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Some Electrical Integrating Circuits and their use in the Measurement of Low Frequency Vibration Amplitudes

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Summary.—The note investigates the possibility of making low frequency vibration measurements by the use of electronic acceleration measuring equipment in conjunction with electrical doubly integrating circuits. It is shown that by this method many of the disadvantages associated with the use of seismic displacement units can be obviated particularly over the frequency range 2 to 40 c.p.s. Three electrical integrating methods are discussed, the correct circuit conditions for the integration of periodic sinusoidal, rectangular and triangular waveforms are derived.

A description is given of an existing acceleration measuring equipment incorporating two of the described integration networks; its sensitivity, frequency response and methods of increasing these factors are discussed in detail.

1. *Introduction.*—Dynamic testing of structures frequently necessitates the measurement of small displacements in the frequency range of, say, two to many hundreds of cycles per second; this often has to be done under conditions where no fixed reference point is available, *e.g.* flight. To-date, displacement measuring units have been widely used for the purpose despite their fragility, bulk and large amplitude and phase errors at low frequencies. The purpose of this note is to show that by the use of present day acceleration pick-ups, amplifiers, double integrating circuits and galvanometer recorders it is possible to obviate many of the difficulties encountered in the use of displacement units at these low frequencies.

2. *Present Position of Vibration Measuring Equipment.*—2.1. *Displacement Measuring Equipment.*—Many different makes of this type of unit are available, most of them consisting of a magnet and coil, one elastically suspended with respect to the other which is rigidly fixed to the vibrating structure. Movement of the support produces relative movement of coil and magnet giving an output proportional to velocity of movement. This output voltage is integrated electrically, amplified and indicated on a cathode-ray tube or galvanometer. The response of such a system to a constant exciting amplitude of changing frequency is shown in Fig. 1, while the phase of the output relative to that of the input, assuming the electrical integration to be accurate, is shown in Fig. 2. At frequencies below a certain value, depending on the damping factor, response is variable in amplitude and phase. Taking an optimum value of damping of approximately 0·3 critical it can easily be shown that response cannot be considered to be independent of amplitude and phase (5 per cent error) at frequencies below four times the natural frequency, and as it is difficult to reduce the undamped natural frequency of seismic units below 7 to 10 c.p.s. this gives a lower limit of the usable frequency range of 30 to

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40 c.p.s. It is true that a single frequency sinusoidal vibration can be recorded and measured accurately at lower frequencies by using frequency and phase corrections, but in practice this proves difficult due to the presence of harmonics in the exciting waveform and in any case renders accurate measurement of low frequency-transients impossible.

It is thus seen that it is difficult to interpret the results given by displacement measuring pick-ups in the frequency range 2 to 40 c.p.s. An alternative method of measuring periodic vibration in this range, also applicable to some extent to low frequency unidirectional transients, is investigated below.

2.2. *Acceleration Measuring Equipment*.—During the past few years considerable development both in U.S.A. and Great Britain has taken place in apparatus for the measurement of low frequency accelerations. By the use of sensitive pick-up units, alternating current bridges, amplifying circuits and galvanometer recorders, it is possible to measure, accurately and simultaneously at several points, accelerations in the amplitude range $\pm 6g$ in the frequency range 0 to 70 c.p.s. It is proposed to investigate the possibility of using this type of apparatus together with suitable additional circuits for the measurement of displacements of 1 to $0 \cdot 001$ in. in the frequency range 2 to 40 c.p.s.

It is necessary first of all to consider the behaviour of simple electrical networks to periodic waveforms and follow this by consideration of their behaviour to similar waveforms when coupled to electronic amplifiers.

Three standard waveforms are reviewed, it being considered that the errors involved in the integration of other waveforms can be estimated from those given by

(a) Sinusoidal waveform
(b) Periodic rectangular waveform
(c) Periodic trinagular waveform.

3. *Electrical Integration*.—This can be performed in several ways using L, C and R networks in conjunction with vacuum tube circuits. The only basic circuit considered here is the R-C circuit shown in Fig. 3. This can be used separately or in conjunction with amplifiers of different types, the following cases being considered:—

(1) Integration of periodic sinusoidal, square and triangular waveforms using a series R-C circuit.
(2) Integration of periodic sinusoidal, square and triangular waveforms using an amplifier feedback through an R-C differentiating circuit.
(3) Integration of sinusoidal waveforms using amplifier with positive feedback.

Other circuits for the integration of unidirectional signals of long duration are outside the scope of this note.

The advantages and disadvantages of each of the methods (1), (2) and (3) are discussed in the light of application to the present problem of double integration of an electrical signal proportional to acceleration. Full theory is not given in the main text, but added as Appendices.

3.1. *R-C Circuit*.—3.11. *Sinusoidal waveform*.—The circuit is shown in Fig. 3. If E and v are the input and output voltages respectively, f the frequency of the exciting waveform and ω its circular frequency ($\omega = 2\pi f$), then it can be shown that considering maximum amplitudes only and assuming relatively low input and high output impedances

$$v = \frac{-\sin \phi}{CR} \int E \, dt$$

where ϕ is the phase angle between input and output voltages and is given by $\tan \phi = -\omega CR$.

Thus if f_1 is the lower frequency limit and

$$\omega_1 CR = 2\pi f_1 CR \geqslant 5 \quad \ldots \quad (1)$$

then 90 deg $> \phi >$ 78·5 deg

or $\quad 1 > \sin\phi > 0{\cdot}981$

so that with an error of less than 2 per cent

$$V \simeq \frac{1}{CR} \int E\, dt.$$

To this accuracy then, the peak value of the output can be considered proportional to the integral of that of the input. The accuracy of course increases with increase of frequency.

The angle of lag at the lower frequency limit between the integrated output and the original input waveform is 78·5 deg instead of the correct value of 90 deg. This phase error of course decreases with increase of frequency. Presence of this phase error, which expressed as a percentage of a complete cycle is 3 per cent, does not matter where maximum amplitudes only are concerned. It does become important however, when relative values of input and output voltage are considered at any particular instant, *e.g.*, the deviation of the output voltage from a value proportional to the integral of the input voltage at the instant the input voltage is a maximum, is 4 per cent of the current value. If phased points at other parts of the cycle are considered the errors are larger still.

In the main, however, it is only maximum amplitudes with which we are concerned, with perhaps a knowledge of phase angle in order to determine at what points maxima of integrated waves of different frequencies occur with respect to one another. It is thus concluded that sufficient accuracy of integration of amplitudes of periodic sinusoidal waveforms can be obtained if

$$CR = \frac{5x2T}{2\pi} = 1{\cdot}6T$$

at the largest periodic time $2T$ it is required to integrate. Using this value the attenuation will be a factor $\sqrt{26}$ at the lower frequency limit.

