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The Scope and Accuracy of Vortex Lattice Theory

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The Scope and Accuracy of Vortex Lattice Theory

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Summary.—The report gives an outline of the development of the principles on which potential problems in lifting-plane theory are solved by the use of a vortex lattice for the purpose of computing downwash. The conditions of convergence necessary for an accurate solution are defined, and the main purpose of the report is to show that those connected with the lattice have been, or can easily be satisfied.

Published solutions by this method have been mainly concerned with spanwise load grading and local aerodynamic centre and examples are given here of earlier checks on accuracy for rectangular and triangular wings, and a yawed infinite wing, based either on an alteration of the lattice spacing or on comparison with downwash obtained by surface integrals. The study of accuracy is now advanced by a comparison based on exact values calculated from surface integrals given by W. P. Jones, and applied to a rectangular and a sweptback wing. The downwashes obtained from the lattice are shown to converge to the exact values, but by a comparison of two solutions for the sweptback wing it is shown that the beneficial coupling effect of the lattice makes it unnecessary to obtain individual downwash values to great accuracy, at least for spanwise load grading and aerodynamic centre calculations.

Trial calculations reveal that there would be no difficulty in extending the convergence to detailed pressure distribution or other properties of any thin wing, but it is desirable to give prior attention to the main effects of wing thickness and viscosity.

1. *Introduction.*—Recent criticism of the work which has been completed on the calculation of aerodynamic wing loading by a theory involving the use of a lattice for computing downwash, coupled with the fact that the planning of extensive programmes of research may depend on an accurate judgment of this work, makes it advisable for a clear statement of the objective to be rendered. A previous note[1] on the accuracy of the calculations was issued in May, 1946, but with the completion of more work it is possible to state the case with greater knowledge.

2. *Outline of Theory.*—The work which has so far been completed has been based on the following developments :—

(a) A generalisation of the work originated by Blenk[2] on a rectangular wing, and extension to arbitrary plan form and to include wing twist, discontinuities due to deflected flaps, effects of compressibility, and so on.

(b) The solution of potential problems, and many non-potential problems as, for example, those which include the effects of viscosity, to a prescribed degree of accuracy, by the use of a network instead of a continuous medium for the evaluation of downwash.

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These two lines of advance are being dealt with concurrently in a programme which has so far been mainly directed to an exploration of whether problems expressible in terms of vortex sheets can be solved by the use of a network of vortices simple enough to make the work economically suitable for general use. Favourable circumstances are met at the outset in that lifting line theory, in which the chordwise distributed vorticity is reduced to a single line vortex for the purpose of downwash calculation, has been in use for years with marked success, and it is reasonable to anticipate that improved accuracy is obtainable by building on the foundations of lifting line theory without going so far as to lose the essential simplicity of the representation.

2.1. At the outset, it was necessary to make a choice between spending a great amount of time in carrying to the limit one typical case, with complete proofs as to accuracy, or of proceeding on the assumption that the developments were valid and covering, in consequence, much more ground. Because of the importance of the application of the work to design problems, the latter choice was made, and there is no doubt about the correctness of the decision, for it has led to considerable advances in general knowledge and the provision of a framework of general information from which it is apparent that a rigid proof of accuracy for a simple case would be useless from the point of view of the general application of the theory. The difficulty in providing positive proof of accuracy is that the simpler forms of corroborative evidence, *e.g.*, wind tunnel tests, are not pure potential solutions, the provision of which for a complicated case is an exceedingly difficult matter.

2.2. Early in the work, comparisons were made between solutions obtained by using exact integrals of downwash and by the lattice, and, after a suitable procedure for allowing for the discontinuity in plan had been developed, the effort was concentrated on covering a wide range of thin wing solutions with a complication sufficient only to ensure that the spanwise load grading and local aerodynamic centre were given to good accuracy. From the coefficients given with these solutions the pressure distribution over the surface could be worked out, but this was obviously a case where the user must take the risk of unspecified accuracy. It was known some years ago that the two chordwise terms used for the calculations were inadequate to give the detailed pressure distribution to great accuracy in the region of a discontinuity, and, so far, a solution to the required degree of complication to give final covergence has not been worked out. The writer has satisfied himself by trial calculations that the same method taken to more terms will give a convergent pressure distribution, but there is little point in finishing a calculation of this nature until the main effects of wing thickness and perhaps viscosity have been investigated, as these are always present in any application of the theory.

2.3. For most of the work which has so far been published, the lattice is used as a quick method of evaluation of downwash, and, as it will be shown below that this process is convergent, the work conforms to the now established practice of lifting plane theory in whatever form this may be expressed.

The use of a lattice, however, will be shown below to offer certain advantages in saving of labour, because it introduces a beneficial coupling between the downwash values.

For wings with deflected flaps, the use of a lattice offers in addition a means of simplifying the solutions so that these can be effected rapidly when the detailed pressure distribution is not required, but only accuracy in the spanwise load grading and local aerodynamic centre.

3. *Development of the Downwash Calculations.*—At an early stage it was decided to standardise the principle of representing a continuous circulation in the spanwise direction by a system of rectangular vortices for purposes of downwash calculation, by making the strength of each rectangular vortex the same as the value of the continuous circulation at the midpoint of the transverse vortex. It was an obvious move to try out this sytem by comparison with lifting line theory, with which agreement is essential to form a proper foundation for the work, and this procedure has the advantage that the downwash associated with this theory is defined by a simple mathematical formula.

As a result of a study of the induced downwash, it was found that a corrector vortex was required at the position 0·9625 and, with this included in the pattern, the downwash was correct for all practical purposes up to $0 \cdot 8s$, but not beyond, when a spacing of 20 to the span was used. It also appeared that $0 \cdot 9s$ could be reached by using precisely the same pattern with the spacing halved, and that this method of extension was repeatable. The way in which the whole scheme becomes practical when the corrector vortex is added is shown in Fig. 1, in which are plotted the induced downwash as calculated by the lattice compared with the same calculated mathematically for the 19-vortex pattern, *i.e.*, excluding the two end corrector vortices, and the 21-vortex pattern. The details of calculation can be readily deduced from the information given in R. & M. 1910 (1943)[3] and 2591 (1947)[4].

The results of a further investigation are given in Table 1, which compares the values of induced downwash corresponding to lifting-line theory, as obtained by a lattice of vortices for several spanwise gradings of $K/4sV$, with the exact values. It will be noted that the 21-vortex pattern is reasonably good to $\eta = 0 \cdot 8$, even for $\eta^{10}(1 - \eta^2)^{1/2}$, but that a considerable general improvement is shown when the 41 pattern is used. The general convergence in the spanwise direction with respect to number of vortices seems to be well established.

3.1. The convergence in the chordwise direction is based on two-dimensional considerations, and the sequence is as in Fig. 2. The idea is that any chordwise function can be represented for purposes of downwash calculation and not necessarily for loading, by replacing the continuous function by a finite number of line vortices spaced at even distances along the chord, *e.g.*, 4 at 1, 3, 5 and 7 eighths of chord, and so on. The magnitudes of the line vortices are then calculated by making the two-dimensional values of the downwash due to the line vortices the same as those due to the continuous distribution at stations midway between the line vortices, the spare condition being used to make the total vorticity correct. The first approximation involves one line vortex only, and the analysis must therefore be limited to the term $\cot \theta/2$. The line vortex of equal total magnitude is placed at the half-chord, and the control point, for which the downwash due to the line vortex equals that due to the distributed vorticity will be at 1·0 chord. Because the downwash due to $\cot \theta/2$ is uniform, it is possible to improve the representation by moving the vortex and control point forward until the vortex coincides with the centre of pressure (C.P.) of the continuous distribution, *i.e.*, the vortex is now placed at 1/4 chord and the control point at 3/4 chord. As the solution is based on the use of the continuous distribution, and the representation of line vortices is used only for the determination of downwash, this variation would have no effect on a wing of constant chord, and only a minor effect in the general case. The first approximation is correct for two-dimensional flow, but there is an appreciable error when the flow is three-dimensional.

3.2. The second approximation involves a limited number of vortices, say 4 or 6, or more, and it will be seen that, for the 4-vortex system as shown in Fig. 2, for which the magnitudes are given in Table 7 of R. & M. 2591[4], the C.P.'s of the system for the terms $\cot \theta/2$, $\sin \theta$, and $\sin 2\theta$, are 0·313, 0·5, and ∞ instead of the values 0·25, 0·5 and ∞ given by the continuous loading. The wing loading system does not, of course, depend directly on these values, which are incidental to a system designed for the computation of downwash, but it is of interest to note that the C.P. for the $\cot \theta/2$ term of the lattice could be altered from 0·313 to 0·25 by moving the system forward by 0·063 chord. A variation of this kind would seriously impair the symmetry of the arrangement and the ease of application of the method, but would lead only to a trivial variation in the solution. Where 12 chordwise vortices are used (see Table 12) the C.P. of the $\cot \theta/2$ lattice is at 0·271, a much nearer approach to the quarter chord. With the second approximation, the two-dimensional values of downwash are accurate, while the errors in the three-dimensional values are greatly reduced. The number of vortices in the second approximation requires adjustment depending on the plan of the wing.

When the number of vortices has been increased sufficiently, *e.g.*, the approximation n, it will be found that the C.P. will be the same for the continuous and line vortex systems, and that the values of downwash will be accurate for both two and three dimensions. The success of the application depends, however, upon a close approximation to the limit having been reached with a few vortices, and there is abundant evidence that a considerable amount of useful and accurate work can be effected with the 4 or 6-vortex pattern, although in some special and advanced cases a greater number will be necessary.

4. *General Scheme of Convergencies.*—The following convergencies must be nearly simultaneously complete if an accurate solution for a wing is to be obtained :—

(a) Convergence of solution with respect to lattice spacing spanwise.

(b) Convergence with respect to lattice spacing chordwise.

(c) Convergence with respect to number and position of control points.

(d) Convergence with respect to the number of unknown coefficients or loading functions used in the solution.

The first two depend upon varying the size of the mesh of the lattice ; the fourth upon the use of a Fourier series or other special functions by which the vortex sheet representing the wing is defined ; and the third to a certain extent upon the other three.

The degree of complexity of the solution which is required to establish convergence of the four items will vary with the nature of the problem to be treated and with the characteristics in question. For example, a wing of circular plan form will require much less work than a wing with discontinuities in plan, and the effect of deflecting flaps will be greatly to increase the complication. Also, experience has shown that there is a variation in rate of convergences : for example, the grading of spanwise circulation converges quickly, and often involves only one chordwise term, cot $\theta/2$; and a considerable amount of evidence has been collected which supports the view that the local aerodynamic centre can be adequately defined by the use of two chordwise terms only. Much valuable information can be obtained by a study of the circulation and aerodynamic centre, but trial calculations have shown that a reasonably accurate calculation of pressure distribution in the neighbourhood of a discontinuity in plan would require at least three or four terms and there is no doubt that an approach to the accurate solution in the vicinity of a corner would require a large number of terms.

4.1. The method of obtaining covergence can vary considerably. For instance, the number of control points can, and should in many cases, be greater than the number of unknown coefficients, and, by subsequent normalisation of the equations connecting the unknowns, the effect of any error at an individual control point reduced considerably. In fact, it appears that the averaging effect of this process is often particularly valuable in assisting the convergence of the solution.

