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Wing-Fuselage Flutter of Large Aeroplanes

By

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Summary.—A general theoretical method is described which takes into account a large number of degrees of freedom and is based on the design data for the aeroplane.

The problem specifically investigated is the symmetrical flutter of a particular aircraft. Twelve degrees of freedom are assumed to cover pitching and translational motion of the whole aeroplane, flexure and torsion of the wings, and fuselage vertical bending. The tailplane is regarded as rigid. In the case considered, estimates indicate that the lowest critical speed is well above the maximum design speed of the aeroplane. The influence of the additional degrees of freedom associated with movements of the control surfaces is not considered.

1. *Introductory Remarks*.—In the past, flutter calculations have usually been restricted to a few degrees of freedom only, corresponding to the first four or five resonance modes at most. Nowadays, however, aeroplanes have much wider speed ranges and flutter at high speed might involve the higher as well as the lower modes of vibration*. There is, therefore, a real need for a method of calculation of both resonance modes and critical flutter speeds which takes into account a large number of degrees of freedom and is preferably based on the design data for the aeroplane. The treatment outlined in this report may be regarded as a development of a method which has already been proposed in connection with the flutter of wings with wing engines.†

PART I

General Theory

2. *Wing Co-ordinates and Displacements*.—The reference axes are OX along the centre-line of the fuselage, OY at right angles to OX in the plane of the wing, and OZ normal to the plane of the wing (*see* Fig. 1). All distances from these axes are expressed in terms of the mean chord c of the wing as unit of length. The wings (tip to tip) are divided symmetrically into $2s-1$ chordwise strips of width $2l_1c$, l_2c, .. l_sc with centre-lines at the spanwise positions h_1c, .. $\pm h_2c$, .. $\pm h_sc$, where the centre-line of the centre strip lies along the OX-axis ($h = 0$). For the starboard wing the centre-lines defined by h_1, h_2, .. h_s cut the OY-axis at H_1 ($\equiv 0$), H_2, .. H_s.

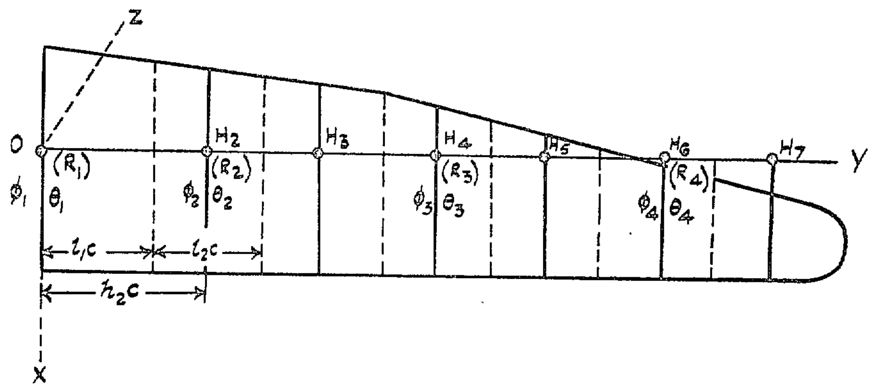

FIG. 1

* The influence of shear is neglected in the calculations of resonance frequencies.

† Section 2 of Ref. 1.

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The displacements of the wing strip at h_i relative to the root chord are defined by the co-ordinates $\varphi(h_i)$, $\theta(h_i)$, where $c\varphi(h_i)$ represents the downward displacement at H_i due to wing flexure, and where $\theta(h_i)$ denotes the change in the incidence of the strip due to twist of the wing. Then, if $c\varphi$ denotes the downward displacement of the root chord at O, and if θ_1 represents the change in incidence, the total angular displacements $\Phi(h_i)$ $\Theta(h_i)$ of the strip at h_i will be given by

$$\Phi(h_i) = \varphi_1 + \varphi(h_i), \Theta(h_i) = \theta_1 + \theta(h_i) . \qquad (1)$$

Reference sections are chosen at η_1 ($\equiv h_1$), η_2, η_3, . . η_n to coincide with some of the h sections. The corresponding reference displacement co-ordinates are φ_1, θ_1, and φ_2, θ_2; φ_3, θ_3; . . . φ_n, θ_n relative to φ_1, θ_1. The reference centres R_1, R_2, . . . R_n coincide with some of the H points as shown in Fig. 1 for the particular case of $s = 7$ and $n = 4$.

The downward displacement of a point P at x ($\equiv c\xi$), y ($\equiv c\eta$) is then given by

$$z(\eta) = c\{\Phi(\eta) + \xi\Theta(\eta)\} , \qquad (2)$$

where $\Phi(\eta)$, $\Theta(\eta)$ represent the absolute displacements of the section η. By the introduction of modal functions, the absolute displacements can be expressed in the form

$$\left.\begin{aligned} \Phi(\eta) - \varphi_1 = \varphi(\eta) &= \sum_{j=2}^{n} \varphi_j f_j(\eta) + \sum_{j=2}^{n} \theta_j g_j(\eta) , \\ \Theta(\eta) - \theta_1 = \theta(\eta) &= \sum_{j=2}^{n} \varphi_j G_j(\eta) + \sum_{j=2}^{n} \theta_j F_j(\eta) , \end{aligned}\right\} \qquad (3)$$

where $\varphi(\eta)$, $\theta(\eta)$ represent relative displacements due to bending and twisting of the wing. The modal functions f, g, G, F, satisfy the following relations when $i \geqslant 2$, $j \geqslant 2$, namely,

$$\left.\begin{aligned} f_j(\eta_j) = 1 ;\ f_j(\eta_i) = 0, i \neq j ; \\ F_j(\eta_j) = 1 ;\ F_j(\eta_i) = 0, i \neq j ; \end{aligned}\right\} \qquad (4)$$

and $G_j(\eta_i) = 0 = g_j(\eta_i)$ at all reference sections. When $j = 1$,

$$f_1(\eta) = 1 = F_1(\eta) ;\ g_1(\eta) = 0 = G_1(\eta) \qquad (5)$$

for all values of η.

In matrix notation, the total displacements of the strips h_i are given by

$$\{\Phi(h_1), .. \Phi(h_s), \Theta(h_1), .. \Theta(h_s)\} = R\{\varphi_1, .. \varphi_n, \theta_1, .. \theta_n\} , \qquad (6)$$

where

$$\left.\begin{aligned} R &= \begin{bmatrix} f & g \\ G & F \end{bmatrix} , \\ f &= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 1 & f_2(h_2) & \cdots & f_n(h_2) \\ \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdots & \cdots & \cdot \\ 1 & f_2(h_s) & \cdots & f_n(h_s) \end{bmatrix} , \\ g &= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & g_2(h_2) & \cdots & g_n(h_2) \\ \cdot & \cdots & \cdots & \cdot \\ \cdot & \cdots & \cdots & \cdot \\ 0 & g_2(h_s) & \cdots & g_n(h_s) \end{bmatrix} \end{aligned}\right\} \qquad (7)$$

with similar expressions for F and G. The relative displacements are given by

$$\{\varphi(h_2), \ldots \varphi(h_s), \theta(h_2), \ldots \theta(h_s)\} = R_w \{\varphi_2, \ldots \varphi_n, \theta_2, \ldots \theta_n\}, \quad \ldots \quad \ldots \quad (8)$$

where

$$R_w \equiv \begin{bmatrix} f_w & g_w \\ G_w & F_w \end{bmatrix} \quad \ldots \quad (9)$$

The submatrices f_w, g_w, G_w, F_w, correspond to f, g, G, F, respectively with the first row and column of each omitted. Thus

$$f_w = \begin{bmatrix} f_2(h_2) & \ldots & f_n(h_2) \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ f_2(h_s) & \ldots & f_n(h_s) \end{bmatrix} \quad \ldots \quad (10)$$

3. *Fuselage Co-ordinates and Displacements.*—The fuselage is represented by a system of masses m_r elastically connected.

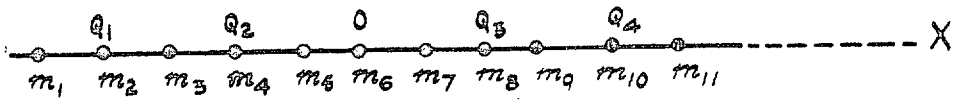

Fig. 2.

Let $x_r\ (\equiv c\xi_r)$ represent the distance of any mass m_r from the origin O, and let the displacement of m_r due to bending of the fuselage in the vertical plane be denoted by $c\chi_r$. The total displacement of m_r, when the translational displacement $c\varphi_1$ and the pitching displacement θ_1 of the whole aeroplane as a rigid body are taken into account, is then given by $c\Psi_r$, where

$$c\Psi_r = c\varphi_1 + c\xi_r\theta_1 + c\chi_r. \quad \ldots \quad (11)$$

If the fuselage is represented by s masses, the displacement of each mass will be given by

$$\{\Psi\} = D\{\varphi_1, \theta_1\} + \{\chi\}, \quad \ldots \quad (12)$$

where $\{\Psi\} \equiv \{\Psi_1, \Psi_2, \ldots \Psi_s\}$ and $\{\chi\} \equiv \{\chi_1, \chi_2, \ldots \chi_s\}$.

The matrix D in equation (12) is defined by

$$D = \begin{bmatrix} 1 & \xi_1 \\ 1 & \xi_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & \xi_s \end{bmatrix} \quad \ldots \quad (13)$$

If the relative angular displacements* $q_1, q_2, q_3, \ldots q_N$ at the reference points Q_1, Q_2, Q_N, are taken as reference co-ordinates, the displacement χ_r of m_r is expressible in the form

$$\{\chi_1, \chi_2, \ldots \chi_s\} = \begin{bmatrix} \mu & 0 \\ 0 & 0 \\ 0 & \nu \end{bmatrix} \{q_1, q_2, \ldots q_N\} \quad \ldots \quad (14)$$

The submatrices μ and ν refer to the front and the rear parts of the fuselage respectively, and the row of null elements corresponds to the zero value of χ for the mass at the origin. The total displacements can be expressed in the form

$$\{\Psi\} = U\{t\} \quad \ldots \quad (15)$$

where $\{t\} \equiv \{\varphi_1, \theta_1, q_1, q_2, q_N\}$, and where the matrix U is given by equations (12), (13) and (14).

* Actual downward displacements due to bending are cq_1, cq_2, etc.

4. *Inertial Coefficients**.—(a) *Wing*.—Let the mass distribution of the ith strip of wing be concentrated on its centre line h_i, and let the mass, mass-moment about OY, and moment of inertia about OY be $M_0 m_e(h_i)$, $M_0 c p_e(h_i)$ and $M_0 c^2 q_e(h_i)$ respectively, where M_0 is a convenient unit of mass. The kinetic energy of the ith strip† of wing is then given by

$$2K(h_i) = M_0 c^2 [m_e(h_i)\dot{\Phi}(h_i)^2 + 2p_e(h_i)\dot{\Phi}(h_i)\dot{\Theta}(h_i) + q_e(h_i)\dot{\Theta}(h_i)^2]$$

$$= M_0 c^2 [\dot{\Theta}'(h_i), \dot{\Theta}'(h_i)] \begin{bmatrix} m_e(h_i) & p_e(h_i) \\ p_e(h_i) & q_e(h^i) \end{bmatrix} \begin{bmatrix} \dot{\Phi}(h_i) \\ \dot{\Theta}(h_i) \end{bmatrix}$$

and the kinetic energy T_w of one whole wing with half the fuselage and half the tailplane is given by

$$2T_w \equiv M_0 c^2 [\dot{\Phi}', \dot{\Theta}'] \begin{bmatrix} m_e & p_e \\ p_e & q_e \end{bmatrix} \begin{bmatrix} \dot{\Phi} \\ \dot{\Theta} \end{bmatrix} \quad \ldots \quad (16)$$

where $\overset{\circ}{\Phi}$, $\overset{\circ}{\Theta}$ denote columns of the total velocities and where m_e, p_e, q_e, are diagonal matrices. By the use of equation (6), equation (16) yields

$$2T_w = M_0 c^2 \dot{w}' A_r w, \quad \ldots \quad (17)$$

where

$$w \equiv \{\varphi, \theta\},$$

$$A_r \equiv R' \begin{bmatrix} m_e & p_e \\ p_e & q_e \end{bmatrix} R \equiv \begin{bmatrix} m & p \\ p' & q \end{bmatrix}, \quad \ldots \quad (18)$$

and R is defined by equation (6).

(b) *Fuselage*.—The inertial coefficients corresponding to φ_1, θ_1, q_1, q_2, .. q_N which define the mode of displacement of the fuselage can be derived in the same way as those for the wing. The fuselage and tailplane are represented by a system of masses m_r, and the total kinetic energy T_f of these masses is given by

$$2T_f = M_0 c^2 \sum_{r=1} m_r \dot{\psi}_r^2,$$

$$= M_0 c^2 \dot{\psi}' m_f \dot{\psi},$$

$$= M_0 c^2 \dot{t}' A_f \dot{t},$$

where, by equation (15),

$$A_f \equiv U' m_f U, \quad \ldots \quad (19)$$

and m_f represents a diagonal matrix of the fuselage masses. The inertial coefficients corresponding to φ_1, θ_1 in A_f have already been included in equation (18) and hence the total kinetic energy $2T$, of the whole aeroplane can be expressed as

$$2T = M_0 c^2 [\dot{w}' A_r \dot{w} + \tfrac{1}{2} \dot{t}' A_f \dot{t}] = M_0 c^2 \dot{W}' A^* W, \quad \ldots \quad (20)$$

where $\bar{A}_f$ is the same as A_f except for the coefficients of $\dot{\varphi}_1^2$, $\dot{\varphi}_1 \dot{\theta}_1$ and $\dot{\theta}_1^2$ which are assumed to be zero. Hence A^* represents the inertial matrix corresponding to the reference co-ordinates denoted by the column matrix

$$W \equiv \{\varphi_1, \varphi_2, \ldots \varphi_n, \theta_1, \theta_2, \ldots \theta_n, q_1, q_2 \ldots q_N\}.$$

$$\equiv \{\varphi, \theta, q\}. \quad \ldots \quad (21)$$

5. *Aerodynamic Generalised Moments*.—(a) *Wings*.—Let us first consider the aerodynamic forces acting on the ith strip of the wing. The downward force Z_i at H_i and the pitching moment M_i about OY can be expressed in the form

* Aerodynamic inertial effects are included.

† The entre strip, $i = 1$, includes half the fuselage and half the tailplane which are assumed to be rigid at this stage.

$$\left.\begin{aligned} -\frac{cZ_i}{\rho c^2 c_i l_i V} &= c\sigma_{1i}\dot{\Phi}(h_i) + \sigma 2_i c_i \dot{\Theta}(h_i) + V\sigma_{3i}\Theta(h_i)\,, \\ -\frac{M_i}{\rho c c_i^2 l_i V} &= c\sigma_{4i}\dot{\Phi}(h_i) + \sigma_{5i} c_i \dot{\Theta}(h_i) + V\sigma_{6i}\Theta(h_i)\,, \end{aligned}\right\} \quad \ldots \quad \ldots \quad (22)$$

where

$$\begin{aligned} &\sigma_{1i} = l_{\dot{z}},\ \ \sigma_{2i} = l_{\dot{\theta}} - hl_{\dot{z}},\ \ \sigma_{3i} = l_{\theta}, \\ &\sigma_{4i} = -m_{\dot{z}} - hl_{\dot{z}},\ \ \sigma_{6i} = -m_{\theta} - hl_{\theta}\,, \\ &\sigma_{5i} = -m_{\dot{\theta}} - h(l_{\dot{\theta}} - m_{\dot{z}}) + h^2 l_{\dot{z}}\,, \end{aligned}$$

and where hc_i represents the distance behind the leading edge of the point H_i. It should be noted that h is in general dependent on the spanwise position of the strip. At the centre section, equation (22) give the force and moment on half the strip since its width is taken to be $2l_1 c$. The values assumed for the fundamental derivatives are

$$\left.\begin{aligned} &l_{\dot{z}} = 1\cdot5, \qquad l_{\dot{\theta}} = 1\cdot4, \qquad l_{\theta} = 1\cdot6 \\ &m_{\dot{z}} = -0\cdot375,\ m_{\dot{\theta}} = -0\cdot7,\ m_{\theta} = -0\cdot4, \end{aligned}\right\} \quad \ldots \quad (23)$$

as given in R. & M. 1782[2]. They can, however, be replaced by two-dimensional frequency dependent derivatives if that is desirable.

Let $L_1, L_2, \ldots L_n$ and $M_1, M_2, \ldots M_n$ represent the flexural and torsional generalised aerodynamic moments corresponding to the displacements φ, θ of one wing. Then, the amount of work done when the wing is given the further small displacements $\delta\varphi$, $\delta\theta$ is $[L_1, L_2, \ldots L_n]\,\delta\varphi + [M_1, M_2, \ldots M_n]\,\delta\theta$

$$\begin{aligned} &= \sum_{i=1}^{s} cZ_i \delta\Phi(h_i) + \sum_{i=1}^{s} M_i \delta\Theta(h_i)\,, \\ &= \sum_{i=1}^{s} [cZ_i, M_i]\{\delta\Phi(h_i), \delta\Theta(h_i)\}\,, \\ &= \sum_{i=1}^{s} [cZ_i, M_i]\begin{bmatrix} f(h_i), & g(h_i) \\ G(h_i), & F(h_i) \end{bmatrix}\{\delta\varphi, \delta\theta\}\,. \end{aligned} \quad \ldots \quad \ldots \quad (24)$$

Hence, the generalised moments are given by

$$[L, M] = \sum_{i=1}^{s} [cZ_i, \quad M_i]\begin{bmatrix} f(h_i), & g(h_i) \\ G(h_i), & F(h_i) \end{bmatrix},$$

or, more conveniently, by

$$\begin{aligned} [L, M] &= \sum_{i=1}^{s} \begin{bmatrix} f'(h_i), & g'(h_i) \\ G'(h_i), & F'(h_i) \end{bmatrix}\{cZ_i, M_i\}\,, \\ &= R'\{cZ, M\}, \end{aligned} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (25)$$

where R' is the transposed of R, and where $\{cZ, M\}$ represents the column of the cZ_i and M_i terms.

Now, from equation (22),

$$\begin{bmatrix} -cZ_i \\ -M_i \end{bmatrix} = \rho l_i c^4 V \begin{bmatrix} \sigma_{1i} k_i, & \sigma_{2i} k_i^2 \\ \sigma_{4i} k_i^2, & \sigma_{5i} k_i^3 \end{bmatrix} \begin{bmatrix} \dot{\Phi}(h_i) \\ \dot{\Theta}(h_i) \end{bmatrix} + \rho l_i c^3 V^2 \begin{bmatrix} 0, & \sigma_{3i} k_i \\ 0, & \sigma_{6i} k_i^2 \end{bmatrix} \begin{bmatrix} \Phi(h_i) \\ \Theta(h_i) \end{bmatrix} \quad \ldots \quad \ldots \quad (26)$$

where $k_i \equiv c_i/c$, c_i being the local chord.

Let

$$B_0 \equiv \begin{bmatrix} \sigma_1 K_1, & \sigma_2 K_2 \\ \sigma_4 K_2, & \sigma_5 K_3 \end{bmatrix}, C_0 \equiv \begin{bmatrix} 0, & \sigma_3 K_1 \\ 0, & \sigma_6 K_2 \end{bmatrix}, \quad \ldots \quad (27)$$

where K_1, K_2, K_3 and $\sigma_1, \sigma_2, \ldots \sigma_6$ are diagonal matrices defined by

$$K_n = \begin{bmatrix} l_1 k_1^n, & 0 & 0 & \ldots & 0 \\ 0 & l_2 k_2^n & 0 & \ldots & 0 \\ 0 & 0 & l_3 k_3^n & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & l_s k_s^n \end{bmatrix}, \quad \ldots \quad (28)$$

and

$$\sigma_j \equiv \begin{bmatrix} \sigma_{j1} & 0 & 0 & . & . & 0 \\ 0 & \sigma_{j2} & 0 & . & & 0 \\ 0 & 0 & \sigma_{j3} & . & & 0 \\ . & . & . & . & & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & \sigma_{js} \end{bmatrix} \quad \ldots \quad (29)$$

Then, by equations (25), (26) and (27),

$$\{L, M\} = -\rho c^4 V R' B_0 R \dot{w} - \rho c^3 V^2 R' C_0 R w, \quad \ldots \quad (30)$$

where $w \equiv \{\varphi, \theta\}$. When the motion is simple harmonic, $\dot{w} = ipw$, and equation (30) yields

$$\{L, M\} = -\rho V^2 c^3 (C + i\omega B) w, \quad \ldots \quad (31)$$

where $C \equiv R' C_0 R$, $B \equiv R' B_0 R$ and $\omega \equiv pc/V$. Formulae (30) and (31) give the aerodynamic moments on one wing only, and do not include the aerodynamic moments on the tailplane. The contributions due to the tailplane are considered in the next section.

(b) *Tailplane.*—The tailplane is replaced by an equivalent rigid rectangular plane of the same span and area. In the estimation of the aerodynamic moments the classical derivatives defined by equation (23) are used. Let Z_t ($\equiv c\varphi_t$), θ_t denote the displacement of the equivalent tailplane referred to its leading edge, and let Z_t, M_t represent the corresponding downward force and pitching moment. The displacements φ_t, θ_t depend on φ_1, θ_1 and the fuselage reference co-ordinates $q_m, q_{m+1}, \ldots q_N$ corresponding to reference points behind the origin.* They can be expressed in the form

$$\{\varphi_t, \theta_t\} = \alpha_t \{\varphi_1, \theta_1, q_m, q_{m+1}, \ldots q_N\}, \quad \ldots \quad (32)$$

where α_t is a $\bar{2}$, $N-m+3$ matrix. Let L_{1t}, M_{1t}, N_m, N_{m+1}, $\ldots N_N$ represent the generalised aerodynamic moments. Then as in section 4a, and by the use of equation (32)

$$\begin{aligned} [L_{1t}, M_{1t}, N_m, \ldots N_N]\delta \{\varphi_1, \theta_1, q_m, \ldots q_N\} &= cZ_t\, \delta\varphi_t + M_t\, \delta\theta_t \\ &= [cZ_t, M_t] \{\delta\varphi_t, \delta\theta_t\} \\ &= [cZ_t, M_t]\, \alpha_t \delta \{\varphi_1, \theta_1, q_m, \ldots q_N\}, \end{aligned}$$

and hence

$$\{L_1, M_{1t}, N_m, \ldots N_N\} = [cZ_t, M_t]\, \alpha_t = \alpha_t' \{cZ_t, M_t\}. \quad \ldots \quad (33)$$

* The motion of the tailplane is independent of the co-ordinates $q_1 \ldots q_{m-1}$, of the front part of the fuselage.

Then, by equation (26), $\{cZ_t, M_t\}$ can be expressed in the form

$$- \{cZ_t, M_t\} = 2\rho c^4 V\beta_0\{\dot{\varphi}_t, \dot{\theta}_t\} + 2\rho c^3 V^2 \gamma_0 \{\varphi_t, \theta_t\}, \quad (34)$$

where β_0 and γ_0 are $\overline{2, 2}$ matrices. Substitution in equation (33) then yields

$$\{L_{1t}, M_{1t}, N_m, \ldots N_N\} = - 2\rho c^4 V\beta_t \dot{w}_t - 2\rho c^3 V\gamma^2 \gamma_t w_t , \quad (35)$$

where

$$\left. \begin{aligned} w_t &\equiv \{\varphi_1, \theta_1, q_m \ldots q_N\} , \\ \beta_t &\equiv \alpha_t'\beta_0\alpha_t, \\ \gamma_t &\equiv \alpha_t'\gamma_0\alpha_t. \end{aligned} \right\} \quad (36)$$

Formula (35) gives the generalised aerodynamic moments due to the motion of the whole tailplane. The factor 2 is omitted when only half is being considered.

6. *Elastic Coefficients.*—(a) *Wing.*—To determine the elastic constants for the wing, the centre strip $h_i = \eta_1 = 0$ is assumed fixed so that $\varphi_1 = \theta_1 = 0$. The displacements at any other section are then given by equation (8), namely,

$$\{\varphi(h_i), \theta(h_i)\} = R_w \{\varphi_2, \varphi_3, \ldots \varphi_n, \theta_2, \theta_3, \ldots \theta_n\} ;$$

where

$$R_w \equiv \begin{bmatrix} f_w & g_w \\ G_w & F_w \end{bmatrix}$$

The elements of the matrix R_w are chosen to satisfy the conditions imposed by the elastic characteristics of the wing. In order to find the displacements due to a distribution of forces and moments at the reference sections, use is made of the following flexibility coefficients:—

- α_{ij} Linear displacement at R_i due to unit flexural force applied at R_j,
- β_{ij} Linear displacement at R_i due to unit twisting moment at section η_j,
- δ_{ij} Twist at section η_i due to unit flexural force at R_j,
- γ_{ij} Twist at section η_i due to unit moment at η_j.

By the Reciprocal Theorem, $\alpha_{ij} = \alpha_{ji}$, $\beta_{ij} = \delta_{ji}$ and $\gamma_{ij} = \gamma_{ji}$.