For double integration two stages will be required, separated by some high impedance such as a vacuum tube stage. Using the above value of CR at the lowest frequency concerned, the error in integration will be 4 per cent, this decreasing with increase in frequency. The attenuation will be a factor 26 at the lower frequency limit and the phase angle at this frequency approximately 160 deg instead of the theoretical value of 180 deg. This error is considerably smaller than that given by a seismic displacement unit used at low frequencies (2 c.p.s. upwards).

3.12. *Periodic rectangular waveform.*—The response of the circuit of Fig. 3, to the periodic rectangular waveform of amplitude E and periodic time $2T$, as shown in Fig. 4(a) is considered in Appendix I. For the output waveform to be proportional to the integral of that of the input, with a maximum error in the peak value of 2·0 per cent, the following condition has to be satisfied,

$$CR \geqslant 25T. \quad \ldots \quad (2)$$

For the maximum values of the input and output waveforms the attenuation factor is 25.

For double integration two stages will be required separated by a high impedance. Under these conditions the error will be 5 per cent with an attenuation factor of approximately 625.

3.13. *Periodic triangular waveforms.*—Response of the *R-C* circuit to a periodic triangular waveform of the shape shown in Fig. 4b is also considered in Appendix I. It is shown that for the output voltage from the *R-C* circuit to be proportional to the integral of the input voltage with an error of less than 2 per cent of the maximum value, the following condition has to be satisfied,

$$CR \geqslant 15T. \qquad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \qquad (3)$$

When the maximum values of the input and output voltages the attenuation factor is 30.

For double integration two stages will be required separated by a relatively high impedance. The error in double integration will be 4 per cent and the attenuation factor approximately 900.

The more stringent results obtained for the rectangular and triangular waveforms, as compared with those for the sinusoidal waveform, are explained by the fact that in the latter *only* the relative maximum amplitudes have to be considered whereas in the former, due to different shape of original and integrated waveforms, relative values have to be considered at identical instants of time.

3.2. *Integration Using Amplifier and Feedback Through an R-C Differentiating Circuit.*—Fig. 5 shows a schematic diagram of a circuit in which, when certain conditions are satisfied the output voltage v will be the integral of the input voltage over a specified frequency range. Part of the output voltage is fed back into the amplifier input through an *R-C* differentiating network. The feedback factor is thus proportional to frequency, so the gain of the amplifier, if the feedback factor is large compared with the original input, will be inversely proportional to the feedback factor, *i.e.*, inversely proportional to frequency—the required condition for the integration of sinusoidal amplitudes. The necessary circuit conditions for different waveforms are now obtained.

3.21. *Sinusoidal waveform.*—Let A be the amplifier gain with no feedback present.

Let $E \sin \omega t$ be the amplifier input voltage.

β be the feedback factor, *i.e.*, the fraction of the amplifier output voltage feedback into the input.

f_1 and f_2 the lower and upper frequency limits.

Considering maximum amplitudes only, it can easily be shown that for any frequency f between f_1 and f_2 the output from the *R-C* feed-back circuit is

$$2\pi fCRE$$

with an error of less than $+2$ per cent of the correct value provided

$$2\pi f_2 CR \geqslant 0 \cdot 2 \qquad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \qquad (4)$$

i.e., at the highest frequency under consideration.

The feedback circuit can thus be considered as a differentiating network giving

$$\beta \simeq 2\pi fCR \qquad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \qquad (5)$$

with an error of less than 2 per cent over the frequency range f_1 to f_2.

The gain of the amplifier with feedback is given by the expression

$$\frac{A}{(1 + |A_\beta|^2 - 2|A\beta| \cos \phi)^{1/2}}$$

where ϕ is the phase angle between the input and the feedback voltage. If condition (5) is satisfied $\phi \simeq 90$ deg (assuming no phase shift in the amplifier itself) and

$$\text{Gain} = \frac{A}{(1 + |A_\beta|^2)^{1/2}} \simeq \frac{1}{\beta} \qquad \ldots \quad (6)$$

with an error less than -2 per cent of the correct value if

$$A_\beta \geqslant 5 \qquad \ldots \quad (7)$$

at the frequency f_1, *i.e., at the lowest frequency under consideration.*

Thus if conditions (4) and (7) are satisfied

$$\text{Gain of amplifier} \simeq \frac{1}{2\pi fCR}$$

with an error of -2 per cent and $+2$ per cent at the lower and upper frequency limits respectively. (*N.B.* When these conditions are satisfied the gain of the amplifier at the lowest frequency is $A/5$.) It is thus seen that there is an upper and lower frequency limit to the range over which the output can be considered proportional to the integral of the input voltage. If f_1 and f_2 are these lower and upper limits respectively, the conditions to be satisfied are:—

$$2\pi f_1 CRA \geqslant 5 \qquad \ldots \quad (8)$$

$$2\pi f_2 CR \leqslant 0 \cdot 2. \qquad \ldots \quad (9)$$

In the particular case under consideration in this note $f_1 = 2$ and $f_2 = 40$ c.p.s. and conditions (8) and (9) reduce to

$$CR \leqslant 0 \cdot 00079 \text{ seconds with a minimum value of } A = 500.$$

Thus for single integration by this method over the frequency range 2 to 40 c.p.s. with an error not greater than 2 per cent at the extremes of the range, the gain of the amplifier without feedback will have to have a minimum value of 500 and the CR value of a maximum of $0 \cdot 0008$ sec giving a resultant overall amplification of 100.

Phase Angle Considerations.—The above results have been derived on the assumption that the feedback voltage is 90 deg out of phase with the input voltage. For the values of CR, f and A given in the previous paragraph this is not strictly true especially at the upper frequency limit of integration. It is shown in Appendix III however, that using the above-mentioned values of CR and A, the overall amplitude error does not exceed 2 per cent while the phase angle between input and integrated output voltage varies from approximately 88 deg at the lower frequency limit of 2 c.p.s. to approximately 79 deg at the upper frequency limit of 40 c.p.s. The phase error of a double integrating circuit consisting of the above circuit preceded by an *R-C* integrating stage, these obeying the conditions of sections 3.21 and 3.11, respectively, are also considered in Appendix III. It is shown that under these conditions, the errors are $9 \cdot 5$ deg (lead) at the lower frequency limit of 2 c.p.s. and $11 \cdot 5$ deg lag at 40 c.p.s.

