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A Review of the Essentials of Impact Force Theories for Seaplanes and Suggestions for Approximate Design Formulæ

By

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Summary.—Classical theories of impact of seaplanes on water have been based on the assumption of a transfer of momentum to a hypothetical associated mass of water attached to the seaplane, such that the total momentum of the two remains constant. Recent developments of the theory show that this treatment fails to take account of momentum shed to the wake formed behind a seaplane when it has forward speed, *i.e.*, it neglects the planing forces.

This report reviews the essential theory and assumptions underlying recent work, and puts forward an approximate design formula for the maximum deceleration during a main step impact which is directly a function of the initial impact conditions. It has the form

$$\left(\frac{dV_n}{dt}\right)_{\max} = -A\left(\frac{K\rho}{M}\right)^{1/3} V_{n0}^2 .$$

where V_{n0} is the velocity normal to the keel at first impact, the factor A is uniquely determined by the ratio of the flight path angle to the attitude, K is a function of the geometry and attitude of the step, which depends on the assumptions made in defining the associated mass, ρ is the density of the fluid and M the mass of the seaplane. Values of the constants are given in generalized curves in Figs. 3, 4 and 5.

1. *Introduction.*—All theories which have been evolved to date for determining the forces acting on seaplane hulls or floats during the course of an impact with the water have been based on the assumption of a transfer of momentum from the hull or float to a hypothetical associated mass of water.

In any impact, a downwards velocity is imparted to the water particles in contact with the hull and if the hull has an appreciable forward speed then water particles moving downwards will be left behind to form a wake. The present position in the development of the theory is to assume therefore that in all impacts downwards momentum is transferred directly from the body to an associated mass of water 'attached' to it (and therefore moving with it), and in an oblique impact some of this momentum will be shed in the wake behind the body because of the forward speed of the latter. Thus forward speed will have two effects. The first will be to affect the rate of growth of the associated mass attached to the body, the second to leave behind an increasing amount of momentum in the wake† and both these effects must be taken into

* R.A.E. Report Aero. 2230—received 11th February, 1948.

† The validity of the use of associated mass methods for dealing with the motion of a body through a free surface is examined in a later report. (R. & M. 2681).

account when setting up the equation for the conservation of momentum. The relative importance of these two divisions of the momentum of the water varies with the flight path angle. When the resultant velocity is normal to the keel and the attitude of the hull is small, then the attached associated mass retains all of the transferred momentum. Retaining the same hull attitude and decreasing the flight path angle causes momentum to be shed in the wake in increasing amounts and lessens the rate of growth of the associated mass until in the limit the pure planing case is reached when all the transferred momentum finds its way to the wake.

The 'classical' impact theory as developed by Von Kármán[5] and Wagner[6] treats of the first case when the associated mass retains all of the transferred momentum. Wagner also deals with 'sliding' or planing motions but later writers have not in general taken any account of this portion of his work when modifying the impact theory to take account of forward speed. Thus E. T. Jones (R. & M. 1932) and Pabst[8] only make allowance for forward speed by taking the velocity normal to the keel as the effective parameter instead of the velocity normal to the water surface as in the classical case. Neither makes any allowance either for the planing force or for the effect of forward speed on the rate of growth of the associated mass.

McCaig[7] goes a step further by including the effect of forward speed on the rate of growth of the associated mass but still neglects the planing force. In recent work however, both effects have been included in theories developed in England by Crewe (R. & M. 2513), in America by Mayo[1] and Benscoter[2] and in the Netherlands.[9] All of these investigations have produced formulæ for estimating the forces acting during impact but the use of different symbols and variations in the methods of application make it difficult to compare their relative merits.

At the same time, Johnstone[15] has made a modification of pure planing theory which allows for increasing immersion, but neglects the impact force. Here again, comparison is made difficult not only by differences in notation but also by the complete difference in derivation.

The aim of the present report is to review the essentials of these later impact force theories (Refs. 1 and 2, and R. & M. 2513), to point out where differences arise in them and to develop approximate formulæ*, expressed in terms of the physically significant factors, which will cover the useful range of landing conditions.

Comparison of the later theories (References 1 and 2 and R. & M. 2513) with those of Jones (R. & M. 1932), McCaig[7] and Johnstone[15] is made by means of these approximate formulæ since it is considered that this approach leads to the clearest physical comparison (as given above).

The review is restricted to the straight-sided wedge without chine immerson or angular velocity, the case considered in the classical theory.

2. *The General Theory of Impact of a Plane-Faced Wedge*.—2.1. *Nature of the Forces Acting*.—The first problem is to determine the nature of the forces acting in an oblique impact. This can be done most easily by assuming that associated mass methods as developed for motion in an unbounded fluid will give a sufficiently good approximation to the motion through a free surface, provided suitable correction factors are applied

Mayo[1], Benscoter[2] and the Dutch[9] each deal with the problem by this method and make the additional assumption that the three-dimensional oblique impact case can be broken down into the sum of a series of two-dimensional cases.

2.1.1. *Two-dimensional Impact*.—The treatment of the two-dimensional case (vertical drop of an infinitely long wedge of constant cross-section at zero trim) is then made in accordance with Von Kármán's[5] and Wagner's[6] assumptions, *i.e.*, that all the momentum of the wedge is transferred to and retained by a fictitious associated mass of water. The 'associated mass' in this case was assumed by Von Kármán[5] to be half of that obtained when a flat plate moves in an unbounded fluid, the width of the plate being the wetted width of the wedge. Thus it is the mass of half a circular cylinder of water on the wetted width of the wedge as diameter.

*A subsequent report[16] gives improved formulæ and curves recommended for use in design.

If the mass of the wedge is M and if, for convenience, we define the associated mass as μM, then the momentum equation will read

$$MV_{n0} = MV_n + \mu MV_n \quad \ldots \quad (1)$$

where V_n is the penetration velocity, normal to the keel and V_{n0} is the value of V_n at first impact.

The resultant upwards force on the wedge, arising from changes in momentum, is normal to the keel and is given by :—

$$F_n = -\frac{d}{dt}(MV_n)$$

$$= \frac{d}{dt}(\mu M . V_n) \quad \ldots \quad (2)$$

from equation 1.

Also, we can write $\mu M = \rho \bar{K} z^2$ (per unit length) where z is the depth of penetration, see Fig. 1, ρ is the density of fluid and $\bar{K}$ is a factor which depends on the geometry of the wedge and allows both for splash-up and for finite deadrise angle β. Splash-up is the rise of displaced fluid up the sides of the wedge and a correction for it, to the value of the associated mass, was first introduced by Wagner from consideration of the flow past a flat plate.

Substituting for μM in equation (2) we obtain

$$F_n = \frac{d}{dt}(\rho \bar{K} z^2 V_n) \quad \ldots \quad (3)$$

$$= \frac{d}{dt}\left(\rho \bar{K} z^2 \frac{dz}{dt}\right)$$

since $V_n = dz/dt$ in the two-dimensional case, and solutions of this equation will give the motion of the wedge.

In this treatment we have neglected any forces due to viscosity and buoyancy as being small compared with the inertia forces.

2.1.2. *Three-dimensional impact.*—Turning now to the three-dimensional case, *i.e.*, the oblique impact of a plane-faced wedge at finite trim without chine immersion, conditions during the impact are as shown in Fig. 2a. The case is for simplicity restricted in the first place to the wedge with a straight transverse discontinuity (or step on a hull). The problem is to determine the distribution of the momentum after transfer from the wedge to the fluid.

As in the two-dimensional case it is assumed in the first place that the effects of gravity and viscosity can be ignored and that the attitude τ remains constant during the impact. It follows that if we divide the fluid into sections of spacing dx by fixed planes normal to the keel, then, as the wedge moves through them, the flows in the various sections can be treated as independent of each other, and essentially two-dimensional. Also the velocity component parallel to the keel (V_T) will remain constant and will have no effect on the normal forces. Thus in each section we assume that a force equation of the form of equation (3) will apply, *i.e.*,

$$F = \frac{d}{dt}(\rho \bar{K} z^2 dx\, V_n) . \quad \ldots \quad (3)$$

The total force on the wedge at any time defined by immersion z can then be obtained by considering either (1) the time rate of change of the total momentum imparted to the fluid from the beginning of the impact, or (2) the summation of the force elements acting on the wedge. The first method seems the more obvious physically and is given below. It is assumed explicitly that the momentum of the fluid at any time is divided between the associated mass at the wedge and the wake formed since first impact. The second method makes no explicit assumptions about the wake momentum and corresponds to that used by Mayo[1] and Benscoter[2]. Details of it are given in Appendix I.

The origin of co-ordinates is defined as fixed in space at the point O where the step s entered the water at time $t = 0$, (*see* Fig. 2a). The x-axis is taken parallel to the keel and the z-axis normal to it. The flight path angle is taken as γ, its initial value as γ_0 and the velocity components as shown in Fig. 2a.

We assume at time t the momentum associated with an element dx of the fluid is that due to the two-dimensional motion of a wedge with an immersion z, *i.e.*,

$$\text{momentum} = \rho \bar{K} z^2 \, dx \, V_n.$$

Then the total momentum of the water at time t is given by

$$\int_{-\infty}^{+\infty} \rho \bar{K} z^2 V_n \, dx.$$

When

$x > x_s + L$ (*i.e.*, ahead of the wedge) the fluid is unaffected and $V_n = 0$.

$x_s + L > x > x_s$, (*i.e.*, under the wedge) the fluid is given a normal velocity V_n, where $V_n = V_n(t)$ and is independent of x. The wetted length is here assumed to be that due to intersection with the undisturbed water surface.

$x_s > x > 0$ (*i.e.*, behind the wedge) the fluid is moving with normal velocity V_n, where $V_n = V_n(x)$ is independent of t. Provided there is a straight transverse step to the wedge, $V_n(x)$ is the final velocity imparted to the fluid in the plane at x when the step of the wedge passed through this plane at some earlier time t'. The associated mass of each section in this region is defined by z_s', the step depth normal to the keel at time t';

$x < 0$. The water is unaffected and $V_n = 0$.

The total momentum of the fluid at time, t, is therefore

$$\int_0^{x_s} \rho \bar{K} z^2 V_n(x) \, dx + \int_{x_s}^{x} \rho \bar{K} z^2 V_n(t) \, dx$$

or expressed as a function of time where

$$\frac{dx}{dt} = V_T \text{ and } K = \frac{\bar{K} \cot \tau}{3}$$

the total momentum is

$$\int_0^t \rho \bar{K} (z_s')^2 V_n(t') V_T \, dt' + \rho K z_s^3 V_n(t).$$

By analogy with the two-dimensional case, $\rho K z_s^3$, which is the mass of a half cone of water defined by the intersection of the wedge with the water surface, will be taken as the associated mass of water and denoted by μM, where M is the mass of the wedge.