The displacements z, θ at the reference sections* due to any system of forces Z and moments M applied at the reference sections are then given by

$$\begin{bmatrix} z \\ \theta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \beta' & \gamma \end{bmatrix} \begin{bmatrix} Z \\ M \end{bmatrix} \equiv \Phi \begin{bmatrix} Z \\ M \end{bmatrix}, \quad (37)$$

where α, β, γ, β' ($\equiv \delta$) denote the $\overline{n-1, n-1}$ matrices of the flexibility coefficients. Inversion of equation (37) yields

$$\{Z, M\} = E\{z, \theta\} ,$$

where

$$E = \Phi^{-1} = \begin{bmatrix} e & n \\ n' & d \end{bmatrix} \quad (38)$$

The corresponding stiffness and flexibility matrices for the angular co-ordinates $\varphi_2, \ldots \varphi_n, \theta_2, \ldots \theta_n$ in the Lagrangian form are given by

$$\varepsilon\bar{\bar{E}} = \begin{bmatrix} c^2 e & cn \\ cn' & d \end{bmatrix} = \varepsilon \begin{bmatrix} \bar{e} & \bar{n} \\ \bar{n}' & \bar{d} \end{bmatrix}, \quad (39)$$

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* The reference section at the centre ($i = 1$) is omitted.

and

$$\varepsilon^{-1}\,\bar{\Phi} = \begin{bmatrix} \alpha/c^2 & \beta/c \\ \beta'/c & \gamma \end{bmatrix} = \varepsilon^{-1}\begin{bmatrix} \bar{\alpha} & \bar{\beta} \\ \bar{\beta}' & \bar{\gamma} \end{bmatrix},$$

where ε is a typical stiffness and $\bar{E}$, $\bar{\Phi}$ are non-dimensional stiffness and flexibility matrices. If

$$J_{n-1} \equiv \begin{bmatrix} I_{n-1} & 0 \\ 0 & cI_{n-1} \end{bmatrix}$$

where I_{n-1} is the unit matrix of order $n-1$, then equations (37), (38) and (39) yield

$$\left.\begin{aligned} \bar{\Phi} &= \frac{\varepsilon}{c^2} J_{n-1}\Phi J_{n-1}, \\ \text{and} \qquad E &= \frac{\varepsilon}{c^2} J_{n-1}\bar{E} J_{n-1} \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (40)$$

The distortion of matrices f_w, g_w, G_w, F_w are chosen in such a way that, when the reference sections are loaded in a general manner, the wing displacements at the sections $h_2, h_3,..h_s$, as given by equation (8), accord with those required by measurement or direct calculation. The flexibility coefficients required are defined as follows:—

$\alpha(\eta_j)$ Column of linear displacements at $H_2, H_3,..H_s$ due to unit flexural force at R_j,

$\beta(\eta_j)$ Column of linear displacements at $H_2, H_3,..H_s$ due to unit moment at section η_j,

$\delta(\eta_j)$ Column of twists at sections $h_2, h_3,..h_s$ due to unit flexural force at R_j,

$\gamma(\eta_j)$ Column of twists at sections $h_2, h_3,..h_s$ due to unit moment at section η_j.

The reference section $h_1 = \eta_1 = 0$ is assumed to be fixed and only the relative displacements of the wing are considered. If forces $Z_2, Z_3,..Z_n$ and twisting moments $M_2, M_3,..M_n$ are applied at the remaining $n-1$ reference sections, the angular displacements of the h sections will be given by

$$\{c\varphi(h), \theta(h)\} = S\{Z, M\}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (41)$$

where S denotes the $\overline{2S-2,\ 2n-2}$ matrix defined by

$$S \equiv \begin{bmatrix} \alpha_0 & \beta_0 \\ \delta_0 & \gamma_0 \end{bmatrix} = \begin{bmatrix} \alpha(\eta_2)\ \alpha(\eta_3) \ldots \alpha(\eta_n) & \beta(\eta_2),\ \beta(\eta_3) \ldots \beta(\eta_n) \\ \delta(\eta_2)\ \delta(\eta_3) \ldots \delta(\eta_n), & \gamma(\eta_2)\ \gamma(\eta_3) \ldots \gamma(\eta_n) \end{bmatrix} \qquad \ldots \quad \ldots \quad (42)$$

The displacements defined by equation (41) are also given by

$$J_{s-1}\{c\varphi(h), \theta(h)\} = R_w J_{n-1}\Phi\{Z, M\}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (43)$$

where J_{s-1} is similar to J_{n-1}, but of higher order since $s>n$. Equations (41) and (42) then yield

$$J_{s-1}S = R_w J_{n-1}\Phi, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (44)$$

and, by the use of equation (40), it can be deduced that

$$\begin{aligned} R_w &= J_{s-1}S\Phi^{-1}J_{n-1}{}^{-1} = J_{s-1}SEJ_{n-1}{}^{-1} \\ &= \frac{\varepsilon}{c^2}(J_{s-1}SJ_{n-1})\bar{E} = \bar{\Phi}_0\bar{E}, \end{aligned} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (45)$$

where

$$\bar{\Phi}_0 = \frac{\varepsilon}{c^2}(J_{s-1}SJ_{n-1})$$

$$= \varepsilon \begin{bmatrix} \frac{\alpha_0}{c^2} & \frac{\beta_0}{c} \\ \frac{\delta_0}{c} & \gamma_0 \end{bmatrix} = \begin{bmatrix} \bar{\alpha}_0 & \bar{\beta}_0 \\ \bar{\delta}_0 & \bar{\gamma}_0 \end{bmatrix} . \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (46)$$

When R_w is known, the matrix R defined by equation (7) can readily be determined. The displacements at the h sections corresponding to the reference co-ordinates φ, θ are then given by equation (6).

(b) *Fuselage.*—The relative displacements χ of the masses m_r due to bending of the fuselage are given by equation (14). If χ_n represents the zero displacement of the mass m_n at the origin, and if $q_1, q_2, \ldots q_{m-1}$ and $q_m, q_{m+1}, \ldots q_N$ denote the reference co-ordinates for the front and the rear parts of the fuselage respectively, equation (14) can be replaced by

$$\begin{aligned} \{\chi_1, \chi_2, \ldots \chi_{n-1}\} &= \mu\{q_1, q_2, \ldots q_{m-1}\}, \\ \{\chi_{n+1}, \chi_{n+2}, \ldots \chi_s\} &= \nu\{q_m, q_{m+1}, \ldots q_N\}. \end{aligned} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (47)$$

The procedure for the determination of μ and ν is similar to that described in the previous section for the derivation of R_w, but, in this case, since only fuselage bending is involved, it is much simpler. For the front part of the fuselage the deflection cq produced by loads $P_1, P_2, \ldots P_{m-1}$ at the reference points $Q_1, Q_2, \ldots Q_{m-1}$ are given by

$$\{cq\} = \alpha_f\{P\}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (48)$$

where α_f denotes the $\overline{m-1}$, $m-1$ matrix of the flexibility coefficients corresponding to unit loads Inversion of equation (48) gives

$$\{P\} = \alpha_{f-1}\{cq\}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (49)$$

and hence the stiffness matrix $\varepsilon\bar{E}_f$ corresponding to the generalised angular co-ordinates $q_1, q_2, \ldots q_{m-1}$ can be expressed as

$$\varepsilon\bar{E}_f = c^2\alpha_f^{-1}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (50)$$

Similarly, for the rear part of the fuselage the stiffness matrix $\varepsilon\bar{E}_b$ is given by

$$\varepsilon\bar{E}_b = c^2\alpha_b^{-1} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (51)$$

where α_b is the corresponding $\overline{s-m}$, $s-m$ matrix of the flexibility coefficients referred to the co-ordinates $q_m, q_{m+1}, \ldots q_N$.

The deflections $c\chi_1, c\chi_2, \ldots c\chi_{n-1}$ at the $n-1$ points of the front part of the fuselage are given by

$$c\{\chi_1, \chi_2, \ldots \chi_{n-1}\} = S_f\{P\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (52)$$

where S_f is the $\overline{n-1}$, $m-1$ matrix of the deflections corresponding to unit loads applied at the reference points. By the use of equations (47), (49) and (52), it can be shown that

$$S_f\alpha_f^{-1}\{cq\} = c\mu\{q\}. \qquad \ldots \quad \ldots \quad \ldots \qquad (53)$$

Hence, for the front part of the fuselage,

$$\mu = S_f\alpha_f^{-1}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (54)$$

Similarly, it can be shown that

$$\gamma = S_b\alpha_b^{-1} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (55)$$

for the rear part. When the matrices α_f, α_b S_f, S_b have been determined by measurement or calculation, μ and ν can then be readily derived from equations (54) and (55) respectively. Since

D is already known, the matrix U of equation (15) can then be found. The stiffness matrix for the whole fuselage referred to the co-ordinates $q_1, q_2, \ldots q_N$ can be expressed in the form

$$\varepsilon \bar{E}_f = c^2 \begin{bmatrix} \alpha_f^{-1} & 0 \\ 0 & \alpha_b^{-1} \end{bmatrix}, \qquad (56)$$

where ε is the typical stiffness used in equation (39).

7. *Dynamical Equations.*—The equations of motion corresponding to the generalised co-ordinates W defined by equation (21) are obtained by the use of equations (20), (31), (35), (39) and (56). The kinetic energy $2T$ of the whole aeroplane is given by equation (20) in the form

$$2T = M_0 c^2 \dot{W}' A^* \dot{W}. \qquad (57)$$

Since no stiffnesses are associated with the co-ordinates φ_1, θ_1, the stiffness matrix $\varepsilon \bar{E}_w$ for one wing corresponding to the displacement co-ordinates $\varphi_1, \varphi_2, \ldots \varphi_n, \theta_1, \theta_2 \ldots \theta_n$ is, by equation (39) expressible in the form

$$\varepsilon \bar{E}_w = \varepsilon \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \bar{e} & 0 & \bar{n} \\ 0 & 0 & 0 & 0 \\ 0 & \bar{n}' & 0 & \bar{d} \end{bmatrix}, \qquad (58)$$

where the null elements correspond to φ_1, θ_1. The stiffness matrix $\varepsilon \bar{E}_f$ for the fuselage is given by equation (56) and, for convenience, is expressed in the form

$$\varepsilon \bar{E}_f = \varepsilon \begin{bmatrix} \bar{e}_1 & 0 \\ 0 & \bar{d}_1 \end{bmatrix}. \qquad (59)$$

Hence the stiffness matrix $2\varepsilon E^*$ for the whole aeroplane corresponding to all the co-ordinates φ, θ, q is given by

$$2\varepsilon E^* = \varepsilon \begin{bmatrix} 2\bar{E}_w & 0 \\ 0 & \bar{E}_f \end{bmatrix} \qquad (60)$$

The generalised aerodynamic moments for the whole aeroplane are $2(L_1, L_2, \ldots L_n, M_1, M_2, \ldots M_n)$ corresponding to the wing displacements $\varphi_1, \varphi_2\, \varphi_n, \theta_1, \theta_2\, \theta_n$ as given by equation (30), and $L_{1t}, M_{1t}, N_m, \ldots N_N$ corresponding to $\varphi_1, \theta_1, q_m, \ldots q_n$ due to the tailplane. It is then clear that the aerodynamic moments corresponding to φ, θ, q, for the whole aeroplane can be expressed in the form (see equation (30))

$$2\{L^*, M^*, N^*\} \equiv -2\rho c^3 V^2 C^* W - 2\rho c^4 V B^* \dot{W}, \qquad (61)$$

where $2L_1^*$, $2M_1^*$ now include the tailplane contributions L_{1t}, M_{1t}. The dynamical equations are then expressible in the form

$$M_0 c^2 A^* \ddot{W} + \rho c^4 V B^* \dot{W} + (\rho c^3 V^2 C^* + \varepsilon E^*) W = 0 \qquad (62)$$

where A^*, B^*, C^* and E^* are $2n+N$, $2n+N$ matrices. Now, if $W = ke^{\lambda' t}$ where k represents a matrix column of the amplitudes, equation (62) yields on substitution

$$[a\lambda^2 + b\lambda + c + ey]k = 0, \qquad (63)$$

where

$$\left.\begin{aligned} a &= \frac{M_0 A^*}{\rho c^3}, \quad b = B^*, \quad c = C^*, \\ y &= \frac{\varepsilon}{\rho V^2 c^3}, \quad e = E^*, \quad \lambda = \frac{\lambda' c}{V}. \end{aligned}\right\} \qquad (64)$$

For simple harmonic motion of frequency $p/2\pi$, $\lambda = i\omega$, where $\omega = pc/V$, and equation (63) gives

$$[- a\omega^2 + ib\omega + c + ey]\, k = 0. \qquad \ldots \quad \ldots \quad \ldots \quad (65)$$

The resonance modes in still air are given by equation (65), when $c = b = 0$, *i.e.*, by

$$[a - ez]\, k = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (66)$$

where

$$z \equiv \frac{y}{\omega^2} = \frac{\varepsilon}{\rho c^5 p^2}.$$

It should be noted that the aerodynamic inertia terms are included in the matrix a of the inertial coefficients.

PART II

Application of Theory

8. *Numerical Application.*—The theory developed in the previous sections is applied to the case of a large transport aeroplane, the numerical work being based entirely on the design data issued by the aircraft company concerned. Symmetrical motion only is considered and the displacements are defined in terms of 12 reference co-ordinates. The degrees of freedom assumed are (a) translational and pitching motion of the whole aeroplane, (φ_1, θ_1), (b) wing flexure relative to $\varphi_1(\varphi_2, \varphi_3, \varphi_4)$, (c) wing twist relative to $\theta_1(\theta_2, \theta_3, \theta_4)$, (d) bending of front fuselage (q_1, q_2), and (e) bending of rear fuselage (q_3, q_4) with the tailplane assumed rigid. The wing is divided into strips as shown in Fig. 1, and the fuselage is represented by a number of elastically connected masses as in Fig. 2. In Tables 1a, b the dimensions of the wing strips, positions of the fuselage masses, and the appropriate inertial values are listed. The flexibility coefficients were determined by the method outlined in the Appendix, approximate formulae to represent the given stiffness distributions EI and C of the wings and B of the fuselage being used (*see* Figs. 4, 5). Full details of these formulae and the flexibility matrices S, S_f and S_b derived are given in Table 2a. The corresponding displacement matrices R and U are given in Table 2b. It is then possible to calculate the matrices a, b, c, e of equation (63) corresponding to the reference co-ordinates

$$W \equiv \{\varphi_1, \varphi_2, \varphi_3, \varphi_4, \quad \theta_1, \theta_2, \theta_3, \theta_4, \quad q_1, q_2, q_3, q_4\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (67)$$

The value $\varepsilon = 10^7$ is assumed for the typical stiffness,

For the particular aeroplane considered, the wing strip with centre-line at ch_4 contains the centre of gravity of the fuel tank (*see* Fig. 1). Since φ_3, θ_3 refer to the displacement of this particular strip and since φ_3, θ_3 are relative to φ_1, θ_1 respectively, it is more convenient to arrange the reference co-ordinates in the form

$$W_0 = \{\varphi_1, \theta_1, \varphi_3, \theta_3, \varphi_2, \varphi_4, \theta_2, \theta_4, q_1, q_2, q_3, q_4\} \qquad \ldots \quad \ldots \quad (68)$$

The dynamical equations (65) must be rearranged accordingly and expressed in the form

$$(- a_0\omega^2 + ib_0\omega + c_0 + e_0 y)\, k_0 = 0, \qquad \ldots \quad \ldots \quad \ldots \quad (69)$$

where the element in the first two rows and columns of e_0 are null, and where only the 4×4 partitioned matrix in the top left-hand corner of a_0 is influenced by the amount of fuel in the tank. The matrices a_0, b_0, c_0 and e_0 are given in Table 3. Since e_0 is symmetrical, equation (69) can be reduced to the diagonal form in y by a transformation of the type $0u0'$, where $0'$ is the transpose of 0. Let $k_0 \equiv 0' k_1$, and let

$$u_1 k_1 \equiv (- a_1\omega^2 + ib_1\omega + c_1 + I(2)y)\, k_1 = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (70)$$

be the transformed system of equations, where $I(2)$ here denotes the 12th order unit matrix with the first two diagonal elements replaced by zeros, and where

$$a_1 \equiv 0a_0 0', \; b_1 \equiv 0b_0 0', \; c_1 \equiv 0c_0 0'.$$

The matrices $0'$, a_1, b_1, c_1 are given in Table 4.

Flutter is possible when

$$| u_1 | \equiv | - a_1\omega^2 + ib_1\omega + c_1 + I(2)y| = 0 \quad \ldots \quad (71)$$

for real values of ω and y. The resonance frequencies and modes are given by

$$[a_1 - I(2)z]k_1 = 0 \quad \ldots \quad (72)$$

where $z \equiv y/\omega^2$.

9. *Evaluation of the Stability Determinant.*—In partitioned form, the matrix u_1 of equation (70) can be expressed as follows:—

$$u_1 \equiv [- a_1\omega^2 + ib_1\omega + c_1 + I(2)y],$$

$$\equiv \begin{bmatrix} A(\bar{2}, 2) & B_0(\bar{2}, 2) & B_1(\bar{2}, 4) & C(\bar{2}, 2) & D(\bar{2}, 2) \\ E_0(\bar{2}, 2) & P_0(\bar{2}, 2) & P_1(\bar{2}, 4) & 0 & 0 \\ E_1(\bar{4}, 2) & P_2(\bar{4}, 2) & P_3(\bar{4}, 4) & 0 & 0 \\ F(\bar{2}, 2) & 0 & 0 & Q(\bar{2}, 2) & 0 \\ G(\bar{2}, 2) & 0 & 0 & 0 & R(\bar{2}, 2) \end{bmatrix} \quad \ldots \quad (73)$$

where only the diagonal terms in P_0, P_3, Q and R involve y. Let

$$\sigma \equiv \begin{bmatrix} I_2 & 0 & -B_1P_3^{-1}, & -CQ^{-1}, & -DR^{-1} \\ 0 & I_2 & -P_1P_3^{-1} & 0 & 0 \\ 0 & 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & I_2 & 0 \\ 0 & 0 & 0 & 0 & I_2 \end{bmatrix} \quad \ldots \quad (74)$$

where P_3^{-1}, Q^{-1}, R^{-1} denote the reciprocals of the matrices P_3, Q and R respectively, and where I_n denotes the unit matrix of nth order.

Premultiplication of u_1 by σ yields

$$\sigma u_1 = \begin{bmatrix} A-B_1P_3^{-1}E_1-CQ^{-1}F-DR^{-1}G & B_0-B_1P_3^{-1}P_2 & 0 & 0 & 0 \\ E_0-P_1P_3^{-1}E_1 & P_0-P_1P_3^{-1}P_2 & 0 & 0 & 0 \\ E_1 & P_2 & P_3 & 0 & 0 \\ F & 0 & 0 & Q & 0 \\ G & 0 & 0 & 0 & R \end{bmatrix} \quad \ldots \quad (75)$$

Since $| \sigma | = 1$, and $| \sigma u_1 | = | \sigma | \, | u_1 |$, relations (73), (74) and (75) yield the reduced form Δ of $| u_1 |$, namely,

$$\Delta = | P_3 | \times | Q | \times | R | \begin{vmatrix} A-B_1P_3^{-1}E_1-CQ^{-1}F-DR^{-1}G & B_0-B_1P_3^{-1}P_2 \\ E_0-P_1P_3^{-1}E_1 & P_0-P_1P_3^{-1}P_2 \end{vmatrix} \quad (76)$$

The numerical procedure adopted to find the real values of ω and y for which $\Delta = 0$ was to assign ω and to reduce $| u_1 |$ to the above 4th order form with y kept general. Evaluation of Δ for a range of values of y is then relatively easy, the critical values of ω and y being determined by trial and error.

The reduction of $| u_1 |$ to the 4th order form Δ given by equation (76) involves the reciprocal of the 4th order matrix P_3. When ω is assigned, P_3 can be expressed in the form

$$P_3 = [yI_4 - u_3] \quad \ldots \quad (77)$$

where u_3 is a complex numerical matrix of the 4th order. If

$$| P_3 | = y^4 + p_1y^3 + p_2y^2 + p_3y + p_4 . \quad \ldots \quad (78)$$

it can be shown that

$$\left.\begin{aligned} -p_1 &= D_1, \\ -2p_2 &= D_2 + p_1 D_1, \\ -3p_3 &= D_3 + p_1 D_2 + p_2 D_1, \\ -4p_4 &= D_4 + p_1 D_3 + p_2 D_2 + p_3 D_1 = -4\,|u_3|, \end{aligned}\right\} \quad \ldots \quad \ldots \quad (79)$$

where D_n, $n = 1, 2, 3, 4$, denotes the sum of the diagonal terms in $u_3{}^n$. The reciprocal $P_3{}^{-1}$ is then given by

$$|P_3|P_3{}^{-1} = u_3{}^3 + u_3{}^2(y+p_1) + u_3(y^2+p_1y+p_2) + (y^3+p_1y^2+p_2y+p_3)I \quad \ldots \quad (80)$$

From the above formulae[3], the numerical form of $P_3{}^{-1}$ can readily be deduced for any chosen value of y. The matrices Q and R are of 2nd order and their reciprocals are easily determined.

10. *Normal Modes.*—In some flutter investigations, it is advantageous to transform the first set of reference co-ordinates chosen to a set in which each co-ordinate is associated with a resonance mode of the aeroplane. The equations of motion, such as those given by equation (70), must then be transformed accordingly. Now equation (72) is of the 10th order in z and will, therefore, yield 10 resonance modes and frequencies. Let k_m represent the 12×10 matrix of the modal columns where for the tank empty case, the sum of the squares of the last ten elements in any column is made equal to unity by multiplication by a suitable factor (*see* Table 5). Then

$$k_m{}' I(2) k_m = I_{10}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (81)$$

and equation (72) yields

$$k_m{}' a_1 k_m = d_0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (82)$$

where $k_m{}'$ is the transpose of k_m, and d_0 is the diagonal matrix of the characteristic roots. In order to express equation (70) in terms of the normal modes, write $k_1 = k_m\hat{q}$, where $\hat{q}$ represents the column of the amplitudes of the normal co-ordinates $q(\equiv \hat{q}e^{i\omega t})$. Then, after premultiplication by $k_m{}'$ equation (70) yields

$$[Iy + (k_m{}'c_1k_m) + i\omega(k_m{}'b_1k_m) - d_0\omega^2]\,\hat{q} = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (83)$$

where I is the unit matrix of the 10th order. The use of equation (83) rather than equation (70) is advisable when direct information is required on the influence of any particular resonance mode on the flutter characteristics of the aeroplane.

11. *Range of Calculations.*—(a) *Resonance.*—To check the inertial and stiffness coefficients estimated in this report, the resonance frequencies and modes of the whole aeroplane were calculated for comparison with modes and frequencies determined experimentally by the firm on their comparison with modes and frequencies determined experimentally by the firm on their vibration model*. The solutions of equation (72) were determined by the relaxation method by Dr. L. Fox of the Mathematics Division, N.P.L., and the actual modes of displacement of the aeroplane derived from his results are given in Tables 6a, b, and shown plotted in Figs. 6 to 23. For the empty fuel tank case all the roots of equation (72) were determined, but only the first eight were calculated for the full tank case. As shown by Table 7, reasonable agreement with experiment was obtained. The calculated fundamental frequency is, however, lower than the experimental because of the allowance made for aerodynamic inertia. If this effect were neglected good agreement with the fundamental frequency of the model with empty or full fuel tanks would be obtained. The higher modes of vibration are not affected to the same extent by the inclusion of aerodynamic inertia effects in the calculations. Any effects due to shear, which might be considerable at high frequencies, are not included.

* The aircraft firm concerned made a 1/20 scale stiffness-inertia model with stiffness and mass distributions corresponding to those given by the design data for the proposed aeroplane.

(b) *Flutter*.—The flutter characteristics of the aeroplane with fuel tanks (a) empty and (b) full were investigated for the cases where the fuselage is

(i) flexible and mobile (F, M),

(ii) rigid and mobile (R, M),

(iii) rigid and immobile (R, I).

The determinantal conditions to be satisfied for flutter to occur are respectively, (i) $\Delta(F, M) = 0$; (ii) $\Delta(R, M) = 0$; and (iii) $\Delta(R, I) = 0$; where $\Delta(F, M)$ corresponds to Δ as defined by equation (76) and where

$$\Delta(R, M) = |P_3| \times \begin{vmatrix} A - B_1P_3^{-1}E_1 & B_0 - B_1P_3^{-1}P_2 \\ E_0 - P_1P_3^{-1}E_1 & P_0 - P_1P_3^{-1}P_2 \end{vmatrix},$$

$$\Delta(R, I) = |P_3| \times |P_0 - P_1P_3^{-1}P_2|.$$

The condition of the fuel tanks influences A, E_0, B_0 and P_0 only.

In the flutter calculations, ω and y are varied to cover the practical range and the corresponding values of the stability determinants obtained. In general the value of the determinant is complex, and, initially, its real part was plotted against the imaginary for constant y or ω values. This method of plotting leads, however, to rather complicated curves. Much simpler diagrams are obtained by expressing the determinant considered in the form

$$\Delta = re^{i\theta}$$

and by plotting r against θ. The values of ω and y which give $r = 0$ are the critical values* for which flutter is possible. Sets of curves for the cases considered are given in Figs. 24 to 29. They indicate that the possibility of flutter below 630 m.p.h. is very remote (*see* Figs 26, 29).

Some calculations for the empty tank case were also done on the basis of equation (83) and the use of normal co-ordinates. All the binary combinations of the first six modes were considered, but it was found that only the 1,5; 1,6 and 2,6 combinations gave rise to flutter. The first mode is mainly wing flexure with slight fuselage bending, the second is mainly fuselage bending with some wing flexure, the fifth is a fuselage overtone coupled with wing torsion, and the sixth is an overtone in wing torsion (*see* Figs 6, 7, 10, 11). The calculated critical speeds are extremely high (of the order 1000 m.p.h.) and have no practical significance. The results do indicate, however, that fuselage flexibility and the higher modes of vibration should be taken into account in flutter calculations when the critical speed is likely to be high.

12. *Concluding Remarks*.—This report deals only with the problem of symmetrical flutter of an aeroplane. Antisymmetrical flutter can, however, be treated similarly with very little modification of the basic theory. For the symmetrical case considered, it is shown how the effect of fuselage mobility and flexibility can be taken into account, and how the high order stability determinant which results from the assumption of many degrees of freedom can be reduced to a lower order without much loss of generality. The theoretical speeds were not actually calculated in the case considered, since they are well above the speed range of the aeroplane and in the region where allowance for compressibility effects would be required. Without an exact knowledge of these speeds, it is difficult to estimate the influence of fuselage flexibility and mobility. Theoretically, it is of course possible to modify the design data to give lower critical speeds at which compressibility effects are not important, but such an approach to the problem would be unsatisfactory, and the design data would not correspond to an actual aeroplane. Moreover, calculations on such a basis might be misleading.

* The critical value θ_c of θ is not determined uniquely. It can be θ_c or $\theta_c \pm \pi$ depending on the way the origin is approached along the curve obtained by plotting the real part of Δ against the imaginary for varying ω with $y = y_c$.

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APPENDIX

Flexibility Coefficients

1. *Method of Calculation.*—For the purposes of this section, the wing is divided into two parts as shown in Fig. 3.

