of displacements. Two examples will demonstrate this.

In the first example shown in Fig. 3 two unequal circular holes of radii r_1 and r_2 whose centres lie on the x-axis overlap, meeting at an angle $\theta_1 + \theta_2$. It follows from equation (8) that the horizontal components of P_1 and P_2 at the junction points A and B will cancel out, but the resultant vertical component $P_1 \cos \theta_1 + P_2 \cos \theta_2$ will necessitate a tension member between AB with a section area of

$$\frac{t}{1-\nu}(r_1 \cos \theta_1 + r_2 \cos \theta_2) .$$

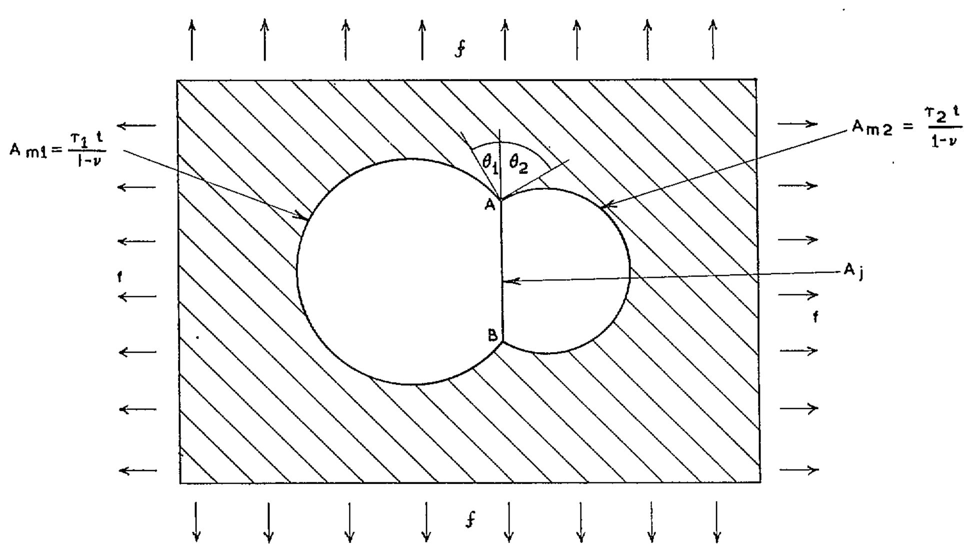

FIG. 3. Two circular holes overlapping.

A different arrangement is shown in Fig. 4a, for in this case the sign of the balancing loads at A, B is reversed and so (because members with negative section areas are not practicable) the members AC and BD must be present. These will again have a section area of $\frac{t}{1-\nu}(r_1 \cos \theta_1 + r_2 \cos \theta_2)$ and loads of magnitude $ft(r_1 \cos \theta_1 + r_2 \cos \theta_2)$ must now be applied externally to them.

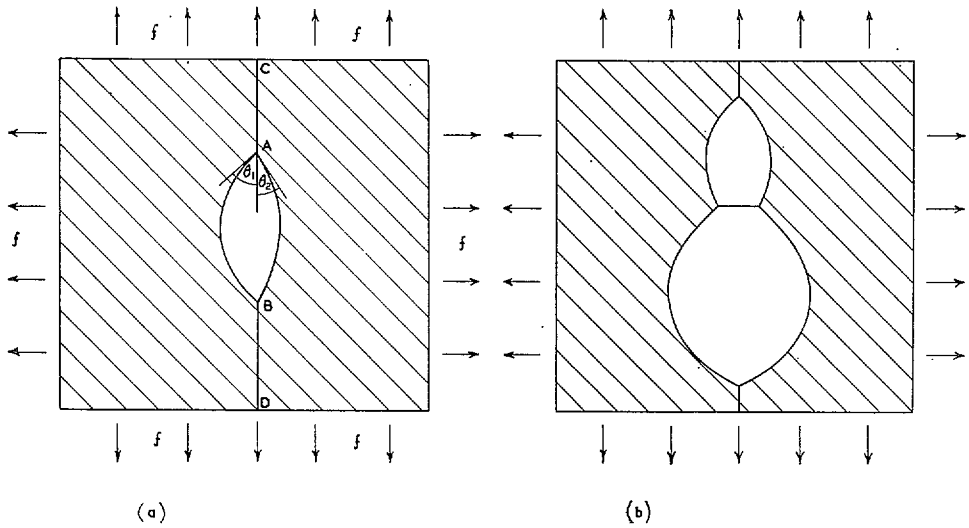

FIG. 4. Holes bounded by circular arcs.

Fig. 4b is a combination of the types represented in Figs. 3 and 4a.

4.2. *Stress in y-direction Twice that in x-direction.*—Such a stress distribution occurs in a thin-walled cylinder subjected to hydrostatic pressure.

Suppose $$\bar{\sigma}_y = 2\bar{\sigma}_x = f\,,$$

then ϕ is given by $$\phi = \frac{f}{2}\left(x^2 + \frac{y^2}{2}\right) + ax + by + c\,,$$

so that the shape for the neutral hole is an ellipse with major and minor axes in the ratio $\sqrt{2} : 1$. If a, b, c are chosen so that the centre of the ellipse is at the origin and the minor axis is $2r$ then the equation for the neutral hole will be

$$\phi \propto x^2 + \frac{y^2}{2} - r^2 = 0\,. \quad \ldots \quad (13)$$

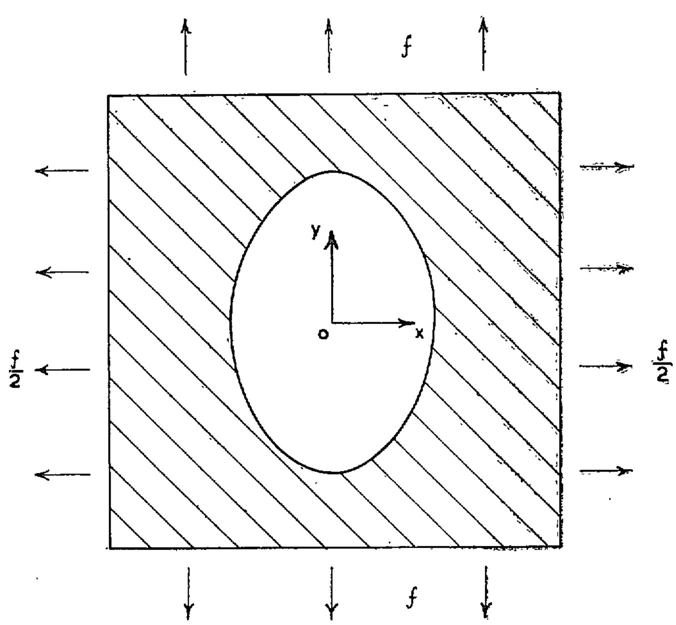

FIG. 5. Elliptical hole with axes $\sqrt{2} : 1$.

Substituting equation (13) in (11) gives

$$\frac{A_m}{rt} = \frac{\sqrt{2}(1 + x^2/r^2)^{3/2}}{1 - 2\nu + 3x^2/r^2} \,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

This variation has been plotted in Fig. 16. It will be noticed that the maximum and minimum values of A_m differ appreciably.

4.3. *Uniform Stress in y-direction Alone.*—Such a stress distribution is probably the one most commonly approximated to in aircraft structures.

If $\bar{\sigma}_y = f$ the most general form for ϕ is

$$\phi = \frac{fx^2}{2} + ax + by + c \,,$$

and so the neutral hole (or cut-out, since the hole is not 'closed') is any parabola of the type

$$\phi \propto x^2 - ry = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

where r is arbitrary and determines the size.

This parabolic shape is well known in connection with the design of certain suspension bridges in which the weight/unit length of span, i.e., $t\bar{\sigma}_y$, is constant.

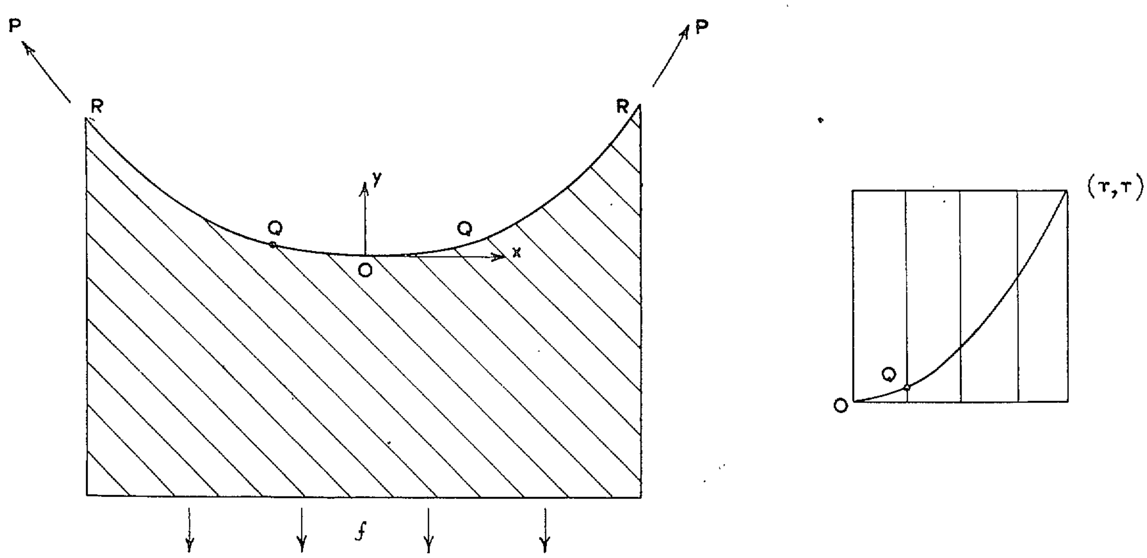

FIG. 6. Ideal parabolic cut-out.

The section area A_m is obtained by substituting equation (15) in equation (11):

$$\frac{A_m}{rt} = \frac{(1 + 4y/r)^{3/2}}{2(4y/r - \nu)} \,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

This expression becomes negative over the range $0 < y < \nu r/4$, from Q to Q in Fig. 6 (say), and so any practical design must utilise those parts of the parabola from Q to R. In fact as A_m is very large in the immediate vicinity of Q it will also be inadvisable, from a weight point of view, to use these parts of the parabola near Q. The inset in Fig. 6 shows the position of Q to scale taking $\nu = \frac{1}{4}$. It will be seen that the 'useless' region is comparatively small; it is due entirely to the Poisson contraction.

The simplest symmetrical arrangement with positive A_m everywhere would be that shown in Fig. 7.

