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Summary.—From the first, civil airworthiness requirements have included climb performance among the safety criteria. Hitherto climb performance standards have been empirical, and magnitudes have been chosen by reference to current aircraft types regarded as satisfactory. A weakness of this empirical type of requirement is that no method is provided for modifying the standards to meet new operating procedures and aircraft design features. To overcome this difficulty, a more rational basis for deriving the climb standards is proposed.

The conception is introduced of a 'datum' performance, below which conditions predisposing to an accident exist, and the level of safety judged by an 'incident rate' which is the frequency with which the operational performance of aircraft falls below this datum. A standard is chosen so that when the aircraft type complies, the incident rate will not exceed some tolerable value. To derive such a standard, account must be taken of the various conditions such as weather and airframe state which affect performance. The standard need only be framed in terms of some of these conditions; the effect of others may be included on a statistical basis by providing an appropriate 'performance margin' over the datum. It is shown how the treatment of the conditions affects the form and efficiency of the standard.

The margin appropriate to a given incident rate is obtained from the distribution function of the climb performance; this function is, in turn, derived from the distribution functions of the conditions treated statistically, and their effect on climb performance in a given aircraft configuration. The effect of engine failure is included by taking account of the probability of engine failure and the associated loss in performance. To simplify the treatment of changes in aircraft configuration, the flights are divided into stages, such as take-off climb, in which the configuration, except for the incidence of engine failure, is sensibly constant. It is then shown that the required standard (datum plus margin) for any stage may be specified in terms of a single case (*i.e.*, number of operative engines); the case chosen is that found to be dominant in incident causation.

Numerical examples are given of the derivation of standards by the method described.

PREFACE

Since the inception of air transport services it has been recognised that regulations are desirable to ensure the aircraft employed are airworthy. These regulations may cover a wide field of design, construction, maintenance and operational standards, and in the present report attention is confined to one aspect of the design and operational standards. It will be apparent that for an aeroplane to be airworthy it must, in various circumstances, be capable of gaining height. Consequently minimum standards of climb performance form an important part of the design and operation sections of airworthiness regulations.

A climb performance standard was, in fact, envisaged by the original I.C.A.N. convention of 1919. Subsequently, in 1926, the first British civil airworthiness requirements were published and the minimum flight performance specified by stipulating that the aircraft must be able to

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reach a given altitude above take-off in less than three minutes. Since that date developments in aircraft design, and the desire to make regulations more representative of operational practice have resulted in the development, notably by the C.A.A. of America, of the relatively complex schedule of flight performance standards of the present time. During this period however, there has been little change in the method of deciding the form and magnitude of these performance standards. The magnitude has usually been decided by fixing a level which could be met by current aircraft types regarded as satisfactory and the form fixed empirically.

The present (1948) proposed international (I.C.A.O.) performance standards were adopted from the American C.A.A. domestic standards; during recent international discussions it has become apparent that there is a considerable body of opinion which feels that, whilst the C.A.A. requirements of 1945 represented the most complete and up-to-date set of regulations then existing, the climb performance standards do not provide the uniform level of safety between aircraft types and operational conditions which is desirable. It is, for instance, considered by some that certain standards are unnecessarily severe on aircraft with high wing-loadings, because the rate of climb required varies at the square of the stalling speed. However, because of the empirical nature of the requirements it is difficult to debate their validity, particularly as minima; a successful solution is even more unlikely to be obtained by international debate of alternative empirical standards which express national ideas. The major difficulty associated with a debate of overall minimum performance standards arises because the only factual data which bear directly on the problem are the accident rates resulting directly from performance deficiencies. Because of the, fortunately, rare occurrence of such accidents the difficulty of rational discussion of the form and magnitude of requirements to cover all stages of flight, and all types of aircraft, becomes apparent.

To enable profitable discussion of the performance standards the prime need, therefore, is to break the problem down into more manageable elements which, individually, can be founded on more readily available factual evidence. In numerous discussions of the existing standards it has been apparent that there is, in general, agreement on the sort of contingencies for which such requirements are intended to provide—for instance, changes in air temperature from standard, and variation of engine power between engines of the same type. Individually such effects can be inevestigated theoretically and experimentally and factual evidence is available, or can be obtained. Because of this, discussion of such elements is more likely to be profitable. We need then a method of combining the effect of the individual contingencies and if we can obtain that, it will be possible to derive rational standards which will be more firmly based on experimental evidence.

This method of approach and the conception of synthesising requirements from the effects of individual contingencies was originated by P. A. Hufton who, in a paper given a limited circulation in October, 1947, outlined the method and gave numerical examples. From subsequent discussions of this, and later papers by the staff of Aeroplane and Armament Experimental Establishment, it was apparent that the broad conception was generally regarded as sufficiently promising to warrant development and the present paper represents the first results of this work.

Our main objective at this stage is to establish a rational approach to the problem of devising the structure of a requirements code and the individual standards. This is discussed qualitatively in Part I and the mathematical development given in Part II. In Part III numerical examples are given to show the practicability of application and the kind of requirements so derived. This has also enabled specific points to be discussed in more detail. In the numerical examples we have used quantitative values based on the best evidence available to us. They are considered to be representative but it is appreciated that finally a much larger collection of statistical data is desirable. If the method, as such, were accepted there seems no reason, certainly on an international scale, why the quantitative values should not be established to the requisite accuracy.

Although originated, and here presented, specifically in connection with climb performance standards for civil aircraft, the approach appears to have more general applications in connection with design requirements.

We wish here to acknowledge the assistance provided throughout this work by the staff of A.A.E.E. who gave valuable criticism, and assistance in the numerical work.

PART I

QUALITATIVE DISCUSSION

1. *Introduction*.—1.1. The ability to manœuvre may be regarded as an essential feature of an airworthy aeroplane and consequently any code of airworthiness requirements must include manœuvrability standards. For this purpose regulations prescribing performance, strength, stability and control standards will, in general, be required. The level, or severity, of such standards will be limited by considerations of safety and economics of operation, and in practice the level chosen will generally represent a compromise between these conflicting considerations.

1.2. In the past, performance standards have been decided empirically. The purpose of this paper is to discuss methods whereby the degree of empiricism could be reduced, as a first step to the complete rationalisation of such requirements.

1.3. The essential performance characteristics in flight which must be achieved by an airworthy aeroplane are

(a) the ability to change altitude,

(b) the ability, separately and in conjunction with (a), to change direction.

It may be shown that a given rate of turn is related to a corresponding gradient of climb in straight flight by certain aerodynamic parameters; consequently, in conjunction with such parameters, it is possible to define the necessary performance characteristics in terms of a climb performance standard. This is, in fact, the traditional method which has been employed since the inception of civil airworthiness requirements some 20 years ago[1].

1.4. The objective, therefore, is to devise a system whereby the minimum safe climb performance may be specified quantitatively. Since the primary consideration is to achieve a satisfactory standard of safety it will be appropriate to commence the discussion with this subject.

1.5. Certain terms are subsequently used with a particular significance and for convenience in reading they have been collected and defined in Appendix I.

2. *Level of Safety*.—2.1. *General*. The first national regulations on aerial navigation[4], produced in 1911, sought to ensure the safety of persons on the ground*; in their subsequent development they have sought to ensure also the safety of passengers carried in the aircraft. We are, therefore, concerned with risks to passengers and third parties.

2.2. In any code of airworthiness requirements there will be an implied level of safety or accident rate since, as in any human activity, we cannot achieve *absolute* safety. In the past, when performance requirements were framed empirically, this aspect was not explicitly considered and we have no established convention for assessing the actual or relative level of safety.

2.3. *Unit of Risk*.—To discuss the level of safety we must first decide on the unit of risk. The most obvious units are time, distance or flights either separately or in conjunction with the number of passengers giving passenger-hours, etc. For our present purpose the use of ' flight ' as a unit of risk is very convenient and, although not essential to the method, will be used in the subsequent development of the argument. It may be noted that existing proposed international airworthiness standards[3] are implicitly based on flight as a unit of risk, since permissible weight is fixed for each flight, and flying time, distance flown, or number of passengers do not enter as parameters. For comparative purposes, therefore, retention of the existing basis is desirable at the present time, although if the method were adopted, it could subsequently be developed on any alternative basis which can be shown to be more logical and which results in requirements capable of specification with the desired degree of simplicity.

* An earlier attempt at international regulations with a similar objective (Paris, 1910) had failed.

2.4. *Datum Performance.*—It will be apparent that, in given circumstances, when the performance falls below a certain level (which we call the datum performance) an unintentional contact with the earth's surface will occur and conditions predisposing to an accident to passengers or third parties exist. The severity of the accident or indeed whether there is an accident at all cannot be deduced from performance characteristics alone since it depends, for instance, on structural and fire-proofing arrangements, ditching characteristics, etc. From the aspect of flight performance characteristics, therefore, the contribution which can be made to the attainment of a satisfactory overall level of safety will be to ensure that forced landings due to performance deficiencies, which are in themselves undesirable irrespective of the consequences, do not occur more often than some tolerable frequency.

2.5. *Incident Rate.*—We propose, therefore, to assess the ' level of safety ' by the frequency (on a flight basis) with which the performance of aircraft falls below a defined datum performance and this frequency we call the *incident rate.* The relationship between the incident and accident rates will depend on the level at which we fix the datum performance and on the aircraft's structural and other characteristics mentioned above; the conception of datum performance and incident rate does, however, provide a convenient yardstick for comparing relative average levels of safety implicit in codes of airworthiness performance requirements.

2.6. Since the consequences of an incident may depend on numerous features of the aircraft design, the final aim of the rationalisation would be to eliminate such discrepancies by:

(a) regulations to ensure equal protection to passengers in the event of an incident, or

(b) a variation in the tolerable incident rate depending on the risks attendant on an incident.

These aspects are possible developments of the method but beyond the scope of the present report; in the subsequent discussion, therefore, the incident rate is assumed to be independent of aircraft characteristics.

3. *The Structure of a Requirements Code.*—3.1. In framing the requirements our problem is virtually to define the minimum* performance, for any aircraft within the scope of the requirements, throughout any flight; to do this we must account for all conditions such as weather and airframe state which affect the performance. For example, air temperature is a ' condition ' affecting performance and we may

(a) treat this as a statistical variable by specifying the performance at standard temperature and include an allowance in the performance level to cater for its variability, or

(b) treat this as a parameter by specifying the performance at the actual temperature prevailing.

3.2. An important condition meriting special consideration is the aircraft configuration. Alteration of undercarriage position, for instance, will affect performance by a varying amount for different aircraft types. Since, as stated above, our problem is to define the performance required throughout the flight, we must devise a method of defining the performance with undercarriage both up and down and, neglecting other conditions *pro tem.*, we may

(a) specify the performance required in one configuration making allowance for the change and the variability of the change in drag to the other condition, so treating undercarriage drag as a statistical variable, or

(b) frame the requirement in one configuration in terms (*inter alia*) of undercarriage drag so treating it as a parameter, or

(c) define the required performance in each of a series of flight stages so chosen that the undercarriage position in each is constant.

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* Minimum is here used in a statistical sense, the significance of which will be apparent later.

3.3. We may summarise this aspect by saying that we have the choice of treating any condition as a statistical variable or a parameter or so specifying the requirements that the condition for each is constant. The general structure of the requirements code will depend on these choices. Treatment of conditions as statistical variables results in standards which may be specifiied simply, at the expense of the accuracy with which they ' fit ' individual aircraft types and circumstances. Further, if the statistical variable treatment is used to deal with an attribute of aircraft design (*e.g.*, undercarriage drag) which is likely to vary considerably between types, it follows that some aircraft will be favoured and others penalised directly by the manner in which the requirements have been framed, and this is fundamentally undesirable. Treatment of conditions as paramenters is the more attractive, technically, because of the better fit, but administrative convenience will usually limit the number which can be so treated. The introduction of parameters to define all configuration changes is a case in point and such a treatment would be unwieldy with modern aircraft types; consequently (b) above is not an administratively feasible way of dealing with such changes. Treatment of the type (c) above, however, results in a number* of cases each of relatively simple definition and is to be preferred for ease of specification and interpretation; as will be discussed later (section 11), this approach is in some ways approximate but the practical variation in safety level is likely to be small.

3.4. For administrative convenience, therefore, division of the aircraft's flying life into ' flight stages ' based on configuration is necessary and the stages at present[3] used, *i.e.*, take-off climb, en route, approach and landing are suitable, as a major configuration change occurs between each of these with present types.

From this decision the main structure of the requirements code will be fixed and it is then necessary to decide how individual cases within a stage should be handled. Before this aspect can be discussed it is necessary to determine the adequacy of the proposed division of flights and consider the conditions, other than configuration, which affect the performance requirements.

4. *Conditions to be Considered and their Treatment.*—4.1. *Aircraft Characteristics.* In the preceding section we saw that the treatment of aircraft characteristics as statistical variables was fundamentally undesirable. We may reduce the undesirable effects of the statistical variable treatment, and at the same time keep the code within the bounds of administrative feasibility, by grouping aircraft by ' types ' (that is a series of aircraft built to the same design). The design attribute for the type can then be treated as a parameter and the changes of this attribute between individual aircraft of a type as a statistical variable.

Variation in the weight, drag and power all affect performance. Weight is a controllable condition since in operations it may be adjusted for each flight to ensure that the aircraft meets the requirements. Therefore, we use the estimated weight as a parameter and treat the difference between the estimated and actual weights (error) as a stastistical variable.

The standard drag and power for a type of aircraft in a given configuration will be treated as parameters. Since the basic drag varies between individual aircraft of a type and the basic drag of a given aircraft varies due to inaccuracies in setting flaps, cowl gills, etc., these changes from standard will be treated as statistical variables. Similarly, variations in power will occur due to changes in basic power between individual engines of a type, errors in setting the controls and instrument errors, all of which must also be treated as statistical variables.

4.2. *Atmospheric.*—The atmospheric temperature, pressure, moisture, humidity and large-scale turbulence may all affect an aircraft's performance. The method of accounting for the effects of atmospheric temperature was considered by a special committee of I.C.A.O. in 1947 and the opinion of most delegations was that temperature should be treated as a parameter[2].

For flight cases not in the proximity of an aerodrome it was, however, admitted that there were considerable meteorological difficulties in achieving this at the present time. In the present

* The relative complexity of modern airworthiness codes, as compared with those of 20 years ago, is primarily due, of course, to the increase in the number of configuration changes possible during a flight.

paper, therefore, temperature will be treated as a parameter* for flight cases in the proximity of an aerodrome (*e.g.*, take-off) but otherwise as a statistical variable.

For take-off and landing cases, a reasonable approximation to the actual pressure may be obtained by taking the pressure at a height in the International Standard Atmosphere equal to the actual geographical height; thus by introducing height as a parameter it is only necessary to treat pressure changes at constant geographical height as a statistical variable.

There is little experimental evidence on the effects of humidity and moisture on engine power[2] and pending the further studies proposed in that reference no action on these variables is proposed†; the requirements resulting from this treatment may not, therefore, be applicable to conditions of high humidity for certain power plants.

There is no regular procedure extent for measuring or predicting turbulence to enable this condition to be treated as a parameter and we must, therefore, treat this as a statistical variable.

It may be noted that certain combinations of the atmospheric conditions are conducive to airframe and power plant icing. There is, however, at the present time, inadequate data to assess frequency of occurrence and severity of such conditions and this item is reserved for further study.

4.3.—*Pilotage.*—In considering the variability introduced by the pilot it must be remembered that we are here concerned with keeping the frequency of incidents arising from inadequate performance within an acceptable figure, and that we are not legislating for gross errors of pilotage which must be dealt with in other ways. The types of variation considered admissible in the present connection are minor errors in setting engine and flap controls and in keeping to the intended forward speed and these must be treated as statistical variables.

5. *Performance Margins.*—5.1. In the preceding discussion (section 2.5), we have seen that to provide a given level of safety we must so specify our requirements that the frequency with which the actual performance falls below the datum performance does not exceed the tolerable incident rate. We are therefore concerned with determining the margin of performance above the datum necessary to achieve this, and the specified requirement will be the sum of the datum and margin performance.

5.2. From our discussion of the conditions to be considered and their treatment (section 4) it was seen that some conditions should be treated as parameters (*e.g.*, standard drag for a type) and others as statistical variables (*e.g.*, control setting errors). The performance margins are needed to cater for those conditions treated as statistical variables and the magnitude of the margins will depend on the combined effect on performance of the variability of those conditions. The method by which these margins are determined in the general case is given in Part II of this paper but we may here consider, by way of example, a very simple case where there is only one statistical variable.

An increase in drag of 1 per cent will cause a reduction in gradient of climb of (D/W) per cent where D is the total drag and W the weight. Now for aircraft built to the same design, variations in drag will occur between individual aircraft. If we know the distribution of drag about the mean for the family of aircraft then we can determine that there is a certain probability, 1 in 10^x, that the drag will be more than y per cent above the mean. Since we are considering this variable only it follows that there is a probability of 1 in 10^x that the gradient of climb will be reduced, below that at the mean drag, by more than (yD/W) per cent. If 1 in 10^x is our tolerable incident rate then a performance margin of (yD/W) per cent above the datum will be needed in a requirement framed in terms of mean drag (a parameter).

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* In the parlance of I.C.A.O. we adopt temperature accountability.

† A programme of tests to determine the effects of humidity is under consideration.

6. *Required Division of Flights.*—6.1. From the consideration of configuration changes (section 3.4) it was concluded that the flight should be divided into four main stages:

(i) Take-off climb.

(ii) En route.

(iii) Approach.

(iv) Landing climb.

For stage (iv) it will only be in the event of an aircraft being baulked when attempting to land, that a climb performance standard will be necessary.

Since it is the intention to associate a performance standard with each of these stages it follows that ideally each of the following should be constant in any stage.

(a) Datum performance.

(b) Parameters.

(c) Statistical variables.

(d) Effect of parameters and statistical variables on performance.

Considering these conditions in turn, for a given condition of engine operation (*e.g.*, all engines operative, one engine inoperative, etc.)

(a) The datum performance may, for airworthiness requirements, be taken as constant in the stages chosen.

(b) Some or all of the parameters (*e.g.*, standard drag) change between the four stages above, but are sensibly constant during each stage, with the exception of weight which varies continuously during the en-route stage.

(c) The statistical variables associated with the aircraft characteristics will be sensibly constant during each of the four stages but the variability of atmospheric conditions is fundamentally on a space-time basis. For the short period stages (take-off, approach, landing) we may reasonably assume that atmospheric conditions remain constant during a stage and in these cases treat the variability on a flight-stage basis. In the en-route case both time and location vary appreciably and the treatment required needs further consideration (section 6.2).

(d) The effect of the variables on performance can be shown (Part II, equation (31)) to be primarily a function of drag/weight ratio or wing loading and aircraft dimensions and the effect of a given variable will, therefore, be sensibly constant during each of the stages except for the en-route case where weight variation may cause a known progressive change.

6.2. Since in the en-route stage variation of weight and atmospheric conditions may occur, the treatment is not so simple as in the short period stages, and is further complicated when we consider the effects of engine failures. So far an accurate method of treatment capable of practical use has not been obtained and, in order not to delay the publication of the general method, upper and lower limit requirements for this stage will be derived pending further investigation and their derivation is discussed later (section 8.3).

6.3. Therefore, we conclude, with reservations in respect of en-route, that the stages dictated by configuration changes will be adequate for their intended purpose. Having discussed the reasons, largely administrative, for choosing a particular structure for the requirements code, we may consider how the individual standards, or requirements, may be derived. In the above discussion we have, however, considered a constant condition of engine operation and the effect of engine failure must first be examined.

7. *Engine Failure.*—7.1. *Assessment.*—The incidence of engine failure may be expected to depend on the running time at various powers, the number of flights (which defines the frequency of certain operations such as refuelling) and the frequency of inspections and other operations on the engine installation. If we are to avoid excessive complication in the specification of the final requirements it will be apparent that a simple basis must be used for the definition of engine failure probability; the practical alternatives appear to be by flights or flying time. It is not apparent from any fundamental considerations which is preferable and it seems possible that the probability may be a function of both. For instance, reliability may be expected to be influenced by the time spent at maximum power which will be related to the number of flights and not total flying time. The available data from five operators are given in Appendix IX but these are too limited in scope for the point to be resolved.

The present international (and national) requirements, however, do not vary with flight endurance and consequently the probability of engine failure is implicitly taken on a flight basis. To avoid more changes than are essential, and particularly one which might be unacceptable administratively, we propose, in the present paper, to continue on the existing lines and assess engine reliability on a flight basis.

A detailed analysis of engine failures appears to be an investigation which might well be undertaken by I.C.A.O.

7.2. *Effect on Performance.*—It will be shown later (Part II, section 9.1) that the performance with all engines operative is related to that with any number of engines inoperative by simple parameters (*e.g.*, aircraft weight and drag); the change in performance is thus a manageable function.

8. *The Stage Requirements.*—8.1. By the method outlined in section 5 we are able, for a particular *case* (stage and condition of engine operation), to determine the performance margin appropriate to a given incident rate. In the short period stages where aircraft weight and drag may be assumed constant we have, as noted in the previous section, an easily manageable performance relationship between the various cases. Further, there is also a known relationship between the probabilities of various numbers of engines being inoperative. Consequently as will be shown later (Part II, section 9) a stage incident probability can be defined by a single flight case.

8.2. The baulked-landing stage is different from the remaining stages, because it will not be entered on every flight; in determining the performance margin for this stage the probability of entering the stage must, therefore, be included.

8.3. For the en-route stage, we are faced with greater difficulties than in the short period stages due to the weight variation. To derive the upper and lower limits already mentioned, we propose to assume that all en-route engine failures occur during a short period in the stage. This enables us to use the simple treatment discussed above and if we associate the requirement so derived with the starting weight we have an upper limit because this postulates all engine failures at the start of the flight; similarly the lower limit will be given by associating the requirement with the landing weight which will be optimistic.

Before considering the complete flight we will at this juncture, consider how the datum performance and incident rate may be decided.

9. *Fixing the Datum Performance.*—Strictly from the airworthiness (as opposed to operational) aspect, level flight with the ability to make moderate turns would represent the datum performance, since when the performance falls below this level a forced landing will normally*

* From the current (I.C.A.O.) airworthiness cruising altitude of 5,000 feet, there is a choice of landing point within a radius, inversely proportional to gradient of descent, and being up to roughly 100 miles for 1 per cent. If this radius included a suitable aerodrome an accident may be avoided. This, probably small, effect, however, is ignored, the result being pessimistic.

follow. It is felt, however, that such an interpretation of the purpose of airworthiness requirements is unrealistic and, whilst the ability to maintain level flight with moderate turns appears a reasonable datum for the airworthiness en-route case, small positive gradients are considered desirable for the flight conditions in the proximity of the ground. It is, therefore, proposed that the datum performance for each stage be fixed *a priori* and in the numerical application of the method (Part III) values will be used which appear to be of the right order.

10. *The Tolerable Incident Rate.*—10.1. The order of the currently tolerable incident rate may be determined by considering the broad policy of existing requirements codes.

10.2. The first feature of interest in the present proposed code of international standards[3] is that whilst single-engined aircraft are excluded from Transport Category A, twin-engined aircraft, as such, are not. Now if Π represents the probability of an effectively complete engine failure in a given stage, the probability of both engines of a twin being inoperative at the same time is Π^2 if the failures are assumed independent. In the event of total engine failure the aircraft will adopt an angle of descent of at least $(D/L)_{\min}$. Thus within our assumption of independence the existing code allows a performance below our datum to occur on a probability of Π^2. The implied incident rate therefore, is greater than Π^2 and may be less than Π.

10.3. The second relevant feature of the present airworthiness standards is that the flight cases, except baulked landing, are specified for the condition of one engine inoperative; for en-route, an ' all engines operative ' case is also given. The baulked-landing case (all engines operative) does not contribute to the present discussion because to determine its implications we would have to assume a numerical value for the probability of an aircraft being baulked. In the remaining stages we have a one engine inoperative case in each. It will subsequently be shown that such a case is dominant in incident causation over a restricted range of incident rate and that for values much outside this range the incident contribution from the case is insignificant. Consideration of Figs. 1 to 4 (the details and derivation of which are discussed later) shows this in particular cases, but the general inference is not affected within the practical range of the numerical values. If the existing requirements code is realistic in its broad conception, one would expect the cases specified to be those in which performance deficiencies were most likely to cause incidents. In any particular case the limits implied can be deduced; in the general case, for two- and four-engined aircraft, with which we are most concerned, investigation has suggested that, for practical purposes, the lower limit varies from about $\Pi^{3/4}$ to $\Pi^{7/8}$ and the upper limit from about $\Pi^{7/4}$ to Π^2. In the existing requirements a case with all engines operative en-route is specified in addition to that with one engine inoperative; from our present discussion only one case per stage is necessary for airworthiness purposes, and we may also note that the all-engine operative case will only be dominant for larger values of the incident rate.