4.2. It should now be quite clear to the reader that when a solution is published the meaning to be attached to it is that the work has been carried out to a degree of complication sufficient to establish to a reasonable degree of accuracy certain properties, of which the circulation and the local aerodynamic centre are two of the most important. It should not be assumed that the solutions may not later be improved, if the necessity arises, by extending the work to give accuracy in respect to further properties of the wing, when the pressure distribution at selected stations is required to specified accuracy. It should be remembered that there is no hope of finality in this work, as the analysis could gradually be extended to questions which are merely academic and could have no possible practical application, for instance, the behaviour of the solution in the immediate vicinity of a corner on the wing plan.

Because of the indefinite nature of the convergences and the fact that finality in the provision of potential solutions is quite impossible, the main effort has been put into making maximum use of the application of the work. Published solutions which claim to have reached a certain degree of convergence have been checked by internal evidence and by any external evidence which bears on the problem, and the writer has so far found nothing which can be used to prove that the developments are invalid. A final point is that there is little merit in over-elaboration of a solution for a thin wing when it is known that all wings have a finite thickness.

4.3. We now present some of the latest evidence as well as evidence which has been collected at intervals during the development of the work from the original rectangular wings. The object of providing this evidence was to show that the calculations were proceeding on the right lines and not that a hypothetical ' exact solution ' had yet been reached.

5. *Examples of Convergence : Rectangular Wing.*—The solution for a rectangular wing of aspect ratio 6 was originally calculated by the use of the 84-vortex lattice using 6 control points, located at $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ chord and $\eta = 0\cdot2$ and $0\cdot8$. The same wing was also considered some years ago by Argyris using 6 control points in very nearly the same position, but with the downwash at each point calculated from the formula applicable to the surface integral instead of by a network or simpler means. In this work Argyris corrected some of Blenk's[2] results which were known to be slightly in error. The two results are now given in Table 2 and Fig. 8, and, although neither solution has been taken to the limit, and so complete agreement is not to be expected, the agreement was close enough to support the conclusion that the downwash for a straight wing could be calculated by the use of a network of vortices.

5.1. Although calculations of downwash at an individual point are of limited use in assessing the accuracy of the overall solutions, because they involve too narrow an issue, they can be used to show that the calculations of downwash by the lattice method are convergent. It is possible to make an advance in the work by the use of exact mathematical formulæ, given by W. P. Jones[5] in R. & M. 2225, by which the values of the downwash at the mid-chord points of a rectangular wing of aspect ratio 6 can be calculated for vorticity distributions corresponding to the most important terms of the series normally used. The accuracy of the values given by Jones was a little uncertain and they have now been recomputed to greater accuracy by the Staff of the Mathematics Division of the Laboratory, who have provided the exact values now given in Table 3. The values corresponding to vortex sheets $2V(1-\eta^2)^{1/2}\cot\theta/2$, $2V\eta(1-\eta^2)^{1/2}\cot\theta/2$, $2V\eta^2(1-\eta^2)^{1/2}\cot\theta/2$, and $2V\eta^4(1-\eta^2)^{1/2}\cot\theta/2$ for the mid-chord point at $\eta = 0\cdot2$, $0\cdot5$, and $0\cdot8$ are given in this table. In order to show how the lattice calculations converge, work has been completed which includes the use of 4, 6 or 8 vortices to the chord and 21 or 41 to the span. In R. & M. 2591[4] it is shown how, after a certain side displacement from the control point, the number of vortices is reduced to one for the $\cot\theta/2$ or $\sin\theta$ term, and the effect of varying the number of columns with the full number of vortices has now also been studied. It is most convenient to define the system by the semi-range, expressed as a proportion of the span, covered on each side of the control point by the columns with the full vortex system, for example, 9 columns of 21 vortex $\equiv$ 4 columns on each side $\equiv 0\cdot2$ span semi-range.

5.2. Before considering final details of the convergence revealed by the figures of Tables 3 and 4, we consider some points of interest. The downwash factors are usually obtained direct from the critical tables (R. & M. 2461[6]) to three places of decimals, and an examination of the integrals has shown that there would be no appreciable advantage in increasing the places of decimals to four in the close range over which the full number of vortices are used. Outside this range, where one only vortex per station is used, error assessment indicates that there is a slight advantage in using four places of decimals, and, in order to allow this to be done accurately, additional tables of downwash have been included as an Appendix. These tables are supplementary to the ' Tables of complete downwash due to a rectangular vortex' (R. & M. 2461[6]). All future work will therefore use three places of decimals within the close range, and four places outside this.

We now consider the magnitude of the range over which the full number of vortices are used. If examination be made of items 2, 3 and 4, items 7, 8 and 9, and other similar sets of calculations given in Tables 3 and 4, it will be seen that the computed downwash has for all practical purposes converged to its final value at semi-range $0\cdot3$, and all future work will adopt this value as the standard for either 21 or 41 vortex work.

5.3. A selection of results for points 1, 2 and 3 has been plotted in Figs. 3 and 4. It will be seen that for points 1 and 2 the lattice with 21 spanwise vortices will give a very close approximation to the true values of the downwash, while for point 3 it may be necessary to go to 41 spanwise vortices if an accuracy of closer than 1 per cent is required. The results establish without doubt that the lattice process converges rapidly for the rectangular wing. It seems that for this wing all the accuracy normally required is obtainable by using not more than 8 vortices chordwise, but further work on another wing described below with 12 vortices will show that greater accuracy is obtainable if required.

6. *Examples of Convergence : Infinite Yawed Wing.*—Calculated values of the downwash at selected points on an infinite wing of constant chord at 45 deg yaw have been made with various patterns of lattice to compare with the exact values. The four cases treated are shown in Fig. 5 and include patterns equivalent to the 21-vortex solution ; 84-vortex solution ; and 126-vortex solutions with $2y_v = 0 \cdot 25$ chord, which is the same as a spacing of $0 \cdot 1s$ with aspect ratio 5, and $2y_v = 0 \cdot 3\dot{3}$ chord, equivalent to a spacing of $0 \cdot 1s$ with aspect ratio $6 \cdot \dot{6}$. The latter case has been included as the pattern changes its character when y_v is varied.

The values of downwash were computed by summing the effects of the individual vortices by the standard method and using the standard chordwise factors, the spanwise loading function being taken as unity. The summation was carried out to the limit of the published tables ($y = 76$) and the limits for an infinite number of terms estimated by using the condition, which is easily verified for an unyawed vortex, that the limit is approached hyperbolically. It would be possible to examine the limit more closely by pure mathematical analysis but it is thought that the addition accuracy would be so small as not to warrant the time spent on it.

6.1. For an infinite wing with circulation $K = \frac{1}{2}\pi Vc$, the true value of w/V over the wing is $1/\sqrt{2}$ or $0 \cdot 707$. In Table 5 are given the values of w/V for the same circulation obtained by summation of the effects of the lattice of vortices. The result given by the single vortex is a trifle high, which is consistent with the known fact that solutions by this pattern usually give a slightly low value of the circulation. The other values which include points at $0 \cdot 5$, $0 \cdot 75$ and $0 \cdot 8\dot{3}$ chord are extremely close to the true value, and better agreement is hardly to be expected considering that the limits of the sums of the downwashes due to the lattice pattern are obtained by a method which is not exact.

This work shows that there is no fundamental error for a sweptback wing due to the use of a lattice orientated parallel to the wing direction.

7. *Examples of Convergence : Triangular Wing.*—Further work has been carried out on the triangular wing for which a number of solutions have been described in R. & M 2591[4]. In order to remove the variations near the tip which, although of no practical importance, tend to mask the degree of convergence, a comparison is now made of two solutions (a) 126-vortex, 6-point solution, with control points at $\eta = 0 \cdot 2$, $0 \cdot 6$ and $0 \cdot 8$ and at $0 \cdot 5$ and $0 \cdot 8\dot{3}$ chord and (b) 328-vortex, 12-point solution, with control points at $\eta = 0 \cdot 2$, $0 \cdot 5$, $0 \cdot 7$, and $0 \cdot 8$ and at $0 \cdot 25$, $0 \cdot 5$ and $0 \cdot 75$, chord. The solutions were standard solutions with the application of the centre line correction, and the additional condition was used that the local a.c. at the tip is at $0 \cdot 25$ chord. It has been explained in R. & M. 2591[4] that this condition is a likely one in that it agrees with the assumption that the curvature of flow at the tip remains finite. The two solutions are given in Table 6, and Fig. 9 and show that, as far as the load coefficient and local aerodynamic centre are concerned, the solutions have converged for all practical purposes with respect to the alteration of lattice spacing and number of control points within the limits of the loading functions used. The slightly low value of $dC_L/d\alpha$ for the 126-vortex solution is usual and is corrected by the formula given in R. & M. 2591.

8. *Examination of V-Wing.*—A considerable amount of work, both theoretical and in the wind tunnel, is being carried out on the wing defined in report[7] 11609, *i.e.*, a V-wing of aspect ratio 3 with uniform chord, and 45 deg of sweepback. This wing provides a more difficult case than a straight wing as it involves the important discontinuity in direction of trailing edge at

the wing centre, the effect of which it is required to establish. An examination is the more desirable in that it is intended to use this wing for a more detailed investigation into pressure distribution.

A plan of the wing is given in Fig. 6, showing the five points at $\eta = 0$, $0 \cdot 2$ and $0 \cdot 8$ at which the accuracy of the lattice calculations of downwash is to be gauged. Because of the time involved in calculating the accurate surface integrals, it has been necessary to limit the investigation to the spanwise distribution represented by $(1 - \eta^2)^{1/2}$, and to one or two terms chordwise, but the results given for the rectangular wing in section 5 leave no doubt that this is sufficient to establish the general accuracy.

8.1. Exact values of the downwash w/V have been calculated by a surface integral for points 2, 3, 4, 5 for the vortex distribution $2(1 - \eta^2)^{1/2} \cot \theta/2$, and for point 1 for the distribution $2(1 - \eta^2)^{1/2} [\cot \theta/2 - \sin \theta]$. The variation for point 1 is necessary because the $(1 - \eta^2)^{1/2}$ loading function does not contain the singularity which is present in the complete loading functions for a wing with discontinuity, the effect of which is to cause the downwash at points on the discontinuity to remain finite.

In the absence of the special spanwise loading functions it therefore becomes necessary to introduce the condition $k = 0$ at the control point, k being the intensity of vorticity, but it is important to note that this does not invalidate the demonstration of convergence.

The evaluation of the surface integrals was undertaken by the Mathematics Division of the Laboratory, the required formulæ being given by W. P. Jones[8] in R. & M. 2145.