The complete momentum equation at time t will now be

$$MV_{n0} = MV_n + \mu M V_n + \int_0^t \rho \bar{K} (z_s')^2 V_n(t') V_T \, dt'. \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

This differs from the two-dimensional case by the addition of the last term which allows for the shedding of fluid with downward velocity from the step into the wake because of forward speed.

The resultant impact force will be normal to the keel and given by

$$F = \frac{d}{dt}\,(\text{momentum})$$

$$= \frac{d}{dt}\,(\mu M.V_n) + \rho \bar{K} z_s^{\,2}\, V_n V_T.$$

$$= \frac{d}{dt}\,(\mu M.V_n) + 3\rho K z_s^{\,2}\, V_n V_T \tan\tau.$$

$$= \frac{d}{dt}\,(\mu M.V_n) + V_n \frac{d(\mu M)}{dz_s}\, V_T \tan\tau. \quad \ldots \quad (5)$$

and since $\frac{dz_s}{dt} = V_n - V_T \tan\tau$

$$F = \mu M \frac{dV_n}{dt} + \frac{d(\mu M)}{dz_s}\, V_n^{\,2}. \quad \ldots \quad (5a)$$

Comparison with the expressions given by Mayo[1], Benscoter[2] and Crewe (R. & M. 2513) is made in Appendix I.

These expressions for the momentum equation and for the force acting on the wedge have been developed assuming a straight transverse step. If the step is not straight, *e.g.*, rounded or of vee-shape in planform, then momentum flows into the wake over a range of values of x, as shown in Fig. 2b for a straight vee step. The assumption of plane parallel flow is not likely to be valid in this region and in addition there is the equivalent of chine immersion. However an approximation to the correct division of momentum transfer might be obtained if a straight transverse step were assumed at a suitable station and the wedge lines were extended to meet this step. The choice of station would best be determined by analysis of experimental results. *A priori*, likely positions for the vee-step, Fig. 2b would be either at the lines $W_1 W_1'$ or $W_2 W_2'$ or at the centroids of the triangles $W_1 W_1'$s or $W_2 W_2'$s.

A further point requiring clarification concerns the position of the splashed up water line PW_2. Some preliminary unpublished tests at the Royal Aircraft Establishment towing tank indicate that on a planing wedge there is a forward splash-up and that the true splashed up water line is along a line $P'W_3$ approximately parallel to PW_2. If so then further modification would be required to the value taken for the associated mass in the momentum equation.

2.2. *Solution of Equation of Motion.*—Equation 5(a) reads

$$F = \mu M \frac{dV_n}{dt} + \frac{d(\mu M)}{dz_s}\, V_n^{\,2}. \quad \ldots \quad (5a)$$

So far we have neglected a possible form drag. Benscoter[2] expresses this in the form

$$F_s = \delta \frac{d(\mu M)}{dz_s}\, V_n^{\,2}$$

where δ is the ratio of the form drag force to the inertia force.

Introducing this form drag, equation (5a) becomes

$$F = \mu M \frac{dV_n}{dt} + (1 + \delta) \frac{d(\mu M)}{dz_s}\, V_n^{\,2}$$

$$= -\, M \frac{dV_n}{dt}$$

i.e.

$$(1 + \mu) \frac{dV_n}{dt} + (1 + \delta) \frac{d\mu}{dz^s}\, V_n^{\,2} = 0. \quad \ldots \quad (6)$$

The solution of this equation has normally been made assuming constant horizontal velocity (V_H), instead of constant velocity parallel to the keel (V_T), so that the results can be compared with experimental results from tank tests.

If the horizontal velocity is kept constant, then while the resultant water force will still be normal to the keel the resultant acceleration will be vertical, hence

$$\left(M\frac{dV_n}{dt}\right)\Big/ - \cos\tau = F\cos\tau$$

which can be transformed to

$$(1 + \mu')\,dr + (1 + \delta)\,\frac{(r + 1)^2}{r}\cos^2\tau\,d\mu' = 0 \quad \ldots \quad (7)$$

where $r = \dfrac{V_v}{V_H \tan\tau} = \dfrac{\tan\gamma}{\tan\tau}$

and $\mu' = \mu\cos^2\tau.$

or, the associated mass is closely a function of the parameter r only ($\cos^2\tau$ is usually nearly unity).

Equation (7) can be obtained from Crewe's generalized equation (R. & M. 2513)

$$\frac{s\,d\mu'}{(1 + \mu')} + \frac{w\,dw}{w^2 + qw + r} = 0$$

by putting

$w = r$
$r = 1$ (assumes a fully developed wake)
$q = 2$ (viscosity forces etc. neglected)
$s = (1 + \delta)\cos^2\tau.$

Integrating equation (7),

$$\log(1 + r) + \frac{1}{1 + r} + (1 + \delta)\cos^2\tau\log(1 + \mu')$$
$$= \log(1 + r_0) + \frac{1}{1 + r_0} \quad \ldots \quad (8)$$

where subscript zero refers to initial condition.

Differentiating equation (6) with respect to z_s gives as conditions for maximum acceleration (denoting the values by the subscript m)

$$\mu_m' = \frac{2r_m}{r_m[1 + 6(1 + \delta)\cos^2\tau] + 6(1 + \delta)\cos^2\tau}$$

$$r_m = \frac{6\mu_m'\,(1 + \delta\cos^2\tau)}{2 - [1 + 6\,(1 + \delta)\cos^2\tau]\,\mu_m'} \quad \ldots \quad (9)$$

Equations (8) and (9) admit graphical solutions to obtain curves of μ_m and r_m against r_0.

The results of Benscoter[2] and Crewe (R. & M. 2513) are given in Fig. 3.

2.3. *Maximum Deceleration.*—So far we have obtained the values of the associated mass and velocity at the time of maximum deceleration in terms of the initial velocity conditions only.

The deceleration is given by

$$(1 + \mu')\frac{dV_n}{dt} + (1 + \delta)\frac{d(\mu')}{dz_s}V_n^2 = 0.$$

Putting

$$\mu' M = K z_s^3\cos^2\tau$$

and

$$V_n = (1 + r)\;V_H\sin\tau,$$

$$\frac{dV_n}{dt} = -\left(\frac{1 + \delta}{1 + \mu'}\right)\frac{3\mu'}{z_s}\,(1 + r)^2\,V_H^2\sin^2\tau.$$

where $r_0 = \tan \gamma_0 / \tan \tau$.

Details of the solution are given in Appendix III, and it should be noted that as r_0 tends to infinity equation (13) tends to

$$V_n = \frac{V_{n0}}{1 + \mu}.$$

which is the two-dimensional relation.

3.1. *Associated Mass and Velocity at Instant of Maximum Deceleration.*—Since $\cos^2 \tau \simeq 1$, equation (13) can be replaced by

$$\frac{1 + r}{1 + r_0} = \frac{1}{1 + \mu} (1 - \mu/r_0) \quad \text{.. .. ..} \quad (14)$$

If $r_0 \geqslant 1$, then comparison with the general solution for r_m (the value of r at the instant of maximum deceleration) shows that if μ_m be taken equal to its ultimate value of 2/7 and μ_m/r_0 neglected, then

$$\frac{1 + r_m}{1 + r_0} = \frac{7}{9} \quad \text{.. .. ..} \quad (14a)$$

to within one per cent, which means that

$$\frac{(V_n)_m}{V_{n0}} = \frac{7}{9} \quad \text{.. .. ..} \quad (13a)$$

to the same order (when $r_0 \geqslant 1$).

Now when τ is small and δ is negligible equation (9) becomes (generally)

$$\mu_m = \frac{2r_m}{7r_m + 6}. \quad \text{.. .. ..} \quad (15)$$

and therefore from (14a)

$$\mu_m = \frac{2\,(7r_0 - 2)}{49r_0 + 40} \quad \text{.. .. ..} \quad (15a)$$

when $r_0 \geqslant 1$.

The close agreement given by these approximate formulæ for r_m and μ_m when $r_0 \geqslant 1$ is shown in Fig. 3, where points calculated from equations (14a) and (15a) are denoted by crosses.

When $r_0 < 1$, μ_m is small and neglecting $\mu_m{}^2$ we obtain from equations (14) and (15) the relation

$$\mu_m = \frac{2r_0{}^2}{7r_0{}^2 + 10r_0 + 2}. \quad \text{.. .. ..} \quad (15b)$$

Also

$$r_m = \frac{3r_0{}^2}{5r_0 + 1}. \quad \text{.. .. ..} \quad (14b)$$

Values of μ_m and r_m calculated from equations (15b) and (14b) are denoted by circles in Fig. 3 and are in good agreement with the exact theory solutions of Crewe (R. & M. 2513) and Benscoter[2] from $r_0 = 1$ right down to $r_0 = 0$.

3.2. *Formulæ for Maximum Deceleration.*—Substituting from equation (13) in equation (12) we obtain

$$\frac{dV_n}{dt} = -\frac{1}{1 + \mu} \frac{d\mu}{dz_s} \frac{V_{n0}{}^2}{(1 + \mu)^2} (1 - \mu/r_0)^2.$$

Since

$$\mu M = \rho K z_s{}^3$$

$$\frac{dV_n}{dt} = -V_{n0}{}^2 \frac{3\mu^{2/3}}{(1 + \mu)^3} \left(\frac{K\rho}{M}\right)^{1/3} (1 - \mu/r_0)^2.$$

At the instant of maximum deceleration, this becomes

$$\left(\frac{dV_n}{dt}\right)_{\max} = - V_{n0}{}^2 \frac{3\mu_m{}^{2/3}}{(1+\mu_m)^3}\left(\frac{K\rho}{M}\right)^{1/3}\left(1 - \frac{\mu_m{}^2}{r_0}\right)$$

$$= - A\left(\frac{K\rho}{M}\right)^{1/3} V_{n0}{}^2 \qquad \text{(16)}$$

where $$A = \frac{3\mu_m{}^{2/3}}{(1+\mu_m)^3}\left(\frac{1-\mu_m}{r_0}\right)^2 \qquad \text{(17)}$$

and will be called the deceleration factor.

If $r_0 > 1$ from equation (15a)

$$\mu_m = \frac{2(7r_0 - 2)}{49r_0 + 40}$$

and Figs. 5a and 5b show that the resulting values of A approximate to the values derived from Crewe's (R. & M. 2513) solution down to $r_0 = 0{\cdot}5$, within the limits of the error introduced by neglecting the form drag force.