10.4. A detailed examination of the third point which assists us in our choice of incident rate must be deferred until the numerical application is made, but Fig. 6, disregarding the numerical values for the present, will indicate qualitatively this aspect. In a given condition of engine operation there are appreciable ' steps ' in the curve of performance required against incident rate (level of safety), as shown in Fig. 6, for aircraft with a small number of engines. The physical explanation for this is that the loss in performance when an engine is cut is large compared with the random variation of performance due to the contingencies we have already discussed*. Consequently as the dominant case changes with reduction in incident rate, there will be a period when increases in performance have little effect on the incident rate†. This suggests that in any particular case there may be an economic limit to the stage incident rate. This will be closely associated with the upper limit from the ' dominant case ' considerations already discussed. In fact, this step (subsequent to the one engine inoperative case being dominant) occurs at an incident probability of $\frac{1}{2}n(n-1)\Pi^2$ where n is the total number of engines. For

* In other words there is not much overlap in performance between a ' good ' aircraft favourably situated with $n - 1$ engines operative and a ' bad ' aircraft unfavourably situated with n engines operative, provided n is small.

† This is further discussed in Part III.

given values of n and Π, we may express this in the form Π^x for comparison with the ranges obtained above. For a twin, of course, this limit will be Π^2 irrespective of the value of Π; for a four-engined aircraft the limit would, for instance, be $\Pi^{1 \cdot 8}$ for $\Pi = 10^{-4}$.

10.5. From these three aspects, therefore, we have ways of determining the range of the currently tolerable incident rate, in terms of Π^x, and the implications of changes. We may call Π^x the 'relative incident rate'; it has no absolute significance but is a convenient fiction to bridge the gap between our past experience with empirical, and proposed use of rational, standards. Having once bridged this gap we would, of course, convert this relative incident rate into an absolute (numerical) value by substituting a numerical value for Π based on an adequate collection of existing data; thereafter we should use this constant numerical value throughout, irrespective of the Π value, until the time was opportune for further revision of the incident rate.

11. *Consideration of the Complete Flight.*—11.1. We have already discussed how a standard may be derived for a given flight stage to ensure that the stage incident rate does not exceed some acceptable value; we are, however, concerned with the probability of an incident per flight.

The accurate compounding of the stage incident probabilities to obtain the flight incident probability, although technically feasible, is not a very practicable proposition because, for a given flight incident rate, the performance required in the various stages will be related; no readily usable relationships exist however between aircraft's available performance in the various stages, as already discussed.

11.2. Now if the performance at each stage were statistically independent and if the aircraft just complied with the requirement at each stage, the overall incident probability would be very nearly the sum of the stage incident probabilities when the probabilities and number of stages are small. Thus with x stages each with the same stage incident rate (Q), the flight incident rate would be very nearly xQ.

On the other hand, if the performance in each stage were completely dependent statistically and/or the aeroplane had appreciable reserve performance in some stages, the flight incident rate would be very nearly equal to Q.

11.3. The real situation lies between these simple extremes because:—

(a) some of the variables will cause dependence between stages* whereas others can result in independence,† and

(b) more important, the weight for a given aircraft-flight will be determined by the 'critical' flight stage and in the remaining stages a surplus performance over that required will be available. In practice, this surplus can reduce the incident rate in the non-critical case appreciably.

Investigation of these conditions suggests that the overall incident rate may, in practice, be of the order of twice the stage incident rate and a numerical example is given in Part III. Now with the very small tolerable incident rates with which we are concerned (in the order of one in a million, to anticipate subsequent discussion) an uncertainty in the order of 2 is not of great importance and the formulation of requirements independently for the different stages is a justifiable approximation.

12. *Summary.*—To summarise the qualitative discussion the proposed procedure for the derivation of airworthiness performance standards is as follows:—

(a) From the aspects of practicability, technical accuracy and administrative simplicity, relevant conditions are treated as parameters or statistical variables or the structure of the code so arranged to retain the condition constant in stages.

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* If the engine power is basically low for a given aircraft then this will reduce the performance in all cases.

† For instance, mis-setting of controls.

(b) Datum performance levels for each flight stage (decided from (a)) are fixed *a priori*.

(c) A tolerable incident rate is decided to implement broad policy considerations in relation to the desired level of safety.

(d) Performance margins are calculated, from a knowledge of the variability of the statistical variables and their combined effect on performance, to ensure that the performance does not fall below the datum more frequently than the tolerable incident rate.

(e) Combination of the datum and margin performance gives the requirement for a given case to be specified in terms of those conditions treated as parameters.

(f) From knowledge of engine failure rates and the relation between performance in the various cases, a stage incident probability and hence the requirement for a stage can be based on a single flight case.

(g) Administrative and not technical feasibility limits the requirements code to the specification of independent standards for the separate stages and this will result in a variation in flight incident rate in practice, within limits considered to be acceptable.

PART II

MATHEMATICAL ANALYSIS

1. *Introduction*.—1.1. In this Part the mathematical background of the procedure discussed qualitatively in Part I is developed. A general analytical expression for the climb gradient for aircraft with reciprocating* engines is first developed. This follows well-known lines with the exception of the section dealing with the effect of atmospheric gustiness on performance. A suitable analytical expression for this effect has not, to our knowledge, been previously published†, and a theoretical analysis is made in section 3.

1.2. Using the general expression for climb gradient it is then shown how a performance standard may be derived for a given flight case (stage and condition of engine operation) for a prescribed case incident probability and datum performance. The relation between the case and stage incident probabilities is then derived. The final equations (43) and (44) are such as might be written instinctively, but because of their importance a detailed derivation is given.

1.3. Finally, having established the basic equations their practical application to the derivation of performance standards is considered.

1.4. A list of symbols and definitions of terms is given in Appendix I.

2. *Expression for the Climb Gradient in Calm Air*.—2.1. *Performance Equation*.—The gradient of climb γ of an aircraft is given in terms of the thrust T, drag D and weight W by the basic performance equation

$$\gamma = \frac{T - D}{W}. \qquad \qquad (1)$$

In this equation T and D are functions of both aircraft characteristics and atmospheric conditions and we shall proceed to express γ in terms of these variables. In the work that follows, a ' bar ' over the symbol denotes the ' standard ' value of the variables, *i.e.*, its value at standard

* For convenience in presentation, the changes necessary with turbo-jet engines are dealt with in Appendix VIII.

† Lanchester discussed this problem (*Aerodonetics*, 1908) in connection with soaring flight of birds.

atmospheric conditions or, where applicable, for the aircraft type as distinct from the individual aircraft concerned. In the case of weight, $\bar{W}$ denotes the estimated weight, based on the tare weight for the particular aircraft plus the estimated load.

2.2. *The Thrust Term.*—Consider first the thrust term

$$\frac{T}{W} = \frac{\eta P \sqrt{\sigma}}{W V_i} ,$$

where P is the engine power,

η the propeller efficiency,

σ the relative air density,

and V_i the equivalent air speed.

In terms of the standard values of the variables,

$$\frac{T}{W} = \frac{\bar{T}}{\bar{W}} \cdot \frac{\eta}{\bar{\eta}} \cdot \frac{P}{\bar{P}} \left(\frac{\sigma}{\bar{\sigma}}\right)^{1/2} \left(\frac{\bar{W}}{W}\right) \cdot \left(\frac{\bar{V}_i}{V_i}\right).$$

We will assume that $\eta \propto C_p^{\,a} J^b$ (*see* Appendix III), where $C_p \propto P/\sigma N^3$ is the propeller power coefficient, $J \propto V_i/\sqrt{(\sigma)} N$ is the propeller advance ratio and N denotes the engine speed.

Then

$$\frac{\eta P}{\bar{\eta}\bar{P}} = \left(\frac{P}{\bar{P}}\right)^{1+a} \left(\frac{\bar{\sigma}}{\sigma}\right)^{a+b/2} \left(\frac{V_i}{\bar{V}_i}\right)^{b} \left(\frac{\bar{N}}{N}\right)^{b+3a} .$$

We will also assume* that $P \propto p^c \theta^d N^f B_{\bar{n}}^{\,g}$, where p, θ denote the atmospheric pressure and temperature and B is the manifold pressure.

Let $P_{\bar{n}}$ denote the value of P at $\bar{\theta}$, $\bar{p}$, $\bar{N}$, $\bar{B}$,
and $B_{\bar{n}}$ the value of B at $\bar{\theta}$, $\bar{p}$, $\bar{N}$.

$P_{\bar{n}}$ and $B_{\bar{n}}$ are then the standard values of power and manifold pressure for the particular aircraft; they will in general differ from the standard values $\bar{P}$ and $\bar{B}$ for the aircraft *type*.

Then

$$\frac{P}{\bar{P}} = \frac{P_{\bar{n}}}{\bar{P}} \left(\frac{p}{\bar{p}}\right)^{c} \left(\frac{\theta}{\bar{\theta}}\right)^{d} \left(\frac{N}{\bar{N}}\right)^{f} \left(\frac{B_{\bar{n}}}{\bar{B}}\right)^{g}.$$

Hence, writing $\dfrac{\bar{T}}{\bar{W}} = \dfrac{\bar{D}}{\bar{W}} + \bar{\gamma}$,

$$\frac{T}{W} = \left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right) \left(\frac{P_{\bar{n}}}{\bar{P}}\right)^{1+a} \left(\frac{p}{\bar{p}}\right)^{c(1+a)} \left(\frac{\theta}{\bar{\theta}}\right)^{d(1+a)} \left(\frac{N}{\bar{N}}\right)^{f(1+a)-b-3a} \left(\frac{B_{\bar{n}}}{\bar{B}}\right)^{g(1+a)}$$

$$\times \left(\frac{\sigma}{\bar{\sigma}}\right)^{1/2-a-b/2} \left(\frac{V_i}{\bar{V}_i}\right)^{b-1} \left(\frac{W}{\bar{W}}\right)^{-1}$$

$$= \left[\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right] \left(\frac{P_{\bar{n}}}{\bar{P}}\right)^{1+a} \left(\frac{V_i}{\bar{V}_i}\right)^{b-1} \left(\frac{B_{\bar{n}}}{\bar{B}}\right)^{g(1+a)} \left(\frac{p}{\bar{p}}\right)^{h} \left(\frac{\theta}{\bar{\theta}}\right)^{k} \left(\frac{N}{\bar{N}}\right)^{m} \left(\frac{W}{\bar{W}}\right)^{-1} \quad .. \quad (2)$$

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* This form differs from that normally assumed in this country but is more convenient for the present purpose. For discussion, *see* Appendix II.

where

$$h = c + ca - a + \tfrac{1}{2} - \frac{b}{2} \, ,$$

$$k = d + da + a - \tfrac{1}{2} + \frac{b}{2} \, ,$$

$$m = f(1 + a) - b - 3a \, .$$

2.3. *The drag term.*—Consider now the drag term in equation (1)

$$\frac{D}{W} = C_{DZ} \; . \; \frac{\rho_0 S}{2} \frac{V_i^2}{W} + \frac{1}{\pi A} \; \frac{2}{\rho_0 S} \; \frac{W}{V_i^2} \, ,$$

where

C_{DZ} is the profile-drag coefficient,

ρ_0 is the sea-level atmospheric density,

A is the equivalent aspect ratio,

and S is the wing area.

In terms of the lift coefficient C_L and the standard values,

$$\frac{D}{W} = C_{DZ} \frac{1}{\bar{C}_L} \left(\frac{V_i}{\bar{V}_i}\right)^2 \left(\frac{\bar{W}}{W}\right) + \frac{1}{\pi A} \; \bar{C}_L \left(\frac{\bar{V}_i}{V_i}\right)^2 \left(\frac{W}{\bar{W}}\right) .$$

Denote D at $\bar{V}_i$, $\bar{W}$ by $D_{\bar{m}}$; this represents the standard drag of the individual aircraft and will in general differ from $\bar{D}$, its value for the type.

Writing $\bar{V}_i^2 = r\bar{V}_{is}^2$, where $\bar{V}_{is}$ is the stalling speed

$$\frac{D_{\bar{m}}}{\bar{W}} = C_{DZ} \frac{r}{\bar{C}_{L\,\max}} + \frac{1}{\pi A} \frac{\bar{C}_{L\,\max}}{r} \, .$$

Hence

$$\frac{D}{W} = \left[\left(\frac{\bar{D}}{\bar{W}}\right)\left(\frac{D_{\bar{m}}}{\bar{D}}\right) - \frac{\bar{C}_{L\,\max}}{\pi A r}\right] \left(\frac{V_i}{\bar{V}_i}\right)^2 \left(\frac{\bar{W}}{W}\right) + \frac{\bar{C}_{L\,\max}}{\pi A r} \left(\frac{\bar{V}_i}{V_i}\right)^2 \left(\frac{W}{\bar{W}}\right) . \quad \ldots \quad \ldots \quad (3)$$

2.4. *The climb gradient.*—Thus, from equations (2) and (3)

$$\gamma = \left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right) \left[\left(\frac{P_{\bar{n}}}{\bar{P}}\right)^{1+a} \left(\frac{V_i}{\bar{V}_i}\right)^{b-1} \left(\frac{B_{\bar{n}}}{\bar{B}}\right)^{g(1+a)} \left(\frac{p}{\bar{p}}\right)^{h} \left(\frac{\theta}{\bar{\theta}}\right)^{k} \left(\frac{N}{\bar{N}}\right)^{m} \left(\frac{W}{\bar{W}}\right)^{-1}\right]$$

$$- \left(\frac{\bar{D}}{\bar{W}}\right) \left[\left(\frac{D_{\bar{m}}}{\bar{D}}\right) \left(\frac{V_i}{\bar{V}_i}\right)^2 \left(\frac{W}{\bar{W}}\right)^{-1}\right]$$

$$- \left(\frac{\bar{C}_{L\,\max}}{\pi A r}\right) \left[\left(\frac{V_i}{\bar{V}_i}\right)^{-2} \left(\frac{W}{\bar{W}}\right) - \left(\frac{V_i}{\bar{V}_i}\right)^2 \left(\frac{W}{\bar{W}}\right)^{-1}\right] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

Equation (4) expresses the climb gradient γ in terms of

(a) the parameters for the aircraft type $\dfrac{\bar{D}}{\bar{W}}$, $\bar{\gamma}$, A, r, $\bar{C}_{L\,\max}$,

(b) the characteristics of the individual aircraft and pilotage $\dfrac{P_{\bar{n}}}{\bar{P}}$, $\dfrac{D_{\bar{m}}}{\bar{D}}$, $\dfrac{W}{\bar{W}}$, $\dfrac{V_i}{\bar{V}_i}$, $\dfrac{B_{\bar{n}}}{\bar{B}}$, $\dfrac{N}{\bar{N}}$,

(c) the atmospheric conditions $\dfrac{p}{\bar{p}}$, $\dfrac{\theta}{\bar{\theta}}$.

3. *The Contribution of Gustiness to Climb Gradient.*—When the aircraft is flying through gusty air, the angle of climb will be affected through changes in lift and induced drag. This section derives an additional term to be added to the expression of equation (4) when gusty air conditions are included.

3.1. *The Equations of Motion.*—The aircraft is flying at an angle of climb γ to the horizontal with speed V and we shall suppose that the upward component of the velocity of the air is εV, where ε is small. Also, suppose

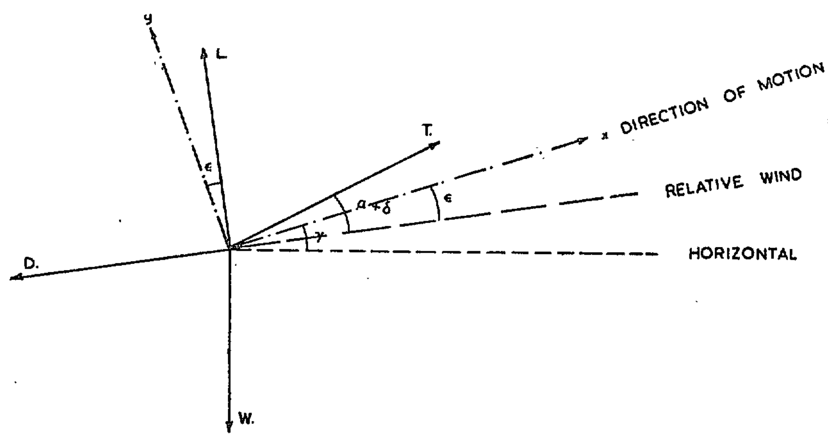
α incidence measured from the no-lift line,

δ inclination of the thrust to the no-lift line,

ng normal acceleration of the aircraft.

The equations of motion will be referred to axes x and y along and at right-angles to the direction of motion; note that these axes rotate with angular velocity of magnitude $d\gamma/dt$. We shall neglect second powers of α, δ, ε, γ, etc., so that the sines of these angles will be equated to the angles themselves and the cosines equated to unity.

The forces on the aircraft are then as represented in the diagram.



The equation of motion along the x-axis is

$$T + L\varepsilon - D - W\gamma = \frac{W}{g}\frac{dV}{dt} \qquad \text{(5)}$$

and along the y-axis,

$$L + D\varepsilon + T(\alpha + \delta - \varepsilon) - W = Wn \qquad \text{(6)}$$

Also, since the axes are rotating with speed $d\gamma/dt$,

$$ng = V\frac{d\gamma}{dt} \qquad \text{(7)}$$

We can simplify these equations further by using the fact that T and D are small (less than $0 \cdot 1$, en route) compared with W or L and so neglect the thrust and drag term in equation (6). Rewriting equation (6) using (7), we get

$$L - W = \frac{WV}{g} \frac{d\gamma}{dt} . \qquad (8)$$

3.2. *The Gust Pattern.*—The gust pattern through which the aircraft flies determines the way in which the vertical component εV and the horizontal component ηV of the velocity of the air as encountered by the aircraft vary with time. A general form which has zero mean is

$$\left.\begin{aligned} \varepsilon &= \Sigma\, \varepsilon_s \cos s\lambda t \\ \eta &= \Sigma\, \eta_s \cos s\mu t \end{aligned}\right\} . \qquad (9)$$

A special case of this form is, of course, a train of sine formation, *e.g.*,

$$\varepsilon = \varepsilon_1 \cos \lambda t . \qquad (10)$$

To reduce the algebra, the solution will be worked out for the single term given by equation (10); the result for the general expression (9) can then be deduced from this particular solution by addition, as the equations are linear in ε. The effect of the η component will be discussed in section 3.4.

3.3. *The Flying Technique.*—When flying through gusty air, the pilot will endeavour to keep the motion of the aircraft as steady as possible. He cannot readily fly at constant air speed, as he can only control this indirectly, but he can keep his speed on the average to some assigned figure, at the same time keeping the aircraft steady by maintaining as far as possible a constant attitude in space. The flying technique chosen is, therefore, that the aircraft is controlled by the pilot so that

(i) the air speed is on the average kept to some assigned value,

(ii) the attitude is held constant at a value appropriate to the air speed of (i).

These two assumptions, expressed in mathematical form give us

$$\frac{\lambda}{2\pi} \int_0^{2\pi/\lambda} V_a \, dt \doteq V_0, \text{ say} \qquad (11)$$

$$\left.\begin{aligned} \alpha + \gamma - \varepsilon &= \text{constant} \\ &= \bar{\alpha} + \bar{\gamma} \end{aligned}\right\} \qquad (12)$$

where $\bar{\alpha}$ and $\bar{\gamma}$ are the mean values of α and γ, and V_a denotes the true air speed and is given by

$$V = V_a + \eta V .$$

3.4. *A Restatement of the Equations.*—The equations (5), (8) and (12) define the motion of the aircraft, where ε is given by (10) and V is subject to (11). Of the variables in these three equations T, L, and D can be expressed as functions of α and V_a, and hence of α and V. The equations, therefore, can be put in terms of the four variables α, γ, V and t and we shall proceed to do this.

For thrust, from equation (2), $T \propto V_a^{b-1}$ and b has been taken as $0 \cdot 4$ so that we may write

$$\frac{T}{W} = \tau \left(\frac{V_a}{V_0}\right)^{-0 \cdot 6} . \qquad (13)$$

For lift, since the motion is unsteady, there will be a lag (the Wagner effect) between the lift and the corresponding incidence and the simple lift-incidence relation is no longer true. If α_L is the incidence appropriate to L in steady flight, we may put

$$\frac{L}{W} = \left(\frac{V_a}{V_0}\right)^2 \cdot \frac{\alpha_L}{\alpha_S}, \qquad (14)$$

where α_S is the incidence for steady flight in calm air at a speed V_0. α_L will depend on α and the conditions of flight and will be discussed later.

For drag, put

$$\frac{D}{W} = \frac{D_0}{W} + \frac{C_L}{\pi A} \cdot \frac{L}{W}$$

$$= k_1 \left(\frac{V_a}{V_0}\right)^2 + k_2 \left(\frac{V_a}{V_0}\right)^2 \frac{\alpha_L^{\,2}}{\alpha_S}. \qquad (15)$$

Note that, taking the slope of the lift curve as $2\pi A/(2 + A)$, where A is the aspect ratio,

$$k_2 = \frac{2\pi A}{2 + A} \Big/ \pi A = \frac{2}{2 + A}. \qquad (16)$$

We are here assuming that the unsteady induced drag is related to unsteady lift by the usual expression.

The equations (5), (8) and (12) can now be rewritten as

$$\left.\begin{aligned} \tau \left(\frac{V_a}{V_0}\right)^{-0\cdot 6} + \left(\frac{V_a}{V_0}\right)^2 \frac{\alpha_L \varepsilon}{\alpha_S} - k_1 \tau \left(\frac{V_a}{V_0}\right)^2 - k_2 \left(\frac{V_a}{V_0}\right)^2 \frac{\alpha_L^{\,2}}{\alpha_S} - \gamma &= \frac{1}{g}\frac{dV}{dt} \\ \left(\frac{V_a}{V_0}\right)^2 \frac{\alpha_L}{\alpha_S} - 1 &= \frac{V}{g}\frac{d\lambda}{dt} \\ \alpha + \gamma - \varepsilon &= \bar{\alpha} + \bar{\gamma} \end{aligned}\right\} \qquad (17)$$

It can be verified that the Greek letters are all of small order (about 10^{-1}), k_1 about unity and k_2 about $\frac{1}{5}$.

From the first equation (17), dV/dt is of order $g\alpha$, which is small compared with V; since the applied disturbances ε and η are oscillatory, the solution of V will be oscillatory about the mean V_0 and we can put

$$V = V_0(1 + v)$$

where v is of order $g\alpha/\lambda V_0$ and is small, provided λ is not less than about $0\cdot 1$.

It follows that

$$V_a = V_0(1 - \eta + v)$$

and on substitution in equation (17), and expanding the V_a terms in terms of $\eta - v$, the terms involving η and v will all be of second order and can be neglected to the first order of approximation. To this approximation, therefore, we can write $V_a = V_0$ and the horizontal component of the gusts is of no significance. The first equation then becomes

$$\tau + \frac{\alpha_L \varepsilon}{\alpha_S} - k_1 \tau - k_2 \frac{\alpha_L^{\,2}}{\alpha_S} - \gamma = \frac{1}{g}\frac{dV}{dt} \quad ..$$

For corresponding flight in still air, we have

$$\tau - k_1\tau - k_2\,\alpha_S - \gamma_S = 0$$

which on subtraction gives

$$\left.\begin{aligned} \frac{\alpha_L \varepsilon}{\alpha_S} - k_2\frac{\alpha_L^2 - \alpha_S^2}{\alpha_S} - (\gamma - \gamma_S) &= \frac{1}{g}\frac{dV}{dt} \\ \text{The other equations are} \qquad \frac{\alpha_L}{\alpha_S} - 1 &= \frac{V_0}{g}\frac{d\gamma}{dt} \\ \alpha + \gamma - \varepsilon &= \bar{\alpha} + \bar{\gamma} \end{aligned}\right\} \qquad (18)$$

3.5. *Solution of the Equations.*—For the gust form $\varepsilon = \varepsilon_1 \cos \lambda t$, the solution of the equations (18) may be taken in the form

$$\left.\begin{aligned} \alpha - \bar{\alpha} &= \alpha_1' \cos \lambda t - \alpha_1'' \sin \lambda t \\ \gamma - \bar{\gamma} &= \gamma_1' \cos \lambda t - \gamma_1'' \sin \lambda t \end{aligned}\right\} \qquad (19)$$

This solution describes the motion after the aircraft has settled down in the gust train. The complete solution includes an exponential term $e^{-gt/V_0\alpha S}$ which relates to the initial conditions and becomes unimportant as t departs from its initial value.