3.22. *Periodic rectangular waveform.*—The conditions for the correct integration of the waveform of Fig. 4a by the circuit of Fig. 5 are obtained in Appendix III. If $2T_1$ and $2T_2$ are the lower and upper limits of the periodic times considered and ω_1 and ω_2 the corresponding circular frequencies the general conditions to be satisfied, for errors less than 2 per cent are

$$CR = T_2/100 \qquad \ldots \quad (10)$$

$$A = 175 T_1/T_2 \qquad \ldots \quad (11)$$

In the particular case under consideration, $2T_1 = 0\cdot5$ seconds and $2T_2 = 0\cdot025$ second giving

$$CR \leqslant 1\cdot25 \times 10^{-4} \text{ seconds}$$

$$A \geqslant 3500$$

and the amplification with feedback at the lower frequency limit is approximately 920.

3.23. *Periodic triangular waveform.*—The conditions for the correct integration of the waveform of Fig. 4b by the circuit under consideration are also derived in Appendix II. Briefly the conditions to be satisfied for errors less than 2 per cent are:—

$$CR \leqslant T_2/157 \quad \ldots \quad (12)$$

$$A \geqslant 250T_1/T_2. \quad \ldots \quad (13)$$

In our particular case where the 'frequency' range 2 to 40 c.p.s. is under consideration the minimum conditions to be satisfied become:—

$$CR = 0\cdot8 \times 10^{-4} \text{ seconds}$$

$$A = 5000$$

and the effective amplification with feedback is 1000.

3.3. *Integration by Use of Amplifier and Positive Feedback.*—3.31. *Sinusoidal waveform.*—A variation of the method described in section 3.1 has been developed and used by the Anglo-Iranian Oil Company Ltd.[2]. It consists of the *R-C* circuit described above, an amplifier and feedback network arranged so as to feedback a small fraction of the output signal in the same phase as that of the input, thus employing positive feedback. It is possible in this way, with a given *R-C* value, to achieve the same degree of accuracy of integration of a given frequency with less attenuation than that given by the use of the *R-C* circuit alone. The reason for this is, briefly, as follows:—

It is seen from section 3.1 that the amplitude of the output voltage v from *R-C* of circuit Fig. 3 is given by

$$v = \frac{-\sin\phi}{CR}\int E\,dt$$

where $\tan\phi = -\omega CR$.

For small values of ω, the output is obviously less than it should be for proportionality to the integral of the input voltage. If, however, the correct voltage in the same phase is fed back from the output of the accompanying amplifier into the input circuit, the accuracy of integration at a given frequency can be increased. The adjustment is critical and is only exact at one particular frequency. As, however, the error decreases with increase in frequency, *i.e.*, $\sin\phi$ approximates nearer to unity, it is only necessary to make the adjustment to the required accuracy at the lowest frequency under consideration.

It can be shown that with this circuit, using a positive feedback of 10 per cent of the input voltage and allowing an error of adjustment of ± 25 per cent from the correct value, the same accuracy of integration can be obtained as that given by a simple *R-C* circuit of twice the attenuation.

Phase error is somewhat less than that given by the straightforward *R-C* circuit, the phase angle being a few degrees nearer 90 deg.

4. *Discussion of Results and Their Application to the Present Problem.*—4.1. *Comparison of the Above-described Three Methods of Integration.*—The positive feedback method has the big disadvantage, especially where two stages are required for double integration, that the positive feedback decreases the stability of the associated amplifier; its use is only advocated when the input signal to be integrated is so small that attenuation considerations are of extreme importance.

Both the straightforward *R-C* circuit and that incorporating an amplifier with differentiated feedback through an *R-C* circuit have, for the same accuracy of integration, the same order of attenuation. The latter has the disadvantage that it has an upper frequency limit and requires one or more additional stages of amplification. As, however, an amplifier is required in any case, the added complication is not excessive but has the advantage of stabilising the amplifier and so making it independent of valve characteristics etc.

The straightforward *R-C* is somewhat more easily applied in practice, necessitating the use only of a simple *R-C* network in front of an existing amplifier.

Where double integration is required, over a range of frequencies where, as in the present case, the upper limit can be fixed, a combination of the *R-C* circuit and amplifier incorporating feedback through an *R-C* differentiating circuit has much to recommend it. With this arrangement the necessary high impedance between the two stages of integration is provided, the amplifier stabilised, and as shown in Appendix III the phase errors are less than those obtained in section 3.11 for the double *R-C* circuit. If two simple *R-C* circuits are used at least one extra stage of amplification will be required, both from the point of view of loss of signal and substitution of a high impedance between the integrating stages. This method would give no amplifier stabilisation.

For double integration it would be impracticable to employ two stages of an amplifier each with the appropriate degree of differentiated feedback, because of the large number of valve stages required.

4.2. *Application to the Integration of Periodic and Transient Waveforms.*—As expected the circuit conditions for the accurate integration of periodic rectangular and triangular waveforms, though practical, are not so easily attained as those for sinusoidal waveforms of comparative periodicity. In a large class of measurements, however, *e.g.*, resonance testing, the waveforms concerned do approximate to a sinusoidal form, thus allowing the less severe conditions of section 3.11 and 3.21 to apply. It is these conditions with which the remainder of this note is concerned, the results of sections 3.12, 3.13, 3.22 and 3.23 being applicable for amplifier design when periodic waveforms approximating to rectangular or triangular shape have to be considered. It is pointed out that the results of section 3.12 and 3.13 are applicable to the integration of a unit step function and one increasing linearly with time, this justifying their inclusion in this note.

4.3. *Practical Circuit.*—A double integrating circuit on the above lines has been developed by Messrs. Miller Ltd., Pasadena, U.S.A. The complete instrument, of which only the integrating stages are shown in Fig. 6, consists of a bridge circuit energised by voltage at 2000 c.p.s., high frequency amplifier, phase discriminating demodulator, integrating circuits and cathode follower output circuit feeding into a galvanometer whose sensitivity is independent of frequency up to approximately 70 c.p.s. A ' push pull ' variable inductance type acceleration pick-up unit of range $\pm 12g$ forms the two variable arms of the bridge.

This circuit as shown in section 3.21 has an upper and lower frequency limit for accurate integration, so the *R-C* values are arranged that they can be changed when frequencies below 8 c.p.s. and above 2 c.p.s. have to be integrated. This allows the use of component values giving less attenuation above 8 c.p.s. than if the same *R-C* values were used throughout the whole of frequency range from 2 c.p.s. upwards. This method has of course the disadvantage that one has to know approximately the frequency of the vibration under measurement. In some applications, such as resonance testing where the structure is excited at known frequencies, this is practical.

The following table gives the accuracy of the double integration of sinusoidal waveforms of different frequencies, obtained both by the use of the foregoing theory and also practically by use of a vibration table.

2 to 8 c.p.s. displacement position			8 c.p.s. upward displacement position		
Frequency	Error (Theory) (per cent)	Error (Practical) (per cent)	Frequency	Error (Theory) (per cent)	Error (Practical) (per cent)
2 c.p.s.	−7	−10	8 c.p.s.	−10	−9
8 c.p.s.	+3	+1	40 c.p.s.	+3	+1

The agreement is good considering that the linearity of the pick-up calibration may depart from linearity by as much as 1 per cent and that a nominal figure of 70 had to be used for the voltage gain of the integrating stage.