If $r_0 < 1$, then Fig. 5b, shows that taking

$$\mu_m = \frac{2r_0{}^2}{7r_0{}^2 + 10r_0 + 2}$$

as in equation (15b) gives agreement within the same limits right down to $r_0 = 0{\cdot}05$. Fig. 5a shows that the same agreement is obtained at large values of r_0, but it should be noted that in these cases the corresponding values of μ_m and r_m will be considerably in error (*cf.* Fig. 3).

Fig. 5b also shows that when $r_0 < 0{\cdot}25$ then a good approximation to A is given by

$$A = \frac{r_0}{(1+r_0)^2}. \qquad \text{(18)}$$

With this substitution equation (16) becomes (since $V_{n0}/(1 + r_0) \doteq V_{v0}/r_0$ when τ is small),

$$\left(\frac{dV_n}{dt}\right)_{\max} = -\left(\frac{K\rho}{M}\right)^{1/3} V_{v0}V_H \tan\tau \qquad \text{(16a)}$$

$$\doteq -\left(\frac{K\rho}{M}\right)^{1/3} V_0{}^2 r_0 \tan\tau$$

a solution which is useful because it does not require knowledge of μ_m.

Thus the whole range of r_0 can be covered with sufficient accuracy by an expression of the form of equation (16), *i.e.*

$$\left(\frac{dV_n}{dt}\right)_{\max} = - A\left(\frac{K\rho}{M}\right)^{1/3} V_{n0}{}^2$$

which depends only on the initial conditions and the geometry of the wedge.

V_{n0} is the physically significant velocity component at first impact. The factor A allows for the effect of forward velocity on the maximum deceleration and is uniquely determined by $\tan\gamma/\tan\tau$ to a first approximation and $(\rho K/M)^{1/3}$ depends on the geometry and attitude of the wedge. For example, the effect of deadrise appears only in the factor $(\rho K/M)^{1/3}$, and the variation of maximum deceleration with deadrise will therefore depend on the assumptions made for the associated mass factor K.

4. *Comparison with Classical Theory and the Importance of the Planing Force.*—With the exception of Johnstone's theory[15], the classical theory assumed that the momentum of the

float plus that of an associated mass of water remained constant during the impact period, *i.e.*, they neglected the momentum shed in the wake and obtained, as in section 2 of this report.

$$MV_{n0} = MV_n + \mu MV_n$$

instead of the general equation

$$MV_{n0} = MV_n + \mu MV_n + \int_0^t \frac{d(\mu M)}{dz_s} V_n V_T \tan \tau \, dt$$

and therefore obtained

$$\frac{dV_n}{dt} = - \frac{V_n}{1 + \mu} \frac{d\mu}{dt}.$$

Interpretations of $d\mu/dt$ have also varied. If the resultant velocity is normal to the keel, *i.e.*, there is no tangential component of velocity, then

$$\frac{d\mu}{dt} = V_n \frac{d\mu}{dz_s}$$

but generally

$$\frac{d\mu}{dt} = (V_n - V_T \tan \tau) \frac{d\mu}{dz_s},$$

i.e., there is an effect of the tangential velocity of the rate of growth of the associated mass. However some writers (*e.g.*, E. T. Jones in R. & M. 1932 and Pabst[8]) also neglected this effect and assumed that in general

$$\frac{d\mu}{dt} = V_n \frac{d\mu}{dz_s}.$$

Johnstone[15] on the other hand obtained his results for maximum impact accelerations from a consideration of steady planing forces only. He assumed that the effect of flight path angle was equivalent to planing at an increased incidence, and obtained reasonable results based on measurements of planing forces on wedges at high speeds. It is implicitly assumed that there is no acceleration effect on the flow past a planing surface. The difficulties of defining splash-up, associated mass and distribution of momentum are avoided by the use of an empirically determined associated mass factor which is dependent only on deadrise angle. The estimation of wetted areas would still however require knowledge of the splash-up factor.

4.1. *The Effect of Neglecting Both the Planing Forces and the Effect of Forward Speed on the Rate of Growth of the Associated Mass* $d\mu/dt = V_n \, d\mu/dz_s$ *as in R. & M.* 1932).—If planing forces are neglected

$$V_n = \frac{V_{n0}}{1 + \mu}$$

and since

$$\frac{d\mu}{dt} = V_n \frac{d\mu}{dz_s}$$

$$\frac{dV_n}{dt} = - \frac{V_n^2}{1 + \mu} \frac{d\mu}{dz_s}$$

$$= - V_{n0}^2 \frac{3\mu^{2/3}}{(1 + \mu)^3} \left(\frac{K\rho}{M}\right)^{1/3}.$$

This has its maximum value when $\mu = \mu_m = 2/7$, when $\left(\frac{dV_n}{dt}\right)_{\max} = -0{\cdot}61\ V_{n0}^{2}\left(\frac{K\rho}{M}\right)^{1/3}.$

From the present theory (equation 15) we have

$$\left(\frac{dV_n}{dt}\right)_{\max} = -V_{n0}^{2}\left(\frac{K\rho}{M}\right)^{1/3}\frac{3\mu_m^{2/3}}{(1+\mu_m)^3}\left(1-\frac{\mu_m}{r_0}\right)^2$$

$$= -AV_{n0}^{2}\left(\frac{K\rho}{M}\right)^{1/3}.$$

The variation of A with r_0 is shown in Fig. 5a. Its value is progressively reduced from the value of 0·61 of Jones' assumptions as $r_0 (= \tan\gamma/\tan\tau)$ decreases from infinity to zero, *i.e.*, as the ratio of the planing impact force to the pure impact force is increased. Jones' formula would thus give over large values of maximum acceleration for small values r_0, although the discrepancy will be less than ten per cent for $r_0 > 8$.

This latter condition covers most of his model test data, and explains the agreement he secured.

4.2. *The Effect of Neglecting Planing Forces but Including the Effect of Forward Speed on the Rate of Growth of the Associated Mass* ($d\mu/dt = (V_n - V_T\tan\tau)\,d\mu/dz_s$, *as in McCaig*[7]).—

If planing forces are neglected

$$V_n = \frac{V_{n0}}{1+\mu}$$

but we now have

$$\frac{d\mu}{dt} = (V_n - V_T\tan\tau)\frac{d\mu}{dz_s}$$

hence

$$\frac{dV_n}{dt} = -\frac{V_n}{1+\mu}(V_n - V_T\tan\tau)\frac{d\mu}{dz_s}$$

$$= -\frac{3\mu^{2/3}}{(1+\mu)^3}V_{n0}^{2}\left\{1-(1+\mu)\frac{V_T\tan\tau}{V_{n0}}\right\}\left(\frac{K\rho}{M}\right)^{1/3}$$

$$\simeq -\frac{3\mu^{2/3}}{(1+\mu)^3}V_{n0}^{2}\left\{1-\frac{1+\mu}{1+r_0}\right\}\left(\frac{K\rho}{M}\right)^{1/3} \text{ if } \tau \text{ small}$$

$$= -\frac{3\mu^{2/3}}{(1+\mu)^3}\left(1-\mu/r_0\right)V_{n0}V_{v0}\left(\frac{K\rho}{M}\right)^{1/3} \quad \ldots \quad \ldots \quad \ldots \quad (19)$$

which is equivalent to McCaig's expression for dV_n/dt in Ref. 7. This form is discussed by Crewe (R. & M. 2513) and μ_m is shown to be given by

$$\mu_m = \mu_{m0} = \frac{2r_m}{7r_m+3}$$

as against

$$\mu_m = \frac{2r_m}{7r_m+6}$$

when planing forces are included.

Fig. 6a shows the difference in value between μ_m and μ_{m0}.

The effect on maximum deceleration of neglecting the planing forces is shown in Fig. 6b. For this purpose we write equation (19) in the form

$$\frac{dV_n}{dt} = - A_0 V_{n0}^2 \left(\frac{K\rho}{M}\right)^{1/3} \quad \ldots \quad (20)$$

where

$$A_0 = \frac{3\mu_{m0}^{2/3}}{(1 + \mu_{m0})^3}\left(1 - \frac{\mu_m}{r_0}\right)\frac{r_0}{1 + r_0}$$

and compare A_0 with A of equation (16) (where planing forces are included) over a range of r_0.

The values are also given in the following table. It must be remembered here that the deceleration factors A and A_0 are for different immersions and $\mu_m \simeq \mu_{m0}$. The actual planing force component included in the factor A is obtained in Appendix IV.

r_0 =	$\frac{1}{2}$	1	2	4	6	8	10
A_0 =	0·131	0·0243	0·357	0·455	0·500	0·527	0·543
A =	0·293	0·401	0·481	0·537	0·561	0·572	0·580
A_0/A =	0·45	0·61	0·74	0·85	0·89	0·92	0·94

The figures in the last row of the above table show that the error introduced by neglecting the wake term may be up to fifty per cent at the small values of r_0 associated with normal good landings.

4.2.1. *Further Notes on McCaig's Approximate Formula.*—McCaig[7] assumes that, in equation (19), μ/r_0 is negligible for design stress cases and hence obtains the constant value of $\mu_m = 2/7$. Substituting back in equation (19) we obtain for the maximum deceleration

$$\left(\frac{dV_n}{dt}\right)_{\max} = - 0{\cdot}61\ V_{n0}V_{v0}\left(\frac{K\rho}{M}\right)^{1/3} \quad \ldots \quad (21)$$

which is the equivalent of equation (3) of Ref. 7. Compared with Jones' result, V_{n0}^2 has been replaced by $V_{n0} \times V_{v0}$ which allows for the effect of forward velocity.

With this form and the use of suitably chosen correction factors in the associated mass factor K, he obtains[12] excellent agreement with experimental acceleration results and with Mayo's theoretical results down to small values of r_0. The approximate $V_n \times V_v$ form has also been found to give reasonable agreement with maximum pressure results[14].

Associated Mass.—McCaig's value of the associated mass factor ρK is discussed in Appendix II. He gives in Ref. 7 a form equivalent to

$$\rho K = \rho K_2 = \rho\frac{\pi^3}{24}\cot^2\beta\cot\tau\left(1 - \frac{\beta}{\pi}\right)\left(1 - \frac{3\pi}{4}\frac{\tan\tau}{\tan\beta}\right).$$

Comparison[12] of his form with experimental results showed that best agreement was obtained if the aspect ratio correction factor $(1 - 3\pi/4 \times \tan\tau/\tan\beta)$ were neglected.