Substituting for α and γ from equation (19) in the last equation (18), we get,

$$\alpha_1' \cos \lambda t - \alpha_1'' \sin \lambda t + \gamma_1' \cos \lambda t - \gamma_1'' \sin \lambda t = \varepsilon_1 \cos \lambda t\,,$$

which, on equating coefficients, gives

$$\left.\begin{aligned} \alpha_1' + \gamma_1' &= \varepsilon_1 \\ \alpha_1'' + \gamma_1'' &= 0 \end{aligned}\right\} \qquad (20)$$

It is shown in Appendix VI that the relation between L and α, *i.e.*, the relation between α_L and α_1, for a rigid aircraft in this type of motion is given by

$$\alpha_L - \bar{\alpha} = C' \cos \lambda t - D' \sin \lambda t \qquad (21)$$

where

$$\left.\begin{aligned} C' &= \tfrac{1}{4}\varepsilon_1\{A'(1 + 3\cos\omega) - 3B'\sin\omega\} - \gamma_1'A' - \gamma_1''(B - \tfrac{1}{4}\omega) \\ D' &= -\tfrac{1}{4}\varepsilon_1\{3A'\sin\omega + B'(1 + 3\cos\omega)\} + \gamma_1'(B - \tfrac{1}{4}\omega) - \gamma_1''A' \end{aligned}\right\} \qquad (22)$$

and $\omega = c\lambda/V_0$ and A' and B' are tabulated against ω in the Appendix VI.

Substituting from equations (19) and (21) in the second equation (18) gives

$$\bar{\alpha} + C' \cos \lambda t - D' \sin \lambda t - \alpha_S = -\frac{1}{\mu}(\lambda\gamma_1' \sin \lambda t + \lambda\gamma_1'' \cos \lambda t) \qquad (23)$$

writing $\mu = \dfrac{g}{V_0\alpha_S}$; equating coefficients, we get

$$\left.\begin{aligned} C' &= -\frac{\lambda}{\mu}\gamma_1'' \\ D' &= \frac{\lambda}{\mu}\gamma_1' \\ \bar{\alpha} &= \alpha_S \end{aligned}\right\} \qquad (24)$$

Combining equations (22) and (24) gives two simultaneous equations in γ_1' and γ_1'', the solution of which can be shown to be

$$\left.\begin{aligned} \gamma_1' &= \frac{\frac{1}{4}\varepsilon_1}{A'^2 + (B' - \frac{1}{4}\omega - \omega\zeta)^2}\Big[(A'^2 + B'^2)(1 + 3\cos\omega) - 3\omega A'(\tfrac{1}{4} + \zeta)\sin\omega \\ &\quad - \omega B'(\tfrac{1}{4} + \zeta)(1 + 3\cos\omega)\Big] \\ \gamma_1'' &= -\frac{\frac{1}{4}\varepsilon_1}{A'^2 + (B' - \frac{1}{4}\omega - \omega\zeta)^2}\Big[3(A'^2 + B'^2)\sin\omega + \omega A'(\tfrac{1}{4} + \zeta)(1 + 3\cos\omega) \\ &\quad - 3\omega B'(\tfrac{1}{4} + \zeta)\sin\omega\Big] , \end{aligned}\right\} \quad \ldots \quad (25)$$

where $\zeta = \frac{\lambda}{\mu} \Big/ \omega = \frac{2w}{g\rho ac}$, a function of the aircraft and height only.

These values of γ_1' and γ_1'' when substituted in equation (19) give the solution for γ for the oscillatory part of the motion. The variation of α follows from equation (20) and is

$$\alpha = \alpha_S + (\varepsilon_1 - \gamma_1')\cos\lambda t + \gamma_1''\sin\lambda t \quad \ldots \quad (26)$$

and α_L is from equations (21) and (24),

$$\alpha_L = \alpha_S - \frac{\lambda}{\mu}(\gamma_1''\cos\lambda t + \gamma_1'\sin\lambda t) \quad \ldots \quad (27)$$

which enables the lift to be determined from equation (14).

3.6. *Effect of Simple Gust Train on Climb Gradient.*—The mean angle of climb $\bar{\gamma}$ is derived from the first equation (18); integrating this equation over a cycle gives

$$\bar{\gamma} - \gamma_S = \frac{\lambda}{2\pi}\int_0^{2\pi/\lambda}\left(\frac{\alpha_L\varepsilon}{\alpha_S} - k_2\frac{\alpha_L^2 - \alpha_S^2}{\alpha_S}\right)dt\ .$$

Substitute for α_L from (27) and use the fact that on integration only terms in $\sin^2\lambda t$ and $\cos^2\lambda t$ remain; then

$$\bar{\gamma} - \gamma_S = \frac{\lambda}{2\pi\alpha_S}\left[\int_0^{2\pi/\lambda}(-\frac{\lambda}{\mu}\gamma_1''\varepsilon_1\cos^2\lambda t)\,dt - k_2\int_0^{2\pi/\lambda}\frac{\lambda^2}{\mu^2}(\gamma_1''^2\cos^2\lambda t + \gamma_1'^2\sin^2\lambda t)\,dt\right]$$

$$= -\frac{\lambda}{2\mu\alpha_S}\left\{\gamma_1''\varepsilon_1 + k_2\frac{\lambda}{\mu}(\gamma_1'^2 + \gamma_1''^2)\right\}. \quad \ldots \quad (28)$$

Now γ_1' and γ_1'' are of the form

$$\gamma_1' = \varepsilon_1 f_1(\zeta, \omega)$$
$$\gamma_1'' = \varepsilon_1 f_2(\zeta, \omega) ,$$

where $f_1(\zeta, \omega)$, $f_2(\zeta, \omega)$ can be determined numerically for a given ζ from the equations (25) for γ_1' and γ_1'' and tabulated values of A' and B' against ω.

Hence the expression (28) for $\bar{\gamma} - \gamma_S$ is of the form

$$\bar{\gamma} - \gamma_S = \frac{\varepsilon_1^2}{\alpha_S} f_3(\zeta\ ,\ \omega\ ,\ k_2)$$

$$= \frac{V_{G1}^2}{w} f_4\left(\frac{c}{\nu}\ ,\ \rho\ ,\ a\ ,\ \frac{w}{c}\right) \quad \ldots \quad (29)$$

since $\varepsilon_1 = \frac{V_{G1}}{V_0}$, $\alpha_S = \frac{w}{\frac{1}{2}\rho V_0^2 a}$, $\zeta = \frac{2w}{g\rho ac}$, $\omega = 2\pi\frac{c}{\nu}$, $k_2 = 1 - \frac{a}{2\pi}$.

In Fig. 5, $(\bar{\gamma} - \gamma_S)/(V_{G1}^2/w)$ is plotted against c/ν for sea-level density an aspect ratio of 8, and for $w/c = 2$ and 4; it will be seen that the curve is not very sensitive to the value of w/c. Also from the form of the expressions, the result will vary little over practical values of the aspect ratio. In deducing this result, it has been assumed (in section 3.4) that λ was not less than about $0{\cdot}1$, *i.e.*, that c/ν was not less than about $0{\cdot}1c/2\pi V_0$, which is of order $0{\cdot}002$; the curve has, therefore, not been continued to zero c/ν, but we may note that it passes through the origin and reaches a maximum of about $0{\cdot}2$ at $\nu = 30c$. As c/ν increases, the curve appears to tend asymptotically to zero; in deriving the relation, it has been assumed (Appendix VI, section 3) that ν is not less than about $6c$, so the curve has not been continued beyond this point. However, for very small ν, the ordinate will be nearly zero, and if intermediate values (ν of order c) were required, they could be deduced from a more exact treatment taking into account the curvature of the gust field.

The curve of Fig. 5 thus shows the effect of gustiness on the mean angle of climb over a practical range of values of c/ν. The result is an increase in angle of climb, which has a maximum value of about $0{\cdot}002V_G^2/w$ at a critical value of $\nu \simeq 30c$.

3·7. *Effect of General Gust Pattern on Mean Climb Gradient.*—The result given in the preceding section was deduced from the simple gust pattern $\varepsilon = \varepsilon_1 \cos \lambda t$ of (10). The more general form (9),

$$\varepsilon = \sum_s \varepsilon_s \cos s\,\lambda t$$

$$= \frac{1}{V_0} \sum_s V_{Gs} \cos \frac{2\pi s V_0 t}{\nu}$$

will result in an expression for angle of climb of the form

$$\bar{\gamma} - \gamma_S = \frac{1}{w} \sum_s V_{Gs}^2 f_4\left(\frac{cs}{\nu}, \rho, a, \frac{w}{c}\right) \qquad \ldots \quad (30)$$

which is simply the sum of terms given by equation (29).

Equation (30) shows that, whatever the gust pattern, provided it has zero mean, gustiness results in an increase in climb gradient, which is a function of the gust pattern (defined by V_G and ν), the wing loading w, the wing mean chord c, the lift-curve slope a and the atmospheric density ρ. Fig. 5 indicates that most of the contribution to $\bar{\gamma} - \gamma_S$ comes from the components of the gust pattern of wavelength about $30c$, and those of wavelength less than $5c$ or greater than $200c$ are relatively unimportant.

4. *Method of Deriving a Standard for a Given Flight Case.*—4.1. *General Expression for Climb Gradient.*—Combining equations (4) and (30), we obtain a general expression for the climb gradient of the aircraft. Putting, in equation (30), $w = \bar{w}\ .\ W/\bar{W}$ and expressing ρ as a function of $\bar{\sigma}$, $p/\bar{p}$ and $\theta/\bar{\theta}$, we can express this general equation in the form

$$\gamma = \left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right) f\left(\frac{P_{\bar{n}}}{\bar{P}}, \frac{W}{\bar{W}}, \frac{V_i}{\bar{V}_i}, \frac{B_{\bar{n}}}{\bar{B}}, \frac{N}{\bar{N}}, \frac{p}{\bar{p}}, \frac{\theta}{\bar{\theta}}\right) - \frac{\bar{D}}{\bar{W}} f\left(\frac{D_{\bar{m}}}{\bar{D}}, \frac{W}{\bar{W}}, \frac{V_i}{\bar{V}_i}\right)$$

$$- \frac{\bar{C}_{L\,\max}}{\pi A r} f\left(\frac{W}{\bar{W}}, \frac{V_i}{\bar{V}_i}\right) + \sum_s V_{Gs}^2 f\left(\bar{w}, c, A, \bar{\sigma}, \frac{W}{\bar{W}}, \frac{P}{\bar{P}}, \frac{\theta}{\bar{\theta}}, \frac{\nu}{s}\right) \qquad \ldots \quad (31)$$

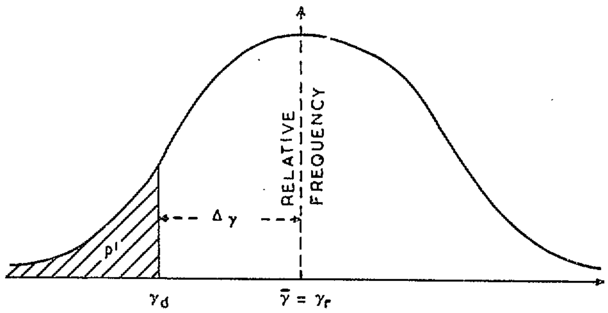
where, throughout this section, f denotes 'a function of'.

4.2. *The Performance Margin and Incident Probability.*—The variables $(P_{\bar{n}}/\bar{P})$, $(D_{\bar{m}}/\bar{D})$, $(W/\bar{W})$, $(V_i/\bar{V}_i)$, $(B_{\bar{n}}/\bar{B})$, $(N/\bar{N})$, $(p/\bar{p})$, $(\theta/\bar{\theta})$, V_{Gs}^2, ν, which we shall denote by x_i, will be treated as statistical variables (*see* Part I, section 4). To allow for the variability of the statistical variables, a performance margin γ_m over the datum γ_d must be provided as we have already seen from the discussion in Part I. The derived standard will then be in the form of a requirement stating that $\bar{\gamma}$ must not be less than $\gamma_d + \gamma_m$; write $\gamma_d + \gamma_m = \gamma_r$, the required angle of climb. This means that when $\bar{\gamma} = \gamma_r$, there is a certain probability p' that γ will be less than γ_d; p' is the incident probability. Our problem now, therefore, is to derive the margin γ_m (and so the required gradient γ_r) to give a prescribed incident probability p'.

4.3. *The Required Climb Gradient.*—Rewrite (31) in terms of the x_i in the form

$$\gamma = f\left(x_i\,,\,\bar{\gamma}\,,\,\frac{\bar{D}}{\bar{W}}\,,\,\bar{C}_{L\,\max}\,,\,\bar{w}\,,\,c\,,\,A\,,\,r\,,\,\bar{\sigma}\right). \qquad (32)$$

We may note from the more detailed form of this relation (equations (4) and (30)) that $\gamma = \bar{\gamma}$ when the x_i have their standard value $\bar{x}_i$. Then if the probability distributions of the x_i about their standard values $\bar{x}_i$ as mean are known, equation (32) will enable us to determine the distribution of γ about its mean $\bar{\gamma}$. The distribution function of γ is derived in general terms in section 4.4.



From this distribution, the probability that $\gamma < \gamma_d$ can be deduced, *i.e.*, the probability of getting a deviation as large as $\Delta\gamma = \gamma_d - \bar{\gamma}$ from the mean. If $\bar{\gamma} = \gamma_r$, this probability must equal the incident probability p'. It follows that having chosen p', we can deduce $\Delta\gamma$, which will be a function of the parameters of the population, p', and $\bar{\gamma}$, where $\bar{\gamma} = \gamma_r$. It follows that

$$\gamma_r = \gamma_d - \Delta\gamma\ . \qquad (33)$$

Equation (33) determines the requirement γ_r, the process being carried out by successive approximation since $\Delta\gamma$ is itself a function of γ_r.

γ_r is thus derived in terms of γ_d, p' and the parameters $(\bar{D}/\bar{W})$, $\bar{C}_{L\,\max}$, $\bar{w}$, c, A, r, $\bar{\sigma}$.

4.4. *Derivation of the Probability Distribution of* γ.—Equation (32) gave the general relation between the angle of climb γ and the statistical variables x_i, which are independent. Before we can deduce the distribution of γ from the distributions of the individual x_i, we must express the deviation $\Delta\gamma$ as a sum of the contributions due to the Δx_i, where $\Delta x_i = x_i - \bar{x}_i$. This can be done by expanding $\Delta\gamma$ by Taylor's theorem, giving

$$\Delta\gamma = \Sigma\left(\overline{\frac{\partial f}{\partial x_i}}\right)\Delta x_i + \frac{1}{2!}\left\{\Sigma_i\left(\overline{\frac{\partial^2 f}{\partial x_i^2}}\right)\Delta x_i^2 + 2\Sigma_{i>j}\left(\overline{\frac{\partial^2 f}{\partial x_i\,\partial x}}\right)\Delta x_i\,\Delta x_j\right\}$$
$$+\frac{1}{3!}\left\{\Sigma_i\left(\overline{\frac{\partial^3 f}{\partial x_i^3}}\right)\Delta x_i^3 + \dots\right\} + \dots$$

where the bar denotes the value of the differential coefficient at mean values of the x_i.

This series converges fairly rapidly as either the Δx_i or the higher derivatives, as discussed in Appendix IV, are small in practice. In fact, no large error is introduced if only the first term is retained, giving

$$\Delta\gamma = \sum_i \frac{\overline{\partial f}}{\partial x_i} \Delta x_i$$

$$= \sum_i \Delta\gamma_i, \text{ say.}$$

Those $\Delta\gamma_i$ which are normally distributed can be simply combined to give a normal distribution $\Delta\gamma_a$ due to these where, since the Δx_i (and hence the $\Delta\gamma_i$) are statistically independent, the variance of $\Delta\gamma_a$ is given by

$$\text{var } \Delta\gamma_a = \Sigma \text{ var } \Delta\gamma_i \qquad (34)$$

the sum being taken only over the normally distributed $\Delta\gamma_i$.

The remaining $\Delta\gamma_i$ must then be combined successively.

Suppose $\Delta\gamma_r$ is the first non-normal $\Delta\gamma_i$ to be combined. We want the distribution function of $(\Delta\gamma_a + \Delta\gamma_r)$ and this is given by

$$F_{a+r}(\Delta\gamma_{a+r}) = \int_{-\infty}^{\infty} F_r(\Delta\gamma_{a+r} - x)F_a'(x)\, dx\,, \qquad (35)$$

writing $(\Delta\gamma_a + \Delta\gamma_r)$ as $\Delta\gamma_{(a+r)}$ and

where $F_r(x)$ is the distribution function of $\Delta\gamma_r$

$F_a(x)$ is the distribution function of $\Delta\gamma_a$

$$F'(x) = \frac{dF(x)}{dx}\,.$$

In general, it will not be possible to evaluate the integral on the right-hand side of equation (35) algebraically and the rather tedious process of graphical integration will have to be used, choosing sufficient values of $\Delta\gamma_{a+r}$ to establish the form of the function $F_{a+r}(\Delta\gamma_{a+r})$ over the range required.

Having obtained the distribution function of $\Delta\gamma_{a+r}$, the next variable $\Delta\gamma_s$ must be combined by the same process, to give the distribution function of

$$\Delta\gamma_{a+r+s} = \Delta\gamma_{a+r} + \Delta\gamma_s$$

from

$$F_{a+r+s}(\Delta\gamma_{a+r+s}) = \int_{-\infty}^{\infty} F_s\Delta(\gamma_{a+r+s} - x)F'_{a+r}(x)dx\,.$$

The process is continued until all the $\Delta\gamma_i$ have been included, and the distribution function $F(\Delta\gamma)$ of $\Delta\gamma = \Sigma\Delta\gamma_i$ is found.

5. *The Case Incident Probability.*—5.1. By the method of section 4 a requirement may be derived when the values of the parameters, the distribution of the variables x_i and the datum performance γ_d are fixed; such a requirement would, therefore, be applicable to a given stage and condition of engine operation* where these values will be sensibly constant.

The probability p' from which the requirement was derived is, therefore, the probability of getting an incident in a given stage in a given condition of engine operation. We may define p_r'

* *e.g.*, one engine inoperative.

as the probability of an incident in the case of r engines inoperative; p_r' will be referred to as the case incident probability. Since we have assumed that all the parameters and variables remain constant throughout the stage, the incident, if it occurs, will come when the aircraft enters the stage or when an engine becomes inoperative in the course of the stage. p_r', therefore, may more precisely be defined as the probability of an incident on entering the stage when r engines are inoperative on entry to the stage.

We are interested in the stage incident probability, rather than the case incident probability, and if we know the state of engine operation on entering the stage and the probability π of an engine failing during the stage, we can derive the stage incident probability Q from the various case incident probabilities p_r of the stage. This derivation will now be made.

6. *Exact Expression for the Stage Incident Probability.*—6.1. The following notation will be used:—

- p_r' the probability of an incident on entering the stage with r engines inoperative, on entry to the stage,
- p_r the probability of entering the stage with r engines inoperative,
- π_{rs} the probability of s or more engines failing during the stage when r engines are inoperative on entering the stage,
- π the probability of any one engine failing during the stage,
- n total number of engines.

6.2. *The Probability of an Incident in a Stage.*—Suppose the aircraft enters a particular stage with a performance low enough to result in an incident with all engines operative (the probability of this incident being p_0'); this incident can only occur if the aircraft is in this 'performance condition', so that the probability of entering the stage in this performance condition is p_0'. Since an incident will follow whether 0, 1, 2, . . . n engines are inoperative, the probability of an incident in this performance condition is

$$p_0'(p_0 + p_1 + p_2 + \ldots + p_n) .$$

Now suppose the aircraft enters the stage with performance low enough to result in an incident if one engine is inoperative but not low enough to result in an incident with all engines operative; the probability of this condition is $(p_1' - p_0')$. An incident will occur on entering the stage if 1, 2 . . . n engines are inoperative, so the probability of this happening in this condition is

$$(p_1' - p_0')(p_1 + p_2 + \ldots + p_n) .$$

In addition an incident will occur during the stage if the aircraft enters with all engines operative and an engine fails subsequently; the probability of this happening is $p_0\pi_{01}$, so the probability of an incident this way is

$$(p_1' - p_0')p_0\pi_{01} .$$

Therefore, the total probability of an incident in this condition is

$$(p_1' - p_0')(p_0\pi_{01} + p_1 + p_2 + \ldots + p_n) .$$

By a similar argument, if the aircraft enters the stage with performance low enough to result in an incident with two engines inoperative, but not with one, the combined probability of an incident either on entering the stage or during the stage is

$$(p_2' - p_1')(p_0\pi_{02} + p_1\pi_{11} + p_2 + \ldots + p_n) .$$

Summing these probabilities up to the case of the aircraft entering the stage with performance high enough only to produce an incident if all n engines are inoperative, the total probability of an incident occurring either on entry or during the stage is

$$
\begin{aligned}
Q = & \; p_0'(p_0 + p_1 + p_2 + \ldots + p_n) \\
& + (p_1' - p_0')(p_0\pi_{01} + p_1 + p_2 + \ldots + p_n) \\
& + (p_2' - p_1')(p_0\pi_{02} + p_1\pi_{11} + p_2 + \ldots + p_n) \\
& + \ldots \\
& + (p_n' - p_{n-1}')(p_0\pi_{0n} + p_1\pi_{1n-1} + p_1\pi_{2n-2} + \ldots + p_n) \,. \qquad \ldots \quad \ldots \quad (36)
\end{aligned}
$$

6.3. *Expressions for the* π_{rs}.—We can relate the π_{rs} to the probability π of any one engine failing during the stage by using the binomial expansion, if we assume that failures of individual engines are independent; π_{rs} will be the sum of the coefficients of x^s, x^{s+1}, ... x^n in the expression

$$\{\pi x + (1 - \pi)\}^{n-r}$$

i.e.,

$$
\begin{aligned}
\pi_{rs} = & \; \frac{(n - r)(n - r - 1) \ldots (n - r - s + 1)}{s!} \pi^s (1 - \pi)^{n-r-s} \\
& + \ldots \\
& + (n - r)\, \pi^{n-r-1} (1 - \pi) \qquad \ldots \quad \ldots \quad \ldots \quad (37) \\
& + \pi^{n-r} \,.
\end{aligned}
$$

This gives, for example,

$$\pi_{00} = \pi_{10} = \ldots = \pi_{r0} = 1$$

$$\pi_{01} = 1 - (1 - \pi)^n$$

$$\pi_{11} = 1 - (1 - \pi)^{n-1}$$

$$\pi_{0s} = \frac{n(n - 1) \ldots (n - s + 1)}{s\,!} \pi s (1 - \pi)^{n-s} + \ldots + n\pi^{n-1}(1 - \pi) + \pi^n \,.$$

6.4. *Relation Between Successive Stages*.—The probability, p_r, of entering a stage with r engines inoperative is clearly related to the probabilities of incidents and engine failures on the previous stages. Let the prefix t denote the stage number. We shall find an expression for ${}_{t+1}p_r$ in terms of the p_r, π_{rs} and p_r' of the t-th stage, *i.e.*, in terms of ${}_tp_r$, ${}_t\pi_{rs}$ ${}_tp_r'$. We assume that the performance condition of the aircraft on entering the $(t + 1)$-th stage is independent of its performance condition on leaving the t-th stage; so we simply have to consider the probability of the aircraft reaching the end of the t-th stage without incident and with r engines inoperative at the end. The effect of the assumption of independence will, in practice, be pessimistic because some degree of dependence exists (*see* Part I, section 11.3).

Consider the t-th stage.

If the aircraft gets through with all engines operative it must enter the stage with all engines operative and no engines must fail during the stage; the probability of this is ${}_tp_0(1 - {}_t\pi_{01})$. In addition if it is not to have an incident, its performance must not be so low as to give an incident with all engines operative, *i.e.*, the probability of entering the t-th stage in this condition is $(1 - {}_tp_0')$. Hence the probability of coming out of the t-th stage with all engines operative is

$$ {}_{t+1}p_0 = (1 - {}_tp_0')\,{}_tp_0(1 - {}_t\pi_{01}) \, . $$

To get ${}_{t+1}p_1$, we use a similar argument; this time, it may either enter the stage with all engines operative and one only fails during the stage (probability ${}_tp_0({}_t\pi_{01} - {}_t\pi_{02})$) or it may enter the stage with one engine inoperative and none may fail during the stage (probability ${}_tp_1(1 - {}_t\pi_{11})$). The performance condition on entering the stage must be such as not to result in an incident if one (or no) engines are inoperative, the probability of which is $1 - {}_tp_1'$. Hence

$$ {}_{t+1}p_1 = (1 - {}_tp_1')\{{}_tp_0({}_t\pi_{01} - {}_t\pi_{02}) + {}_tp_1(1 - {}_t\pi_{11})\} \, . $$

In general,

$$ {}_{t+1}p_r = (1 - {}_tp_r')\{{}_tp_0({}_t\pi_{0r} - {}_t\pi_{0r+1}) + {}_tp_1({}_t\pi_{1r-1} - {}_t\pi_{1r}) + \ldots + {}_tp_r(1 - {}_t\pi_{r1})\} \, . \qquad (38) $$

For the first stage, it may be assumed that the aircraft enters with all engines operative, so that

$$ {}_1p_0 = 1 $$

$$ {}_1p_r = 0 \text{ for } r > 0 \, . $$

Hence

$$ {}_2p_r = (1 - {}_1p_r')({}_1\pi_{0r} - {}_1\pi_{0r+1}) $$

$$ \begin{aligned} {}_3p_r = (1 - {}_2p_r')[&(1 - {}_1p_0')(1 - {}_1\pi_{01})({}_2\pi_{0r} - {}_2\pi_{0r+1}) \\ &+ (1 - {}_1p_1')({}_1\pi_{01} - {}_1\pi_{02})({}_2\pi_{1r-1} - {}_2\pi_{1r}) \\ &+ \ldots \\ &+ (1 - {}_1p_r')({}_1\pi_{0r} - {}_1\pi_{0r+1})(1 - {}_2\pi_{r1})] \end{aligned} \qquad (39) $$

and so on, using formula (38) as a reduction formula.