The sensitivity of the apparatus is somewhat difficult to express concisely. The amplifier circuit in front of the integrators responds to a signal proportional to acceleration so that its load, for a given amplitude of vibration of the pick-up, depends upon the frequency of excitation. The output from the integrating stage, however, remains at a steady value. It is thus possible for the initial amplifier stages to become overloaded before the latter. This occurs for an input signal corresponding to an acceleration of $1 \cdot 6g$ and it is indicated by the flashing of a neon lamp. For the above reason it is difficult to specify the sensitivity of the instrument on the basis of amplitude of vibration against galvanometer deflection. It can however, be put as follows:—

At 40 c.p.s. the maximum permissible deflection of the galvanometer spot is 1·5 cm, this being given for an accelerometer vibration of 0·010 in. peak to peak. Assuming the gain is adjusted to give the maximum permissible deflection all displacements greater than 0·010 in. can be measured with an error of 3·5 per cent or less (depending upon the frequency) in the range 8 to 40 c.p.s. In the 2 to 8 c.p.s. position displacements greater than 0·050 in. can be measured with an error of 5 per cent or less.

4.4. *Increase of Sensitivity and Accuracy of Existing Circuit.*—The sensitivity and accuracy of the equipment described above is enough for many applications but where greater accuracy of integration and sensitivity is required this can be obtained as follows:—

4.41. *Increase of accuracy of integrating circuit.*—This can be done by altering the respective R-C values and amplification factor A of the double integrating circuit to conform with the conditions given in equations (1), (8) and (9). This will reduce the overall errors due to double integration in the range 2 to 40 c.p.s. to $\pm$ 4 per cent. Change of R-C values will increase the attenuation but this will be offset by the necessary increase of the amplification of the second integrating stage. The introduction of this extra stage introduces little difficulty as direct coupling is not required.

4.42. *Increase in sensitivity.*—The above-described apparatus has a pick-up with an acceleration range of $\pm 12g$ and a useful frequency range of 0 to 120 c.p.s. The maximum acceleration involved in the measurement of vibration amplitude of from $\pm$ 5 in. amplitude at 2 c.p.s. to 0·010 in. at 40 c.p.s. is $2 \cdot 5g$.

If the stiffness of the pick-up spring mass system is reduced so as to give the same movement and hence the same total inductance change, for $\pm 3g$ acceleration as originally occurred for $\pm 12g$, then the overall sensitivity of the instrument will be increased above that given in section 5.2 at the *lower frequencies of acceleration* in the range 2 to 40 c.p.s. Increase of sensitivity at the upper end of the frequency range will not be obtained for the reason detailed above in paragraph 4.3, that the initial stages of the amplifier respond to a voltage proportional to acceleration and not amplitude of movement. This modification to the pick-up will, of course, reduce the natural frequency of the pick-up by a factor two and the upper limit of its useful frequency range to 60 c.p.s. This limit is still adequate for the measurement of vibration amplitudes in the frequency range 0 to 40 c.p.s.

5. *Conclusions*.—5.1. The accurate measurement of low frequency vibration amplitudes (frequency range 2 to 40 c.p.s.), by the use of seismic displacement units presents difficulties for the following reasons:—

(a) Amplitude errors are large below 30 c.p.s.

(b) Phase errors are large below 30 c.p.s.

(c) The necessary flexibility of seismic displacement pick-ups renders them prone to mechanical failure under the rather severe working conditions met in practice.

(d) Their bulk precludes their use on light structures.

5.2. Vibration amplitude measurements in the low frequency range 2 to 40 c.p.s. can be made accurately by the use of present day low frequency acceleration measuring equipment and doubly integrating the resulting electrical signal before recording. The conditions for the double integration of sinusoidal, rectangular and triangular periodic waveforms are derived; while those necessary for the two latter waveforms are more stringent than for the former, all are considered practical. The necessary conditions for waveforms approximating to a sinusoidal shape are satisfied to a large extent by existing light-weight acceleration pick-ups, carrier amplifiers and low frequency galvanometer recorders.

5.3. Where double integration of electrical signals over a limited frequency range is required the circuit arrangement consisting of an *R-C* integrating circuit followed by an amplifier with feedback through an *R-C* differentiating circuit, thus forming a second integrating stage, has several advantages over other circuit arrangements. With such an arrangement it is practicable to keep errors of integration down to a few per cent with much smaller phase errors than those possible with a seismic displacement unit working under the same conditions.

5.4. The above-mentioned existing circuit has an amplitude error of double integration of -9 per cent and $+1$ per cent at the extremes of the considered frequency range, 2 c.p.s. and 40 c.p.s., respectively. Sensitivity is such that over the frequency range 8 to 40 c.p.s. all displacements greater than 0·010 in. can be measured with an error of 3·5 per cent or less, the actual figure depending upon the particular frequency. In the frequency range 2 to 8 c.p.s. displacements greater than 0·050 in. can be measured with an error of 5 per cent or less. Phase errors are approximately 1 deg (lag) at 2 c.p.s. and 13 deg (lag) at 40 c.p.s.

5.5. By adjustment of circuit values to conform to those derived in section 3.21 the frequency errors can be reduced to approximately ± 5 per cent with phase errors of between 9 deg (lead) at 2 c.p.s. and 12 deg (lag) at 40 c.p.s. The consequent loss in sensitivity can be restored by making the integrating stages contain two valve stages instead of one as at present. Increase of sensitivity at the lower end of the frequency range can also be obtained by reducing the range of the pick-up from its present value of $\pm 12g$ to the maximum acceleration values likely to be encountered during vibration tests, namely $3g$.

REFERENCES

1 *Reference data for Radio Engineers.* "Federal Telephone and Radio Corporation. U.S.A.". Ch. 7, p. 161.

2 The Standard Sunbury Indicator. *Engineer,* 13th December, 1935.

APPENDIX I

Conditions for the Correct Integration of Periodic Rectangular and Triangular Waveforms by a Series R-C Circuit

The necessary conditions for the correct integration of the waveform of Fig. 4a by the circuit of Fig. 3 of the text can easily be obtained either by:—

(a) Considering the response of the circuit to a suddenly applied voltage in the range $0 \leqslant t \leqslant 2T$.

(b) Fourier analysing the square waveform and applying the results of section 3.11 of the main text of the note.