8.2. The true surface integrals, and values obtained by lattices of varying complexity are given for the five points in Tables 7, 8 and 9. The major effort has been connected with the points at the half chord because previous experience has shown that accuracy is more difficult to obtain the further forward the control points are placed. The figures for points 1, 2, and 4 are plotted in Fig. 6. It will be seen that it is inadvisable generally to use a semi-range of less than $0 \cdot 3$, and, taking this value as the criterion, the lattice gives an answer exceedingly close to the exact value for point 2 when the 21/6, 21/8, 21/12 or 41/12 lattices are used. As regards point 4, this has been shown by past experience to be one of the most difficult points to deal with as it is near the tip and in the region of rapidly decreasing circulation. The results, however, show that the lattice calculations are convergent, there being a considerable improvement from the 21/6 value of $0 \cdot 909$ to the 41/12 value of $0 \cdot 945$, the true value being $0 \cdot 960$. There is no doubt that the calculations with an 81 lattice would give still closer agreement.

8.3. Finally, the values for point 1 obtained by the standard process are all high and do not appear to be converging to the exact value. As a result of this disagreement, the use of the process at a discontinuity was examined and it was decided that a slight modification was justifiable and necessary. The modification is shown in Fig. 7 which is concerned with that part of the wing in the neighbourhood of the discontinuity. The use of the standard lattice is equivalent in one respect to representing the area by the stepped diagram as shown. The rectangles represent the average area of the wing satisfactorily except that the one enclosing the discontinuity is too far forward. The modification consists in shifting this rectangle backwards as a unit with its enclosed pattern of vortices and control points to make the areas A and B equal. In the general, or non-symmetrical case, the shift backwards would be governed by averaging on the area CDEFGH.

The success of this modification is shown by the results given in Table 9, and in Fig. 6, where the figures of items 4, 5 and 12 relating to the revised lattice, with the centre rectangle only shifted back $0 \cdot 5$ semiwidth of vortex, indicate satisfactory convergence to the true value of w/V.

8.4. Although convergence is not complete for points at the discontinuity and near the tip, it is at least certain that the values of w/V given by the latest lattice of 41 spanwise and 12 chordwise are very much nearer the exact values than those obtained by the lattice of 21 spanwise and 6 chordwise.

For point 1, the figures in the order 21/6 (original lattice, 0·2 semi-range), 41/12 (modified lattice, 0·3 semi-range), and exact, are 2·317, 2·163, 2·190 ; for point 2, 2·082, 2·089 and 2·085 ; and for point 4, 0·905, 0·945, and 0·960.

The investigation can, if required, be carried to a further degree of accuracy, but the following work will show that it is doubtful whether such a step will be necessary.

9. *Simplification of Downwash Calculations : Spanwise.*—Before making a final assessment of the accuracy of solutions based on the 21/6 lattice which have previously been published, an account is given of a simplification of the work by employing the principle of interpolation. The simplification is such that it is easily possible to use the 41/12 lattice, with the advantage of superior accuracy, and the demonstration of this gives an opportunity of expounding a quicker method of setting out the calculations.

Consider firstly a control point at $\eta = 0$, or at the wing centre. It is obvious from published work, for example, Table 3 of R. & M. 2591[4] that the magnitude of the downwash factors rapidly diminishes as the distance from the control point increases, and the simplification is based on the omission of some of the smaller factors and their replacement by proportions of the factors retained.

A study of the rate of decrease of the factors showed that standard interpolation formulæ were not valid, except when the factors were very small. A successful method of interpolation was obtained by polynomial interpolation on the factors multiplied by y^2, where y is the displacement from the control point less one vortex width. The details of interpolation must be omitted due to lack of space, but there is no new principle involved and the final results only are given. In Table 10 are given the location of the spanwise vortices retained, and the factors to be applied to the downwash to compensate for the missing Stations. The results are given for $\eta = 0$, 0·2, 0·6 and 0·8 and apply to any lattice of 41 spanwise. For $\eta = 0$, only one half the wing need be included.

Plans have been made for further simplification by working on the chordwise distribution in a similar way, but this development must be omitted from the present report.

9.1 An example is now given of the details of the calculation of downwash (the short process) for point 1. By the use of a table of distances as shown in R. & M. 2591[4], the downwash factors for 12 chordwise vortices, or 1 chordwise vortex near the tip, can be written down direct as shown in Table 11, being taken from critical and other tables from relative distances given on the calculating machine by a simple calculation. This table is converted direct to the composite factors for $\cot \theta/2$, and $\sin \theta$ of Table 13 by using the appropriate factors given previously in Table 7 of R. & M. 2591[4], and now augmented in Table 12 by the addition of figures for a 12 point solution. Another column of Table 13 gives $(1 - \eta^2)^{1/2}$ increased by the interpolation factors of Table 10, and the downwash w/V corresponding to the vortex sheet $2V(1 - \eta^2)^{1/2}$ $[\cot \theta/2 - \sin \theta]$ is $\Sigma(1 - \eta^2)^{1/2}$ (factors) $[\cot \theta/2 - \sin \theta]$ multiplied by a factor related to the lattice, in this case 20/3. This integral becomes 2·173 as against the value of 2·163 obtained by the long process, and the exact value 2·190.

A selection of values obtained by the long and short processes, is given in Table 14 for comparison with the exact values for points 1 to 5, and it will be seen that in every case the long can be replaced by the short without sacrificing any appreciable accuracy.

10. *Accuracy of Published Solutions.*—It is clear from these results that the figures given by the lattice of 41 spanwise and 12 chordwise, even if not exact, are very much closer to the exact values than those given by the previously used lattice of 21 spanwise and 6 chordwise, and this is true whether or not the short method of calculation is used. For example, in the order 21/6, 41/12, and exact, the figures for point 1 are 2·317, 2·173, and 2·190 ; for point 2, 2·082, 2·104, and 2·085 ; and for point 4, 0·905, 0·940 and 0·960.

Hence, a measure of the accuracy of the solutions already published and obtained by the use of the 21 lattice, can be found by computing further solutions based on calculations by the 41/12 lattice. The comparison is for the moment limited to the two quantities which have been the main objective of the work so far completed, *i.e.*, the load grading and the aerodynamic centre, and it is not intended for this to be the final verdict on pressure distribution. Two eight point solutions have been calculated for the sweptback wing of Fig. 6, the first using the 21/6 lattice with semi-range $0 \cdot 2$, and the second the equivalent of the 41/12 lattice with semi-range $0 \cdot 3$, the values of the downwash being obtained however by the short process as described in section 9. For these solutions, the control points were located at $\eta = 0, 0 \cdot 2, 0 \cdot 6$ and $0 \cdot 8$ and at $0 \cdot 5$ and $0 \cdot 83$ chord and the loading functions used were $(1-\eta^2)^{1/2}$, $\eta^2(1-\eta^2)^{1/2}$, $\eta^4(1-\eta^2)^{1/2}$, with the special function $P = 0 \cdot 65 P_a + 0 \cdot 35 P_b$, as described in R. & M. 2596[9], in order to allow for the centre-line correction. The values of the latter function used in the calculations, and applicable to the lattice of 41 spanwise, are given in Table 16.

10.1. The two solutions are given in Table 15 and have been plotted in Fig. 10 from which it will be seen that the agreement is remarkably good, the only difference being a slight variation in the curve of local aerodynamic centre. It is therefore concluded that the answer would be little different if the downwash integrals had been exact. Although the degree of agreement is surprising in view of the changes in individual downwash values, it supports other evidence that published solutions based on the 21/6 integrals are sufficiently accurate. It is clear that the lattice representation adds something beneficial to lifting plane theory, presumably due to a favourable coupling between the downwash values, even though each is not necessarily given to close accuracy.

11. *Conclusions.*—The report has provided corroborative evidence that published potential solutions of spanwise load grading and local aerodynamic centre do not suffer appreciable inaccuracy on account of any defect in calculations of downwash at control points, and are therefore almost as if the downwashes had in fact been calculated to great exactitude by surface integrals.

Trial calculations by the writer have shown that the same methods and loading functions are likely to give convergence is detailed pressure distribution, but three or four terms instead of two may be needed in the region of a discontinuity. As the application of the work to practical problems must invariably involve wing thickness and viscosity, there is little point in paying minute attention to thin wing solutions. The effort should be put into finding the simplest variation to allow for wing thickness, concentrating first on load grading and local aerodynamic centre after which the detailed pressure distribution can receive attention.

In conclusion the writer wishes to acknowledge the help received from Dr. E. T. Goodwin and Mr. F. W. J. Olver of Mathematics Division, both in respect to advice received in discussion of some of the mathematical problems involved, including the use of statistics and interpolation, and in the general implication of the work of computing the surface integrals. Acknowledgments are also due to Mr. W. P. Jones, who has helped the writer considerably in elucidating some of the problems and difficulties concerned in the use of his surface integrals.

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* and see Appendix to this present R. & M.

APPENDIX

Tables supplementary to R. & M. 2461

Value of −F

±X	Y = 26		Y = 28		Y = 30		Y = 32	
	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve
0	0·00296	0·00297	0·00256	0·00256	0·00222	0·00222	0·00196	0·00196
5	0·00240	0·00353	0·00210	0·00301	0·00186	0·00259	0·00166	0·00225
10	0·00190	0·00403	0·00170	0·00341	0·00152	0·00293	0·00137	0·00254
15	0·00148	0·00445	0·00135	0·00376	0·00123	0·00322	0·00113	0·00278
20	0·00115	0·00478	0·00107	0·00404	0·00099	0·00346	0·00092	0·00299
25	0·00091	0·00502						
30	0·00072	0·00521	0·00069	0·00442	0·00065	0·00380	0·00062	0·00329
40	0·00048	0·00545	0·00046	0·00465	0·00044	0·00401	0·00043	0·00348
50	0·00033	0·00560	0·00033	0·00478	0·00032	0·00413	0·00031	0·00360
60	0·00024	0·00569	0·00024	0·00487	0·00023	0·00422	0·00023	0·00368
80	0·00015	0·00578	0·00014	0·00497	0·00014	0·00431		
100	0·00010	0·00583	0·00009	0·00502	0·00010	0·00435	0·00009	0·00382

±X	Y = 34		Y = 36		Y = 38		Y = 40	
	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve
0	0·00173	0·00173	0·00154	0·00154	0·00138	0·00138	0·00125	0·00125
5	0·00148	0·00198	0·00134	0·00175	0·00120	0·00157	0·00110	0·00140
10	0·00124	0·00222	0·00113	0·00196	0·00103	0·00174	0·00095	0·00155
15	0·00103	0·00243	0·00095	0·00214	0·00088	0·00189	0·00081	0·00169
20	0·00085	0·00261	0·00079	0·00230	0·00074	0·00203	0·00069	0·00181
30	0·00059	0·00287	0·00055	0·00254	0·00053	0·00224	0·00050	0·00200
40	0·00041	0·00305	0·00040	0·00269	0·00038	0·00239	0·00037	0·00213
50	0·00030	0·00316	0·00029	0·00280	0·00028	0·00249	0·00027	0·00223
60	0·00022	0·00324	0·00022	0·00287	0·00022	0·00255	0·00021	0·00229
100	0·00009	0·00337	0·00009	0·00300	0·00009	0·00268	0·00009	0·00241