Mayo's value for ρK (see Appendix II) is

$$\rho K = \rho K_1 = 0{\cdot}82\,\rho\frac{\pi^3}{24}\cot\tau\cot^2\beta\left(\frac{1 - 2\beta}{\pi}\right)^2\left(\frac{\tan\beta}{\beta}\right)^2\left(1 - \frac{\tan\tau}{2\tan\beta}\right)$$

$$= 0{\cdot}82\,\rho\frac{\pi}{6}\cot\tau\left(\frac{\pi}{2\beta} - 1\right)^2\left(1 - \frac{\tan\tau}{2\tan\beta}\right).$$

Mayo has introduced[1] a factor 0·82 in order to obtain agreement with measured results. If the form drag factor δ be included in the force equation (as in Ref. 11) then the factor becomes 0·75.

The variation in associated mass factor ρK between Mayo (K_1) and McCaig (K_2) is shown in Fig. 7b for a wedge with deadrise angle of $22\frac{1}{2}$ deg. Over the whole range of attitudes K_2 is twenty to thirty per cent greater than K_1.

Deceleration Factor.—McCaig's equation for maximum deceleration (21) may be written as

$$\left(\frac{dV_n}{dt}\right)_{\max} = -A_0' V_{n0}^2 \left(\frac{\rho K2}{M}\right)^{1/3}$$

where $$A_0' = 0{\cdot}61 \frac{r_0}{r_0 + 1} .$$

Fig. 7a compares the deceleration factors A and A_0' over a range of r_0 and shows that McCaig's values are always less than Mayo's values. At the same time, however, McCaig's approximate factor A_0', which neglects μ/r_0, is in better agreement with A than is the factor A_0, which does not neglect μ/r_0 as is shown in Fig. 6b. These two figures (6b and 7a) would give the comparison between the methods if the associated mass factor were the same in all.

Maximum Load Factor.—Finally the variation in the resulting maximum load factors is shown in Fig. 7c. This shows that the differences in A and ρK cancel out to give good agreement between Mayo and McCaig for small values of r_0.

4.3. *The Effect of Neglecting Impact Forces and Using Steady Planing Force Expression (Johnstone*[15]*).*—The explicit planing force component of the total impact force is from equation (5)

$$F_p = 3\rho K z_s^2 \; V_n V_T \tan\tau$$

$$= 3\rho K h^2 \sec^2\tau \, h^2 \, V_n V_T \tan\tau \quad \ldots \quad (23)$$

$$\simeq 3\rho K h^2 \, V_H^2 \tau^2 \left\{1 + r - \tau^2 r^2\right\} \text{ if } \tau \text{ is small.} \quad \ldots \quad (23a)$$

In pure planing ($\gamma = 0$) and

$$F_p = 3\rho K \tan^2\tau \, h^2 \, V_H^2 \quad \ldots \quad (24)$$

and the lift component $L = F_p \cos\tau \simeq F_p$ if τ is small.

Johnstone takes (in the notation of the present report) the pure planing lift to be given by

$$L = a . \frac{\rho}{2} \tau \, h^2 \, V_H^2 \quad \ldots \quad (25)$$

and determines a from experimental evidence as a function of deadrise only.

Comparison of equation (24) and (25) gives

$$a = 6 K \tan\tau .$$

Substituting Benscoter's[2] form for K (section 2.3) above

i.e. $$K = \alpha_1^3 \, \alpha_2^3$$

where $\alpha_1^3 = 0{\cdot}82 \, \pi/_6 \cot^2\beta \cot\tau$ (section 2·3 and Appendix II).

and α_2^3 combines the splash-up, deadrise and aspect ratio correction factors.

we have $$a = 0{\cdot}82 \, \pi \cot^2\beta \; \alpha_2^3 . \quad \ldots \quad (26)$$

If we now substitute in this equation the empirical values of a given by Johnstone[15], we obtain the following values of α_2^3

$\beta =$	10	15	20	25	30
$\alpha_2^3 =$	1·0	1·18	1·11	1·16	1·27

whereas Benscoter[2] estimated $\alpha_2{}^3$ to lie between 0·9 and 1·3 for normal landings, taking the mean value of 1·1 for general use. Thus the 'associated mass' given by Johnstone's values of a will be of the right order for determining impact forces.

Having empirically determined what is effectively an associated mass factor, Johnstone develops three formulæ for impact force, which can be conveniently expressed by

(1) $L = 3\rho K h^2 V_H{}^2 \tau^2$. This assumes that the impact force is identical with the steady planing force experienced at the same attitude and horizontal speed, *i.e.*, that flight path angle has no effect. Comparison with equation (23a) shows that this is only the case when γ is negligible, *i.e.*, for extremely small flight path angles.

(2) $L = 3\rho K h^2 V_H{}^2 \tau^2 (1 + \gamma)$. This assumes that the effect of flight path angle is equivalent to planing at an increased attitude with an associated mass factor dependent on τ only. Comparison with equation (23a) shows that this formula will give a good approximation to the planing force component of the total impact force provided $\tau^2 \gamma^2$ is negligible, *i.e.*, $(V_v / V_H)^2$ negligible. It is shown in Appendix IV that this component can be eighty per cent of the total force at the time of peak deceleration for small flight path angles.

(3) $L = 3\rho K h^2 V_H{}^2 \tau^2 (1 + \gamma)(1 + 2\gamma)$. In addition to the assumption of the second formula this represents an attempt to allow for the effect of flight path angle on associated mass. The resulting force will be greater than the planing impact force, *cf.* equation (23a), and can give fair agreement with total impact force but whether the acceleration effects can be legitimately considered in terms of equivalent planing forces by an effective change of incidence is a moot theoretical point.

5. *Conclusions.*—1. The 'classical' impact theory as developed by Van Karman[5], Wagner[6] and later writers neglect the momentum shed in the wake and therefore obtained the momentum equation

$$MV_{n0} = MV_n + \mu M . V_n$$

where M is the mass of the wedge

μM is the associated mass of water

V_n is the velocity component normal to the keel

and V_{n0} is the value of V_n at first impact.

When the momentum shed in the wake is taken into account, the correct form for the momentum equation at time t becomes

$$MV_{n0} = MV_n + \mu M . V_n + \int_0^t \frac{d(\mu M)}{dz_s} V_n V_T \tan \tau \, dt$$

where $d(\mu M)/dz_s V_T \tan \tau \, dt$ is the amount of associated mass shed from an equivalent straight transverse step into the wake in time dt.

2. Based on the correct form for the momentum equation, Benscoter[2], Mayo[1] and Crewe (R. & M. 2513) put forward general theories which result in a formula for the maximum impact deceleration of a plane faced wedge of the form

$$\left(\frac{dV_n}{dt}\right)_{\max} = -\frac{(1+\delta)}{(z_s)_m} \frac{3\mu_m'}{1+\mu_m'} \frac{(1+\gamma_m)^2}{1+\gamma_0{}^2 \tan^2 \tau} V_0{}^2 \sin^2 \tau .$$

where $z_s = \sqrt[3]{\frac{\mu M}{K\rho}} =$ draft at step normal to keel

$V =$ resultant velocity

$\gamma =$ flight path angle relative to the water surface

τ = attitude of wedge to water surface

$r = \tan \gamma / \tan \tau$

subscript $_0$ refers to values at first impact

subscript m refers to values at instant of maximum deceleration

δ = form drag force/inertia force is a function of the deadrise angle β.

μ_m and r_m can be determined graphically as function of r_0 and δ. The values for $\delta = 0$ are given in Fig. 3.

The value of ρK (and hence of z_s) depends on the assumptions made for the value of the associated water mass μM. Using Mayo's form for μM, which includes an empirical correction factor to give agreement with experimental data, Crewe produced the curves for $(1/K)^{1/3}$ reproduced in Fig. 4.

3. An approximate formula for maximum impact deceleration, which can be expressed explicitly in terms of the initial conditions, is given in the present report as

$$\left(\frac{dV_n}{dt}\right)_{\max} = - A \left(\frac{K\rho}{M}\right)^{1/3} V_{n0}^{\ 2}$$

where
$$A = \frac{3\mu_m^{\ 2/3}}{(1 + \mu_m)^3}(1 - \mu_m/r_0)^2 < 0{\cdot}61$$

and
$$\mu_m = \frac{2(7r_0 - 2)}{49r_0 + 40} \text{ when } r_0 > 1$$

$$= \frac{2r_0^{\ 2}}{7r_0^{\ 2} + 10r_0 + 2} \text{ when } r_0 < 1.$$

Values of μ, K and A are given in Figs. 3, 4 and 5.

In this expression, V_{n0} is the physically significant velocity component at first impact. The factor A allown for the effect of forward speed on the impact force and, as before $(\rho K/M)^{1/3}$ depends on the assumptions made for the associated water mass.

4. It is shown that the planing force, defined by rate of increase of momentum in the wake may account for eighty per cent or more of the total impact force at small values of r_0 (< 1).

5. Of the earlier theories which neglected the momentum shed in the wake.

(*a*) The only allowance made by E. T. Jones (R. & M. 1932) for the effect of forward speed was in taking the velocity component normal to the keel as the fundamental velocity parameter. The value of A does not decrease with decrease of flight path angle and his results are shown only to be of value for $r_0 > 8$.

(*b*) McCaig[7] included the effect of forward speed on the rate of growth of associated mass and obtained the product $V_{n0} \times V_{v0}$ as his fundamental parameter. In terms of $V_{n0}^{\ 2}$ the constant A then decreases with decrease of flight path angle and gives a useful first approximation for the effect of forward speed, but as an approximation to the complete theory his full formula is only justified theoretically for values of $r_0 > 6$. His approximate formula, however, is a good empirical approximation for small values of r_0, by virtue of his choice of associated mass factor ρK.

6. Johnstone's theory[15] considers the impact forces entirely in terms of momentum shed in the wake, and in effect makes allowance for the pure impact force by considering it as a planing force resulting from an increase of incidence and associated mass proportional to the flight path angle. The wake momentum treatment is only justifiable theoretically for values of $r_0 < 0{\cdot}5$, but by using empirical planing force data reasonable results can be obtained for total impact force.

List of Symbols

(*a*) *Geometrical*

β	Deadrise angle
τ	Attitude of wedge, relative to undisturbed water surface
γ	Angle of descent, γ_0 = value of γ at first impact
r	$\tan \gamma / \tan \tau$, r_0 = value of r at first impact
h, z, z_s, *and* L	See Fig. 2 for definitions
O	point of first impact on water surface
ox, oz	axis fixed in space, parallel to and perpendicular to keel.