Following method (a), the voltage v appearing across C for $0 \leqslant t \leqslant T$ is given by:—

$$v = E\,(1 - e^{-t/CR}) \quad \ldots \quad (1)$$

$$= \frac{Et}{CR}\left[1 - \frac{t}{2!CR} + \frac{t^2}{3!C^2R^2} - \frac{t^3}{4!C^3R^3} + \right]. \quad \ldots \quad (2)$$

If $CR \geqslant 25T$ then (2) can be written as

$$v \doteqdot Et/CR$$

with an error less than 2·0 per cent of the maximum value of the output voltage, *i.e.*, $E/25$. The same result can be shown to hold for the negative half-cycle.

The necessary condition for the integration of this waveform (error 2 per cent) is thus

$$CR = 25T$$

giving an attenuation factor of 25 between the maximum values of the input and output waveforms.

Periodic Triangular Waveform.—Response of the R-C circuit to a periodic triangular waveform of the shape shown in Fig. 4b of the text is now considered. Over the region OA the input voltage at any instant t can be represented by the equation

$$e = Et/T \quad \ldots \quad (3)$$

where $2T$ is the periodic time and E the maximum value of the input voltage. The output voltage v is given by

$$v = E\left[\frac{t}{T} - \frac{CR}{T}(1 - e^{-t/CR})\right]. \quad \ldots \quad (4)$$

Let $CR = xT$ then (4) can be reduced to

$$v = \frac{Et^2}{2xT^2}\left[1 - \frac{t}{3xT} + \frac{t^2}{12x^2T^2} - \frac{t^3}{60x^3T^3} + \ldots\right] \quad \ldots \quad (5)$$

The quantity in the bracket is a converging series in the range $o \leqslant t \leqslant T$ if $x > 1$. If $x \geqslant 15$ then (5) reduces to

$$v \simeq \frac{Et^2}{2xT^2} = \frac{Et^2}{2CRT} \quad \text{.. .. ..} \quad (6)$$

with an error of less than 2 per cent.

Equation (6) is proportional to the integral of (3) this being the required condition. It can be proved that the same condition holds for the negative half-cycle BC, namely, that for the output voltage from the R-C circuit to be proportional to the integral of the input voltage with an error of less than 2 per cent of the maximum value, then

$$CR \geqslant 15T. \quad \text{.. .. ..} \quad (7)$$

For the maximum values of the input and output voltages the attenuation factor is then 30.

APPENDIX II

Conditions for Correct Integration of Periodic Rectangular and Triangular Waveforms by an Amplifier with Feedback through an R-C Differentiating Circuit

The necessary conditions for the correct integration of the waveform of Fig. 4a by the circuit of Fig. 5 of the main text are now obtained.

By Fourier analysis this waveform can be expressed as a function of time f(t) where:—

$$f(t) = \frac{4}{\pi}\left[\sin \omega t + \frac{\sin 3\omega t}{3} \;\ldots\; \frac{\sin\,(2n-1)\omega t}{2n-1}\right] \quad \text{.. .. ..} \quad (1)$$

where $\omega = 2\pi/2T = 2\pi f$.

Let $2T_1$ and $2T_2$ be the lower and upper limits of periodic times considered and ω_1 and ω_2 the corresponding circular frequencies.

Using the results obtained for sinusoidal waveforms in paragraph 3.2 of the main text it is seen that the conditions for true integration with an error less than ± 2 per cent from the true proportional value, are

$$(2n-1)\omega_2 CR \leqslant 0{\cdot}2 \quad \text{.. .. ..} \quad (2)$$

$$A\omega_1 CR \geqslant 5 \quad \text{.. .. ..} \quad (3)$$

The gain of the amplifier for the $(2n-1)$th harmonic is $1/(2n-1)\omega CR$ for any frequency $\omega_1 \leqslant \omega \leqslant \omega_2$ and if these conditions are satisfied the output of the amplifier is

$$4/\omega\pi CR\left[\cos \omega t + \frac{\cos 3\omega t}{3^2} \;..+\; \frac{\cos\,(2n-1)\omega t}{(2n-1)^2}\right] \quad \text{.. .. .. ..} \quad (4)$$

This is proportional to the integral of expression (1), *i.e.*, of the periodic square wave. The required accuracy of integration will be obtained by taking $n = 4$ since the series is convergent and the fifth term only contributes 1 per cent to the total of the first four terms. We thus have

$$CR = T_2/100$$

$$A = 175T_1/T_2.$$

In our particular case

$$2T_1 = 0{\cdot}5 \text{ sec}, \quad T_1 = 0{\cdot}25 \text{ sec}$$

$$2T_2 = 0{\cdot}025 \text{ sec}, \quad T_2 = 0{\cdot}012 \text{ sec}$$

giving

$$CR \leqslant 1{\cdot}25 \times 10^{-4} \text{ sec}$$

$$A \geqslant 3500$$

and the amplification with feedback at the lowest frequency is approximately 920.

Periodic Triangular Waveform.—The periodic triangular waveform of Fig. 4b of the text can be expressed as

$$f(t) = \frac{1}{\pi}\left[\sin \omega t - \frac{\sin 2\omega t}{2} + \frac{\sin 3\omega t}{3} (-1)^{(n-1)} \frac{\sin n\omega t}{n}\right] \quad \ldots \quad (5)$$

where $\omega = \pi/T$ and $T_1 \leqslant T \leqslant T_2$, giving a range of periodicities.

If the above sinusoidal components are fed into the circuit under consideration the output is:—

$$-\frac{1}{\pi\omega}\left[\cos \omega t - \frac{\cos 2\omega t}{2^2} + \frac{\cos 3\omega t}{3^2}\right] \quad \ldots \quad (6)$$

with an error of less than 2 per cent providing (*vide* section 3.21)

$$n\omega CR \leqslant 0{\cdot}2 \quad \text{when } \omega = \pi/T_2$$

$$A\omega CR \geqslant 5 \quad \text{when } \omega = \pi/T_1$$

(6) is a converging series and it is sufficient to take ten terms into consideration as the eleventh constitutes less than 2 per cent to the total value. Thus to the required accuracy the condition to be satisfied becomes

$$CR \leqslant T_2/157 \quad \ldots \quad (7)$$

$$A \geqslant 250T_1/T_2. \quad \ldots \quad (8)$$

In our particular case where the 'frequency' range 2 to 40 c.p.s. is under consideration the minimum conditions to be satisfied become:—

$$CR = 0{\cdot}8 \times 10^{-4} \text{ sec}$$

$$A = 5000$$

and the effective amplification with feedback is 1000.