±X	Y = 42		Y = 44		Y = 46		Y = 48	
	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve
0	0·00114	0·00114	0·00104	0·00104	0·00094	0·00094	0·00087	0·00087
5	0·00100	0·00127	0·00092	0·00115	0·00084	0·00105		
10	0·00087	0·00140	0·00081	0·00126	0·00075	0·00114	0·00069	0·00105
20	0·00065	0·00162	0·00060	0·00147	0·00057	0·00132	0·00053	0·00121
30	0·00047	0·00180	0·00045	0·00162	0·00043	0·00146	0·00041	0·00133
40	0·00035	0·00192	0·00034	0·00173	0·00033	0·00156	0·00031	0·00143
60	0·00020	0·00207	0·00020	0·00187	0·00020	0·00169	0·00019	0·00155
100	0·00009	0·00218	0·00009	0·00198	0·00009	0·00180	0·00009	0·00165

Value of —F (continued)

	Y = 50		Y = 52		Y = 54		Y = 56	
±X	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve
0	0·00080	0·00080	0·00074	0·00074	0·00068	0·00068	0·00064	0·00064
10	0·00064	0·00096	0·00060	0·00088	0·00056	0·00081	0·00052	0·00076
20	0·00050	0·00110	0·00048	0·00100	0·00045	0·00092	0·00042	0·00086
30	0·00039	0·00121						
40	0·00030	0·00130	0·00029	0·00119	0·00028	0·00109	0·00027	0·00101
60	0·00019	0·00141	0·00018	0·00130	0·00018	0·00119	0·00017	0·00111
100	0·00008	0·00152	0·00008	0·00140	0·00008	0·00129	0·00008	0·00120

	Y = 58		Y = 60		Y = 62		Y = 64	
±X	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve
0	0·00060	0·00060	0·00056	0·00056	0·00052	0·00052	0·00049	0·00049
10	0·00049	0·00070						
20	0·00040	0·00079	0·00038	0·00073	0·00036	0·00068	0·00034	0·00064
40	0·00026	0·00093	0·00025	0·00086	0·00024	0·00080	0·00023	0·00075
60	0·00017	0·00102	0·00016	0·00095	0·00016	0·00088	0·00016	0·00082
100	0·00008	0·00111	0·00008	0·00103	0·00008	0·00096	0·00008	0·00090

	Y = 66		Y = 68		Y = 70		Y = 72	
±X	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve
0	0·00046	0·00046	0·00044	0·00044	0·00041	0·00041	0·00038	0·00038
20	0·00033	0·00059	0·00031	0·00056	0·00030	0·00052	0·00028	0·00049
40	0·00022	0·00070	0·00021	0·00066	0·00021	0·00061	0·00020	0·00057
60	0·00015	0·00077	0·00015	0·00072	0·00014	0·00068	0·00014	0·00063
100	0·00008	0·00084	0·00008	0·00079	0·00007	0·00075	0·00007	0·00070

	Y = 74	
±X	X +ve	X −ve
0	0·00036	0·00036
20	0·00027	0·00046
40	0·00019	0·00054
60	0·00013	0·00060
100	0·00007	0·00066

Values of —4 F

	$Y = 53$		$Y = 61$		$Y = 69$		$Y = 77$	
$\pm\frac{1}{4}X$	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve
0	0·00285	0·00285	0·00215	0·00215	0·00168	0·00168	0·00135	0·00135
5	0·00184	0·00386	0·00148	0·00282	0·00121	0·00215	0·00101	0·00169
10	0·00113	0·00457	0·00097	0·00333	0·00084	0·00232	0·00073	0·00197
20	0·00047	0·00523	0·00044	0·00386	0·00041	0·00295	0·00038	0·00232
40	0·00014	0·00556	0·00014	0·00416	0·00014	0·00322	0·00013	0·00257
60	0·00007	0·00563						
100	0·00003	0·00567	0·00003	0·00427	0·00003	0·00333	0·00002	0·00268

	$Y = 85$		$Y = 93$		$Y = 101$		$Y = 109$	
$\pm\frac{1}{4}X$	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve
0	0·00110	0·00110	0·00092	0·00092	0·00078	0·00078	0·00068	0·00068
5	0·00085	0·00136	0·00073	0·00112	0·00063	0·00094	0·00055	0·00080
10	0·00064	0·00157						
15			0·00042	0·00143	0·00038	0·00119	0·00035	0·00100
20	0·00035	0·00186						
40	0·00013	0·00208						
50			0·00009	0·00176	0·00008	0·00149	0·00008	0·00127
100	0·00002	0·00219	0·00002	0·00183	0·00002	0·00155	0·00002	0·00133

	$Y = 117$		$Y = 125$		$Y = 133$		$Y = 141$	
$\pm\frac{1}{4}X$	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve
0	0·00058	0·00058	0·00051	0·00051	0·00045	0·00045	0·00040	0·00040
5	0·00049	0·00068	0·00043	0·00059	0·00038	0·00052		
10							0·00029	0·00051
15	0·00032	0·00085	0·00029	0·00073	0·00027	0·00063		
25							0·00017	0·00063
50	0·00008	0·00109	0·00008	0·00094	0·00008	0·00082		
100	0·00002	0·00115	0·00002	0·00100	0·00002	0·00088	0·00002	0·00078

	$Y = 149$		$Y = 157$		$Y = 165$		$Y = 173$	
$\pm\frac{1}{4}X$	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve
0	0·00036	0·00036	0·00032	0·00032	0·00030	0·00030	0·00026	0·00026
10	0·00027	0·00045	0·00024	0·00041				
15					0·00019	0·00040	0·00018	0·00035
25	0·00016	0·00056	0·00015	0·00050				
50					0·00007	0·00052	0·00006	0·00047
100	0·00002	0·00070	0·00002	0·00063	0·00002	0·00057	0·00002	0·00051

APPENDIX—*continued*

Values of $-4\,F$ (*continued*)

	$Y = 181$		$Y = 189$		$Y = 197$		$Y = 205$	
$\pm\frac{1}{4}X$	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve
0	0·00024	0·00024	0·00022	0·00022	0·00020	0·00020	0·00019	0·00019
30	0·00011	0·00038	0·00010	0·00035	0·00010	0·00031	0·00009	0·00029
100	0·00002	0·00047	0·00002	0·00043	0·00002	0·00039	0·00002	0·00036

	$Y = 213$		$Y = 221$		$Y = 229$		$Y = 237$	
$\pm\frac{1}{4}X$	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve
0	0·00018	0·00018	0·00016	0·00016	0·00016	0·00016	0·00014	0·00014
30	0·00009	0·00026	0·00009	0·00024	0·00008	0·00023	0·00008	0·00020
100	0·00002	0·00033	0·00002	0·00031	0·00002	0·00029	0·00002	0·00026

	$Y = 245$		$Y = 253$		$Y = 261$		$Y = 269$	
$\pm\frac{1}{4}X$	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve
0	0·00014	0·00014	0·00012	0·00012	0·00012	0·00012	0·00011	0·00011
30	0·00007	0·00020	0·00007	0·00018	0·00007	0·00016	0·00007	0·00018
100	0·00002	0·00025	0·00002	0·00023	0·00002	0·00021	0·00002	0·00020

	$Y = 277$		$Y = 285$		$Y = 293$		$Y = 301$	
$\pm\frac{1}{4}X$	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve	X +ve	X −ve
0	0·00010	0·00010	0·00010	0·00010	0·00010	0·00010	0·00009	0·00009
30	0·00006	0·00015	0·00006	0·00014	0·00006	0·00013	0·00006	0·00012
100	0·00002	0·00019	0·00002	0·00018	0·00002	0·00017	0·00002	0·00016

TABLE 1

Comparison of Induced Downwash Calculated by Lattice of Vortices for Several Gradings of $K/4sV$ with the Exact Values for Lifting-Line Theory

η	Places of decimals in factors	Spacing in terms of semi-span	Number of vortices	$(1-\eta^2)^{1/2}$		$\eta(1-\eta^2)^{1/2}$		$\eta^2(1-\eta^2)^{1/2}$		$\eta^4(1-\eta^2)^{1/2}$		$\eta^6(1-\eta^2)^{1/2}$		$\eta^8(1-\eta^2)^{1/2}$		$\eta^{10}(1-\eta^2)^{1/2}$	
				Lattice	Exact	Lattice	Exact	Lattice	Exact	Lattice	Exact	Lattice	Exact	Lattice	Exact	Lattice	Exact
0·2	4	0·1	21	1·003	1·000												
0·5	4	0·1	21	1·003	1·000												
0·7	4	0·1	21			1·406	1·400	0·978	0·970	0·360	0·340	0·001	−0·023	−0·151	−0·174	−0·192	−0·211
0·8	3	0·1	21	1·007	1·000												
0·8	4	0·1	21	1·004	1·000	1·606	1·600	1·431	1·420	0·991	0·963	0·553	0·508	0·233	0·177	0·025	−0·034
0·8	4	0·05	41	1·006	1·000			1·426	1·420			0·523	0·508			−0·017	−0·034
0·9	4	0·05	41	1·006	1·000											1·155	1·106

TABLE 2

Rectangular Wing, Aspect Ratio 6 *: Six Point Solutions,* $\eta = 0 \cdot 2$ *and* $0 \cdot 8$

Blenk-Argyris, based on surface integrals		Based on 84-vortex lattice	
a_0	0·06713	a_0	0·06768
a_1	−0·00156	a_1	−0·00095
a_2	0·00009	a_2	−0·00018
c_0	0·03196	c_0	0·03468
c_1	−0·02312	c_1	−0·02668
c_2	−0·00474	c_2	−0·00502
$dc_L/d\alpha$	4·231	$dc_L/d\alpha$	4·295
a.c.	$0 \cdot 2391\bar{c}$ behind L.E.	a.c.	$0 \cdot 2393\bar{c}$ behind L.E.

