R. & M. No. 2619
(10,025, 10,317, 9,906, 10,852)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

The Asymptotic Theory of Boundary-layer Flow with Suction

Part I The Theory of Similar Velocity Distributions
Part II Flow with Uniform Suction
Part III Flow with Variation of Suction Velocity

By

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LONDON : HER MAJESTY'S STATIONERY OFFICE
1952

PRICE 12*s* 6*d* NET

The Asymptotic Theory of Boundary-layer Flow with Suction

(Parts I, II, and III)

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Reports and Memoranda No. 2619
September, 1947

General Summary.—The subject of this report is the steady two-dimensional flow of a boundary layer over a permeable surface through which the fluid is withdrawn at a known rate of suction. This rate of suction is assumed, in accordance with the hypotheses of the boundary layer, to be small compared with the stream velocity, and of order $R^{-1/2}$, where R is the Reynolds number. It is supposed here that the suction is relatively large, though still of the same order, and in these circumstances the three following conditions hold approximately.

(i) The boundary-layer thickness is inversely proportional to the velocity of suction.

(ii) The velocity distribution within the boundary layer is the "asymptotic suction profile"

$$\frac{u}{U} = 1 - e^{-\frac{v_0 y}{\nu}}$$

where U is the velocity outside the boundary layer and v_0 is the suction velocity.

(iii) The skin friction is equal to $\rho U v_0$, where ρ is the density of the fluid.

These give the initial approximation to the behaviour of the boundary layer. Using a method of successive approximation we can then find a series in inverse powers of v_0 which formally satisfies the boundary layer equations and represents the solution either exactly or asymptotically for large values of v_0. In the terms of this series the effects of varying stream velocity or suction velocity appear.

Part I deals with the similar solutions of the boundary-layer equations, Part II with an arbitrary pressure distribution but constant suction velocity, and Part III with the general problem. Thus the results of Parts I and II can be obtained from Part III, but they are of interest in themselves. Attempts are made in both Parts I and II to find when separation occurs, but only rough estimates can be made as the series do not converge well. In Part II the theory is applied to the flow over a porous circular cylinder in a uniform stream, and also to the use of suction round the nose of an aerofoil to prevent stalling at high incidence.

The only previous work on this approach appears to be a report by Pretsch[1], which according to Mangler[2] contains a study of the similar profiles on the same lines as Part I. The report by Pretsch has not been examined, and it is therefore not known if his results agree with those given here. A special case of Part I is in course of publication[3].

CONTENTS

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Part I. The Theory of Similar Velocity Distributions.

Summary.—Generalising an earlier report[3], we here consider the similar solutions of the boundary-layer equations. These are found when the external velocity is of the form $U = cx^m$ and the suction velocity of the form $v_0 = kx^{\frac{1}{2}(m-1)}$. In these circumstances the equation of motion can be reduced to an ordinary non-linear equation whose boundary conditions determine the magnitude of the suction velocity. This non-linear equation was given originally by Falkner and Skan[4] for $v_0 = 0$, and numerical solutions were obtained with the aid of a differential analyser by Hartree[5], after a simple transformation. We take the equation in Hartree's form and by the method of R. & M. 2298[3], expand the solution in an asymptotic series for large suction quantities. The first approximation to the solution gives the velocity distribution within the boundary layer as that discovered by Griffith and Meredith[6] and three more terms of the series have been found. The series for the skin friction, the displacement and momentum thicknesses, and their ratio H are obtained from the velocity distribution. Numerical results are given in the tables and graphs for various amounts of suction.

1. *Introduction.*—The solution of the boundary layer equations without suction when $U = cx^m$ was first given by Falkner and Skan[4] who reduced the equation of motion to an ordinary third-order non-linear equation. The velocity distributions at different sections of the boundary layer are similar, and it was shown by Goldstein[7] that except for $U = ae^{kx}$ this is the only such solution.

These results can be extended to give solutions which involve boundary-layer suction through a permeable surface. This fact is pointed out briefly by Preston[8] and more fully by Goldstein[9], while Thwaites[10] has made a more detailed examination of the conditions for similar solutions with suction. The earliest investigations of these similar solutions were by Mangler[11] and Hoistein[12], referred to by Mangler[2] in the A.V.A. Monograph on boundary layers. Schlichting and Bussmann[13] considered the particular case $m = 0$, the flat plate with suction proportional to $x^{-1/2}$ which was also investigated by Thwaites[14].

The equation of motion of the boundary layer is, in the usual notation,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (1)$$

and u and v are derived from the stream function by the equations

$$\left.\begin{aligned} u &= \frac{\partial\psi}{\partial y}, \\ v &= -\frac{\partial\psi}{\partial x}. \end{aligned}\right\} \qquad (2)$$

When the external velocity distribution is

$$U = cx^m \qquad (3)$$

the partial differential equation (1) is reduced to an ordinary equation by the substitutions

$$\psi = (c\nu)^{1/2}\, x^{(m+1)/2} f(\eta) = (U\nu x)^{1/2} f(\eta), \qquad (4)$$

$$\eta = \left(\frac{c}{\nu}\right)^{1/2} x^{m-(1)/2} y = \left(\frac{U}{\nu x}\right)^{1/2} y. \qquad (5)$$

Then

$$u = Uf'(\eta), \qquad (6)$$

$$v = -\frac{1}{2}\left(\frac{U\nu}{x}\right)^{1/2}\Big[(m+1)\,f(\eta) + (m-1)\,\eta f'(\eta)\Big], \qquad (7)$$

and equation (1) becomes

$$mf'^2 - \tfrac{1}{2}(m+1)\,ff'' = m + f'''. \qquad (8)$$

The boundary conditions are obtained by considering u and from equation (6) we have

$$\left.\begin{array}{l} f'(0) = 0 \\ f'(\infty) = 1 \end{array}\right\}. \qquad (9)$$

Also

$$v(x, 0) = -\tfrac{1}{2}(m + 1)f(0) \left(\frac{U\nu}{x}\right)^{1/2} \qquad (10)$$

so that $f(0)$ determines the magnitude of the suction velocity.

Hartree[3] made the transformation

$$F = \left(\frac{m+1}{2}\right)^{1/2} f, \qquad (11)$$

$$Y = \left(\frac{m+1}{2}\right)^{1/2} \eta, \qquad (12)$$

and studied equation (8) in the form

$$F''' + FF'' + \beta(1 - F'^2) = 0, \qquad (13)$$

where
$$\beta = \frac{2m}{m+1}. \qquad (14)$$

The boundary conditions for equation (13) are

$$\left.\begin{array}{l} F'(0) = 0 \\ F'(\infty) = 1 \end{array}\right\} \qquad (15)$$

and if

$$F(0) = K \qquad (16)$$

we have

$$v(x, 0) = -v_0 = -K\left(\frac{m+1}{2}\right)^{1/2}\left(\frac{U\nu}{x}\right)^{1/2}, \qquad (17)$$

where v_0 is the suction velocity.

If c is negative, or $m < -1$, the analysis must be modified in order to make the square roots real, and we find in place of equation (13) the equation

$$-F''' + FF'' + \beta(1 - F'^2) = 0 \qquad (18)$$

with boundary conditions given by equation (15), and we take

$$F(0) = -K \qquad (19)$$

in order that positive values of K shall give suction. When both $c < 0$, $m < -1$ we have equations (13) and (16) again. Equation (18) cannot have a solution which satisfies the boundary condition at infinity unless $\beta < 0$. These further solutions are due to Mangler[11]. The two cases $m = -1$ can be discussed better directly and will not be considered here, though the asymptotic method can be applied to them. They correspond formally to $\beta = \pm\infty$ in equation (13).

It may be observed that when $m = 1$, $\beta = 1$ and v_0 is constant. This case represents the flow near the stagnation point of a blunt nosed body and also is a solution of the full viscous equations when the bounding surface is a plane wall.

As was pointed out above, the velocity distributions in the boundary layer are similar at different sections also when

$$U = ae^{kx}. \qquad (20)$$

By writing

$$\psi = \left(\frac{2U\nu}{k}\right)^{1/2} F(Y) \qquad (21)$$

where

$$Y = \left(\frac{Uk}{2\nu}\right)^{1/2} y, \qquad \text{(22)}$$

we find

$$\left.\begin{aligned} u &= UF'(Y) \\ v &= -(\tfrac{1}{2}Uk\nu)^{1/2}[F(Y) - YF'(Y)] \end{aligned}\right\}, \qquad \text{(23)}$$

so that the equation of motion becomes

$$F''' + FF'' + 2(1 - F'^2) = 0, \qquad \text{(24)}$$

which is equation (13) with $\beta = 2$. The boundary conditions are those of equations (15) and (16) with

$$v_0 = K(\tfrac{1}{2}Uk\nu)^{1/2}. \qquad \text{(25)}$$

Hence the results which will be obtained for $\beta = 2$ can be interpreted to this case. If a or k were negative, we should similarly find equation (18) with $\beta = 2$, an impossible case.

We proceed to investigate the asymptotic behaviour of the solution of Hartree's equation when β is fixed and K is large.

2. *Transformation of the Equation.*—When Y is small, and K is large, $F = K$ and F' is negligible compared with 1. Hence equation (13) is approximately

$$F''' + KF'' + \beta = 0 \qquad \text{(26)}$$

This integrates on multiplying by e^{KY} to give

$$F'' = Ae^{-KY} - \frac{\beta}{K},$$

where A is a constant of integration. If $Y = O(1/K)$ and we suppose that A is not small, the second term may be neglected, and on integration we get

$$F' = -\frac{A}{K}e^{-KY} + B,$$

where B is a constant. But since $F'(0) = 0$, $B = A/K$ and

$$F' = B(1 - e^{-KY}). \qquad \text{(27)}$$

This rather crude argument suggests that in order to make further progress it will be necessary to take KY instead of Y as the independent variable.

Therefore, make the following transformation, which is designed to get K out of the boundary conditions and into the equation.

Let

$$\zeta = KY = \frac{v_0 y}{\nu}, \qquad \text{(28)}$$

and

$$F = K + \frac{1}{K}\phi(\zeta). \qquad \text{(29)}$$

Then we have

$$F'(Y) = \phi'(\zeta) \qquad \text{(30)}$$

and equation (13) becomes

$$K^2\phi''' + \left(K + \frac{1}{K}\phi\right)K\phi'' + \beta(1 - \phi'^2) = 0,$$

$$i.e., \ \phi''' + \phi'' + \frac{1}{K^2}[\phi\phi'' + \beta(1 - \phi'^2)] = 0. \qquad \text{(31)}$$

The boundary conditions are

$$\left.\begin{aligned}\phi(0) = \phi'(0) &= 0,\\ \phi'(\infty) &= 1.\end{aligned}\right\} \qquad (32)$$

When we consider equation (18) we write

$$F = -K + \frac{1}{K}\phi(\zeta) \qquad (33)$$

in place of equation (29) and find

$$\phi''' + \phi'' - \frac{1}{K^2}\left[\phi\phi'' + \beta(1 - \phi'^2)\right] = 0. \qquad (34)$$

This is merely equation (31) with the sign of K^2 changed. We can, therefore, confine our attention to equation (31) in the subsequent analysis, and the results for equation (34) will then follow.

When K is large, equation (31) is approximately

$$\phi''' + \phi'' = 0, \qquad (35)$$

the solution of which, with the boundary conditions of equation (32), is

$$\left.\begin{aligned}\phi' &= 1 - e^{-\zeta}\\ \phi &= \zeta - 1 + e^{-\zeta}\end{aligned}\right\} \qquad (36)$$

Now

$$\frac{u}{U} = F'(Y) = \phi'(\zeta)$$

$$= 1 - \exp\left(\frac{v_0 y}{\nu}\right), \qquad (37)$$

which is the velocity distribution first given by Griffith and Meredith[6].

This argument can be extended to find the correction terms giving the deviation of the velocity distribution from the Griffith-Meredith profile or, what is exactly equivalent, to find a development of the solution as an asymptotic series in inverse powers of K.

3. *Asymptotic Series for the Solution.*—To find an asymptotic series which satisfies equation (31) formally we assume that

$$\phi = \phi_0 + \frac{\phi_1}{K^2} + \frac{\phi_2}{K^4} + \ldots \qquad (38)$$

Then by substituting in equation (31) and equating to zero the coefficients of the various powers of K we obtain the following set of differential equations, which may be solved in turn:—

$$\phi_0''' + \phi_0'' = 0, \qquad (39)$$

$$\phi_1''' + \phi_1'' + \phi_0\phi_0'' + \beta(1 - \phi_0'^2) = 0, \qquad (40)$$

$$\phi_2''' + \phi_2'' + \phi_0\phi_1'' + \phi_1\phi_0'' - 2\beta\phi_0'\phi_1' = 0, \qquad (41)$$

$$\phi_3''' + \phi_3'' + \phi_0\phi_2'' + \phi_1\phi_1'' + \phi_2\phi_0'' - \beta(\phi_1'^2 + 2\phi_0'\phi_2') = 0, \qquad (42)$$

etc.