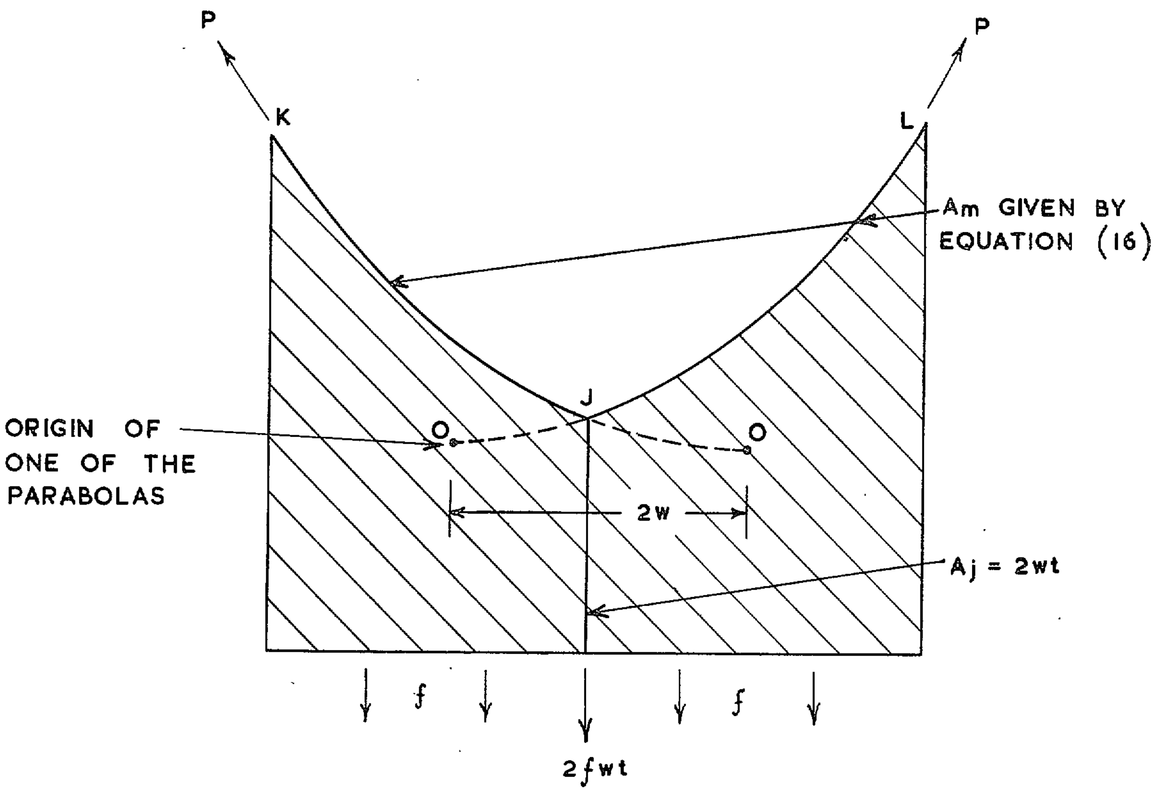

FIG. 7. Cut-out bounded by parabolic arcs.

At the junction point J the horizontal components of the P's will cancel out, but to offset the resultant vertical component $2fwt$, given by equation (8) it will be necessary to introduce the tension member as shown.

Suppose now the panel of Fig. 7 represents the lower surface of an aircraft wing in the neighbourhood of a cut-out. The externally applied loads at K, L would have no horizontal components so that we should have to induce such components by modifying the structure. This could be accomplished by inserting a compression member between K and L, but it must be pointed out that the member KL would be fairly inefficient since its stress would be only νf. Such an arrangement is shown in Fig. 8.

It will be noticed that this type of cut-out reinforcement could only be applied efficiently to a wing of 3-spar construction.

The booms of constant section area A_f shown in Fig. 8 can clearly be added to the rest of the structure without altering its constant stress character. The booms can also be regarded as special cases of the general problem considered in this report. For example, if we take the origin on the line LL' and take

$$\phi = \frac{f}{2}\left(x^2 + \frac{2A_f x}{t}\right)$$

a suitable boundary of a neutral 'hole' is the line LL' (since $\phi = 0$) and the appropriate section area of the reinforcing member, as determined from equation (11), is A_f.

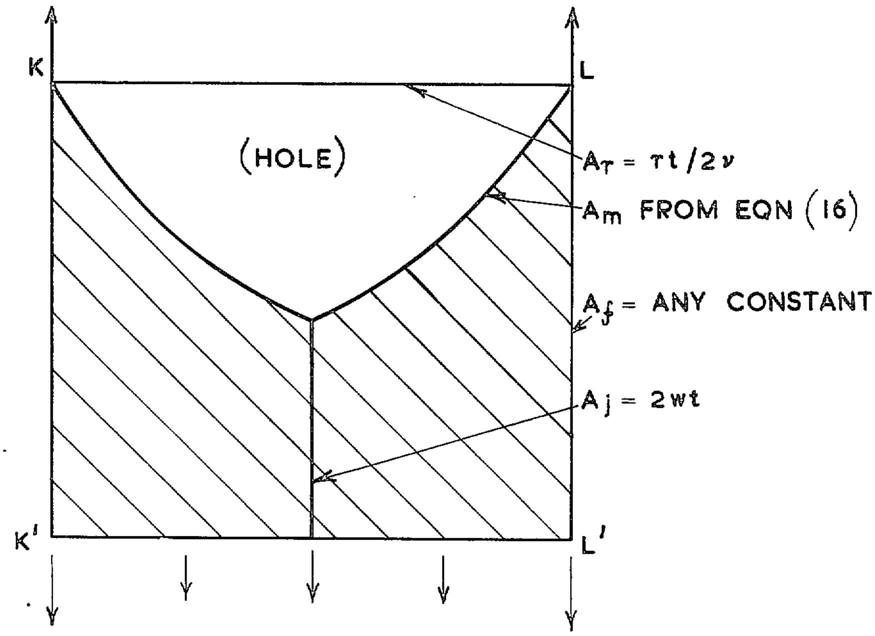

FIG. 8. Cut-out bounded by parabolic arcs and compression member.

4.3.1. *Subsidiary forms based on the parabola.*—There are other types of neutral hole or cut-out that can be made on these principles. They will not be discussed here, but their basic forms are shown in Fig. 17. The last two types shown there utilise the 'useless' part of the parabola and the sheet and cut-out have therefore to be interchanged. These last types are very inefficient from the weight point of view.

4.4. *Equal Bending Stresses in Two Directions.*—Such a stress distribution, though not of great practical importance, is considered as it brings out a number of fresh points concerning neutral holes in general. The stress distribution differs from those so far considered in that the stresses in the sheet are not constant.

The most general form for ϕ is

$$\phi \propto x^3 + y^3 + ax + by + c ,$$

so that the shape of a neutral hole is, in general, a cubic.

Owing to the more complex form for ϕ the coefficients a, b, c no longer refer directly to the position and size of the hole. Here, however, the coefficients will be chosen specially so that the stress function may be factorised. The advantage of this is that it will be possible to find a 'closed form' for the shape of the neutral hole. We take

$$\begin{aligned} \phi &\propto x^3 + y^3 - r^2(x + y) \\ &\equiv (x + y)(x^2 - xy + y^2 - r^2) , \end{aligned} \qquad \ldots \quad \ldots \quad \ldots \quad (17)$$

and choose the shape of the neutral hole from the second factor only, *i.e.*,

$$x^2 - xy + y^2 - r^2 = 0 . \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (18)$$

Equation (18) represents an ellipse with major and minor axes in the ratio $\sqrt{3} : 1$ and inclined at 45 deg to the x-axis (*see* Fig. 18).

The cross-section area of the reinforcement, obtained from equations (11) and (17), is given by

$$A_m = \frac{t(5r^2 - 3xy)^{3/2}}{6\{(4 - \nu)r^2 - 3xy\}}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

which remains practically constant at its mean value of $0 \cdot 50rt$.

5. *Weight Efficiencies*.—Holes and cut-outs which represent exactly the sheet which has been removed are necessarily 100 per cent efficient as regards strength and stiffness. We need therefore discuss the efficiency from the weight point of view alone.

A suitable criterion for the weight efficiency is

$$\frac{\text{weight of material removed}}{\text{weight of reinforcement inserted}} = \eta_w, \text{ say.}$$

This expression does not afford a direct comparison with a hole which is not neutral. A hole which is not neutral will cause stress concentrations which would lower the strength of the complete structure unless the thickness of the sheet surrounding the hole were increased. Such an increase in sheet thickness should be taken into account when comparing efficiencies in any particular case, though this cannot be done in a generalised form.

Before discussing the comparative efficiencies of different neutral holes a few facts will be listed below. The material is taken to have unit density and for numerical results ν will be taken as $\frac{1}{4}$. The notation for reinforced sheet is given in Appendix III.

5.1. *Circular hole* (section 4.1).—

Material removed $= \pi r^2 t$.

Reinforcement inserted $= 2\pi r\left(\frac{rt}{1 - \nu}\right)$,

therefore $$\eta_w = \frac{1 - \nu}{2} = 37 \cdot 5 \text{ per cent.}$$

5.1.1. *Circular holes overlapping at any re-entrant angle* (as in Fig. 3).—

$$\eta_w = \frac{1 - \nu}{2} = 37 \cdot 5 \text{ per cent.}$$

5.1.2. *Hole bounded by circular arcs* (as in Fig. 4a).—Assuming that a continuation of the members CA, BD formed part of the 'uncut' structure it will be found that

$$\eta_w = \frac{1 - \nu}{2} + \left(\frac{1 + \nu}{2}\right) \frac{\sin \theta_1 \sin \theta_2 \sin (\theta_1 + \theta_2)}{\theta_1 \sin^2 \theta_2 + \theta_2 \sin^2 \theta_1}$$

$$= 77 \cdot 3 \text{ per cent if } \theta_1 = \theta_2 = \frac{\pi}{4}.$$

5.1.3. *Circular hole*; sheet reinforced by two orthogonal sets of closely spaced stringers such that $X = Y$.

Material removed $= \pi r^2 t(1 + 2X)$.

Reinforcement inserted (*see* Appendix I)

$$= \frac{2\pi r^2 t}{1 - \nu}\{1 + X(1 - \nu)\},$$

therefore

$$\eta_w = \frac{(1 - \nu)(1 + 2X)}{2\{1 + X(1 - \nu)\}}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (20)$$

Typical values are:

$X = \frac{1}{2}$, $\eta_w = 54\cdot5$ per cent

$X = 1$, $\eta_w = 64\cdot2$ per cent

$X = \infty$ (*i.e.*, no sheet), $\eta_w = 100$ per cent.

5.2. *Elliptical Hole with Axes* $\sqrt{2} : 1$ (section 4.2).

Material removed $= \sqrt{2} \,.\, \pi r^2 t$.

Reinforcement inserted $= \int A_m$ around the ellipse $x^2 + \frac{y^2}{2} = r^2$,

$$= \frac{2\sqrt{2}}{9}\left(7 + \frac{25}{2\sqrt{7}}\right)\pi r^2 t,$$

therefore $\eta_w = 38\cdot4$ per cent.

5.2.1. *Elliptical hole with axes* $\sqrt{2}:1$; sheet reinforced by sets of stiffeners in x- and y-directions.