6.5. *The Stage Incident Probability.*—Substitution of the expressions (37) for the π_{rs} and the expressions (39) for the ${}_tp_r$ in equation (36) will give a general expression for the stage incident probability Q_t in terms of the ${}_tp_r'$ and π_t, where π_t is the value of π in the t-th stage.

7. *An Approximate Expression for the Stage Incident Probability.*—7.1. *Basis of Approximation.*—This general expression is somewhat intractable, and it can be simplified by making use of the fact that π and p_0' are small and that p_0' will be of the order π^2 or less (Part I, section 10). Also p_1' is of order π for a twin-engined aircraft and smaller for $n > 2$; p_2' is of order unity for a twin-engined aircraft and smaller for $n > 2$; and so on. For approximation, the most unfavourable case is p_0' of order π^2, p_1' of order π and p_r' $(r > 1)$ of order unity, and in particular more favourable cases, further approximation may be carried out.

7.2. *Approximate Expressions for π_{rs}.*—Neglecting terms of the third order and higher, the expressions π_{rs} of equation (37) for the probabilities of different combinations of engine failures can be reduced to

$$ \pi_{00} = \pi_{10} = \ldots = \pi_{r0} = 1 $$

$$ \pi_{01} = n\pi - \frac{n(n-1)}{2\,!}\pi^2 $$

$$\pi_{r1} = (n - r)\pi - \frac{(n - r)(n - r - 1)}{2!}\pi^2$$

$$\pi_{02} = \frac{n(n - 1)}{2!}\pi^2$$

$$\pi_{r2} = \frac{(n - r)(n - r - 1)}{2!}\pi^2$$

$$\text{all } \pi_{rs}\ (s > 2) = 0 . \qquad \ldots \qquad (40)$$

By neglecting the third-order terms and higher we are, in effect, saying that we are prepared to neglect flight conditions when more than two engines fail during a stage.

7.3. *Approximate Probability of an Incident in a Stage.*—The expression (36) for the probability of an incident reduces to

$$Q = p_0\left(p_0' + p_1'n\pi + p_2'\frac{n(n - 1)}{2!}\pi^2\right)$$
$$+ p_1\left[p_1'\{1 - (n - 1)\pi\} + p_2'\{(n - 1)\pi - (n - 1)(n - 2)\pi^2\} + p_3'\frac{(n - 1)(n - 2)}{2!}\pi^2\right]$$
$$+ p_2\left[p_2'\{1 - (n - 2)\pi + \frac{(n - 2)(n - 3)}{2!}\pi^2\} + p_3'\{(n - 2)\pi - (n - 2)(n - 3)\pi^2\}\right.$$
$$\left. + p_4'\frac{(n - 2)(n - 3)}{2!}\pi^2\right]$$
$$+ \ldots$$
$$+ p_n p_n' . \qquad \ldots \qquad (41)$$

7.4. *Approximation Expressions for the* $_tp_r$.—The relations (39) between the p_r in successive stages* reduce to

Stage 1

$$\left.\begin{aligned} {}_1p_0 &= 1 \\ {}_1p_r &= 0 \text{ for } r > 0 \end{aligned}\right\} . \qquad \ldots \qquad (42a)$$

Stage 2

$$\left.\begin{aligned} {}_2p_0 &= 1 - n\pi_1 + \frac{n(n - 1)}{2!}\pi_1^2 - {}_1p_0' , \\ &\qquad \text{where } \pi_t \text{ is the value of } \pi \text{ in the } t\text{-th stage} \\ {}_2p_1 &= n\pi_1 - n(n - 1)\,\pi_1^2 - {}_1p_1'n\pi_1 \\ {}_2p_2 &= \frac{n(n - 1)\pi_1^2}{2!}(1 - {}_1p_2') \\ {}_2p_r &= 0,\ r > 2 . \end{aligned}\right\} . \qquad \ldots \qquad (42b)$$

* We here use the stages discussed in Part I, that is, take-off climb, en route, approach and baulked landing.

Stage 3

$$\left.\begin{aligned}
{}_3p_0 &= 1 - n(\pi_1 + \pi_2) + \frac{n(n-1)}{2!}(\pi_1^2 + \pi_2^2) + n^2\pi_1\pi_2 - ({}_1p_0' + {}_2p_0') \\
{}_3p_1 &= n(\pi_1 + \pi_2) - n(n-1)(\pi_1^2 + \pi_2^2) - n(2n-1)\pi_1\pi_2 \\
&\quad - {}_1p_1'n\pi_1 - {}_2p_1'n(\pi_1 + \pi_2) \\
{}_3p_2 &= \frac{n(n-1)}{2!}(1 - {}_2p_2')\{(\pi_1 + \pi_2)^2 - {}_1p_2'\pi_1^2\} \\
{}_3p_r &= 0 \text{ for } r > 2
\end{aligned}\right\} \quad \ldots \quad (42c)$$

For the fourth stage, baulked landing, the probabilities, p_r, of entering the stage in the varying conditions must include a factor p_b which is the probability of a baulked landing arising. For simplicity, we will assume that p_b is independent of the number of engines inoperative. This would not be expected in practice and the effect of this simplification is examined later (Part III).

Stage 4

$$\left.\begin{aligned}
{}_4p_0 &= p_b\Big[1 - n(\pi_1 + \pi_2 + \pi_3) + \frac{n(n-1)}{2!}(\pi_1^2 + \pi_2^2 + \pi_3^3) \\
&\quad + n^2(\pi_2\pi_3 + \pi_3\pi_1 + \pi_1\pi_2) - ({}_1p_0' + {}_2p_0' + {}_3p_0')\Big] \\
{}_4p_1 &= p_b[n(\pi_1 + \pi_2 + \pi_3) - n(n-1)(\pi_2^2 + \pi_1^2 + \pi_3^2) \\
&\quad - n(2n-1)(\pi_2\pi_3 + \pi_3\pi_1 + \pi_1\pi_2) - {}_1p_1'n\pi_1 - {}_2p_1'n(\pi_1 + \pi_2) \\
&\quad - {}_3p_1'n(\pi_1 + \pi_2 + \pi_3)] \\
{}_4p_2 &= p_b\frac{n(n-1)}{2!}(1 - {}_3p_2')[(\pi_1 + \pi_2 + \pi_3)^2 \\
&\quad - {}_2p_2'\{(\pi_1 + \pi_2)^2 - {}_1p_2'\pi_1^2\} - {}_1p_2\pi_1^2] \\
{}_4p_r &= 0 \text{ for } r > 2
\end{aligned}\right\} \quad \ldots \quad (42d)$$

In all stages it will be seen that $p_r = 0$ when $r > 2$ so that the final effect of the assumptions made in section 7.1 is the neglect of any flight case where more than two engines are inoperative at one time. This is in accordance with currently accepted practice. If cases involving a greater number of inoperative engines were considered necessary, terms of higher order than the second would be required.

7.5. *Final Equations.*—The expressions (42) for the ${}_tp_r$ can now be substituted in the incident probability (41) to give the incident probability on each stage in terms of the ${}_tp_r'$ and π_t.

For stage 1, the incident probability

$$Q_1 = {}_1p_0' + {}_1p_1'n\pi_1 + {}_1p_2'\frac{n(n-1)}{2!}\pi_1^2 .$$

For stage 2,

$$Q_2 = {}_2p_0' + {}_2p_1'n(\pi_1 + \pi_2) + {}_2p_2'\frac{n(n-1)}{2!}\{(\pi_1 + \pi_2)^2 - {}_1p_2'\pi_1^2\} .$$

For stage 3,

$$Q_3 = {}_3p_0' + {}_3p_1'n(\pi_1 + \pi_2 + \pi_3) + {}_3p_2' \frac{n(n-1)}{2!} \Big\{(\pi_1 + \pi_2 + \pi_3)^2$$

$$- {}_2p_2'(\pi_1 + \pi_2)^2 - {}_1p_2'(1 - {}_2p_2')\pi_1^2\Big\} \;..$$

For stage 4,

$$Q_4 = p_b \Big[{}_4p_0' + {}_4p_1'n(\pi_1 + \pi_2 + \pi_3 + \pi_4) + {}_4p_2' \frac{n(n-1)}{2!} (\pi_1 + \pi_2 + \pi_3 + \pi_4)^2$$

$$- {}_3p_2'(\pi_1 + \pi_2 + \pi_3)^2 - {}_2p_2'(1 - {}_3p_2')(\pi_1 + \pi_2)^2 - {}_1p_2'(1 - {}_2p_2')(1 - {}_3p_2')\pi_1^2\Big].$$

These expressions for Q_r may be simplified by writing

$$\Pi_r = \Sigma\, \pi_r,$$

giving

$$Q_1 = {}_1p_0' + {}_1p_1'n\Pi_1 + {}_1p_2' \frac{n(n-1)}{2!} \Pi_1^2 \qquad .. \quad .. \quad .. \quad .. \quad .. \quad (43a)$$

$$Q_2 = {}_2p_0' + {}_2p_1'n\Pi_2 + {}_2p_2' \frac{n(n-1)}{2!} (\Pi_2^2 - {}_1p_2'\Pi_1^2) \qquad .. \quad .. \quad .. \quad .. \quad (43b)$$

$$Q_3 = {}_3p_0' + {}_3p_1'n\Pi_3 + {}_3p_2' \frac{n(n-1)}{2!} [\Pi_3^2 - {}_2p_2'\Pi_2^2 - {}_1p_2'(1 - {}_2p_2')\Pi_1^2] \qquad .. \quad (43c)$$

$$\frac{Q_4}{p_b} = {}_4p_0' + {}_4p_1'n\Pi_4 + {}_4p_2' \frac{n(n-1)}{2!} [\Pi_4^2 - {}_3p_2'\Pi_3^2 - {}_2p_2'(1 - {}_3p_2')\Pi_2^2$$

$$- {}_1p_2'(1 - {}_2p_2')(1 - {}_3p_2')\Pi_1^2] . \qquad .. \quad .. \quad .. \quad .. \quad .. \quad .. \quad (43d)$$

8. *Expression for Flight Incident Probability.*—For the complete flight, the incident probability is, with these approximations

$$\Sigma\, Q_r = {}_1p_0' + {}_2p_0' + {}_3p_0' + p_b\, {}_4p_0'$$

$$+ n({}_1p_1'\Pi_1 + {}_2p_1'\Pi_2 + {}_3p_1'\Pi_3 + p_b\, {}_4p_1'\Pi_4)$$

$$+ \frac{n(n-1)}{2!} [{}_1p_2'(1 - {}_2p_2')(1 - {}_3p_2')(1 - p_b\, {}_4p_2')\Pi_1^2$$

$$+ {}_2p_2'(1 - {}_3p_2')(1 - p_b\, {}_4p_2')\Pi_2^2 + {}_3p_2'(1 - p_b\, {}_4p_2')\Pi_3^2 + p_b\, {}_4p_2'\Pi_4^2] . \qquad .. \quad (44)$$

These three terms represent the summed probabilities of incidents with zero, one and two engines inoperative respectively.

9. *Stage Requirements.*—9.1. *The stage Incident Probability in Terms of One Case.*—The expressions (43a to 43d) give the stage incident probabilities in terms of the ${}_tp_r'$, *i.e.*, the case incident probabilities. But since there is a relation between the performance γ_r with r engines inoperative and γ_{r+1} with $r + 1$ engines inoperative, there will be a relation between p_r' and p_{r+1}'.

With all engines operative, we have

$$\gamma_0 = \frac{T}{W} - \frac{D}{W} \; .$$

For an aeroplane with n engines, with one engine inoperative, and with the same control settings on the operative engines after engine failure

$$\gamma_1 = \frac{T}{W}\left(1 - \frac{1}{n}\right) - \frac{D + \Delta D}{W}$$

where ΔD is the change in drag due to one engine being inoperative. Hence the loss in performance due to the engine cut is

$$\Delta\gamma = \frac{1}{n}\left(\frac{D}{W} + \gamma_0\right) + \frac{\Delta D}{W} .$$

Thus, if γ_m is the performance margin above the datum γ_d

$$\gamma_{m1} = \gamma_{m0} - \frac{1}{n}\left(\frac{D}{W} + \gamma_{m0} + \gamma_d\right) - \frac{\Delta D}{W} . \qquad (45)$$

Since the probability of an incident in a given configuration is a unique function of the performance margin

$$p_1' = f\left[p_0', n, \frac{D}{W}, \gamma_d, \frac{\Delta D}{W}, \Pi\right] \qquad (46)$$

and similarly for p_2'.

Therefore, it follows that the stage incident probability (Q) (equation (43)) may be obtained as a function of any one of the case incident probabilities (p_r') for that stage, the parameters n, Π, D/W, $\Delta D/W$, γ_d the distribution function of γ and the p_2' terms from previous stages. These latter terms cannot, in general, be accounted for* unless the specified performance is made a function of the realised performance in other stages; as discussed below (section 10) this is not a practical proposition. Such terms, therefore, are neglected, which is pessimistic, and we can thus define the required stage incident rate in terms of one case only which in some instances would be a simplification compared with the present code. From the above equations, however, the relative importance of the various cases is not apparent and this question will now be considered.

9.2. *The Dominant Case in a Stage.*—From the expression for stage incident probability we may, by use of equation (46), calculate the relative contributions from the various conditions of engine operation in terms of the parameters mentioned above. As an illustration we will consider a four-engined aircraft and assume γ normally distributed with standard derivation $0 \cdot 3$ per cent and $\gamma_d = 0$.

The results are given in Figs. 1 to 4. Fig. 1 is representative of cases where the propellers of inoperative engines are feathered (*e.g.*, en route) where $\Delta D/W$ will be approximately zero; Fig. 2 is for the same data with a term in $\Delta D/W$ introduced and represents propeller windmilling, whilst Figs. 3, 4 show the effects of changing D/W and the engine reliability respectively. In Fig. 1, comparable curves for a twin-engined aircraft are also shown.

The most important features, which are unaffected by the numerical values chosen, are that the contributions to the stage incident rate are, for practical purposes, confined to two cases and at a particular incident rate arise predominantly from one case.

For given conditions, therefore, we are able to calculate the dominant case on which to base our requirements.

It will subsequently be shown that the parameters Π, n, D/W and γ_d enter in the derivation of the performance requirements for other reasons; consequently when $\Delta D/W$ may be neglected,

* There are particular exceptions. For instance on twin-engined aircraft p_2' terms are always unity.

the exact interpretation of equations (43) may be applied without any increased complexity in the specification of the final requirements. When $\Delta D/W$ cannot be neglected (*e.g.*, propeller windmilling) this additional parameter may enter in the specification of the requirements.

10. *The Flight Incident Probability.*—It is apparent from equation (44) that, for a given flight incident probability, the incident probabilities for the various flight stages as we know them now are inter-related. Thus if a given aircraft had a general excess of performance in the t-th stage, ${}_tp_0'$ etc., would be small and greater incident probabilities (a lower standard of performance) could be accepted in the other stages. Such a principle is not, however, admitted in any existing code of airworthiness requirements, where the performance required in a given stage is the same whether the aircraft's performance is ' border-line ' in all other stages or whether it has large reserves of performance. Now it is clear that an attempt to operate a requirements code based on the rigorous equation (44), which includes the interrelation of ' stage ' requirements, would be laborious and almost certainly not a practical possibility in actual operations. Administratively some simplification, therefore, is essential and technically this is feasible without much loss of precision by considering stages independently. A qualitative argument has already been given in Part I (section 11) and a numerical example to support this is given in Part III to show that a reasonable approximation to a prescribed flight incident probability Q can be obtained by using this as the stage incident probability for each stage.

PART III

NUMERICAL APPLICATION AND DISCUSSION

1. *Introducion.*—In this part a numerical application is given to show the method and the standards so derived, and enable a more detailed examination of specific points to be made. The actual numerical values used were obtained from the most readily available data; they are considered sufficiently representative to achieve the defined objectives and serve as a basis of discussion. If the method were adopted, a more comprehensive collection and analysis of data would be desirable.

2. *Datum Performance.*—2.1. The following datum performances are assumed throughout:—

Take-off climb, Baulked-landing climb } 0·5 per cent gradient with 15 deg banked turn,

En-route, Approach } level flight with 15 deg banked turn.

2.2. For a steady turn with bank β the corresponding gradient in level flight at the same speed is given by

$$\gamma = k' \frac{D}{W} \tan^2 \beta, \text{ where } k' \text{ is the ratio of induced to total drag,}$$

and for a 15 deg banked turn

$$\gamma = 0 \cdot 07 k' \frac{D}{W} . \qquad (1)$$

3. *Variability of Statistical Variables.*—3.1. The variables other than gusts are assumed to be normally distributed and the standard deviations used are given in Appendix V.

3.2. The standard deviation of basic power was obtained from bench test results on 150 engines of seven types.

3.3. Variation in basic manifold pressure may result from

(i) inaccurate setting of the throttles (if manual), or	Engine throttled,
(ii) inaccurate functioning of the manifold pressure control (if automatic), or	
(iii) variations in the supercharger compression ratio	Engine at full throttle.

The evidence is very limited but the suggested figure of 1·6 per cent for the standard deviation appears representative.

3.4. Considering the drag variation, an analysis of speed measurements on 55 aircraft of the same type gave a standard deviation of drag of 1·1 per cent which is in agreement with our general experience. The aircraft did not have controllable cooling flaps and if we assume a standard deviation of 2 deg in setting, the corresponding deviation of drag for a twin-engined aircraft is estimated to be 0·2 per cent and less for aircraft with more engines. This gives a total standard deviation of drag, with flaps up, of 1·12 per cent. With flaps fully down there will be a variation in angle due to the positioning of the stop and in intermediate positions there will usually be greater errors. We assume standard deviations of flap position of 1 deg and 2 deg respectively. An investigation of various typical flap arrangements suggests a value of $(\Delta D)/D$ of about 1·2 per cent per degree variation and the compounding of these values with the other variations gives the appropriate values.

3.5. In the weight term we are concerned with errors in the estimated weight for which there may be two main causes.

(a) Variation in passenger weight when passengers are not individually weighed.

(b) Variation in fuel weight when a mean fuel density figure is used.

For these effects a standard deviation of 1 per cent is considered adequate.

3.6. The aircraft and engine speed standard deviations have been estimated at 2·1 per cent and 0·5 per cent respectively. The total aircraft speed error was compounded from assumed standard deviations of 2 per cent, 0·5 per cent and 0·25 per cent respectively for inaccurate flying, instrument error and A.S.I. system.

3·7. From consideration of the frequency of occurrence of high temperatures and the areas over which these occur, a world-wide standard deviation of air temperature equal to 10 deg C is suggested and from a limited examination of meteorological records a standard deviation of pressure, at a given station, of 1 per cent is suggested as representative. Both these assumptions should be examined by a competent meteorological authority; the value assumed for the temperature variation in particular is important and has a considerable bearing on the en-route case where temperature is not treated as a parameter. The production of a temperature frequency distribution on a world-wide basis would involve considerable work, but would be valuable in many other connections. In the meantime the Meteorological Office have stated unofficially that they are, in the absence of special investigation, unable to suggest a better figure than the value of 10 deg C assumed in the preliminary papers.

3.8. From the theoretical treatment of the effect of gusts on performance in Part II it was found that, irrespective of the gust pattern, provided it has a zero velocity mean, the effect on gradient of climb is favourable with the simple, but representative, assumptions made on the flying technique. Because of this and the lack of knowledge on the space-time structure of gusts the effect of gustiness on performance has not been included in the numerical examples. Flight tests are proceeding to check whether neglect of the effect of gusts is justified and it is hoped that the results will be available in the near future. These tests will also be used to check the validity of the assumptions made in the theoretical treatment. It is proposed to discuss this subject further in a separate report.

4. *Performance Margins*.—4.1. Using the assumed standard deviations discussed in the previous paragraphs we may, by use of equations (31) and (34) of Part II, obtain the distribution function of gradient of climb in terms of $\bar{D}/\bar{W}$, $\bar{C}_{L\max}/\pi Ar$, n' and $\bar{\gamma}$ (Appendix V).

Now $\bar{C}_{L\max}/\pi Ar = k'\bar{D}/\bar{W}$, where k' is the ratio of induced to total drag and we write this function as $\bar{M}$. For a given configuration the effect of variations of k', within the practical range, are numerically small so, for simplicity, k' is assumed constant for all types in a given stage.

4.2. Starting from an assumed required gradient of climb ($\bar{\gamma}$) we may, therefore, for a given stage and for an aircraft with a given number of engines, determine the distribution function of γ in terms of $(\bar{D}/\bar{W})$; for any given probability level we may then, by successive approximation as explained in Part II, section 4.3, obtain a value for $\bar{\gamma}$ in terms of $\bar{D}/\bar{W}$.

4.3. In most cases the convergence is rapid; for the en-route cases, owing to the dominance of the temperature effect, this is not so and a different calculation technique, suggested by K. J. Lush, is used.

In this case the variance is expressed as

$$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2 \left[A + B\left(\frac{\bar{\gamma}}{\bar{D}/\bar{W} + \bar{\gamma}}\right) + C\left(\frac{\bar{\gamma}}{\bar{D}/\bar{W} + \bar{\gamma}}\right)^2\right],$$

where A, B and C are constants and the first term in the square bracket is, for the en-route case, large compared with the others. It is then sufficiently accurate to calculate the value of the square bracket term for a mean value of $\bar{D}/\bar{W}$ and an estimated $\bar{\gamma}$, no second approximation being necessary, that is we write:—

$$\text{Var}\,\gamma = A'(\bar{D}/\bar{W} + \bar{\gamma})^2 .$$

Hence for a given probability and datum performance, an explicit expression for the performance required is obtained in terms of $\bar{D}/\bar{W}$.

5. *Engine Failure Rates*.—5.1. The engine failure statistics provided by various operators for this investigation are given in Appendix IX. These are not necessarily complete engine failures and to quote from Operator II, 'Engine failures in this summary include all cases where full power was not available to the pilot for whatever reason and does not necessarily imply that the engine itself was defective'.

5.2. To check the basis of assessment of the failure rates, it is necessary to have data for a given aircraft type employed on stages of differing average duration. This occurs only in the case of aircraft B. A statistical analysis showed that the variation of probability of failure per flight with average flight duration was not significant. Thus the proposed basis of assessment (by flights) is not inconsistent with the data available; the sample available is, of course, very small and a much larger collection is needed for more definite indications to be obtained. As discussed in Part I, the use of engine failure probability on a flight basis is a continuation of present (implicit) practice. If the collection of more data indicates a change to be desirable, no fundamental difficulty in application is foreseen.

5.3. Considering the engine failures during take-off, Operator II stated that, except in one case, the take-off was abandoned after engine failure. Operator I stated that in the one case reported the take-off was continued. The other operators did not give this information for the remaining four cases and we will make the pessimistic assumption that they were all continued.

5.4. From the summation of the available data the following values for the cumulative probability of an engine being inoperative (Π_r) are obtained*.

Stage	*Probability of an engine being inoperative per engine-flight*
Take-off climb	$0 \cdot 238 \times 10^{-3}$
En route	$0 \cdot 692 \times 10^{-3}$
Approach	$0 \cdot 692 \times 10^{-3}$
Baulked landing	$0 \cdot 692 \times 10^{-3} p_b$

6. *Tolerable Incident Rate.*—6.1. The conditions governing the choice of incident probability have been discussed qualitatively in Part I, and in the subsequent numerical examples a value of $1 \cdot 0 \times 10^{-5}$* has been taken. This value was selected after consideration of the ranges over which the one engine inoperative case is dominant for the first three stages, typical calculations being given in Figs. 1 to 4. The numerical values used are typical for the cases under consideration, and investigation has shown that the range of Π^x is not greatly influenced by practical variations in D/W, Π or standard deviation. The minimum range is approximately $\Pi^{7/8}$ to $\Pi^{7/4}$ and so in conjunction with our other criterion of excluding single and including twin-engined aircraft our final range of choice was Π to $\Pi^{7/4}$. Numerically, the minimum range is given by taking Π appropriate to take-off and $\Pi^{7/4}$ appropriate to en route, that is $0 \cdot 238 \times 10^{-3}$ to $2 \cdot 95 \times 10^{-6}$. The value $1 \cdot 0 \times 10^{-5}$ was chosen arbitrarily as the smallest 'round number' in this range. The other criterion in the choice of incident rate is most conveniently dealt with in the discussion of the derived requirements (section 10.4).