The boundary conditions are

$$\left.\begin{aligned}\phi_0(0) = \phi_0'(0) &= 0,\\ \phi_0'(\infty) &= 1,\end{aligned}\right\} \qquad (43)$$

and for $r \geqslant 1$

$$\phi_r(0) = \phi_r'(0) = \phi_r'(\infty) = 0. \qquad (44)$$

The solution of (39) is

$$\left.\begin{aligned} \phi_0'' &= e^{-\zeta} \\ \phi_0' &= 1 - e^{-\zeta} \\ \phi_0 &= \zeta - 1 + e^{-\zeta} \end{aligned}\right\} \qquad \qquad (45)$$

Now ϕ' is shown by equation (40) to be a linear function of β. In fact, if

$$\phi_1 = \tfrac{1}{4}(\phi_{10} + \beta\phi_{11}), \qquad \qquad (46)$$

the equations for ϕ_{10} and ϕ_{11} are

$$\phi_{10}'' + \phi_{10}'' + 4\,\phi_0\phi_0'' = 0, \qquad \qquad (47)$$

$$\phi_{11}''' + \phi_{11}'' + 4(1 - \phi_0'^2) = 0. \qquad \qquad (48)$$

Equation (47) is

$$\phi_{10}''' + \phi_{10}'' + 4e^{-\zeta}\,(\zeta - 1 + e^{-\zeta}) = 0,$$

which integrates on multiplying by $e\zeta$ to give

$$\phi_{10}'' = -\,(2\zeta^2 - 4\zeta + A_{10})e^{-\zeta} + 4e^{-2\zeta},$$

and

$$\phi_{10}' = B_{10} + (2\zeta^2 + A_{10})e^{-\zeta} - 2e^{-2\zeta}$$

where A_{10} and B_{10} are constants of integration.

The boundary conditions give us $B_{10} = 0$ and $A_{10} = 2$. Therefore, we have

$$\left.\begin{aligned} \phi_{10}'' &= -(2\zeta^2 - 4\zeta + 2)e^{-\zeta} + 4e^{-2\zeta}, \\ \phi_{10}' &= (2\zeta^2 + 2)e^{-\zeta} - 2e^{-2\zeta}. \end{aligned}\right\} \qquad \qquad (49)$$

Integrating again,

$$\phi_{10} = C_{10} - (2\zeta^2 + 4\zeta + 6)e^{\zeta} + e^{-2\zeta}$$

and since $\phi_{10}(0) = 0$, $C_{10} = 5$. Thus

$$\phi_{10} = 5 - (2\zeta^2 + 4\zeta + 6)e^{-\zeta} + e^{-2\zeta}. \qquad \qquad (50)$$

Similarly equation (48) is

$$\phi_{11}''' + \phi_{11}'' + 8e^{-\zeta} - 4e^{-2\zeta} = 0$$

and we obtain on integration

$$\left.\begin{aligned} \phi_{11}'' &= -\,(8\zeta - 10)e^{-\zeta} - 4e^{-2\zeta} \\ \phi_{11}' &= (8\zeta - 2)e^{-\zeta} + 2e^{-2\zeta} \\ \phi_{11} &= 7 - (8\zeta + 6)e^{-\zeta} - e^{-2\zeta}. \end{aligned}\right\} \qquad \qquad (51)$$

We see from equation (41) that ϕ_2 is quadratic in β, and on writing

$$\phi_2 = \tfrac{1}{8}(\phi_{20} + \beta\phi_{21} + \beta^2\phi_{22}), \qquad \qquad (52)$$

we obtain differential equations for ϕ_{20}, ϕ_{21}, ϕ_{22} which give

$$\left.\begin{aligned} \phi_{20}' &= -\,(\zeta^4 + 6\zeta^2 - 2\zeta + 18\tfrac{1}{3})e^{-\zeta} \\ &\quad + (4\zeta^2 + 8\zeta + 20)e^{-2\zeta} - \tfrac{5}{3}\,e^{-3\zeta}, \\ \phi_{21}' &= -\,(8\zeta^3 + 6\zeta^2 + 26\zeta + 3)e^{-\zeta} - (4\zeta^2 - 8\zeta)e^{-2\zeta} + 3e^{-3\zeta}, \\ \phi_{22}' &= -\,(16\zeta^2 + 24\zeta - 17\tfrac{1}{3})e^{-\zeta} - (16\zeta + 16)e^{-2\zeta} - \tfrac{4}{3}e^{-3\zeta}. \end{aligned}\right\} \qquad \qquad (53)$$

ϕ_3 is cubic in β, and by putting

$$\phi_3 = \tfrac{1}{16}\,(\phi_{30} + \beta\phi_{31} + \beta^2\phi_{32} + \beta^3\phi_{33}) \qquad \qquad (54)$$

we find from equation (42) that

$$\left.\begin{aligned}
\phi_{30}' &= (\tfrac{1}{3}\zeta^6 + 5\zeta^4 + 3\tfrac{1}{3}\zeta^3 + 54\tfrac{1}{3}\zeta^2 - 21\tfrac{7}{9}\zeta + 253\tfrac{13}{27})e^{-\zeta} \\
&\quad - (4\zeta^4 + 16\zeta^3 + 72\zeta^2 + 156\zeta + 282\tfrac{2}{3})e^{-2\zeta} \\
&\quad + (5\zeta^2 + 13\tfrac{1}{3}\zeta + 30\tfrac{4}{9})e^{-3\zeta} - 1\tfrac{7}{27}e^{-4\zeta}, \\
\phi_{31}' &= (4\zeta^5 + 7\zeta^4 + 52\tfrac{2}{3}\zeta^3 + 111\zeta^2 + 203\tfrac{1}{3}\zeta + 255\tfrac{5}{9})e^{-\zeta} \\
&\quad + (4\zeta^4 - 16\zeta^3 - 48\zeta^2 - 216\zeta - 235\tfrac{1}{3})e^{-2\zeta} \\
&\quad - (9\zeta^2 + 4\zeta + 23\tfrac{4}{9})e^{-3\zeta} + 3\tfrac{2}{9}e^{-4\zeta}, \\
\phi_{32}' &= (16\zeta^4 + 56\zeta^3 + 158\tfrac{2}{3}\zeta^2 + 379\tfrac{7}{9}\zeta - 159\tfrac{22}{27})e^{-\zeta} \\
&\quad + (32\zeta^3 + 48\zeta^2 + 116\zeta + 190\tfrac{2}{3})e^{-2\zeta} \\
&\quad + (4\zeta^2 - 25\tfrac{1}{3}\zeta - 28\tfrac{1}{9})e^{-3\zeta} - 2\tfrac{20}{27}e^{-4\zeta}, \\
\phi_{33}' &= (21\tfrac{1}{3}\zeta^3 + 112\zeta^2 + 154\tfrac{2}{3}\zeta - 181\tfrac{8}{9})e^{-\zeta} \\
&\quad + (64\zeta^2 + 192\zeta + 159\tfrac{1}{3})e^{-2\zeta} \\
&\quad + (16\zeta + 21\tfrac{7}{9})e^{-3\zeta} + \tfrac{7}{9}e^{-4\zeta}.
\end{aligned}\right\} \quad \ldots \quad (55)$$

An alternative derivation of the asymptotic series is to expand F in powers of β as

$$F = F_0 + \beta F_1 + \beta^2 F_2 + \ldots\ldots, \quad \ldots \quad \ldots \quad \ldots \quad (56)$$

to obtain the equations giving the functions F_r from equation (13) as

$$\left.\begin{aligned}
F_0''' + F_0 F_0''' &= 0, \\
F_1''' + F_0 F_1'' + F_0'' F_1 + 1 - F_0'^2 &= 0, \\
F_2''' + F_0 F_2'' + F_0'' F_2 + F_1 F_1'' - 2 F_0' F_1' &= 0. \\
\text{etc.}
\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (57)$$

and then to investigate the asymptotic behaviour of the functions F_r by means of equations (57).

The functions ϕ_{10}, ϕ_{20}, ... which alone are involved when $\beta = 0$ are identical with those referred to as ϕ_1, ϕ_2, . . . in R. & M. 2298[3], and the series for $\beta = 0$ is the same as that given there, if allowance is made for the different definitions of $\phi(\zeta)$ and K.

4. *Properties of the Boundary Layer.*—We are now in a position to obtain series for the skin friction, the displacement and momentum thicknesses of the boundary layer, and the form parameter H. We have for the skin friction

$$\frac{\tau_0}{\rho U^2} = \frac{\nu}{U^2}\left(\frac{\partial u}{\partial y}\right)_{y=0} = \frac{v_0}{U}\left(\frac{\partial(u/U)}{\partial(v_0 y/\nu)}\right)_{y=0} = \frac{v_0}{U}\phi''(0). \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (58)$$

The skin friction is therefore roughly proportional to the suction velocity, and

$$\frac{\tau_0}{\rho U v_0} = \phi''(0)$$

$$= 1 + \frac{\phi_{10}''(0) + \beta\phi_{11}''(0)}{4K^2} + \frac{\phi_{20}''(0) + \beta\phi_{21}''(0) + \beta^2\phi_{22}''(0)}{8K^4} + \ldots$$

$$= 1 + \frac{2 + 6\beta}{4K^2} - \frac{6\tfrac{2}{3} + 24\beta + 21\tfrac{1}{3}\beta^2}{8K^4} + \frac{61\tfrac{1}{9} + 255\tfrac{8}{9}\beta + 344\tfrac{2}{9}\beta^2 + 157\tfrac{4}{9}\beta^3}{16K^6} + O(K^{-8}). \quad (59)$$

The displacement thickness is

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

$$= \int_0^\infty \left(1 - \phi'(\zeta)\right) d\zeta \; \frac{\nu}{v_0}$$

and so

$$\frac{v_0\delta^*}{\nu} = \int_0^\infty \left\{ e^{-\zeta} - \frac{\phi_1'}{K^2} - \frac{\phi_2'}{K^4} - \ldots \right\} d\zeta$$

$$= 1 - \frac{\phi_1(\infty)}{K^2} - \frac{\phi_2(\infty)}{K^4} - \ldots\ldots\ldots\ldots$$

$$= 1 - \frac{5 + 7\beta}{4K^2} + \frac{39\frac{8}{9} + 87\beta + 51\frac{1}{9}\beta^2}{8K^4}$$

$$- \frac{524\frac{13}{18} + 1450\frac{11}{108}\beta + 1393\frac{1}{18}\beta^2 + 477\frac{73}{108}\beta^3}{16K^6} + O(K^{-8}). \quad \ldots \quad \ldots \quad (60)$$

Similarly the momentum thickness is given by

$$\frac{v_0\theta}{\nu} = \int_0^\infty \phi'(\zeta)(1 - \phi'(\zeta))\, d\zeta$$

$$= \frac{1}{2} - \frac{3\frac{1}{3} + 3\frac{2}{3}\beta}{4K^2} + \frac{30\frac{5}{9} + 57\frac{1}{2}\beta + 27\frac{11}{18}\beta^2}{8K^4}$$

$$- \frac{433\frac{31}{45} + 1086\frac{94}{135}\beta + 916\frac{41}{180}\beta^2 + 263\frac{449}{540}\beta^3}{16K^6} + O(K^{-8})\ . \quad \ldots \quad \ldots \quad (61)$$

These three quantities are connected by the momentum equation

$$U\frac{d\theta}{dx} = -(\delta^* + 2\theta)\frac{dU}{dx} - v_0 + \frac{\nu}{U}\left(\frac{\partial u}{\partial y}\right)_{y=0}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (62)$$

and by using this we obtain a check on the work, and an additional term in the series for $\phi''(0)$ In fact

$$\frac{\tau_0}{\rho U v_0} = 1 + \frac{1 + 3\beta}{2K^2} - \frac{3\frac{1}{3} + 12\beta + 10\frac{2}{3}\beta^2}{4K^4}$$

$$+ \frac{30\frac{5}{9} + 127\frac{17}{18}\beta + 172\frac{1}{9}\beta^2 + 78\frac{13}{18}\beta^3}{8K^6}$$

$$- \frac{433\frac{41}{45} + 2045\frac{29}{270}\beta + 3453\frac{7}{270}\beta^2 + 2573\frac{31}{270}\beta^3 + 741\frac{137}{270}\beta^4}{16K^8} + O(K^{-10}) \quad (63)$$