Material removed $= \sqrt{2} \,.\, \pi r^2 t(1 + X + Y)$

and from equation (33) of Appendix III

$$A_m = \sqrt{2} \,.\, Kt(x^2 + r^2)^{3/2} \div \{(4\alpha - \varepsilon)x^2 + (\varepsilon - 2\nu)r^2\} \qquad \ldots \quad \ldots \quad \ldots \quad (21)$$

so that the reinforcement inserted is

$$\frac{\sqrt{2} \,.\, \pi K r^2 t Z(e)}{\varepsilon - 2\nu}$$

where

$$Z(e) = e(5 - 2e) + 2(1 - e)^2\left(\frac{e}{1 + e}\right)^{1/2}$$

and

$$e = \frac{\varepsilon - 2\nu}{4\alpha - \varepsilon},$$

therefore

$$\eta_w = (1 + X + Y)(\varepsilon - 2\nu)/KZ(e). \qquad \ldots \quad \ldots \quad \ldots \quad (22)$$

Typical values are:

$$X = Y = \tfrac{1}{2}\,, \quad \eta_w = 57\cdot 1 \text{ per cent}$$

$$X = \tfrac{1}{2}Y = \tfrac{1}{2}\,, \quad \eta_w = 61\cdot 1 \text{ per cent.}$$

When X and Y are both large compared with unity (*i.e.*, the sheet thickness may be neglected) equation (22) becomes

$$\eta_w = \frac{\lambda(4 + 9\lambda + 4\lambda^2)}{(1 + \lambda^2)(2 + \lambda)^2}\,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (23)$$

where $\lambda = \sqrt{(Y/X)}$.

Expression (23) has a maximum value of unity when $Y = 2X$; all members would be equally stressed in this case.

5.3. *Parabolic Cut-out.* (Section 4.3).

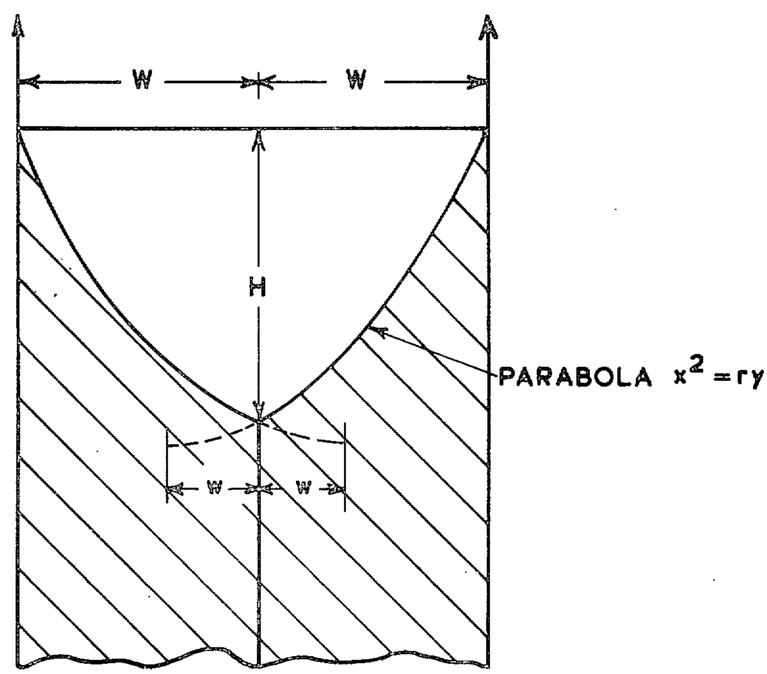

FIG. 9. Notation for parabolic cut-out.

If we assume that a continuation of the central boom formed part of the uncut structure the material removed is

$$\frac{4W^3t}{3r}(1 + 3\mu + 3\mu^2)$$

where

$$\mu = \frac{w}{W} = \frac{rH}{2W^2} - \frac{1}{2}\,,$$

and the material inserted (including the compression member) =

$$\text{Material removed} + \frac{W^2 t(1+\nu)^2 \Gamma}{4\nu} \quad \ldots \quad (24)$$

where

$$\Gamma = 4\left(\frac{r}{W}\right) + \sqrt{\nu}\left(\frac{r}{W}\right)^2 \log \frac{(H/W + W/r - \sqrt{\nu})(H/W - W/r + \sqrt{\nu})}{(H/W - W/r - \sqrt{\nu})(H/W + W/r + \sqrt{\nu})},$$

a non-dimensional quantity proportional to the increase in structure weight.

It follows that

$$\eta_w = \frac{1}{1 + \dfrac{3(1+\nu)^2 r\Gamma}{16\nu W(1 + 3\mu + 3\mu^2)}} \quad \ldots \quad (25)$$

This can be made nearly unity by making H/r sufficiently large. But for this type of cut-out the problem is usually not one in which an efficient 'hole' is required. Instead, the parabolic cut-out may be merely a means for diffusing two concentrated loads uniformly into the sheet and 3-boom structure. A least value is therefore required not for η_w but for Γ (*see* equation (24)) which is proportional to the increase in structure weight.

For any particular value of H/W, which defines approximately the size of the parabola, there is a value of r/W which makes Γ a minimum. These values and the corresponding values of w/W and η_w are given in Table 1 below. The ordinary figures are for $\nu = \frac{1}{4}$, those in brackets for $\nu = \frac{1}{2}$.

TABLE 1

Parameters Corresponding with Minimum Γ

H/W	1·0	1·4	1·73	2·0	2·5
$\Gamma_{\min}$	14·6	6·5 (10·6)	4·3 (6·1)	3·4 (4·5)	2·4
r/W	2·22	1·20 (1·60)	0·87 (1·08)	0·71 (0·84)	0·53
w/W	0·61	0·38 (0·61)	0·27 (0·44)	0·21 (0·34)	0·17
η_w per cent	9·4 per cent	20·6 per cent (22·3 per cent)	30·6 per cent (33·5 per cent)	38·3 per cent (42·7 per cent)	51·0 per cent

5.3.1. *Parabolic Cut-out*; sheet reinforced by closely spaced stringers (Y).

The case when the sheet is reinforced by two sets of closely spaced stringers does not admit of ready analysis. In most practical cases, however, the reinforcement parallel to the direction of the applied loads (stringers) is much greater than that at right-angles to the applied loads (ribs). If this small lateral stiffness is neglected it will be found that the material removed and the material inserted are each $(1 + Y)$ times their previous (unreinforced) values. Table 1 is therefore applicable to this case.

5.4. *Elliptical hole with axes* $\sqrt{3} : 1$. (Section 4.4).

$$\text{Material removed} = \frac{2\pi r^2 t}{\sqrt{3}} .$$

$$\text{Reinforcement inserted} = \frac{\pi r^2 t}{3\sqrt{3}} \left\{ 5 + \nu + \frac{(1 + \nu)^2}{\sqrt{\{(1 - \nu)(5 - \nu)\}}} \right\} ,$$

therefore $\eta_w = 99$ per cent.

For a material with ν less than $0 \cdot 22$, η_w actually exceeds 100 per cent, in other words the weight of reinforcement inserted would be slightly less than the weight of material removed.

Further examples in which the weight of the reinforcement is less than the weight of material removed (sometimes by a very large amount) are given in Appendix IV. The stress distributions corresponding to all highly efficient neutral holes are characterised by the comparatively low stresses developed in that part of the sheet which is to be removed.

5.5. *Discussion of Efficiencies*.—The following general conclusions concerning neutral holes can be drawn from the preceding sections.

(*a*) When the stresses in the sheet are constant the weight efficiency of neutral holes (as defined on page 11) is never greater than unity so that neutral holes could not be used for lightening purposes.

(*b*) When the stresses in the sheet are not constant the weight efficiency could be greater than unity, so that in this case neutral holes could be used for lightening purpose. The stress distributions corresponding to such highly efficient neutral holes are characterised by the comparatively low stresses developed in that part of the sheet which is to be removed.

Referring to the three types of neutral hole considered here in which the sheet stresses were constant, it is found that:

(*c*) The weight efficiency of the parabolic cut-out becomes comparable with those for the circular and $\sqrt{2} : 1$ elliptical holes for values of H/W about 2. For values of H/W less than about 2 the parabolic cut-out soon becomes inefficient. It must be remembered though that the figures quoted in Table 1 include the compression member; it is possible that for other reasons a heavy rib boom may be needed where the compression member would be. If this were so the efficiency figures would be increased considerably, roughly speaking 70 per cent of the increased weight being due to the compression member.

(*d*) The weight efficiency of circular and $\sqrt{2} : 1$ elliptical holes in reinforced sheet is higher than in unreinforced sheet, though this statement by itself may be misleading ; for example, in the circular hole the higher efficiency is due, apart from the factor $(1 - \nu)$, to the lower efficiency of the basic structure. Efficiency of 100 per cent is obtained when the sheet is vanishingly light and the stringers are such that each is strained the same amount.

(*e*) The cross-sectional areas of the reinforcing members, and hence the weight efficiency, depend appreciably on Poisson's ratio. This is especially so in the $\sqrt{2} : 1$ elliptical and parabolic cases.

6. *Conclusions*.—A theory has been developed which shows that if a plane sheet is loaded in any one particular way there is, in general, a large variety of reinforced holes that may be made in the sheet which do not affect the stress distribution in the remainder of the sheet. Such reinforced holes necessarily have exactly the same stiffness and at least the same strength as the replaced sheet and are here defined as neutral holes. If the stress distribution in the sheet is prescribed by an Airy stress function ϕ the shape for such neutral holes is given by a curve of the form ϕ (to which may be added any linear function of x and y) $= 0$; or, with certain modifications, the hole may be bounded by arcs of such curves. The cross-section area of the reinforcement is a function of ϕ and Poisson's ratio. The theory is equally applicable to stringer reinforced sheet ; the only difference lies in the expression for the reinforcement cross-section area. Apart from some trivial exceptions a hole which is neutral for one type of loading will not be so for another.

In addition to the usual assumptions made in the theory of elasticity it is assumed that the bending stiffness of the reinforcement is negligible compared with its tensile stiffness. This assumption is fully justified in Appendix I, where in addition it is shown that a negligible bending stiffness is in fact necessary if the reinforcement is to have least weight.

Four cases only have been considered in detail. These are:

(*a*) A circular hole—for equal direct stress in all directions (as occurs in a thin-walled sphere under pressure).

(*b*) An elliptical hole with axes in the ratio $\sqrt{2} : 1$—for direct stresses in the direction of the axes in the ratio $2 : 1$ (as occurs in a thin-walled cylinder under pressure).

(*c*) A parabolic cut-out—for direct stress in one direction only.

(*d*) An elliptical hole inclined at 45 deg with axes in the ratio $\sqrt{3} : 1$—for equal bending stresses in two directions.

In all these cases the hole shapes and the cross-section areas of the reinforcing members are reasonable from an engineering point of view, though the parabolic cut-out necessitates a 3-boom type of construction. Other shapes and stress distributions are given in Appendix IV.

When the stresses in the sheet are constant the weight efficiencies of neutral holes are normally about 40 per cent, though higher efficiencies are obtained if the sheet is reinforced by stringers. 100 per cent efficiency is obtained when the sheet is vanishingly light and the stringers are such that each is strained the same amount.

When the stresses in the sheet are not constant the weight of the reinforcement for a neutral hole may be less than the weight of material removed. Such neutral holes, in which the weight efficiency exceeds 100 per cent, are characterised by the comparatively low stresses developed in that part of the sheet which is to be removed.

A model of a parabolic cut-out has been tested and good agreement obtained with the theory.