7. *The Turbo-jet Engine.*—Up to this stage the treatment has dealt with the conventional reciprocating engine/propeller power plant for ease of presentation. With turbo-jet engines the principles are unaffected and only the power laws and engine failure probabilities call for re-examination. The first is dealt with in Appendix VIII and representative numerical values derived. On the second aspect, engine reliability, no data have yet been accumulated. In the absence of better information we have used the same values as for reciprocating engines. On this basis, it is only for the en-route case, where temperature is not treated as a parameter, that different standards are necessary.

We may note here that, when operating statistics become available, if a material difference in reliability compared with the reciprocating engine is shown, it will be possible, by the present method, to adjust the standards to provide a similar level of safety to that of reciprocating engined aircraft; such a procedure would be difficult with the present empirical requirements.

8. *Details of the Calculations.*—8.1. Sample calculations typical of the two methods used are given in Appendix VII. For the en-route, approach and baulked-landing cases, $\Delta D/W$ (equation (45), Part II) has been assumed zero because it is normal to consider the propeller of the inoperative engine feathered in these cases; the expression for stage incident rate, neglecting terms from previous stages, (*e.g.*, equation (43a), Part II) has been used.

8.2. For the take-off climb case it has already been explained (Part II, section 9.2) that, in general, an extra parameter may be necessary to deal with the case of a windmilling propeller. For the chosen incident probability this was not necessary because, to the limits of accuracy of the calculation, $p_0' = 0$ and $p_2 = 1 \cdot 0$ for the case of $\Delta D/W = 0$ and consequently this will be true whatever the drag of the windmilling propellers.†

8.3. For the baulked-landing case we have assumed the probability of a baulked landing to be 1 in 400 landings. This figure was suggested as a pessimistic value by the Operations Branch of the Ministry of Civil Aviation.

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* *See* Addendum.

† The case is based on one engine inoperative and consequently with a positive value of $\Delta D/W$ an incident is less likely all engines operative and more likely two engines inoperative.

8.4. The standards for each stage were determined for several values of D/W over representative ranges as follows, and for the values of k' indicated:

	$\bar{D}/\bar{W}$	k'
Take-off climb	0·06 to 0·14	0·7
En route	0·05 to 0·08	0·7
Approach	0·08 to 0·16	0·6
Baulked landing	0·10 to 0·18	0·5

Two- and four-engined aircraft are considered since these are the most common types in use.

8.5. For the take-off and landing cases, to avoid the complication of different standards above and below full throttle height, the pessimistic assumption has been made that an equal number of take-offs and landings are made in each condition; the variance is then given by the mean of the variances for the two conditions. For the en-route case the difference in the standards was sufficiently large to warrant the quotation of both. The reason for these separate standards and also that for turbo-jet engined aircraft is primarily due to the treatment of temperature as a statistical variable.

9. *The Derived Standards.*—9.1. In all cases the gradient of climb required approximates very closely to a linear function of D/W, over the range considered, and is here expressed in that form:—

Case	*Percentage gradient of climb required* — *Two-engined aircraft*	*Four-engined aircraft*	
(1) Take-off climb	$(0{\cdot}5 + 12{\cdot}7\,D/W)$	$(0{\cdot}5 + 13{\cdot}0\,D/W)$	One engine inoperative.
(2) En route, below F.T.H.	$16{\cdot}9\,D/W$	$18{\cdot}1\,D/W$	One engine inoperative.
above F.T.H.	$22{\cdot}0\,D/W$	$24{\cdot}3\,D/W$	One engine inoperative.
turbo-jets	$26{\cdot}2\,D/W$	$29{\cdot}1\,D/W$	One engine inoperative.
(3) Approach	$13{\cdot}4\,D/W$	$13{\cdot}7\,D/W$	One engine inoperative.
(4) Baulked landing	$(0{\cdot}5 + 12{\cdot}0\,D/W)$	$(0{\cdot}5 + 12{\cdot}3\,D/W)$	All engines operative.

9.2. The configuration assumptions implicit in these requirements are flaps up for en route, full down for baulked landing and at an intermediate position for take-off and approach.

10. *Discussion of Standards.*—10.1. *General.* Summarising previous discussion, compliance with the above standards would be specified at:

(a) Height: 5,000 ft for en route, and for the remaining cases at a height in International Standard Atmosphere equal to the geographical height.

(b) Temperature: International Standard Temperature for en route, and ambient temperature for the remaining cases.

(c) Weight: Take-off weight for (1) and landing weight for (3) and (4). The upper and lower limits of the en-route case will be given by associating the standard with take-off and landing weight respectively.

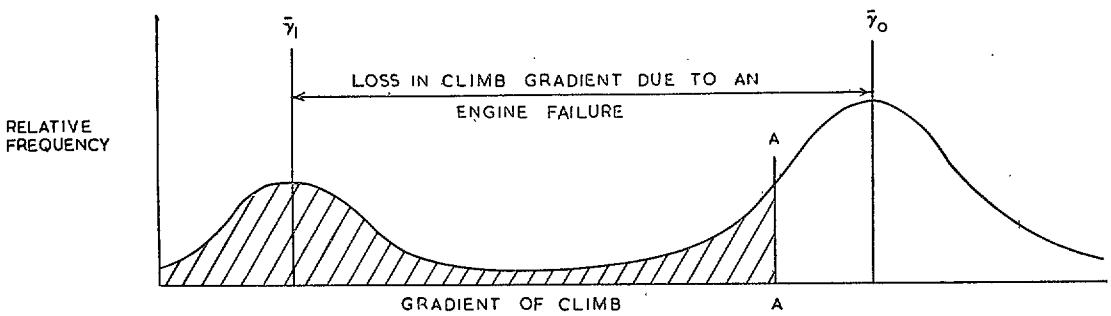
(d) Drag/weight ratio: The mean for the aircraft type in the appropriate configuration.

10.2. *Drag/Weight Ratio.*—In the present method, the required performance is expressed as a function of D/W instead of stalling speed, which is used in the existing proposed international standards[3]. The drag/weight ratio can be estimated during the design stage and can be determined in flight tests by measuring gradients of descent. These flight tests would be made at the appropriate speeds and configurations; engines which would, in the climb tests, be inoperative would be feathered or windmilling as appropriate and the remaining propellers run at zero thrust. In the absence of the necessary instrumentation it is considered that windmilling in coarse pitch would be an acceptable interpretation.

10.3. *Variation in Datum Performance.*—No particular merit is claimed for the datum performance gradients used and it is probable that in subsequent discussions other values will be considered.

To examine the effect of this the take-off climb standard for a four-engined aircraft has been reworked for a large datum gradient (3·5 per cent) instead of the 0·5 per cent previously used. The resulting standard was $\gamma = (3 \cdot 5 + 13 \cdot 8D/W)$ per cent. The effect on that part of the requirement to cater for the statistical variables is thus comparatively small; in rounding off the coefficients to give the formal standard, this could be done so that changes in datum gradient can be incorporated directly.

10.4. *Variation in Stage Incident Rate.*—The effect on performance of varying the stage incident rate is most readily calculable for the en-route case and the effect is shown in Figs. 6 and 7. It will be seen that the variation is comparatively slow where the one engine inoperative case is dominant and a small increase in performance increases the level of safety appreciably. The curve is, however, of stepped formation and there comes a point where large increases in performance have a relatively insignificant effect on the level of safety. The following relative frequency diagram may assist in an understanding.



The diagram shows the scatter in performance about the means $\bar{\gamma}_0 \bar{\gamma}_1$, etc., which arises for reasons already discussed. With the numerical values which we have assumed, and for aircraft with a small number of engines, the scatter is, for individual cases, sufficiently small compared with the performance loss when an engine cuts that, imagining the distributions as separate, the 'tails' overlap at small frequencies. Now if A–A represents the datum performance, the incident rate is represented by the shaded area to the left. As the aircraft performance increases, A–A moves relatively to the left. It will thus be seen that there will be a region where the area to the left of A–A (*i.e.*, incident rate) changes slowly with climb gradient. Now the cases shown in Figs. 6 and 7 are where temperature is included as a statistical variable, and the variance or scatter is greatest. From this aspect, therefore, the 'steps' will be more pronounced in the other cases where the variance of climb gradient is smaller. In cases where the propeller is windmilling the loss in climb gradient due to an inoperative engine will be increased and this will increase the magnitude of the steps. If we now consider the variance, an increase will make the step less defined in any particular case, but this would, of course, be accompanied by an increase in the magnitude of the gradients of climb required. Compare for instance Figs. 6 and 7. Judging by the requirements derived the variances are of the right order (section 10.6).

This point has possibly been laboured but it appears to be one of fundamental importance which is not generally appreciated. It is also a striking illustration of the possible pitfalls of arbitrary requirements when we consider the implications of the proposed I.C.A.O. standards superimposed on Figs. 6 and 7.

This treatment enables a number of other problems to be studied such as the lines of aircraft and engine development most likely to be profitable in the improvement of safety and it is proposed to deal with these in a subsequent report.

For our present purpose, however, it will be seen that this knowledge provides a powerful method of fixing a tolerable incident-rate. The relative value which has been taken (approximately $\Pi^{1\cdot6}$) can apparently only be decreased slightly without incurring an appreciable economic penalty for existing types of aircraft; the term 'economically attainable' incident-rate may not be inappropriate to use for the probability level at which the step occurs.

We have here, of course, considered only part of the curve concerned with the transition from the one to two engines inoperative cases being dominant. In the complete picture there will be a step for each transition and the probability levels at which these occur will be given by the coefficients of terms of the form ${}_tp_r$ in equations (43a) to (43d), Part I, that is of the form $n\Pi$, $[n(n-1)/2!]\Pi^2$, etc. It is proposed to extend the discussion of this subject in a subsequent report.

10.5. *Variation of Engine Failure Rate.**—The data on engine failures is somewhat limited and, as already stated, probably includes cases which were not total failures.

When the one engine inoperative case is dominant

$$Q = ap_1'n\Pi, \text{ where } 2 > a > 1$$

and we have taken $Q = \Pi^{1\cdot6}$

$$p_1' = \frac{\Pi^{0\cdot6}}{an} .$$

Thus, if the number of total failures has been overestimated, p_1' will be too large and the standards too low for a given *relative* incident rate. If we assumed that, in a large scale and more detailed investigation, the probabilities of complete failure were reduced to one-tenth (considered amply pessimistic), then, retaining the same *relative* incident probability (Π^x) the numerical change in the requirements, would be, for instance,

Take-off climb: from $(0\cdot5 + 13D/W)$ per cent to $(0\cdot5 + 14\cdot6D/W)$ per cent.

En route: from $24\cdot9D/W$ per cent to $27\cdot9D/W$ per cent.

Thus it will be seen that for a given *relative* incident probability, the requirements deduced are not likely to be greatly changed by probable changes in the engine failure rate used.

This being so we can, even in the absence of very accurate data on engine failure rates, obtain a close approximation to the requirements by deciding the relative incident rate needed to maintain the existing standard of safety. This relative incident rate, as we have already noted, is a convenient fiction to bridge the gap between our past experience with empirical standards and proposed use of rational standards. Having bridged this gap we may discard this conception and fix, for future work, an absolute value for the tolerable incident rate by determining a value for Π from a large-scale collection of data. The above relation for p_1' will then be untenable and future improvements in engine reliability could, for instance, be utilised to improved safety or economics of operation or any desired compromise between these conflicting requirements.

10.6. *Comparison with Proposed I.C.A.O. Standards.*—As already explained, the primary objective of this report is to establish a method rather than numerical values. Since, however, illustrative numerical requirements have been derived, it is inevitable that comparison with the proposed I.C.A.O. standards[3] will be made. A comparison between the 'upper limit'† en-route standard for an aircraft with four piston engines and the existing one engine inoperative I.C.A.O.

* *See* Addendum.

† That is, compliance at take-off weight as with the I.C.A.O. requirements.

requirements is given in Fig. 8. To convert the gradients to rates of climb a climbing speed of $1 \cdot 2V_{S1}$ has been assumed. Since the I.C.A.O. standard is, rather surprisingly, expressed in terms of stalling speed in the landing configuration it has also been assumed that the ratio of the flaps up and flaps down stalling speed is $1 \cdot 2$.

A standard which has been the subject of much controversy is that for take-off climb with one engine inoperative. For this case a comparison has been obtained by deriving a standard, by the present method, including the temperature term for a four-engined aircraft. This gives a standard to be met at mean temperature, that is on the same basis as the I.C.A.O. requirement.

The value so derived was

$$\text{Gradient of climb required} = (0 \cdot 5 + 19D/W) \text{ per cent*}$$

and is plotted in Fig. 9.

The proposed I.C.A.O. standards which have been used as a basis for comparison were taken from the American C.A.A. standards which were based on operating experience with aircraft such as the Lockheed 18, D.C. 3 and D.C. 4. Such aircraft had a flaps-down stalling speed in the order of 70–80 m.p.h.† Without going into great detail it is apparent that in this range there is a reasonable measure of agreement in the numerical values and that it is in the region extrapolated from this operating experience where the standards diverge.

It is of interest to note that with the assumed value for the probability of an aircraft being baulked, the resulting case for specification is that with all engines operative, as in the existing requirements.

11. *The Use of 'Declared' Temperature.*—11.1. At the international discussion[2] on temperature accountability it was proposed that, as an optional alternative to the use of ambient temperature in the take-off case, operators should be permitted to use a standard temperature declared in advance for the particular airfield and time. For the approach and landing cases the use of ambient temperature does not appear operationally feasible and the 'declared' temperature system would be necessary in a requirements code taking explicit account of air temperature.

11.2. The way in which the determination of declared temperature could be fitted in with the present method of deriving the standards is investigated in Appendix X and the result given in Fig. 10. The value of the declared temperature may be deduced from this curve, knowing the mean temperature and local standard deviation of temperature for the place and period covered by the declaration ; both these values could be easily determined by the responsible Meteorological Authority. From Fig. 10, it appears that the declared temperature would be up to about 7 deg C above the mean.

11.3. As shown in Appendix X, the value of the declared temperature depends on the flight case considered; since it is the take-off case where the ambient and declared systems are alternative, the declared temperature relationship has been derived for this case. At this stage it would be premature to pursue this aspect further in detail, as values depend on the performance standards finally agreed. We may note, however, that for the approach and baulked-landing standards derived in this report, the declared temperature would be approximately correct. If great accuracy were required, it appears feasible to adjust the approach or baulked landing (which would in practice be based on declared temperature) to the relationship derived for take-off.

12. *The Operational Standards.*—12.1. In this report we have considered airworthiness requirements, that is, those performance standards which must be met by an aircraft irrespective of the terrain over which it is employed. The operational standards are those which enable a

* With section 9.1, this shows a comparison of requirements with and without temperature accountability.

† Earlier American requirements limited the permissible stalling speed to 80 m.p.h.

given aircraft to be fitted to a particular route and are beyond the scope of this report. No difficulties are, however, apparent in the operational application of the take-off and landing requirements.

12.2. The en-route case is more complex. In the existing proposed operational standards[5] a rate of climb is specified at 1000 ft above obstructions. This is objectionable for several reasons, one being that the performance required to clear a given obstacle safely depends on the preceding flight history. If an aeroplane were operated at a considerably higher altitude than the obstructions, then it would be quite feasible, after engine failure, for it to cross a given obstruction with a negative rate of climb and subsequently assume level flight.* The problem from this and other aspects is largely one of operational planning and cannot be considered here; the principles of the present method could, however, be applied to this type of problem.

We may note that the quoted en-route cases have been derived for 5,000 ft; for application to operational standards, if such were necessary, the variation in the requirement with height would have to be calculated.

13. *Flight Incident Rate in a Particular Case.*—13.1. The standards have been derived for a given stage incident rate, and we here examine briefly a particular case to show that the resulting flight incident rate is likely to be of the same order.

Attention has been confined to one twin-engined aircraft, for which there was flight test data on climb performance available in a suitable form.

The flight incident probability has been derived from equation (44), Part II, and the upper limit (*i.e.*, starting weight requirement) for en route has been used.

13.2. International standard sea-level conditions for take-off and landing, and a 200-ml range have been assumed. For the standards of section 9.1 the critical condition is the take-off climb. At the limiting weight defined by this case the available performance margin gradients in the other cases are

En route (starting weight)	3·53 per cent
Approach	5·03 per cent
Baulked landing	19·0 per cent.

From the known performance and drag-weight ratios, the respective variances of γ , and hence the incident probabilities, can be calculated. Terms of the form ${}_tp_2'$ are all unity (total loss of power) and

$$\begin{aligned}\Sigma Q_r &= Q_1 + Q_2 + Q_3 + Q_4 \\ &= 1\cdot0 \times 10^{-5} + 0\cdot0422 \times 10^{-5} + (< 10^{-9}) + 0\cdot034 \times 10^{-5} \\ &= 1\cdot08 \times 10^{-5},\end{aligned}$$

that is 1·08 times our stage incident rate. The contribution Q_4 arises almost entirely from the one engine inoperative condition; the value for the probability of a baulked landing assumed (1 in 400) is, however, intended to be applicable to normal operations and it is highly probable that greater precautions would be taken to avoid an aircraft with an inoperative engine being baulked. Consequently, Q_4 is overestimated and the ratio of flight to stage incident rate is likely to be nearer 1·05.

13.3. This ratio will depend on the conditions at take-off and landing; conditions which increase the disparity between the 'compliance weight' for the cases will reduce the ratio and *vice versa.* It may be noted that where the weight is limited by considerations other than the climb requirements (*i.e.*, strength, take-off or landing distances) the ratio may be less than unity. The

* A practical case of a particular aircraft and route has been noted by a British operator.

(pessimistic) upper limit will be 4·0 given when an aircraft is used on a journey such that it just complies with every requirement. This is a most improbable condition and it is thought in practice the ratio is unlikely to exceed 2·0, that is, roughly, two cases equally critical with appreciable reserves on the compliance weight in the other cases.

13.4. This sort of variation in level of safety is thus the price paid for the administrative simplicity of independently specified cases. It is not considered to be of material importance in view of the comparable variations due to other causes, but was an aspect raised in discussion of the preliminary papers on which amplification was requested.

14. *Final Discussion and Further Developments.*—14.1. In this report we have considered the application of statistical methods to the formulation of climb performance standards for civil aircraft and there are, of course, other applications for this type of method. A preliminary treatment of take-off distance requirements for a particular case has, for instance, already been made[9], and other problems, not confined to performance requirements, can be treated.

14.2. We have here carried the procedure through to a derivation of detailed standards of the form now current nationally and internationally. By so doing it should not be inferred that the procedure of specifying detailed standards is considered one which should necessarily continue. The specific standards have been derived because certain broad policies in relation to the desired level of safety were assumed. Certainly for international standards it would be worth considering whether these should not be broadly framed (quantitatively) to specify more objectively the required level of safety, leaving the detailed implementation of such standards to the national requirements codes which would be derived, from the international standards, by the methods we have discussed. Such broadly framed international requirements could, of course, be accompanied by detailed standards of the type now current as an 'acceptable interpretation' of the broad policy.

14.3. It will be apparent that, for a given level of safety the level of performance required can, in general, be reduced as the engine reliability is improved. By the methods discussed this can be assessed quantitatively and there are several possible developments in this field which it is proposed to examine later. At present, for instance, the up-rating of the engines of an aircraft type would, other factors permitting, enable the aircraft to operate at higher weights; we may now consider whether similar advantages should not be available to a type which, after sufficient operational service, showed an engine reliability well above average.

Another interesting point concerns the categorisation of civil aircraft for airworthiness purposes. It is the intention, nationally, to specify certain categories to enable the certification of aircraft which do not comply, for instance, with one engine inoperative performance requirements; up till now it has been common to base such categorisation on a stalling speed limitation. A low limit on stalling speed is a possible way of reducing the consequences of an incident, but the subject of engine reliability appears worthy of investigation and may be a material factor in the admission of aircraft to the lower performance categories.

14.4. In discussions of unpublished papers which preceded the preparation of this report, the view was advanced in some quarters that there was inadequate statistical data on which to base an approach of this type. On an international scale, this is not obviously true except for certain atmospheric variables and it would seem, moreover, that this argument is applicable with even more force to the existing system of empirical standards. The suggestion is advanced that inaccurate certification standards are equally as objectionable as inaccurate design methods and that more effort should, in fact, be devoted to the collection of the necessary data so that in course of time the accuracy with which standards on a rational basis are derived may be improved.

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No.	Author	Title, etc.
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4	—	Aerial Navigation Act, 1911.
5	I.C.A.O.	Final Report of the Operations Division Second Session, 1947. Doc. 3030.
6	H. Templeton	The Technique of Flutter Investigations. R.A.E. Tech. Memo. Structures 8. (Unpublished.)
7	Ministry of Supply	Schedule of Type Approval Tests for Aero-engines. Specification D.E.D. 2000.
8	D. Cameron	British Performance Reduction Methods for Modern Aircraft. R. & M. 2447. January, 1948.
9	Watson and Poole	An Investigation of Rational Take-off Requirements for a Typical Aircraft. AAEE/ICAO Note 3. Published as I.C.A.O. Doc. 6570.

The following standard work on mathematical statistics is also relevant :—

The Advanced Theory of Statistics. Kendall (C. Griffin & Co., Ltd.).

APPENDIX I

Nomenclature and Definition of Terms

Symbols

A	Equivalent aspect ratio or a constant (Part III, section 4.3, and Appendix VII)
A'	A function of ω, *see* Appendix VI, or a constant (Part III, section 4.3)
a	Slope of lift curve or a constant
B	Manifold pressure or a constant (Part III, section 4.3, and Appendix VII)
B'	A function of ω, *see* Appendix VI
b	A constant
C	Engine compression ratio or a constant (Part III, section 4.3, and Appendix VII)
C'	Function defined in Appendix VI, equation (14)
C_{Di}	Induced drag coefficient
C_{Dz}	Profile drag coefficient
C_L	Lift coefficient
C_p	Propeller power coefficient
c	Wing chord or a constant
D	Drag
D_0	Profile drag in steady airflow
D'	Function defined in Appendix VI, equation (14)
d, f, g, h	Constants
J	Propeller advance ratio

k' Induced drag/total drag

k A constant

$k_1 = D_0/T$

$k_2 = 2/(2 + A)$

L Lift

$\bar{M} = \bar{C}_{L\max} / \pi A r$

N Engine speed

n Number of engines

n' Number of operative engines

P Power

p Pressure or probability

p_b The probability of a baulked landing

p_e Exhaust back pressure

p_r The probability of entering a stage with r engines inoperative

p_r' The probability of an incident on entering a stage, with r engines inoperative when the stage is entered

Q Incident probability

$r = (\bar{V}_i/\bar{V}_{is})^2$

S Wing area or standard deviation (Appendix X)

s A constant

T Thrust

t Ratio of deviation to standard deviation, a tabulated function for the normal distribution

V Aircraft true speed

V_a Aircraft true air speed

V_G Vertical gust velocity

V_i Aircraft equivalent air speed. With suffix 'S' means stalling speed

V_{S0} Stalling speed in the landing configuration

V_{S1} Stalling speed in the configuration appropriate to the case considered

V_0 Aircraft mean true air speed

V_w Normal velocity of wing

W Weight

w Wing loading

z Rise in temperature of charge before inlet valve closes (Appendix II) or displacement (Appendix VI)

α Incidence measured from zero lift

β Angle of bank or a constant (Appendix VII)

γ Gradient of climb

γ_d Datum gradient of climb

γ_m Performance margin (gradient)

γ_r Performance required (gradient)

γ_s Gradient of climb in still air

δ Inclination of thrust line to no-lift line

ε Vertical gust velocity/V

$\zeta = 2w/g\rho ac$

η Propeller efficiency or horizontal gust velocity/V

θ Air temperature

λ Circular frequency of vertical gust component $= 2\pi V_0/\nu$

μ Circular frequency of horizontal gust component

ν Gust wave length

π Probability of engine failure or the usual mathematical constant

π_{rs} The probability of s or more engines failing during a stage when r engines are inoperative on entering the stage

π_t The probability of any one engine failing during stage t

$\Pi_r = \Sigma \pi_r$

ρ Air density

σ Relative air density

τ T/W in calm air

ω $c\lambda/V_0$

Definition of Terms

Certain terms are used with a particular significance and for convenience in reading they are here collected and definitions given.

Condition: In Part I means a state or contingency affecting the climb performance.

Configuration: As in I.C.A.O. documents, *i.e.*, 'a term referring to the position of the various elements affecting the aerodynamic characteristics of the aeroplane (*e.g.*, wing flaps and landing gear)'.

Datum performance: The performance level in a given stage below which conditions predisposing to an accident to passengers or third parties exists.

Declared temperature: As in Ref. 2. A standard temperature declared in advance by the appropriate authority for a particular airfield and period.

Flight stage: A subdivision of a flight for convenience in specifying requirements. In the present connection primarily based on aircraft configuration.