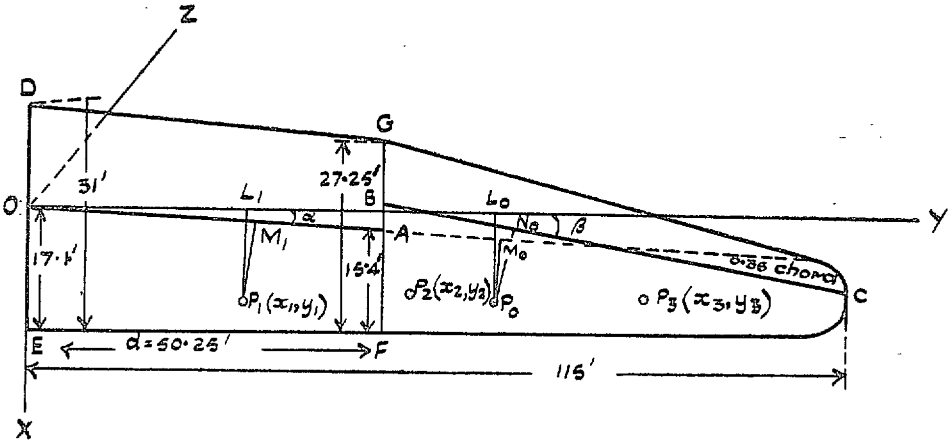


FIG. 3.

Let OA and BC be the flexural axes of the inner and outer parts of the wing respectively. From any point P on the wing draw perpendiculars PL, PM, PN to OY, OA and BC as shown in the above diagram, and let

$$OL = y; \quad LP = x; \quad OM = u, \quad MP = v; \quad BN = \xi, \quad NP = \eta \qquad \text{.. .. (1)}$$

When P corresponds to P_1, P_2, etc., the co-ordinates x, y, u, v, ξ, η of equations (1) have the appropriate suffix.

It is assumed that, when a load is applied at any point of the outer part, the wing is such that the displacements of the inner part are the same as if the outer part were rigid. Hence a load W applied at P_0 would have the same effect at P_1 as a load W at M_0 and a twisting moment Wv_0 about OA. The load, however, must be such that it only produces small displacements. The displacements at P_2 and P_3 due to a load W at P_0 are made up of two parts, (1) the displacements due to W at M_0 and the moment Wv_0 when the outer section is assumed to be rigid, and (2) the additional displacements due to the flexibility of the outer section. The additional displacement of any point on the outer section is obtained by replacing the load W at P_0 by W at N_0 and a twisting moment $W\eta_0$ about BC with the inner section now assumed to be rigid.

(a) *Downward Load W at P_0.*—The displacement z_1 of P_1 is expressible in the form

$$z_1 = B(M_1) + v_1\theta(M_1), \qquad (2)$$

where $B(M_1)$ denotes the displacement due to bending of OA and where $\theta(M_1)$ denotes the twist about OA at M_1. The displacements $B(M_1)$, $\theta(M_1)$ are given by

$$\left.\begin{aligned} EI\frac{\partial^2 B}{\partial u^2} &= W(u_0-u) \\ \text{and} \quad C\frac{\partial \theta}{\partial u} &= Wv_0 \end{aligned}\right\} \qquad (3)$$

where EI and C are known functions of $u \cos \alpha$. If $B(A)$ and $\theta(A)$ denote the displacements at A as given by equation (3), the displacement z_2 of P_2 is given by

$$z_2 = B(A) + (u_2-u_A)\left(\frac{\partial B}{\partial u}\right)_{u=u_A} + v_2\theta(A) + B_2' + \eta_2\theta_2', \qquad (4)$$

where B_2', θ_2' denote the additional displacements due to the flexibility of the outer portion of the wing. These displacements are given by

$$\left.\begin{aligned} EI\frac{\partial^2 B'}{\partial u^2} &= W(\xi_0-\xi) \\ \text{and} \quad C\frac{\partial \theta'}{\partial u} &= W\eta_0, \end{aligned}\right\} \qquad (5)$$

where EI and C are now regarded as functions of $d + \xi \cos \beta$ minus the distance from the centre-line.

The downward displacement at P_3 is given by

$$\left.\begin{aligned} z_3 = B(A) &+ (u_3-u_A)\left(\frac{\partial B}{\partial u}\right)_{u=u_A} + v_3\theta(A) \\ &+ B'(N_0) + (\xi_3-\xi_0)\left(\frac{\partial B'}{\partial \xi}\right)_{\xi=\xi_0} + \eta_3\theta'(N_0) \end{aligned}\right\} \qquad (6)$$

(b) *Downward Load W at P_1.*—The displacement at any point P_0 of the outer section is represented by

$$z_0 = B(M_1) + (u_0-u_1)\left(\frac{\partial B}{\partial u}\right)_{u=u_1} + v_0\theta(M_1), \qquad (7)$$

where $B(M_1)$ and $\theta(M_1)$ are given by

$$\left.\begin{aligned} EI\frac{\partial^2 B}{\partial u^2} &= W(u_1-u) \\ \text{and} \quad C\frac{\partial \theta}{\partial u} &= Wv_1. \end{aligned}\right\} \qquad (8)$$

Equation (7) is also applicable for points of the inner part of the wing to the right of P_1. The deflection at any point P to the left of P_1 is simply given by

$$z = B + v\theta, \quad \ldots \quad (9)$$

where B and θ are given by equation (8).

(c) *Applied Couple.*—The cross-sections of the wing at right-angles to OA for the inner part and BC for the outer part are assumed to be rigid in all applications of the above displacement formulae. Hence the aerodynamic chordwise sections must be distorted when the wing is displaced. The mean angle of twist θ_i of a particular chordwise section of chord c_i corresponding to a unit load at a particular point is, however, expressed as follows in terms of the resulting downward displacements $z_i(0{\cdot}25)$ and $z_i(0{\cdot}75)$ at $0{\cdot}25c_i$ and $0{\cdot}75c_i$ behind the leading edge of of the section considered.,

$$0{\cdot}5c_i\,\theta_i = z_i(0{\cdot}75) - z_i(0{\cdot}25). \quad \ldots \quad (10)$$

Therefore, by the use of formulae (1) to (10) one can estimate the downward displacement at any point and the mean twist of any aerodynamic section due to a downward load applied at a particular point. It follows that the downward displacements corresponding to a couple applied to an aerodynamic section of chord c_s can best be calculated by representation of the couple by an up-load at $0{\cdot}25c_s$ and an equal down-load at $0{\cdot}75c_s$. Unit loads applied at these points correspond to a couple $0{\cdot}5c_s$, and therefore, the twist θ_i' at the ith section per unit moment at the sth section is given by

$$\theta_i' = \frac{z_i'(0{\cdot}75) - z_i'(0{\cdot}25)}{0{\cdot}25c_s c_i}, \quad \ldots \quad (11)$$

where $z_i'(0{\cdot}25)$, $z_i'(0{\cdot}75)$ represent the resultant downward displacements at $0{\cdot}25c_i$, $0{\cdot}75c_i$ respectively due to unit up-load at $0{\cdot}25c_s$ and unit down-load at $0{\cdot}75c_s$.

In practice the use of these formulae for wing displacements is complicated by the fact that EI and C are seldom simple functions of the distance from the centre-line of the aeroplane. Corresponding formulae for the fuselage involve EI only as torsion is not considered.

TABLE 1a

Distribution of Mass, Mass-Moment and Moment of Inertia

(Aerodynamic effects included)

(slug, foot units)

i	ch_i*	c_i/c	l_i	m_e	p_e	q_e
1	0	1·311‡	0·8225	1389	111·4	4876
2	27·5	1·253	0·7359	856·7	−217·6	107·8
3	43·125	$1{\cdot}202_5$	0·6169	239·1	11·12	12·8s
4†	62·375	1·040	1·0498	178·3 (1419)†	26·83 (79·46)†	15·35 (43·15)†
5	82·0	0·8134	0·6494	$62{\cdot}06_5$	17·98	7·953
6	97·0	0·6405	0·6494	41·07	16·04	7·243
7	109·1	0·5009	0·3983	15·44	6·930	3·338

* Mean chord c = 23·1 ft.

† Terms in brackets correspond to full tank case.

‡ Mean chord of half the centre strip.

TABLE 1b

Representation of Fuselage by Elastically Connected Masses

x	ξ	m^f lb	m_f slugs
−69·9	−3·026	3300	102·5
−55·9	−2·420	6940	215·5
−41·9	−1·814	7340	228·0
−26·9	−1·165	9820	305·0
−13·4	−0·5801	7980	247·8
0	0	7080	219·9
16·6	0·7186	11680	362·7
32·6	1·411	7160	222·4
47·1	2·039	4100	127·3
62·1	2·688	2080	64·60
79·6	3·446	9077	281·9†

† Including aerodynamic mass of tailplane (30·35)

TABLE 2a

Flexibility Matrices used in Calculations

$$S \equiv \begin{bmatrix} 108{\cdot}095 & 320{\cdot}326 & 530{\cdot}976 & -0{\cdot}88297 & -0{\cdot}88859 & -0{\cdot}88516 \\ 203{\cdot}191 & 761{\cdot}721 & 1330{\cdot}790 & -1{\cdot}26466 & -2{\cdot}42748 & -2{\cdot}41811 \\ 320{\cdot}326 & 1483{\cdot}996 & 2859{\cdot}635 & -1{\cdot}72359 & -5{\cdot}34704 & -4{\cdot}32808 \\ 439{\cdot}713 & 2263{\cdot}671 & 5071{\cdot}944 & -2{\cdot}19060 & -6{\cdot}04254 & -13{\cdot}59959 \\ 530{\cdot}975 & 2859{\cdot}635 & 7376{\cdot}137 & -2{\cdot}54901 & -7{\cdot}20483 & -38{\cdot}53748 \\ 604{\cdot}399 & 3339{\cdot}221 & 9224{\cdot}275 & -2{\cdot}83772 & -8{\cdot}14129 & -47{\cdot}24610 \\ -0{\cdot}88297 & -1{\cdot}72359 & -2{\cdot}54901 & 1{\cdot}17434 & 1{\cdot}17996 & 1{\cdot}17490 \\ -0{\cdot}88545 & -3{\cdot}20065 & -4{\cdot}57991 & 1{\cdot}18008 & 2{\cdot}04837 & 2{\cdot}04740 \\ -0{\cdot}88859 & -4{\cdot}34704 & -7{\cdot}20483 & 1{\cdot}17996 & 3{\cdot}04892 & 3{\cdot}07222 \\ -0{\cdot}88387 & -4{\cdot}32177 & -17{\cdot}78595 & 1{\cdot}17973 & 3{\cdot}07274 & 5{\cdot}22694 \\ -0{\cdot}88516 & -4{\cdot}32808 & -38{\cdot}53748 & 1{\cdot}17940 & 3{\cdot}07222 & 9{\cdot}13875 \\ -0{\cdot}88073 & -4{\cdot}30596 & -38{\cdot}60085 & 1{\cdot}17997 & 3{\cdot}07361 & 9{\cdot}22931 \end{bmatrix} \times 10^{-8}$$

$$S_f = \begin{bmatrix} 560{\cdot}3 & 153{\cdot}6 \\ 406{\cdot}4 & 118{\cdot}3 \\ 256{\cdot}5 & 83{\cdot}03 \\ 118{\cdot}3 & 42{\cdot}21_5 \\ 32{\cdot}18 & 14{\cdot}03_5 \end{bmatrix} \times 10^{-8}; \qquad S_b = \begin{bmatrix} 25{\cdot}00 & 68{\cdot}25 \\ 76{\cdot}30 & 241{\cdot}5 \\ 127{\cdot}3 & 484{\cdot}0 \\ 180{\cdot}0 & 823{\cdot}5 \\ 241{\cdot}5 & 1337 \end{bmatrix} \times 10^{-8}$$

Wing Stiffness Distributions

$EI = (71-0{\cdot}88y)\times 10^8$ lb ft^2 . . . . $0 \leqslant y \leqslant 50{\cdot}25$ ft
$= 27\times 10^8$, ... $50{\cdot}25 \leqslant y \leqslant 61{\cdot}2$
$= (70{\cdot}38-0{\cdot}7692y)\times 10^8$, ... $61{\cdot}2 \leqslant y \leqslant 85$
$= 0{\cdot}16667\,(115-y)\times 10^8$, ... $85 \leqslant y \leqslant 115$
$C = (27-0{\cdot}2552y)\times 10^8$, ... $0 \leqslant y \leqslant 43{\cdot}5$
$= (-44{\cdot}00+1{\cdot}393y)\times 10^8$, ... $43{\cdot}5 \leqslant y \leqslant 50{\cdot}25$
$= (68{\cdot}07-0{\cdot}8372y)\times 10^8$, ... $50{\cdot}25 \leqslant y \leqslant 61{\cdot}0$
$= (63{\cdot}688-1{\cdot}08205y+0{\cdot}0046539y^2)\times 10^8$, . . . $61 \leqslant y \leqslant 115$

Fuselage Stiffness Distribution

$B = 4{\cdot}1765\times 10^8\,(76{\cdot}90+x)$ $-76{\cdot}9 \leqslant x \leqslant -42{\cdot}9$
$= 14{\cdot}350\times 10^9$ $-42{\cdot}9 \leqslant x \leqslant 0$
$= 0{\cdot}87725\times 10^8\,(165{\cdot}29+x)$ $0 \leqslant x \leqslant 17{\cdot}1$
$= 2{\cdot}7802\times 10^8\,(76{\cdot}445-x)$ $17{\cdot}1 \leqslant x \leqslant 67{\cdot}1$
$= 1{\cdot}3333\times 10^8\,(88{\cdot}100-x)$ $67{\cdot}1 \leqslant x \leqslant 88{\cdot}1$

TABLE 2b

Displacement Matrices used in Calculations

$R \equiv$

1·0	0	0	0	0	0	0	0
1·0	1·0	0	0	0	0	0	0
1·0	0·92468	0·37827	−0·03361	0	0·01178	0·00013	−0·00752
1·0	0	1·0	0	0	0	0	0
1·0	−0·43474	0·90307	0·37172	0	−0·00794	−0·02382	0·02917
1·0	0	0	1·0	0	0	0	0
1·0	0·42800	−0·81119	1·5406	0	0·0942	−0·07259	0·06579
0	0	0	0	1·0	0	0	0
0	0	0	0	1·0	1·0	0	0
0	0·05872	−0·03733	0·01070	1·0	0·54138	0·46133	0·00051
0	0	0	0	1·0	0	1·0	0
0	0·01104	−0·01400	0·00635	1·0	−0·00821	0·65420	0·35400
0	0	0	0	1·0	0	0	1·0
0	0·01202	−0·01273	0·00544	1·0	0·00054	−0·01593	1·0160

$U \equiv$

1·0	−3·026	1·635	−0·8812	0	0
1·0	−2·420	1·0	0	0	0
1·0	−1·814	0·4053	0·7761	0	0
1·0	−1·165	0	1·0	0	0
1·0	−0·5801	−0·04687	0·4330	0	0
1·0	0	0	0	0	0
1·0	0·7186	0	0	0·3878	−0·1900
1·0	1·411	0	0	1·0	0
1·0	2·039	0	0	1·220	0·1416
1·0	2·688	0	0	0·9563	0·4432
1·0	3·446	0	0	0	1·0

TABLE 3

Matrices used in Original Form of Determinantal Equation $|a_0\lambda^2 + b_0\lambda + c_0 + e_0 y| = 0$

a_0 =

94·90 (137·2)	−0·9314 (0·8641)	10·63 (52·95)	1·400 (3·196)	36·11	2·735	−7·139	1·040	7·912	8·510	9·896	5·487
−0·9314 (0·8641)	171·6 (172·6)	1·400 (3·195)	0·8611 (1·809)	−7·208	1·134	3·914	0·4853	−20·29	−7·953	15·31	18·43
10·63 (52·95)	1·400 (3·195)	9·303 (51·63)	1·323 (3·119)	1·840	−0·05475	0·08126	0·002673	0	0	0	0
1·400 (3·196)	0·8611 (1·809)	1·323 (3·119)	0·7330 (1·661)	0·005620	0·06283	0·1067	0·04179	0	0	0	0
36·11	−7·208	1·840	0·005620	36·74	−0·2393	−7·119	−0·05752	0	0	0	0
2·735	1·134	−0·05475	0·06283	−0·2393	2·960	−0·007867	1·078	0	0	0	0
−7·139	3·914	0·08126	0·1067	−7·119	−0·007867	3·812	−0·002598	0	0	0	0
1·040	0·4853	0·002673	0·04179	−0·05752	1·078	−0·002598	0·4459	0	0	0	0
7·912	−20·29	0	0	0	0	0	0	8·998	−1·382	0	0
8·510	−7·935	0	0	0	0	0	0	−1·382	9·695	0	0
9·896	15·31	0	0	0	0	0	0	0	0	8·964	0·7966
5·487	18·43	0	0	0	0	0	0	0	0	0·7966	5·071

b_0 =

9·018	11·02	2·494	1·749	2·247	1·355	1·311	0·8773	0	0	−0·8665	2·056
4·819	22·78	−0·1417	0·6027	−0·4221	0·3791	0·4611	0·3491	0	0	−3·177	7·207
2·536	1·682	2·628	1·545	−0·034015	−0·1200	0·1382	−0·001089	0	0	0	0
−0·1225	0·5849	−0·03906	0·4995	−0·09974	−0·01340	0·06166	0·02356	0	0	0	0
2·188	1·460	−0·05347	0·1646	2·568	0·04392	1·302	0·001447	0	0	0	0
1·341	0·9430	−0·1289	0·07711	0·03775	1·449	−0·01183	0·8780	0	0	0	0
−0·3645	0·4693	−0·04610	0·06416	−0·3409	0·006165	0·4046	0·036774	0	0	0	0
0·3438	0·3617	−0·05762	0·03996	0·02406	0·3868	−0·002894	0·3249	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
−0·0135	−0·2382	0	0	0	0	0	0	0	0	0·2418	−0·1544
1·560	6·172	0	0	0	0	0	0	0	0	−1·007	2·145

Terms in brackets correspond to the full-tank case—all other terms are the same for both cases.

$$
c_0 = \begin{bmatrix}
0 & 9{\cdot}619 & -0{\cdot}06020 & 2{\cdot}842 & 0{\cdot}08287 & 0{\cdot}01980 & 2{\cdot}111 & 1{\cdot}290 & 0 & 0 & -2{\cdot}089 & 1{\cdot}217 \\
0 & 5{\cdot}139 & 0{\cdot}004138 & -0{\cdot}08756 & -0{\cdot}008647 & -0{\cdot}037249 & -0{\cdot}3990 & 0{\cdot}3301 & 0 & 0 & -7{\cdot}213 & 4{\cdot}200 \\
0 & 2{\cdot}704 & -0{\cdot}02438 & 2{\cdot}460 & 0{\cdot}03206 & 0{\cdot}008300 & 0{\cdot}2406 & 0{\cdot}005311 & 0 & 0 & 0 & 0 \\
0 & -0{\cdot}1307 & 0{\cdot}002890 & -0{\cdot}07563 & -0{\cdot}004914 & -0{\cdot}07519 & -0{\cdot}04824 & -0{\cdot}007123 & 0 & 0 & 0 & 0 \\
0 & 2{\cdot}333 & -0{\cdot}03718 & 0{\cdot}2594 & 0{\cdot}06141 & 0{\cdot}01005 & 2{\cdot}066 & 0{\cdot}01113 & 0 & 0 & 0 & 0 \\
0 & 1{\cdot}431 & 0{\cdot}009109 & 0{\cdot}1788 & 0{\cdot}006932 & 0{\cdot}004228 & -0{\cdot}02503 & 1{\cdot}277 & 0 & 0 & 0 & 0 \\
0 & -0{\cdot}3888 & 0{\cdot}003422 & -0{\cdot}04648 & -0{\cdot}005322 & -0{\cdot}039930 & -0{\cdot}3430 & 0{\cdot}033776 & 0 & 0 & 0 & 0 \\
0 & 0{\cdot}3668 & -0{\cdot}002326 & 0{\cdot}03522 & 0{\cdot}001846 & 0{\cdot}001060 & -0{\cdot}005332 & -0{\cdot}3370 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -0{\cdot}0145 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0{\cdot}0183 & -0{\cdot}0106 \\
0 & 1{\cdot}663 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2{\cdot}010 & 1{\cdot}223
\end{bmatrix}
$$

$$
e_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 30{\cdot}59 & 2{\cdot}882 & -48{\cdot}90 & -8{\cdot}519 & -1{\cdot}417 & -1{\cdot}919 & 0 & 0 & 0 & 0 \\
0 & 0 & 2{\cdot}882 & 7{\cdot}352 & -3{\cdot}920 & -0{\cdot}9410 & -5{\cdot}524 & -1{\cdot}888 & 0 & 0 & 0 & 0 \\
0 & 0 & -48{\cdot}90 & -3{\cdot}920 & 156{\cdot}0 & 7{\cdot}893 & 4{\cdot}861 & 1{\cdot}783 & 0 & 0 & 0 & 0 \\
0 & 0 & -8{\cdot}519 & 0{\cdot}9410 & 7{\cdot}893 & 3{\cdot}534 & 0{\cdot}1950 & 0{\cdot}7948 & 0 & 0 & 0 & 0 \\
0 & 0 & -1{\cdot}417 & -5{\cdot}524 & 4{\cdot}861 & 0{\cdot}1950 & 14{\cdot}09 & 0{\cdot}06620 & 0 & 0 & 0 & 0 \\
0 & 0 & -1{\cdot}919 & -1{\cdot}888 & 1{\cdot}783 & 0{\cdot}7948 & 0{\cdot}06620 & 1{\cdot}834 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 27{\cdot}54 & -72{\cdot}05 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -72{\cdot}05 & 247{\cdot}5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 81{\cdot}64 & -14{\cdot}75 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -14{\cdot}75 & 4{\cdot}659
\end{bmatrix}
$$

TABLE 4

Matrices used in Calculations (= (70))

1·0	0	0	0	0	0	0	0	0	0	0	0
0	1·0	0	0	0	0	0	0	0	0	0	0
0	0	0·52780	0	0	0	0	0	0	0	0	0
0	0	—0·035725	0·55096	0	0	0	0	0	0	0	0
0	0	0·11393	0·00039	0·085442	0	0	0	0	0	0	0
0	0	1·10710	0·00922	—0·18934	0·56010	0	0	0	0	0	0
0	0	—0·01414	0·21313	—0·02686	—0·00661	0·26643	0	0	0	0	0
0	0	—0·03555	0·55511	—0·00004	—0·24249	—0·009617	0·73841	0	0	0	0
0	0	0	0	0	0	0	0	0·39006	0	0	0
0	0	0	0	0	0	0	0	0·11349	0·06355	0	0
0	0	0	0	0	0	0	0	0	0	0·16913	0
0	0	0	0	0	0	0	0	0	0	0·53531	0·46328

$a_1 =$

94·90	—0·9314	12·52	—0·1335	2·759	1·327	—1·912	0·7680	4·052	0·5408	4·611	2·542
(137·2)	(0·8641)	(34·79)	(0·8560)								
—0·9314	171·6	0·9677	1·586	—0·9357	0·4916	1·038	0·3584	—8·815	—0·5042	12·46	8·538
(0·8641)	(172·6)	(1·881)	(2·108)								
12·52	0·9677	6·130	0·8519	—0·1008	1·377	—0·2325	0·7930	0	0	0	0
(34·79)	(1·881)	(17·85)	(1·356)								
—0·1335	1·586	0·8519	0·5877	—0·2790	0·2952	0·2283	0·2067	0	0	0	0
(0·8560)	(2·108)	(1·356)	(0·8755)								
2·759	—0·9357	—0·1008	—0·2790	0·4174	—0·2699	—0·1869	0·1543	0	0	0	0
1·327	0·4916	1·377	0·2952	—0·2699	0·6622	—0·01249	0·3660	0	0	0	0
—1·912	1·038	—0·2325	0·2283	—0·1869	—0·01249	0·2706	—0·003678	0	0	0	0
0·7680	0·3584	0·7930	0·2067	—0·1543	0·3660	—0·003678	0·2431	0	0	0	0
4·052	—8·815	0	0	0	0	0	0	1·372	0·03566	0	0
0·5408	—0·5042	0	0	0	0	0	0	0·03566	0·03915	0	0
4·611	12·46	0	0	0	0	0	0	0	0	1·854	1·320
2·542	8·538	0	0	0	0	0	0	0	0	1·320	1·088

Terms in brackets correspond to the tank-full case—all the other terms are the same for both cases.