(*b*) *Velocities*

V	resultant velocity at time t.
V_T, V_n	components of velocity parallel to and perpendicular to the keel
V_H, V_v	components of velocity parallel to and perpendicular to undisturbed water surface (horizontal and vertical if water is calm).

Subscripts zero refer to velocities at first impact.

(*c*) *Forces and Pressures*

F	resultant impact force.
F_n	component of impact force perpendicular to keel
$F =$	F_n if viscosity and gravity forces are neglected
F_p	planing force

(*d*) *Masses*

M	Mass of wedge
μM	Associated mass of water
K	Associated mass factor, given by $\mu M = \varrho K z_s^3$
K	includes factors ξ_1 and ξ_2 where
ξ_1	is factor to allow for deadrise angle
ξ_2	is factor to allow for aspect ratio of the wetted area.

Subscripts m refer to values at instant of maximum deceleration.

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APPENDIX I.

An Alternative Development of the Force and Momentum Expressions of Section 2.1, *and Comparison with the Formulæ Given by Mayo*[1], *Benscoter*[2] *and Crewe* (*R. & M.* 2513)

This method is the same in essence as that employed by Mayo[1] and Benscoter[2], and consists in summing the elements of force given by equation (3) over the wedge length. Take co-ordinates as shown in Fig. 2a. The element of force

$$\delta F = \frac{d}{dt}(\rho \bar{K} z_s\, dx \times V_n)$$

$$= \rho \bar{K} z^2\, dx\, \frac{dV_n}{dt} + 2\rho \bar{K} z \frac{dz}{dt}\, dx \times V_n$$

(assuming that $\bar{K}$ is independent of z, which is only the case for a plane faced wedge)

Since we are considering a fixed plane in the fluid then

$$\frac{dz}{dt} = \frac{dZ}{dt} = V_n,$$

hence $$\delta F = \rho \bar{K} z^2\, dx\, \frac{dV_n}{dt} + 2\rho \bar{K} z\, dx \times V_n^2 .$$

The only sections giving reaction are those in contact with the wedge at time t. Therefore the resultant force

$$F = \rho \bar{K} \frac{dV_n}{dt} \int_{x_s}^{x_s+L} z^2\, dx + \rho \bar{K} V_n^2 \int_{x_s}^{x_s+L} 2z\, dx$$

$$= \frac{\rho \bar{K} z_s^3 \cot \tau}{3} \cdot \frac{dV_n}{dt} + \bar{K} z_s^2 \cot \tau \; V_n^2.$$

As in the first method (section 2.1) define $\mu M = \rho K z_s^3$

where $K = \dfrac{\bar{K} \cot \tau}{3}$, then

$$F = \rho K z_s^2 \frac{dV_n}{dt} + 3\rho K z_s^2 \times V_n^2$$

$$= \mu M \times \frac{dV_n}{dt} + \frac{d(\mu M)}{dz_s} \times V_n^2$$

which is the same as given by equation (5a) of the main text.

Now μM is the associated mass of water below the wedge at time t and by definition must be taken as moving with the wedge,

hence $$\frac{d}{dt}(\mu M) = \frac{d(\mu M)}{dz_s} \times \frac{dz_s}{dt}.$$

If the resultant velocity is normal to the keel then

$$\frac{dz_s}{dt} = \frac{dZ}{dt} = V_n$$

and from equation (4a) we obtain

$$F = \frac{d}{dt}(\mu M \times V_n)$$

which exactly corresponds to the two-dimensional equation.

Normally however there is a velocity component parallel to the keel, in which case

$$\frac{dz_s}{dt} = \frac{dh}{dt} \sec \tau$$

$$= V \sin \gamma \times \sec \tau \; (\textit{see } \text{Fig. } 2)$$

$$= V \sin \left[(\gamma + \tau) - \tau \right] \sec \tau$$

$$= V_n - V_T \tan \tau$$

in which case

$$\frac{d}{dt}(\mu M) = \frac{d(\mu M)}{dz_s} (V_n - V_T \tan \tau)$$

hence from equation (4a)

$$F = \mu M \frac{dV_n}{dt} + V_n \frac{d(\mu M)}{dt} + V_n \times \frac{d(\mu M)}{dz_s} V_T \tan \tau$$

$$= \frac{d}{dt}(\mu M \times V_n) + V_n \frac{d(\mu M)}{dz_s} V_T \tan \tau$$

which is the same result as obtained by the first method (equation (5) of the main text).

The expression

$$\frac{d(\mu M)}{dz_s} \tan \tau = 3\rho K {z_s}^2 \tan \tau$$
$$= \rho \bar{K} {z_s}^2$$

is the associated mass of a unit section of fluid at the step.

Hence

$$V_n \; V_T \frac{d(\mu M)}{dz_s} \tan \tau = \rho \bar{K} z_s V_T V_n$$

gives the rate at which momentum is shed from the step into the wake because of forward speed.

Thus, physically, the force is equal to the sum of the time rate of change of the momentum of the associated mass plus the rate of shedding of fluid with downward velocity to the wake, or it may be said to be composed of a pure impact force plus a planing impact force.

Summarising the results, either method gives the generalised momentum equation at time t during the impact as

$$MV_{n0} = MV_n + \mu M . V_n + \int_0^t \rho \bar{K} (z_s')^2 \, V_n(t') V_T \, dt$$

where the associated mass $\mu M = \rho K {z_s}^3$

and $\bar{K} = 3 K \tan \tau$

while the resultant upwards force on the wedge is normal to the keel and is given by

$$F = \frac{d}{dt} (\mu M . V_n) + \; V_n \frac{d(\mu M)}{dz_s} \; V_T \tan \tau .$$

Comparison of the Form of the Force Equation with those of Benscoter[2], *Mayo*[1] *and Crewe* (*R. & M.* 2513).—(*a*) *Benscoter*[2].—Allowing for differences in notation, equation (5a) is the equivalent of Benscoter's equation (51), which reads

$$F_u = m\ddot{z} + m'\dot{z}^2 .$$

Here, F_u refers to the inertia force, m is the associated mass and z is the space co-ordinate Z of the present report.

$$\dot{z} = V_n, \; \ddot{z} = \frac{dV_n}{dt}$$

and m' corresponds to $\frac{d(\mu M)}{dz_s}$.

(*b*) *Mayo*[1].—Similarly, equation (5) is the equivalent of Mayo's equation (22), which reads

$$F_n = \frac{Ky^3 \frac{dV_n}{dt}}{3 \sin \tau \cos^2 \tau} + \frac{Ky^2 \; {V_n}^2}{\sin \tau \cos \tau}$$

Here, K is the two-dimensional associated mass factor $\rho \bar{K}$ of the present report, and y is the step depth h of the present report.

Equation (5) above is the equivalent of Mayo's equation (24), which reads

$$F_n = \frac{Ky^3 \frac{dV_n}{dt}}{3 \sin \tau \cos^2 \tau} + \frac{Ky^2 \, \dot{y} V_n}{\sin \tau \cos^2 \tau} + \frac{Ky^2 \; V_p V_n}{\cos^2 \tau}$$

where V_p (in Mayo's notation) is the velocity component parallel to the keel.

(*c*) *Crewe* (*R. & M.* 2513).—Crewe gives an expression for the inertia force of the form

$$F = M \frac{d}{dt}(\mu V_n) + B_T h^{x-1} V_T V_n$$

taking $\mu M = Kh^x$

instead of $\mu M = \rho K z^3$.

This amounts to taking a different value of the two-dimensional associated mass factor $\bar{K}$ in the planing force term to that in the pure impact force term. Crewe discusses this point on pp. 38 and 39 of R. & M. 2513. Theoretically it depends on the validity of the strip theory method used by Benscoter and Mayo (and in the present report) in their approach to the problem. However, from experimental evidence it seems justifiable to make use of the same factor in each term, and this simplifies Crewe's equation to that of Benscoter or Mayo,—*i.e.*, equation (5) of this report—always assuming that there is no time lag factor in the build-up of the planing forces.

APPENDIX II.

Summary of the Values of the Associated Mass Factor ρK at Present in Use.

The value of the associated mass depends on the geometry of the wedge and on the total wetted area which has to be considered. The classical approach is to build up the three-dimensional associated mass as the sum of a series of two-dimensional values, introducing a correction factor for aspect ratio to allow for the escape of fluid around the perimeter of the wetted area.

The two-dimensional associated mass was assumed by Von Kármán[5], from the theory of motion of a flat plate, to be the mass of a semi-cylinder of water on the wetted width of the wedge as diameter. He took the wetted width to be the intersection of the wedge with the undisturbed water surface. Wagner[6] modified this assumption by basing his associated mass on the splashed-up wetted width, where splash-up is the rise of displaced water along the sides of the float. He obtained the splash-up by consideration of the two-dimensional flow around the edges of a flat plate.

Since both values are based on the theory of motion of a flat plate, a further correction had to be introduced to allow for the effect of deadrise angle on the motion.

Detailed consideration follows.

Basic Value.—The two-dimensional basic (Von Kármán) value for the associated mass is, as stated above, assumed to be the mass of a semi-cylinder of water diameter equal to the width of the wedge at the undisturbed water surface.

Thus, by integration, a three-dimensional basic value for a wedge at attitude τ (as in Fig. 2) is the mass of a half cone of water determined by the intersection of the wedge with the undisturbed water surface (provided the chines are not immersed). Thus the basic value for the associated mass factor ρK, defined by $\mu M = \rho K z_s^3$, is

$$\rho \frac{\pi}{6} \cot^2 \beta \cot \tau = \rho K_0 \text{ say.}$$

Correction for Splash-up.—As calculated by Wagner[6], splash-up will make the wetted beam of a plane-faced wedge $\pi/2$ times that intersected by the undisturbed water surface. This result is backed by experimental evidence from the Royal Aircraft Establishment towing tank (unpublished). If the associated mass is based on the splashed-up wetted width, then the basic value ρK_0 must be multiplied by the factor $\pi^2/4$. Further Royal Aircraft Establishment Tank unpublished experimental evidence would indicate that there is also a splash forward (*see* section 2.1), but this so far has not been taken into account when estimating associated mass values.

Correction for Deadrise Angle.—The factor for deadrise angle may be denoted by ξ_1 and has been given different values.

Kreps[3] advanced the value

$$\xi_1 = \frac{2 \tan \beta}{\pi}\left[\frac{\Gamma(1/2 + \beta/\pi)\,\Gamma(1 - \beta/\pi)}{\Gamma(3/2 - \beta/\pi)\,\Gamma\beta(/\pi)} - 1\right]$$

$$\simeq 1 - 2\beta/\pi \text{ for practical values of } \beta.$$

This value has since been used by McCaig[7] and Russian writers.