Flight case: Flight in a given stage with a given number of engines inoperative.

Incident rate: The frequency with which the performance falls below the datum. This is used generally or in connection with flights or a particular stage; also 'economically attainable incident rate' being an incident rate at which the performance required increases rapidly with reduction in the rate.

APPENDIX II

Engine Power and Relation to Independent Variables

1. *Introduction.*—The factors affecting the power of a piston engine are numerous and their influences complex. As a result, precise theoretical prediction of the variation of brake power with, for instance, air temperature and air pressure is impracticable.

Approaches to the problem have usually been either

(a) purely empirical, a linear or exponential relation being assumed and the constants determined experimentally, or

(b) semi-theoretical, the form of the required relationship being deduced from a very simplified theoretical treatment and suitable constants deduced experimentally.

Often both techniques are employed in different sections of the same treatment.

2. *Relations in Current Use.*—2.1. The British method[7] of computing engine power at critical altitude from test-bed measurements is of type (b); it is assumed that brake power is proportional to charge flow and it is deduced that at constant engine speed

$$P \propto \frac{B - p_e/C}{\theta_i + z} \qquad \ldots \quad (1)$$

where

P	brake power,
B	absolute manifold pressure,
p_e	exhaust back pressure,
C	compression ratio,
θ_i	absolute intake temperature,
z	rise of temperature of charge before the inlet valve closes.

z, which is assumed constant, is determined empirically. A value of 127 is taken as being a mean for all supercharged engines; the variation of supercharger compression ratio with air temperature is assumed linear, and the coefficient for it determined individually for each engine type. Above the critical altitude it is merely assumed that the engine power is linear with regard to air density, the slope of the line being such that the power at zero density is negative and numerically equal to the friction power, as given by a semi-empirical formula.

2.2. In correcting aircraft performance to standard conditions British practice has been to use the slightly different relation[8],

$$P + 0\cdot 1P_{\bar{n}} \propto \frac{B - p_e/C}{\theta + 127} \qquad \ldots \quad (1a)$$

where θ is the ambient air temperature and the term $0\cdot 1P_{\bar{n}}$ represents an allowance for friction power not proportional to indicated power.

2.3. In American practice, corrections for air pressure are not normally needed, but it is assumed that at constant manifold pressure

$$P \propto \theta_i^d$$

and if there is no reliable information on the particular engine type, d is taken as $-0\cdot 5$.

3. *The Proposed Relation.*—3.1. It is convenient for our present purpose to use a relation of the form:

$$\frac{P}{P_{\bar{n}}} = \left(\frac{p}{\bar{p}}\right)^c \left(\frac{\theta}{\bar{\theta}}\right)^d \left(\frac{N}{\bar{N}}\right)^f \left(\frac{B_{\bar{n}}}{\bar{B}}\right)^g$$

where

$P_{\bar{n}}$ power at $\bar{p}$, $\bar{\theta}$, $\bar{N}$ and $\bar{B}$,

$B_{\bar{n}}$ manifold pressure at $\bar{p}$, $\bar{\theta}$ and $\bar{N}$,

p ambient air pressure,

θ ambient air temperature,

N engine speed.

The term $(B_{\bar{n}}/\bar{B})$ provides for mis-setting of the throttles (below critical altitude) or engine to engine variation in the supercharger characteristics (above critical altitude). Any variation of manifold pressure with air temperature or pressure (on engines at full throttle) is covered by the θ and p terms respectively.

3.2. *Manifold Pressure.*—The power of a given engine is not quite proportional to manifold pressure because of the influence of the exhaust back pressure, but it is sufficiently precise for present purposes to ignore the influence of back pressure and assume a g of $1 \cdot 0$. The variation of B with which we are concerned is quite small, so the form of relation proposed may be expected to fit adequately.

3.3. *Air Temperature.*—There is little difference between the forms

$$P \propto \frac{1}{\theta + z}$$

and

$$P \propto \theta^d$$

over a range of even $\pm$ 20 per cent in θ, if z and d are so chosen as to give the same slope at standard temperature. For instance, $z = 127$ and $d = -0 \cdot 69$ would give the same value of $dP/d\theta$ at 288 deg. At extreme temperatures they would give the following ($P_{\bar{\theta}}$ being the value of P at the standard temperature $\bar{\theta}$).

Relation z or d	$\frac{\theta}{\bar{\theta}}$	$\frac{P}{P_{\bar{\theta}}}$
z	1·2	0·88
	0·8	1·16
d	1·2	0·88
	0·8	1·17

The standard value of 127 was agreed some years ago and the latest information suggests that as a representative mean value a somewhat larger figure would be more appropriate. For the present work, therefore, we propose to take a value of d equal to $-0 \cdot 6$ at constant manifold pressure.

Above the critical altitude, allowance must be made for the change of supercharger compression ratio with air temperature by taking

$$d = -0 \cdot 6 + 1 \cdot 0 \frac{\theta}{B} \frac{\partial B}{\partial \theta} \cdot$$

Between sea level and 10,000 ft it is sufficiently accurate to take, as a representative value

$$d = -1{\cdot}2 \ (\text{for } \frac{1}{B}\frac{\partial B}{\partial \theta} = -0{\cdot}002) .$$

3.4. *Engine Speed.*—We are only concerned with small changes of engine speed due to imprecise setting or operation of the engine controls, so the proposed form of relation may, as with manifold pressure, be assumed to fit adequately.

The appropriate value of the coefficient f varies with engine speed and supercharger gear ratio, and between types. It is different above and below the critical altitude. For the range of engine speed with which we are concerned, if the manifold pressure is constant but the engine is not a very long way below the critical altitude, f will vary between about 0·5 and 0·0. A representative value is 0·3. A representative value above the critical altitude is 1·3.

3.5. *Air Pressure.*—Below the critical altitude the variation of power with air pressure is small, and may be ignored. Above the critical altitude air pressure has a large influence on power, which will be closely represented by taking c equal to 1·0.

3.6. *Combined Result.*—Combining the numerical values quoted above gives us

$$\frac{P}{P_{\bar{n}}} = \left(\frac{\theta}{\bar{\theta}}\right)^{-0\cdot6} \left(\frac{N}{\bar{N}}\right)^{0\cdot3} \left(\frac{B_{\bar{n}}}{\bar{B}}\right)^{1\cdot0} \qquad \text{below critical altitude,}$$

and

$$\frac{P}{P_{\bar{n}}} = \left(\frac{p}{\bar{p}}\right)^{1\cdot0} \left(\frac{\theta}{\bar{\theta}}\right)^{-1\cdot2} \left(\frac{N}{\bar{N}}\right)^{1\cdot3} \left(\frac{B_{\bar{n}}}{\bar{B}}\right)^{1\cdot0} \qquad \text{above critical altitude.}$$

APPENDIX III

Propeller Efficiency and Relationship to Independent Variables

At the relatively low forward speeds with which we are concerned, compressibility effects may normally be ignored and the efficiency of η of a propeller is a function of the non-dimensional quantities C_p and J, where

$$C_p = \frac{P}{\rho n^3 D^5} ,$$

$$J = V/nD$$

n being the rotational speed and D the diameter.

As with the engine power, it is convenient to assume a relationship of the form

$$\eta \propto C_p^{\,a} J^b .$$

This form is algebraically convenient, and it has the additional advantage that with it, it is possible, by expressing η, C_p and J as ratios of the values under standard conditions, to avoid generalisations about their absolute values and to consider only the likely ranges of a and b. These are roughly —0·1 to —0·4 and 0·3 to 0·5 respectively. Representative values are —0·2 and +0·4. We will therefore assume that

$$\eta \propto C_p^{-0\cdot2} J^{0\cdot4} .$$

It follows with this assumption that

$$\frac{C_p}{\eta}\frac{\partial \eta}{\partial C_p} = -0{\cdot}2 .$$

$$\frac{J}{\eta}\frac{\partial \eta}{\partial J} = 0{\cdot}4 .$$

APPENDIX IV

Approximations to the Series Representing the Effect of Statistical Variables on Performance

1. *Introduction*.—In Part II, section 4.4, it was stated that the effect on climb gradient of changes Δx_i of the statistical variables from their standard values x_i was given by

$$\Delta\gamma = \sum_i \left(\overline{\frac{\partial f}{\partial x_i}}\right) \Delta x_i + \frac{1}{2!}\left\{ \sum_i \left(\overline{\frac{\partial^2 f}{\partial x_i^2}}\right) \Delta x_i^2 + 2 \sum_{i>j} \left(\overline{\frac{\partial^2 f}{\partial x_i \partial x_j}}\right) \Delta x_i \, \Delta x_j \right\}$$

$$+ \frac{1}{3!}\left\{ \sum_i \left(\overline{\frac{\partial^3 f}{\partial x_i^3}}\right) \Delta x_i^3 + \ldots \right\} + \text{higher order terms} \quad \ldots \quad \ldots \quad (1)$$

the expressions for the partial derivatives being obtained from equation (31) of Part II. In the numerical examples of Part III we assume that, with the exception of gusts, the x_i are normally distributed. The numerical treatment is considerably simplified if second order and higher partial derivatives are neglected, because then the contributions $\Delta\gamma_i$ due to each Δx_i will also then be normally distributed. The errors introduced by such an assumption are here examined by an approximate method, for all variables except gustiness. We are primarily concerned with the one engine inoperative case and here p_1' is approximately $Q/n\Pi$, that is $3 \cdot 6 \times 10^{-3}$.

The treatment used here is to consider the contribution to $\Delta\gamma$ of each Δx_i in turn and investigate the error in this contribution arising from the neglect of all but the first order term $\frac{\partial f}{\partial x_i} \Delta x_i$. The expression for the contribution $\Delta\gamma_k$ due to Δx_k follows from equation (1) by putting $\Delta x_i = 0$, $(i \neq k)$, giving

$$\Delta\gamma_k = \left(\overline{\frac{\partial f}{\partial x_k}}\right) \Delta x_k + \frac{1}{2!}\left(\overline{\frac{\partial^2 f}{\partial x_k^2}}\right) \Delta x_k^2 + \frac{1}{3!}\left(\overline{\frac{\partial^3 f}{\partial x_k^3}}\right) \Delta x_k^3 + \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

The result will be approximate only, but should give an idea of the order of the error incurred by this desirable simplification.

2. *Numerical Values*.—The numerical values of Δx_k substituted in equation (2) are based on a probability level of 10^{-3}, Δx_k being deduced from its standard deviation, assumed in Part III. Numerical values for the indices in the engine power and propeller relationships are taken from Appendices II and III. Representative values for $\bar{D}/\bar{W}$, $\bar{\gamma}$ and $\bar{M}$ of $0 \cdot 10$, $0 \cdot 02$ and $0 \cdot 06$ are assumed except for the temperature variable which enters only in en-route cases and here $\bar{D}/\bar{W}$ is taken as $0 \cdot 07$.

3. *Drag*.—For this variable no approximation is involved since it follows from equation (31) of Part II that second and higher order derivatives are zero.

4. *Temperature*.—From equation (31), Part II:

$$\left(\overline{\frac{\partial f}{\partial x}}\right) = \left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right) k$$

$$\left(\overline{\frac{\partial^2 f}{\partial x^2}}\right) = \left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right) k(k-1) \,,$$

etc.

Thus

$$\Delta\gamma = \left(\frac{\bar{D}}{\bar{W}} + \gamma\right)\left[\Delta x \, k + \frac{1}{2!} \Delta x^2 (k)(k-1) + \frac{1}{3!} \Delta x^3 (k)(k-1)(k-2) \ldots \right]$$

The error by neglecting terms in Δx^2 and higher is therefore

$$\Delta \gamma_\varepsilon = \left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)\left[(1 + \Delta x)^k - 1 - \Delta x\, k\right].$$

With a standard deviation of temperature of 10 deg C we are concerned with values of Δx not exceeding $0 \cdot 111$.

The error in climb gradient will be approximately

0·0025 above full throttle height

0·0010 below full throttle height .

These errors are tolerable for the present treatment and their effect is pessimistic. A more accurate treatment using the actual distribution could be used later if desired by the method given at the end of section 4.4 of Part II. The final numerical values would first have to be agreed, however, before the additional labour would be justified. The errors enter only in the en-route case which is not subject to temperature accountability.

5. *Weight.*—In this case

$$\Delta \gamma_\varepsilon = (\bar{\gamma} + \bar{M}) \frac{\Delta x^2}{(1 + \Delta x)}$$

and with a standard deviation of weight of 1 per cent the error in climb gradient will be less than 0·0001 which is negligible.

6. *Speed.*—In this case

$$\Delta \gamma_\varepsilon = \left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)\left[(1 + \Delta x)^{b-1} - 1 - \Delta x\,(b - 1)\right]$$
$$- \left(\frac{\bar{D}}{\bar{W}} - \bar{M}\right) \Delta x^2$$
$$- \bar{M}\,(1 - \Delta x)^{-2} - 1 + \Delta x\ .2$$

and with a standard deviation of speed of 2·1 per cent the error in climb gradient will be 0·0005 which is also negligible.

7. *Other Variables.*—The other variables appear in equation (31) of Part II in a similar form to that of the temperature variable and may be similarly treated, the errors in these instances being smaller than the cases quoted above.

APPENDIX V

Effect of Variables other than Gusts on Performance

Variable (x_i)	$\partial f/\partial x_i$* Algebraic	$\partial f/\partial x_i$* Numerical †	Standard deviation of x_i	Variance of $\Delta\gamma_i \times 10^4$
1. Basic relative power	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)(1 + a)$	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right) \times 0{\cdot}8$	$\frac{0{\cdot}011}{\sqrt{n'}}$	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2 \times \frac{0{\cdot}774}{n'}$
2. Basic relative manifold pressure	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right) g(1 + a)$	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right) \times 0{\cdot}8$	$\frac{0{\cdot}016}{\sqrt{n'}}$	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2 \times \frac{1{\cdot}638}{n'}$
3. Relative engine speed	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right) m$	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)$ × 1·24 above F.T.H. 0·44 below F.T.H.	$\frac{0{\cdot}005}{\sqrt{n'}}$	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2 \times \frac{0{\cdot}385}{n'}$ above F.T.H. $\frac{0{\cdot}048}{n'}$ below F.T.H.
4. Relative pressure	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right) h$	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)$ × 1·3 above F.T.H. 0·5 below F.T.H.	0·010	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2$ × 1·69 above F.T.H. 0·25 below F.T.H.
5. Relative temperature	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right) k$	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)$ × − 1·46 above F.T.H. − 0·98 below F.T.H.	0·0360‡	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2$ × 27·58 above F.T.H. 12·43 below F.T.H.
6. Basic relative drag	$-\left(\frac{\bar{D}}{\bar{W}}\right)$	$-\left(\frac{\bar{D}}{\bar{W}}\right)$	0·0112 flaps up 0·0265 flaps intermediate 0·0164 flaps down	$\left(\frac{\bar{D}}{\bar{W}}\right)^2$ × 1·25 flaps up 7·01 flaps intermediate 2·69 flaps down
7. Relative speed	$\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)(b - 1) - 2\frac{\bar{D}}{\bar{W}} + 4\bar{M}$	$-0{\cdot}6\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right) - 2\frac{\bar{D}}{\bar{W}} + 4\bar{M}$	0·021	$\left[0{\cdot}6\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right) + 2\left(\frac{\bar{D}}{\bar{W}}\right) - 4\bar{M}\right]^2 \times 4{\cdot}41$
8. Relative weight	$-(\bar{\gamma} + 2\bar{M})$	$-(\bar{\gamma} + 2\bar{M})$	0·010	$(\bar{\gamma} + 2\bar{M})^2$

* From equation (4), Part II. † Constants from Appendices II and III. ‡ At 5,000 feet.

APPENDIX VI

The Relation between Lift and Incidence in Gusty Air

1. *Introduction.*—In the problem considered in section 3 of Part II the aircraft is flying through a gust train, the vertical speed εV of the air being defined by $\varepsilon = \varepsilon_1 \cos \lambda t$. In this motion the aircraft incidence and direction of motion are also varying sinusoidally according to the relations

$$\left.\begin{aligned} \alpha - \bar{\alpha} &= \alpha_1' \cos \lambda t - \alpha_1'' \sin \lambda t \\ \gamma - \bar{\gamma} &= \gamma_1' \cos \lambda t - \gamma_1'' \sin \lambda t \end{aligned}\right\} \quad \ldots \quad (1)$$

The classic relation between lift and incidence cannot be applied to this unsteady motion as there will be a lag (the Wagner effect) between the lift and the incidence. This appendix derives the relation between the lift and the incidence for this particular type of motion.

We are indebted to the Structures Department, R.A.E., for the suggestion that the solution of this problem could be derived from results used in flutter investigations, and in particular to Dr. Jordan for assistance in applying these results to the present problem.

This flutter theory considers the case of an aerofoil oscillating with sinusoidal motion in both translation and rotation in a steady air stream. This is applied in section 3 to the problem of an aircraft moving steadily through vertical gusts of sinusoidal velocity distribution and the motion of the aircraft itself is added to this solution in section 4. In this treatment, it is assumed that the aircraft is rigid.

2. *Expression for Lift of Oscillating Aerofoil (from Flutter Theory).*—The expression for the lift per unit span of an infinite aspect ratio aerofoil oscillating sinusoidally in translation z_0 and rotation α_0 in a uniform airstream of speed V_0 is given as

$$\frac{L}{\rho c V_0^2} = (-\omega^2 l_z'' + i\omega l_z' + l_z)\frac{z_0}{c} + (-\omega^2 l_\alpha'' + i\omega l_\alpha' + l_\alpha)\alpha_0 \quad \ldots \quad (2)$$

where

$$\left.\begin{aligned} z_0 &= (z_0' + iz_0'')e^{i\lambda t} \\ \alpha_0 &= (\alpha_0' + i\alpha_0'')e^{i\lambda t} \end{aligned}\right\} \quad \ldots \quad (3)$$

and the derivatives are defined by

$$\begin{aligned} l_z'' &= \tfrac{1}{4}\pi & \quad l_\alpha'' &= \tfrac{1}{8}\pi \\ l_z' &= \pi A' & \quad l_\alpha' &= \tfrac{1}{4}\pi(1 + 3A' - 4B'/\omega) \\ l_z &= \pi\omega B' & \quad l_\alpha &= \tfrac{1}{4}\pi(4A' + 3\omega B')\,, \end{aligned}$$

where A', B' are tabulated against $\omega = c\lambda/V_0$, c being the mean chord of the aerofoil.

The wing motion is referred to the leading edge of the aerofoil and z_0 is measured positive downwards. Equations (2) and (3) are given in complex notation. This notation, used for relating purely harmonic quantities, reduces the algebra. Equation (2) expresses the lift through the relation $L = L^* e^{i\lambda t}$, where L^* is a complex quantity independent of t. The physical quantity corresponds in general either to the real or imaginary part; in this report, the real part is used, as is evident from equation (1).

An extract from the tables for A' and B' is given below

ω	A'	B'
0	1·0000	0
0·2	0·8319	0·1723
0·5	0·6925	0·1852
1·0	0·5979	0·1507
1·5	0·5590	0·1213
∞	0·5000	0

3. *Application of Flutter Theory to Aircraft Moving Steadily in Gust Train.*—The field of the gusts is given by

$$\varepsilon = \varepsilon_1 \cos \lambda t$$

at the leading edge of the wing. At distance x behind the leading edge,

$$\varepsilon = \varepsilon_1 \cos \lambda\,(t - x/V_0)$$

since the aircraft is moving with speed V_0 through the gusts. In complex notation,

$$\varepsilon = \left(\varepsilon_1 \cos \frac{\lambda x}{V_0} - i\varepsilon_1 \sin \frac{\lambda x}{V_0}\right) e^{i\lambda} \quad \ldots \quad (4)$$

Consider now the case of the wing oscillating in a steady air stream. The position of the wing is defined by the co-ordinates z_0 and α_0 of equation (3). At a point x behind the loading edge, the normal velocity V_w of the wing is

$$V_{w1} = \frac{d}{dt}(z_0 + x\alpha_0)$$

and the component of the velocity of the air stream in this direction is

$$V_{w2} = V_0 \sin \alpha_0 \simeq V_0\alpha_0 .$$

The total velocity of this point of the wing relative to the air is then

$$\begin{aligned} V_w &= V_{w1} + V_{w2} \\ &= \frac{d}{dt}(z_0 + x\alpha_0) + V_0\alpha_0 \\ &= i\lambda(z_0 + x\alpha_0) + V_0\alpha_0 . \quad \ldots \quad (5) \end{aligned}$$

The velocity V_w is positive downwards and if this is equated to the upward gust velocity εV_0 of equation (4), we shall find the z_0 and α_0 for the motion in flutter equivalent to the motion through the gust train. Comparing equations (4) and (5) gives:

$$V_0(\varepsilon_1 \cos \lambda x/V_0 - i\varepsilon_1 \sin \lambda x/V_0)e^{i\lambda t} = i\lambda(z_0 + x\alpha_0) + V_0\alpha_0 . \quad \ldots \quad (6)$$

Equating coefficients of $e^{i\lambda t}$ in equations (5) and (6) gives

$$\left.\begin{aligned} \varepsilon_1 \cos \frac{\lambda x}{V_0} &= -\frac{\lambda}{V_0} z_0'' - \frac{\lambda}{V_0} x\alpha_0'' + \alpha_0' \\ \varepsilon_1 \sin \frac{\lambda x}{V_0} &= \frac{\lambda}{V_0} z_0' + \frac{\lambda}{V_0} x\alpha_0' + \alpha_0'' \end{aligned}\right\} \quad \ldots \quad (7)$$

Equations (7) give two equations for the four unknowns $z_0', z_0'', \alpha_0', \alpha_0''$, and can thus be satisfied for two values of x; choose for convenience the leading and trailing edge $x = 0$ and $x = c$. This gives exact equivalence at the leading and trailing edge, but there will not be exact equivalence all along the chord owing to the curvature of the gust field; provided that the gust wavelength ν is greater than about $6c$, the error will not be serious.

Putting $x = 0$ and $x = c$ in equations (7) and writing $\omega = c\lambda/V_0$ gives

$$\begin{aligned} \varepsilon_1 &= -\omega z_0''/c + \alpha_0' \\ 0 &= \omega z_0'/c + \alpha_0'' \end{aligned}$$

and

$$\begin{aligned} \varepsilon_1 \cos \omega &= \varepsilon_1 - \omega\alpha_0'' \\ -\varepsilon_1 \sin \omega &= \omega\alpha_0' . \end{aligned}$$

Hence

$$\left.\begin{aligned} \alpha_0' &= -\frac{\varepsilon_1}{\omega} \sin \omega \\ \alpha_0'' &= \frac{\varepsilon_1}{\omega} (1 - \cos \omega) \\ z_0'/c &= -\frac{\varepsilon_1}{\omega^2} (1 - \cos \omega) \\ z_0''/c &= -\frac{\varepsilon_1}{\omega^2} (\omega + \sin \omega) \end{aligned}\right\} \quad \ldots \quad (8)$$

give the contribution to α_0 and z_0 due to *steady* motion of the aircraft through the gust train.

4. *Additional Terms due to the Motion of the Aircraft.*—In addition to the oscillation of the air stream considered in para. 4, the aircraft itself is moving in space with oscillatory motions both in attitude and displacement and these give an additional contribution to the α_0 and z_0 of (3): call these terms α_a and z_a to distinguish them.

The attitude of the aircraft in space is $\alpha + \gamma - \varepsilon$ as will be seen from the figure of section 3.1 of Part II; the oscillatory part of the motion in attitude* is

$$\alpha + \gamma - \varepsilon - (\bar{\alpha} + \bar{\gamma})$$

so that the extra contribution* to α_0 is

$$\left.\begin{aligned} \alpha_a' &= \alpha_1' + \gamma_1' - \varepsilon_1 \\ \alpha_a'' &= \alpha_1'' + \gamma_1'' \end{aligned}\right\} \quad \ldots \quad (9)$$

* This is, in fact, zero for the motion considered (*see* equation (12) of Part II), but is included here for the sake of generality.

since equation (1) expressed in complex form is

$$\alpha - \bar{\alpha} = (\alpha_1' + i\alpha_1'')e^{i\lambda t}$$
$$\gamma - \bar{\gamma} = (\gamma_1' + i\gamma_1'')e^{i\lambda t} .$$

The vertical displacement is defined by the vertical component of velocity $-V_0\gamma$ (measured positive downwards); the oscillatory part of this is $-V_0(\gamma - \gamma)$, so that

$$dz_a/dt = - V_0(\gamma - \bar{\gamma}) = - V_0(\gamma_1' + i\gamma_1'')e^{i\lambda t}$$

or

$$z_a' = - \frac{V_0}{\lambda}\gamma_1''$$

$$z_a'' = \frac{V_0}{\lambda}\gamma_1'$$

i.e.,

$$\left.\begin{aligned} \frac{z_a'}{c} &= - \frac{\gamma_1''}{\omega} \\ \frac{z_a''}{c} &= \frac{\gamma_1'}{\omega} . \end{aligned}\right\} \quad \ldots \quad (10)$$

Equations (9) and (10) give the contribution to α_0 and z_0 due to the oscillatory motion of the aircraft itself.