η	C_{LL}/C_L	Local a.c.	η	C_{LL}/C_L	Local a.c.
0	1·182	0·247	0	1·180	0·249
0·05	1·182	0·247	0·05	1·179	0·248
0·10	1·180	0·247	0·10	1·178	0·248
0·15	1·177	0·246	0·15	1·174	0·248
0·20	1·173	0·246	0·20	1·170	0·247
0·25	1·167	0·245	0·25	1·165	0·246
0·30	1·159	0·244	0·30	1·158	0·245
0·35	1·149	0·243	0·35	1·148	0·244
0·40	1·137	0·242	0·40	1·136	0·242
0·45	1·122	0·240	0·45	1·121	0·241
0·50	1·103	0·239	0·50	1·103	0·239
0·55	1·079	0·238	0·55	1·080	0·238
0·60	1·051	0·236	0·60	1·052	0·236
0·65	1·015	0·234	0·65	1·017	0·234
0·70	0·972	0·233	0·70	0·973	0·232
0·75	0·917	0·231	0·75	0·920	0·230
0·80	0·849	0·229	0·80	0·852	0·227
0·85	0·761	0·227	0·85	0·764	0·225
0·90	0·644	0·225	0·90	0·646	0·223
0·95	0·472	0·223	0·95	0·474	0·221
1·00	0·000	0·221	1·00	0·000	0·218

a.c. abbreviation for aerodynamic centre
L.E. ,, ,, leading edge

TABLE 3

Comparison of Computed Values of w/V at Mid-chord Positions on a Rectangular Wing of Aspect Ratio 6

Vortex sheet $k = 2V(1 - \eta^2)^{1/2} \cot \theta/2$

Item	Integration	Point 1	Point 2	Point 3
1	W. P. Jones, exact surface integral	1·270	1·163	0·931
2	Lattice 21 to span, 4 to chord, semi-range 0·2	1·259	1·152	0·914
3	Lattice 21 to span, 4 to chord, semi-range 0·4	1·264	1·156	0·918
4	Lattice 21 to span, 4 to chord, semi-range 0·5	1·264	1·156	0·918
5	Lattice 21 to span, 6 to chord, semi-range 0·2	1·263	1·157	0·920
6	Lattice 21 to span, 6 to chord, semi-range 0·4	1·267	1·159	0·923
7	Lattice 21 to span, 8 to chord, semi-range 0·2			0·920
8	Lattice 21 to span, 8 to chord, semi-range 0·4			0·923
9	Lattice 21 to span, 8 to chord, semi-range 0·5			0·923
10	Lattice 41 to span, 4 to chord, semi-range 0·1	1·238	1·135	0·908
11	Lattice 41 to span, 4 to chord, semi-range 0·2	1·263	1·156	0·920
12	Lattice 41 to span, 4 to chord, semi-range 0·3	1·267	1·159	0·922
13	Lattice 41 to span, 8 to chord, semi-range 0·2			0·926
14	Lattice 41 to span, 8 to chord, semi-range 0·3			0·928

Vortex sheet $k = 2V\eta(1 - \eta^2)^{1/2} \cot \theta/2$

Item	Integration	Point 1	Point 2	Point 3
15	W. P. Jones, exact surface integral	0·316	0·738	1·003
16	Lattice 21 to span, 4 to chord, semi-range 0·2	0·313	0·729	0·982
17	Lattice 21 to span, 4 to chord, semi-range 0·4	0·313	0·729	0·983
18	Lattice 21 to span, 4 to chord, semi-range 0·5	0·313	0·729	0·983
19	Lattice 21 to span, 6 to chord, semi-range 0·2	0·314	0·733	0·989
20	Lattice 21 to span, 6 to chord, semi-range 0·4	0·314	0·732	0·989
21	Lattice 41 to span, 4 to chord, semi-range 0·2	0·313	0·730	0·983
22	Lattice 41 to span, 4 to chord, semi-range 0·3	0·314	0·730	0·984
23	Lattice 41 to span, 8 to chord, semi-range 0·2			0·993
24	Lattice 41 to span, 8 to chord, semi-range 0·3			0·994

TABLE 4

Comparison of Computed Values of w/V at Mid-chord Positions on a Rectangular Wing of Aspect Ratio 6

Vortex sheet $k = 2V\eta^2(1 - \eta^2)^{1/2} \cot \theta/2$

Item	Integration	Point 1	Point 2	Point 3
1	W. P. Jones, exact surface integral	−0·088	0·287	0·856
2	Lattice 21 to span, 4 to chord, semi-range 0·2	−0·082	0·287	0·838
3	Lattice 21 to span, 4 to chord, semi-range 0·4	−0·080	0·288	0·838
4	Lattice 21 to span, 4 to chord, semi-range 0·5	−0·080	0·288	0·838
5	Lattice 21 to span, 6 to chord, semi-range 0·2	−0·083	0·288	0·845
6	Lattice 21 to span, 6 to chord, semi-range 0·4	−0·082	0·288	0·845
7	Lattice 41 to span, 4 to chord, semi-range 0·1	−0·086	0·282	0·836
8	Lattice 41 to span, 4 to chord, semi-range 0·2	−0·083	0·288	0·839
9	Lattice 41 to span, 4 to chord, semi-range 0·3	−0·081	0·288	0·839
10	Lattice 41 to span, 8 to chord, semi-range 0·2			0·847
11	Lattice 41 to span, 8 to chord, semi-range 0·3			0·848

Vortex sheet $k = 2V\eta^4(1 - \eta^2)^{1/2} \cot \theta/2$

Item	Integration	Point 1	Point 2	Point 3
12	Lattice 21 to span, 4 to chord, semi-range 0·2	−0·053	−0·010	0·558
13	Lattice 21 to span, 4 to chord, semi-range 0·4	−0·052	−0·010	0·558
14	Lattice 21 to span, 4 to chord, semi-range 0·5	−0·052	−0·010	0·558
15	Lattice 21 to span, 6 to chord, semi-range 0·2	−0·053	−0·011	0·563
16	Lattice 21 to span, 6 to chord, semi-range 0·4	−0·052	−0·011	0·563
17	Lattice 41 to span, 4 to chord, semi-range 0·1	−0·053	−0·015	0·552
18	Lattice 41 to span, 4 to chord, semi-range 0·2	−0·053	−0·012	0·553
19	Lattice 41 to span, 4 to chord, semi-range 0·3	−0·052	−0·012	0·553
20	Lattice 41 to span, 8 to chord, semi-range 0·2			0·559
21	Lattice 41 to span, 8 to chord, semi-range 0·3			0·559

TABLE 5

Downwash at Points on an Infinite Yawed Wing obtained by Summation of Effects of a Lattice of Vortices

Description	Point	w/V
True value for all points	—	0·707
1 vortex on chord, chord $= 8y_v$ ..	1	0·722
4 vortices on chord, chord $= 8y_v$ ..	2	0·704
	3	0·705
6 vortices on chord, chord $= 8y_v$ ..	4	0·706
	5	0·705
6 vortices on chord, chord $= 6y_v$ ..	6	0·705
	7	0·706

TABLE 6

Comparison of Two Solutions for a Triangular Wing

	126 vortex, 6 point		328 vortex, 12 point	
$dc_L/d\alpha$	2·513		2·590	
η	Load coeff.	Local a.c.	Load coeff.	Local a.c.
0	1·328	0·298	1·335	0·302
0·05	1·326	0·297	1·333	0·302
0·10	1·320	0·297	1·327	0·301
0·15	1·310	0·295	1·317	0·299
0·20	1·296	0·293	1·303	0·297
0·25	1·278	0·291	1·285	0·294
0·30	1·256	0·288	1·262	0·291
0·35	1·228	0·285	1·235	0·288
0·40	1·196	0·282	1·202	0·284
0·45	1·158	0·278	1·163	0·279
0·50	1·115	0·274	1·119	0·275
0·55	1·065	0·270	1·068	0·270
0·60	1·008	0·266	1·009	0·266
0·65	0·944	0·262	0·943	0·261
0·70	0·873	0·258	0·868	0·257
0·75	0·792	0·254	0·784	0·253
0·80	0·701	0·252	0·690	0·250
0·85	0·598	0·250	0·584	0·248
0·90	0·478	0·248	0·462	0·247
0·95	0·328	0·248	0·314	0·247
1·00	0·000	0·250	0·000	0·250

TABLE 7

Computed Values of w/V for a Sweptback Wing of Aspect Ratio 3 Corresponding to the Vortex Sheet $k = 2V(1 - \eta^2)^{1/2} \cot \theta/2$

Item	Integration	Point 2
1	Surface integral using formula given by W. P. Jones ..	2·085
2	Lattice 21 to span, 4 to chord, semi-range 0·2	2·120
3	Lattice 21 to span, 4 to chord, semi-range 0·3	2·128
4	Lattice 21 to span, 6 to chord, semi-range 0·2	2·082
5	Lattice 21 to span, 6 to chord, semi-range 0·3	2·086
6	Lattice 21 to span, 8 to chord, semi-range 0·2	2·077
7	Lattice 21 to span, 8 to chord, semi-range 0·3	2·080
8	Lattice 21 to span, 12 to chord, semi-range 0·2	2·087
9	Lattice 21 to span, 12 to chord, semi-range 0·3	2·088
10	Lattice 41 to span, 8 to chord, semi-range 0·2	2·099
11	Lattice 41 to span, 8 to chord, semi-range 0·3	2·104
12	Lattice 41 to span, 12 to chord, semi-range 0·2	2·084
13	Lattice 41 to span, 12 to chord, semi-range 0·3	2·089

TABLE 8

Computed Values of w/V for a Sweptback Wing of Aspect Ratio 3 Corresponding to the Vortex Sheet $k = 2V(1 - \eta^2)^{1/2} \cot \theta/2$

Item	Integration	Point 4
1	Surface integral using formula given by W. P. Jones ..	0·960
2	Lattice 21 to span, 6 to chord, semi-range 0·2	0·905
3	Lattice 21 to span, 6 to chord, semi-range 0·3	0·909
4	Lattice 21 to span, 8 to chord, semi-range 0·2	0·917
5	Lattice 21 to span, 8 to chord, semi-range 0·3	0·920
6	Lattice 21 to span, 12 to chord, semi-range 0·2	0·931
7	Lattice 21 to span, 12 to chord, semi-range 0·3	0·934
8	Lattice 41 to span, 8 to chord, semi-range 0·2	0·941
9	Lattice 41 to span, 8 to chord, semi-range 0·3	0·945
10	Lattice 41 to span, 12 to chord, semi-range 0·2	0·943
11	Lattice 41 to span, 12 to chord, semi-range 0·3	0·945

TABLE 9

Computed Values of w/V for a Sweptback Wing of Aspect Ratio 3 Corresponding to the Vortex Sheet $k = 2V(1 - \eta^2)^{1/2}$ $(\cot \theta/2 - \sin \theta)$

Item	Integration	Point 1		
		$2V(1 - \eta^2)^{1/2} \cot \theta/2$	$2V(1 - \eta^2)^{1/2} \sin \theta$	Difference
1	Surface integral using formula given by W. P. Jones			2·190
2	Lattice 21 to span, 6 to chord, semi-range 0·2 ..	3·875	1·558	2·317
3	Lattice 21 to span, 6 to chord, semi-range 0·3 ..	3·874	1·555	2·319
4	Revised lattice 21 to span, 6 to chord, semi-range 0·2	3·461	1·337	2·124
5	Revised lattice 21 to span, 6 to chord, semi-range 0·3	3·459	1·335	2·124
6	Lattice 21 to span, 8 to chord, semi-range 0·2 ..	3·826	1·548	2·278
7	Lattice 21 to span, 8 to chord, semi-range 0·3 ..	3·825	1·547	2·278
8	Lattice 21 to span, 12 to chord, semi-range 0·2	3·813	1·554	2·259
9	Lattice 21 to span, 12 to chord, semi-range 0·3	3·809	1·552	2·257
10	Lattice 41 to span, 8 to chord, semi-range 0·3 ..	4·362	2·034	2·328
11	Lattice 41 to span, 12 to chord, semi-range 0·3	4·274	2·000	2·274
12	Revised lattice 41 to span, 12 to chord, semi-range 0·2	3·903	1·740	2·163
13	Revised lattice 41 to span, 12 to chord, semi-range 0·3	3·901	1·738	2·163

TABLE 10

Simplification of Downwash Calculations by Omission of Vortices, and Factorial Treatment of those Retained

* Position of control point

$\eta = 0$		$\eta = 0{\cdot}2$	
Station retained	Factor	Station retained	Factor
0*	1	0·98125	1
0·05	2	0·80	5·0302
0·10	2	0·60	5·4023
0·15	2	0·40	1·5675
0·20	3·0154	0·35	1
0·40	13·1458	0·30	1
0·60	2·1118	0·25	1
0·80	13·7270	0·20*	1
0·98125	2	0·15	1
		0·10	1
		0·05	1
		0·00	1·5077
		−0·20	6·4717
		−0·40	2·1195
		−0·60	3·5571
		−0·80	6·3440
		−0·98125	1
Total	41	Total	41