By taking the quotient of the series (60) and (61) we find that

$$H = \frac{\delta^*}{\theta} = 2 + \frac{1\frac{2}{3} + \frac{1}{3}\beta}{2K^2} - \frac{15\frac{2}{3} + 20\frac{7}{9}\beta + 2\frac{8}{9}\beta^2}{4K^4}$$

$$+ \frac{239\frac{137}{270} + 490\frac{307}{540}\beta + 288\frac{2}{5}\beta^2 + 30\frac{103}{540}\beta^3}{8K^6} + O(K^{-8})\ . \quad \ldots \quad \ldots \quad (64)$$

5. *Special Cases :* $\beta = \pm 1$.—When $\beta = 1$, we have $m = 1$ and the external velocity distribution is $U = cx$, which is appropriate to the flow near the stagnation point of a blunt nosed body. The functions ϕ_r simplify and the terms in $e^{-(r+1)\zeta}$ cancel. We have

$$\left.\begin{aligned} \phi_0' &= 1 - e^{-\zeta}, \\ 2\phi_1' &= (\zeta^2 + 4\zeta)e^{-\zeta}, \\ 4\phi_2' &= -(\tfrac{1}{2}\zeta^4 + 4\zeta^3 + 14\zeta^2 + 24\zeta + 2)e^{-\zeta} + 2e^{-2\zeta}, \\ 8\phi_3' &= (\tfrac{1}{6}\zeta^6 + 2\zeta^5 + 14\zeta^4 + 66\tfrac{2}{3}\zeta^3 + 218\zeta^2 + 358\zeta + 83\tfrac{2}{3})e^{-\zeta} \\ &\quad - (4\zeta^2 + 32\zeta + 84)e^{-2\zeta} + \tfrac{1}{3}e^{-3\zeta}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (65)$$

Also

$$\frac{\tau_0}{\rho U v_0} = 1 + \frac{4}{2K^2} - \frac{26}{4K^4} + \frac{409\frac{1}{3}}{8K^6} - \frac{9246\frac{4}{9}}{16K^8} + O(K^{-10}), \quad \ldots \quad (66)$$

$$\frac{v_0\delta^*}{\nu} = 1 - \frac{6}{2K^2} + \frac{89}{4K^4} - \frac{1922\frac{7}{9}}{8K^6} + O(K^{-8}), \quad \ldots \quad (67)$$

$$\frac{v_0\theta}{\nu} = \frac{1}{2} - \frac{7}{4K^2} + \frac{115\frac{2}{3}}{8K^4} - \frac{2700\frac{4}{9}}{16K^6} + O(K^{-8}), \quad \ldots \quad (68)$$

$$H = 2 + \frac{1}{K^2} - \frac{39\frac{1}{3}}{4K^4} + \frac{1048\frac{2}{3}}{8K^6} + O(K^{-8}). \quad \ldots \quad (69)$$

All of these results can be obtained by a different manner of attack, which will be expounded in Part II.

Another interesting case is that of $\beta = -1$, which corresponds to $m = -\frac{1}{3}$. The original equation (13) can then be integrated immediately twice to give

$$2F' + F^2 = Y^2 + 2CY + D, \quad \ldots \quad (70)$$

and this equation has been integrated by Thwaites in terms of the error function, giving the exact values of the skin friction and displacement thickness. This case was first noticed by Mills[15], who considered $c < 0$ in equation (3), corresponding to equation (18) in place of equation (13).

For $\beta = -1$

$$\left.\begin{aligned} \phi_0' &= 1 - e^{-\zeta}, \\ 2\phi_1' &= (\zeta^2 - 4\zeta + 2)e^{-\zeta} - 2e^{-2\zeta}, \\ 4\phi_2' &= -(\tfrac{1}{2}\zeta^4 - 4\zeta^3 + 8\zeta^2 - 2\zeta - 1)e^{-\zeta} \\ &\quad + (4\zeta^2 - 8\zeta + 2)e^{-2\zeta} - 3e^{-3\zeta}, \\ 8\phi_3' &= (\tfrac{1}{6}\zeta^6 - 2\zeta^5 + 7\zeta^4 - 7\tfrac{1}{3}\zeta^3 - 5\zeta^2 + 10)e^{-\zeta} \\ &\quad - (4\zeta^4 - 16\zeta^3 + 20\zeta^2 + 8\zeta + 8)e^{-2\zeta} \\ &\quad + (9\zeta^2 - 12\zeta + 2)e^{-3\zeta} - 4e^{-4\zeta} \end{aligned}\right\} ; \quad \ldots \quad (71)$$

$$\frac{\tau_0}{\rho U v_0} = \left(1 - \frac{2}{K^2}\right)^{1/2}, \quad \ldots \quad (72)$$

$$\frac{v_0\delta^*}{\nu} = K^2 - K(K^2 - 2)^{1/2}, \quad \ldots \quad (73)$$

$$\frac{v_0\theta}{\nu} = \frac{1}{2} + \frac{1}{12K^2} + \frac{1}{12K^4} + \frac{11}{288K^6} + O(K^{-8}), \quad \ldots \quad (74)$$

$$H = 2 + \frac{2}{3K^2} + \frac{5}{9K^4} + \frac{193}{216K^6} + O(K^{-8}). \quad \ldots \quad (75)$$

6. *Numerical Results.*—The functions ϕ_0', ϕ_{10}', ... ϕ_{33}' are tabulated in Table 1, and the functions ϕ_1', ϕ_2', ϕ_3' are given for $\beta = \pm 1$ in Table 2. Table 3 gives the velocity distributions for $K = 2{\cdot}5$ with $\beta = -1$ and 0, and for $K = 5$ and 10 with $\beta = -2, -1, 0, 1, 2$. Some of these velocity distributions are shown in Fig. 1.

If

$$\sigma_1 = K\left(\frac{m+1}{2}\right)^{1/2} \quad \ldots \quad (76)$$

we have

$$v_0 = \sigma_1\left(\frac{U\nu}{x}\right)^{1/2}, \quad \ldots \quad (77)$$

thus generalising the notation employed by Thwaites[14] in the case $m = 0$ to other values of m. Values of $\tau_0/\rho U v_0$, $v_0\delta^*/\nu$, $v_0\theta/\nu$ and H have been calculated for $\sigma_1 = 2{\cdot}5$, 5, 10, and for $K = 2{\cdot}5$, 5, 10, 20, with a wide range of β. They are given in Tables 4 to 7 with the coefficients of the various powers of K in the series. The results are plotted in Figs. 2 to 5 against β for constant K and in Figs. 6 to 9 against m for constant σ_1.

7. *Extrapolation to Separation.*—The series obtained do not behave satisfactorily when separation profiles are approached, because of the singularity at separation. This can be seen readily in the case $\beta = -1$, for which equation (72) shows the behaviour of the skin friction near $K = \sqrt{2}$, where it vanishes. To estimate when separation occurs a method of extrapolation is adopted. Writing

$$z = \frac{1}{K^2}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (78)$$

the skin friction, given by equation (63), is of the form

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots, \qquad \ldots \quad \ldots \quad \ldots \qquad (79)$$

and for $\beta \leq -1$ the coefficients $a_1, a_2, \ldots$ are negative. If we cut off the series for $f(z)$ after the term in z^n we have a polynomial in z which has one positive zero, z_n say. This will be an approximation to the true zero of $f(z)$, and owing to the singularity it is found that z_n can be extrapolated to $n = \infty$ either graphically by plotting against $1/n$ or numerically by assuming for z_n a polynomial form in $1/n$. This method should give the desired value of z with comparatively little error, and works well for $\beta = -1$ when it can be checked against the exact result. Values of K and σ_1 calculated in this way are given in Table 8, and are shown in Figs. 10, 11. They have been compared with results produced by another method, described in Part II, and agree well.

8. *Convergence of the Series.*—The whole of the argument has been purely formal, and it is not known whether the series (38) is in fact convergent. If the three differentiations required to substitute in the equation (31) are permissible, then equation (38) will be the exact solution, but if not it can only represent the solution asymptotically as $K \to \infty$. It was observed that equation (18) cannot have a solution for $\beta > 0$, and hence equation (34) cannot. But the series solution of equation (34) is obtained from equation (38) by changing the sign of K^2, so that equation (38) cannot be convergent if $\beta > 0$. It is plausible that the series is convergent if $\beta < 0$, for sufficiently large values of K, and that the smallest value of K for which it converges gives the separation profile.

TABLE 1

ζ	ϕ_0'	ϕ_{10}'	ϕ_{11}'	ϕ_{20}'	ϕ_{21}'	ϕ_{22}'	ϕ_{30}'	ϕ_{31}'	ϕ_{32}'	ϕ_{33}'
0	0	0	0	0	0	0	0	0	0	0
0·125	0·1175031	0·23497	0·67510	−0·78342	− 2·82013	− 2·50630	7·181	30·068	40·446	18·499
0·25	0·2211992	0·44189	1·21306	−1·47568	− 5·30948	− 4·71283	13·521	56·606	76·129	34·812
0·375	0·3127107	0·62315	1·63202	−2·08928	− 7·50744	− 6·64392	19·123	80·034	107·583	49·168
0·5	0·3934693	0·78057	1·94882	−2·63580	− 9·44779	− 8·31790	24·082	100·725	135·271	61·752
0·75	0·5276334	1·02989	2·33573	−3·56744	−12·66379	−10·95439	32·391	135·170	180·892	82·201
1	0·6321206	1·20085	2·47795	−4·33610	−15·12811	−12·73571	39·003	162·115	215·595	97·140
1·25	0·7134952	1·30417	2·45621	−4·98540	−16·95517	−13·77810	44·347	183·215	241·342	107·330
1·5	0·7768698	1·35077	2·33088	−5·54043	−18·22555	−14·20408	48·750	199·701	259·621	113·427
2	0·8646647	1·31672	1·93133	−6·40494	−19·34551	−13·69420	55·654	222·135	278·125	115·749
2·5	0·9179150	1·17676	1·49101	−6·94206	−18·95262	−12·08885	60·951	233·782	277·937	108·667
3	0·9502129	0·99078	1·10027	−7·13720	−17·50465	−10·04984	65·129	236·708	264·205	96·167
4	0·9816844	0·62206	0·55014	−6·59744	−13·10637	− 6·15647	69·951	221·056	213·221	66·483

TABLE 2

	$\beta = 1$			$\beta = -1$		
ζ	ϕ_1'	ϕ_2'	ϕ_3'	ϕ_1'	ϕ_2'	ϕ_3'
0	0	0	0	0	0	0
0·125	0·22752	−0·76373	6·012	−0·11003	−0·05870	−0·05875
0·25	0·41374	−1·43725	11·317	−0·19279	−0·10988	−0·11054
0·375	0·56379	−2·03008	15·994	−0·25222	−0·15322	−0·15594
0·5	0·68235	−2·55019	20·114	−0·29206	−0·18824	−0·19522
0·75	0·84141	−3·39820	26·916	−0·32646	−0·23226	−0·25548
1	0·91970	−4·02499	32·116	−0·31928	−0·24296	−0·29110
1·25	0·94010	−4·46483	36·015	−0·28801	−0·22604	−0·30352
1·5	0·92041	−4·74626	38·844	−0·24503	−0·18987	−0·29724
2	0·81201	−4·93058	41·979	−0·15365	−0·09420	−0·25658
2·5	0·66694	−4·74794	42·584	−0·07856	−0·00979	−0·22259
3	0·52276	−4·33646	41·388	−0·02737	+0·03970	−0·22132
4	0·29305	−3·23254	35·670	+0·01798	0·04406	−0·27297