APPENDIX III

Phase Considerations of Double Integration by One R-C and One Amplifier with Differentiating Feedback Circuit

As is seen in section 3.11 of the main text the phase angle between input and output voltage of the straight *R-C* integrating circuit differs from the theoretically correct value of 90 deg by $11 \cdot 5$ deg at the lower frequency limit when the condition for sufficiently accurate integration of amplitudes of sinusoidal waveforms (eqn. 1) are satisfied. In addition, when integration obtained by the use of an amplifier circuit utilising feedback through an *R-C* differentiating circuit was considered it was assumed in obtaining condition of section 3.21 that the phase of the feedback voltage was accurately at 90 deg to the original input waveform. This is not strictly true, especially at the upper frequency limit of integration. The effect of this approximation upon the conditions given for accurate integration by this circuit (maximum error 2 per cent), and also the resultant phase angle between input and output voltage when it is preceded by a straight *R-C* circuit (for double integration) are now considered.

The nomenclature of sections 3.11. and 3.21 is used and it is assumed that the pick-up is such that damping is either negligible, giving no phase error, or the optimum value of $0 \cdot 7$ critical damping is obtained, giving a phase angle proportional to frequency and hence at all frequencies a constant time delay which can be easily taken into account. Over the frequency range with which we are concerned (2 to 40 c.p.s.) phase errors in the amplifier, before feedback is employed, can be reduced to negligible proportions.

Effect on Results of the Deviation from 90 *deg of Phase of Feedback Voltage.*—The gain of an amplifier with voltage feedback is given by the expression

$$\frac{A}{(1 + (A\beta)^2 - 2(A\beta)\cos\phi)^{1/2}} \quad \ldots \quad (1)$$

where ϕ is the phase angle between the input and feedback voltage. This expression has been shown to reduce to

$$1/\beta \simeq 1/\omega CR$$

with an error of less than ± 2 per cent over a given frequency range (f_1 to f_2) provided:—

$$A\beta \geqslant 5 \text{ at the lower frequency limit} \quad \ldots \quad (2)$$

$$\omega CR \leqslant 0 \cdot 2 \text{ at the upper frequency limit } (\omega = 2\pi f_2) \quad \ldots \quad (3)$$

and

$$\phi = 90 \text{ deg.} \quad \ldots \quad (4)$$

Actually when the condition $\omega CR \leqslant 0 \cdot 2$ is satisfied the value of ϕ is not 90 deg but some lower value depending upon the frequency. At 2 c.p.s. it is given by $\tan^{-1} 100$. Substituting this value of ϕ in expression (1) and making $A\beta = 5$ shows that the amplifier gain differs by $-1 \cdot 7$ per cent from $1/\beta$ compared with -2 per cent when ϕ is assumed 90 deg.

At the upper frequency limit of 40 c.p.s. where $\tan^{-1} \phi = 5$ if condition (3) is just satisfied, $A = 100$, and the gain of the amplifier as obtained from expression (1) differs by less than $0 \cdot 5$ per cent from the required value $1/\beta$. As shown in section 3.21, β itself is, at this frequency, itself in error by $+ 2$ per cent.

It is thus concluded that the effect of the change of the phase of the feedback voltage from the theoretically correct value of 90 deg does not effect the conclusion of section 3.21.

Phase Angle of Double Integrating Circuit Consisting of One R-C and One Amplifier with Differentiated Feedback Circuit.—When the circuit of section 3.21 is preceded by a straight *R-C* circuit fulfilling the requirements of section 3.11, the resultant phase angle at different frequencies between input and output voltage is easily obtained.

Lower Frequency Limit 2 *c.p.s.*—The feedback voltage phase is at an angle of $\tan^{-1} 100$ to the input voltage and it has five times its magnitude. The resultant input voltage is thus at an angle ϕ_1 given by $\tan^{-1} \phi_1 = 30$, *i.e.*, $\phi_1 = 88$ deg and since the phase change in the amplifier can be neglected, it follows that the phase of the output of the integrating stage, relative to the original input has the same value.

The preceding *R-C* integrating circuit has at this lower frequency limit a phase angle ϕ_2 given by $\phi_2 = \tan^{-1}(-5)$.

The resultant phase angle ϕ is $\phi_1 + \phi_2$

$$\tan \phi = \tan(\phi_1 + \phi_2) = \frac{\tan \phi_1 + \tan \phi_2}{1 - \tan \phi_1 \tan \phi_2}$$

which with the above values gives $\phi = 9 \cdot 5$ deg lead.

Upper Frequency Limit (40 *c.p.s.*).—The feedback voltage (assuming conditions of section 3.21 have been satisfied) is at an angle of $\tan^{-1} 5$ with the original input voltage and has a hundred times, its value. The resultant phase of the amplifier output can thus be considered to be at $\tan^{-1} 5$ to the input without appreciable error.

Thus

$$\phi_1 = \tan^{-1} 5 \simeq 79 \text{ deg.}$$

The *R-C* integrating circuit (assuming the conditions of section 3.11 are satisfied) is such that the phase angle ϕ of output relative to input is given by

$$\phi_2 = \tan^{-1}(-100).$$

The resultant phase angle $\phi = \tan^{-1}(-0 \cdot 2)$

whence $\phi = 11 \cdot 5$ deg (lag).

Reference to Fig. 2, shows that for any practical value of natural frequency these values are considerably less than those given by a seismic type displacement pick-up.