$\eta = 0{\cdot}6$		$\eta = 0{\cdot}8$	
Station retained	Factor	Station retained	Factor
0·98125	1	0·98125	1
0·9	2·9097	0·95	1
0·8	1·0625	0·90	1
0·75	1·0278	0·85	1
0·70	1	0·80*	1
0·65	1	0·75	1
0·60*	1	0·70	1
0·55	1	0·65	1
0·50	1	0·6	1·5077
0·45	1	0·4	6·4717
0·4	1·5077	0·2	1·8053
0·2	6·4717	0	5·4700
0·0	1·8053	−0·2	1·7453
−0·2	5·4700	−0·4	8·75
−0·4	3·8078	−0·6	−2·50
−0·6	1·3750	−0·8	8·75
−0·8	7·5625	−0·98125	1
−0·98125	1		
Total	41	Total	41

TABLE 11

Factors for Integration of Downwash at Point 1. Short Process

0	0·05	0·10	0·15	0·20	0·4	0·6	0·8	0·98125
4·007	−1·325	−0·255	−0·100	−0·048	−0·006	−0·002	−0·0008	−0·0001
4·010	−1·320	−0·248	−0·092	−0·041	−0·005	−0·002		
4·016	−1·310	−0·233	−0·078	−0·033	−0·004	−0·002		
4·032	−1·282	−0·197	−0·058	−0·024	−0·004	−0·001		
4·088	−1·166	−0·127	−0·037	−0·017	−0·003	−0·001		
4·691	−0·501	−0·062	−0·023	−0·012	−0·003	−0·001		
−0·691	−0·104	−0·031	−0·014	−0·008	−0·002	−0·001		
−0·088	−0·038	−0·017	−0·010	−0·006	−0·002	−0·001		
−0·032	−0·019	−0·011	−0·007	−0·005	−0·002	−0·001		
−0·016	−0·011	−0·007	−0·005	−0·004	−0·001	−0·001		
−0·010	−0·007	−0·005	−0·004	−0·003	−0·001	−0·001		
−0·007	−0·005	−0·004	−0·003	−0·002	−0·001	−0·001		

TABLE 12

Factors to Represent Chordwise Functions as Line Vortices

Solution	Position on chord from L.E.	cot $\theta/2$	sin θ	sin 2θ
12 point	0·0417	0·1612	0·0099	0·0182
	0·1250	0·0771	0·0175	0·0262
	0·2083	0·0550	0·0215	0·0250
	0·2917	0·0435	0·0240	0·0201
	0·3750	0·0358	0·0257	0·0128
	0·4583	0·0301	0·0264	0·0044
	0·5417	0·0254	0·0264	−0·0044
	0·6250	0·0215	0·0257	−0·0128
	0·7083	0·0179	0·0240	−0·0201
	0·7917	0·0145	0·0215	−0·0250
	0·8750	0·0110	0·0175	−0·0262
	0·9853	0·0070	0·0099	−0·0182

TABLE 13

Station	cot $\theta/2$	sin θ	$(1-\eta)^{1/2}$ × factors
0	1·61851	0·50000	1
0·05	−0·50407	−0·14293	1·9974
0·10	−0·08956	−0·02330	1·9900
0·15	−0·03288	−0·00823	1·9774
0·2	−0·01525	−0·00388	2·9545
0·4	−0·00211	−0·00069	12·0481
0·6	−0·00079	−0·00030	1·6894
0·8	−0·00040	−0·00015	8·2362
0·98125	−0·00005	−0·00002	0·3854

TABLE 14

Comparison of Short and Long Processes for Downwash Calculation

Vortex Sheet	Solution	Point				
		1	2	3	4	5
$2V(1-\eta^2)^{1/2}(\cot\theta/2-\sin\theta)$	Exact 41/12 Long 41/12 Short	2·190 2·162 2·173				
$2V(1-\eta^2)^{1/2}\cot\theta/2$	Exact 41/12 Long 41/12 Short		2·08 2·089 2·104	2·08 2·114	0·96 0·945 0·940	1·21 1·206
$2V(1-\eta^2)^{1/2}\sin 2\theta$	41/12 Long 41/12 Short				0·925 0·925	
$2V\eta^2(1-\eta^2)^{1/2}\cot\theta/2$	41/12 Long 41/12 Short				0·652 0·652	
$2VP_{65/35}\cot\theta/2$	41/12 Long 41/12 Short				0·769 0·769	
$2VP_{65/35}\sin 2\theta$	41/12 Long 41/12 Short				0·531 0·530	

TABLE 15

Comparison of Two Solutions for Sweptback Wing

η	Solution based on original 21/6 lattice		Solution based on 41/12 lattice, short process	
	C_{LL}	Local a.c.	C_{LL}	Local a.c.
0	1·030	0·340	1·030	0·339
0·05	1·045	0·326	1·046	0·324
0·10	1·072	0·302	1·073	0·300
0·15	1·093	0·286	1·095	0·283
0·20	1·110	0·273	1·112	0·270
0·25	1·122	0·264	1·125	0·261
0·30	1·132	0·258	1·134	0·255
0·35	1·138	0·253	1·141	0·249
0·40	1·142	0·248	1·144	0·244
0·45	1·141	0·243	1·143	0·239
0·50	1·136	0·237	1·136	0·234
0·55	1·124	0·231	1·124	0·228
0·60	1·105	0·223	1·105	0·221
0·65	1·077	0·214	1·076	0·213
0·70	1·037	0·204	1·036	0·203
0·75	0·984	0·191	0·982	0·192
0·80	0·912	0·177	0·910	0·179
0·85	0·817	0·159	0·814	0·163
0·90	0·689	0·138	0·686	0·145
0·95	0·501	0·114	0·498	0·124
1·00	0·000	0·086	0·000	0·099
$dc_L/d\alpha$	2·714		2·671	
a.c.	0·926 $\bar{c}$ behind apex		0·924$\bar{c}$ behind apex	

TABLE 16

Values of P_a and P_b for 0·05 Lattice

η	P_a			P_b			$0{\cdot}65P_a + 0{\cdot}35P_b$		
	True	0·05 lattice	w/V	True	0·05 lattice	w/V	True	0·05 lattice	w/V
0	0·1431	0·1480	1·0	0·2420	0·2451	1·00	0·1777	0·1820	1·0000
0·05	0·1258	0·1268	0·5	0·2305	0·2317	0·75	0·1624	0·1635	0·5875
0·10	0·0989	0·0970	0	0·2073	0·2077	0·50	0·1368	0·1358	0·1750
0·15	0·0836	0·0830	0	0·1795	0·1793	0·25	0·1172	0·1167	0·0875
0·20	0·0737	0·0733	0	0·1531	0·1518	0	0·1014	0·1008	0
0·25	0·0661	0·0659	0	0·1352	0·1347	0	0·0903	0·0900	0
0·30	0·0599	0·0598	0	0·1218	0·1215	0	0·0816	0·0814	0
0·35	0·0547	0·0546	0	0·1107	0·1105	0	0·0743	0·0741	0
0·40	0·0500	0·0500	0	0·1010	0·1009	0	0·0679	0·0678	0
0·45	0·0458	0·0458	0	0·0924	0·0924	0	0·0622	0·0622	0
0·50	0·0420	0·0420	0	0·0846	0·0846	0	0·0569	0·0569	0
0·55	0·0384	0·0384	0	0·0773	0·0773	0	0·0520	0·0520	0
0·60	0·0350	0·0350	0	0·0704	0·0704	0	0·0474	0·0474	0
0·65	0·0318	0·0318	0	0·0638	0·0638	0	0·0430	0·0430	0
0·70	0·0286	0·0286	0	0·0573	0·0573	0	0·0386	0·0386	0
0·75	0·0254	0·0254	0	0·0509	0·0509	0	0·0343	0·0343	0
0·80	0·0221	0·0221	0	0·0443	0·0443	0	0·0299	0·0299	0
0·85	0·0187	0·0187	0	0·0374	0·0374	0	0·0252	0·0252	0
0·90	0·0149	0·0149	0	0·0299	0·0299	0	0·0201	0·0201	0
0·95	0·0103	0·0103	0	0·0206	0·0206	0	0·0139	0·0139	0
0·9625	0·0089		0	0·0178		0	0·0120		0
0·98125	0·0062	0·0062	0	0·0125	0·0125	0	0·0084	0·0084	0
1·00	0·0000	0·0000	0	0·0000	0·0000	0	0·0000	0·0000	0

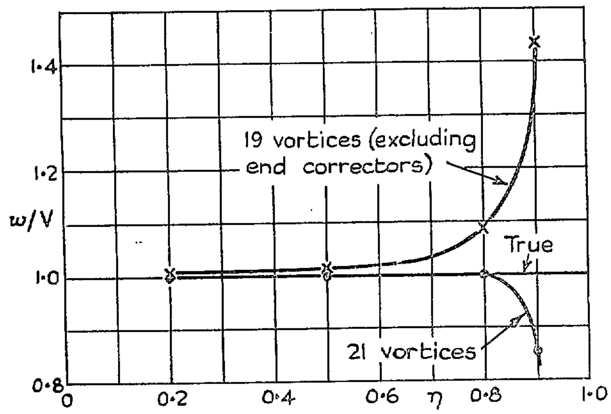

FIG. 1. Induced downwash for $K/4sV = (1 - \eta^2)^{1/2}$ calculated by lattice of vortices.

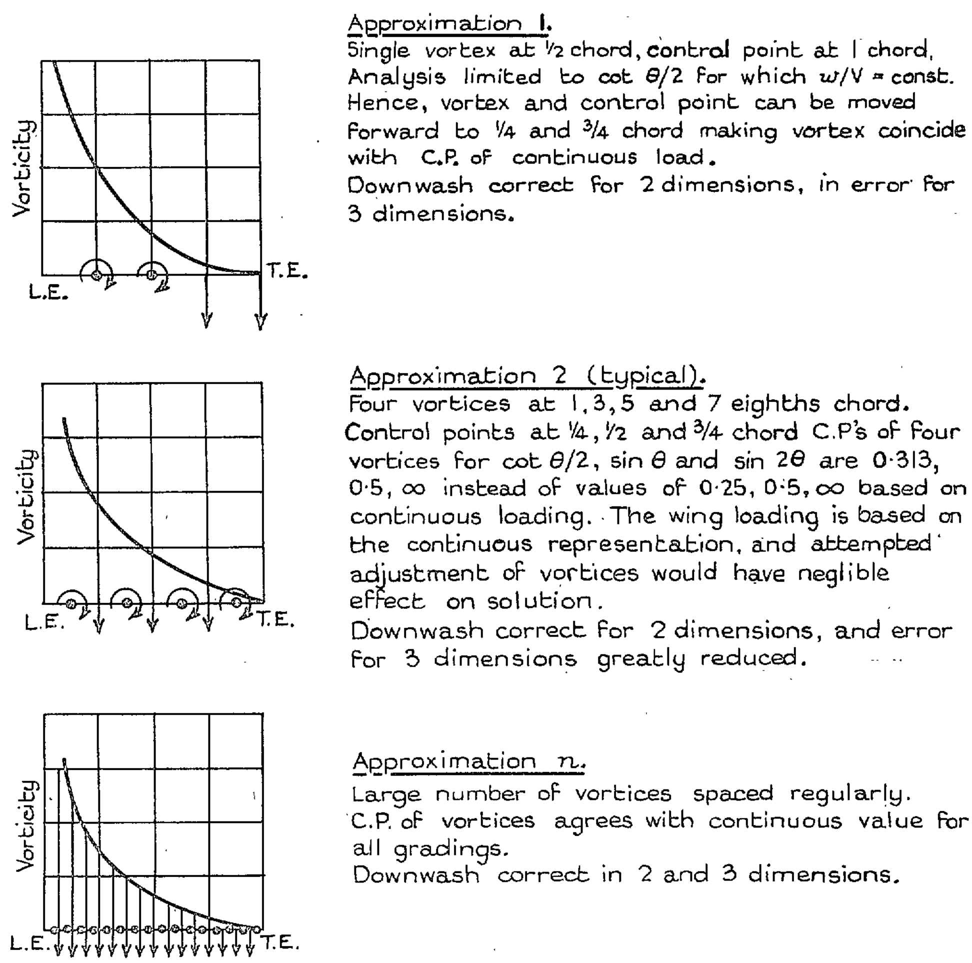

FIG. 2. Sequence of approximations to chordwise load grading for use in downwash calculations.