TABLE 3

Velocity Distributions in the Boundary Layer

Values of u/U for various β, K and ζ

	$\beta = -2$		$\beta = -1$			$\beta = 0$			$\beta = 1$		$\beta = 2$	
ζ	$K = 5$	$K = 10$	$K = 2·5$	$K = 5$	$K = 10$	$K = 2·5$	$K = 5$	$K = 10$	$K = 5$	$K = 10$	$K = 5$	$K = 10$
0	0	0	0	0	0	0	0	0	0	0	0	0
0·125	0·105	0·11465	0·098	0·11300	0·11640	0·124	0·120	0·11808	0·125	0·11971	0·131	0·1213
0·25	0·199	0·21611	0·187	0·21330	0·21926	0·237	0·225	0·22229	0·236	0·22520	0·245	0·2280
0·375	0·283	0·30593	0·268	0·30237	0·31017	0·335	0·319	0·31424	0·333	0·31816	0·345	0·3219
0·5	0·358	0·38546	0·341	0·38147	0·39053	0·421	0·401	0·39539	0·418	0·40006	0·432	0·4045
0·75	0·486	0·51824	0·468	0·51419	0·52435	0·563	0·537	0·53017	0·557	0·53573	0·574	0·5411
1	0·589	0·62241	0·574	0·61894	0·62890	0·674	0·643	0·63507	0·664	0·64095	0·682	0·6466
1·25	0·671	0·70413	0·660	0·70159	0·71059	0·757	0·726	0·71670	0·746	0·72249	0·762	0·7280
1·5	0·738	0·76825	0·732	0·76675	0·77440	0·820	0·789	0·78018	0·808	0·78564	0·823	0·7908
2	0·834	0·85801	0·837	0·85835	0·86312	0·905	0·877	0·86788	0·892	0·87233	0·903	0·8765
2·5	0·896	0·91317	0·904	0·91474	0·91713	0·952	0·928	0·92077	0·939	0·92415	0·947	0·9273
3	0·935	0·94703	0·946	0·94917	0·94994	0·977	0·959	0·95260	0·966	0·95505	0·972	0·9573
4	0·976	0·98042	0·985	0·98246	0·98187	0·995	0·987	0·98316	0·990	0·98433	0·993	0·9854

TABLE 4

		Coefficients of				$\tau_0/\rho U v_0$						
β	m	K^{-2}	K^{-4}	K^{-6}	K^{-8}	$K=2·5$	$K = 5$	$K = 10$	$K = 20$	$\sigma_1=2·5$	$\sigma_1=5$	$\sigma_1=10$
−18	−0·9	−26·5	−810·833	−50702·1	−3994780				0·9277		0·9433	0·98654
−10	−0·833	−14·5	−237·5	−7845	−322953				0·96213		0·94870	0·98774
−6	−0·75	−8·5	−78·8333	−1443·14	−32354·5			0·905	0·97823	0·78	0·95531	0·98925
−4	−0·667	−5·5	−31·5	−345·708	−4540·51		0·69	0·94146	0·98605	0·822	0·96182	0·98074
−3	−0·6	−4	−15·8333	−116·222	−977·725		0·804	0·95829	0·98990	0·850	0·96692	0·99194
−2	−0·5	−2·5	−5·5	−20·833	−89·674		0·8896	0·97443	0·99372	0·8896	0·97443	0·99372
−1·5	−0·429	−1·75	−2·3333	−4·975	−12·810	0·63	0·92591	0·98226	0·99561	0·91459	0·97999	0·99498
−1·25	−0·385	−1·375	−1·25	−1·776	−3·585	0·738	0·94288	0·98612	0·99655	0·92904	0·98288	0·99576
−1	−0·333	−1	−0·5	−0·5	−0·625	0·8247	0·95917	0·98995	0·99750	0·94516	0·98657	0·99666
−0·75	−0·273	−0·625	−0·0833	−0·225	+0·546	0·8973	0·97485	0·99374	0·99844	0.96331	0·99089	0·99773
−0·5	−0·2	−0·25	0	−0·029	+0·056	0·9599	0·99000	0·99750	0·99937	0·98399	0·99600	0·99900
−0·25	−0·111	+0·125	−0·25	+1·012	−6·307	1·014	1·00465	1·00123	1·00031	1·00783	1·00215	1·00055
0	0	0·5	−0·8333	3·819	−27·106	1·06	1·0188	1·00492	1·00124	1·0355	1·00969	1·00248
+0·25	+0·143	0·875	−1·75	9·316	−75·24		1·0326	1·00858	1·00218	1·068	1·01918	1·00494
0·5	0·333	1·25	−3	18·424	−167·97		1·0460	1·01221	1·00311		1·03147	1·00821
0·75	0·6	1·625	−4·5833	32·067	−326·87		1·0589	1·01582	1·00403		1·0480	1·01272
1	1	2	−6·5	51·167	−577·90		1·0714	1·01940	1·00496		1·071	1·01940
1·25	1·667	2·375	−8·75	76·645	−951·33		1·084	1·02294	1·00588		1·11	1·03026
1·5	3	2·75	−11·3333	109·426	−1481·80		1·095	1·02646	1·00680			1·0511
2	∞	3·5	−17·5	200·583	−3174·06		1·117	1·03342	1·00864			

TABLE 5

β	m	Coefficients of K^{-2}	K^{-4}	K^{-6}	$v_0\delta^*/\nu$ $K=2{\cdot}5$	$K=5$	$K=10$	$K=20$	$\sigma_1=2{\cdot}5$	$\sigma_1=5$	$\sigma_1=10$
−18	−0·9	30·25	1879·24	147502				1·090		1·069	1·01661
−10	−0·833	16·25	535·125	22021·7				1·044		1·061	1·01394
− 6	−0·75	9·25	169·7361	3825·24			1·114	1·02425	1·29	1·051	1·01184
− 4	−0·667	5·75	63·7083	847·379		1·4	1·065	1·01479	1·22	1·0415	1·00976
− 3	−0·6	4	29·8611	261·583		1·23	1·043	1·01019	1·17	1·0343	1·00812
− 2	−0·5	2·25	8·7917	39·042		1·107	1·0234	1·00568	1·107	1·0234	1·00568
− 1·5	−0·429	1·375	3·0486	8·013	1·33	1·060	1·01406	1·00346	1·070	1·01612	1·00395
1·25	−0·385	0·9375	1·375	2·764	1·20	1·040	1·00952	1·00235	1·050	1·01175	1·00290
− 1	−0·333	0·5	0·5	+0·625	1·095	1·021	1·00505	1·00125	1·028	1·00676	1·00167
− 0·75	−0·273	+0·0625	0·4236	−1·201	1·016	1·003	1·00067	1·00016	1·005	1.00100	1·00023
− 0·5	−0·2	−0·375	1·1458	−5·514	0·95	0·987	0·99636	0·99907	0·979	0·99427	0·99852
− 0·25	−0·111	−0·8125	2·6667	−15·112	0·88	0·971	0·99213	0·99799	0·952	0·98648	0·99644
0	0	−1·25	4·9861	−32·795	0·8	0·956	0·98797	0·99691	0·915	0·9767	0·99387
+ 0·25	+0·143	−1·6875	8·1042	−61·361		0·942	0·984	0·99583	0·87	0·9649	0·99063
0·5	0·333	−2·125	12·0208	−103·609		0·93	0·980	0·99476		0·950	0·98634
0·75	0·6	−2·5625	16·7361	−162·338		0·92	0·976	0·99370		0·930	0·9804
1	1	−3	22·25	−240·347		0·90	0·972	0·99264		0·90	0·9720
1·25	1·667	−3·4375	28·5625	−340·435		0·89	0·968	0·99158		0·85	0·9589
1·5	3	−3·875	35·6736	−465·400		0·88	0·964	0·99053			0·933
2	∞	−4·75	52·2917	−801·160		0·85	0·957	0·98844			

TABLE 6

β	m	Coefficients of K^{-2}	K^{-4}	K^{-6}	$v_0\theta/\nu$ $K=2{\cdot}5$	$K=5$	$K=10$	$K=20$	$\sigma_1=2{\cdot}5$	$\sigma_1=5$	$\sigma_1=10$
−18	−0·9	15·66667	992·694	78808·4				0·547		0·536	0·50809
−10	−0·833	8·33333	277·083	11415·1				0·523		0·531	0·50715
− 6	−0·75	4·66667	84·9444	1880·62			0·557	0·51223	0·64	0·526	0·50597
− 4	−0·667	2·83333	30·2917	383·667			0·5318	0·50728	0·605	0·5204	0·50481
− 3	−0·6	1·91667	13·3194	106·488		0·61	0·5206	0·50488	0·579	0·5163	0·50389
− 2	−0·5	1	3·25	11·590		0·546	0·51034	0·50252	0·546	0·5103	0·50252
− 1·5	−0·429	0·54167	0·8038	1·580	0·614	0·5231	0·50550	0·50136	0·5266	0·50630	0·50155
− 1·25	−0·385	0·3125	0·2279	0·523	0·558	0·5129	0·50315	0·50078	0·5160	0·50388	0·50096
− 1	−0·333	+0·08333	0·0833	+0·0382	0·5156	0·5035	0·50084	0·50021	0·5047	0·50113	0·50028
− 0·75	−0·273	−0·14583	0·3702	−1·421	0·480	0·4947	0·49858	0·49964	0·4925	0·49796	0·49947
− 0·5	−0·2	−0·375	1·0885	−5·401	0·45	0·4864	0·49635	0·49907	0·479	0·49426	0·49852
− 0·25	−0·111	−0·60417	2·2383	−13·447	0·41	0·479	0·49417	0·49850	0·464	0·48989	0·49736
0	0	−0·83333	3·8194	−27·106		0·471	0·49202	0·49794	0·445	0·4847	0·49593
+ 0·25	+0·143	−1·0625	5·8320	−47·922		0·464	0·48991	0·49738	0·42	0·4782	0·49411
0·5	0·333	−1·29167	8·2760	−77·442		0·457	0·4878	0·49682		0·470	0·49173
0·75	0·6	−1·52083	11·1515	−117·212		0·450	0·4858	0·49627		0·459	0·48849
1	1	−1·75	14·4583	−168·778		0·443	0·4838	0·49571		0·443	0·4838
1·25	1·667	−1·97917	17·8286	−233·685		0·44	0·4818	0·49516		0·41	0·4762
1·5	3	−2·20833	22·3663	−313·480		0·43	0·4799	0·49461			0·462
2	∞	−2·66667	32	−523·915		0·41	0·476	0·49353			

TABLE 7

β	m	Coefficients of			H						
		K^{-2}	K^{-4}	K^{-6}	$K=2{\cdot}5$	$K=5$	$K=10$	$K=20$	$\sigma_1=2{\cdot}5$	$\sigma_1=5$	$\sigma_1=10$
−18	−0·9	−2·16667	−144·417	−11402·7				1·99350		1·995	1·99888
−10	−0·833	−0·83333	−24·1944	−752·115			1·988	1·99775		1·99692	1·99929
−6	−0·75	−0·16667	+1·25	+144·662			1·999	1·99959	1·998	1·99922	1·99979
−4	−0·667	+0·16667	5·3056	119·928		2·023	2·0023	2·00045	2·011	2·00138	2.00029
−3	−0·6	0·33333	5·1667	68·531		2·026	2·00392	2·00087	2·018	2·00303	2·00069
−2	−0·5	0·5	3·5833	21·306		2·027	2·00538	2·00127	2·027	2·00538	2·00127
−1·5	−0·429	0·58333	2·25	6·333	2·18	2·0273	2·00606	2·00147	2·0320	2·00697	2·00169
−1·25	−0·385	0·625	1·4479	2·244	2·147	2·0275	2·00640	2·00157	2·0346	2·00792	2·00194
−1	−0·333	0·66667	+0·5556	0·8935	2·125	2·0276	2·00672	2·00167	2·0373	2·00899	2·00223
−0·75	−0·273	0·70833	−0·4271	2·634	2·113	2·0278	2·00704	2·00177	2·0403	2·01020	2·00257
−0·5	−0·2	0·75	−1·5	7·819	2·11	2·0281	2·00736	2·00187	2·044	2·01165	2·00298
−0·25	−0·111	0·79167	−2·6632	16·802	2·12	2·0285	2·00767	2·00196	2·049	2·01332	2·00347
0	0	0·83333	−3·9167	29·938	2·13	2·0289	2·00797	2·00206	2·056	2·0153	2·00407
+0·25	+0·143	0·875	−5·2604	47·581		2·0294	2·00827	2·00216	2·06	2·0178	2·00483
0·5	0·333	0·91667	−6·6944	70·083		2·030	2·00856	2·00225		2·021	2·00583
0·75	0·6	0·95833	−8·21875	97·799		2·031	2·00886	2·00235		2·025	2·00719
1	1	1	−9·8333	131·083		2·032	2·0091	2·00244		2·032	2·0091
1·25	1·667	1·04167	−11·5382	170·289		2·034	2·0094	2·00253		2·04	2·0122
1·5	3	1·08333	−13·3333	215·769		2·035	2·0097	2·00263		2·06	2·018
2	∞	1·16667	−17·1944	326·971		2·04	2·0102	2·00281			

TABLE 8

Amount of suction for separation profiles

β	m	K_s	σ_{1s}
−18	−0·9	10·85	2·427
−10	−0·833	7·815	2·256
−6	−0·75	5·745	2·031
−4	−0·667	4·392	1·793
−3	−0·6	3·563	1·593
−2	−0·5	2·572	1·286
−1·5	−0·429	2·023	1·082
−1·25	−0·385	1·767	0·980
−1	−0·333	1·414	0·817
−0·1988	−0·0904	0	0
0	0	−0·876	−0·619

Part II. Flow with Uniform Suction

Summary.—In this part the asymptotic theory is used to study the general two-dimensional boundary-layer flow over a porous surface through which there is a constant velocity of suction.