LIST OF SYMBOLS

Ox, Oy	Cartesian co-ordinate axes
t	Thickness of sheet
$\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}_{xy}$	"Mean applied stresses" such that
$t\bar{\sigma}_x$, $t\bar{\sigma}_y$, $t\bar{\tau}_{xy}$	Forces in the stiffened sheet per unit length
σ_x, σ_y, τ_{xy}	Stresses in the sheet, so that if the sheet is not stiffened $\bar{\sigma}_x = \sigma_x$, etc.
ϕ	A stress function (*see* equation (1))
∇^4 =	$\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)^2$
ψ	Angle which tangent to boundary of hole makes with Ox
P	Load in reinforcing member bounding hole
A_m	Section area of reinforcing member bounding hole
ε_m	Strain in reinforcing member bounding hole
E, G	Elastic moduli
ν	Poisson's ratio
a, b, c	Arbitrary constants
f	Particular value of stress
r	Length which determines size of hole
θ	An angle
A_j, A_f, A_r	Section areas of straight reinforcing members
η_w	Weight efficiency, defined in section 5
X, Y	Relative thicknesses of equivalent sheets of X- and Y-members
e, $Z(e)$, λ	Are defined in section 5.2.1
W, H, w	Are defined in Fig. 9
μ =	w/W
Γ	Is defined in section 5.3

Additional symbols used in Appendix I:—

ε	A strain
n	Side ratio of rectangular sectioned reinforcing member
Δ_b =	$\dfrac{\text{Increment of stress due to bending}}{\text{Stress due to uniform tension}}$
Δ_e =	$\dfrac{\text{Bending energy}}{\text{Tensile energy}}$
δ =	$\dfrac{\text{Reinforcement width}}{\text{Radius of hole}}$

Additional symbols used in Appendices II and III are given there.

Additional symbols used in Appendix IV are:—

f_1, f_2	Principal stresses
Q	Out-of-balance load
q	A shear stress
p	Is defined in Fig. 11
l	Is defined in equation (35)
S	Section area of a stringer
r, θ	Polar co-ordinates

LIST OF REFERENCES

No.	*Author*	*Title, etc.*
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2	E. H. Mansfield	Elasticity of a sheet reinforced by stringers and skew ribs, with applications to swept wings. R. & M. 2758. December, 1949.
3	C. Gurney	An analysis of the stresses in a flat plate with a reinforced circular hole under edge forces. R. & M. 1834. 1938.
4	S. Timoshenko	*Theory of Elasticity*. McGraw-Hill Book Co. 1934.

APPENDIX I

The Effect of Bending Stiffness of the Reinforcing member

It is shown here that the assumption of negligible bending stiffness of the reinforcing member is quite justifiable. The circular and $\sqrt{2} : 1$ elliptical holes are considered and it is shown by an approximate method that the energy stored in the reinforcing member due to bending is negligible compared with that due to direct loads. An exact solution for the circular hole is also quoted. ν will be taken as $\frac{1}{4}$ throughout.

(*a*) *Circular hole.*—Due to a uniform strain ε in all directions the change in curvature of the reinforcing member is $-\varepsilon/r$. The cross-sectional area, from equation (12), is $4rt/3$ and therefore the width (in the plane of the sheet) of the reinforcing member, assuming it to be of rectangular section with a width to depth ratio of n, is $\sqrt{(4rtn)}/3$. It follows from engineer's bending theory that the ratio of the increment of stress (or strain) due to bending to the stress (or strain) due to uniform tension is

$$\Delta_b = \sqrt{(nt/3r)}.$$

Thus if the reinforcing member has a square cross-section and $t = 0 \cdot 05$ in., $r = 5$ in., $\Delta_b = 0 \cdot 06$, but if $t = 0 \cdot 1$ in., $r = 5$ in. and $n = 4$, $\Delta_b = 0 \cdot 16$.

The ratio Δ_e of bending energy to tensile energy is obtained by integrating (stress)^2 over the cross-sectional area and is given by

$$\begin{aligned} \Delta_e &= \tfrac{1}{3}\Delta_b^{\,2} \\ &= nt/9r\,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (26) \end{aligned}$$

For the two cases considered above $\Delta_e = 0 \cdot 001$ and $0 \cdot 009$, so that even in the second example the bending energy is less than 1 per cent of the tensile energy.

An exact solution for the circular hole when the reinforcement has a rectangular cross-section has been given by Gurney (R. & M. 1834[3]). For a neutral hole the true section area is

$$\left[1 + (\tfrac{1}{2} - \nu)\delta + \frac{\delta^2}{4 + 2\delta}\right]$$

times the section area given by this report. Here δ is the ratio of the reinforcement width to the hole radius.

It is clear that the factor inside the brackets will not differ appreciably from unity. Further, since ν is never greater than $\frac{1}{2}$ the factor slightly exceeds unity so that for least weight δ should be made as small as possible. For a given reinforcement depth (out of the plane of the sheet) it follows that a reinforcement material with a high E will be slightly more efficient than one with a lower E, assuming that the failing direct strains are the same and the densities are proportional to the E's. For example, high tensile steel will be slightly more efficient than duralumin, though other considerations, such as temperature effects, may alter this conclusion.

The broad conclusions of this last paragraph appear to be true for all neutral holes. The assumption of negligible bending stiffness has been justified and a negligible bending stiffness has in fact been shown to be a necessary quality of the reinforcement if the reinforcement is to have least weight.

(*b*) *Elliptical hole.*—If the strain in the direction of the major axis is 7ε, say, that in the direction of the minor axis is 2ε. From geometry the maximum curvature, at the ends of the major axis, is $\sqrt{2}/r$ and the minimum curvature is $1/2r$ and the changes in these two curvatures are $3\sqrt{(2\varepsilon)}/r$ and $-6\varepsilon/r$ respectively. From equation (14) the section areas of the reinforcing member at these points of maximum and minimum curvature are $2\sqrt{2} . rt$ and $8rt/7$. The corresponding values of Δ_e are therefore

$$\frac{3nt}{2\sqrt{2} . r} \text{ and } \frac{24nt}{343r} . \qquad \qquad (27)$$

The first value for Δ_e is nearly 10 times the corresponding value for the circular case and the second is about $0 \cdot 6$ times. If we take $t = 0 \cdot 1$ in., $r = 5$ in. and $n = 4$ (a severe example) the greatest value for Δ_e will be $0 \cdot 085$; the average Δ_e for the complete structure is about $0 \cdot 02$.

APPENDIX II

Finite Stringer Spacing

When the size of the hole is not large compared with the stringer pitch the analysis of Appendix III will not be valid, but an extended use of the arguments of section 3.1.1 may be made. A simple example will demonstrate this. A sheet is reinforced by stringers in the y-direction of relative section area Y and at pitch p. A uniform loading is applied in the y-direction. What is the shape of the neutral cut-out?

A general form for the stress function in the region between the $(n - 1)$th and nth stringers is, apart from a constant of proportionality,

$$x^2 + a_n x + b_n y + c_n , \qquad \qquad (28)$$

and between the nth and $(n + 1)$th

$$x^2 + a_{n+1} x + b_{n+1} y + c_{n+1} . \qquad \qquad (29)$$

From equation (8) the vertical component of the balancing load is

$$t(a_n - a_{n+1}) = - 2Ypt \quad \left(\text{i.e., } - Ypt \frac{\partial^2 \phi}{\partial x^2}\right) , \qquad \qquad (30)$$

since it must balance the load in the nth stringer. The horizontal component of the balancing load is

$$t(b_{n+1} - b_n) = 0 , \qquad \qquad (31)$$

since there is no horizontal component in the nth stringer.

If $b_n = b_{n+1} =$ etc. $= - r$ and the origin is chosen so that a_n and c_n are zero the curve of equation (28) may be written

$$x^2 = ry$$

and with the restrictions of equations (30) and (31) and the condition of continuity at the nth stringer the curve of equation (29) becomes

$$x^2 + 2pYx = ry + 2p^2Y ,$$

and the curve between the $(n + 1)$th and $(n + 2)$th stringers is given by

$$x^2 + 4pYx = ry + 6p^2Y ,$$

and between the $(n + k - 1)$th and $(n + k)$th stringers by

$$x^2 + 2kpYx = ry + k(k + 1)p^2Y . \qquad \ldots \qquad \ldots \qquad \ldots \qquad (32)$$

The envelope of curves of the type (32) meets successive stringers at points which lie on the parabola

$$x^2(1 + Y) - pYx = ry .$$

APPENDIX III

Rib and Stringer-reinforced Sheet

(*a*) *Skew ribs*.—The results of R. & M. 1834[3] which affect this report are given below.

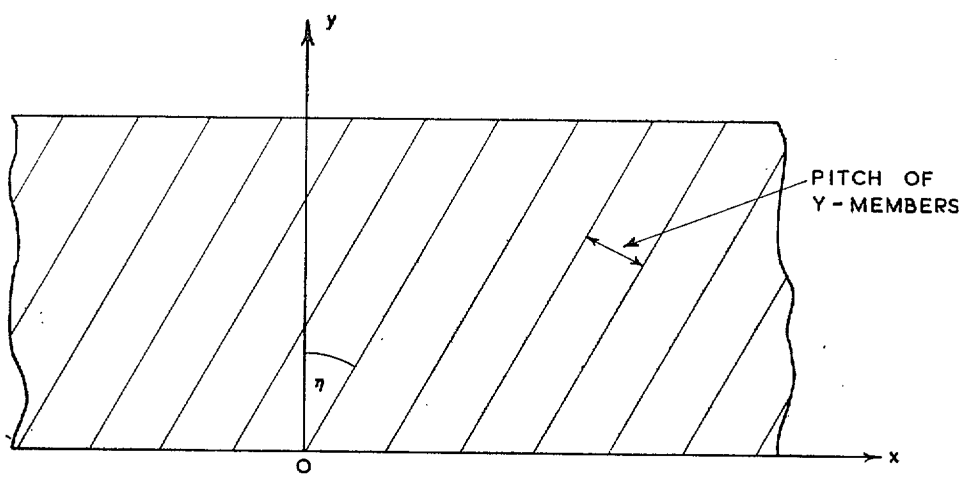

FIG. 10. Figure showing axes for rib and stringer-reinforced case.