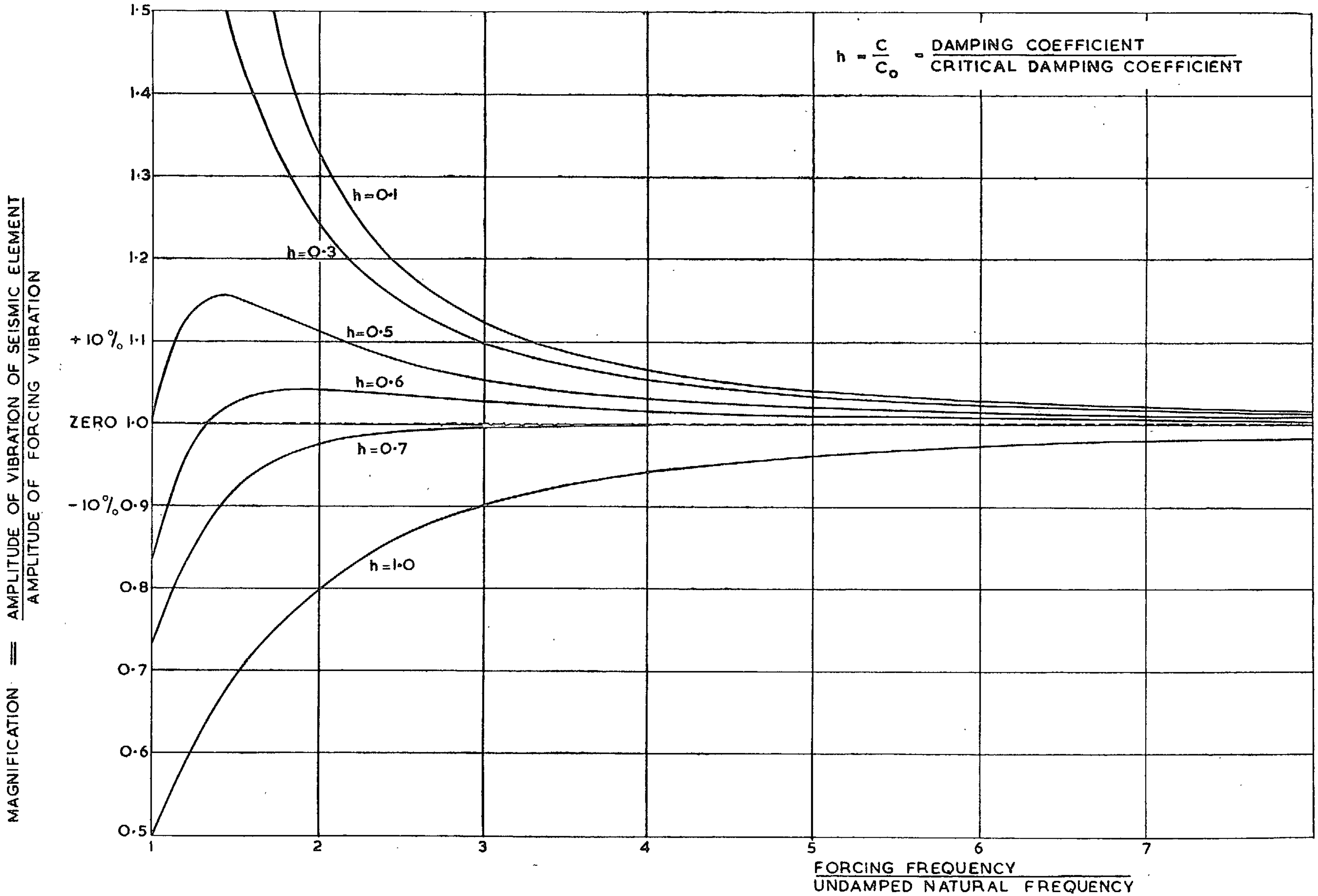

FIG. 1. Seismic pick-up. Variation of amplitude response with frequency for different degrees of damping.

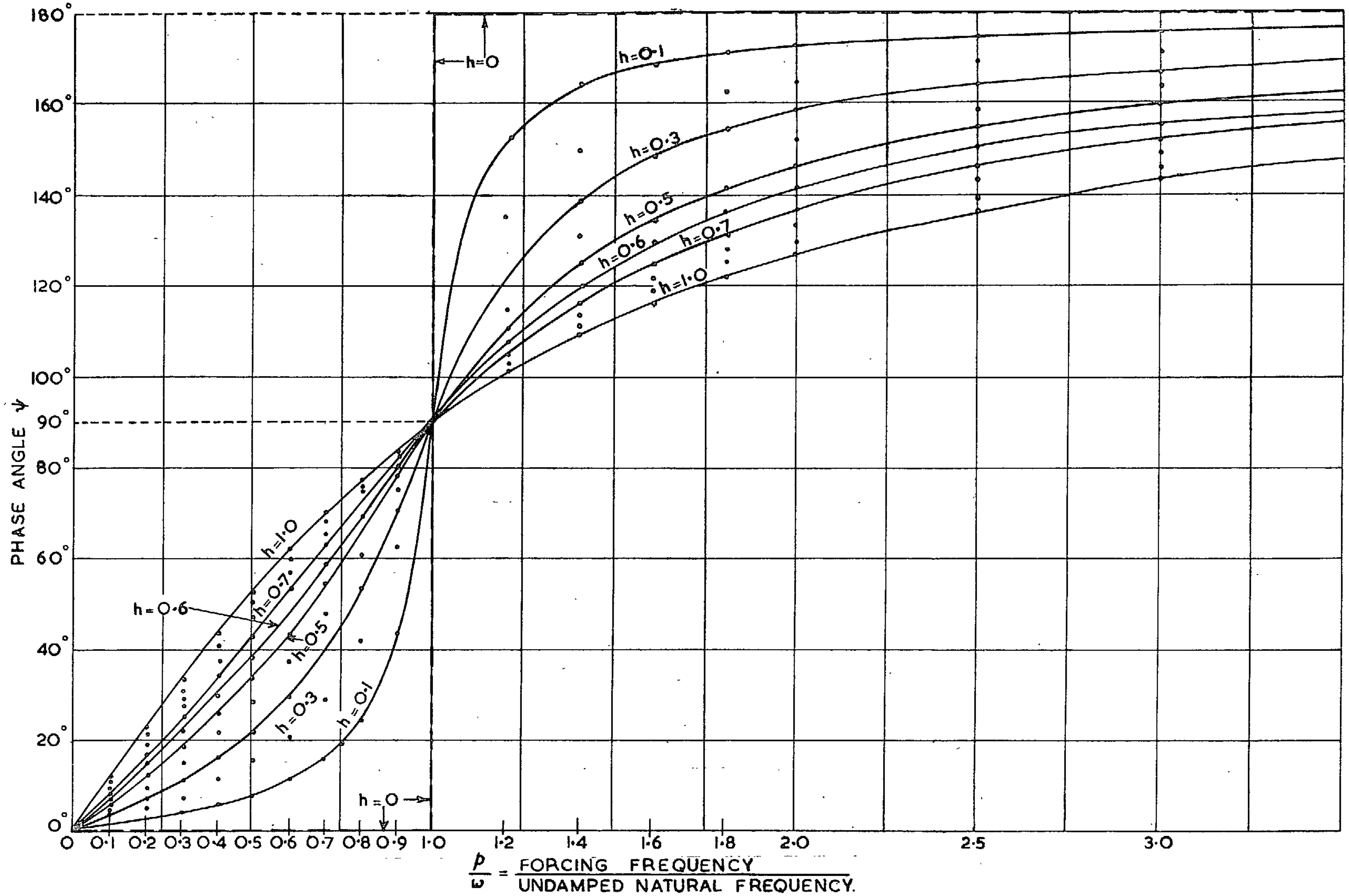

FIG. 2. Variation of phase angle with damping for a mechanical system.