After a preliminary transformation (in section 2) we find in section 3 a series for the velocity. From this the series giving the displacement and momentum thicknesses of the boundary layer and the skin friction are obtained in section 4. In section 5 an application of the asymptotic theory is made to the general method of expansion in series of powers of x, and it is found that the functions involved in this method can be expressed as asymptotic series. The case of a linearly decreasing velocity outside the boundary layer is treated in section 6, with particular reference to the problem of finding the amount of suction necessary to prevent separation. This has been studied previously by Prandtl[16], and by Preston[8], using the momentum equation with assumed separation profiles. It is shown that it is unlikely that any suction velocity will suffice to maintain positive skin friction, though this may not imply separation of the flow. In section 7 the flow past a porous circular cylinder is considered, and section 8 describes how separation calculations can be made for other velocity distributions. Section 9 shows the effect of suction through a porous leading edge in preventing separation of the flow over a thin aerofoil at high incidence, the results for an 8·3 per cent thick symmetrical Joukowski aerofoil being given in Fig. 13. Finally there is a short discussion of some of the singularities which may restrict the application of the method.

1. *Introduction.*—In Part I the boundary-layer flows studied had the property that the velocity distribution was similar at all sections. Consequently it was sufficient to consider a single section of the boundary layer and the problems of Part I were therefore effectively one-dimensional. We now pass to strictly two-dimensional problems and shall consider in this part those in which the suction velocity is constant. After putting the equations of the boundary layer in non-dimensional form we shall be able to make a transformation analogous to that of Part I and obtain a solution in inverse powers of the suction velocity.

The equation of motion of the boundary layer is

$$u \frac{\partial u}{dx} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2}, \qquad (1)$$

where U is the velocity at the edge of the boundary layer, and

$$\left. \begin{aligned} u &= \frac{\partial \psi}{\partial y}, \\ v &= -\frac{\partial \psi}{\partial x}, \end{aligned} \right\} \qquad (2)$$

where ψ is the stream function. We assume that the suction velocity v_0 is large, and then $\partial v/\partial y = -\partial u/\partial x$ is small compared with v_0 so that to the first approximation $v = -v_0$. The terms $u(\partial u/\partial x)$ and $U(dU/dx)$ are bounded, whereas $v(\partial u/\partial y) = -v_0(\partial u/\partial y)$ is large. Hence $\nu(\partial^2 u/\partial y^2)$ is also large and equation (1) reduces to

$$-v_0 \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}. \qquad (3)$$

The boundary conditions for u are

$$\left. \begin{aligned} u &= 0 \text{ at } y = 0, \\ u &= U \text{ at } y = \infty, \end{aligned} \right\} \qquad (4)$$

so that equation (3) gives

$$\frac{u}{U} = 1 - e^{-v_0 y/\nu} . \qquad (5)$$

Then the displacement thickness is

$$\begin{aligned} \delta^* &= \int_0^\infty \left(1 - \frac{u}{U}\right) dy \\ &= \frac{\nu}{v_0}, \end{aligned} \qquad (6)$$

and the skin friction is

$$\left.\begin{aligned}\tau_0 &= \mu\left(\frac{\partial u}{\partial y}\right)_{y=0}\\ &= \rho U v_0 \,.\end{aligned}\right\} \qquad \text{(7)}$$

2. *The Transformation of the Equation.*—Let c be a representative length and U_0 a representative velocity, and let R be the Reynolds number.

$$R = \frac{U_0 c}{\nu}\,. \qquad \text{(8)}$$

We now put equation (1) in non-dimensional form by writing

$$\xi = \frac{x}{c}\,, \qquad \text{(9)}$$

$$\eta = y\left(\frac{U_0}{c\nu}\right)^{1/2}, \qquad \text{(10)}$$

$$\psi = (U_0 c\nu)^{1/2} f(\xi,\eta)\,, \qquad \text{(11)}$$

$$U = U_0 F(\xi), \qquad \text{(12)}$$

so that

$$u = U_0 \frac{\partial f}{\partial \eta}\,, \qquad \text{(13)}$$

$$v = -\left(\frac{U_0\nu}{c}\right)^{1/2} \frac{\partial f}{\partial \xi}. \qquad \text{(14)}$$

Therefore equation (1) becomes

$$\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi \partial \eta} - \frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2} = F(\xi)F'(\xi) + \frac{\partial^3 f}{\partial \eta^3} \qquad \text{(15)}$$

and the boundary conditions are, from equations (4)

$$\left.\begin{aligned}\frac{\partial f}{\partial \eta}(\xi, 0) &= 0,\\ \frac{\partial f}{\partial \eta}(\xi, \infty) &= F(\xi).\end{aligned}\right\} \qquad \text{(16)}$$

For a constant suction velocity v_0 we must have also

$$f(\xi, 0) = K\xi, \qquad \text{(17)}$$

so that

$$v(\xi, 0) = -K\left(\frac{U_0\nu}{c}\right)^{1/2},$$

that is

$$\frac{v_0}{U_0} = KR^{-1/2}. \qquad \text{(18)}$$

To apply the asymptotic method we must first make a preliminary transformation, taking as the independent variable

$$\zeta = K\eta = \frac{v_0 y}{\nu}\,, \qquad \text{(19)}$$

and putting

$$f(\xi, \eta) = K\xi + \frac{1}{K}\,\phi(\xi, \zeta). \quad \ldots \quad (20)$$

This gives

$$\frac{\partial f}{\partial \eta} = \frac{\partial \phi}{\partial \zeta}, \quad \ldots \quad (21)$$

and equation (15) becomes

$$\frac{\partial \phi}{\partial \zeta}\,\frac{\partial^2 \phi}{\partial \xi \partial \zeta} - \left(K + \frac{1}{K}\frac{\partial \phi}{\partial \xi}\right).\,K\frac{\partial^2 \phi}{\partial \zeta^2} = FF' + K^2\frac{\partial^3 \phi}{\partial \zeta^3}$$

or

$$\frac{\partial^3 \phi}{\partial \zeta^3} + \frac{\partial^2 \phi}{\partial \zeta^2} + \frac{1}{K^2}\left(\frac{\partial^2 \phi}{\partial \zeta^2}\,\frac{\partial \phi}{\partial \xi} - \frac{\partial \phi}{\partial \zeta}\,\frac{\partial^2 \phi}{\partial \xi \partial \zeta} + FF'\right) = 0. \quad \ldots \quad (22)$$

The boundary conditions for ϕ are now independent of K, and are

$$\left.\begin{aligned} \frac{\partial \phi}{\partial \zeta}(\xi, 0) &= 0, \\ \frac{\partial \phi}{\partial \zeta}(\xi, \infty) &= F(\xi), \\ \phi\,(\xi, 0) &= 0. \end{aligned}\right\} \quad \ldots \quad (23)$$

3. *The Solution in Asymptotic Series.*—When K is large, equation (22) reduces to

$$\frac{\partial^3 \phi}{\partial \zeta^3} + \frac{\partial^2 \phi}{\partial \zeta^2} = 0, \quad \ldots \quad (24)$$

whence

$$\frac{\partial^2 \phi}{\partial \zeta^2} = A(\xi)e^{-\zeta},$$

$$\frac{\partial \phi}{\partial \zeta} = B(\xi) - A(\xi)e^{-\zeta}.$$

The boundary conditions (23) indicate that

$$A(\xi) = B(\xi) = F(\xi)$$

and so

$$\frac{\partial \phi}{\partial \zeta} = F(\xi)(1 - e^{-\zeta}). \quad \ldots \quad (25)$$

Since

$$\frac{u}{U} = \frac{1}{F(\xi)}\,\frac{\partial \phi}{\partial \zeta}, \quad \ldots \quad (26)$$

we have a different derivation of equation (5). Further approximations can be found for large K by assuming for ϕ a series of the type

$$\phi = \Phi_0 + \frac{\Phi_1}{K^2} + \frac{\Phi_2}{K^4} + \ldots \quad \ldots \quad (27)$$

By substitution of this series into equation (22) we obtain on equating the coefficients of successive powers of K to zero the following set of equations for the functions Φ_r

$$\frac{\partial^3\Phi_0}{\partial\zeta^3} + \frac{\partial^2\Phi_0}{\partial\zeta^2} = 0 \qquad (28)$$

$$\frac{\partial^3\Phi_1}{\partial\zeta^3} + \frac{\partial^2\Phi_1}{\partial\zeta^2} + \frac{\partial^2\Phi_0}{\partial\zeta^2}\frac{\partial\Phi_0}{\partial\xi} - \frac{\partial\Phi_0}{\partial\zeta}\frac{\partial^2\Phi_0}{\partial\xi\,\partial\zeta} + FF' = 0 \qquad (29)$$

$$\frac{\partial^3\Phi_2}{\partial\zeta^3} + \frac{\partial^2\Phi_2}{\partial\zeta^2} + \frac{\partial^2\Phi_0}{\partial\zeta^2}\frac{\partial\Phi_1}{\partial\xi} + \frac{\partial^2\Phi_1}{\partial\zeta^2}\frac{\partial\Phi_0}{\partial\xi} - \frac{\partial\Phi_0}{\partial\zeta}\frac{\partial^2\Phi_1}{\partial\xi\,\partial\zeta} - \frac{\partial\Phi_1}{\partial\zeta}\frac{\partial^2\Phi_0}{\partial\xi\,\partial\zeta} = 0 \qquad (30)$$

$$\frac{\partial^3\Phi_3}{\partial\zeta^3} + \frac{\partial^2\Phi_3}{\partial\zeta^2} + \frac{\partial^2\Phi_0}{\partial\zeta^2}\frac{\partial\Phi_2}{\partial\xi} + \frac{\partial^2\Phi_1}{\partial\zeta^2}\frac{\partial\Phi_1}{\partial\xi} + \frac{\partial^2\Phi_2}{\partial\zeta^2}\frac{\partial\Phi_0}{\partial\xi}$$
$$- \frac{\partial\Phi_0}{\partial\zeta}\frac{\partial^2\Phi_2}{\partial\xi\,\partial\zeta} - \frac{\partial\Phi_1}{\partial\zeta}\frac{\partial^2\Phi_1}{\partial\xi\,\partial\zeta} - \frac{\partial\Phi_2}{\partial\zeta}\frac{\partial^2\Phi_0}{\partial\xi\,\partial\zeta} = 0 \qquad (31)$$

etc.

The boundary conditions for these functions are that Φ_0 must satisfy equation (23), and that the rest satisfy

$$\Phi_r(\xi, 0) = \frac{\partial}{\partial\zeta}\Phi_r(\xi, 0) = \frac{\partial}{\partial\zeta}\Phi_r(\xi, \infty) = 0. \qquad (32)$$

The argument given at the beginning of this section shows that

$$\Phi_0 = F(\xi)\,\phi_0(\zeta), \qquad (33)$$

where

$$\left.\begin{aligned}\phi_0'' &= e^{-\zeta}\\ \phi_0' &= 1 - e^{-\zeta}\end{aligned}\right\} \qquad (34)$$

and since $\phi_0(0) = 0$ we have

$$\phi_0 = \zeta - 1 + e^{-\zeta}. \qquad (35)$$

We can now find $\Phi_1, \Phi_2, \ldots\ldots$ in turn from equations (29), (30), etc. On substituting for Φ_0 in (29) we see that

$$\Phi_1 = \tfrac{1}{2}FF'\phi_1(\zeta) \qquad (36)$$

where

$$\phi_1''' + \phi_1'' + 2(\phi_0\phi_0'' - \phi_0'^2 + 1) = 0. \qquad (37)$$

Inserting the known expressions for ϕ_0, ϕ_0' and ϕ_0'', we have

$$\phi_1''' + \phi_1'' = -2(\zeta + 1)e^{-\zeta}. \qquad (38)$$

This equation is easily integrated after multiplying by e^{ζ}, and we find

$$\phi_1'' = -(\zeta^2 + 2\zeta + A_1)e^{-\zeta},$$

and therefore

$$\phi_1' = B_1 + (\zeta^2 + 4\zeta + 4 + A_1)e^{-\zeta}.$$

By the boundary conditions $B_1 = 0$ and $A_1 = -4$ so that

$$\left.\begin{aligned}\phi_1'' &= -(\zeta^2 + 2\zeta - 4)e^{-\zeta}\\ \phi_1' &= (\zeta^2 + 4\zeta)e^{-\zeta}.\end{aligned}\right\} \qquad (39)$$

Also

$$\phi_1 = C_1 - (\zeta^2 + 6\zeta + 6)e^{-\zeta},$$

and, since $\phi_1(0) = 0$, $C_1 = 6$.