The 'mean applied stresses' $\bar{\sigma}_x$, $\bar{\sigma}_y$, $\bar{\tau}_{xy}$ are defined so that $t\bar{\sigma}_x$, $t\bar{\sigma}_y$, $t\bar{\tau}_{xy}$ are the forces in the stiffened sheet per unit length. They are derived from a stress function ϕ as follows:

$$\bar{\sigma}_x = K \frac{\partial^2 \phi}{\partial y^2}$$

$$\bar{\sigma}_y = K \frac{\partial^2 \phi}{\partial x^2}$$

$$\bar{\tau}_{xy} = - K \frac{\partial^2 \phi}{\partial x \, \partial y}$$

where

$$K = 1 + X + Y + \cos^2 \eta \, XY(1 + \nu)(2 \sin^2 \eta + (1 - \nu) \cos^2 \eta) \, ,$$

$$X = \frac{\text{section area of a stringer (or } X\text{-member)}}{t \times \text{pitch of stringers}}$$

$$Y = \frac{\text{effective section area of a rib (or } Y\text{-member)}}{t \times \text{pitch of ribs}}$$

$$\eta = \text{sweepback angle.}$$

Equation (9) would therefore become

$$P = Kt \left\{ \left(\frac{\partial \phi}{\partial x} \right)^2 + \left(\frac{\partial \phi}{\partial y} \right)^2 \right\}^{1/2} .$$

The stresses in the sheet can be expressed in the form

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & J \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 \phi}{\partial x^2} \\ \dfrac{\partial^2 \phi}{\partial x \, \partial y} \\ \dfrac{\partial^2 \phi}{\partial y^2} \end{bmatrix}$$

where

$$A = \nu X - Y \sin^2 \eta + (1 + \nu) \sin^2 \eta \, Y(\sin^2 \eta + 2\nu X \cos^2 \eta)$$

$$B = \sin 2\eta \, Y(1 + \nu)(\sin^2 \eta + \nu X \cos^2 \eta)$$

$$C = 1 + Y \cos^2 \eta \, \{1 + (1 + \nu) \sin^2 \eta\}$$

$$D = 1 + X + Y \sin^2 \eta \, \{\sin^2 \eta - \nu \cos^2 \eta + 2(1 + \nu)(1 + X) \cos^2 \eta\}$$

$$E = 2 \sin \eta \cos^3 \eta \, Y(1 + \nu)(1 + X)$$

$$F = \cos^2 \eta \, Y(\nu \cos^2 \eta - \sin^2 \eta)$$

$$G = -\sin \eta \cos \eta \, Y\{\cos^2 \eta - \nu \sin^2 \eta + X \cos^2 \eta \, (1 - \nu^2)\}$$

$$H = -1 - X - Y + 2Y \sin^2 \eta \cos^2 \eta \, (1 + \nu) - \cos^4 \eta \, XY(1 - \nu^2)$$

$$J = \sin \eta \cos \eta \, Y(\nu \cos^2 \eta - \sin^2 \eta).$$

These results have to be substituted in equation (10) before $A_m (= P/E \varepsilon_m)$ can be determined.

The compatibility equation is

$$\alpha \frac{\partial^4 \phi}{\partial x^4} + 4\beta \frac{\partial^4 \phi}{\partial x^3 \, \partial y} + 2\gamma \frac{\partial^4 \phi}{\partial x^2 \, \partial y^2} + 4\delta \frac{\partial^4 \phi}{\partial x \, \partial y^3} + \varepsilon \frac{\partial^4 \phi}{\partial y^4} = 0$$

where

$$\alpha = 1 + (1 + \nu)\{X(1 - \nu) + Y \sin^2 \eta(\sin^2 \eta(1 - \nu) - 2\nu^2 \cos^2 \eta X + 2(1 + X) \cos^2 \eta)\}$$

$$\beta = \sin \eta \cos \eta Y(1 + \nu)\{\cos^2 \eta(1 + X) - \nu(\sin^2 \eta + \nu X \cos^2 \eta)\}$$

$$\gamma = 1 + (1 + \nu)\{X + Y + XY \cos^4 \eta(1 - \nu^2) - 3 \sin^2 \eta \cos^2 \eta Y(1 + \nu)\}$$

$$\delta = \sin \eta \cos \eta Y(1 + \nu)(\sin^2 \eta - \nu \cos^2 \eta)$$

$$\varepsilon = 1 + \cos^2 \eta Y(1 + \nu)(1 - \nu + \sin^2 \eta(1 + \nu)) .$$

(*b*) *Ribs normal to stringers.*—Although this case may be obtained from the results given above by putting $\eta = 0$, the expression for A_m simplifies considerably and is given below

$$A_m = tK\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right\}^{3/2}\left[\alpha\frac{\partial^2\phi}{\partial x^2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \varepsilon\frac{\partial^2\phi}{\partial y^2}\left(\frac{\partial\phi}{\partial y}\right)^2 + 2\gamma\frac{\partial^2\phi}{\partial x\,\partial y}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\right.$$
$$\left. - \nu\left\{\frac{\partial^2\phi}{\partial x^2}\left(\frac{\partial\phi}{\partial y}\right)^2 + \frac{\partial^2\phi}{\partial y^2}\left(\frac{\partial\phi}{\partial x}\right)^2 - 2\frac{\partial^2\phi}{\partial x\,\partial y}\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\right\}\right]^{-1} . \quad \ldots \quad (33)$$

$$K = 1 + X + Y + XY(1 - \nu^2)$$

$$\alpha = 1 + X(1 - \nu^2)$$

$$\gamma = 1 + (1 + \nu)\{X + Y + XY(1 - \nu^2)\}$$

$$\varepsilon = 1 + Y(1 - \nu^2)$$

and the compatibility equation becomes

$$\alpha\frac{\partial^4\phi}{\partial x^4} + 2\gamma\frac{\partial^4\phi}{\partial x^2\,\partial y^2} + \varepsilon\frac{\partial^4\phi}{\partial y^4} = 0 .$$

APPENDIX IV

Other Stress Distributions

The cases considered here include:

(*a*) General case of uniform stress distribution, with particular reference to pure shear.

(*b*) Pure bending.

(*c*) Polar co-ordinates.

and

(*d*) Stress distributions with circular neutral holes and constant A_m.

These will not be considered in great detail.

(*a*) *General case of uniform stress distribution, with particular reference to pure shear.*—If the axes Ox, Oy are chosen parallel to the directions of the principal stresses f_1, f_2 the stress function is given by

$$\phi = \tfrac{1}{2}(f_1 y^2 + f_2 x^2) + ax + by + c \qquad \qquad (34)$$

and so the neutral hole is in the form of a conic. Furthermore, if f_1 and f_2 are of the same sign the conic is an ellipse with lengths of axes in the ratio $\sqrt{(f_1/f_2)}$; if f_1 and f_2 are of opposite sign, as in pure shear, the hole is an hyperbola or is bounded by arcs of hyperbolas. For example Figure 11 below shows a neutral hole in a sheet subjected to pure shear. (The figure has been rotated through 45 deg so that the axes are parallel to the directions of the principal stresses.) The hole is symmetrically bounded by four arcs of rectangular hyperbolas and suitable loads Q must be applied at the four junction points.

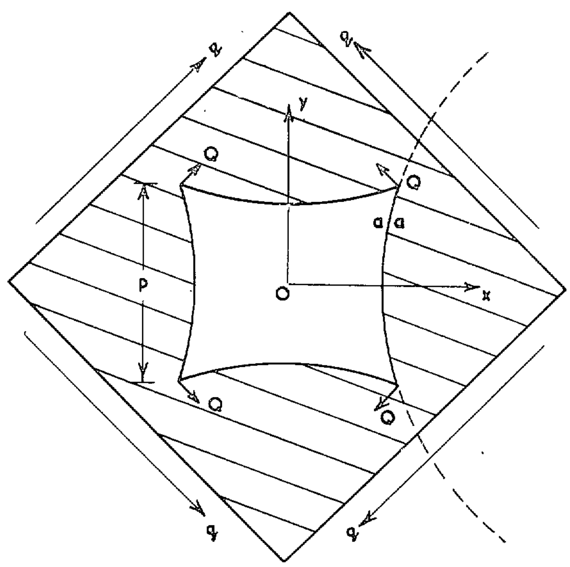

FIG. 11. Hole bounded by hyperbolic arcs.

To determine the out-of-balance loads Q we note that if the hyperbola containing the arc *aa* of Figure 11 has its centre of symmetry at the point $(-l, 0)$ its equation may be written

$$\phi = \frac{q}{2}(x^2 - y^2 + 2lx - pl) = 0 \, . \qquad \qquad (35)$$

Thus for the arc *aa* at the corner point $(\tfrac{1}{2}p, \tfrac{1}{2}p)$:

$$\text{vertical component of load} = \frac{t\,\partial\phi}{\partial x} = qt(p + 2l)/2$$

$$\text{horizontal component of load} = -\frac{t\,\partial\phi}{\partial y} = qtp/2 \, .$$

The loads at the corner points in the other arcs may now be written down from symmetry. The out-of-balance loads Q are of magnitude $\sqrt{2} \, . \, qtl$ acting in the directions shown in Fig. 11. These loads Q could be applied by modifications to the main structure as described below.

Suppose that two equal and orthogonal sets of stringers of cross-section area S and pitch p are attached to the sheet so that intersections of the stringers occur at the four corner points $(\pm\frac{1}{2}p, \pm\frac{1}{2}p)$. If the lengths of stringers between these four corner points are removed the resultant out-of-balance loads will exactly represent the Q loads provided

$$l = \frac{(1 + \nu)S}{t} . \qquad (36)$$

The hole, such as that represented in Fig. 12, will be neutral.

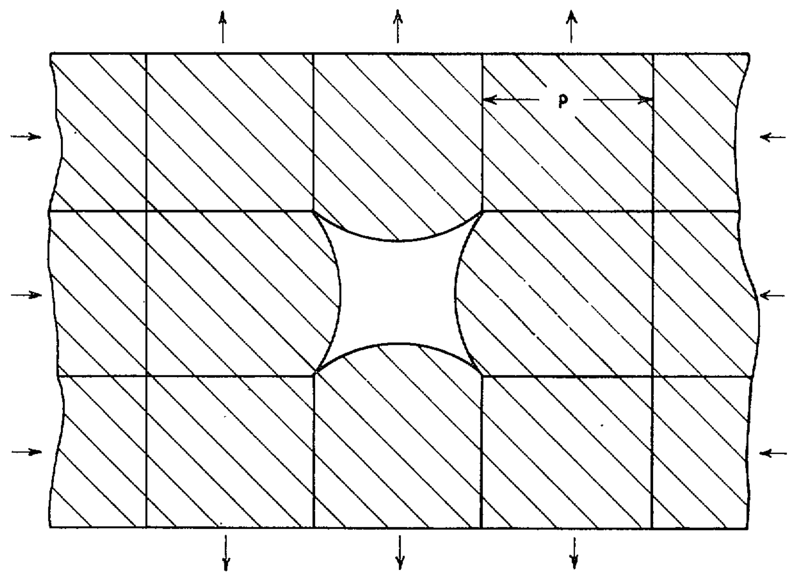

FIG. 12. Neutral hole in stringer-reinforced sheet.

The cross-section area of the reinforcing member *aa* is given by

$$\frac{A_m}{t} = \frac{(2y^2 + l^2 + pl)^{3/2}}{2\sqrt{2} \, . \, (1 + \nu)(l^2 + pl)} . \qquad (37)$$

(*b*) *Pure bending*.—For this case

$$\phi \propto y^3 + a_1 x + b_1 y + c_1 = 0 ,$$

which represents a cubic, is the equation for the neutral hole.

A suitable hole with positive A_m everywhere would be that shown in Fig. 13. The hole is doubly symmetrical being bounded by four equal reinforcing arcs.

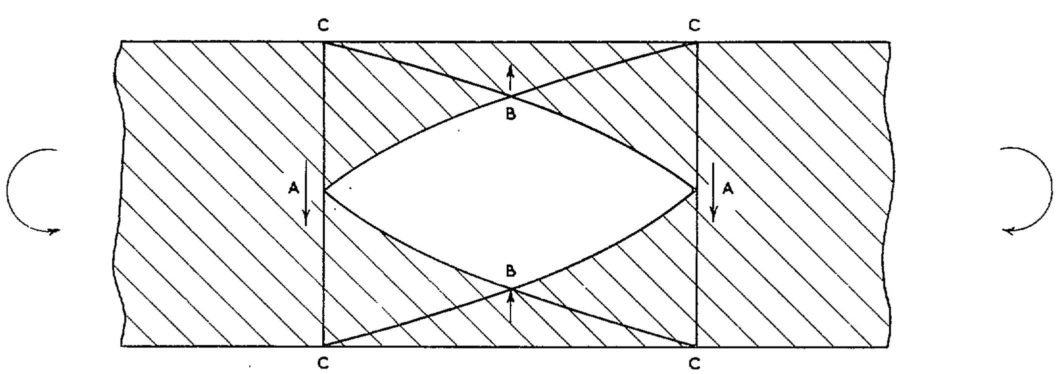

FIG. 13. Neutral hole in beam under pure bending.