5. *Expression for the Non-steady Lift.*—The contributions of equations (8), (9), (10) to z_0 and α_0 must now be substituted in the expression (2) for the lift.

But first, it must be noted that the expression (2) relates to an aerofoil in two-dimensional flow; for a finite aspect ratio wing, a certain allowance must be made by varying the parameter 2π in this expression. It can be shown that for three-dimensional flow this parameter changes from its normal value a at $\omega = 0$ to 2π at high frequencies. The value a will be used in this investigation and this will slightly underestimate the lift force.

Also, the expression (2) contains terms (*e.g.*, l_z'' and l_α''), due to the so-called ' Kelvin's Impulses ', which account for the inertia of the air being moved by the oscillating wing. A good approximation is obtained by omitting these terms in the consideration of the pure gust forces (considered in section 3), but including them in the consideration of the aircraft motion (section 4). The terms to be omitted are those not involving A' or B' (leaving $A' - iB'$ as a factor).

The expression for the lift on the aircraft is therefore

$$C_L/2 \text{ (oscillatory)} = (i\omega l_z' + l_z)z_0/c + \{i\omega(l_\alpha' - \tfrac{1}{8}a) + l_\alpha\}\alpha_0$$
$$+ (- \omega^2 l_z'' + i\omega l_z' + l_z)z_a/c + (- \omega^2 l_\alpha'' + i\omega l_\alpha' + l_\alpha)\alpha_a . \quad \ldots \quad (11)$$

where the derivatives are given by

$$\left.\begin{aligned} l_z'' &= \tfrac{1}{8}a & l_\alpha'' &= \tfrac{1}{16}a \\ l_z' &= \tfrac{1}{2}aA' & l_\alpha' &= \tfrac{1}{8}a(1 + 3A' - 4B'/\omega) \\ l_z &= \tfrac{1}{2}a\omega B' & l_\alpha &= \tfrac{1}{8}a(4A' + 3\omega B') . \end{aligned}\right\} \quad \ldots \quad (12)$$

Now

$$C_L/a \text{ (oscillatory)} = \alpha_L - \bar{\alpha}$$

in the notation of equation (14) of Part II.

Hence, substituting for the derivatives, we obtain

$$\alpha_L - \bar{\alpha} = (A' - iB')\{i\omega z_0/c + (1 + \tfrac{3}{4}i\omega)\alpha_0\}$$
$$+ \{(A' - iB')i\omega - \tfrac{1}{4}\omega^2\} z_a/c + \{(A' - iB')(1 + \tfrac{3}{4}i\omega) - \tfrac{1}{8}\omega^2 + \tfrac{1}{4}i\omega\}\alpha_a . \quad .. \quad (13)$$

Substituting for z_0 and α_0 from equation (8) and for z_a and α_a from equations (9) and (10), we finally derive from the real part of equation (13), after some reduction, the following expression for the unsteady lift

$$\alpha_L - \bar{\alpha} = C' \cos \lambda t - D' \sin \lambda t ,$$

where

$$\left.\begin{aligned} C' &= \tfrac{1}{4}\varepsilon_1\{A'(1 + 3\cos\omega) - 3B'\sin\omega\} - \gamma_1'A' - \gamma_1''(B' - \tfrac{1}{4}\omega) \\ D' &= -\tfrac{1}{4}\varepsilon_1\{3A'\sin\omega + B'(1 + 3\cos\omega)\} + \gamma_1'(B' - \tfrac{1}{4}\omega) - \gamma_1''A' \end{aligned}\right\} . \quad .. \quad (14)$$

In deriving this expression, use has been made of the fact that $\alpha_a = 0$ (*see* Part II, equation (20)).

APPENDIX VII

Sample Calculations

1. *The En-Route Case* (*4-Engine Aircraft*).—1.1. We write

$$\text{var}\,\gamma = A\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2 (1 + B\beta + C\beta^2) ,$$

where

$$\beta = \frac{\bar{\gamma}}{\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)} .$$

Variables		*Variance* × 10^4 (*See* Appendix V)		Contribution to :— *A*	*AB*	*AC*
Power, engine speed and manifold pressure	Above F.T.H.	$\frac{2 \cdot 797}{n'}\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2$		$\frac{2 \cdot 797}{n'}$		
			or		0	0
	Below F.T.H.	$\frac{2 \cdot 460}{n'}\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2$		$\frac{2 \cdot 460}{n'}$		
Temperature	Above F.T.H.	$27 \cdot 58\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2$		27·58		
			or		0	0
	Below F.T.H.	$12 \cdot 43\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2$		12·43		
Drag ..		$1 \cdot 25\left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2 (1 - \beta)^2$		1·25	−2·50	1·25

Variables	Variance $\times 10^4$ (See Appendix V)	Contribution to:— A	AB	AC
Speed ..	$4 \cdot 41 \left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2 (0 \cdot 2 - 0 \cdot 8\beta)^2$	0·18	−1·41	2·82
Weight ..	$1 \cdot 96 \left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2 (1 - 0 \cdot 28\beta)^2$	1·96	−1·10	0·16

Thus, above full throttle height,

$$\text{var}\,\gamma = \left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2 \left[30 \cdot 97 + \frac{2 \cdot 797}{n'} - 5 \cdot 01\beta + 4 \cdot 23\beta^2\right] \times 10^{-4}$$

and below full throttle height,

$$\text{var}\,\gamma = \left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2 \left[15 \cdot 82 + \frac{2 \cdot 46}{n'} - 5 \cdot 01\beta + 4 \cdot 23\beta^2\right] \times 10^{-4} \,.$$

Taking $\beta = 0 \cdot 2$ (as an estimate) and $n' = 3$ we obtain

$$\text{var}\,\gamma = 31 \cdot 07 \left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2 \times 10^{-4} \text{ above F.T.H.}$$

$$= 15 \cdot 81 \left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right)^2 \times 10^{-4} \text{ below F.T.H.}$$

1.2. From equation (43), Part II, we have

$$Q = {}_2p_0' + {}_2p_1' n \Pi_2 + 2p_2' \frac{n(n-1)}{2\,!} (\Pi_2^2 - {}_1p_2' \Pi_1^2) \,.$$

From Part III, section 5.4, $\Pi_2 = 0 \cdot 692 \times 10^{-3}$, $\Pi_1 = 0 \cdot 238 \times 10^{-3}$. Estimate ${}_2p_0' = 0$, ${}_1p_2' = 1 \cdot 0$, ${}_2p_2' = 1 \cdot 0$.

Hence for a four-engined aircraft

$$10^{-5} = {}_2p_1' \, 2 \cdot 768 \times 10^{-3} + 2 \cdot 534 \times 10^{-6} \,. \quad \text{.. .. ..} \quad (1)$$

Therefore ${}_2p_1' = 2 \cdot 698 \times 10^{-3}$.

Corresponding $'t' = 2 \cdot 78$.

Since

$$\bar{\gamma} = \text{datum performance} + \text{performance margin}$$

$$= 0 \cdot 049 \frac{\bar{D}}{\bar{W}} + 2 \cdot 78 (15 \cdot 81)^{1/2} \left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}\right) \times 10^{-2} \text{ (below F.T.H.)}$$

$$\bar{\gamma} = 0 \cdot 180 \frac{\bar{D}}{\bar{W}}$$

1.3. Check on assumed β:

$$\beta = \frac{0 \cdot 180}{1 + 0 \cdot 180} = 0 \cdot 153 \,.$$

This would change the coefficient of var γ to 15·97 and give $\bar{\gamma} = 0\cdot181\bar{D}/\bar{W}$ and no further approximations are needed. It will be noted that the case we have taken is the most sensitive to change in β.

1.4. Check on ${}_2p_0{}' = 0$:

with one engine inoperative

$$\bar{\gamma}_1 = 0\cdot181 \frac{\bar{D}}{\bar{W}}.$$

Hence with all engines operative, assuming in the previous case the propeller of inoperative engine was feathered and $\Delta D/W = 0$, since

$$\bar{\gamma}_0 = \bar{\gamma}_1 + \left(\frac{\bar{D}}{\bar{W}} + \bar{\gamma}_1\right)\frac{1}{n-1}$$

$$\bar{\gamma}_0 = \frac{\bar{D}}{\bar{W}} 0\cdot576 .$$

$$\text{Performance margin} = 0\cdot576 \frac{\bar{D}}{\bar{W}} - 0\cdot049 \frac{\bar{D}}{\bar{W}}$$

$$= 0\cdot527 \frac{\bar{D}}{\bar{W}}$$

$$'t' \simeq \frac{0\cdot527\bar{D}/\bar{W})}{(15\cdot97)^{1/2}\left(\frac{\bar{D}}{\bar{W}} + 0\cdot576\frac{\bar{D}}{\bar{W}}\right) \times 10^{-2}}$$

$= 8\cdot4$ for which the corresponding probability $({}_2p_0{}')$ is of the order 10^{-16}.

1.5. Check on ${}_2p_2{}' = 1\cdot0$:

With two engines inoperative

$$\bar{\gamma}_2 = -0\cdot212 \frac{\bar{D}}{\bar{W}}$$

$$'t' \simeq -\frac{0\cdot261}{(15\cdot97)^{1/2}(1 - 0\cdot212) + 10^{-2}}$$

$= -8\cdot3$ for which the corresponding probability is $0\cdot99999\ldots$

1.6. We may note that the relative contributions to the stage incident rate, are from (1)

All engines operative = 0 per cent

One engine inoperative = 71 per cent

Two engines inoperative = 29 per cent.

2. *Take-off and Landing Cases.*—2.1. The variance of γ is not very sensitive to the value of $\bar{\gamma}$ assumed; for the first approximation a value of $\bar{\gamma}$, for the dominant case, and the incident probabilities in the other cases are estimated. For the case of one engine inoperative the first estimates were $p_0' = 0$ and $p_2' = 1 \cdot 0$. The requirement is then worked for one value of $\bar{D}/\bar{W}$ and expressed in terms of $\bar{D}/\bar{W}$. This relation is then used to obtain the appropriate values of $\bar{\gamma}$ for the second approximation and in practice no further approximations were required.

2.2. A typical calculation for a four-engined aircraft with one engine inoperative is given below: the variance with all engines operative and two engines inoperative are deduced similarly. The method of deriving '*t*' and checking on the assumed probabilities is the same as in section 1.

The full calculation as given here is illustrative and is not necessary in practice.

Case: Approach

Number of Engines: 4 *Number of inoperative Engines*: 1

$$\bar{M} = 0 \cdot 6 \frac{\bar{D}}{\bar{W}}, \quad \gamma_a = 0 \cdot 042 \frac{\bar{D}}{\bar{W}}, \quad X = 1 \cdot 846, \quad Y = 7 \cdot 01$$

	Estimated $\bar{\gamma}$*	0·0110	0·0137	0·0164	0·0192	0·0219
1	$\bar{D}/\bar{W}$	0·08	0·10	0·12	0·14	0·16
2	$(\bar{D}/\bar{W})^2$	0·0064	0·0100	0·0144	0·0196	0·0256
3	$(\bar{D}/\bar{W} + \bar{\gamma})$	0·0910	0·1137	0·1364	0·1592	0·1819
4	$(\bar{D}/\bar{W} + \bar{\gamma})^2$	0·00828	0·0129	0·0186	0·0253	0·0327
5	$0 \cdot 6(\bar{D}/\bar{W} + \bar{\gamma})$	0·0546	0·0682	0·0818	0·0955	0·1091
6	$2 \cdot 0\bar{D}/\bar{W}$	0·160	0·200	0·240	0·280	0·320
7	$4 \cdot 0\bar{M}$	0·192	0·240	0·288	0·336	0·384
8	(5) + (6) − (7)	0·0226	0·0282	0·0338	0·0395	0·0451
9	$(8)^2$	0·00051	0·00080	0·00114	0·00156	0·00203
10	$\bar{\gamma} + 2\bar{M}$	0·1070	0·1337	0·1604	0·1872	0·2139
11	$(\bar{\gamma} + 2\bar{M})^2$	0·01145	0·0179	0·02575	0·03505	0·04575
12	(4) × X	0·0153	0·0238	0·03435	0·0467	0·0604
13	(2) × Y	0·04485	0·0701	0·1009	0·1374	0·1795
14	(9) × 4·41	0·00225	0·00355	0·0050	0·0069	0·30895
15	(var $\Delta\gamma_a \times 10^4$) = (11) + (12) + (13) + (14)	0·07385	0·11535	0·1660	0·22605	0·2946
16	$(15)^{1/2} \times 10^{-2}$	0·002718	0·003398	0·004075	0·004755	0·005427
17	(16) × '*t*'†	0·00758	0·00948	0·01136	0·01326	0·01515
18	γ_a	0·00336	0·00430	0·00504	0·00588	0·00672
19	$\bar{\gamma}$ = (17) + (18)	0·01094	0·01368	0·01630	0·01914	0·02187

* First estimate was $0 \cdot 15\bar{D}/\bar{W}$ which gave $0 \cdot 137\bar{D}/\bar{W}$ for second approximation.
† '*t*' obtained as in section 1.0.

Note.—$X = \Sigma_1^4$ coefficient of variance of $\Delta\gamma_i$ } From Appendix V.
Y coefficient of variance of γ due to drag (x_6) }

APPENDIX VIII

Consideration of Turbo-Jet Engines

1. *Performance Relation.*—For turbo-jet engines the thrust term (T/W) is given by

$$\frac{T}{W}=\frac{\bar{T}}{\bar{W}}\frac{T_{\bar{n}}}{\bar{T}}\left(\frac{p}{\bar{p}}\right)^{1-\frac{1}{2}s}\left(\frac{N}{\bar{N}}\right)^{p}\left(\frac{\theta}{\bar{\theta}}\right)^{-\frac{1}{2}p}\left(\frac{V_i}{\bar{V}_i}\right)^{s}\left(\frac{W}{\bar{W}}\right)^{-1}.$$

Only the first term of equation (4), Part II, is changed and becomes

$$\left(\frac{\bar{D}}{\bar{W}}+\bar{\gamma}\right)\left[\left(\frac{T_{\bar{n}}}{\bar{T}}\right)\left(\frac{p}{\bar{p}}\right)^{1-\frac{1}{2}s}\left(\frac{N}{\bar{N}}\right)^{p}\left(\frac{\theta}{\bar{\theta}}\right)^{-\frac{1}{2}p}\left(\frac{V_i}{\bar{V}}\right)^{s}\left(\frac{W}{\bar{W}}\right)^{-1}\right]$$

For most current engines

p is between 3 and 4,

s is between $-0\cdot1$ and 0 over the speed range in which we are interested.

2. *Variability of Statistical Variables.*—Little data is as yet available on engine to engine variation but analysis of the results of thrust tests on 35 engines of one type gave an estimated standard deviation of thrust of $1\cdot1$ per cent—coincidentally the same as the average value for the power of piston engines. None of the other standard deviations are affected.

3. *Variance of $\Delta\gamma_i$.*—Values of the variance of $\Delta\gamma_i$ have been derived using the mean of the above coefficients and are given below where they differ from those of Appendix V with, for comparison, the values for a piston engine above F.T.H.

	Turbo-jet	*Piston Engine above* F.T.H.
Thrust + engine speed or power + engine speed + manifold pressure	$\left(\frac{\bar{D}}{\bar{W}}+\bar{\gamma}\right)^2\times\frac{4\cdot273}{n'}$	$\left(\frac{\bar{D}}{\bar{W}}+\bar{\gamma}\right)^2\times\frac{2\cdot797}{n'}$
Pressure	$\left(\frac{\bar{D}}{\bar{W}}+\bar{\gamma}\right)^2\times1\cdot103$	$\left(\frac{\bar{D}}{\bar{W}}+\bar{\gamma}\right)^2\times$ $1\cdot69$ above F.T.H. $0\cdot25$ below F.T.H.
Temperature	$\left(\frac{\bar{D}}{\bar{W}}+\bar{\gamma}\right)^2\times39\cdot61$	$\left(\frac{\bar{D}}{\bar{W}}+\bar{\gamma}\right)^2\times$ $27\cdot58$ above F.T.H. $12\cdot43$ below F.T.H.
Speed ..	$4\cdot41\left[0\cdot05\left(\frac{\bar{D}}{\bar{W}}+\bar{\gamma}\right)+2\cdot0\frac{\bar{D}}{\bar{W}}\ 4\cdot0\bar{M}\right]^2$	$4\cdot41\left[0\cdot6\left(\frac{\bar{D}}{\bar{W}}+\bar{\gamma}\right)+2\cdot0\frac{\bar{D}}{\bar{W}}-4\cdot0\bar{M}\right]^2$

APPENDIX IX

Engine Failure Statistics

Operator	Aircraft	Number of engines	Mean hours/ flight	Failures during					Total Engine hours	Total Engine T.O's
				T.O.	Climb	Cruise	Approach	Baulked landing		
I	A	2	2·23	0	8	8	0	0	25,000	11,200
	B	2	1·94	0	2	7	0	0	65,400	33,600
	C	2	0·75	1	6	6	0	0	73,400	97,600
	D	3	1·22	0	1	1	0	0	10,500	8,601
	E	2	1·49	0	0	2	0	0	5,000	3,350
II	F	4	2·85	1	18	67	0	0	165,692	58,188
	G	4	7·77	2	13	53	0	0	397,876	51,264
	H	4	5·96	1	0	2	0	0	3,908	656
	I	2	2·19	6	23	10	0	0	75,694	34,490
	B	2	4·54	1	14	11	0	0	212,100	46,746
	J	4	7·09	0	9	14	0	0	91,508	12,912
III	K	4	2·75	1	21	33	0	0	166,800	60,800
	B	2	1·33	1	6	9	0	0	70,200	52,700
IV	B	2	2·23	1	0	6	0	0	33,398	14,952
	L	4	3·27	0	0	1	0	0	14,224	4,352
V	Not stated	2	1·99	1	0	6	0	0	64,000	32,080
	Not stated	4	5·96	0	1	8	0	0	84,000	14,092
			TOTALS	16	122	244	0	0	1,558,700	537,583

APPENDIX X

Note on the Declared Temperature System

1. Let γ_m be the climb gradient margin required when temperature is treated as a parameter (*i.e.*, the margin associated with the ambient temperature system),

γ_m' be the climb gradient margin required when local temperature is treated as a statistical variable,

both γ_m and γ_m' being based on the (same) given incident probability.

Thus at the declared temperature θ_D the climb gradient $\bar{\gamma}(\theta_D)$ of the aircraft with all other conditions standard must be equal to or greater than $(\gamma_m + \gamma_d)$; hence for an aircraft just meeting the requirement,

$$\bar{\gamma}(\theta_D) = \gamma_m = \gamma_d . \quad \ldots \quad (1)$$

At the local mean temperature, $\bar{\theta}$, this aircraft will have climb gradient $\bar{\gamma}(\bar{\theta})$, where

$$\bar{\gamma}(\bar{\theta}) \simeq \bar{\gamma}(\bar{\theta}_D) + \left(\frac{\overline{\partial \gamma}}{\partial(\theta/\bar{\theta})}\right)\frac{\bar{\theta} - \theta_D}{\theta} . \quad \ldots \quad (2)$$

The climb gradient margin of the aircraft is then $\bar{\gamma}(\bar{\theta}) - \gamma_d$, which compares with the margin γ_m' for the given incident rate. If the standard of safety is not to be lowered, $\bar{\gamma}(\bar{\theta}) - \gamma_d > \gamma_m'$, and therefore the declared temperature involving the minimum operational penalty without increasing the incident rate corresponds to

$$\bar{\gamma}(\bar{\theta}) = \gamma_m' + \gamma_d . \quad \ldots \quad (3)$$

2. From (1), (2) and (3), it follows that this optimum declared temperature θ_{D0} is defined by the relationship

$$\gamma_m' = \gamma_m + \left(\frac{\overline{\partial \gamma}}{\partial(\theta/\bar{\theta})}\right)\frac{\bar{\theta} - \theta_{D0}}{\bar{\theta}} . \quad \ldots \quad (4)$$

Now from Appendix V,

$$\left(\frac{\overline{\partial \gamma}}{\partial(\theta/\bar{\theta})}\right) = k\{\bar{D}/\bar{W} + \bar{\gamma}(\bar{\theta})\}$$

$$= k(\bar{D}/\bar{W} + \gamma_m' + \gamma_d) .$$

Hence from (4),

$$\theta_{D0} = \bar{\theta}\left(1 - \frac{\gamma_m' - \gamma_m}{k(\bar{D}/\bar{W} + \gamma_m' + \gamma_d)}\right) . \quad \ldots \quad (5)$$

Now γ_m and γ_m' are related by

$$\frac{\gamma_m}{S_\gamma} = \frac{\gamma_m'}{(S_\gamma^2 + {}_\theta S_\gamma^2)^{1/2}} ,$$

where S_γ is the standard deviation of γ due to all variables except temperature, and ${}_\theta S_\gamma$ is the standard deviation of γ due to local temperature alone.

Writing $S_\gamma = \gamma_m/t_Q$, where t_Q is a function of the incident probability Q, we have

$$\gamma_m' = t_Q\left(\frac{\gamma_m^2}{t_Q^2} + {}_\theta S_\gamma^2\right)^{1/2} . \quad \ldots \quad (6)$$

Now ${}_\theta S_\lambda = (\bar{D}/\bar{W} + \bar{\gamma}(\bar{\theta}))kS_\theta$ from Appendix V, S_θ being the local standard deviation of $\theta/\bar{\theta}$;

Therefore

$$\gamma_m' = [\gamma_m^2 + t_Q^2\{\bar{D}/\bar{W} + \bar{\gamma}(\bar{\theta})\}^2k^2S_\theta^2]^{1/2} . \quad \ldots \quad (7)$$

Since $\bar{\gamma}(\bar{\theta}) = \gamma_m' + \gamma_d$, equation (7) becomes

$$\gamma_m'^2 = \gamma_m^2 + t_Q^2(\bar{D}/\bar{W} + \gamma_m' + \gamma_d)^2k^2S^2_\theta . \quad \ldots \quad (8)$$

Writing $\bar{D}/\bar{W} + \gamma_d = \phi\gamma_m$, where ϕ is a constant varying little with $\bar{D}/\bar{W}$, (8) gives a quadratic equation in γ_m', the solution of which can be shown to be

$$\gamma_m' = \frac{\psi^2\phi \pm (1 - \psi^2 + \psi^2\phi^2)^{1/2}}{1 - \psi^2}\gamma_m$$

where we have written $\psi = t_QkS_\theta$.

3. Taking the positive solution, which corresponds to the positive sign, substitution in equation (5) gives

$$\theta_{D0} - \bar{\theta} = -\frac{\bar{\theta}}{k}\frac{(\psi^2\phi + \psi^2 - 1) + (\psi^2\phi^2 - \psi^2 + 1)^{1/2}}{\phi + (\psi^2\phi^2 - \psi^2 + 1)^{1/2}} .$$

Now ψ is of the order 10^{-1} and ϕ is of the order 10, so we may approximate this to

$$\theta_{D0} - \bar{\theta} \simeq - \frac{\bar{\theta}}{k} \frac{(\psi^2\phi - 1) + (\psi^2\phi^2 + 1)^{1/2}}{\phi + (\psi^2\phi^2 + 1)^{1/2}} \quad \ldots \quad (9)$$

It has been suggested that the declared temperature should be defined in terms of a (constant) probability of that temperature being exceeded. This means a fixed value of $(\theta_{D0} - \bar{\theta})/S_\theta'$ given by

$$t = \frac{\theta_{D0} - \bar{\theta}}{S_\theta'} = - \frac{1}{k}\left(\frac{\bar{\theta}}{S_\theta'}\right) \frac{(\psi^2\phi - 1) + (\psi^2\phi^2 + 1)^{1/2}}{\phi + (\psi^2\phi^2 + 1)^{1/2}} \quad \ldots \quad (10)$$

where $S_\theta' = \bar{\theta} S_\theta$, the local standard deviation of θ.

4. Remembering that $\psi = t_Q k S_\theta'/\bar{\theta}$ it will be seen that the required probability is a function of

(a) the flight case (t_Q and ϕ),

(b) the type of power plant (k),

(c) the locality ($S_\theta'/\bar{\theta}$).

Since it is only for the take-off case that the ambient and declared systems are alternative, the definition would presumably be 'fitted' to this case. Once the standards have been agreed, t_Q and ϕ will thus be fixed. For this given case, however, it may be shown that the probability varies from nearly 0·5 (that is the declared temperature nearly equal to the mean) to a temperature which will be exceeded on 10 per cent of the occasions. Definition of the declared temperature on a constant probability basis is not, therefore, a very satisfactory procedure.