Therefore

$$\phi_1 = 6 - (\zeta^2 + 6\zeta + 6)e^{-\zeta}. \qquad (40)$$

When we substitute in equation (30) for Φ_0 and Φ_1, we find that Φ_2 must be written as the sum of two terms, namely

$$\Phi_2 = \tfrac{1}{4}(FF'^2\phi_{21} + F^2F''\phi_{22}), \qquad (41)$$

and that the equations for ϕ_{21} and ϕ_{22} are

$$\phi_{21}''' + \phi_{21}'' + 2(\phi_0''\phi_1 + \phi_0\phi_1'' - 2\phi_0'\phi_1') = 0, \qquad (42)$$

$$\phi_{22}''' + \phi_{22}'' + 2(\phi_0''\phi_1 - \phi_0'\phi_1') = 0. \qquad (43)$$

The solution of these equations gives

$$\phi_{21}' = -(\tfrac{1}{2}\zeta^4 + 4\zeta^3 + 14\zeta^2 + 24\zeta + 2)e^{-\zeta} + 2e^{-2\zeta}, \qquad (44)$$

$$\phi_{22}' = -(\tfrac{2}{3}\zeta^3 + 6\zeta^2 + 9)e^{-\zeta} + (2\zeta + 9)e^{-2\zeta}. \qquad (45)$$

Similarly Φ_3 is the sum of three terms

$$\Phi_3 = \tfrac{1}{8}(FF'^3\phi_{31} + F^2F'F''\phi_{32} + F^3F'''\phi_{33}) \qquad (46)$$

where

$$\phi_{31}' = (\tfrac{1}{6}\zeta^6 + 2\zeta^5 + 14\zeta^4 + 66\tfrac{2}{3}\zeta^3 + 218\zeta^2 + 358\zeta + 83\tfrac{2}{3})e^{-\zeta} - (4\zeta^2 + 32\zeta + 84)e^{-2\zeta} + \tfrac{1}{3}e^{-3\zeta}, \qquad (47)$$

$$\phi_{32}' = (\tfrac{2}{3}\zeta^5 + 10\zeta^4 + 56\zeta^3 + 225\zeta^2 + 106\zeta + 504)e^{-\zeta} - (4\zeta^3 + 50\zeta^2 + 256\zeta + 503)e^{-2\zeta} - e^{-3\zeta}, \qquad (48)$$

$$\phi_{33}' = (\tfrac{1}{3}\zeta^4 + 5\tfrac{1}{3}\zeta^3 + 16\zeta^2 + 10\zeta + 36\tfrac{11}{18})e^{-\zeta} - (2\zeta^2 + 20\zeta + 35)e^{-2\zeta} - (\tfrac{1}{3}\zeta + 1\tfrac{11}{18})e^{-3\zeta}. \qquad (49)$$

The function ϕ_{41} which occurs in

$$\Phi_4 = \tfrac{1}{16}(FF'^4\phi_{41} + F^2F'^2F''\phi_{42} + F^3F''^2\phi_{43} + F^3F'F'''\phi_{44} + F^4F''''\phi_{45}) \quad (50)$$

has also been calculated. Its derivative is

$$\phi_{41}' = -(\tfrac{1}{24}\zeta^8 + \tfrac{2}{3}\zeta^7 + 7\zeta^6 + 54\tfrac{2}{3}\zeta^5 + 337\tfrac{2}{3}\zeta^4 + 1510\zeta^3 + 4895\tfrac{2}{3}\zeta^2 + 7805\tfrac{1}{9}\zeta + 3260\tfrac{17}{27})e^{-\zeta} + (4\zeta^4 + 64\zeta^3 + 464\zeta^2 + 1808\zeta + 3280)e^{-2\zeta} - (\zeta^2 + 8\zeta + 19\tfrac{4}{9})e^{-3\zeta} + \tfrac{2}{27}e^{-4\zeta}. \quad (51)$$

From equation (25) we see that the velocity distribution is given by the series

$$\frac{u}{U} = \phi_0' + \frac{F'\phi_1'}{2K^2} + \frac{F'^2\phi_{21}' + FF''\phi_{22}'}{4K^4} + \frac{F'^3\phi_{31}' + FF'F''\phi_{32}' + F^2F'''\phi_{33}'}{8K^6} + \frac{F'^4\phi_{41}' + \ldots\ldots}{16K^8} + \ldots \qquad (52)$$

The functions mentioned in equation (52) are tabulated in Table 9.

It may be noticed that there is a formal resemblance between equation (52) and the corresponding expression in the case of the growth of the boundary layer when the motion is started from rest. The functions which occur in place of ϕ_0', ϕ_1', ϕ_{21}', ... are much more complicated and their calculation is correspondingly more difficult. Consequently, the present series has been taken to higher order terms than was possible in the theory of boundary-layer growth.

4. *Properties of the Boundary Layer.*—We can now find from the series (52) the corresponding series for the displacement and momentum thicknesses and for the skin friction.

The skin friction is

$$\tau_0 = \mu\left(\frac{\partial u}{\partial y}\right)_{y=0} = \rho U_0 v_0 \frac{\partial^2\phi}{\partial\zeta^2}(\xi, 0), \qquad (53)$$

using the relations (13) and (21), so that

$$\frac{\tau_0}{\rho U_0 v_0} = \frac{\partial^2 \Phi_0}{\partial \zeta^2}(\xi, 0) + \frac{1}{K^2}\frac{\partial^2 \Phi_1}{\partial \zeta^2}(\xi, 0) + \frac{1}{K^4}\frac{\partial^2 \Phi_2}{\partial \zeta^2}(\xi, 0) + \ldots$$

$$= F\phi_0''(0) + \frac{FF'\phi_1''(0)}{2K^2} + \frac{FF'^2\phi_{21}''(0) + F^2F''\phi_{22}''(0)}{4K^4} + \ldots$$

$$= F + \frac{4FF'}{2K^2} - \frac{26FF'^2 + 7F^2F''}{4K^4} + \frac{409\tfrac{1}{3}FF'^3 + 355F^2F'F'' + 27\tfrac{8}{9}F^3F'''}{8K^6} + \ldots \quad \ldots \ldots \ldots \ldots \quad (54)$$

The displacement thickness is defined as

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right)dy,$$

and hence

$$\frac{v_0\delta^*}{\nu} = \int_0^\infty \left(1 - \frac{u}{U}\right)d\zeta$$

$$= \int_0^\infty \left\{1 - \phi_0' - \frac{F'\phi_1'}{2K^2} - \frac{F'^2\phi_{21}' + FF''\phi_{22}'}{4K^4} - \ldots \right\} d\zeta$$

$$= 1 - \frac{F'\phi_1(\infty)}{2K^2} - \frac{F'^2\phi_{21}(\infty) + FF''\phi_{22}(\infty)}{4K^4} - \ldots$$

$$= 1 - \frac{6F'}{2K^2} + \frac{89F'^2 + 20FF''}{4K^4} - \frac{1922\tfrac{7}{9}F'^3 + 1386\tfrac{1}{6}FF'F'' + 95\tfrac{1}{27}F^2F'''}{8K^6} + \ldots \quad \ldots \ldots \ldots \ldots \quad (55)$$

Similarly the momentum thickness is

$$\theta = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right)dy$$

so that

$$\frac{v_0\theta}{\nu} = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right)d\zeta$$

$$= \int_0^\infty \left\{\phi_0' + \frac{F'\phi_1'}{2K^2} + \frac{F'^2\phi_{21}' + FF''\phi_{22}'}{4K^4} + \ldots\right\}$$

$$\times \left\{1 - \phi_0' - \frac{F'\phi_1'}{2K^2} - \frac{F'^2\phi_{21}' + FF''\phi_{22}'}{4K^4} - \ldots\right\} d\zeta$$

To evaluate this it is necessary to integrate term by term after multiplication, using the expressions given above for ϕ_0', ϕ_1', etc. The result obtained is

$$\frac{v_0\theta}{\nu} = \frac{1}{2} \quad \frac{3\tfrac{1}{2}F'}{2K^2} + \frac{57\tfrac{5}{6}F'^2 + 13\tfrac{17}{18}FF''}{4K^4} - \frac{1350\tfrac{2}{9}F'^3 + 1012\tfrac{19}{27}FF'F'' + 69\tfrac{61}{72}F^2F'''}{8K^6} + \ldots \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (56)$$

From equations (55) and (56) we find by direct division that

$$H = \frac{\delta^*}{\theta} = 2 + \frac{2F'}{2K^2} \quad \frac{39\tfrac{1}{3}F'^2 + 15\tfrac{7}{9}FF''}{4K^4}$$

$$+ \frac{1048\tfrac{2}{3}F'^3 + 1112\tfrac{7}{27}FF'F'' + 89\tfrac{17}{54}F^2F'''}{8K^6} + \ldots \quad \ldots \ldots \ldots \quad (57)$$

By using the momentum equation

$$U\frac{d\theta}{dx} = -(\delta^* + 2\theta)\frac{dU}{dx} - v_0 + \frac{\tau_0}{\rho U} \quad \ldots \ldots \ldots \quad (58)$$

a check is made on the calculations, and an additional term of the skin friction series is obtained. Equation (58) may be written in the form

$$\frac{\tau_0}{\rho U v_0} = 1 + \frac{F}{K^2}\left\{\frac{d}{d\xi}\left(\frac{v_0\theta}{\nu}\right) + F'\left(\frac{v_0\delta^*}{\nu} + 2\frac{v_0\theta}{\nu}\right)\right\} \qquad \ldots \quad (59)$$

which, with equations (55) and (56), gives

$$\frac{\tau_0}{\rho U v_0} = 1 + \frac{4F'}{2K^2} - \frac{26F'^2 + 7FF''}{4K^4} + \frac{409\tfrac{1}{3}F'^3 + 355FF'F'' + 27\tfrac{8}{9}F^2F'''}{8K^6}$$
$$- \frac{9246\tfrac{4}{9}F'^4 + 16949\tfrac{8}{9}FF'^2F'' + 2025\tfrac{11}{27}F^2F''^2 + 2774\tfrac{7}{27}F^2F'F''' + 139\tfrac{25}{36}F^3F''''}{16K^8}$$
$$+ \ldots \qquad \ldots \quad (60)$$

5. *Applications to the Series Method.*—One of the principal methods of solving the boundary-layer equations in particular cases is that of expansion in a power series in x, starting from the forward stagnation point. This method was devised originally by Blasius[17] and was developed further by Howarth[18]. Each of these authors was only interested in the flow without suction, but the method was used by Bussmann and Ulrich[19] for both suction and blowing in an investigation of the boundary layer on a circular cylinder in a uniform stream. The method requires the numerical solution of a system of differential equations, but we shall see that for large suction velocities this can be replaced by asymptotic series.