Wagner[6] on the other hand put forward the value

$$\xi_1 = \left(1 - 2\beta/\pi\right)^2 \left(\frac{\tan \beta}{\beta}\right)^2$$

which has been used by Mayo[1] and Benscoter[2].

Correction for Aspect Ratio.—A correction for finite aspect ratio is made by a factor ξ_2. Such a factor was empirically determined by Pabst[8] for rectangular plates and takes the form

$$\xi_2 = \frac{\lambda^2 - 0{\cdot}425\,\lambda + 1}{(\lambda^2 + 1)^{3/2}} \quad \text{where } \lambda = \frac{(\text{beam})^2}{\text{wetted area}}$$

$$\simeq 1 - \lambda/2 \quad \text{for } \lambda \leqslant 0{\cdot}7.$$

He then applied this correction to the wedge impact problem by taking λ as the aspect ratio of the triangular area formed by the projection parallel to the keel of the intersection of the wedge with the undisturbed water surface. ($\lambda = 0$ corresponds to the two-dimensional case).

Mayo[1] and Benscoter[2] take $\lambda = \tan \tau/\tan \beta$, *i.e.*, half the aspect ratio of a rectangle on the same base. Crewe (R. & M. 2513) makes the same assumption.

McCaig[7] multiplies this value of λ by three to allow for the fact that the volume of the associated mass is only one-third that of the original half cylinder, and by $\pi/2$ to allow for the splashed-up area.

Combining these factors we obtain

$$\mu M = \rho K z_s^3$$

$$= \frac{\pi^2}{4} \rho K_0 \xi_1 \xi_2 z_s^3$$

which, by substitution of the various values, gives the forms used by Mayo[1], Benscoter[2], Crewe (R. & M. 2513) and McCaig[7], apart from additional constants added to secure agreement with experimental results.

The diversity of formulæ obtained and the necessity for adding arbitrary constants both point to the need for a more rational means of estimating associated mass,* preferably based on three-dimensional concepts rather than by trying to extend further the two-dimensional concept.

APPENDIX III.

An Approximate Solution of the Equation of Motion

If we assume that $\cos^2 \tau \simeq 1$ in normal landings and that the form drag force can be neglected in comparison with the inertia forces, *i.e.*, $\delta = 0$, then the equation of motion (6) becomes

$$\frac{dV_n}{d\mu} + \frac{1}{1 + \mu} \frac{d\mu}{dz_s} V_n^2 = 0$$

*It should be noted that since the original date of this report, such an estimate has been given by Crewe's (Area)^2/Perimeter formula and is used in obtaining the design formulæ of Ref. 16.

which, since $dz_s/dt = V_n - V_T \tan \tau$ is equivalent to

$$\frac{dV_n}{d\mu} = -\frac{V_n^2}{V_n - V_T \tan \tau} \times \frac{1}{1+\mu}.$$

Integrating this equation we obtain

$$\frac{V_n}{V_{n0}} \exp\left\{ V_T \tan \tau \left(\frac{1}{V_n} - \frac{1}{V_{n0}} \right) \right\} = \frac{1}{1+\mu}.$$

Now $(V_T \tan \tau)/V_n$ is not necessarily small if τ is small, but up to the instant of maximum deceleration the exact theory shows that

$$V_T \tan \tau \left(\frac{1}{V_n} - \frac{1}{V_{n0}} \right)$$

is small and hence

$$\exp\left\{ V_T \tan \tau \left(\frac{1}{V_n} - \frac{1}{V_{n0}} \right) \right\} \simeq 1 + V_T \tan \tau \left(\frac{1}{V_n} - \frac{1}{V_{n0}} \right).$$

This approximation is valid within one per cent down to $r_0 = 0{\cdot}5$ ($r_0 = \tan \gamma_0/\tan \tau$) and is ten per cent high at $r_0 = 0{\cdot}2$.

Substituting this approximation in the solution for V_n/V_{n0} we obtain

$$V_n = \frac{V_{n0}^2}{V_{n0} - V_T \tan \tau} \times \frac{1}{1+\mu} - \frac{V_{n0} V_T \tan \tau}{V_{n0} - V_T \tan \tau}.$$

Now τ is small hence

$$\frac{V_T \tan \tau}{V_{n0}} = \frac{1 - r_0 \tan^2 \tau}{1 + r_0}$$

$$\simeq \frac{1}{1 + r_0}.$$

Hence $$V_n = \frac{V_{n0}}{1+\mu}\left(1 - \mu/r_0\right)$$

or $$\frac{1+r}{1+r_0} = \frac{1}{1+\mu}\left(1 - \mu/r_0\right)$$

and these are equations (13) and (14) respectively of the main text.

APPENDIX IV

An Approximate Expression for the Planing Force Component of the Present Theory

The force equation corresponding for oblique impact is given by equation (5), *i.e.*

$$F = \frac{d}{dt}\left(\mu M \frac{dV_n}{dt} \right) + 3\rho K z_s^2 \tan \tau \, V_n V_T$$

where $\mu M = \rho K z_n^3$.

On the right-hand side of this equation the first term corresponds to the pure impact force, the second to the planing force. Strictly speaking it is not possible to make this sharp division since the planing force has an effect on the pure impact force theough the momentum equation.

For some purposes however it is useful to know the magnitude of the planing force occurring explicitly in the above equation. Denote this force by F_p.

Then $$F_p = 3\rho K z_s^2 V_n V_T \tan\tau$$

or $$\frac{F_p}{M} = 3\mu^{2/3}\left(\frac{K\rho}{M}\right)^{1/3} V_n V_T \tan\tau$$

where $$V_n = \frac{V_{n0}}{1+\mu}\left(1 - \mu/r_0\right) \quad \ldots \ldots \ldots \ldots \ldots \ldots \quad (13)$$

and $$V_T \tan\tau \simeq \frac{V_{n0}}{1+r_0}$$

or $$\frac{F_p}{M} = \frac{3\mu^{2/3}}{(1+\mu)^2}\left(\frac{K\rho}{M}\right)^{1/3} V_{n0}^2 \frac{1-\mu/r_0}{1+r_0}*$$

which can be compared with the total force given by

$$\frac{F}{M} = \frac{3\mu^{2/3}}{(1+\mu)^3}\left(\frac{K\rho}{M}\right)^{1/3} V_{n0}^2 \left(1 - \mu/r_0\right)^2.$$

The planing force given by (24) of Section 4.3 will have its maximum value at a different time to the total force, but considering conditions for maximum total force, *i.e.*, $\mu_m = 2r_m/(7r_m + 6)$, we have

$$\frac{F_p}{F} = \frac{(1+\mu_m)}{(1+r_0)(1-\mu_m/r_0)}$$

which $\longrightarrow$ 0 as $r_0 \longrightarrow \infty$

and 0·8 when $r_0 = 0{\cdot}5$.

The value for $r_0 = 0{\cdot}5$ is considerably larger than that given in section 4.2 but it must be remembered that in the present case, a portion of the planing force is already included implicitly in the ‘ pure ’ impact force, where it serves to reduce the value of the latter force.

---

*In pure planing this acceleration would be given by

$$\frac{F_p}{M} = 3\mu^{2/3}\left(\frac{K\rho}{M}\right)^{1/3} V_n^2 \quad \text{where } V_n = V_{n0} = \text{const.}$$

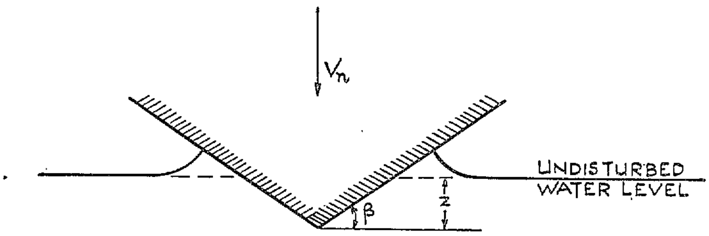

FIG. 1. Normal impact of a wedge with zero attitude (two dimensional case).

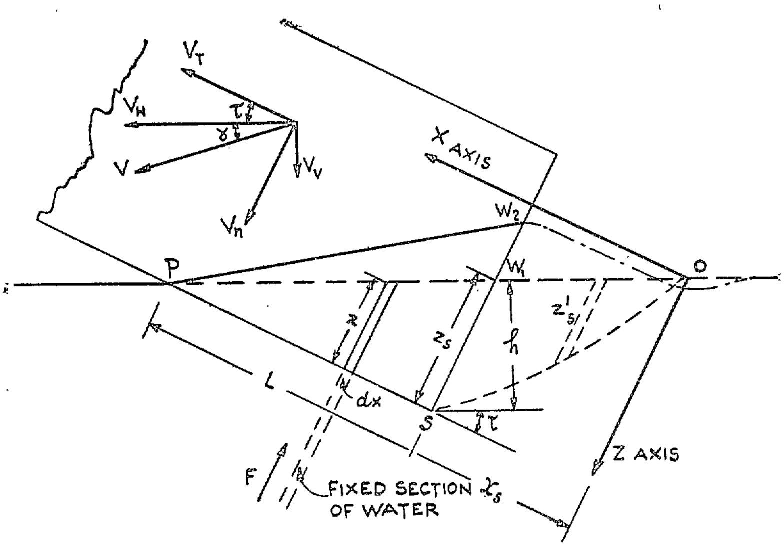

FIG. 2a. Oblique impact of wedge at finite attitude (three-dimensional case). Straight transverse step.

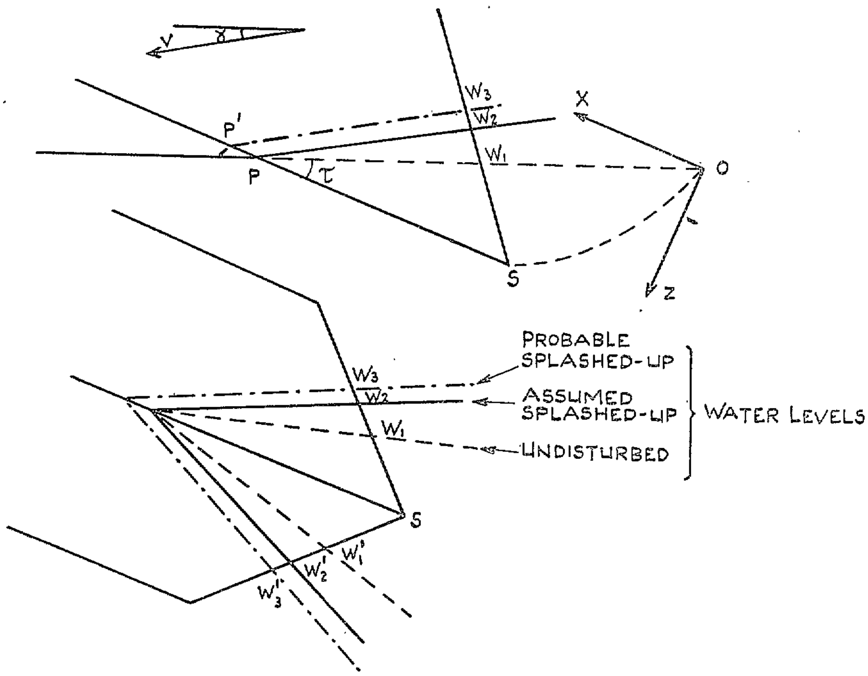

FIG. 2b. Oblique impact of wedge at finite attitude. Vee planform step.