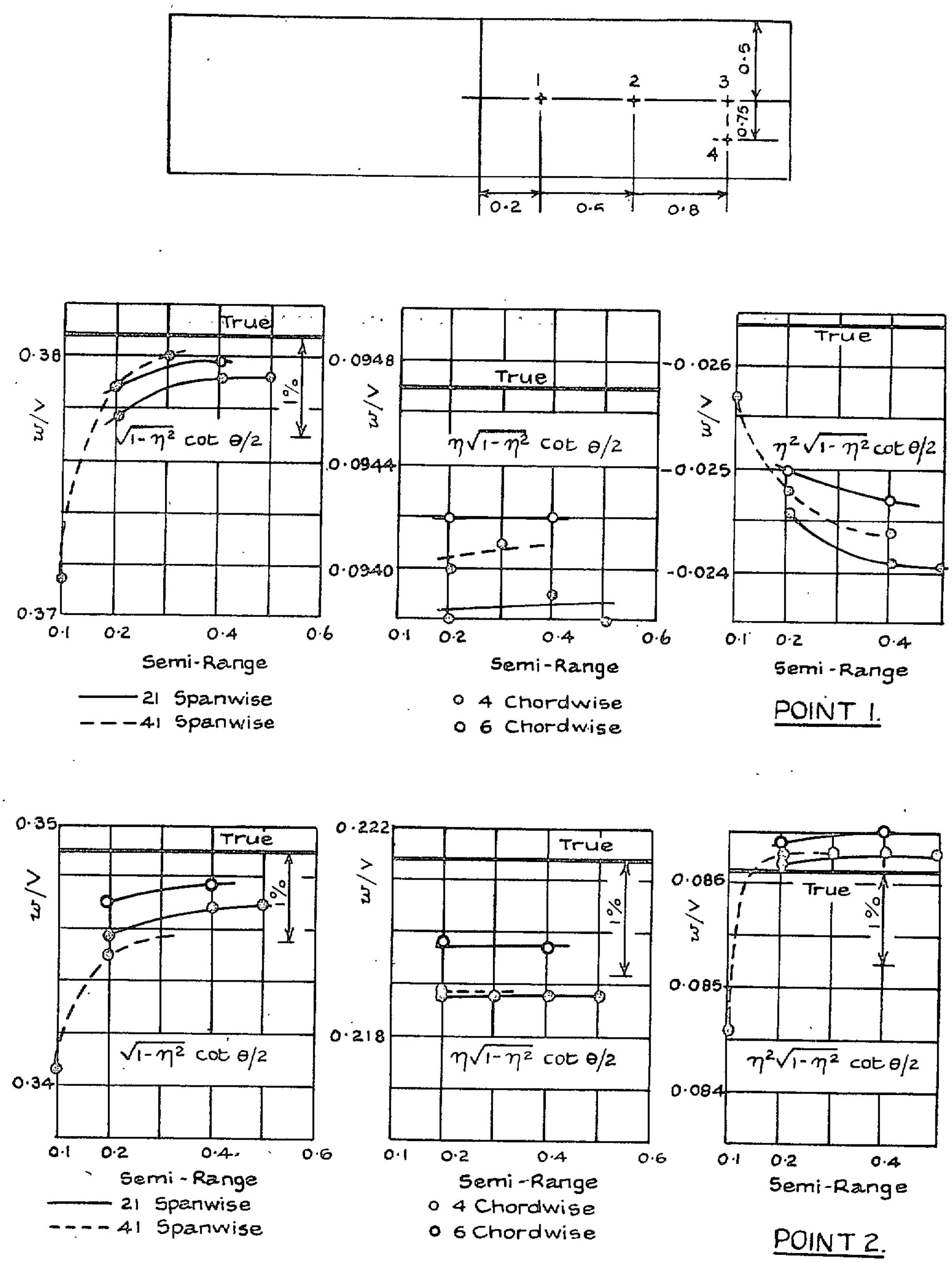

FIG. 3. Rectangular wing aspect ratio 6. Trial points.

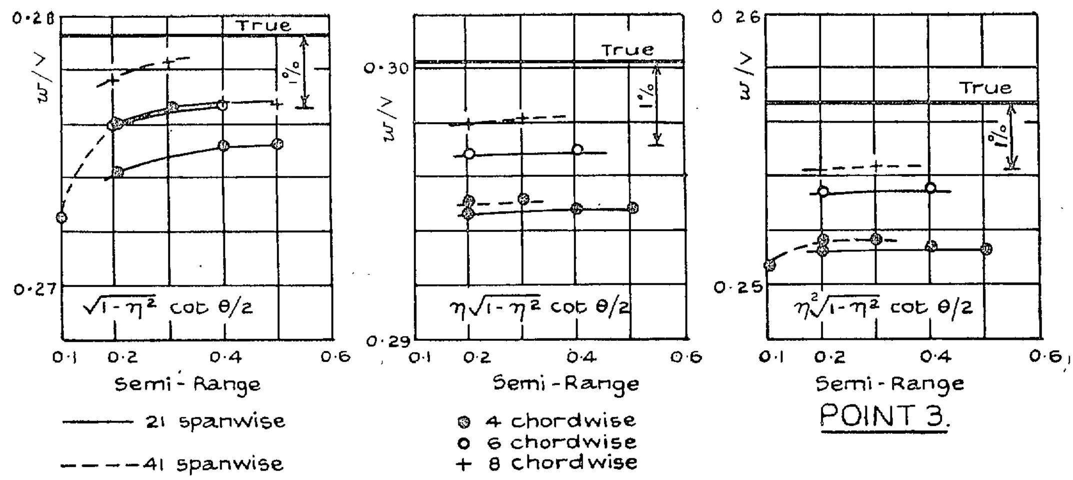

FIG. 4. Rectangular wing aspect ratio 6. Trial points.

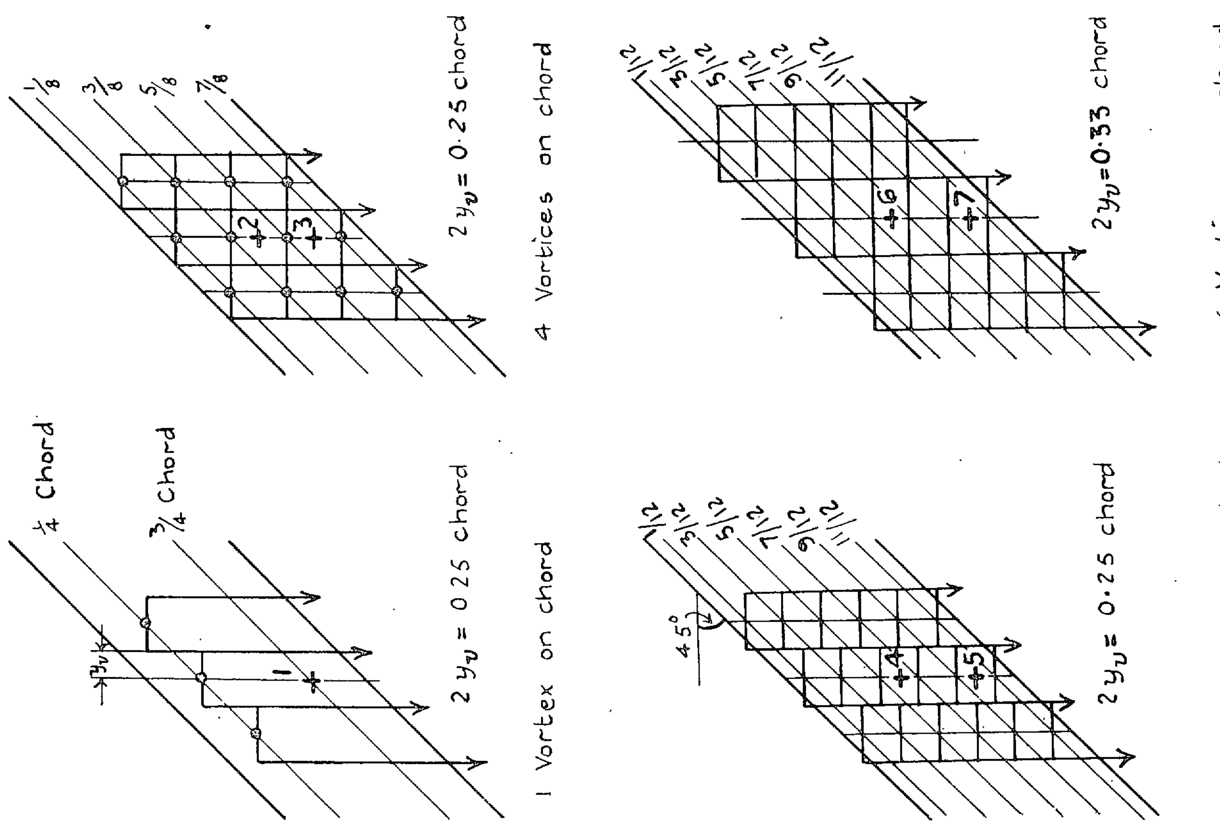

FIG. 5. Calculations of downwash for infinite yawed wing by lattice of vortices.