5. From equation (9) we have, however, a direct relationship for declared temperature which is plotted in Fig. 10. It may be shown that variations of 30 deg in mean temperature ($\bar{\theta}$) are unlikely to effect the declared temperature by more than $\frac{1}{2}$ deg, so the curve has been based on International Standard Temperature at sea level which will be sufficiently accurate in practice.

The curve is based on the turbo-jet power plant, which, being the most sensitive to temperature, gives the worst case. For comparison, an indication of the curves for reciprocating engines (k from Appendix V) is also given; that the declared temperature does not apply equally well to all types of power plant is, of course, inherent in the scheme and generally appreciated.

6. The curve of Fig. 10 is, therefore, a convenient way of specifying the declared temperature. To determine the declared temperature it is only necessary to know the mean temperature and standard deviation of temperature for the airfield and period considered; these would normally be known to, or determinable by, the Meteorological Authority concerned.

ADDENDUM

Further Discussion after Receipt of United States Data on Engine Failures

1. While this report was being prepared, the earlier (unpublished) preliminary notes were discussed with the Civil Aeronautics Board (U.S.A.). As a result of this discussion, the C.A.B. have generously supplied data* on engine failure rates; these statistics covered some 2,800,000 flights and over 600,000,000 hours flying. The data arrived, unfortunately, too late to incorporate the results of this large-scale investigation in the text and numerical working of this report without considerable delay.

2. The following extract from the note* is, however, relevant:

> 'A comparison of the final results of Table V with the results obtained from a similar study in the United Kingdom indicates a marked consistency.'

From the aspect of this report one point requires discussion.

The figure quoted in the U.S. note for the cumulative probability failure en route is $0{\cdot}404 \times 10^{-3}$ while that originally given by the U.K. statistics and quoted in the preliminary papers was $0{\cdot}40 \times 10^{-3}$. The figure used in this report of $0{\cdot}692 \times 10^{-3}$ resulted from additional en-route statistics from one U.K. operator and statistics from two Continental operators; these additional en-route statistics are likely, for several reasons, to be pessimistic for general application and with the limited sample which was available have influenced the mean. However, this disparity is such as would be expected from a limited sample and the original apparently close agreement between the limited U.K. statistics and the more comprehensive U.S. statistics is regarded as coincidental.

3. An aspect covered by the U.S. statistics on which information was not previously available is the ratio of complete effective failures to all failures reported. This was $0{\cdot}5$ to a close approximation. An attempt to deal with this aspect has already been made in Part III, section 10.5 and it will be seen that the assumption made was, in fact, very pessimistic and the derived requirements would be little affected from this aspect and also the change in basic probability already mentioned.

4. These numerical changes do, however, reflect on the absolute numerical value of the tolerable incident rate. On the same *relative* incident rate (Π^x) as we have already used the numerical value would be changed from 10^{-5} to $1{\cdot}38 \times 10^{-6}$.

Since the sharp rise in the 'performance required' curve (Part III, section 10.4) occurs at approximately $[n(n-1)/2!]\Pi^2$ then for a four-engined aircraft, using the U.S. data for total failures, this gives a probability level of $0{\cdot}96 \times 10^{-6}$. It thus appears that the 'economically attainable' stage incident rate is in the order of one in a million.

For an airport catering for 100 departures a day this would mean, even if all aircraft were operated at the critical take-off weight, one take-off incident, due to performance deficiencies, in approximately 30 years. In finally fixing the tolerable incident rate the relative contribution, from this cause, to the total incident rate would need consideration in relation to the economic advantages which may accrue from a reduction in incident rate below that economically attainable.

* 'Frequency of engine failures in scheduled air carrier operation' by James S. Rice, C.A.B. June 3, 1948.

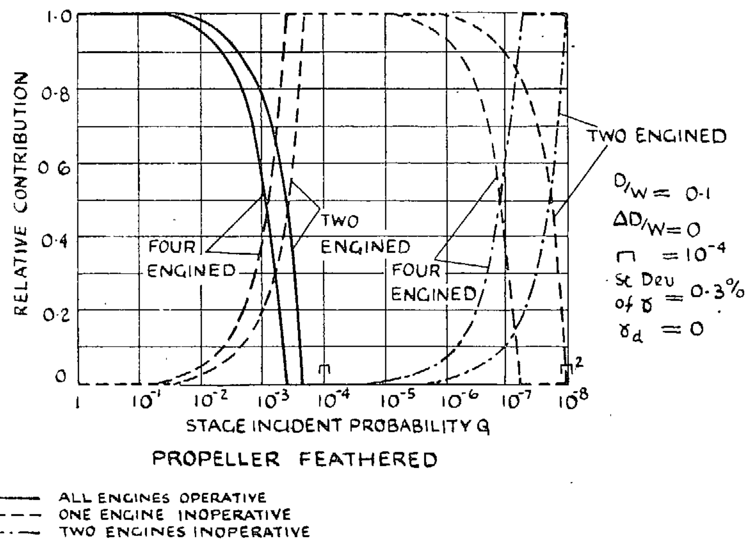


FIG. 1.

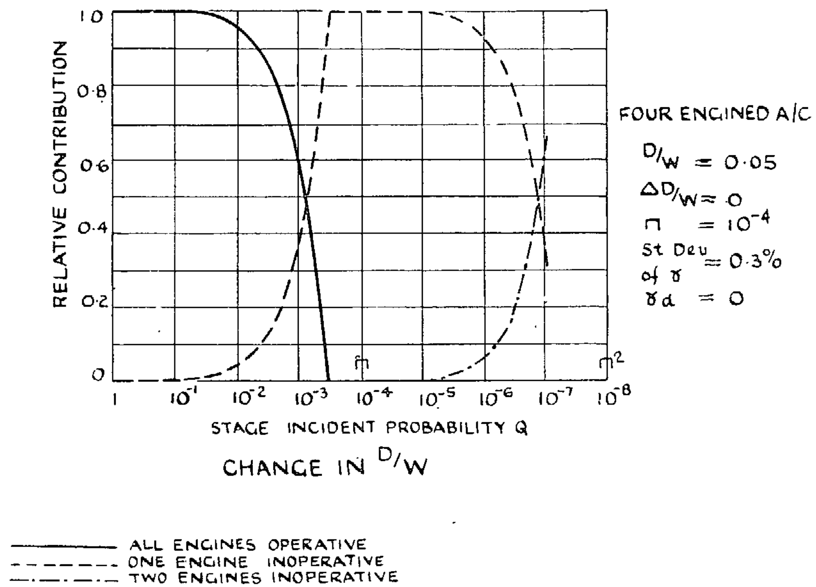


FIG. 3.

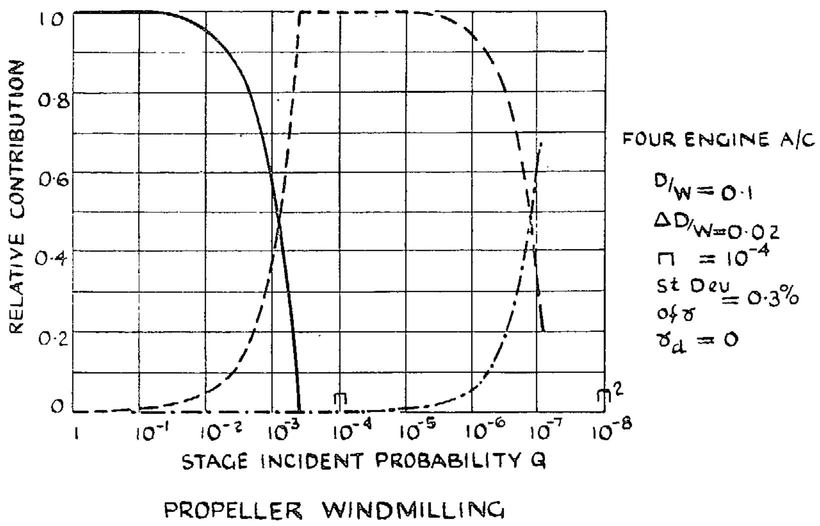


FIG. 2.

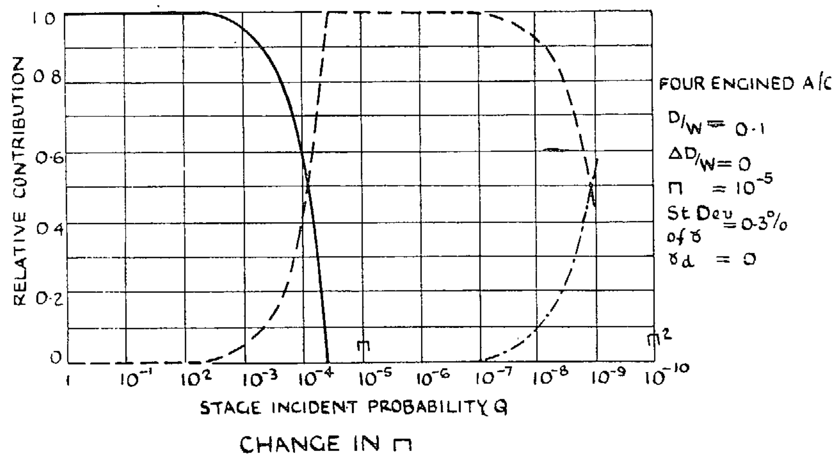
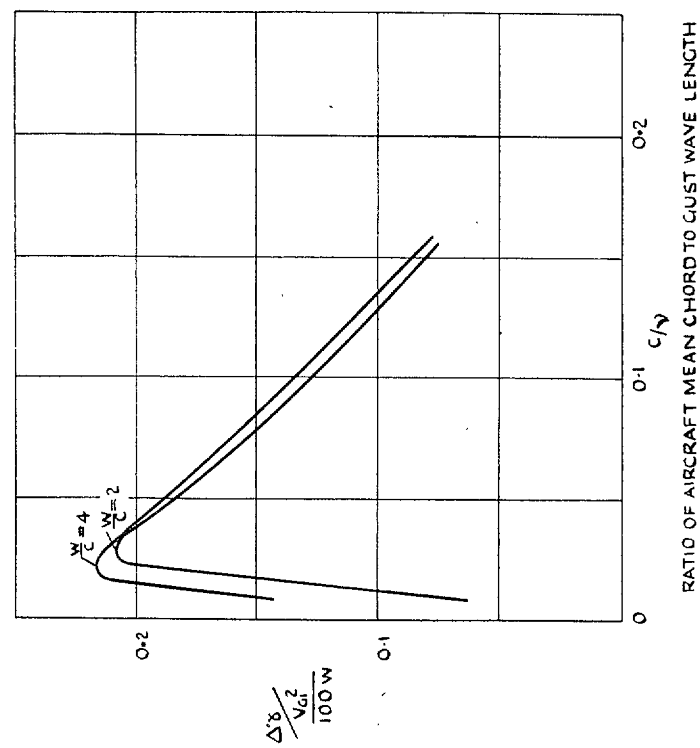


FIG. 4.

Dominant case illustrations.



Δδ = GAIN IN ANGLE OF CLIMB (AT SEA LEVEL CONDITIONS)
W = AIRCRAFT WING LOADING
GUST FORM ASSUMED $V_G = V_{G1} \cos \frac{2\pi V_0 t}{\lambda}$

FIG. 5. Effect of gustiness on climb gradient.

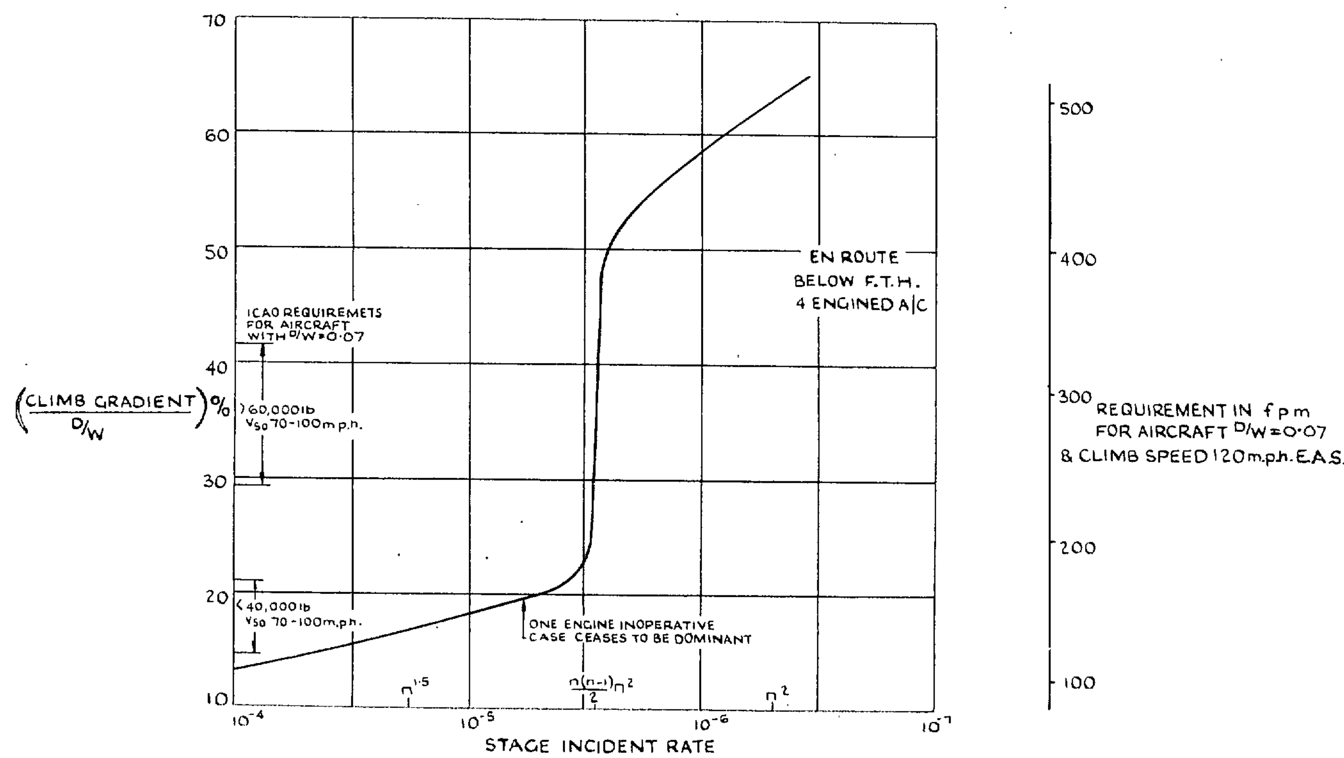


FIG. 6. Typical performance required with one engine inoperative.

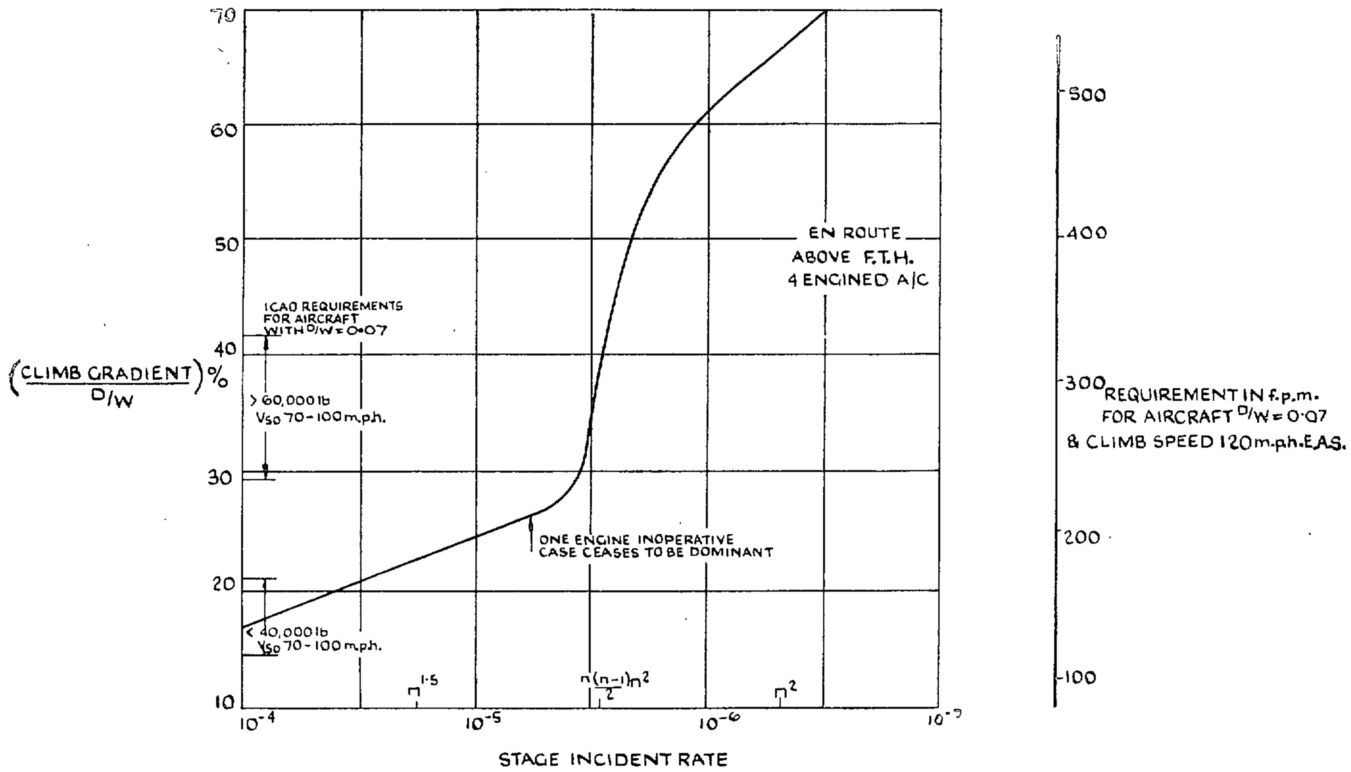


FIG. 7. Typical performance required with one engine inoperative.

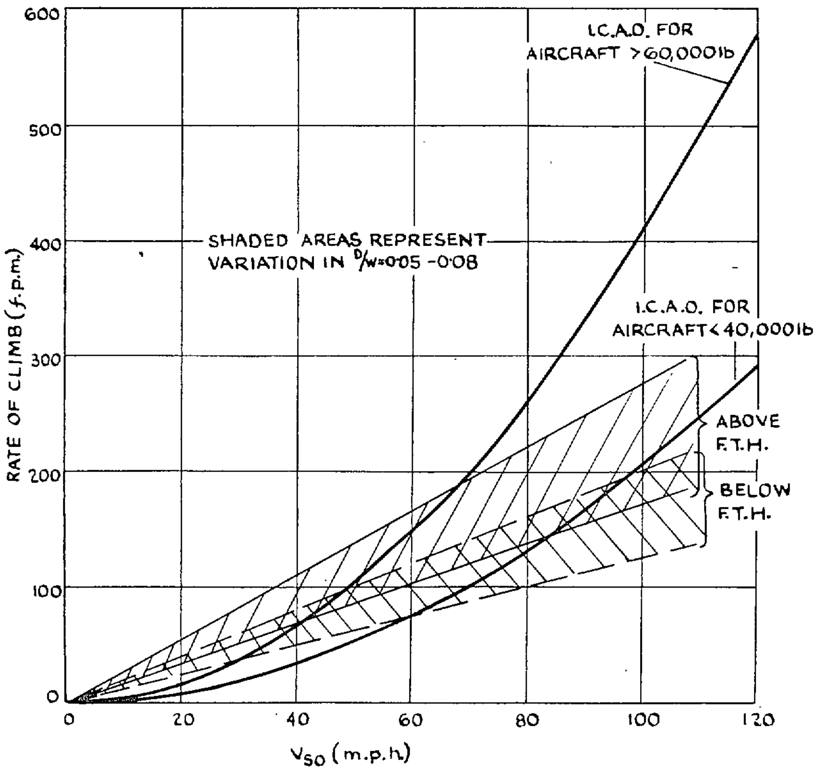


FIG. 8. Comparison of en-route climb standards.

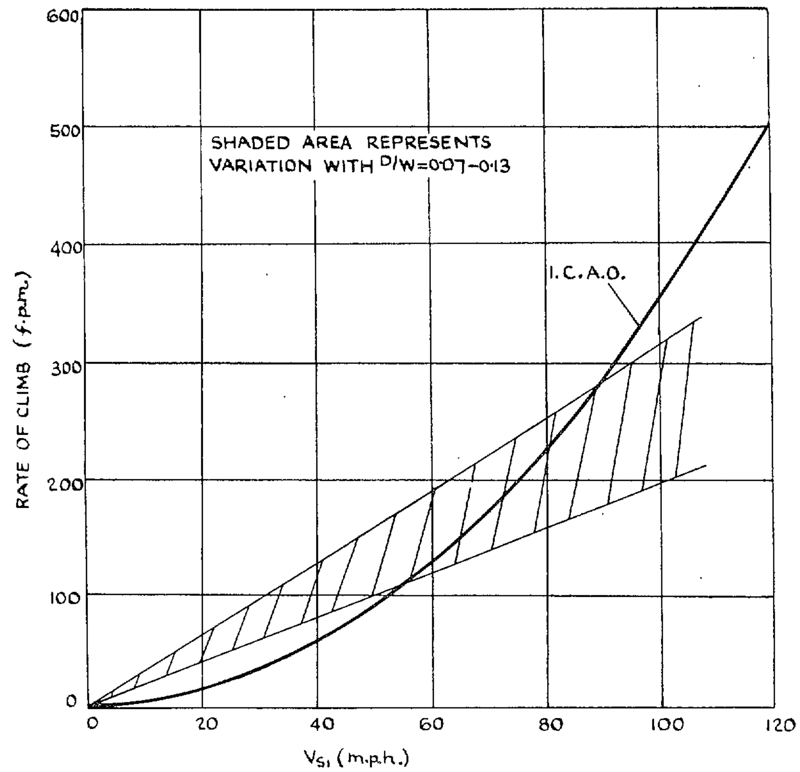


FIG. 9. Comparison of take-off climb standards.

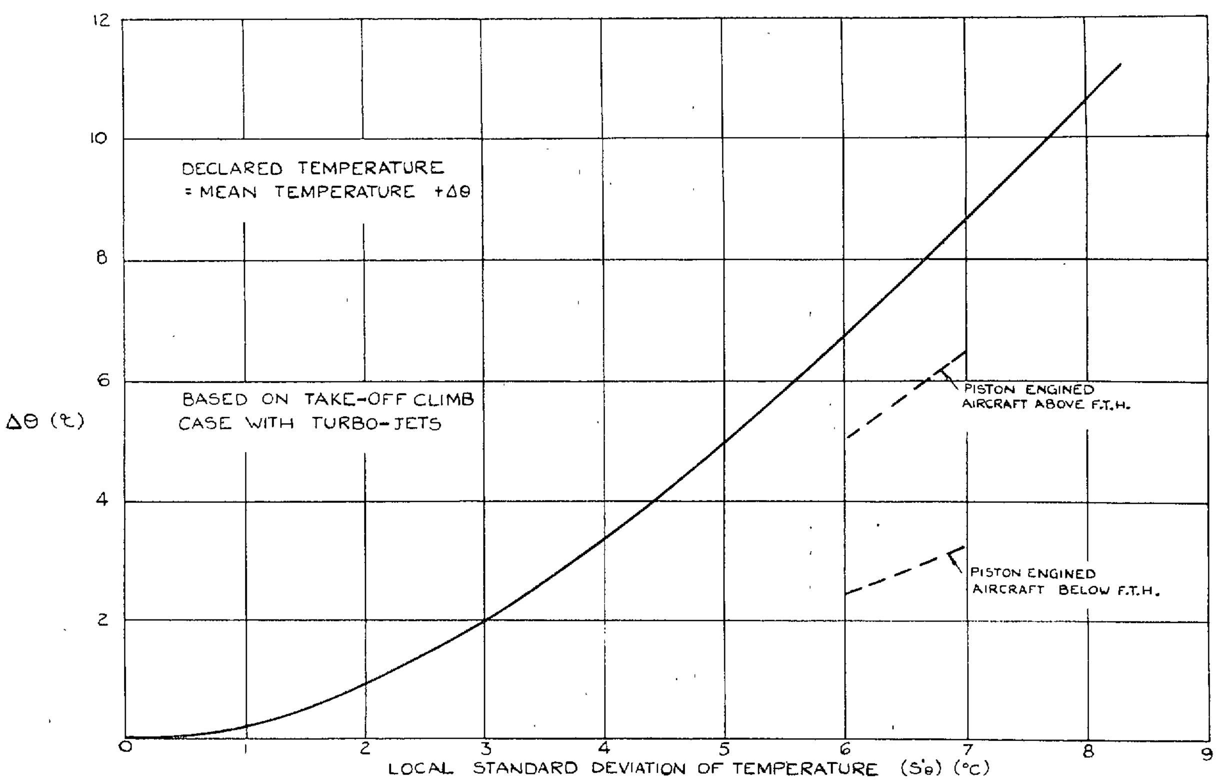


FIG. 10. Declared temperature relationship.

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