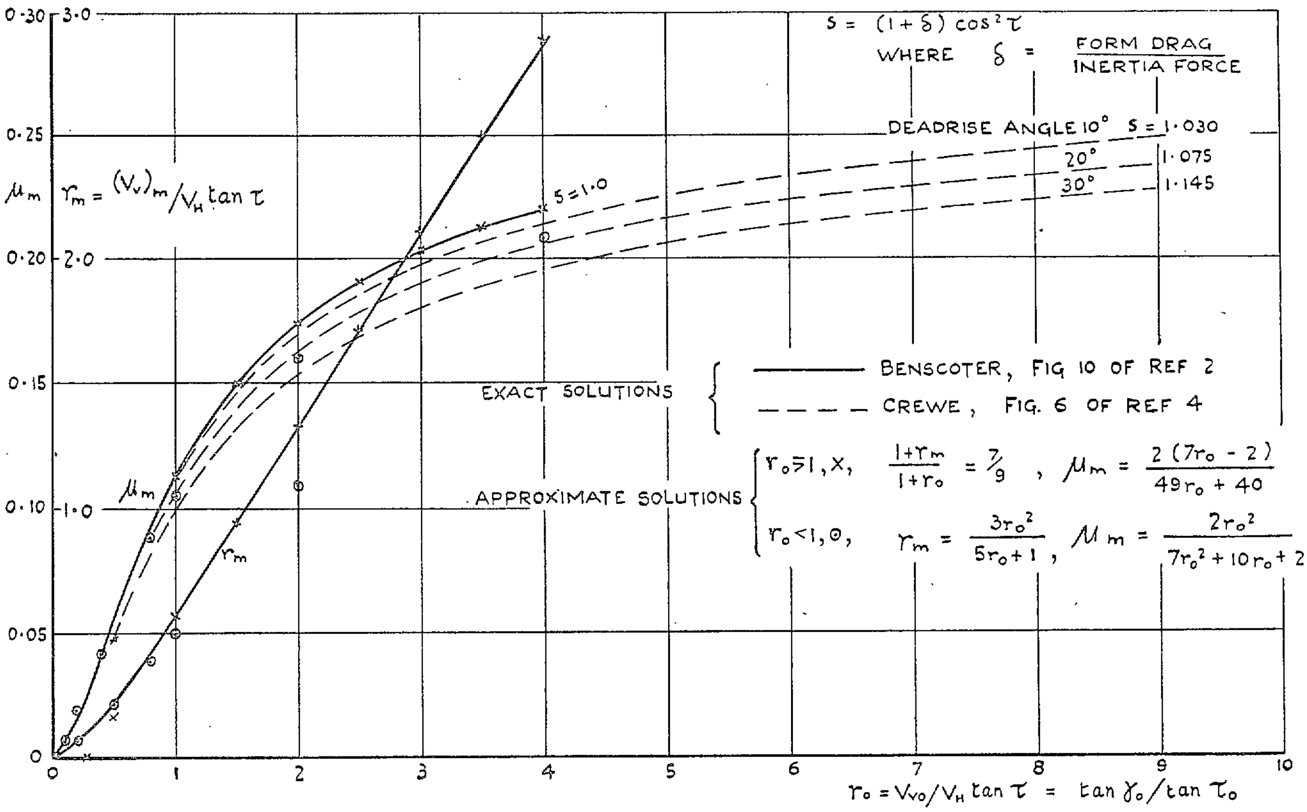

FIG. 3. Associated mass and velocity at instant of peak deceleration.

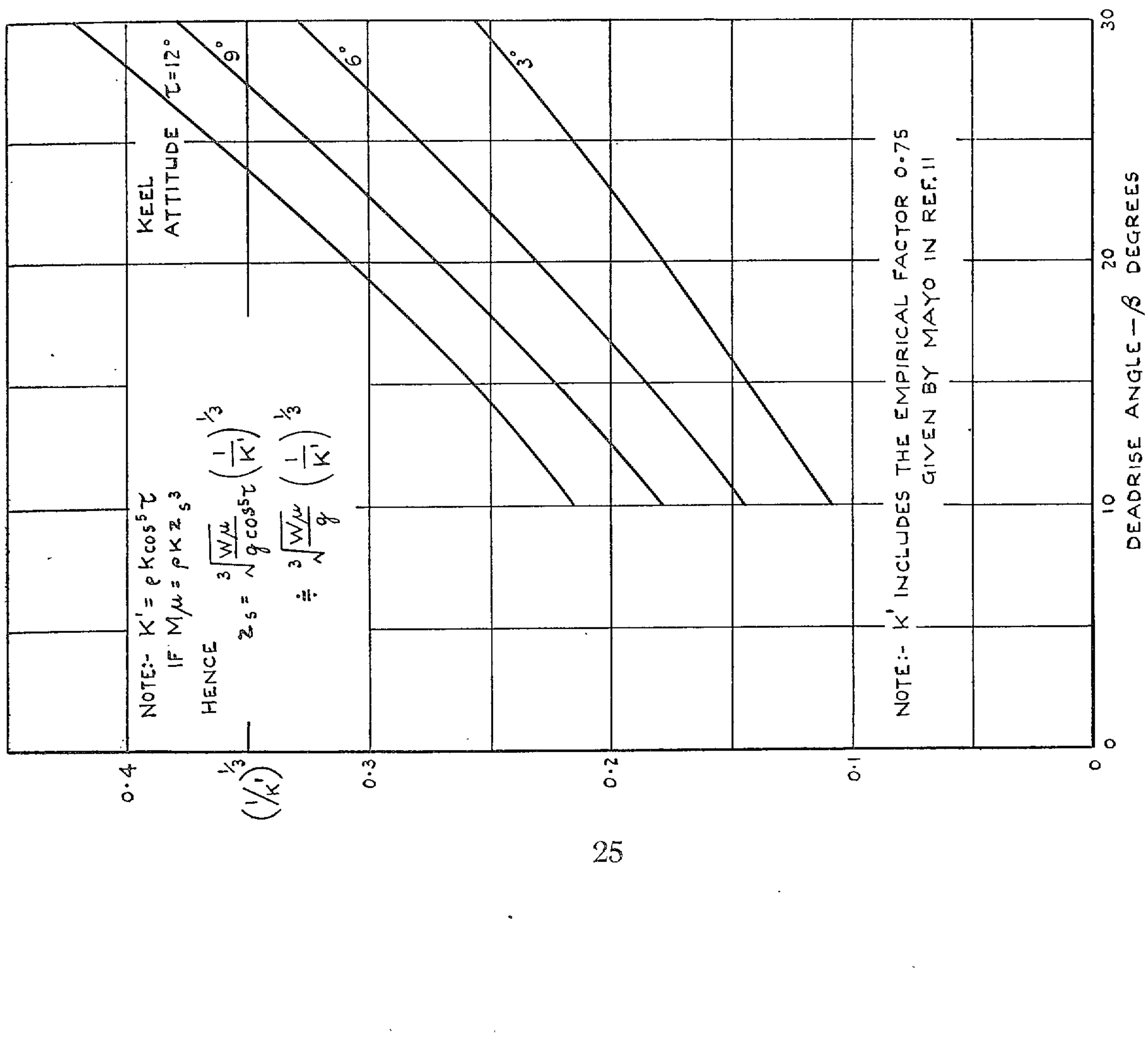

FIG. 4. Curves for estimating draft with respect to undisturbed water level (reproduction of Fig. 14 of Ref. 4).

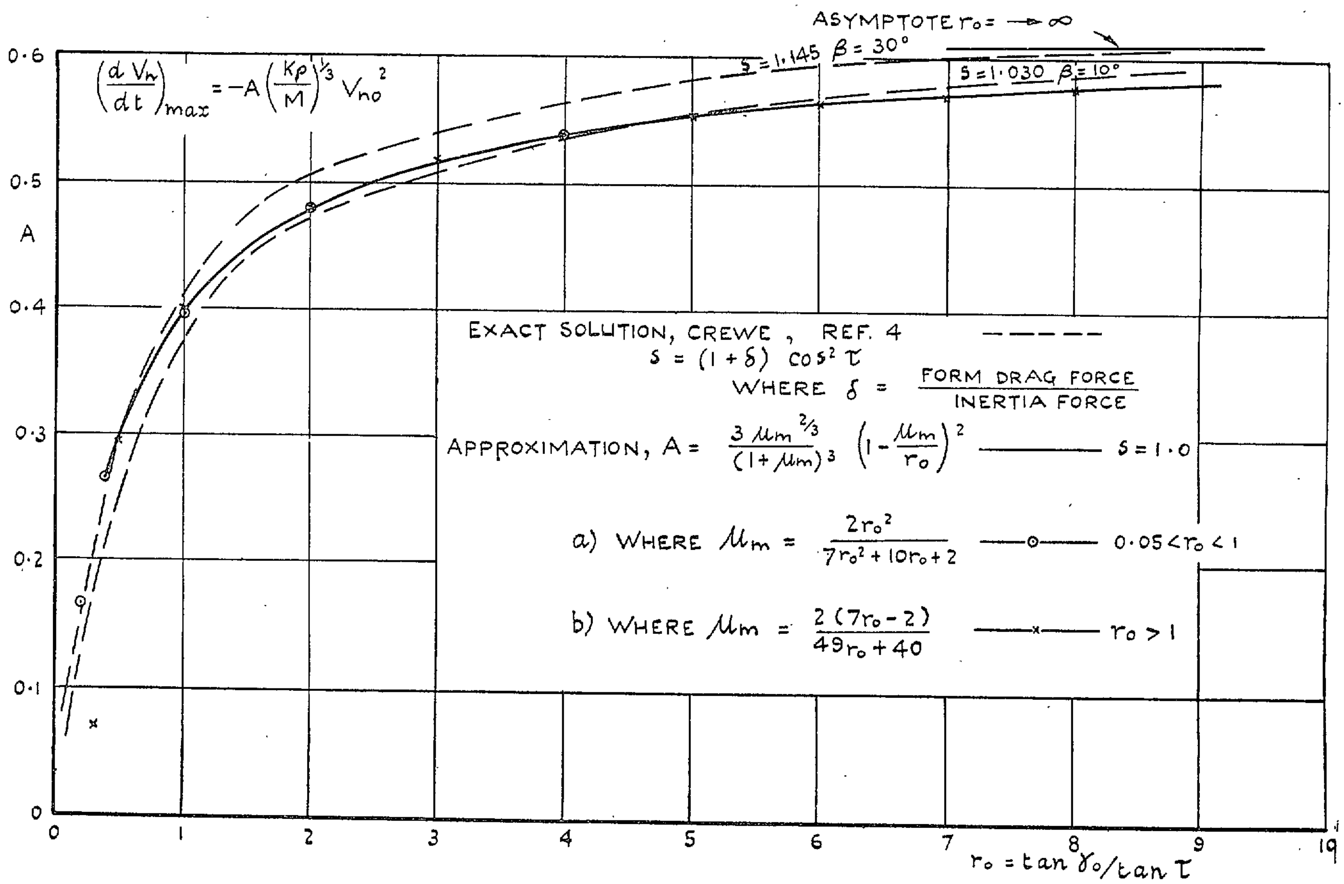

Fig. 5a. Validity of approximate formulæ for maximum deceleration.