$b_1 =$

9·018	11·02	2·838	1·743	−0·09981	0·5375	0·3409	0·6478	0	0	0·9540	0·9525
4·819	22·78	0·2222	0·6275	−0·1202	−0·1246	0·1195	0·2578	0	0	3·321	3·339
2·949	1·973	2·060	1·036	−0·2420	0·5705	0·04520	0·6500	0	0	0	0
0·05889	0·6323	0·1589	0·3111	−0·05489	0·07469	0·02977	0·1488	0	0	0	0
−0·05718	−0·06642	−0·2345	−0·07401	0·06744	−0·1121	0·02894	−0·1227	0	0	0	0
0·6701	0·4374	0·6857	0·2523	−0·1342	0·3019	−0·006263	0·3050	0	0	0	0
−0·1004	0·1216	−0·02069	0·03050	−0·01028	−0·001163	0·02876	−0·002174	0	0	0	0
0·2539	0·2671	0·2605	0·1516	−0·05251	0·1018	−0·002877	0·1772	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0·8328	3·264	0	0	0	0	0	0	0	0	0·5167	0·5201
0·7227	2·859	0	0	0	0	0	0	0	0	0·4533	0·4606

$c_1 =$

0	9·619	−0·1794	2·732	−0·05342	−0·3157	0·5500	0·9526	0	0	0·2982	0·5638
0	5·139	−0·002503	0·04995	0·01010	−0·07781	−0·1095	0·2438	0	0	1·028	1·946
0	3·146	−0·1052	1·623	−0·008485	−0·3099	0·07915	0·9534	0	0	0	0
0	0·06285	−0·004222	0·06982	0·002498	−0·04661	−0·02907	0·1440	0	0	0	0
0	−0·06118	0·005978	−0·09886	−0·004773	0·05719	0·05306	−0·1779	0	0	0	0
0	0·7151	−0·02552	0·4001	0·0$_3$1492	−0·1524	−0·008879	0·4678	0	0	0	0
0	−0·1071	0·001908	−0·02822	0·002382	0·001212	−0·02430	−0·002319	0	0	0	0
0	0·2709	−0·009675	0·1516	0·0$_4$6330	−0·05988	−0·003442	0·1838	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0
0	0·8878	0	0	0	0	0	0	0	0	0·1680	0·3025
0	0·7704	0	0	0	0	0	0	0	0	0·1458	0·2625

TABLE 5

Empty Tank Case

Matrices of Modal Columns

$k_m \equiv$

0·10424	—0·06509	—0·07063	—0·03796	0·03300	—0·01325	0·01107	0·02101	0·01874	0·01437
—0·00256	—0·03972	—0·05192	0·01034	—0·07452	0·00936	0·01429	0·01849	—0·00599	—0·00369
—0·89931	0·07172	0·28114	0·28199	—0·09838	—0·12194	—0·02253	—0·01431	—0·04767	0·00360
—0·19188	0·20302	—0·53404	0·24363	0·44840	0·50121	—0·28834	0·05675	0·11667	—0·15866
0·10626	—0·26940	0·31316	0·28996	—0·08986	0·59428	0·24121	—0·50094	—0·16068	—0·19442
—0·26181	0·19102	—0·09164	—0·70483	—0·07087	0·13055	0·28297	—0·22852	0·23164	—0·42633
—0·01026	0·12760	—0·38814	0·25189	0·27489	—0·39726	0·57596	—0·42460	—0·10053	0·11838
—0·15217	0·11333	—0·09086	—0·38214	0·00611	0·31759	—0·00223	—0·09484	—0·49935	0·67088
0·11033	0·55235	—0·25035	0·25067	—0·72796	0·11118	0·06148	0·05180	—0·07489	—0·05346
0·01147	0·00303	—0·03212	—0·04827	0·12263	—0·11247	0·03149	0·26882	—0·78746	—0·52484
0·16604	0·53862	0·34040	—0·03020	0·21630	—0·19025	—0·47348	—0·49558	—0·08589	—0·07338
0·10704	0·46929	0·43680	0·06860	0·32985	0·30926	0·46627	0·42802	0·11002	0·08634

Full Tank Case

$k_m \equiv$

—0·24318	—0·15463	0·04597	0·02275	—0·01174	—0·03493	0·004	0·058	—	—
—0·00206	—0·10059	0·02197	—0·02482	0·08696	0·02340	0·018	0·052	—	—
1·00000	0·20206	—0·14362	—0·11340	0·05299	0·03754	—0·012	—0·103	—	—
0·13737	0·75707	0·34390	—0·52491	—0·58163	0·24146	—0·257	—0·011	—	—
—0·09143	—0·77387	—0·11350	—0·11150	—0·18634	0·52400	0·813	—1·034	—	—
0·12518	0·67660	0·17493	0·99752	0·07300	0·03786	0·413	—0·280	—	—
0·02904	0·35360	0·09821	—0·47100	0·03389	—0·50900	1·069	—0·752	—	—
0·07253	0·40426	0·11758	0·54025	—0·00944	0·07980	0·197	—0·206	—	—
—0·11470	1·47362	0·01670	—0·40525	0·77524	0·29940	0·113	0·154	—	—
—0·01377	0·01303	0·01598	0·01840	—0·08100	—0·12890	—0·067	0·620	—	—
—0·16031	1·35800	—0·20468	0·09184	—0·26435	—0·11470	—0·649	—1·450	—	—
—0·09717	1·17515	—0·22566	0·06766	—0·47190	0·04665	0·688	1·280	—	—

TABLE 6a

Resonance Modes and Frequencies

Empty Tank Case

$f=$	1·730		3·255		4·840		6·465		7·611	
y	Z_a	θ_a	Z_a	θ_a	Z_a	θ_a	Z_a	θ_a	Z_a	θ_a
0	−0·0853	0·000090	−0·2513	−0·00664	−0·7050	−0·0224	0·0966	−0·00121	−0·2744	0·02707
27·5	−0·0088	0·001225	−0·3083	0·00710	−0·1202	−0·1214	−0·0513	−0·01350	−0·1215	−0·03625
43·125	0·1021	0·001886	−0·2794	0·00928	0·3377	−0·1369	−0·2035	−0·01363	−0·0067	−0·04935
62·375	0·3030	0·002696	−0·1052	0·01163	0·7763	−0·1539	−0·2902	−0·01466	0·1504	−0·06453
82·0	0·5268	0·003376	0·2435	0·01399	1·0	−0·1610	−0·0263	−0·00998	0·4154	−0·06749
97·0	0·8007	0·004457	0·6475	0·01781	0·9959	−0·1727	0·5085	−0·00220	0·7415	−0·07250
109·1	1·0	0·004521	1·0	0·01802	0·9349	−0·1731	1·0	−0·00177	1·0	−0·07258

$f=$	12·152		15·202		17·381		25·041		25·727	
y	Z_a	θ_a	Z_a	θ_a	Z_a	θ_a	Z_a	θ_a	Z_a	θ_a
0	0·0744	−0·0023	0·0694	0·00388	−0·3171	−0·01208	0·1062	−0·00147	−0·0549	0·00061
27·5	−0·1338	0·0012	0·1817	0·02665	0·3532	0·04402	−0·0021	−0·00109	0·0072	−0·00033
43·125	0·0101	−0·0326	0·1439	−0·00295	0·3394	0·01055	−0·0540	0·00641	−0·0114	0·00725
62·375	0·4357	−0·0704	−0·0051	−0·03899	−0·2031	−0·03284	−0·0364	0·01472	−0·0621	0·01507
82·0	0·7793	−0·0891	0·1329	−0·04668	−0·2492	−0·03065	0·1941	−0·02173	0·1306	−0·01947
97·0	0·9659	−0·1212	0·6160	−0·05988	0·3950	−0·02613	0·7455	−0·08920	0·7077	−0·08357
109·1	1·000	−0·1221	1·0	−0·05998	1·0	−0·02554	1·0	−0·09069	1·0	−0·08493

Values of Z_f

x_f	f									
	1·730	3·255	4·840	6·465	7·611	12·152	15·202	17·381	25·041	25·727
−69·9	−0·1396	1·359	−0·463	−0·1810	1·169	−0·1377	−0·0055	0·3340	0·2307	−0·0999
−55·9	−0·1255	0·9526	−0·426	−0·0934	0·591	−0·0419	0·0029	0·0533	0·0229	−0·0094
−41·9	−0·1117	0·552	−0·396	−0·0091	0·042	0·0472	0·0115	−0·2034	−0·1569	0·0686
−26·9	−0·0985	0·170	−0·405	0·0622	−0·376	0·1049	0·0213	−0·3385	−0·1861	0·0792
−13·4	−0·0895	−0·096	−0·490	0·0958	−0·477	0·1030	0·0347	−0·2910	−0·0100	−0·0016
0	−0·0853	−0·251	−0·705	0·0966	−0·274	0·0744	0·0694	−0·3171	0·1062	−0·0549
16·6	−0·0905	−0·262	−0·928	0·0826	0·099	0·1062	−0·0564	−0·0462	0·0494	−0·0263
32·6	−0·1053	−0·116	−0·862	0·0712	0·301	0·1809	−0·3061	0·5542	−0·0240	0·0124
47·1	−0·1251	0·142	−0·517	0·0516	0·306	0·1915	−0·3934	0·8005	0·0595	0·0313
62·1	−0·1518	0·538	0·153	0·0188	0·112	0·1179	−0·2738	0·5907	−0·0513	0·0271
79·6	−0·1913	1·173	1·348	−0·0360	−0·378	−0·0794	0·1432	−0·2676	0·0175	−0·0091

TABLE 6b

Resonance Modes and Frequencies

Full Tank Case

$f=$	1·285		3·234		4·158		5·183	
y	Z_a	θ_a	Z_a	θ_a	Z_a	θ_a	Z_a	θ_a
0	—0·2253	—0·000082	—0·1745	—0·00491	0·7313	0·0151	0·0292	—0·00138
27·5	—0·1269	0·00090	—0·2228	0·00823	0·3189	0·0863	0·0001	—0·01466
43·125	0·0160	0·001125	—0·2026	0·01152	—0·1026	0·1163	—0·0490	—0·01555
62·375	0·2637	0·001520	—0·0541	0·01511	—0·4746	0·1491	—0·0476	—0·01721
82·0	0·5551	0·001894	0·2693	0·01762	—0·2927	0·1610	0·1980	—0·01390
97·0	0·7993	0·002470	0·6584	0·02167	0·3585	0·1800	0·6185	—0·00836
109·1	1·0	0·002508	1·0	0·02189	1·0	0·1810	1·0	—0·00806

$f=$	7·139		9·124		14·151		16·982	
y	Z_a	θ_a	Z_a	θ_a	Z_a	θ_a	Z_a	θ_a
0	—0·0769	0·02465	0·4059	—0·0118	0·0285	0·00556	1·0	0·03881
27·5	—0·1431	—0·00686	—0·1651	0·0380	0·5137	0·06908	—0·7257	—0·08928
43·125	—0·0879	—0·03470	—0·1942	—0·0163	0·4679	0·02133	—0·9366	—0·03478
62·375	0·1063	—0·06673	0·1757	—0·0780	—0·0166	—0·03805	0·0628	0·03704
82·0	0·4101	—0·06989	0·5725	—0·0889	—0·0666	—0·03483	0·4659	0·01716
97·0	0·7398	—0·07450	0·8428	—0·1060	0·4768	—0·02756	—0·1366	—0·02044
109·1	1·0	—0·07459	1·0	—0·1065	1·0	—0·02698	—0·844	—0·02207

Values of Z_f

x_f	f							
	1·285	3·234	4·158	5·183	7·139	9·124	14·151	16·982
—69·9	—0·2760	1·063	—0·206	—0·1553	0·9596	—0·7261	0·1003	—0·884
—55·9	—0·2622	0·749	—0·017	—0·0965	0·5253	—0·2932	0·0323	—0·134
—41·9	—0·2487	0·442	0·171	—0·0399	0·1137	0·1166	—0·0295	0·554
—26·9	—0·2360	0·147	0·369	0·0088	—0·1979	0·4230	—0·0599	0·936
13·4	—0·2278	—0·057	0·543	0·0322	—0·2652	0·4974	—0·0343	0·856
0	—0·2253	—0·175	0·731	0·0292	—0·0769	0·4059	0·0285	1·0
16·6	—0·2341	—0·183	0·834	0·0121	0·2636	0·2898	0·1790	—0·064
32·6	—0·2531	—0·075	0·674	0·0041	0·4339	0·2496	—0·5734	—1·963
47·1	—0·2770	0·113	0·290	0·0032	0·3931	0·1952	—0·6940	—2·778
62·1	—0·3082	0·404	—0·365	0·0084	0·1286	0·1002	—0·4659	—2·033
79·6	—0·3531	0·869	—1·470	0·0227	—0·4729	—0·0626	0·2665	0·931

TABLE 7

Resonance Frequencies (c.p.s.)

Empty Fuel Tanks				Full Fuel Tanks		
		Values given by Aircraft Co.				
Mode No.	Present Report	Theoretical	Experimental†	Mode No.	Present Report	Experimental† Values given by Aircraft Co.
1	1·730*	1·833	1·96	1	1·285*	1·46
2	3·255	3·93	3·25	2	3·234	3·04
3	4·840	4·89	4·91	3	4·157	4·34
4	6·465	6·43	6·40	4	5·183	5·17
5	7·611	7·86	9·3	5	7·139	—
6	12·15	—	—	6	9·124	9·58
7	15·20	—	—	7	14·15	—
8	17·38	—	—	8	16·98	—
9	25·04	—	—	9	—	—
10	25·73	—	—	10	—	—

* Values would be increased by nearly 10 per cent if the aerodynamic inertia effect were neglected

† The experimental values given above were obtained from resonance tests on a 1/20 scale stiffness-inertia model which was constructed to have the specified stiffness and mass distributions.

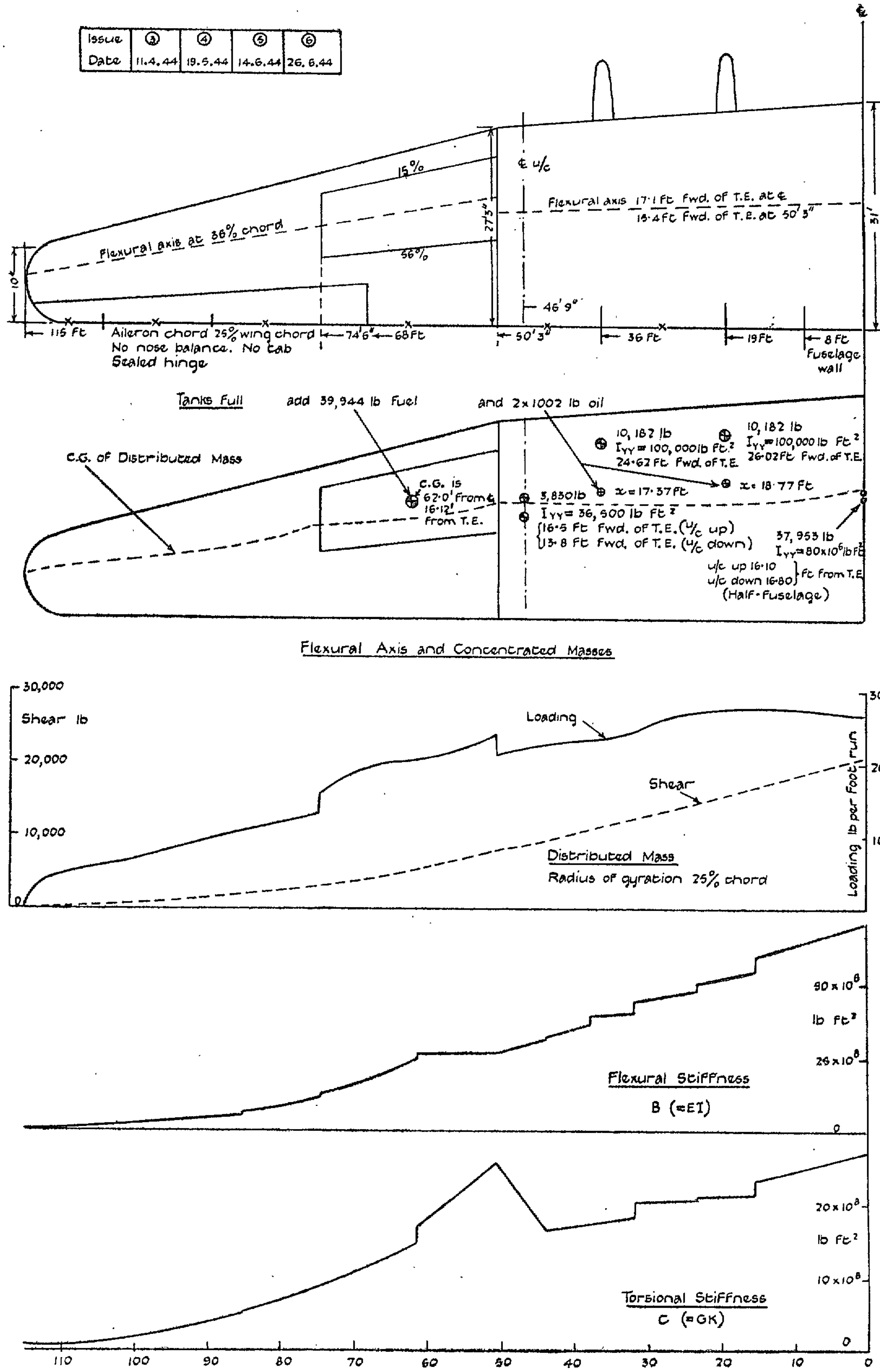


FIG. 4. Wing weight and stiffness.

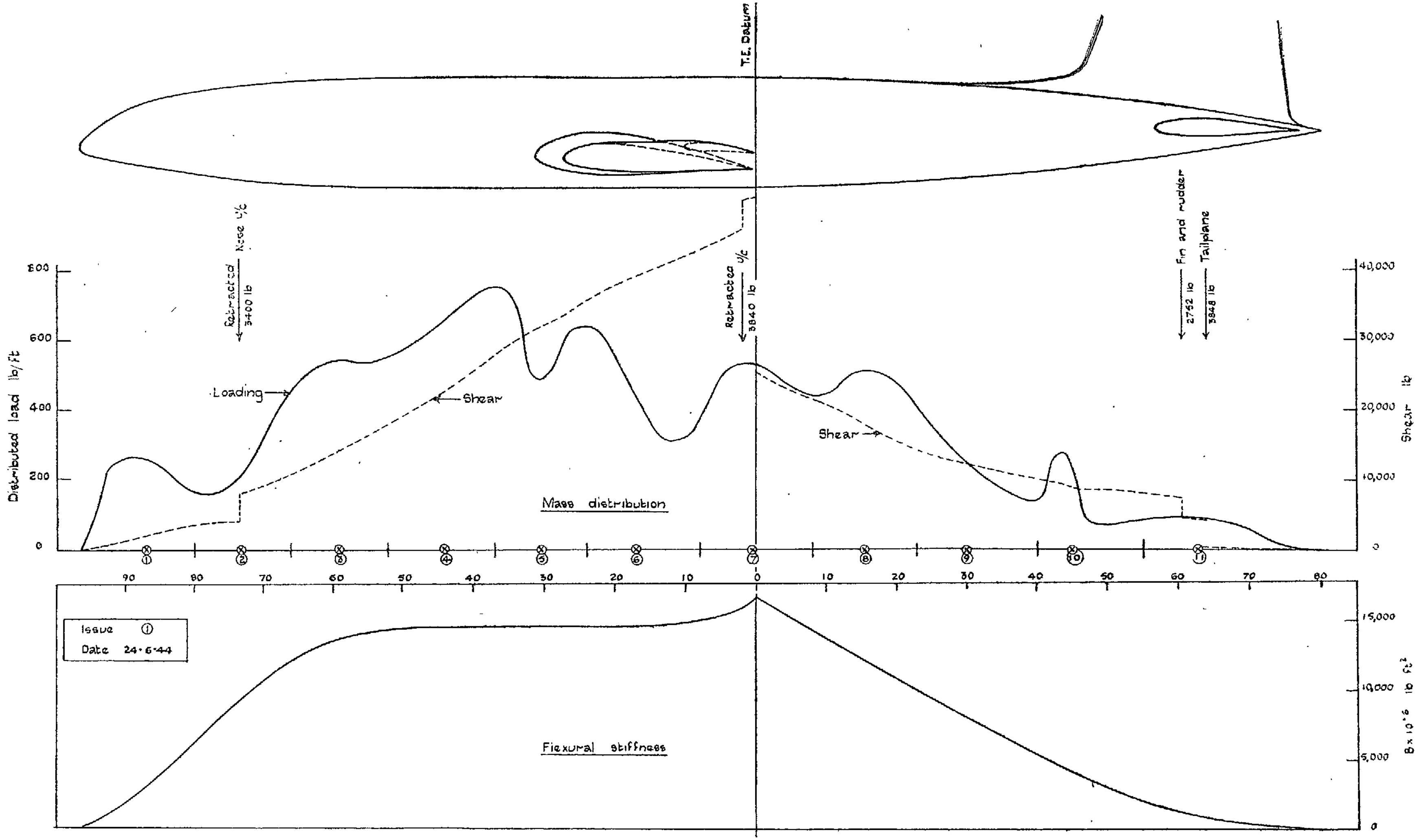


FIG. 5. Fuselage weight and stiffness.

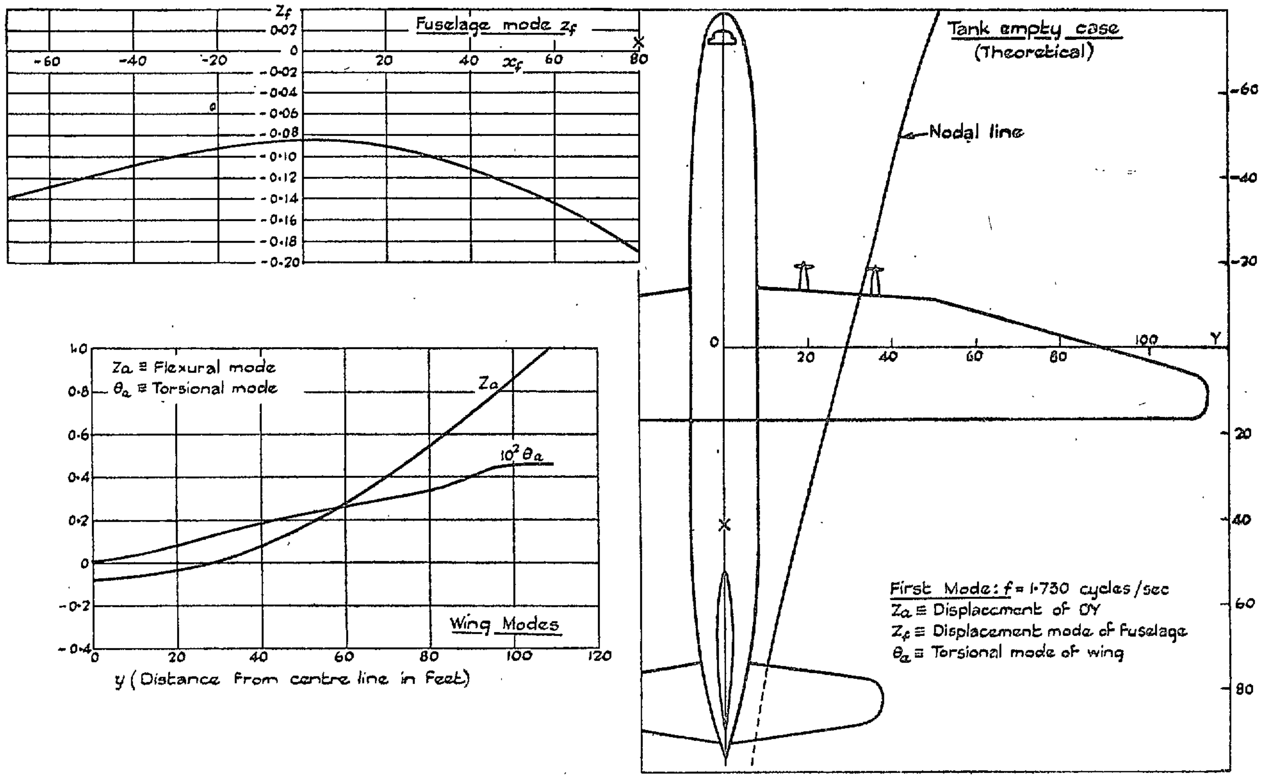

Fusealge mode z_f
z_f
x_f
Tank empty case
(Theoretical)
Nodal line
Z_a ≡ Flexural mode
θ_a ≡ Torsional mode
Z_a
$10^2\theta_a$
Wing Modes
y (Distance from centre line in Feet)
Y
First Mode: f = 1·730 cycles/sec
Z_a ≡ Displacement of OY
Z_f ≡ Displacement mode of fuselage
θ_a ≡ Torsional mode of wing

FIG. 6a.