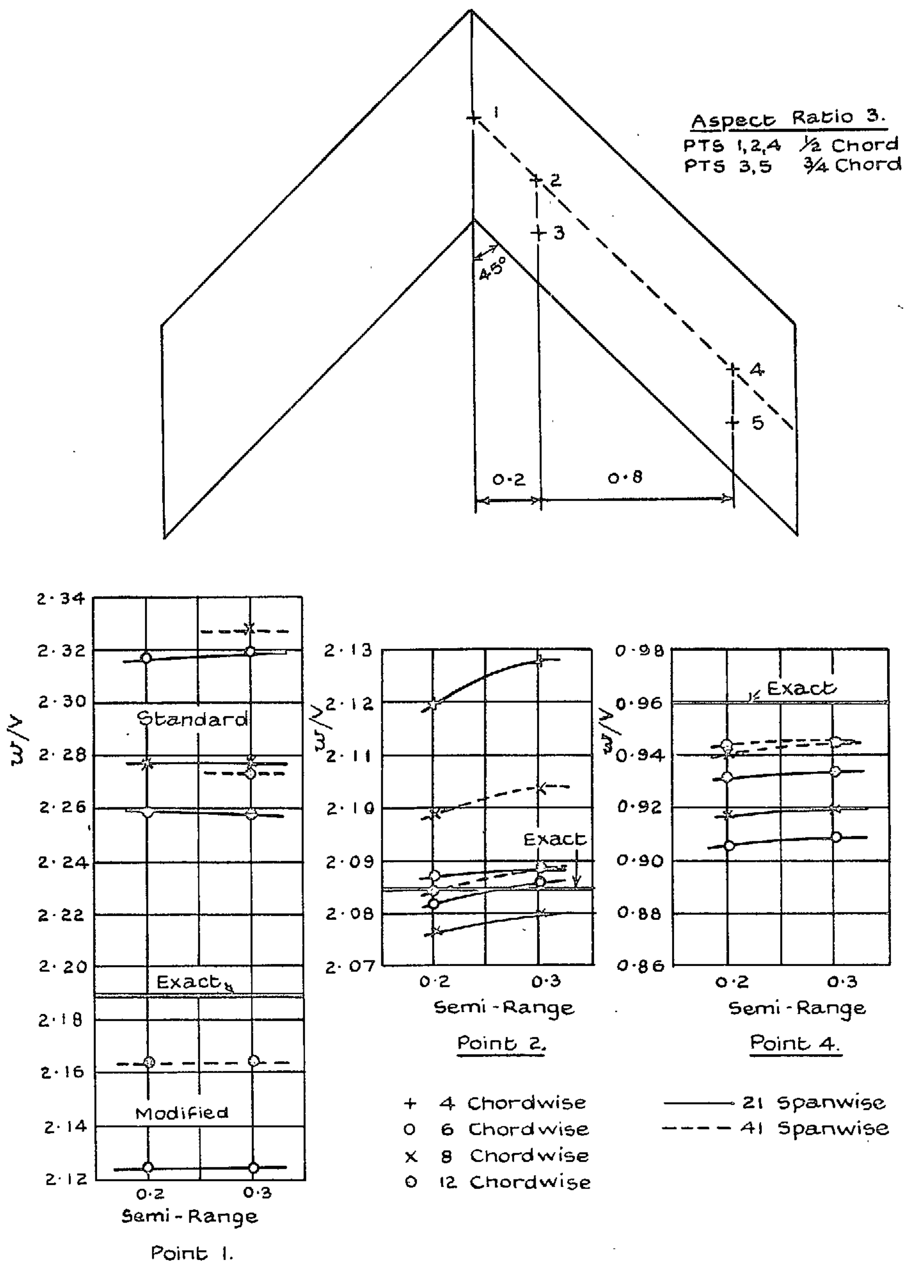

FIG. 6. Trial points on 45 deg swept wing.

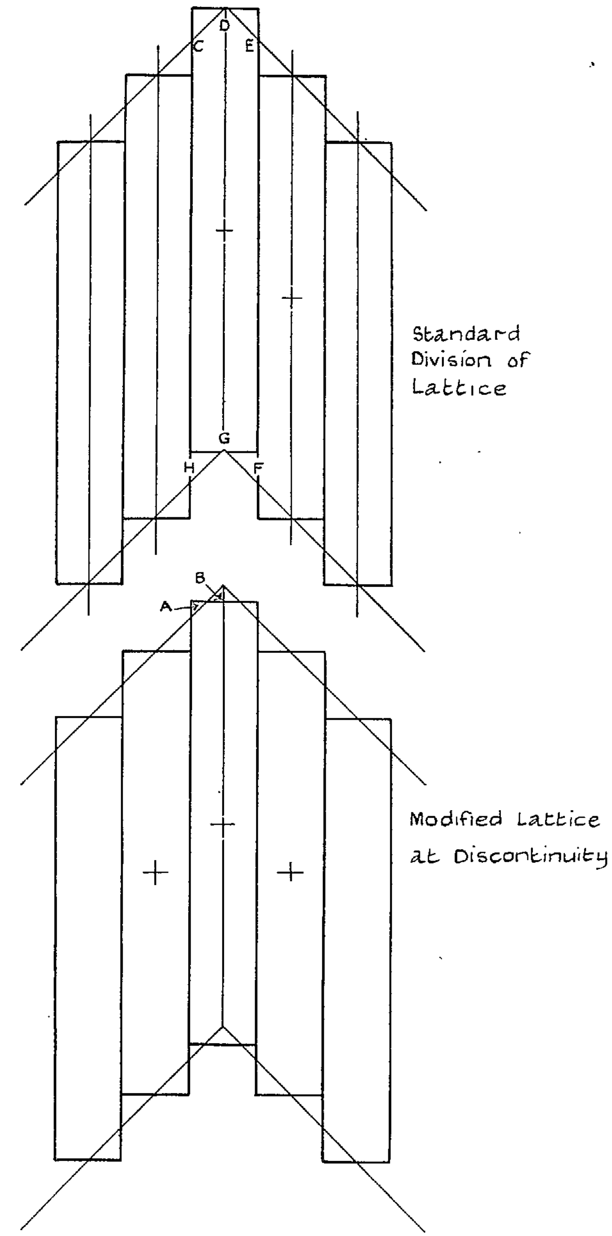

FIG. 7.

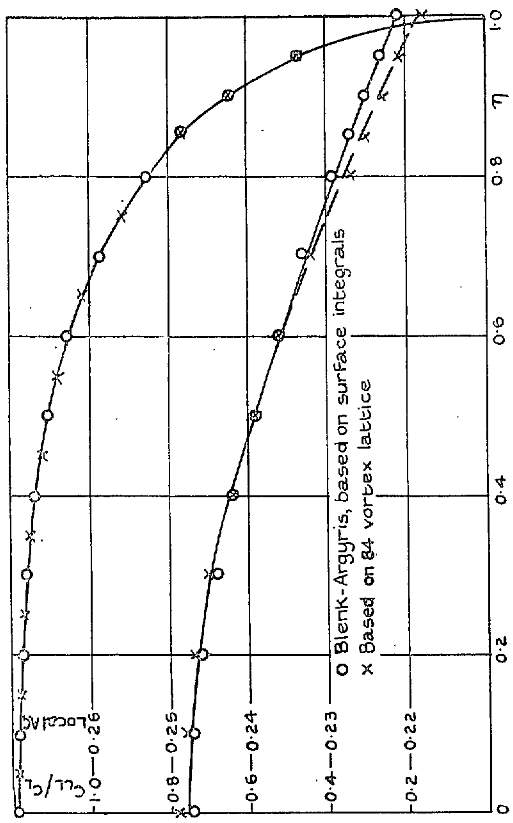

FIG. 8. Comparison of two 6-point solutions for rectangular wing, aspect ratio 6.

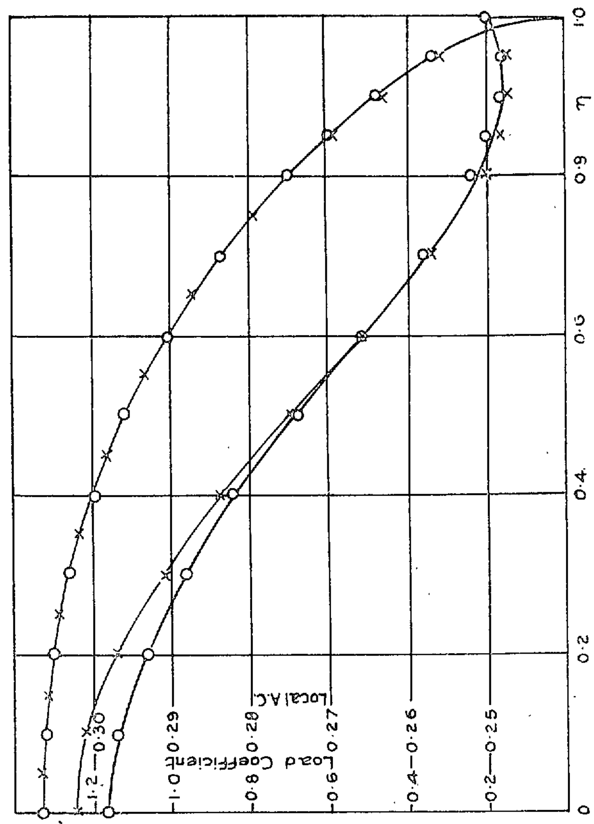

FIG. 9. Comparison of 126-vortex, 6-point and 328-vortex, 12-point solutions for a triangular wing.

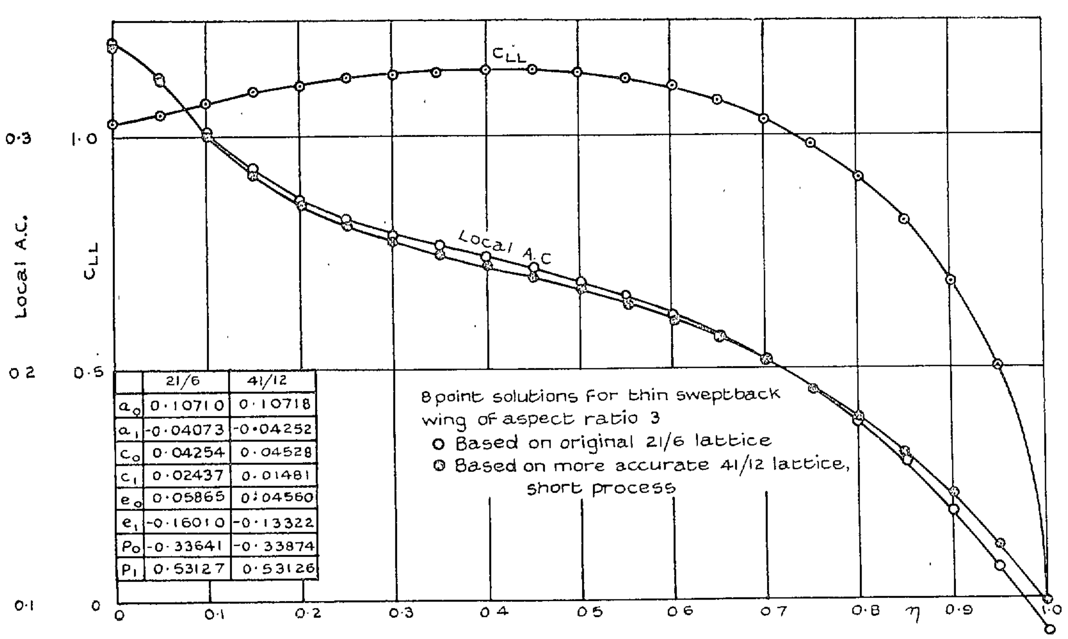

FIG. 10.

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