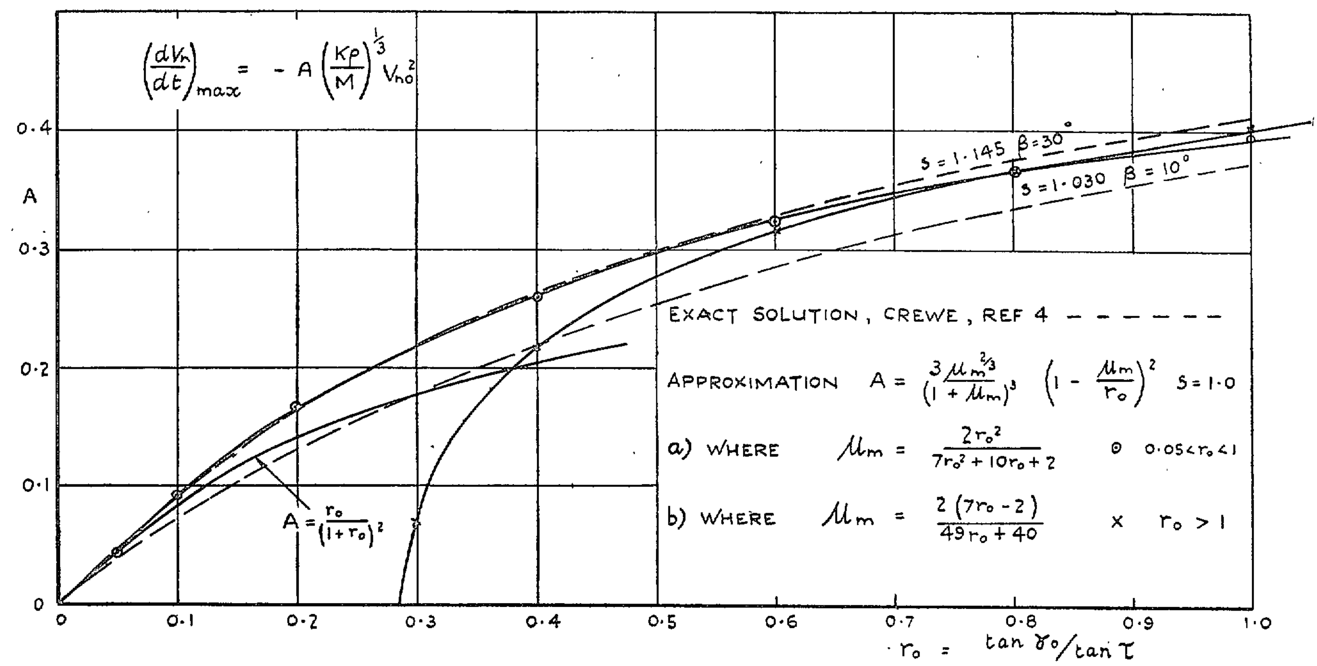

Fig. 5b. Validity of approximate formulæ for maximum deceleration for small values of r_0.

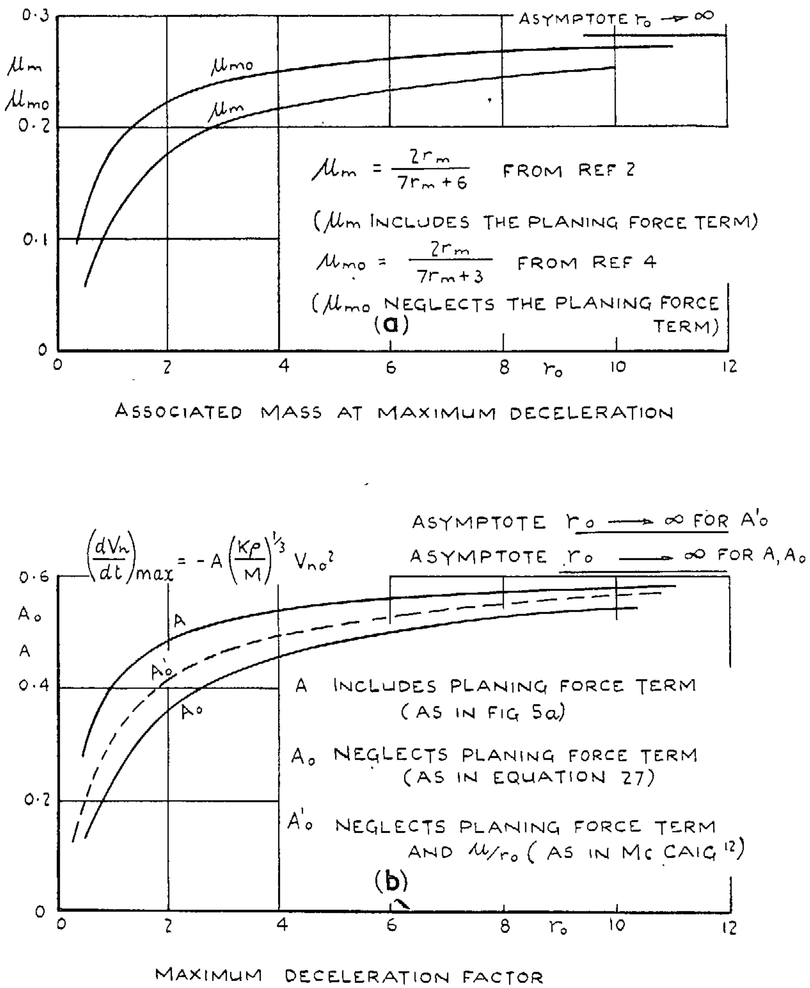

FIG. 6. Effect of neglecting the planing force.

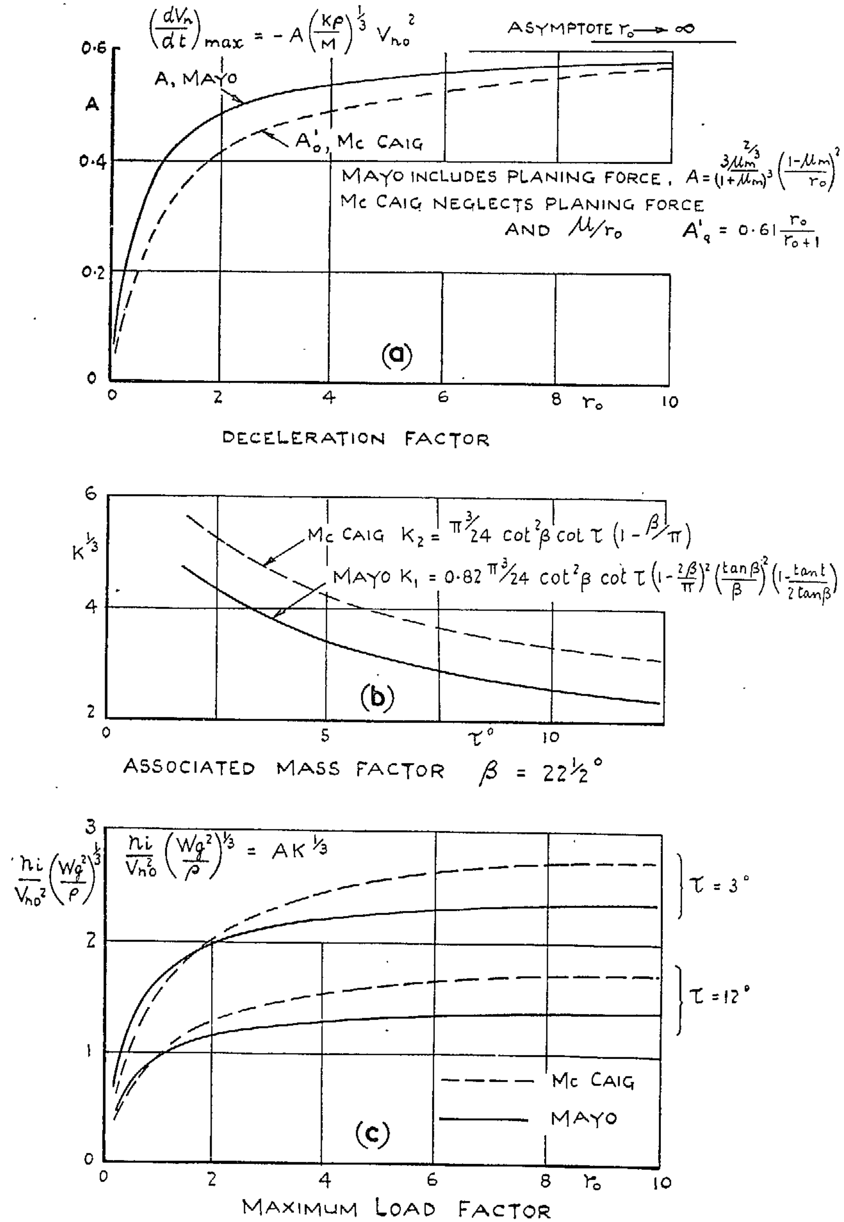

FIG. 7. Comparison of McCaig's with Mayo's theory for $22\frac{1}{2}$ deg wedge.

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