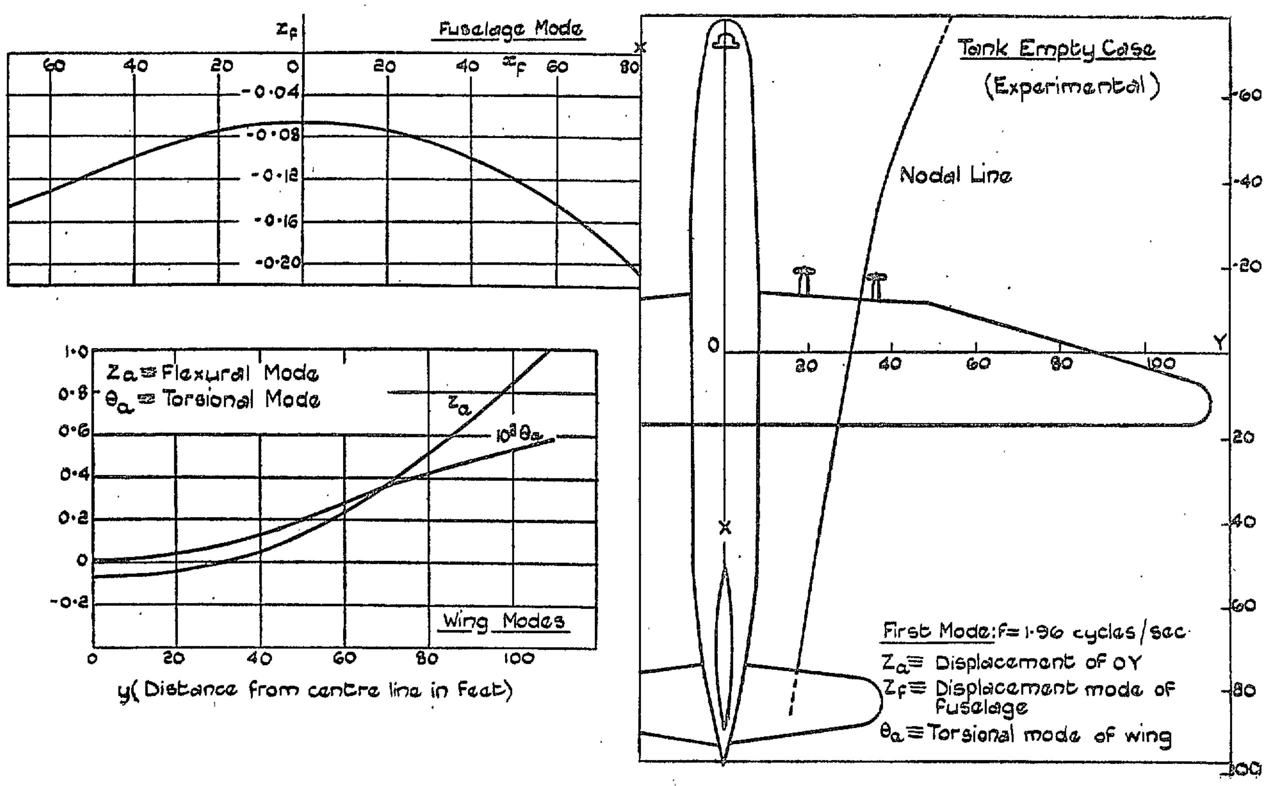
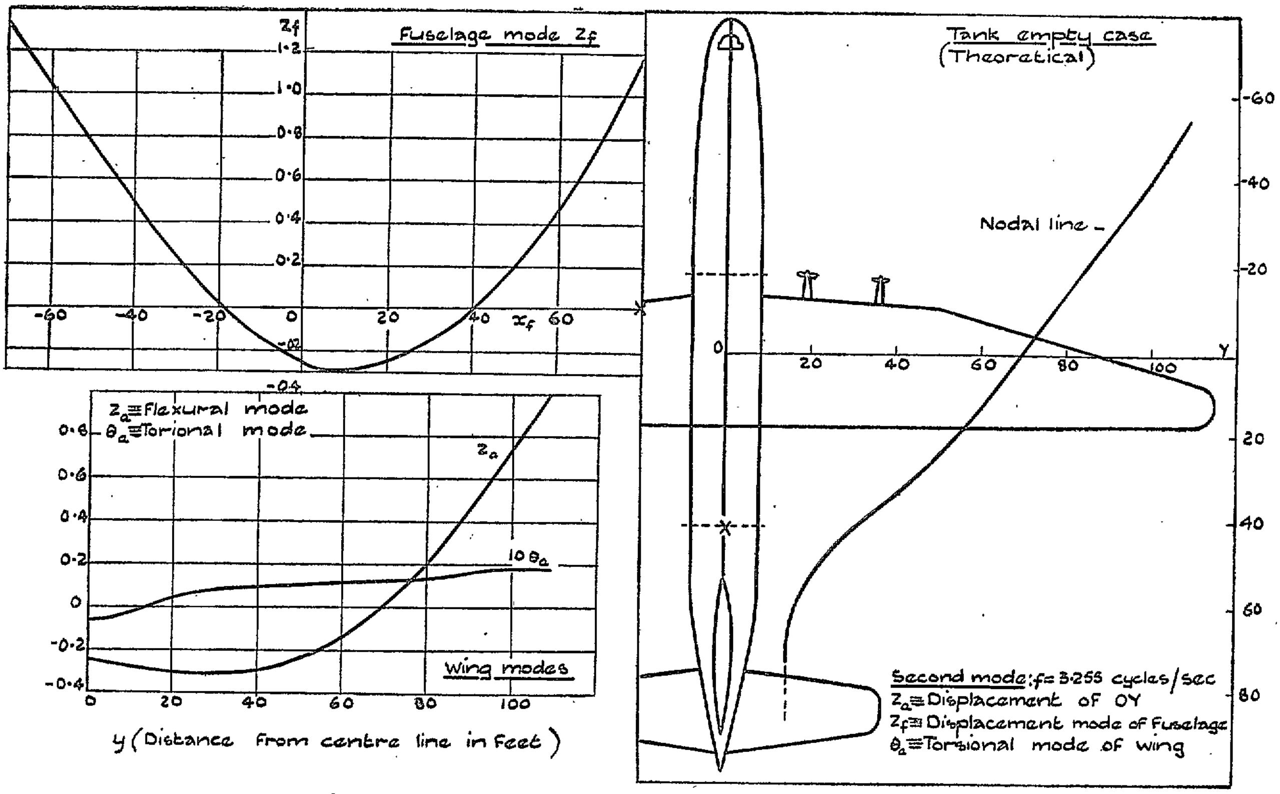
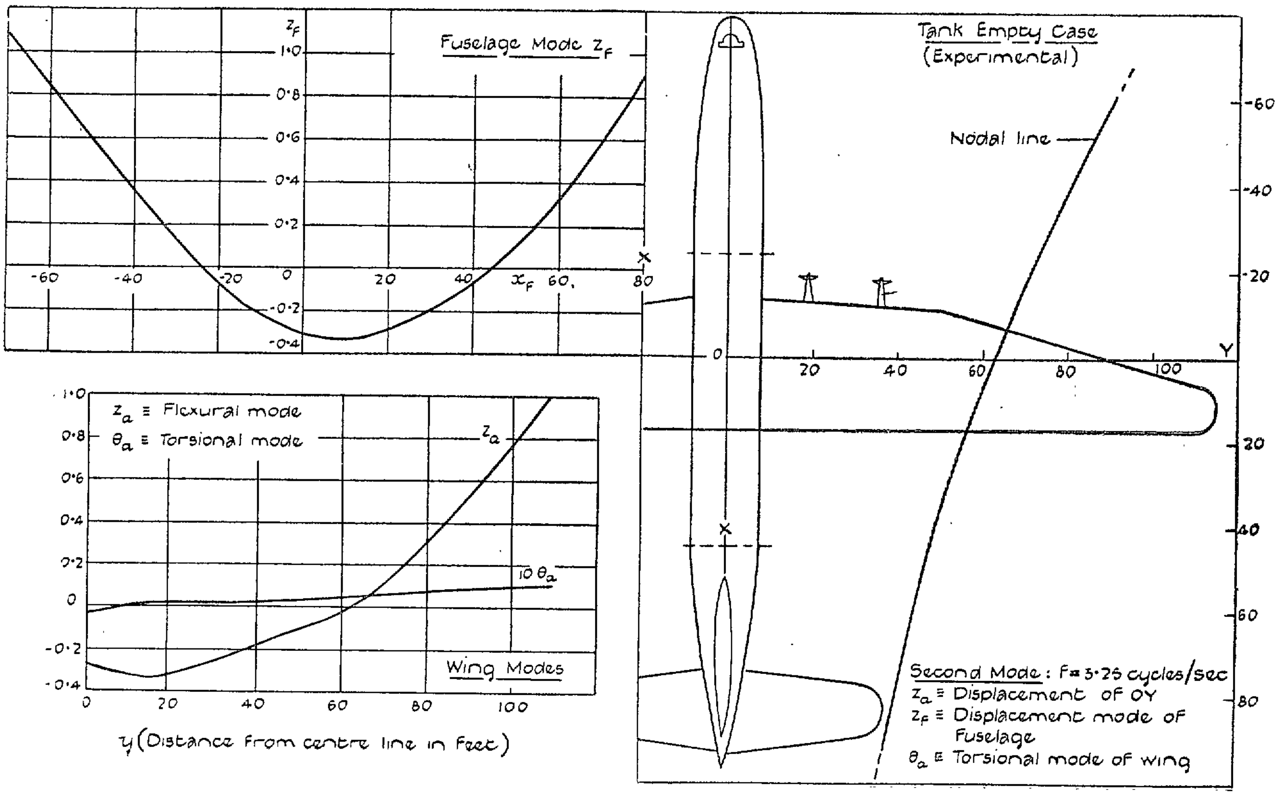

Fuselage Mode
z_f
x_f
Tank Empty Case
(Experimental)
Nodal Line
Z_a ≡ Flexural Mode
θ_a ≡ Torsional Mode
Z_a
$10^3\theta_a$
Wing Modes
y (Distance from centre line in Feet)
Y
First Mode: f = 1·96 cycles/sec
Z_a ≡ Displacement of OY
Z_f ≡ Displacement mode of Fuselage
θ_a ≡ Torsional mode of wing

FIG. 6b.



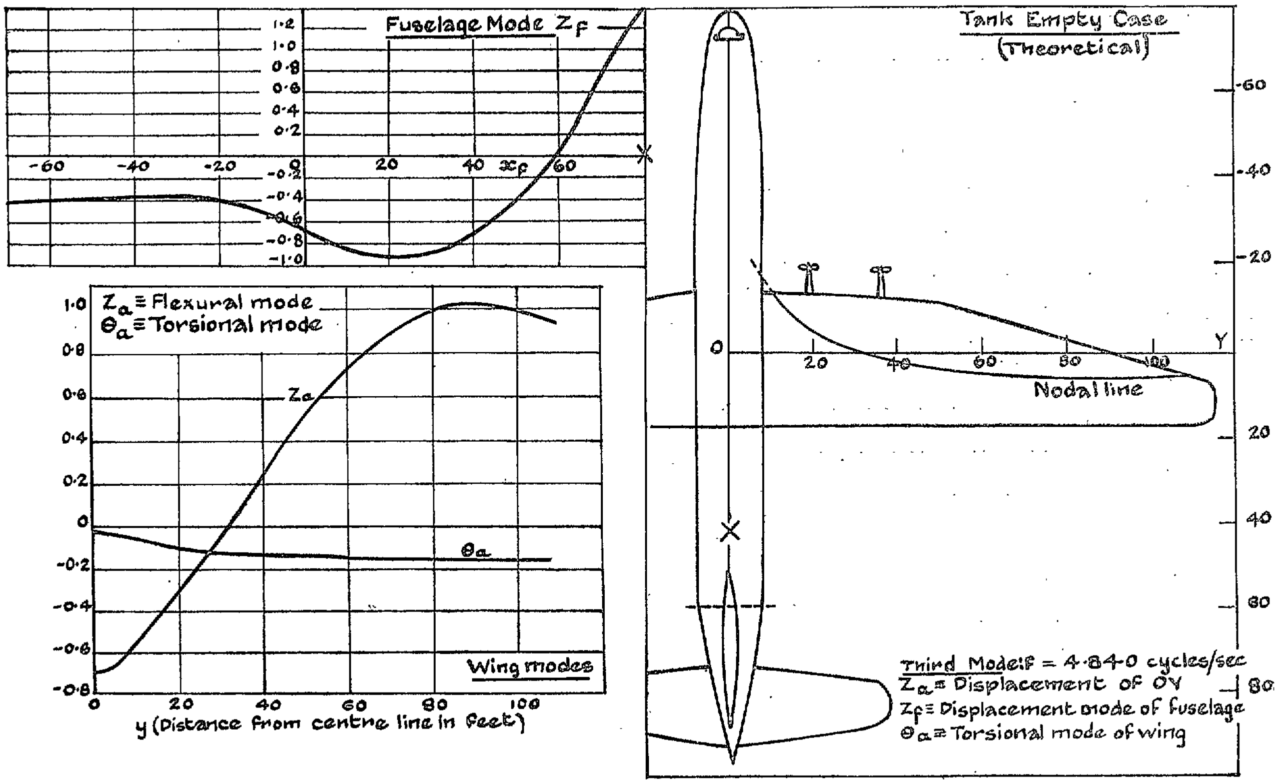
Fuselage mode Z_f
Z_a ≡ Flexural mode
θ_a ≡ Torsional mode
Wing modes
y (Distance from centre line in feet)
Tank empty case (Theoretical)
Nodal line
Second mode: f = 3·255 cycles/sec
Z_a ≡ Displacement of OY
Z_f ≡ Displacement mode of Fuselage
θ_a ≡ Torsional mode of wing

FIG. 7a.



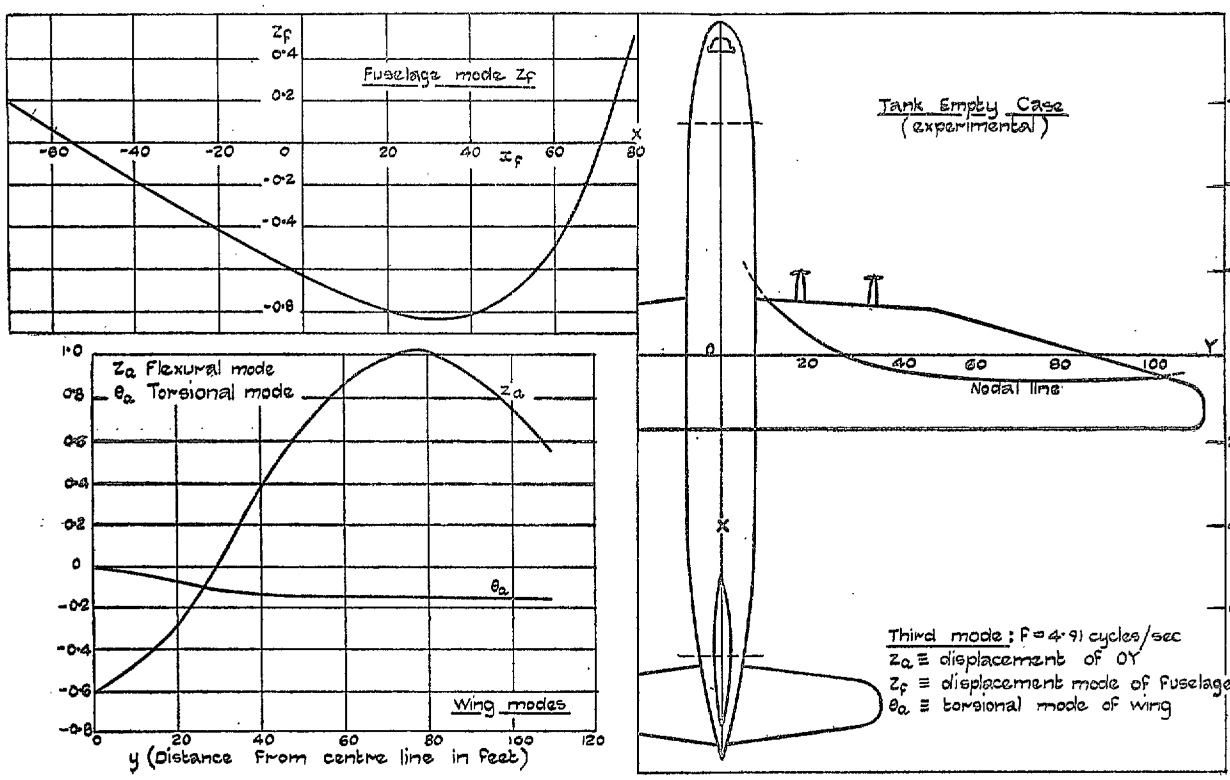
Fuselage Mode Z_F
Z_a ≡ Flexural mode
θ_a ≡ Torsional mode
Wing Modes
y (Distance from centre line in Feet)
Tank Empty Case (Experimental)
Nodal line
Second Mode: F = 3·25 cycles/sec
Z_a ≡ Displacement of OY
Z_F ≡ Displacement mode of Fuselage
θ_a ≡ Torsional mode of wing

FIG. 7b.



Fuselage Mode Z_f
Tank Empty Case (Theoretical)
Z_a ≡ Flexural mode
θ_a ≡ Torsional mode
Wing modes
y (Distance from centre line in feet)
Nodal line
Third Mode: f = 4·840 cycles/sec
Z_a ≡ Displacement of OY
Z_f ≡ Displacement mode of fuselage
θ_a ≡ Torsional mode of wing

FIG. 8a.



Fuselage mode Z_f
Tank Empty Case (experimental)
Z_a Flexural mode
θ_a Torsional mode
Wing modes
y (Distance from centre line in feet)
Nodal line
Third mode: f = 4·91 cycles/sec
Z_a ≡ displacement of OY
Z_f ≡ displacement mode of fuselage
θ_a ≡ torsional mode of wing

FIG. 8b.

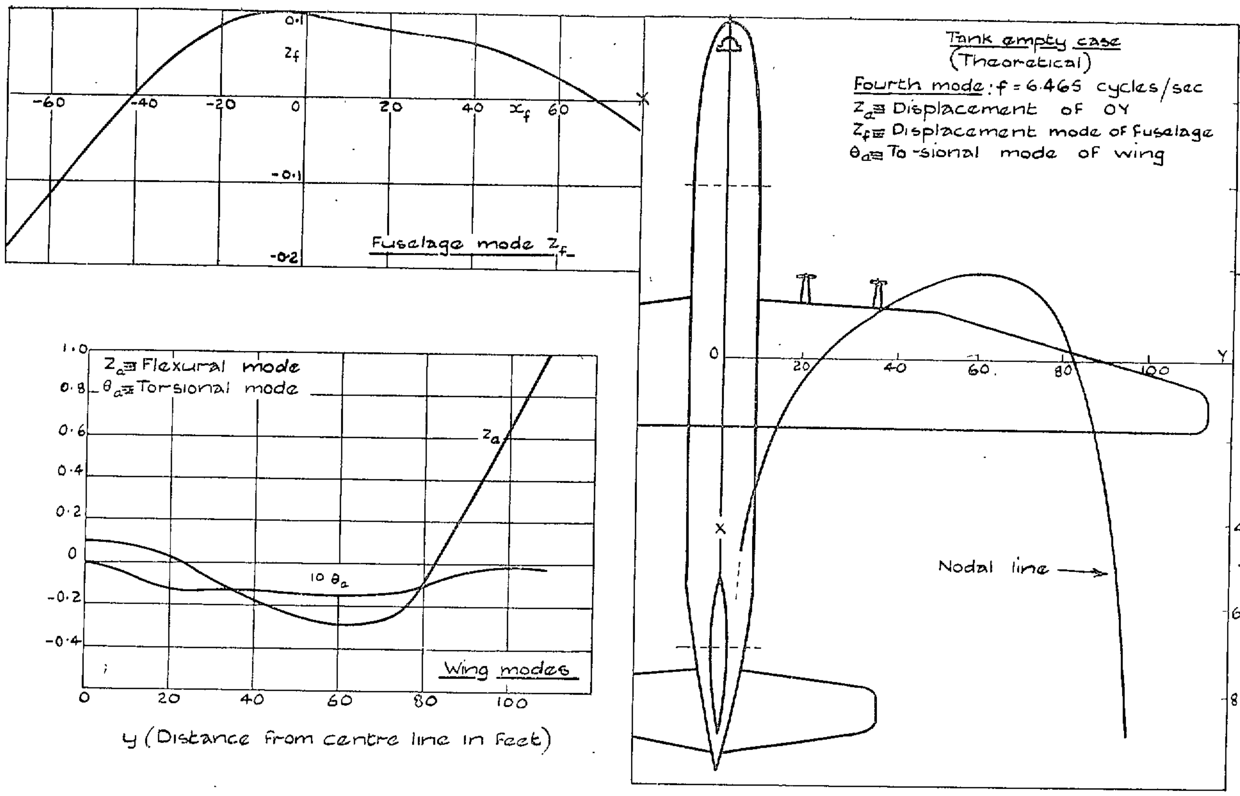

Tank empty case
(Theoretical)
Fourth mode: f = 6.465 cycles/sec
Zₐ ≡ Displacement of OY
Z_f ≡ Displacement mode of fuselage
θₐ ≡ Torsional mode of wing
Fuselage mode Z_f
Zₐ ≡ Flexural mode
θₐ ≡ Torsional mode
Wing modes
y (Distance from centre line in feet)
Nodal line

FIG. 9a.

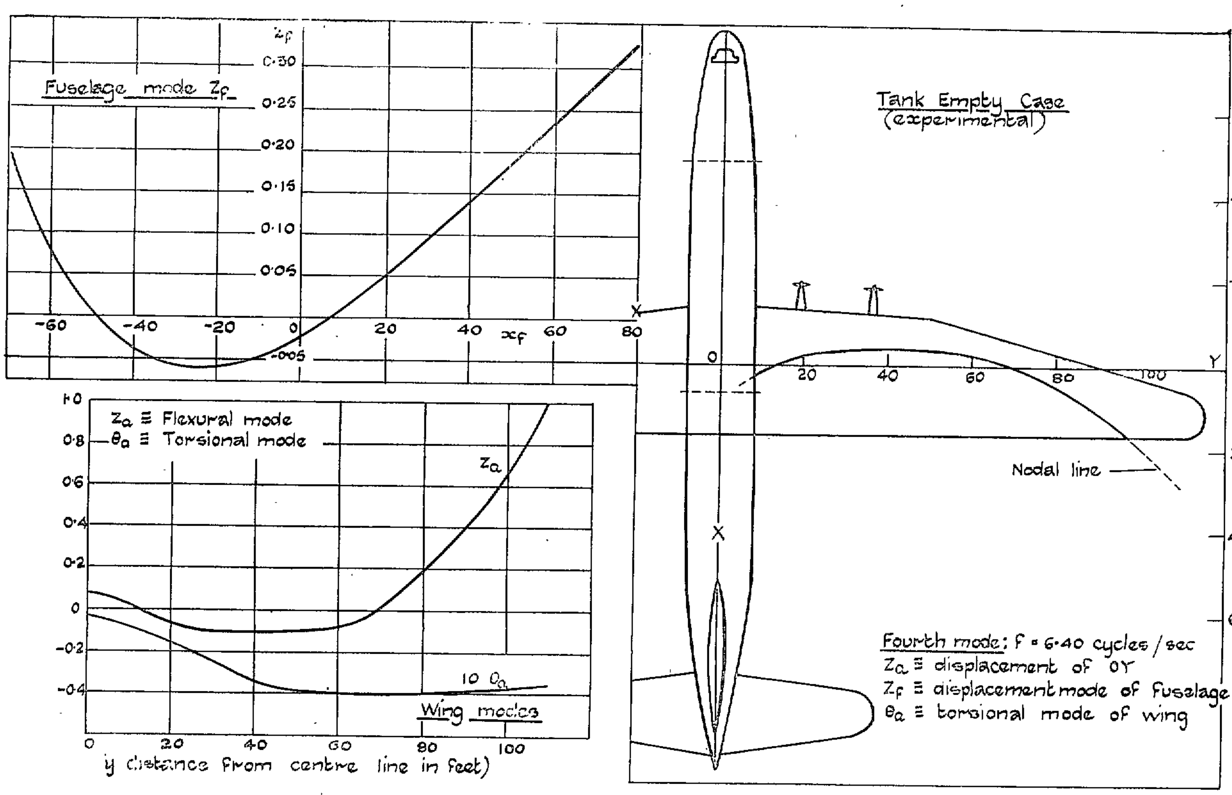
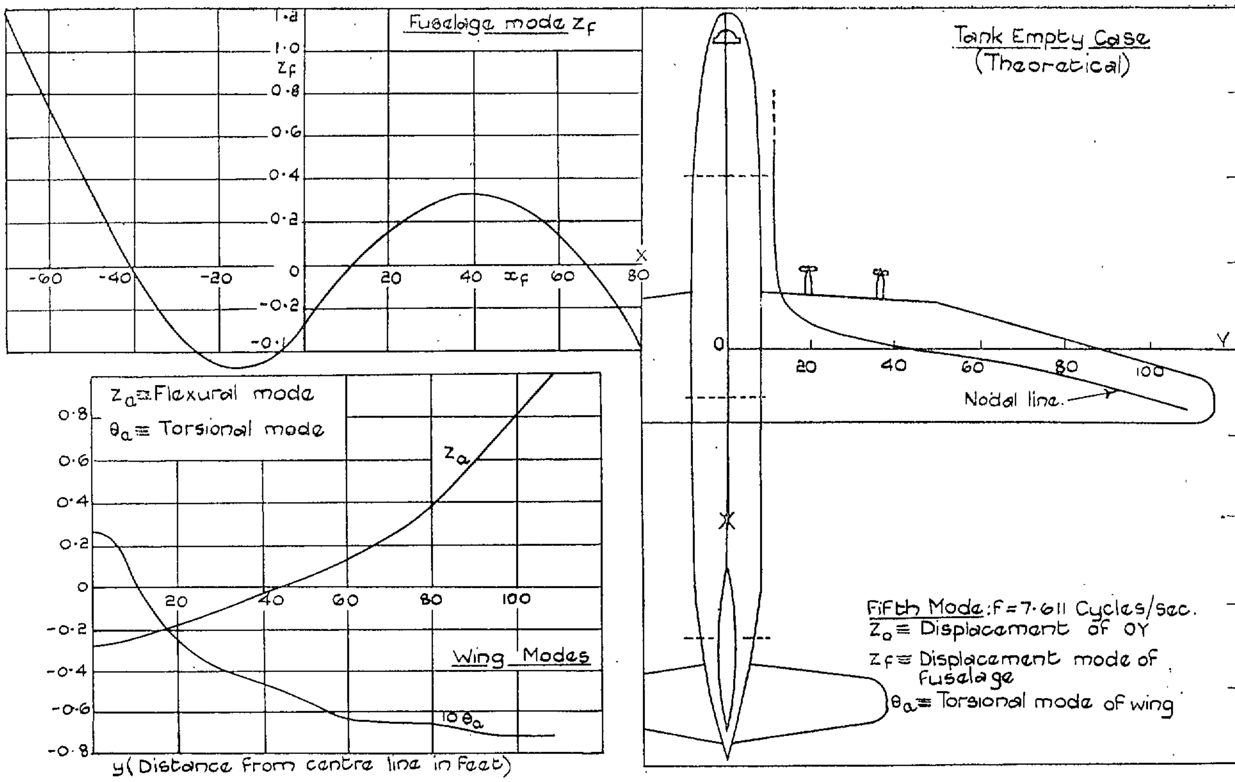
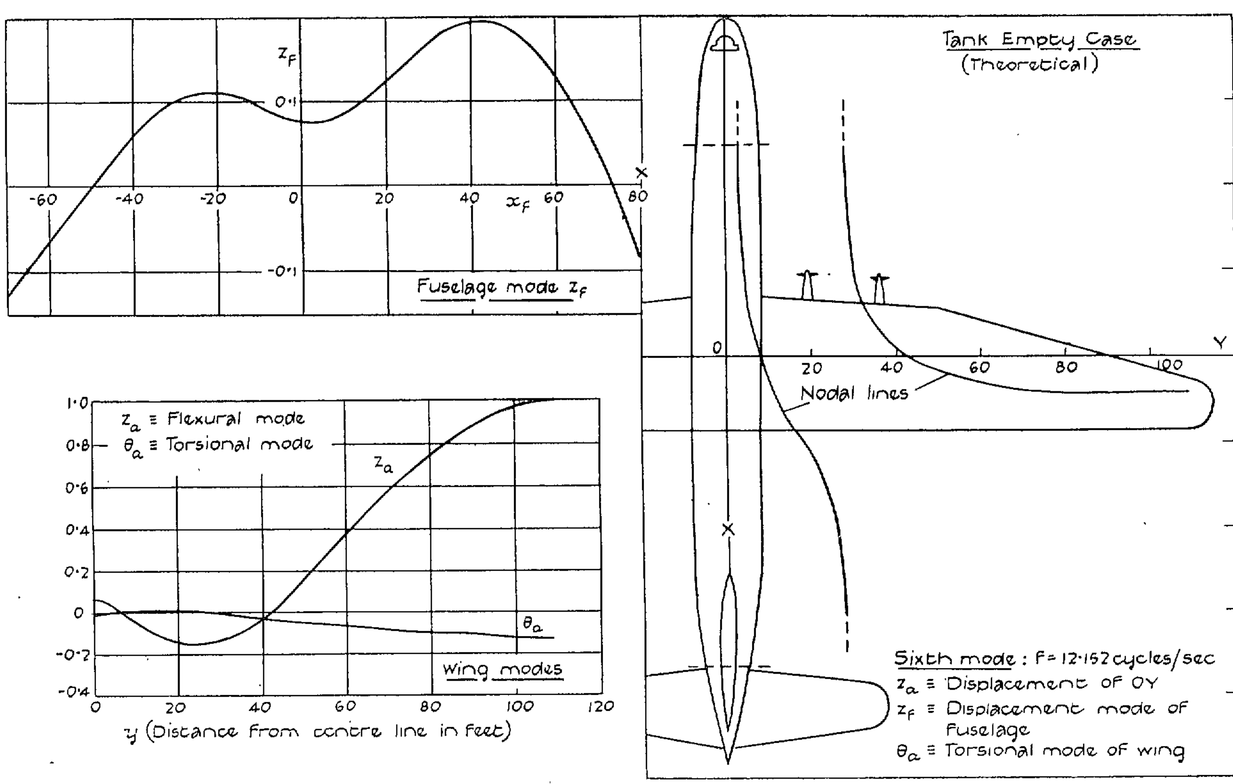

Tank Empty Case
(experimental)
Fuselage mode Z_f
Zₐ ≡ Flexural mode
θₐ ≡ Torsional mode
Wing modes
y distance from centre line in feet)
Nodal line
Fourth mode: f = 6·40 cycles/sec
Zₐ ≡ displacement of OY
Z_f ≡ displacement mode of fuselage
θₐ ≡ torsional mode of wing

FIG. 9b.