We suppose that U can be expanded in a polynomial or power series in x, with origin at the forward stagnation point, and write

$$U = u_1x + u_2x^2 + u_3x^3 + \ldots \qquad \ldots \quad (61)$$

We assume that the stream function has a similar expansion in powers of x, whose coefficients are functions of y, namely

$$\psi = F_1x + F_2x^2 + F_3x^3 + \ldots , \qquad \ldots \quad (62)$$

and from this we find the corresponding series for u and v. By substituting in the equation of motion (1) and equating coefficients of the several powers of x a set of differential equations for the F_r are obtained. The functions F_r can be expressed in terms of universal functions of the non-dimensional variable

$$\eta = \left(\frac{u_1}{\nu}\right)^{1/2} y, \qquad \ldots \quad (63)$$

by means of the relations

$$\left.\begin{aligned} F_1 &= (u_1\nu)^{1/2}f_1 \\ F_2 &= 3u_1^{-1}(u_1\nu)^{1/2}u_2f_2 \\ F_3 &= 4u_1^{-2}(u_1\nu)^{1/2}[u_1u_3g_3 + u_2^2h_3] \\ F_4 &= 5u_1^{-3}(u_1\nu)^{1/2}[u_1^2u_4g_4 + u_1u_2u_3h_4 + u_2^3k_4] \\ &\text{etc.} \end{aligned}\right\} \qquad \ldots \quad (64)$$

The functions f_1, f_2, g_3, . . . satisfy the following equations, given by Howarth,

$$f_1'^2 - f_1f_1'' = 1 + f_1''' \qquad \ldots \quad (65)$$

$$\left.\begin{aligned} 3f_1'f_2' - 2f_1''f_2 - f_1f_2'' &= 1 + f_2''' \\ 4f_1'g_3' - 3f_1''g_3 - f_1g_3'' &= 1 + g_3''' \\ 4f_1'h_3' - 3f_1''h_3 - f_1h_3'' &= \tfrac{1}{2} + h_3''' - \tfrac{9}{2}(f_2'^2 - f_2f_2'') \\ 5f_1'g_4' - 4f_1''g_4 - f_1g_4'' &= 1 + g_4''' \\ 5f_1'h_4' - 4f_1''h_4 - f_1h_4'' &= 1 + h_4''' - \tfrac{12}{5}(5f_2'g_3' - 3f_2''g_3 - 2f_2g_3'') \\ 5f_1'k_4' - 4f_1''k_4 - f_1k_4'' &= k_4''' - \tfrac{12}{5}(5f_2'h_3' - 3f_2''h_3 - 2f_2h_3'') \end{aligned}\right\} \qquad \ldots \quad (66)$$

with the boundary conditions

$$f_1' = f_2' = g_3' = h_3' = g_4' = h_4' = k_4' = \ldots = 0 \text{ at } \eta = 0, \quad (67)$$

$$\left.\begin{array}{l} f_1' = 1, f_2' = \frac{1}{3}, g_3' = \frac{1}{4}, g_4' = \frac{1}{5} \ldots \\ h_3' = h_4' = k_4' = \ldots = 0 \end{array}\right\} \text{ at } \eta = \infty . \quad (68)$$

To obtain flow with uniform suction we have the further boundary conditions

$$f_1 = K, f_2 = g_3 = h_3 = g_4 = h_4 = k_4 = \ldots = 0 \text{ at } \eta = 0 \quad (69)$$

Equation (65) is non-linear, but each of equations (66) is linear. These equations have to be solved numerically for each value of K, where the suction velocity is

$$v_0 = K(u_1\nu)^{1/2} \quad (70)$$

If we choose U_0 and c such that

$$U_0 = u_1 c \quad (71)$$

we can relate the functions f_1, f_2, g_3, h_3, etc. to the functions $\phi_0, \phi_1, \phi_{21}, \phi_{22}$, etc. by substituting (61) into the formulae of section 3. By comparing the coefficients of the constants u_r in the alternative expressions for u we derive the following set of equations, in which the left hand side is a function of η and the right hand side of $\zeta = K\eta$.

$$f_1' = \phi_0' + \frac{\phi_1'}{2K^2} + \frac{\phi_{21}'}{4K^4} + \frac{\phi_{31}'}{8K^6} + \frac{\phi_{41}'}{16K^8} + \ldots \quad (72)$$

$$\left.\begin{array}{l}
3f_2' = \phi_0' + \dfrac{3\phi_1'}{2K^2} + \dfrac{5\phi_{21}' + 2\phi_{22}'}{4K^4} + \dfrac{7\phi_{31}' + 2\phi_{32}'}{8K^6} + \ldots \\
4g_3' = \phi_0' + \dfrac{4\phi_1'}{2K^2} + \dfrac{7\phi_{21}' + 6\phi_{22}'}{4K^4} + \dfrac{10\phi_{31}' + 6\phi_{32}' + 6\phi_{33}'}{8K^6} + \ldots \\
4h_3' = \dfrac{2\phi_1'}{2K^2} + \dfrac{8\phi_{21}' + 4\phi_{22}'}{4K^4} + \dfrac{18\phi_{31}' + 8\phi_{32}'}{8K^6} + \ldots \\
5g_4' = \phi_0' + \dfrac{5\phi_1'}{2K^2} + \dfrac{9\phi_{21}' + 12\phi_{22}'}{4K^4} + \dfrac{13\phi_{31}' + 12\phi_{32}' + 24\phi_{33}'}{8K^6} + \ldots \\
5h_4' = \dfrac{5\phi_1'}{2K^2} + \dfrac{22\phi_{21}' + 16\phi_{22}'}{4K^4} + \dfrac{51\phi_{31}' + 34\phi_{32}' + 18\phi_{33}'}{8K^6} + \ldots \\
5k_4' = \dfrac{4\phi_{21}' + 2\phi_{22}'}{4K^4} + \dfrac{20\phi_{31}' + 10\phi_{32}'}{8K^6} + \ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}\right\} \quad (73)$$

These expansions can be checked by obtaining them directly from the differential equations (65) and (66). In particular if $u_2 = u_3 = \ldots = 0$ we have

$$U = u_1 x \quad (74)$$

$$\text{and } u = Uf_1' \quad (75)$$

which gives us the case $m = 1 (\beta = 1)$ of the flows $U = cx^m$ which were studied in Part I, and the series (72) is identical with that given in Part I, where the series was obtained from equation (65).

6. *The Linearly Decreasing Velocity Distribution.*—A particularly simple form for the velocity outside the boundary layer, and one which is of application to the flow over the rear of an aerofoil or near the rear stagnation point of a cylinder with a rounded trailing edge, is that of a linear decrease in U. In this case F' is a negative constant. Consequently we may write

$$z = \frac{-F'}{2K^2} \quad (76)$$

and we then have

$$\frac{u}{U} = \phi_0' - z\phi_1' + z^2\phi_{21}' - z^3\phi_{31}' + z^4\phi_{41}' - \ldots , \qquad (77)$$

$$\frac{v_0\delta^*}{\nu} = 1 + 6z + 89z^2 + 1922\tfrac{7}{9}z^3 + 52433\tfrac{1}{2}z^4 + \ldots , \qquad (78)$$

$$\frac{v_0\theta}{\nu} = \tfrac{1}{2} + 3\tfrac{1}{2}z + 57\tfrac{5}{6}z^2 + 1350\tfrac{2}{9}z^3 + 38985\tfrac{91}{540}z^4 + \ldots , \qquad (79)$$

$$H = 2 - 2z - 39\tfrac{1}{3}z^2 - 1048\tfrac{2}{3}z^3 - 33782\tfrac{76}{135}z^4 - \ldots , \qquad (80)$$

$$\frac{\tau_0}{\rho U v_0} = 1 - 4z - 26z^2 - 409\tfrac{1}{3}z^3 - 9246\tfrac{4}{9}z^4 - 260807\tfrac{91}{135}z^5 - \ldots . \qquad (81)$$

Equations (78), (79) and (80) contain the additional term arising from ϕ_{41}' not given in equations (55), (56) and (57), and the last term of equation (81) is obtained from the momentum equation (59).

An approximate calculation of the amount of suction which will prevent separation was made by Prandtl[16], who used the momentum equation with Pohlhausen's separation profile, and obtained the result

$$v_0 = 2{\cdot}18 \left(-\nu \frac{dU}{dx}\right)^{1/2} , \qquad (82)$$

which corresponds to

$$K = 2{\cdot}18\,(-F')^{1/2} . \qquad (83)$$

Preston[8] made a similar calculation using the separation profile given by Howarth[20], and obtained

$$K = 1{\cdot}607\,(-F')^{1/2} \qquad (84)$$

We might expect that a better result would be produced by finding the value of z for which equation (81) vanishes. This would then give

$$K = C\,(-F')^{1/2} , \qquad (85)$$

where

$$C = (2z)^{-1/2} . \qquad (86)$$

If we assume all the terms not mentioned in equation (81) are negative, we find $z < 0{\cdot}06606$, so that $C > 2{\cdot}75$ and the method of extrapolation used in Part I gives

$$z = 0{\cdot}025, \quad K = 4{\cdot}5\,(-F')^{1/2} . \qquad (87)$$

This calculation, however, is very doubtful indeed, since series (77) corresponds to a solution of equation (34) of Part I with $\beta = 1$, in just the same way that the forward stagnation point flow equation (74) corresponds to equation (31) of Part I with $\beta = 1$. Since equation (34) of Part I has no solution for $\beta \geqslant 0$ this series cannot converge to it, and the series (77) to (81) cannot represent the flow. It therefore appears that no amount of suction can produce the boundary layer flow envisaged.

It may be expected that if a boundary layer starts to flow along a region where the stream velocity falls linearly, the skin friction will always become negative at some point before the rear stagnation point. This point will depend on the initial velocity profile and the suction velocity. Since a general velocity distribution will give the series (77) to (81) at the rear stagnation point we conclude that the same is true for the flow over any cylinder with a rounded trailing edge. I have suggested elsewhere[21] that with suction the true separation point is not where the skin friction vanishes, but is further downstream, or even may be non-existent if it vanishes close to the rear stagnation point. The effect will depend on the Reynolds number, and the boundary-layer equations will not be adequate to deal with it. It may be easier to prevent separation from the rear at low Reynolds numbers than at high ones.

7. *Flow Past a Circular Cylinder.*—As an illustration of the general theory, we now consider a porous circular cylinder placed in a uniform stream with a constant suction velocity over the surface. This problem was investigated by Bussmann and Ulrich[19], using the method of series expansion starting at the forward stagnation point. They found that when the velocity of suction is

$$v_0 = \left(\frac{U_0 \nu}{d}\right)^{1/2}, \qquad \qquad (88)$$

where U_0 is the main stream velocity and d the diameter of the cylinder, the separation point is 120·9 deg from the front stagnation point, whereas the position without suction is 110 deg, assuming in both cases that the velocity outside the boundary layer is given by the theoretical potential flow.

The suction velocities considered here are much larger, and separation takes place, if at all, only close to the rear stagnation point, though for the reasons given in section 6 the skin friction probably does vanish near the rear of the cylinder.

Take c, the representative length, to be the radius of the cylinder and U_0 the velocity of the stream at infinity. Then the velocity outside the boundary layer is

$$U = 2U_0 \sin (x/c), \qquad \qquad (89)$$

so that

$$F(\xi) = 2 \sin \xi \qquad \qquad (90)$$

and

$$v_0 = K \left(\frac{U_0 \nu}{c}\right)^{1/2}. \qquad \qquad (91)$$

Hence we have

$$\frac{u}{U} = \phi_0'(\zeta) + \frac{\phi_1'(\zeta) \cos \xi}{K^2} + \frac{\phi_{21}'(\zeta) \cos^2 \xi - \phi_{22}'(\zeta) \sin^2 \xi}{K^4} + \frac{\phi_{31}'(\zeta) \cos^3 \xi - (\phi_{32}'(\zeta) + \phi_{33}'(\zeta)) \cos \xi \sin^2 \xi}{K^6} + \ldots, \qquad (92)$$

$$\frac{\tau_0}{\rho U v_0} = 1 + \frac{4 \cos \xi}{K^2} - \frac{26 \cos^2 \xi - 7 \sin^2 \xi}{K^4} + \frac{409\frac{1}{3} \cos^3 \xi - 382\frac{8}{9} \cos \xi \sin^2 \xi}{K^6} - \frac{9246\frac{4}{9} \cos^4 \xi - 19724\frac{4}{27} \cos^2 \xi \sin^2 \xi + 2165\frac{11}{108} \sin^4 \xi}{K^8} + \ldots, \qquad (93)$$

$$\frac{v_0 \delta^*}{\nu} = 1 - \frac{6 \cos \xi}{K^2} + \frac{89 \cos^2 \xi - 20 \sin^2 \xi}{K^4} - \frac{1922\frac{7}{9} \cos^3 \xi - 1481\frac{11}{54} \cos \xi \sin^2 \xi}{K^6} + \ldots, \qquad (94)$$

$$\frac{v_0 \theta}{\nu} = \frac{1}{2} - \frac{3\frac{1}{2} \cos \xi}{K^2} + \frac{57\frac{5}{6} \cos^2 \xi - 13\frac{17}{18} \sin^2 \xi}{K^4} - \frac{1350\frac{2}{9} \cos^3 \xi - 1082\frac{119}{216} \cos \xi \sin^2 \xi}{K^6} + . \qquad (95)$$

$$H = 2 + \frac{2 \cos \xi}{K^2} - \frac{39\frac{1}{3} \cos^2 \xi - 15\frac{7}{9} \sin^2 \xi}{K^4} + \frac{1048\frac{2}{3} \cos^3 \xi - 1201\frac{31}{54} \cos \xi \sin^2 \xi}{K^6} + \ldots \qquad (96)$$

Numerical values are given in Tables 10 and 11 for $K = 5$, 10, 20, and the skin friction, displacement and momentum thicknesses and H are shown in Figs. 12 to 15. It will be noticed in Fig. 15 that H, which is greater than 2 at the forward stagnation point, diminishes towards the rear and becomes less than 2, and by equation (57) a similar phenomenon will happen with any cylinder. This is in opposition to the usual behaviour of H when there is no suction, as it normally increases in a region of adverse pressure gradient. It is seen that, when $K = 5$, H does increase between $\xi = 70$ deg and $\xi = 90$ deg, and it may be that when K is further reduced this becomes more marked, until the onset of separation deprives the later fall of H from having physical meaning. This hypothesis would provide a link between the cases of zero suction and large suction.