The hole will not be neutral unless suitable loads are applied at the junction points A, B in the directions as indicated above. These loads can be exerted by the structure itself if suitable straight members AC and BC are inserted and the beam flanges reduced between CC.

(*c*) *Polar co-ordinates.*—The stress function employed when polar co-ordinates are in use is identical with that employed here. Thus, if ϕ is a stress function in polar co-ordinates the equation determining the shape of a neutral hole will be

$$\phi + ar \cos \theta + br \sin \theta + c = 0 . \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (38)$$

Most of the practical cases of neutral holes have been considered already, but on the structural research side it may sometimes be advantageous to use a bounding arc of a neutral hole to represent exactly a large (or infinite) amount of sheet. For example, a stress function representing a particular state of stress in an infinite wedge is

$$\phi = r^2\theta .$$

If this were to be checked experimentally a finite, and therefore manageable, apex could be cut off from the wedge provided the boundary line suitably reinforced, satisfied equation (38). Such a line would be, say,

$$r = \frac{b \sin \theta}{\theta} \quad (\textit{see } \text{Fig. } 14). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (39)$$

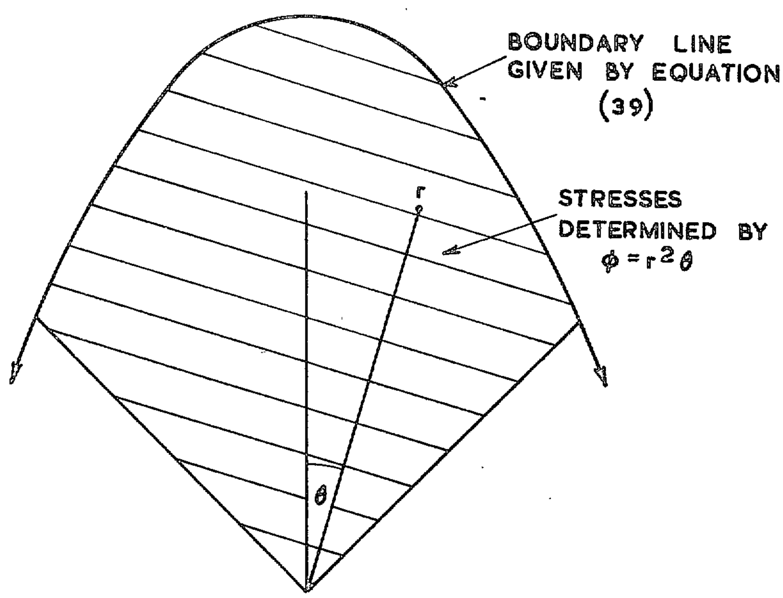

FIG. 14. Example using polar co-ordinates.

The general expression for A_m in polar co-ordinates is

$$\left.\begin{aligned} A_m &= \frac{tP^{3/2}}{Q - \nu R}, \\ \text{where} \qquad & \\ P &= \left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2, \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (40)$$

$$
\left.\begin{aligned}
Q &= \frac{\partial^2\phi}{\partial r^2}\left(\frac{\partial\phi}{\partial r}\right)^2 + \frac{2}{r^2}\frac{\partial^2\phi}{\partial r\,\partial\theta}\left(\frac{\partial\phi}{\partial r}\right)\left(\frac{\partial\phi}{\partial\theta}\right) - \frac{1}{r^3}\frac{\partial\phi}{\partial r}\left(\frac{\partial\phi}{\partial\theta}\right)^2 + \frac{1}{r^4}\frac{\partial^2\phi}{\partial\theta^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2, \\
R &= \frac{1}{r^2}\frac{\partial^2\phi}{\partial r^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2 - \frac{2}{r^2}\frac{\partial^2\phi}{\partial r\,\partial\theta}\left(\frac{\partial\phi}{\partial r}\right)\left(\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r}\left(\frac{\partial\phi}{\partial r}\right)^3 + \frac{2}{r^3}\frac{\partial\phi}{\partial r}\left(\frac{\partial\phi}{\partial\theta}\right)^2 + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\left(\frac{\partial\phi}{\partial r}\right)^2
\end{aligned}\right\} \quad \ldots \quad (40)
$$

(*d*) *Stress distributions with circular neutral holes and constant* A_m.—It was pointed out in section 4.1 that the circular hole with the value for A_m given by equation (12) would be neutral for one particular stress distribution and none other. But the circular shape itself is not confined to that one particular stress distribution. For example, confining attention to plain sheet, any function which satisfies

$$\nabla^4\phi = 0\,, \quad \ldots \quad (41)$$

gives rise to a possible stress distribution. If, therefore, we take

$$\phi = (x^2 + y^2 - r^2)\,F(x,y) \quad \ldots \quad (42)$$

and choose $F(x,y)$ so that equation (41) is satisfied then a stress system will be formed for which the circle is a possible shape for a neutral hole. The section area of the reinforcing member will usually not be constant, but there is a set of possible functions of $F(x,y)$ in which A_m is constant. This set is characterised by $F(x,y)$ being homogeneous in x and y. It is more convenient to use polar co-ordinates in discussing this set. Thus, if c is the radius of the circle and n is the order of homogeneity of $F(x,y)$, equation (42) becomes

$$\phi = (r^2 - c^2)r^n f(\theta) \quad \ldots \quad (43)$$

and it is known[4] that to satisfy equation (41)

$$f(\theta) = a \sin n\theta + b \cos n\theta\,. \quad \ldots \quad (44)$$

Substituting in equation (11) it will be found that

$$A_m = \frac{ct}{(2n + 1 - \nu)}\,, \quad \ldots \quad (45)$$

which is a constant ;

and the corresponding values for the weight efficiency are given by

$$\eta_w = n + \left(\frac{1 - \nu}{2}\right), \quad \ldots \quad (46)$$

so that, except for the special case with $n = 0$ considered in section 4.1, the weight of reinforcement inserted is always appreciably less than the weight of material removed. The reason for this lies in the fact that the sheet which is to be removed is comparatively lightly stressed.

To get a clearer idea of the type of loading necessary, take the case of $n = 2$. The functions $(x^2 - y^2)$ and xy are the possible forms for $F(x,y)$; considering the first alone gives

$$\phi = (x^2 + y^2 - r^2)(x^2 - y^2)\,, \quad \ldots \quad (47)$$

whence

$$\sigma_x = -12y^2 + 2r^2$$

$$\sigma_y = 12x^2 - 2r^2$$

$$\tau_{xy} = 0.$$

Such a state of applied loading consistent with this is shown in Fig. 15.

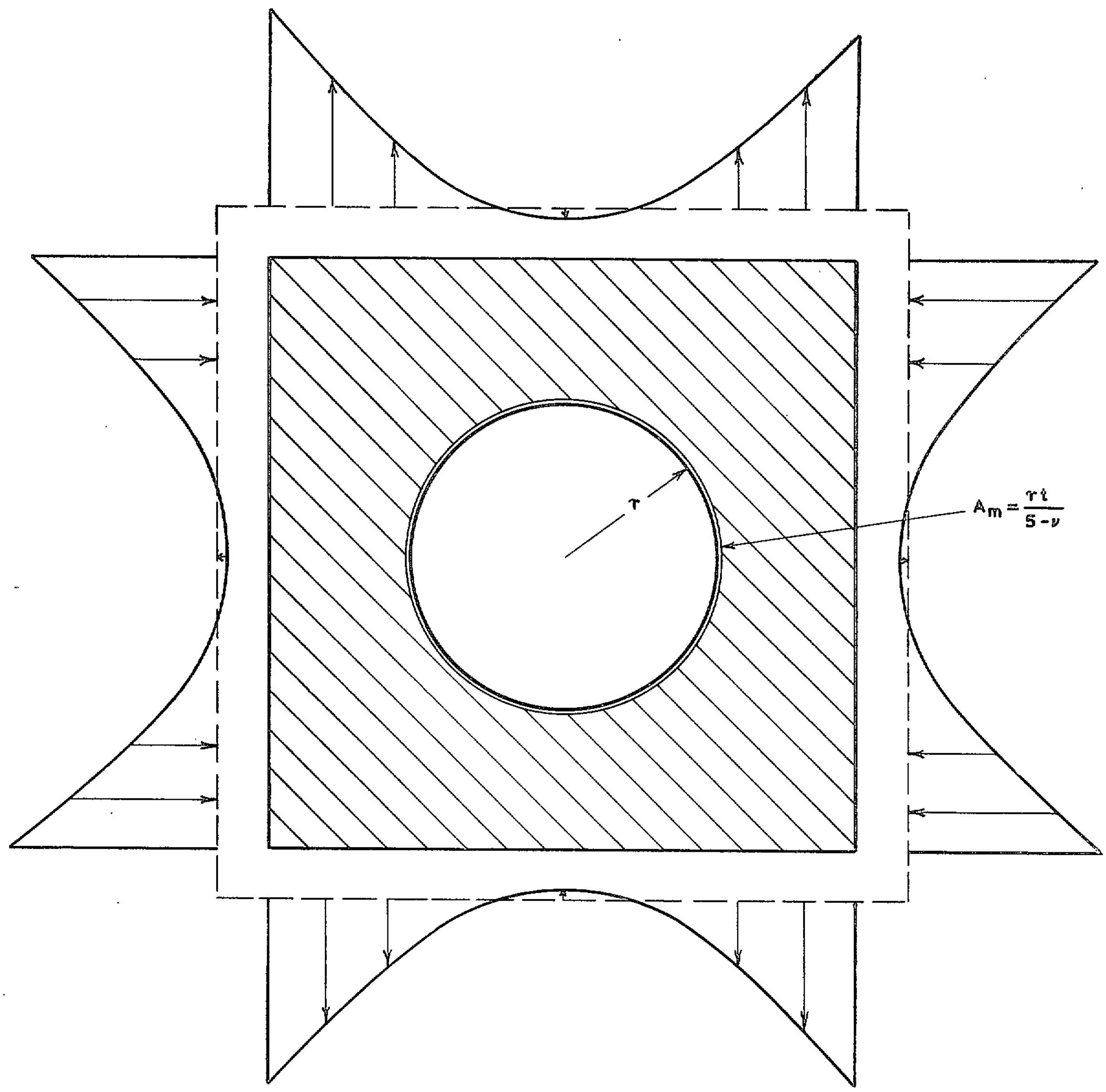

FIG. 15. Loading and neutral hole consistent with equation (47).

APPENDIX V

Test on Model of Parabolic Cut-out

The parabolic cut-out was chosen for the following reasons:

(*a*) The corresponding loading system, pure tension, is particularly common and important.

(*b*) The loading is easy to apply.

(*c*) Additional checks on the stress distribution are possible by strain-gauging the associated straight members, namely the compression member, the central boom and two side booms. This strain-gauging of as many as possible of the booms and reinforcing members is important in that strain-gauge readings from them are more reliable than from the sheet itself.