Fuselage mode z_f
Tank Empty Case
(Theoretical)
z_a ≡ Flexural mode
θ_a ≡ Torsional mode
Wing Modes
$10\theta_a$
y (Distance from centre line in feet)
Nodal line.
Fifth Mode: f = 7·611 Cycles/sec.
z_a ≡ Displacement of OY
z_f ≡ Displacement mode of Fuselage
θ_a ≡ Torsional mode of wing

FIG. 10.



Fuselage mode z_f
Tank Empty Case
(Theoretical)
z_a ≡ Flexural mode
θ_a ≡ Torsional mode
Wing modes
y (Distance from centre line in feet)
Nodal lines
Sixth mode: f = 12·152 cycles/sec
z_a ≡ Displacement of OY
z_f ≡ Displacement mode of Fuselage
θ_a ≡ Torsional mode of wing

FIG. 11.

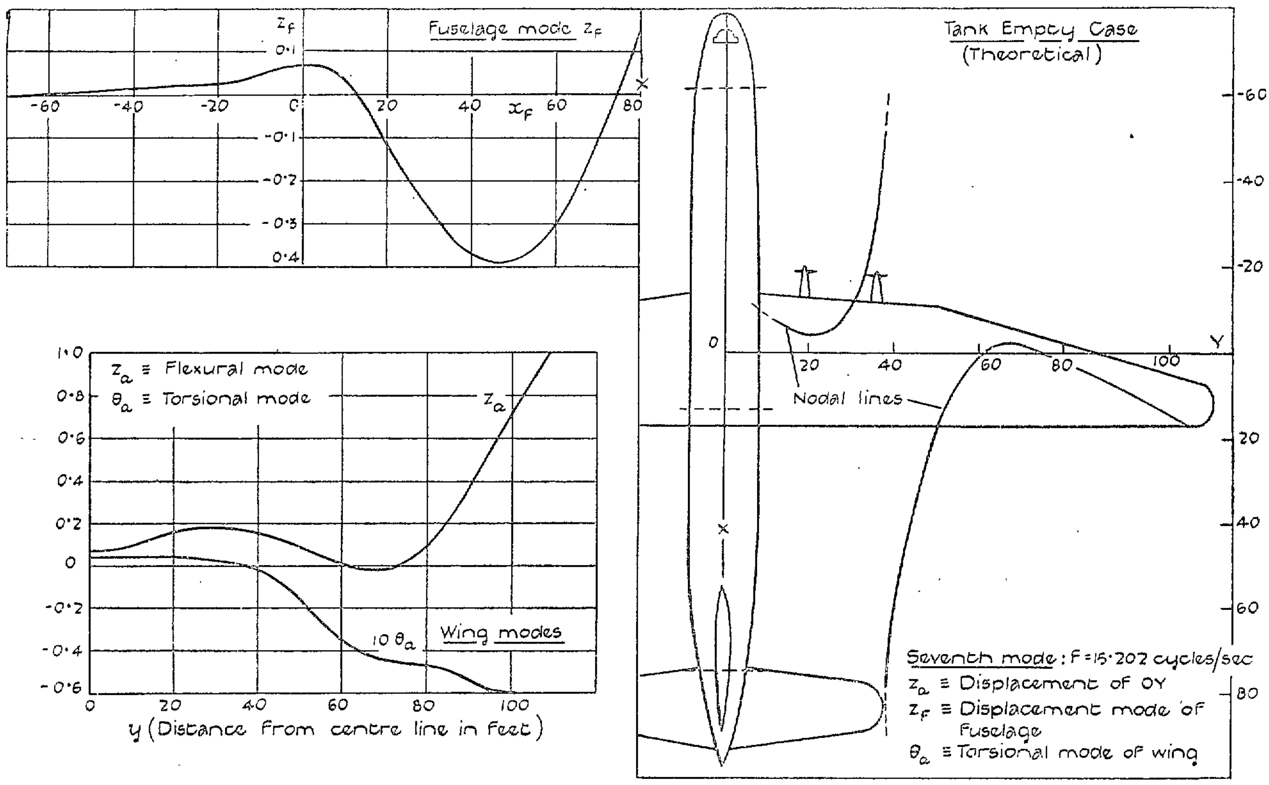

Fuselage mode z_f
z_f
x_f
Tank Empty Case
(Theoretical)
Y
Nodal lines
z_a ≡ Flexural mode
θ_a ≡ Torsional mode
z_a
$10\,\theta_a$
Wing modes
y (Distance from centre line in feet)
Seventh mode: F = 15·202 cycles/sec
z_a ≡ Displacement of OY
z_F ≡ Displacement mode of Fuselage
θ_a ≡ Torsional mode of wing

FIG. 12.

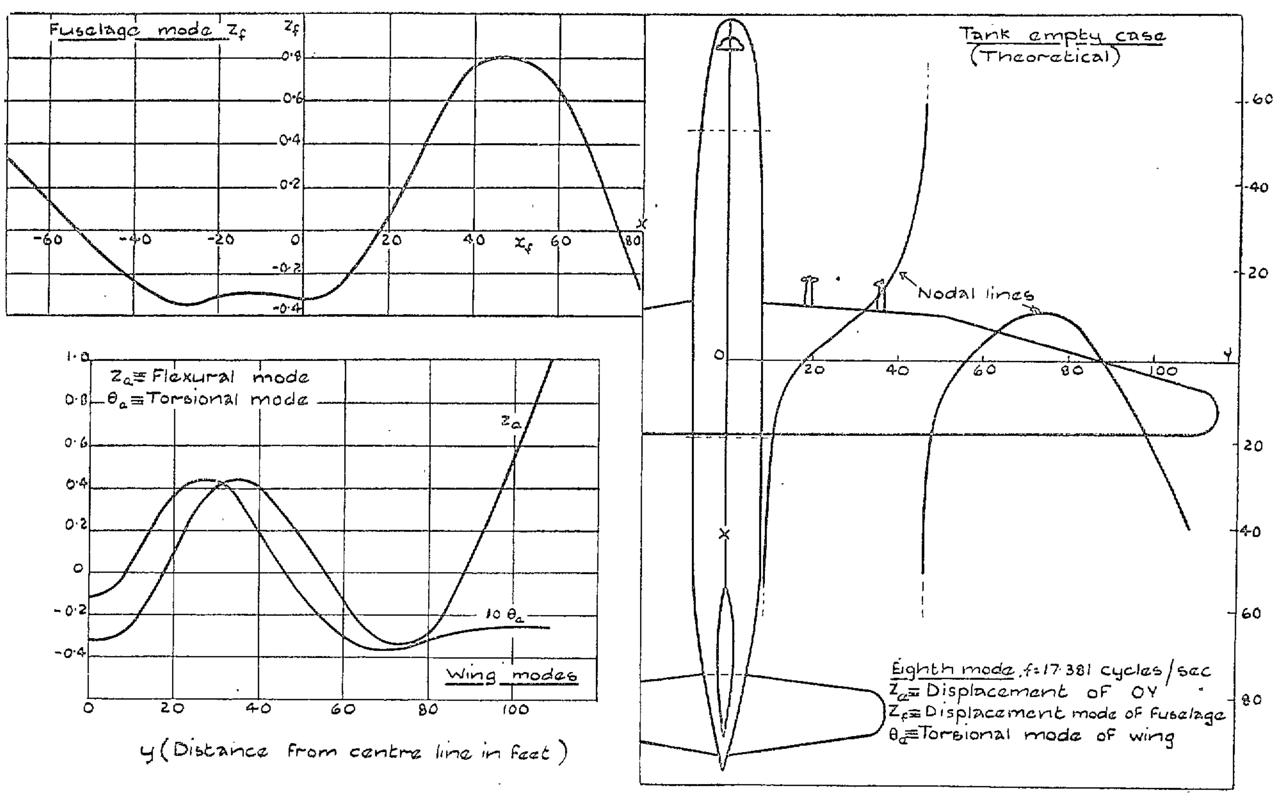

Fuselage mode z_f
z_f
x_f
Tank empty case
(Theoretical)
Y
Nodal lines
z_a ≡ Flexural mode
θ_a ≡ Torsional mode
z_a
$10\,\theta_a$
Wing modes
y (Distance from centre line in feet)
Eighth mode, f = 17·381 cycles/sec
z_a ≡ Displacement of OY
z_f ≡ Displacement mode of Fuselage
θ_a ≡ Torsional mode of wing

FIG. 13.

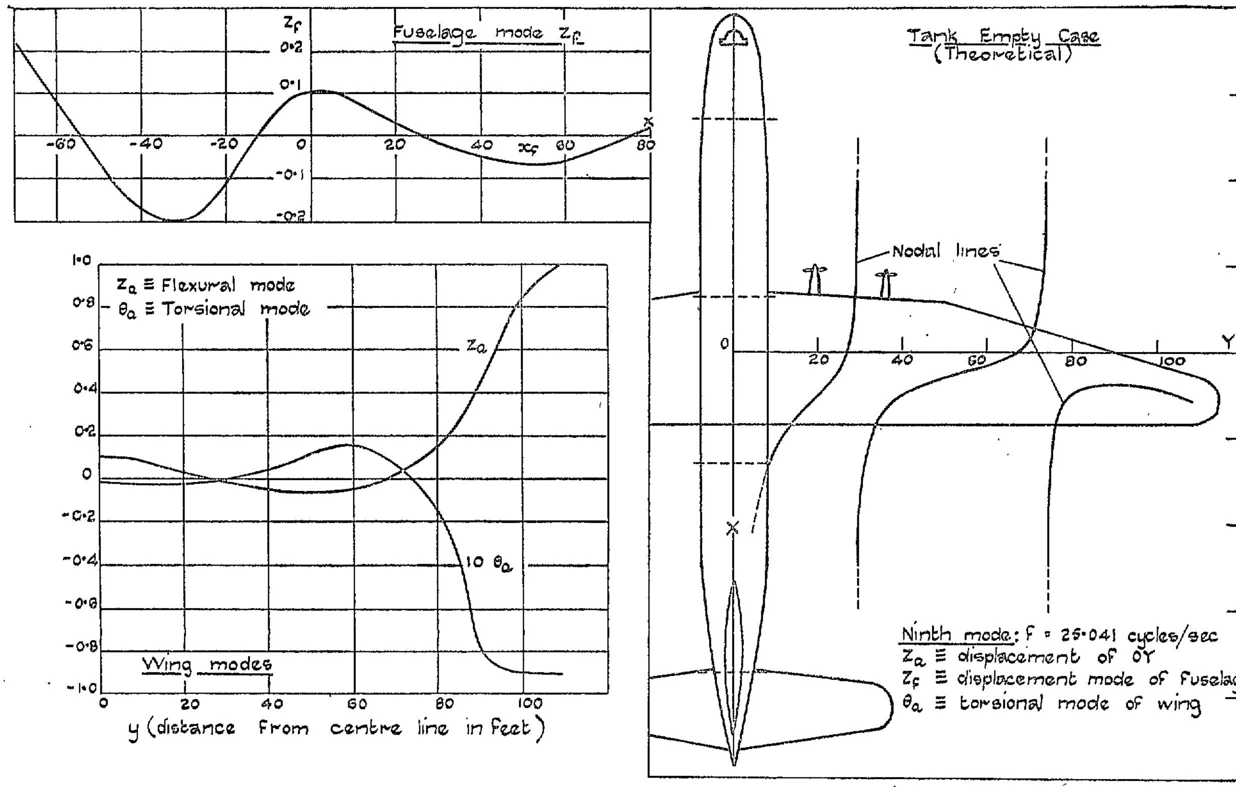

Fuselage mode z_f
Tank Empty Case (Theoretical)
z_a ≡ Flexural mode
θ_a ≡ Torsional mode
Wing modes
y (distance from centre line in feet)
Nodal lines
Ninth mode: f = 25·041 cycles/sec
z_a ≡ displacement of OY
z_f ≡ displacement mode of fuselage
θ_a ≡ torsional mode of wing

FIG. 14.

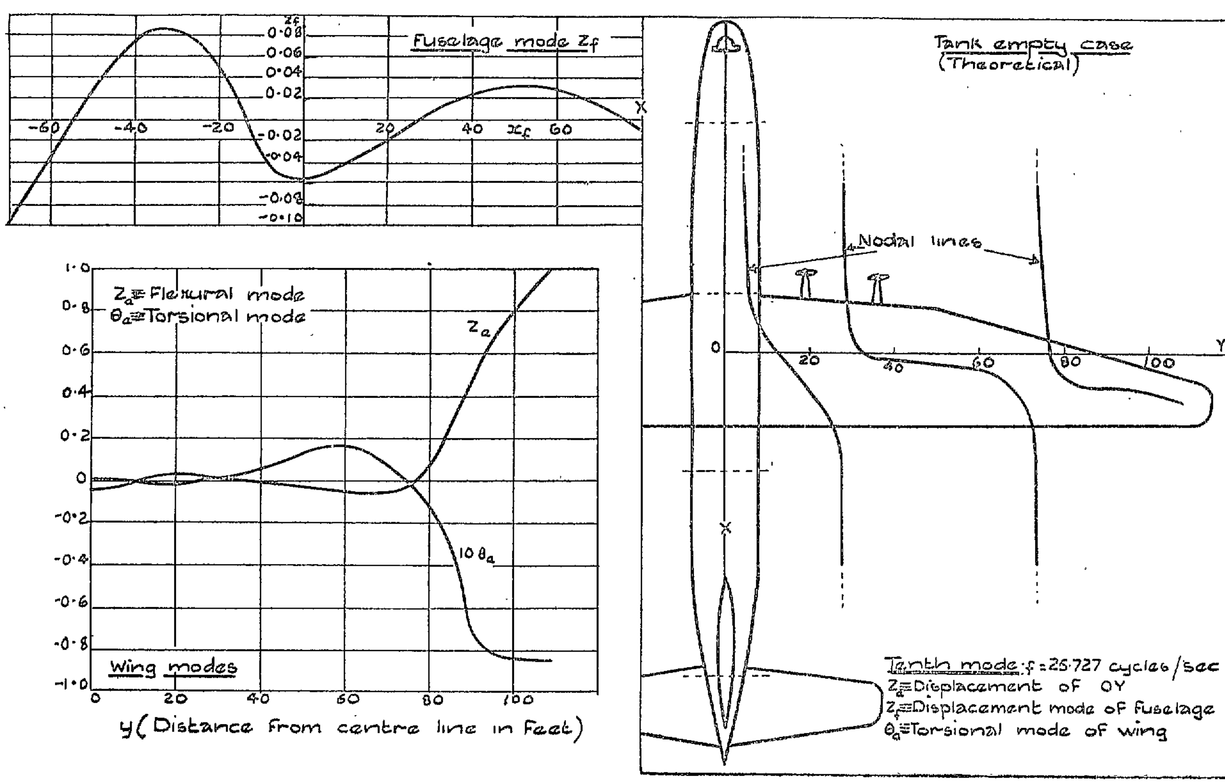

Fuselage mode z_f
Tank empty case (Theoretical)
z_a≡Flexural mode
θ_a≡Torsional mode
Wing modes
y (Distance from centre line in Feet)
Nodal lines
Tenth mode: f = 25·727 cycles/sec
z_a≡Displacement of OY
z_f≡Displacement mode of fuselage
θ_a≡Torsional mode of wing

FIG. 15.

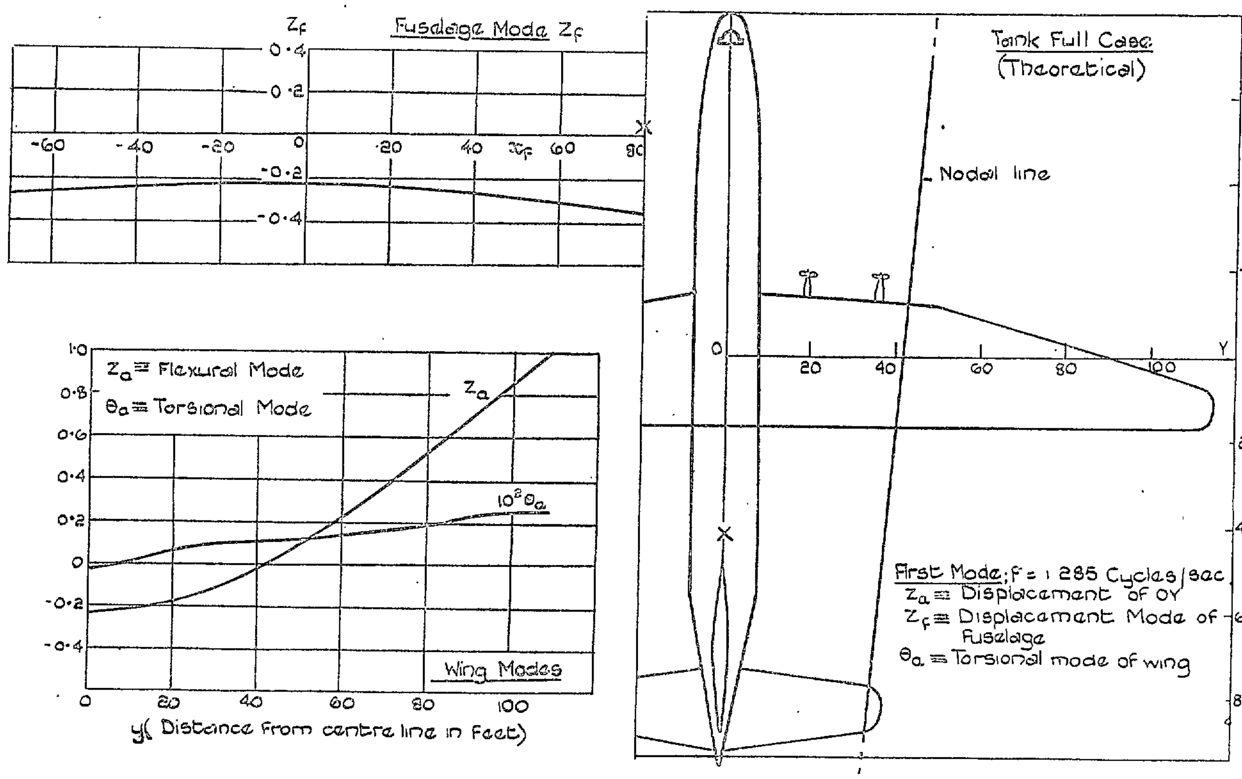


FIG. 16a.

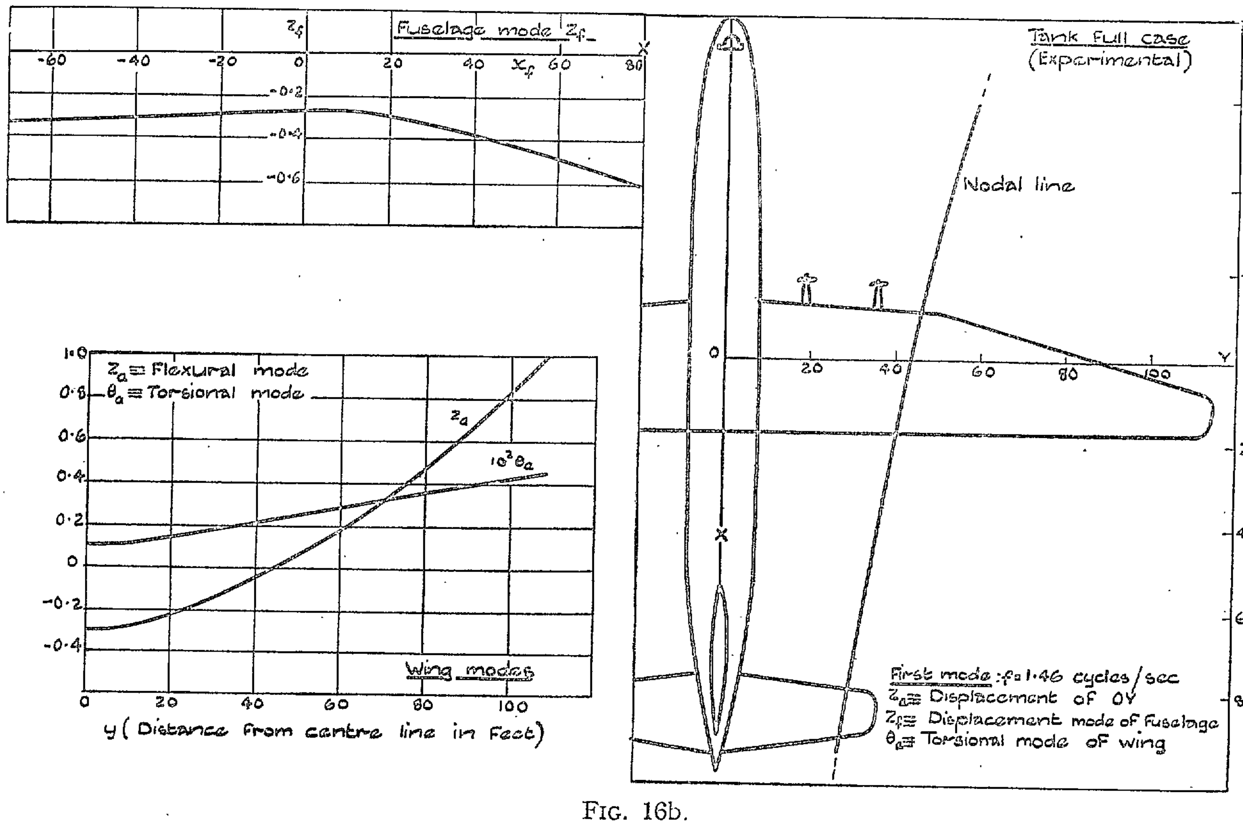


FIG. 16b.

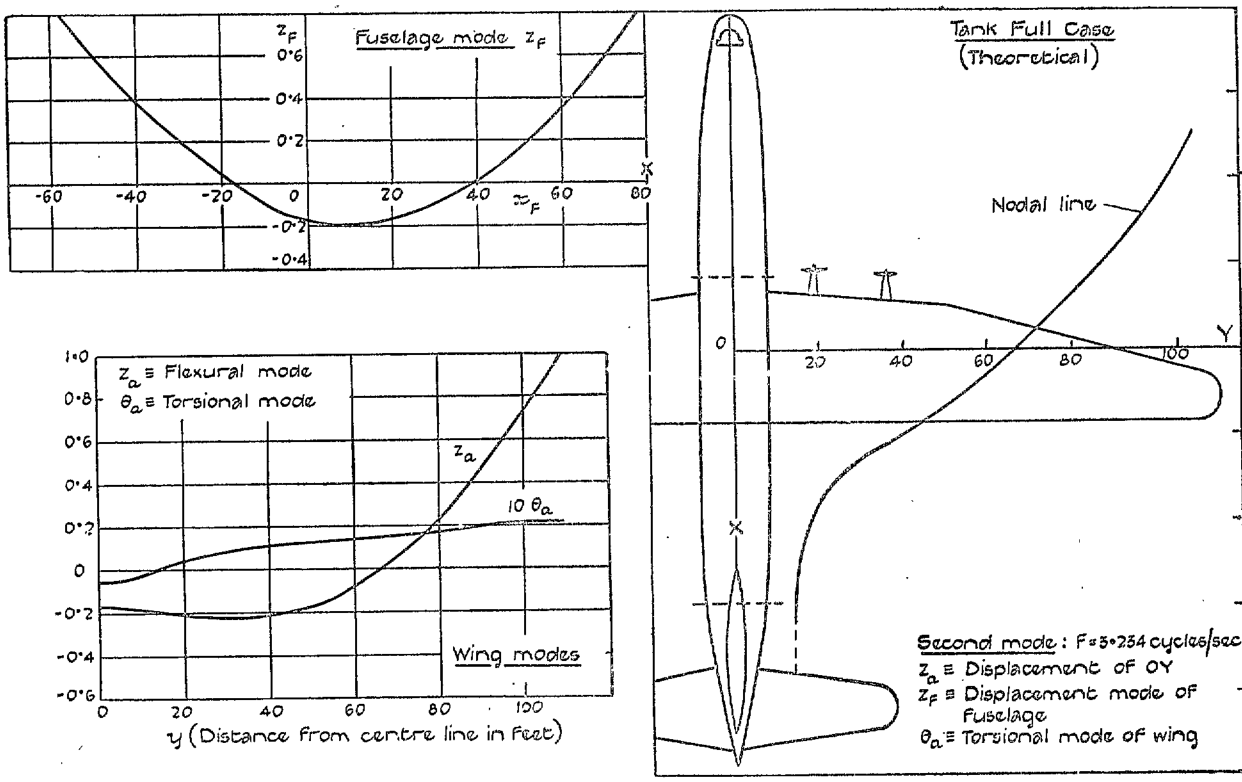


FIG. 17a.