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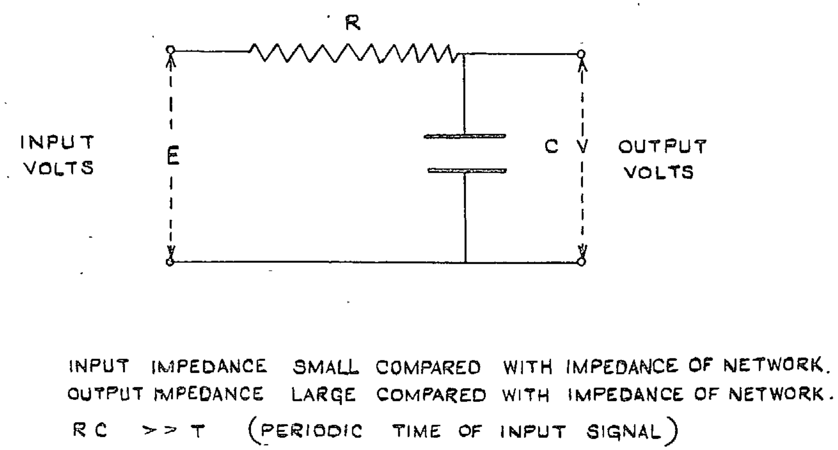

FIG. 3. *R-C* integrating network.

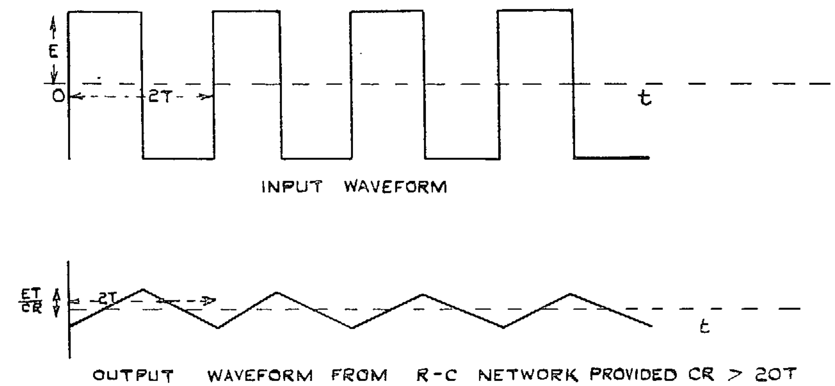

FIG. 4a. Integration of periodic rectangular waveform.

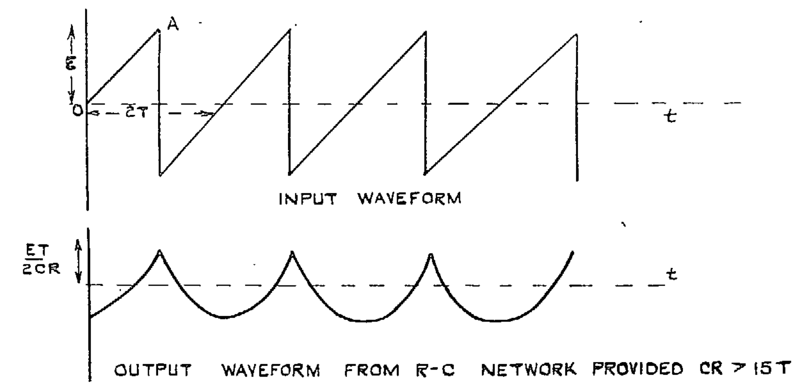

FIG. 4b. Integration of periodic triangular waveform.

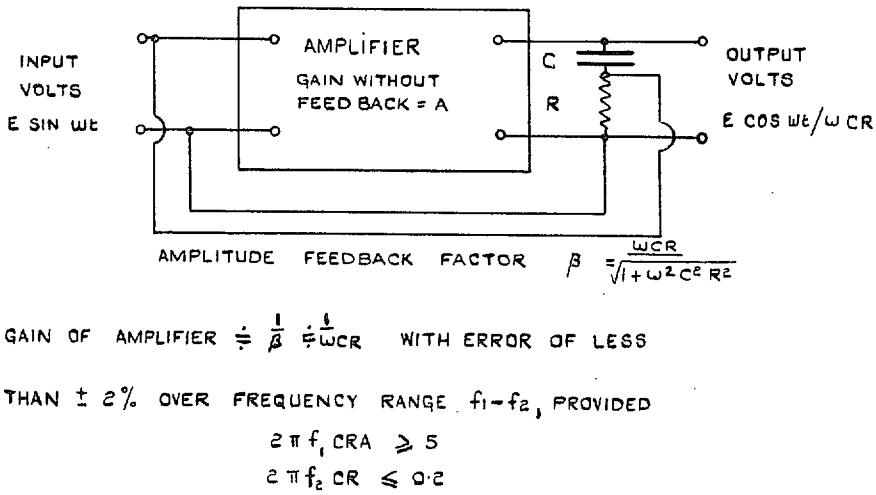

FIG. 5. Electrical integration by the use of amplifier with feedback through *R-C* differentiating circuit.

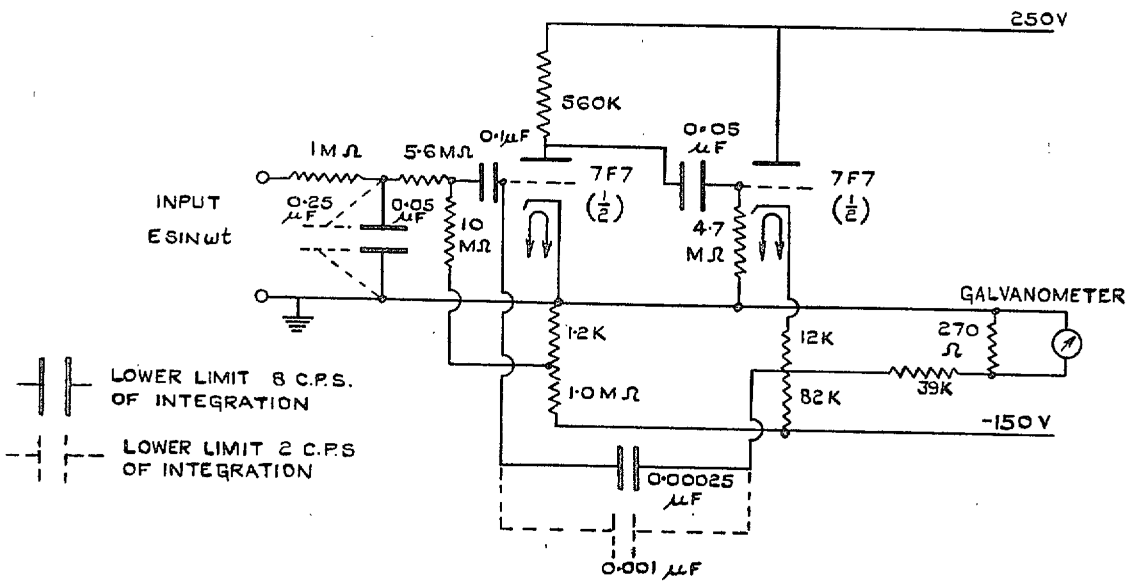

FIG. 6. Double integrating stage of Miller type D amplifier.
(Note: For 12K read 1·2K.)

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