The model was of cellulose nitrate and all joints and connections were rigidly glued. Poisson's ratio for this material is about $\frac{1}{2}$ so that the reinforcing member and central boom were larger, and the compression member smaller, than in a corresponding cut-out in a material in which $\nu = \frac{1}{4}$, say.

The parabolic cut-out and associated 3-boom panel formed one-half of the top surface of a thin-walled box of rectangular cross-section. The box measured $100 \times 16 \times 4$ in., so that the panel width $2W$ was 16 in. The sheet and rib thickness was $0 \cdot 04$ in. and the edge booms A_f were $\frac{1}{2} \times \frac{1}{2}$ in. and the rib pitch was 4 in. There were no stringers and the small lateral stiffening of the ribs was ignored.

The relevant dimensions for the parabolic cut-out were

$$H = 15 \cdot 4 \text{ in. so that}$$

$$H/W = 1 \cdot 92 ,$$

$$w = 10 \cdot 9 \text{ in. so that}$$

$$A_j = 0 \cdot 87 \text{ sq in.}$$

and
$$w/W = 1 \cdot 36 ,$$

$$r = 15 \cdot 4 \text{ in. so that}$$

$$A_r = 0 \cdot 62 \text{ sq in.}$$

and
$$r/W = 1 \cdot 92 .$$

These dimensions differ appreciably from the optimum ones of Table 1, in fact Γ is double its minimum value. However, these dimensions were chosen because the curved reinforcing member has a cross-section area which remains sensibly constant. Its average section area, $1 \cdot 02$ sq in., was taken for convenience in construction. (This section area is actually 3 per cent too high at its highest and 4 per cent too low at the ends.)

The panel was loaded by applying equal bending moments at the two ends of the box. The loading arrangement at one end and the strain-gauge recording apparatus are shown in Fig. 19. The strain-gauge positions are shown in Fig. 20, with the exception of 3 gauges added later to each of the edge booms.

For measuring direct strains each dummy gauge was stuck on at right-angles to the 'active' gauge to form a T. Roughly 50 per cent increased sensitivity was obtained this way, and the shear gauges were now precisely 2 times (instead of E/G times) as sensitive in measuring shear stress as the direct gauges were in measuring direct stress.

The strain-gauge results are plotted in Fig. 21 in such a way that a comparison with the theory may be readily made. The 10 gauges on the various booms and reinforcing members give good agreement, the gauges having an average deviation from theory of only 4 per cent. Twelve gauges on the sheet show appreciably more scatter, as would be expected, and there is a tendency for the readings to decrease with increase in distance from the compression member. This tendency is undoubtedly due to the sheet becoming more slack as the distance increased between its supporting members—edge boom and reinforcing member. It was not possible to increase the total load to such an extent that all slackness was taken up. Of the gauges adjacent to the reinforcing member one was broken and two gave erratic and inconsistent readings ; these were ignored. The sheet readings show an average deviation from theory of 10 per cent.

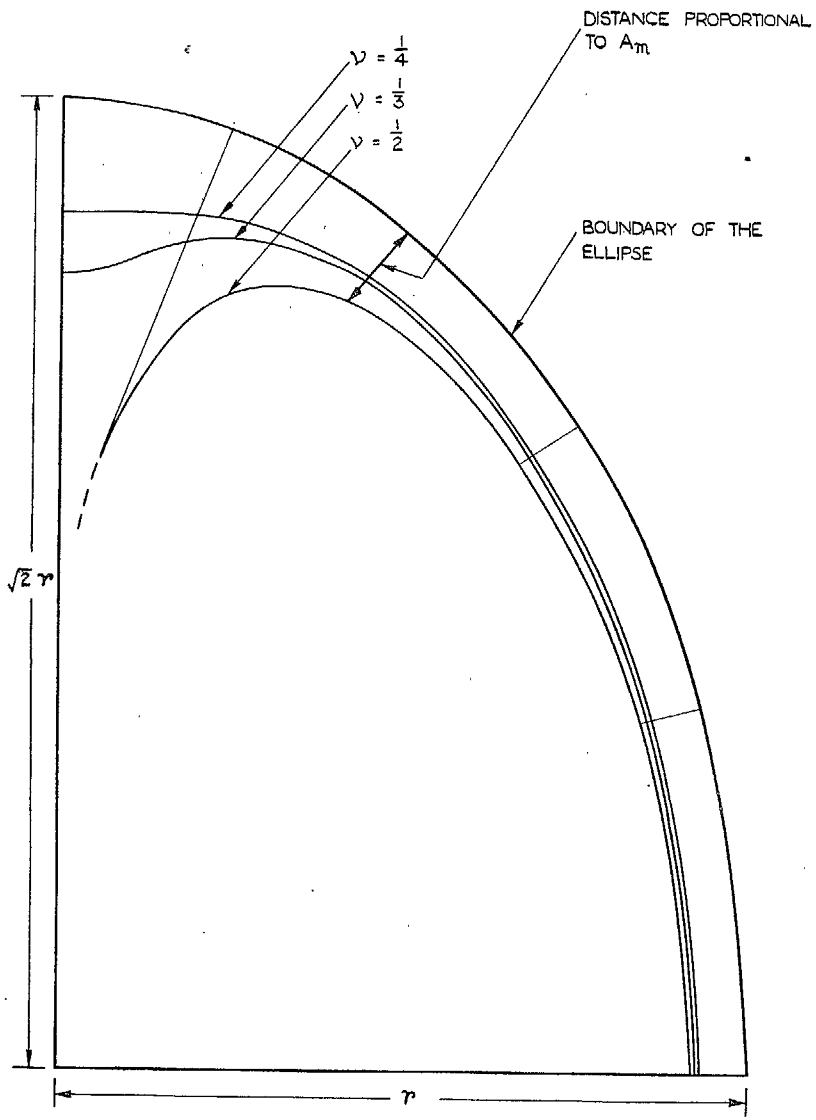

FIG. 16. Relative variation of A_m for the $\sqrt{2}:1$ elliptical hole.

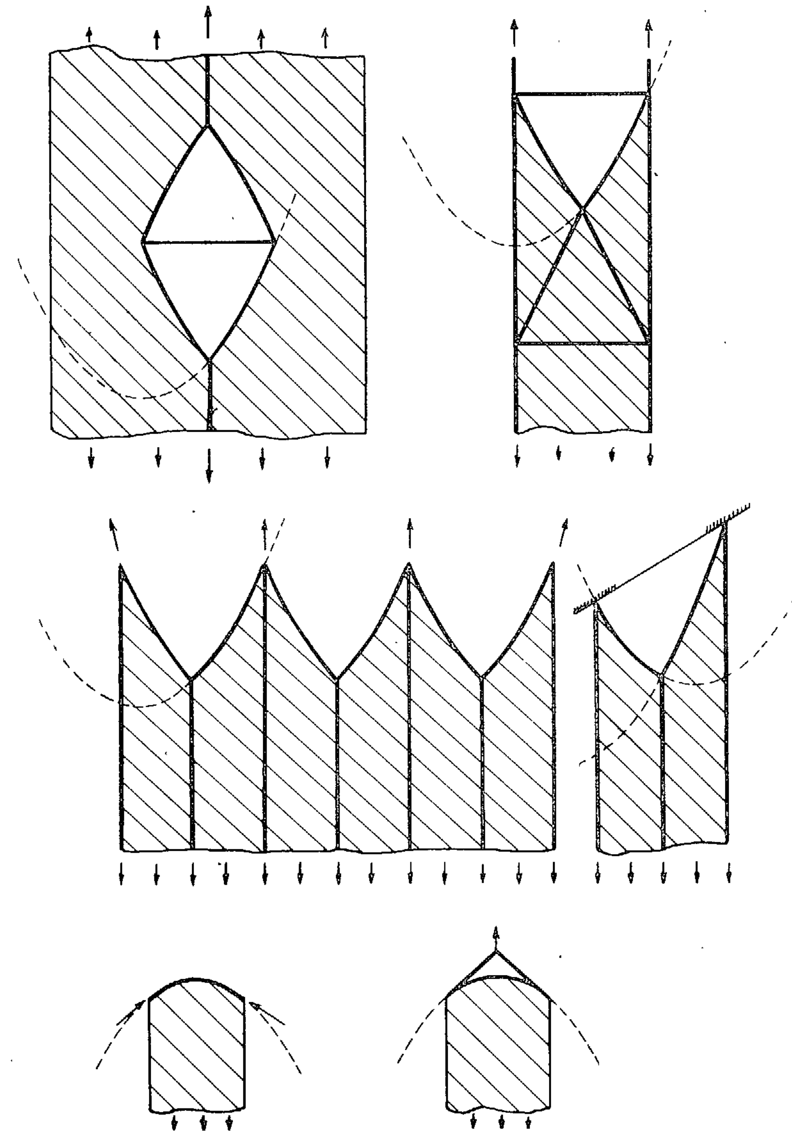

FIG. 17. Subsidiary forms based on the parabola.

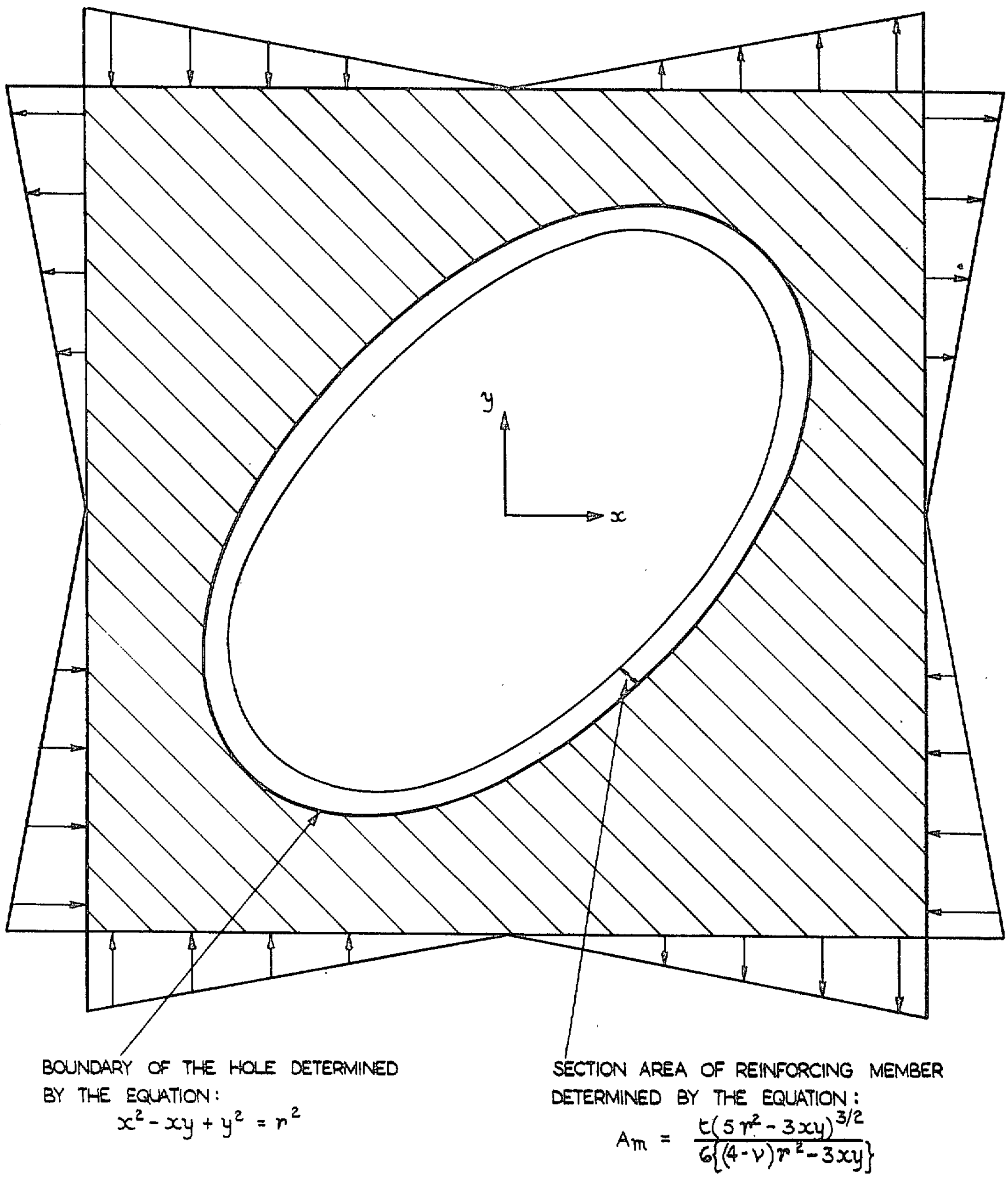

FIG. 18. Neutral hole for equal bending stresses in two directions.