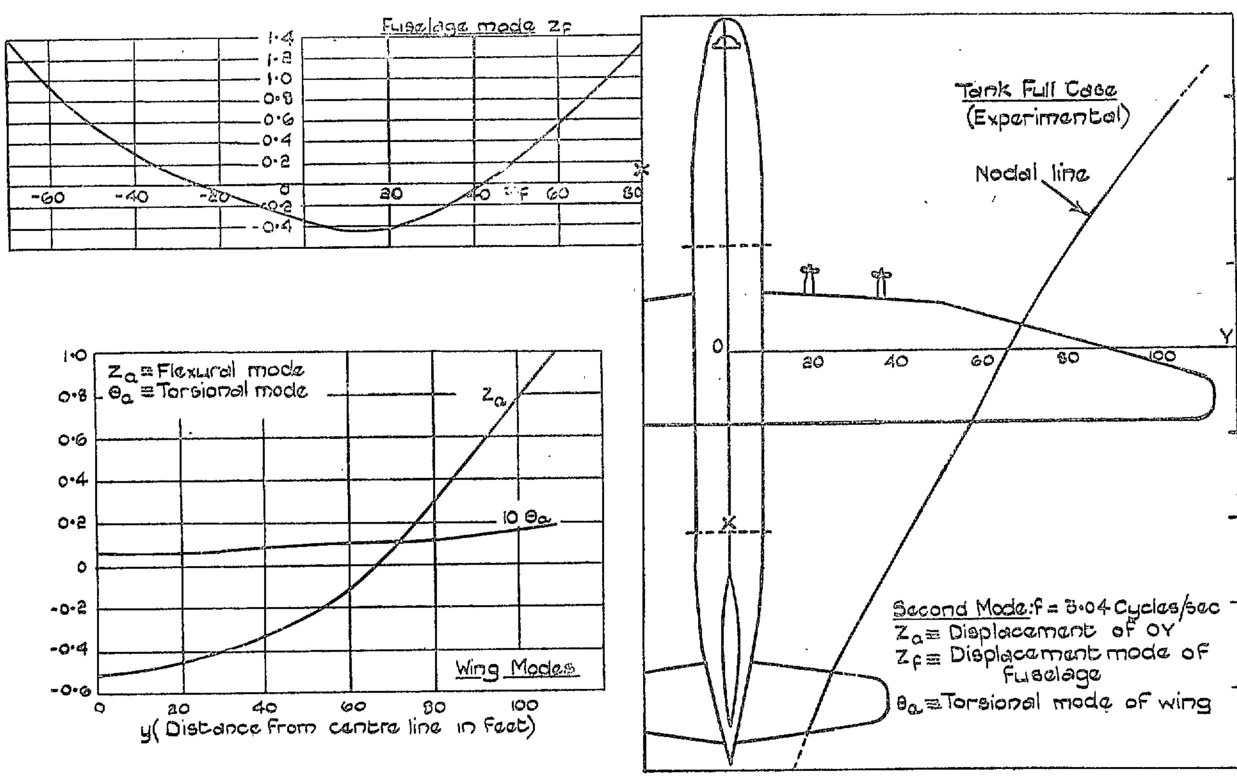


FIG. 17b.

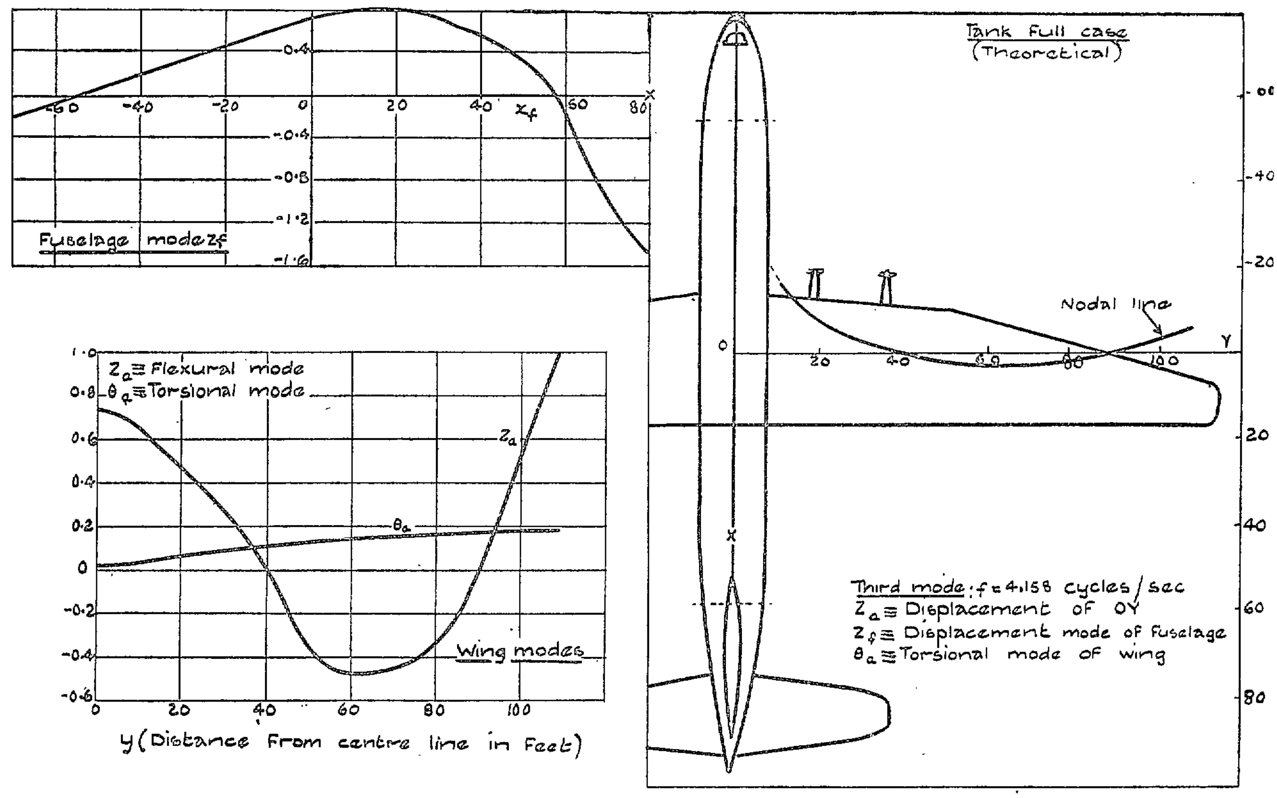


FIG. 18a.

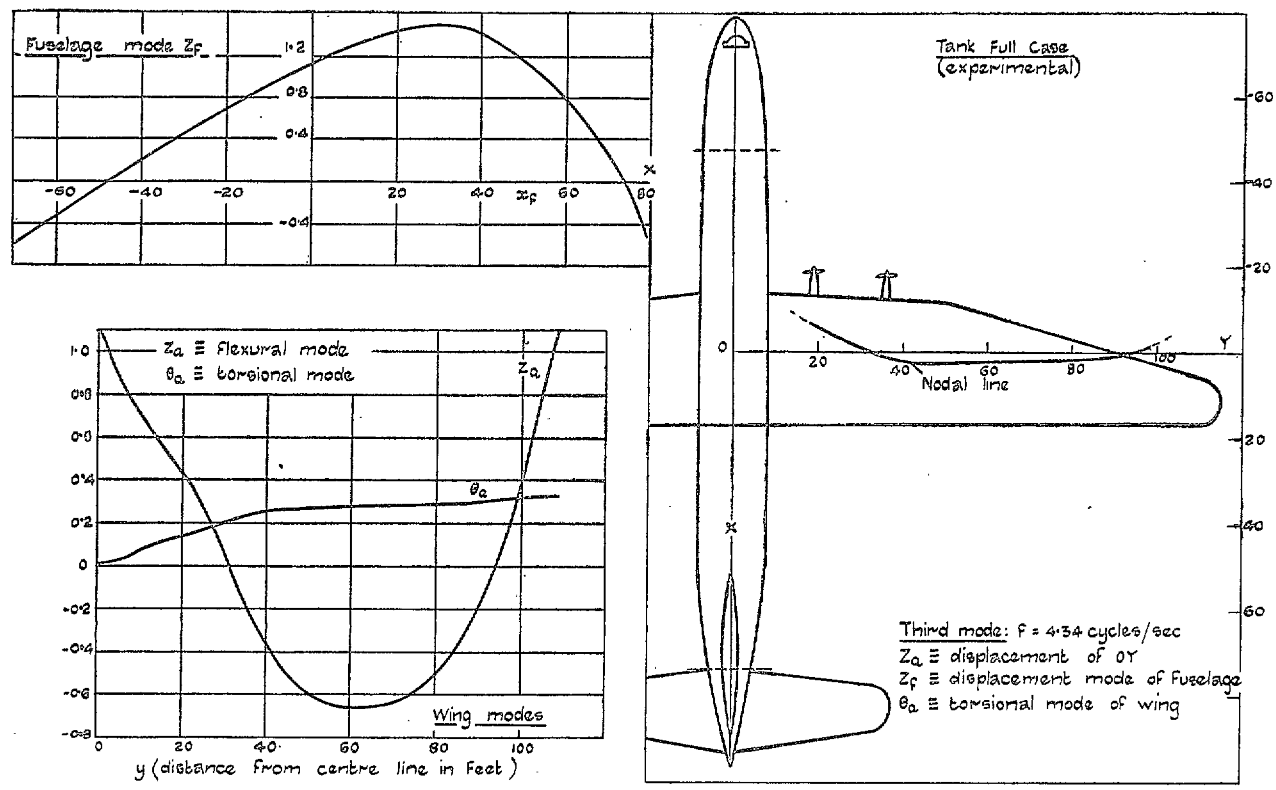


FIG. 18b.

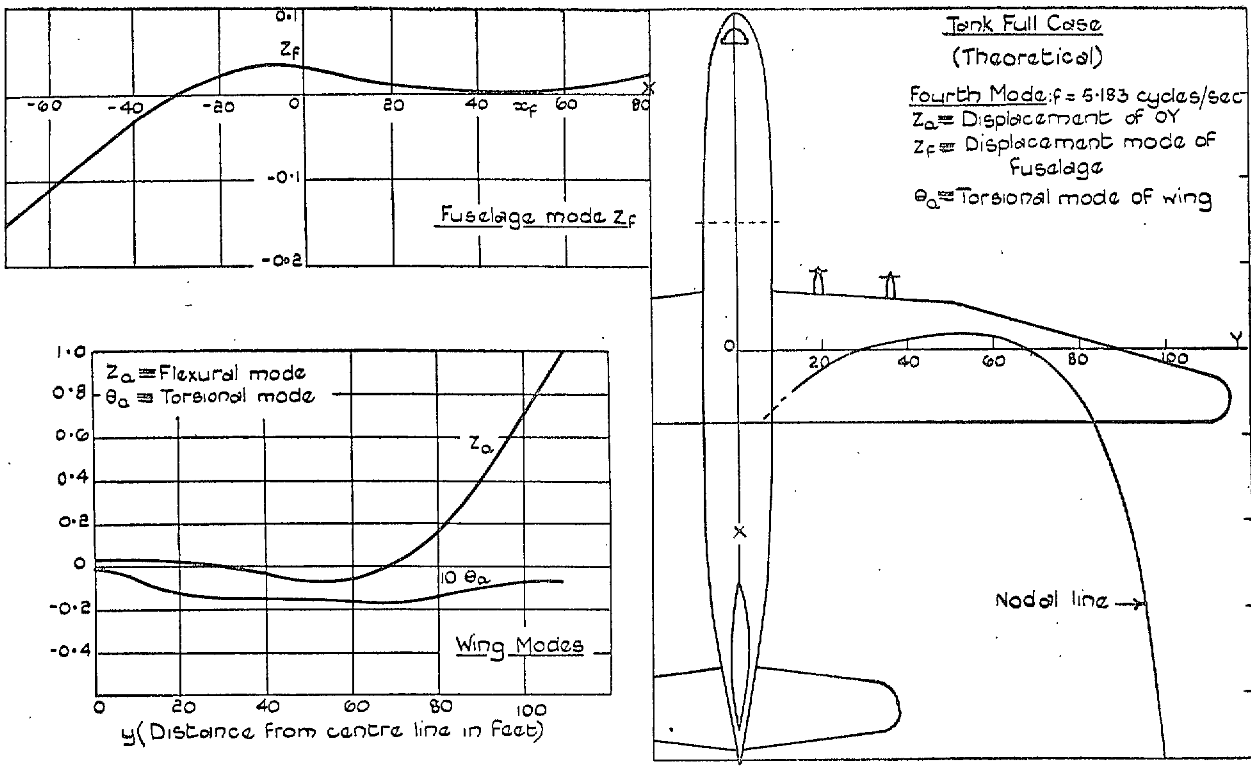


FIG. 19a.

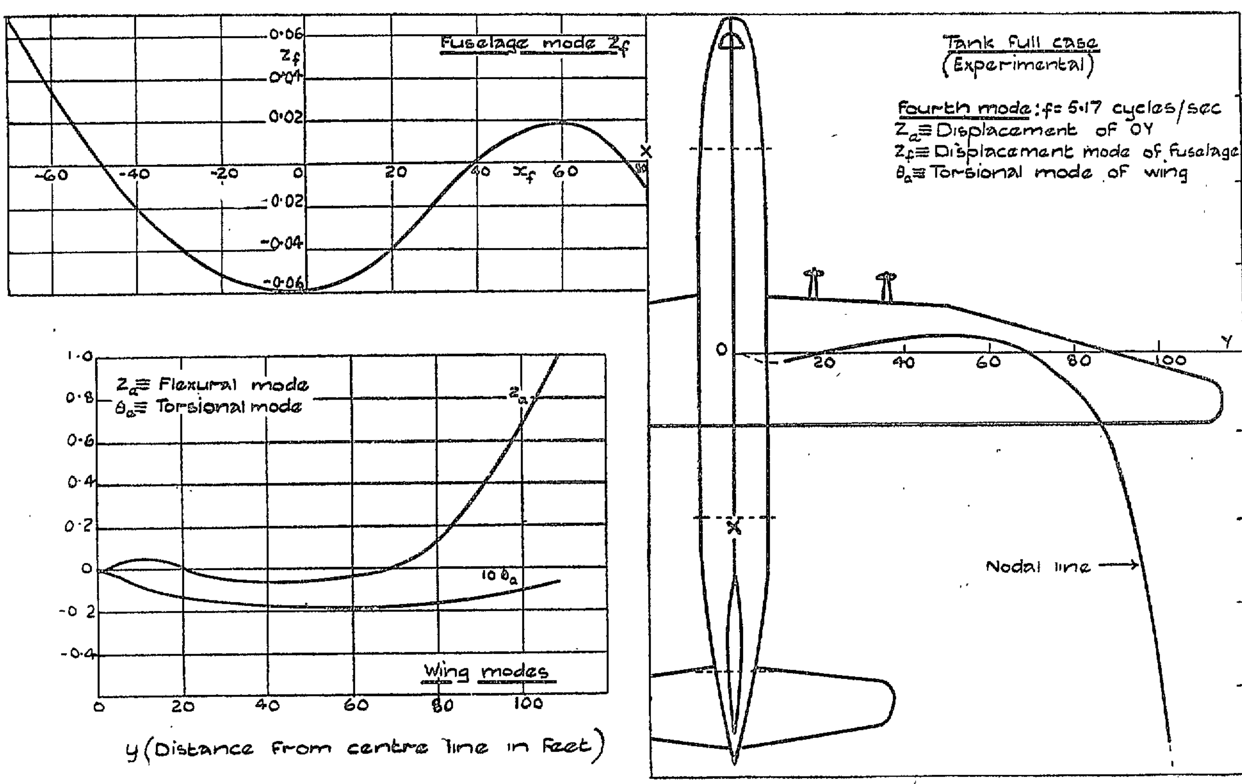


FIG. 19b.

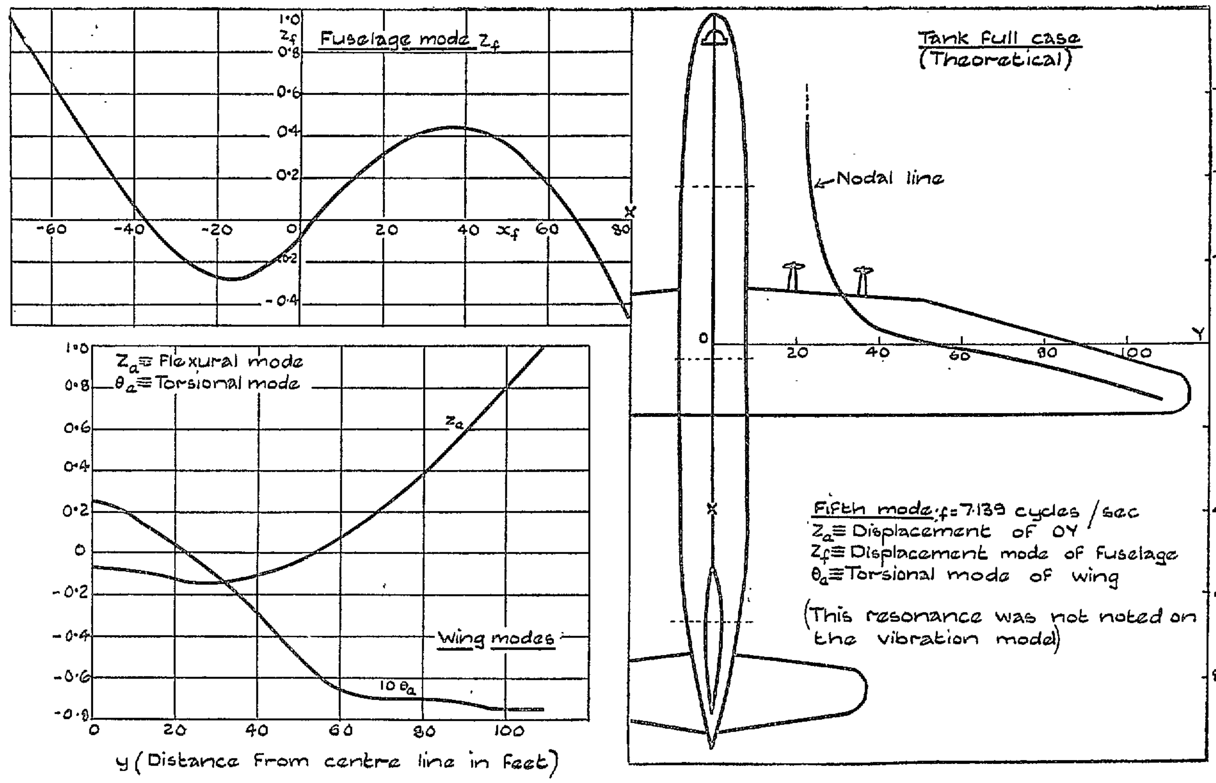


FIG. 20.

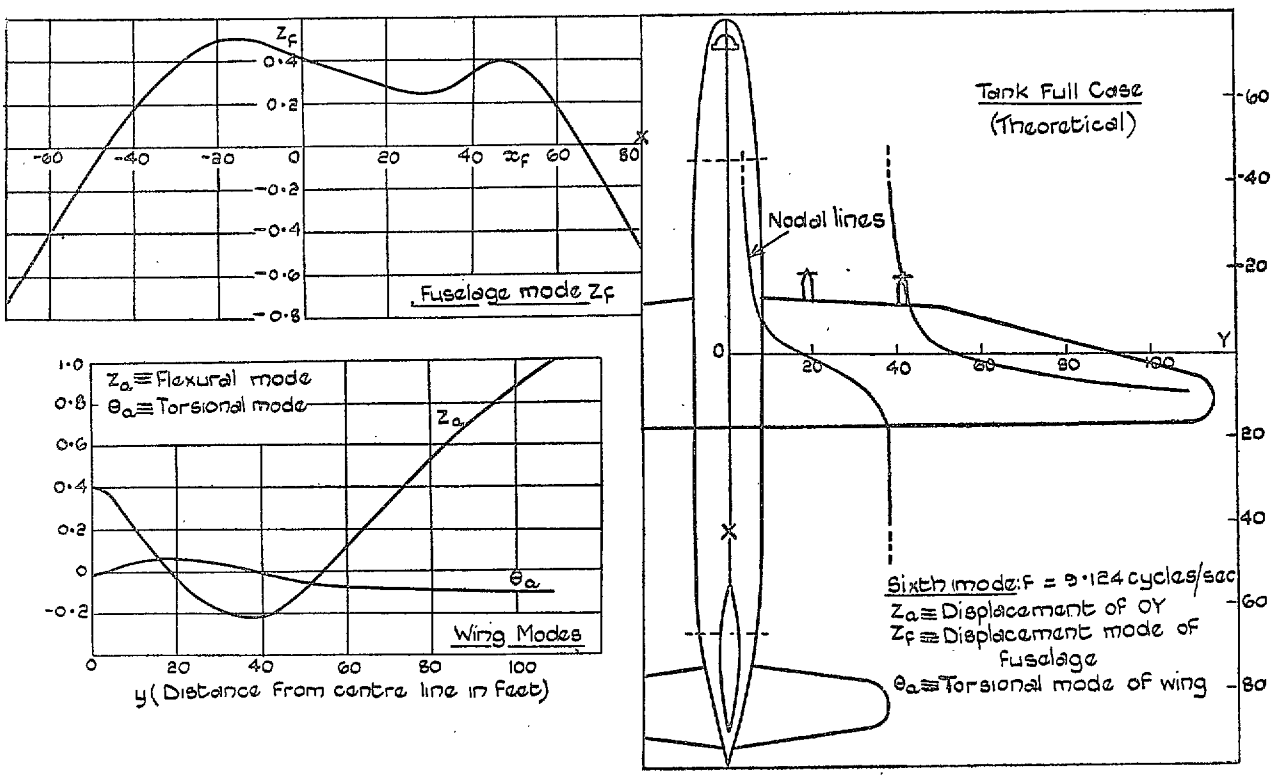


FIG. 21a.

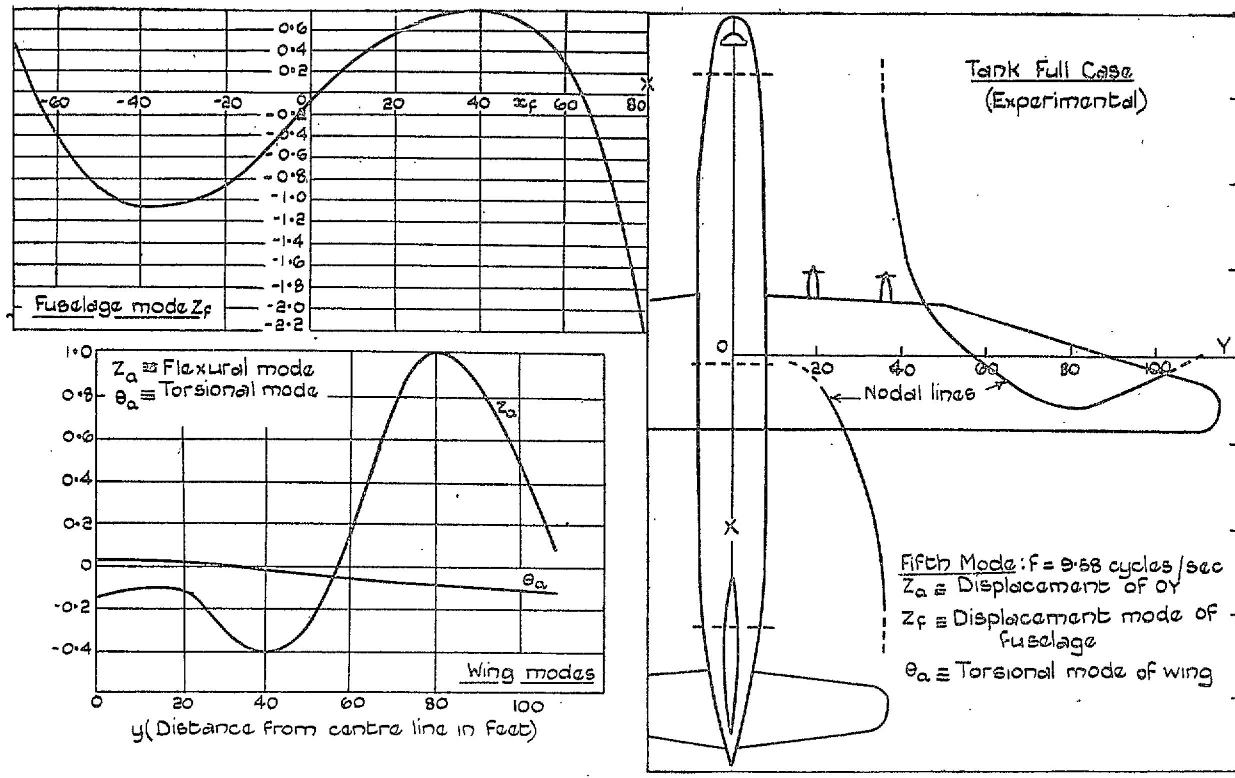


FIG. 21b.

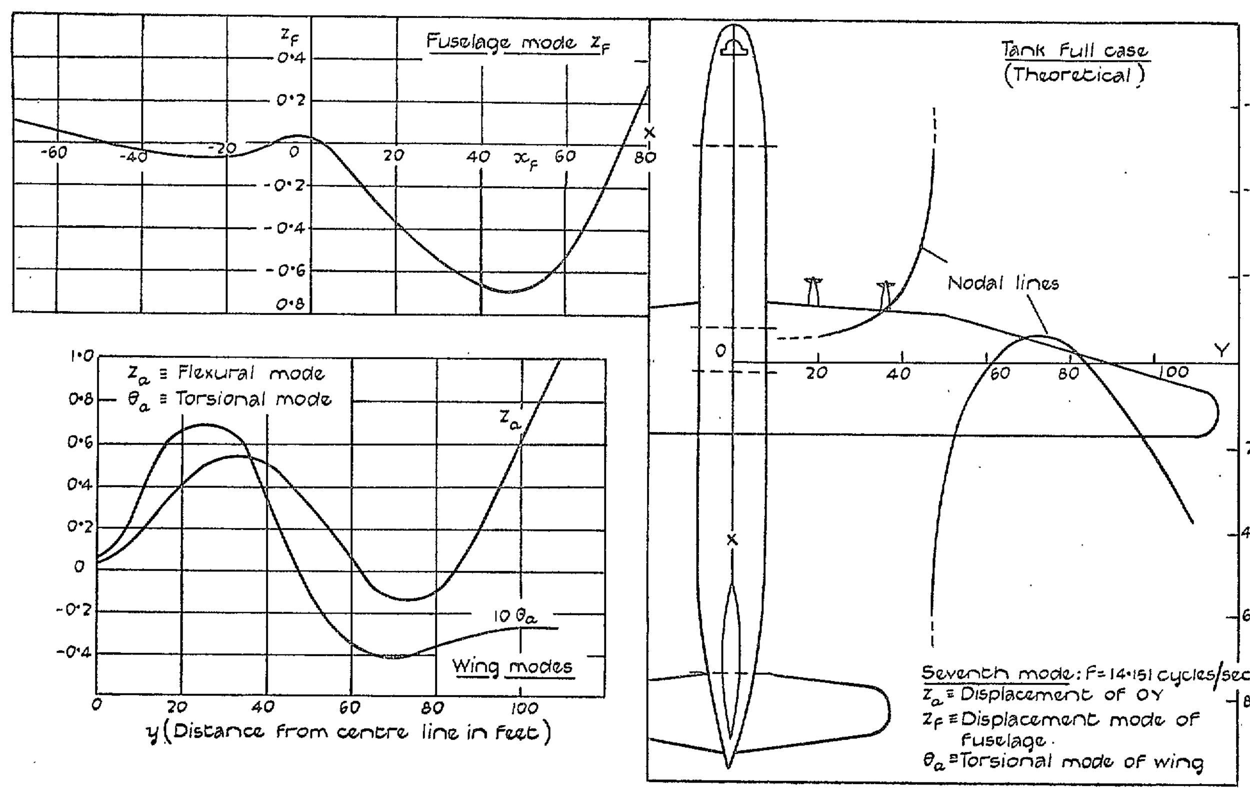
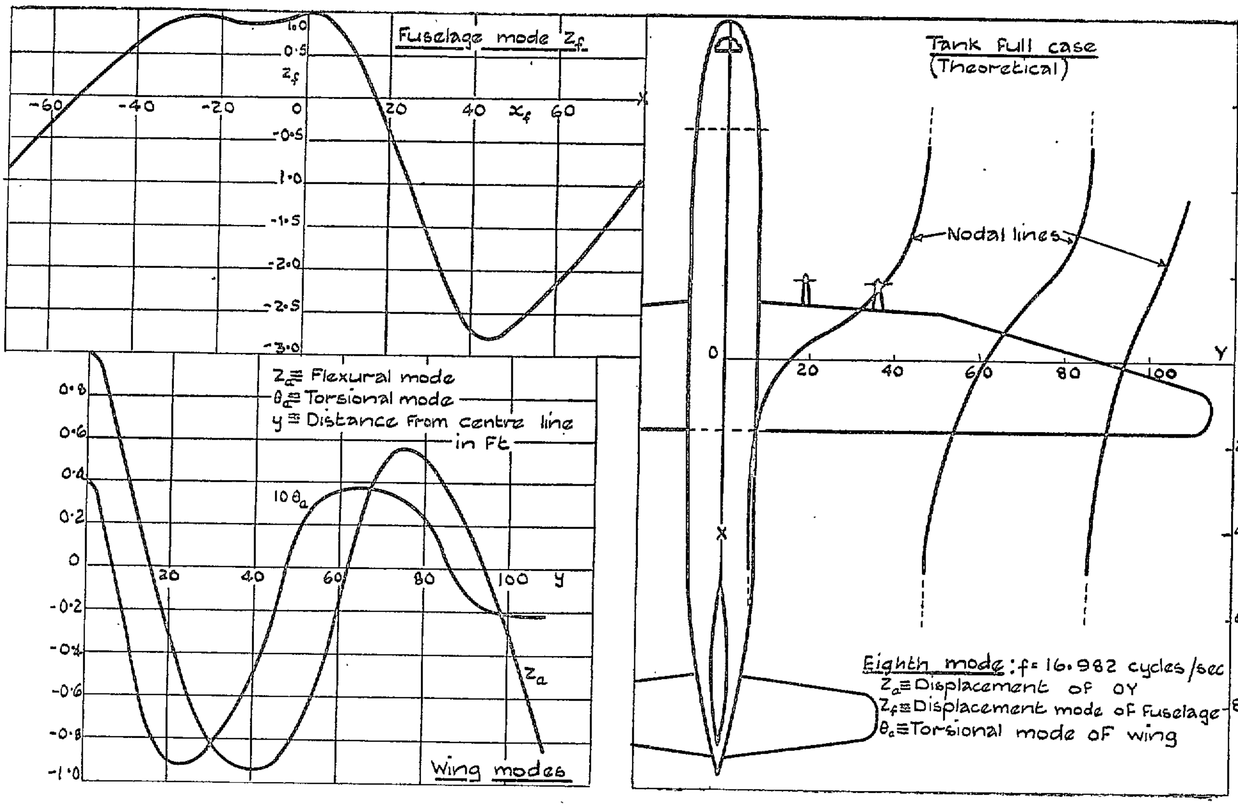
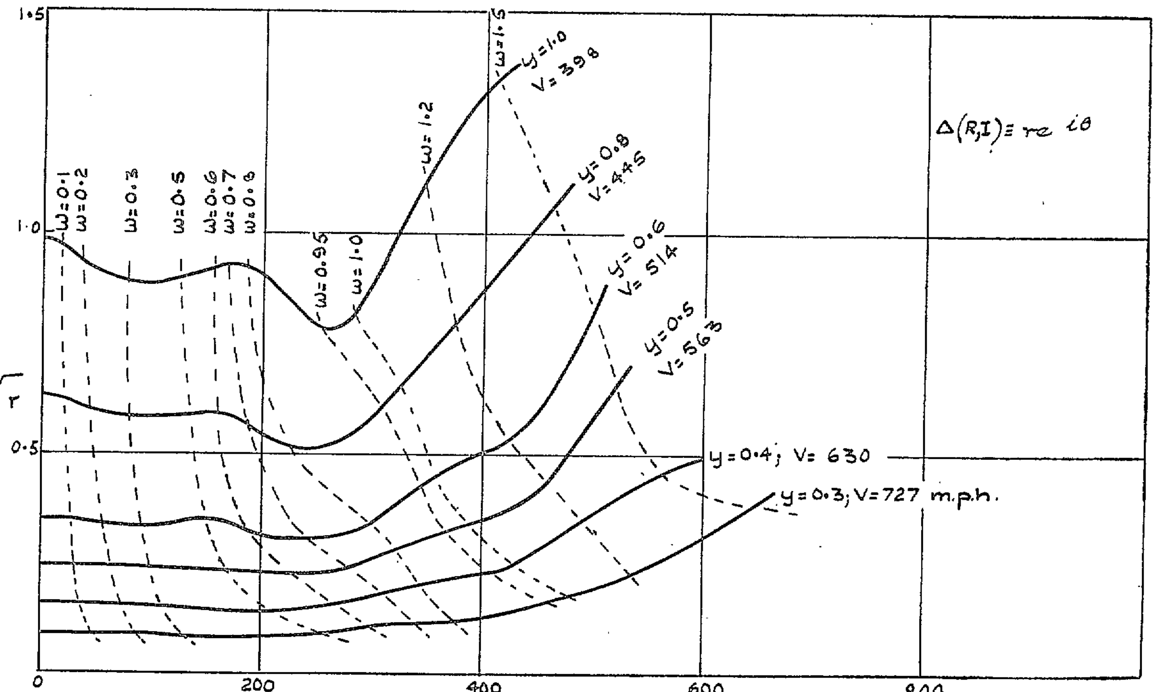


FIG. 22.



Fuselage mode Z_f
Tank full case (Theoretical)
Nodal lines
Z_a ≡ Flexural mode
θ_a ≡ Torsional mode
y ≡ Distance from centre line in Ft
$10\theta_a$
Z_a
Wing modes
Eighth mode: f = 16.982 cycles/sec
Z_a ≡ Displacement of OY
Z_f ≡ Displacement mode of Fuselage
θ_a ≡ Torsional mode of wing

FIG. 23.



$\Delta(R,I) \equiv re^{i\theta}$
$\sqrt[3]{r}$
$\theta°$
y = 1.0 V = 398
y = 0.8 V = 445
y = 0.6 V = 514
y = 0.5 V = 563
y = 0.4; V = 630
y = 0.3; V = 727 m.p.h.
ω = 0.1
ω = 0.2
ω = 0.3
ω = 0.5
ω = 0.67
ω = 0.7
ω = 0.8
ω = 0.95
ω = 1.0
ω = 1.2
ω = 1.5
Fuselage rigid and immobile (tanks empty)

FIG. 24.

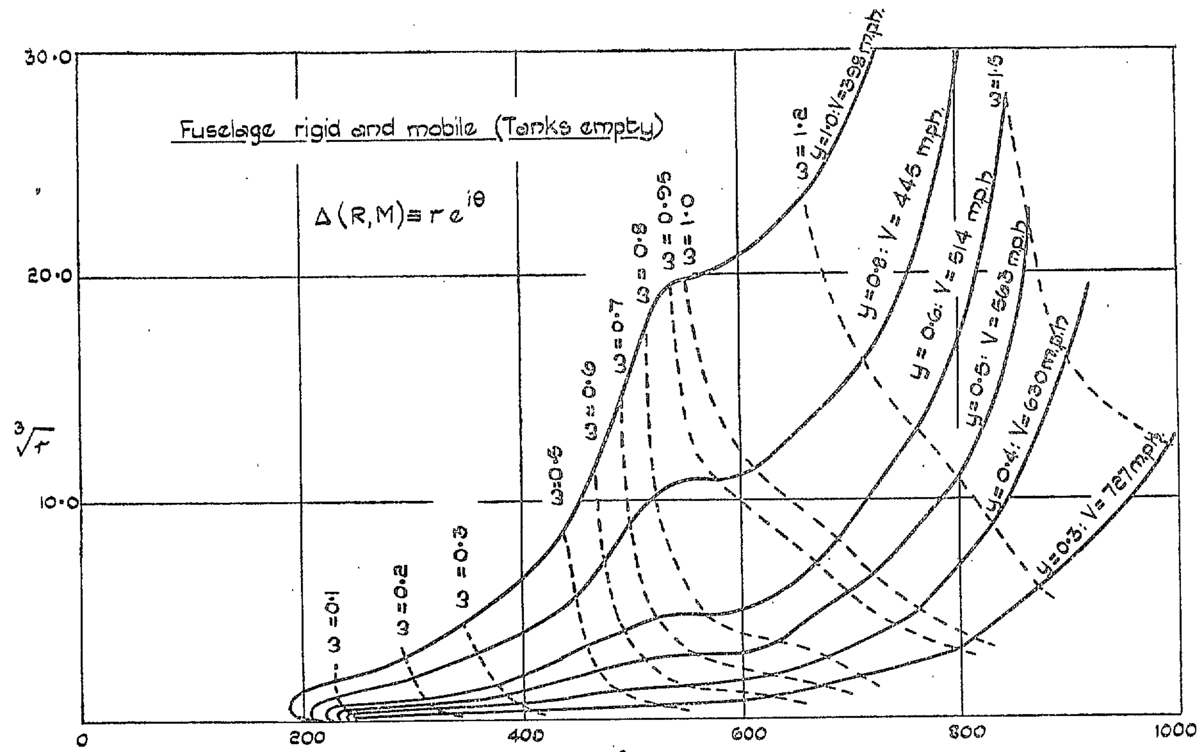

Fuselage rigid and mobile (Tanks empty)
Δ(R,M) ≡ r e^{iθ}
∛r
30·0
20·0
10·0
0
200
400
600
800
1000
θ°
ω = 0·1
ω = 0·2
ω = 0·3
ω = 0·5
ω = 0·6
ω = 0·7
ω = 0·8
ω = 0·95
ω = 1·0
ω = 1·2
ω = 1·5
y = 1·0: V = 398 mph.
y = 0·8: V = 445 mph.
y = 0·6: V = 514 mph.
y = 0·5: V = 563 mph.
y = 0·4: V = 630 mph.
y = 0·3: V = 727 mph.

FIG. 25.

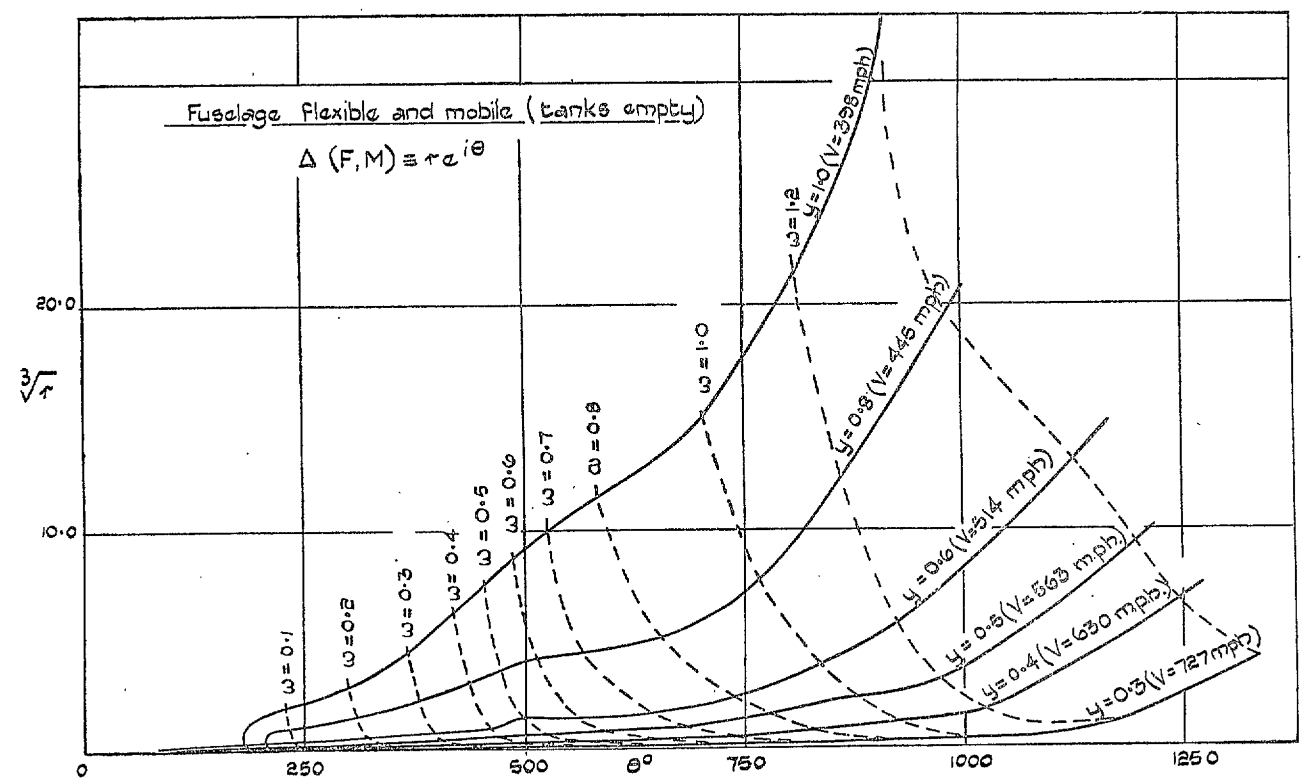

Fuselage flexible and mobile (tanks empty)
Δ (F,M) ≡ r e^{iθ}
∛r
20·0
10·0
0
250
500
θ°
750
1000
1250
ω = 0·1
ω = 0·2
ω = 0·3
ω = 0·4
ω = 0·5
ω = 0·6
ω = 0·7
ω = 0·8
ω = 1·0
ω = 1·2
y = 1·0 (V = 398 mph)
y = 0·8 (V = 445 mph)
y = 0·6 (V = 514 mph)
y = 0·5 (V = 563 m.ph.)
y = 0·4 (V = 630 mph.)
y = 0·3 (V = 727 mph)

FIG. 26.

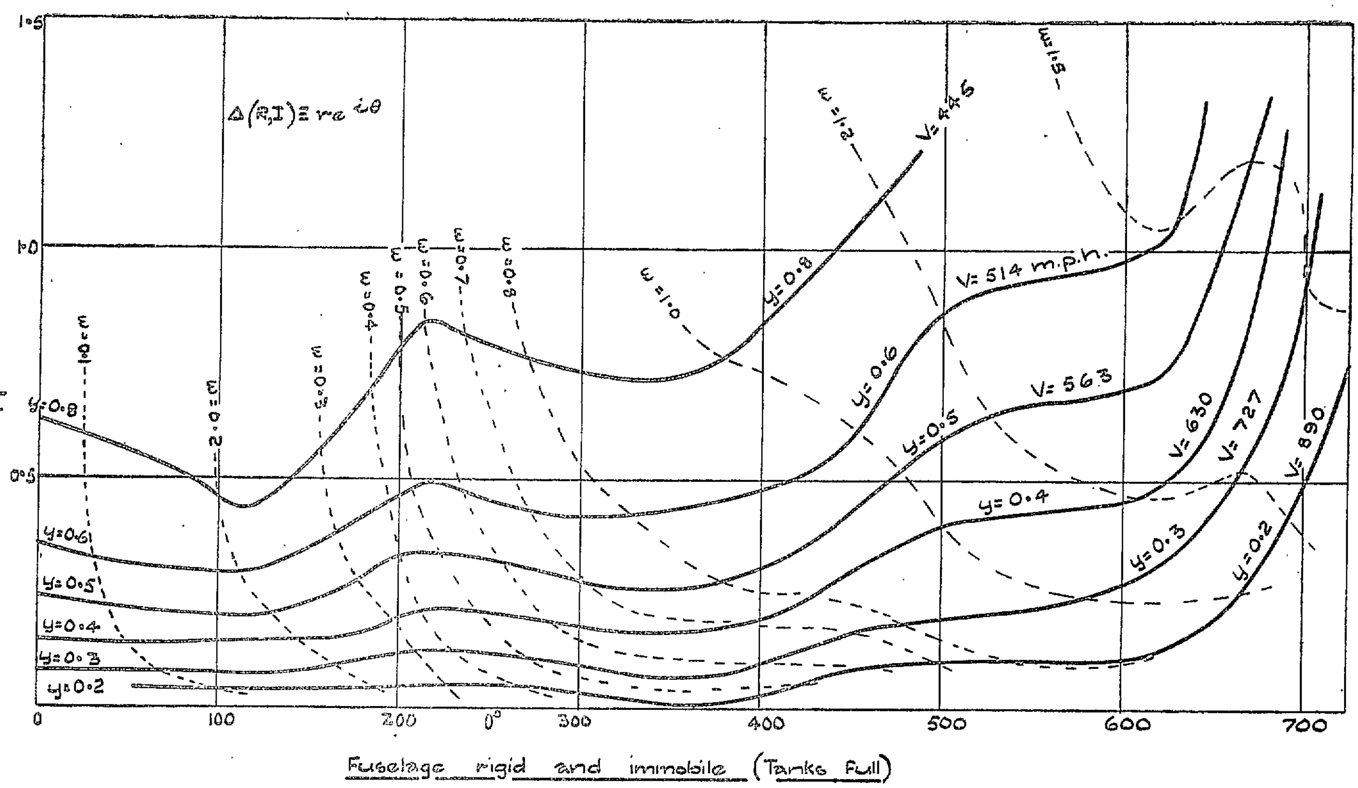

Δ(R,I) ≡ re^{iθ}
V=445
V=514 m.p.h.
V=563
V=630
V=727
V=890
y=0.8
y=0.6
y=0.5
y=0.4
y=0.3
y=0.2
Fuselage rigid and immobile (Tanks Full)

FIG. 27.

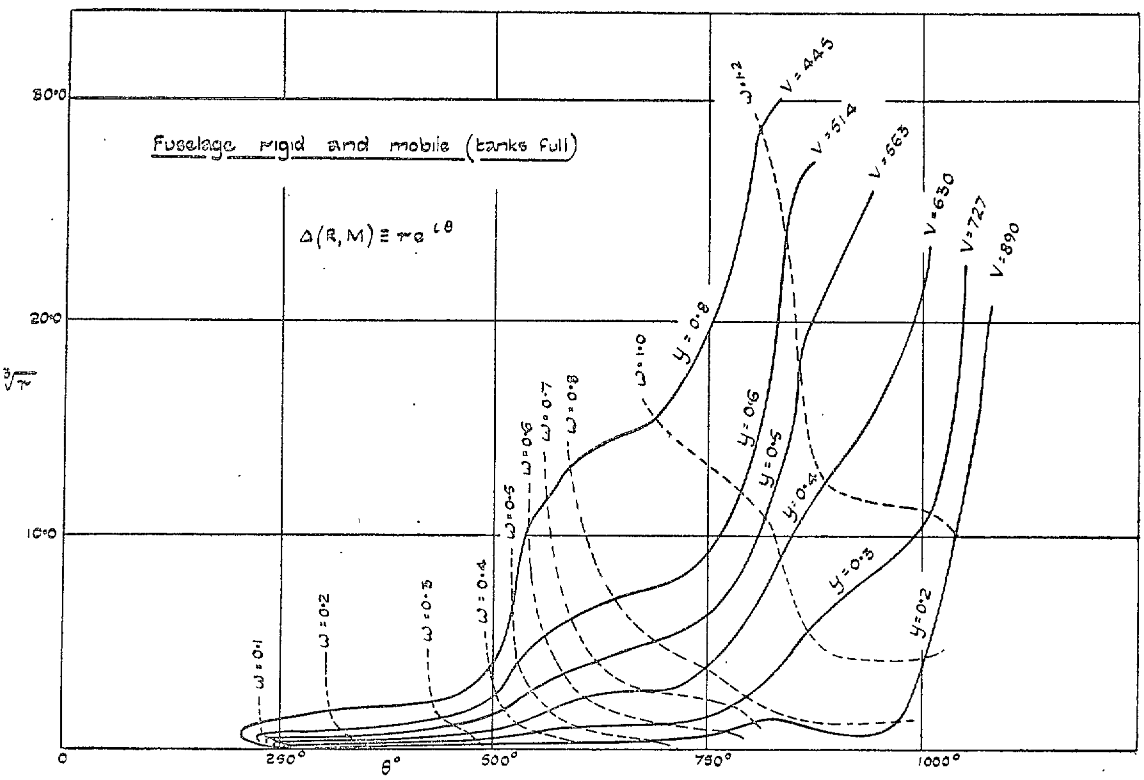
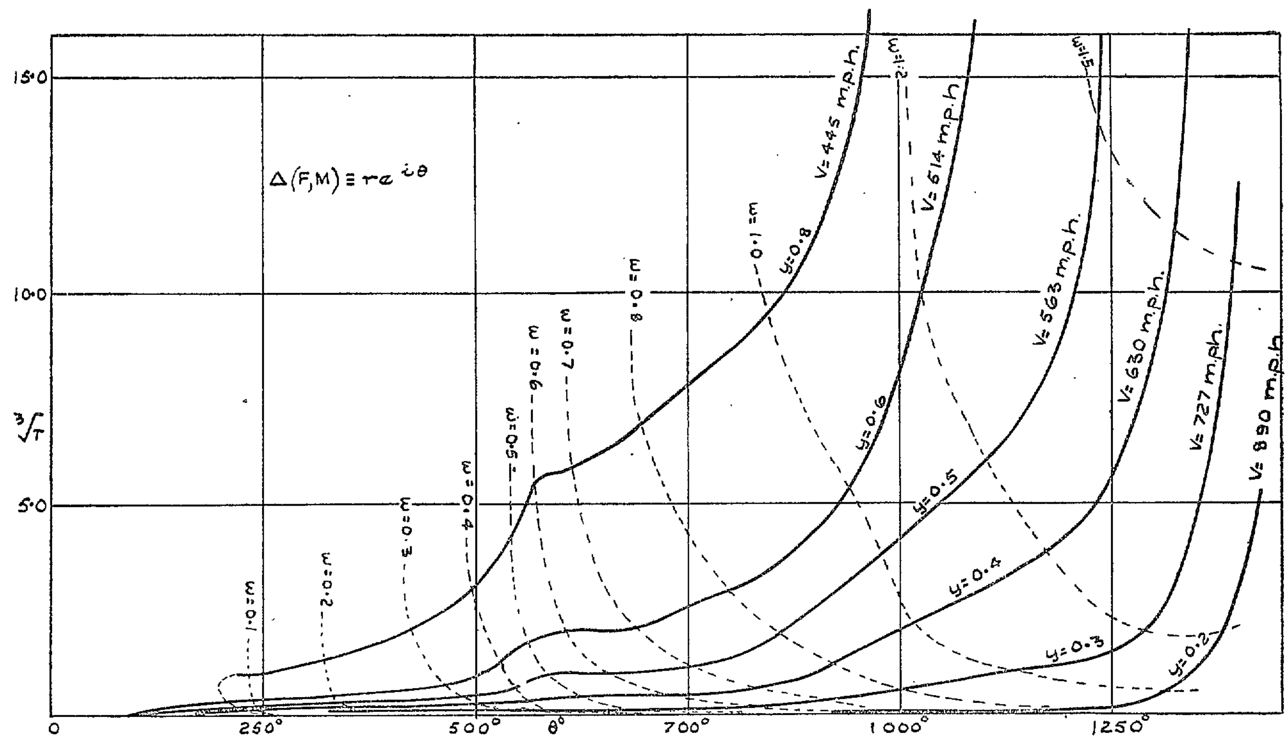

Fuselage rigid and mobile (tanks full)
Δ(R,M) ≡ re^{iθ}
V=445
V=514
V=563
V=630
V=727
V=890
y=0.8
y=0.6
y=0.5
y=0.4
y=0.3
y=0.2

FIG. 28.



Fuselage Flexible and mobile (Tanks Full)

FIG. 29.

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