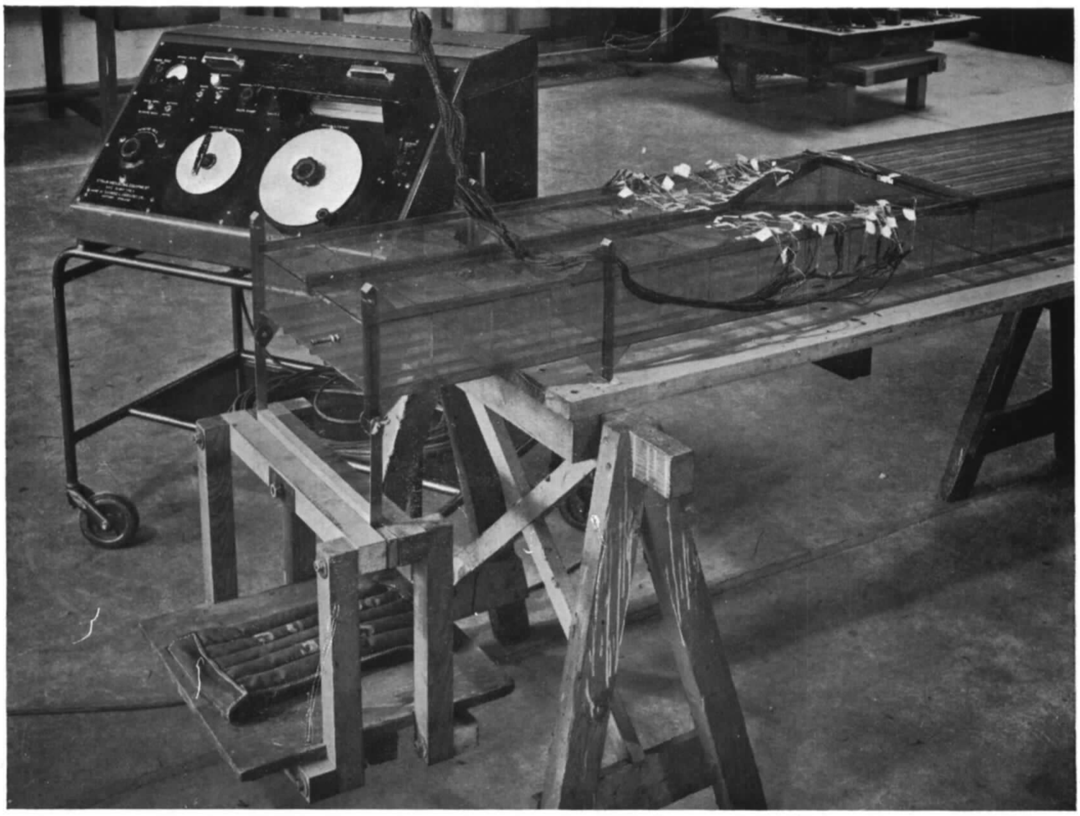

FIG. 19. The parabolic cut-out showing method of loading and recording apparatus.

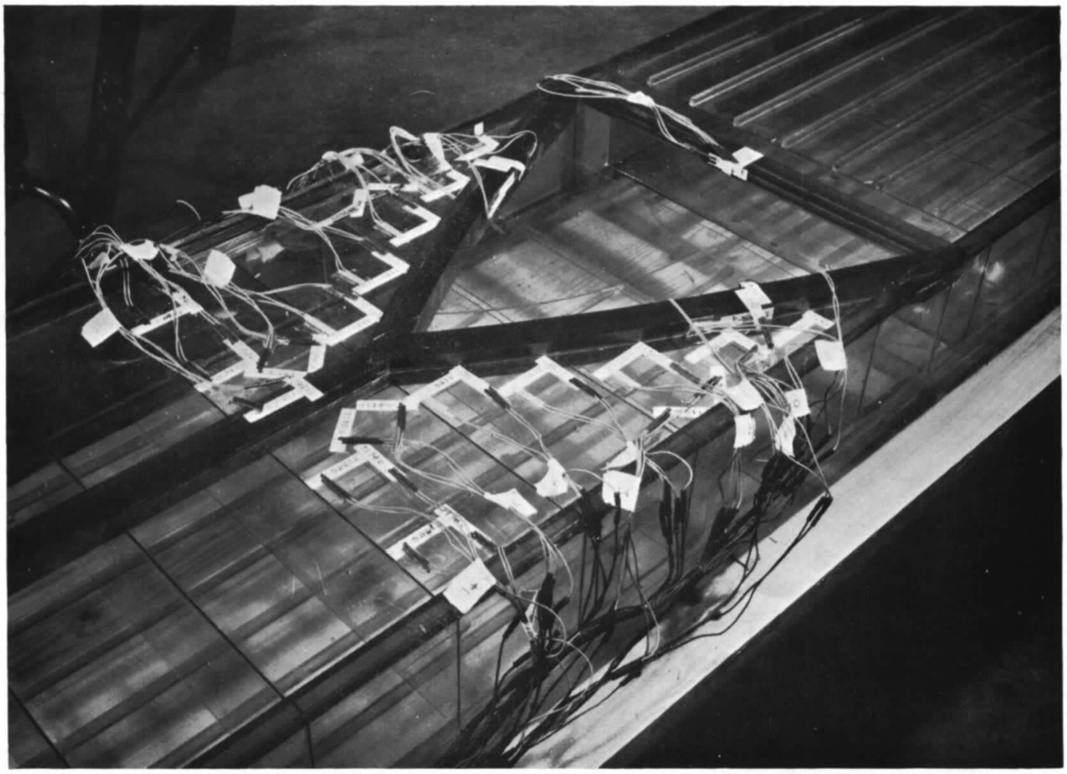

FIG. 20. The parabolic cut-out showing strain-gauge positions.

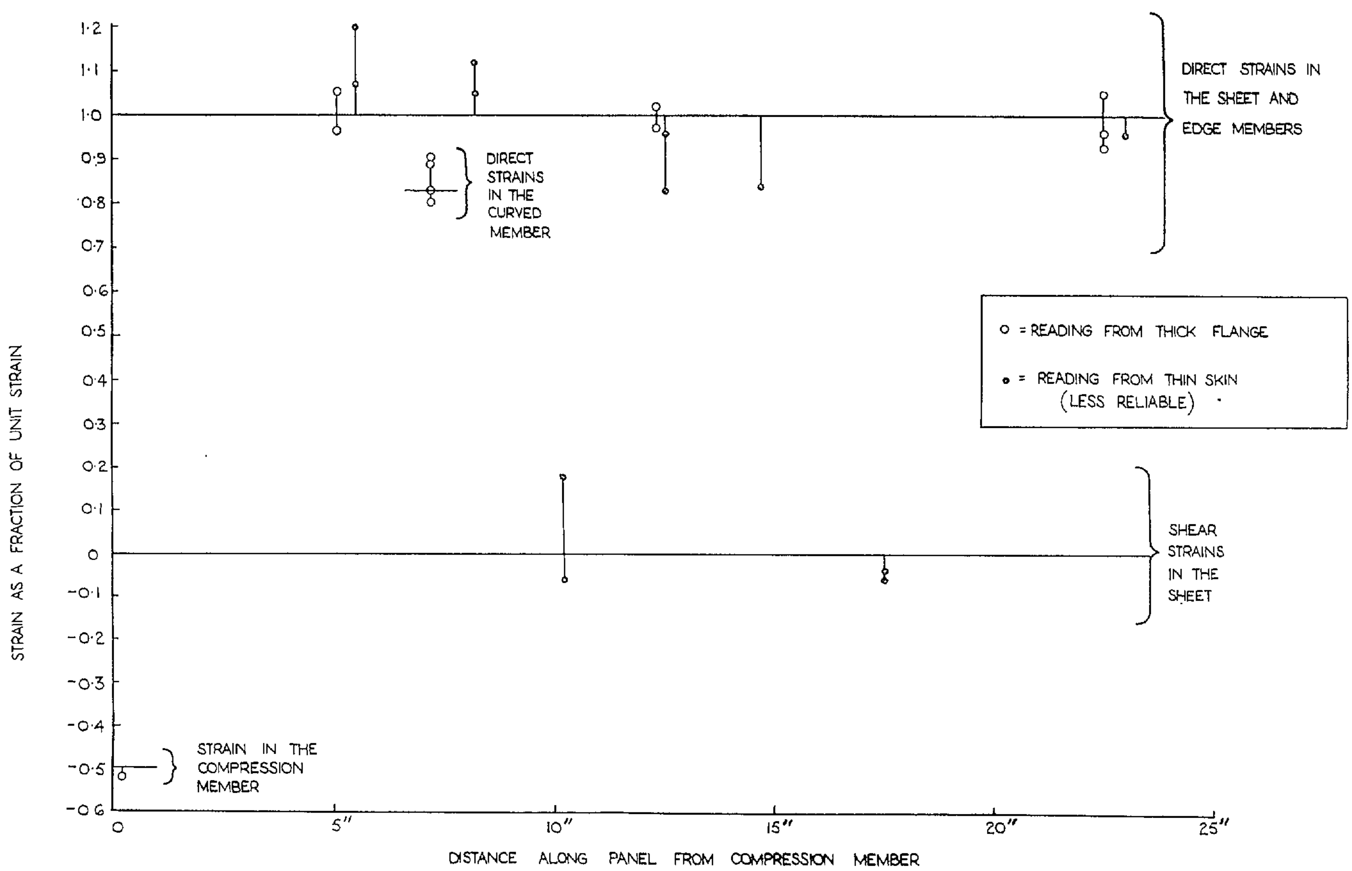

FIG. 21. Experimental results for parabolic cut-out.

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