If $d = 2c$ is the diameter of the cylinder, the quantity coefficient is

$$C_Q = \frac{Q}{U_0 d} = \pi \frac{v_0}{U_0} = \pi K \sqrt{2} \left(\frac{U_0 d}{\nu}\right)^{-1/2}, \quad \ldots \quad (97)$$

where Q is the quantity of fluid sucked in unit time through unit span of the cylinder. If K is given by equation (87) this would be

$$C_Q = 28 \left(\frac{U_0 d}{v}\right)^{-1/2} \quad \ldots \quad (98)$$

8. *Calculation of Separation for a General Velocity Distribution.*—If we consider the skin friction at a fixed point ξ for varying values of K, we may find from equation (60) for what K it vanishes, and so obtain a relation between K and ξ. Then the greatest such value of K will be that required to maintain positive skin friction over the range of ξ considered. If this greatest value occurred at the rear stagnation point, where $F = 0$, we should have the problem considered in section 6, and for the reasons there given we must exclude this case. The calculation, if the relevant derivatives of F are known, may be made by using a formula due to Whittaker, and given in "The Calculus of Observations"[22].

This states that the smallest root of the equation

$$0 = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + a_4 z^4 + \ldots \quad \ldots \quad (99)$$

is

$$z = -\frac{a_0}{a_1} - \frac{a_0^2 a_2}{a_1 \begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix}} - \frac{a_0^3 \begin{vmatrix} a_2 & a_3 \\ a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 \\ a_0 & a_1 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix}} - \frac{a_0^4 \begin{vmatrix} a_2 & a_3 & a_4 \\ a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & a_2 & a_3 \\ a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 \\ 0 & 0 & a_0 & a_1 \end{vmatrix}} - \ldots \quad \ldots \quad (100)$$

In order to apply this we must first calculate the velocity derivatives F', F'', etc., and these can be found accurately only if analytical means are available. For aerofoils this requirement restricts us effectively to those derived from a circle by simple conformal transformations and those designed by Lighthill's method[23]. In either of these cases the modulus of the transformation is known and we may calculate the velocity derivatives in the following manner.

We consider the transformation (of unit modulus at infinity) from the aerofoil in the Z-plane to the unit circle

$$z = e^{i\theta}. \quad \ldots \quad (101)$$

Then c, the chord of the aerofoil, is now a quantity rather less than 4. At incidence α, with the circulation defined by the Kutta-Joukowski condition, the velocity at the surface of the aerofoil is

$$\frac{U}{U_0} = \frac{2[\sin(\theta - \alpha) + \sin \alpha]}{\left|\frac{dZ}{dz}\right|} \quad \ldots \quad (102)$$

Now

$$\left|\frac{dZ}{dz}\right| = \frac{ds}{d\theta} , \qquad \ldots \qquad (103)$$

where s is the distance round the aerofoil in the direction of increasing θ. It is most convenient, however, to have U positive in the direction of increasing x and to have x positive on the upper surface and zero at the leading edge.

Hence

$$x = s_0 - s , \qquad \ldots \qquad (104)$$

where s_0 is the value of s at the leading edge, and with the sign conventions adopted above for U we have

$$F = \frac{2}{s'} [\sin (\theta - \alpha) + \sin \alpha]. \qquad \ldots \qquad (105)$$

In each of the classes of aerofoils considered s' is known as an analytical expression in terms of θ. We also have

$$\xi = \frac{s_0 - s}{c} . \qquad \ldots \qquad (106)$$

Now

$$F' = \frac{dF}{d\theta} \Big/ \frac{d\xi}{d\theta}$$

$$= - \frac{c}{s'} \frac{dF}{d\theta} , \qquad \ldots \qquad (107)$$

Thus

$$F' = \frac{2c}{s'^2} \cos (\theta - \alpha) + \frac{2cs''}{s'^3} [\sin (\theta - \alpha) + \sin \alpha]. \qquad \ldots \qquad (108)$$

Similarly

$$F'' = 2c^2 \left[\frac{\sin (\theta - \alpha)}{s'^3} + \frac{3s''}{s'^4} \cos (\theta - \alpha) - \left(\frac{3s''^2}{s'^5} - \frac{s'''}{s'^4}\right) [\sin (\theta - \alpha) + \sin \alpha] \right] \ldots \quad (109)$$

$$F''' = -2c^3 \left[\frac{\cos (\theta - \alpha)}{s'^4} - \frac{6s''}{s'^5} \sin (\theta - \alpha) - \left(\frac{15s''^2}{s'^6} - \frac{4s'''}{s'^5}\right) \cos (\theta - \alpha) \right.$$

$$\left. + \left(\frac{15s''^3}{s'^7} - \frac{10s''s'''}{s'^6} + \frac{s''''}{s'^5}\right) [\sin (\theta - \alpha) + \sin \alpha] \right]. \qquad \ldots \qquad (110)$$

The s derivatives are to be obtained analytically from the known expression for s'. As an example of this method of calculation we shall consider in section 9 the case of a thin Joukowski aerofoil at a high lift coefficient with suction round the leading edge.

9. *Application to Leading Edge Continuous Suction.*—The principle of employing suction through a porous surface to enable an aerofoil to reach a high lift coefficient has been discussed by Preston[8] and Thwaites[24]. Preston's calculations are based on the use of equation (85) with $C = 1{\cdot}607$, obtained by assuming Howarth's[20] separation profile. His results should, therefore, be multiplied by 2·8 to obtain the corresponding ones given by equation (87). Thwaites' work assumes that the velocity profile will be similar to that of Blasius, and this is also likely to lead to an underestimate since it appears from equation (57) that $H < 2$ in a region of adverse velocity gradient.

As an application of the asymptotic method calculations have been made for an 8·3 per cent thick symmetrical Joukowski aerofoil at high incidences. One of the objects of the calculation was to compare the crude approximation of equation (87) with the more elaborate process of section 8. Consequently for the highest incidence (18 deg) the velocity and its first three derivatives were found from equations (105), (108), (109), (110). F'''' was estimated graphically where required. The coefficients C_1, C_2, C_3, C_4 of K^{-2}, K^{-4}, K^{-6}, K^{-8} in equation (60) were then calculated for some points near the position of maximum velocity gradient. These are all given in Table 12. At $\pi - \theta = 6$ deg the velocity gradient is very close to its maximum value, but the influence of the higher derivatives makes C_3 and C_4 positive there. At $\pi - \theta = 8$ deg the velocity gradient has fallen, and C_3 and C_4 are negative. The effect of the higher derivatives is still sufficiently pronounced to make C_4 comparatively small, and the formula (100) gives for separation

$$\frac{10^2}{K^2} = 0{\cdot}3593038 - 0{\cdot}2567528 - 0{\cdot}0447351 - 0{\cdot}0072263 - \ldots .$$

The rest of the series will probably be negligible or positive, and we find

$$K = 44{\cdot}5.$$

For $\pi - \theta = 12$ deg,

$$\frac{10^2}{K^2} = 0{\cdot}5863177 - 0{\cdot}4383462 - 0{\cdot}0837023 - 0{\cdot}0232541 - \ldots .$$

The remainder of the series probably lies between 0 and $-0{\cdot}06$ and so K lies between 49·4 and 53·4. Finally at $\pi - \theta = 18$ deg. we have $K = 39{\cdot}3$. The maximum of K is, therefore, approximately 52 to 56.

The maximum of $-F'$ is 158, so that equation (87) gives $K = 56{\cdot}6$ Hence the two methods are in close agreement. Consequently, calculations for the incidences of 12 deg and 15 deg were made only by the cruder method. For these we find $K = 40{\cdot}4$ and $K = 48{\cdot}2$ respectively. The velocity distributions near the leading edge are shown in Fig. 16 and Fig. 17 shows the variation of K with C_L.

10. *Conclusion*.—The success of the asymptotic method for the solution of the boundary-layer equations depends on the validity of the expansions employed. Although the series for the velocity, skin friction etc. have been referred to only as being asymptotic, it seems probable that they are in fact convergent for large values of K. It is also likely that the singularity which is expected at the separation point will define the radius of convergence of the series.

There are many other possible singularities where the solution will not hold. Any point where any of the derivatives of F does not exist is a singularity, but the boundary-layer equations cannot be expected to cope with such a point. The most important of the singularities are those which occur when suction starts after the boundary layer has already gone for some distance from its start, or when the flow does not resemble equation (74) at the leading edge. In these cases there will be a region of transition during which the boundary layer accommodates itself to the suction conditions. The present theory resembles that of boundary-layer growth in that the velocity distribution at a particular section does not depend on that at other sections, and is controlled solely by the local values of F and its derivatives, and by the suction velocity. Hence we may expect the effect of the previous development of the boundary layer to become progressively less during the transition region. The final condition will never be exactly attained, but will be approached closely within a short distance for large suction velocity.

The simplest example of a transition region is the case of a flat plate in a uniform stream with constant suction. This problem has been studied extensively by approximate methods, and an exact numerical solution was obtained by Iglisch[25]. At the leading edge the boundary layer is of zero thickness and has Blasius' profile, but ultimately it tends to the asymptotic suction profile, which is approached within a given degree of accuracy in a distance of order $U\nu v_0^{-2}$. A similar problem which has also received attention is that of a flat plate in a stream of falling velocity. An accurate solution without suction was obtained by Howarth[20], and Thwaites[26] made an investigation by an approximate method of the effect of small suction velocities on the position of the separation point. Here also the initial profile is that of Blasius, while the asymptotic method gives the results of section 6. An accurate investigation of this problem would settle some of the difficulties noted in section 6, and give some evidence about the flow near the rear stagnation point.

TABLE 9

Functions used in Calculating the Velocity Distribution

ζ	ϕ_0'	ϕ_1'	ϕ_{21}'	ϕ_{22}'	ϕ_{31}'	ϕ_{32}'	ϕ_{33}'	ϕ_{41}'
0	0	0	0	0	0	0	0	0
0·125	0·1175031	0·45504	− 3·05493	−0·82245	48·097	41·713	3·277	−1086·5
0·25	0·2211992	0·82748	− 5·74899	−1·54733	90·534	78·509	6·167	−2045·1
0·375	0·3127107	1·12758	− 8·12032	−2·18409	127·954	110·936	8·712	−2890·6
0·5	0·3934693	1·36469	−10·20075	−2·74032	160·914	139·455	10·946	−3635·8
0·75	0·5276334	1·68281	−13·59281	−3·63552	215·327	186·306	14·597	−4868·1
1	0·6321206	1·83940	−16·09996	−4·27476	256·927	221·627	17·307	−5815·4
1·25	0·7134952	1·88019	−17·85933	−4·69360	288·120	247·365	19·221	−6534·1
1·5	0·7768698	1·84082	−18·98503	−4·92503	310·750	265·047	20·458	−7067·8
2	0·8646647	1·62402	−19·72232	−4·94975	335·832	281·031	21·293	−7709·8
2·5	0·9179150	1·33388	−18·99177	−4·57767	340·669	277·560	20·517	−7937·6
3	0·9502129	1·04553	−17·34585	−3·99557	331·105	260·881	18·719	−7878·0
4	0·9816844	0·58610	−12·93014	−2·69890	285·356	207·101	13·857	−7212·2

TABLE 10

Velocity distributions within the boundary layer

Values of u/U for varying ξ, ζ and K

ζ	$\xi = 0$			ξ = 30 deg			ξ = 60 deg		
	$K = 5$	10	20	$K = 5$	10	20	$K = 5$	10	20
0	0	0	0	0	0	0	0	0	0
0·125	0·13	0·122	0·11862	0·13	0·121	0·11848	0·13	0·120	0·11807
0·25	0·25	0·229	0·22323	0·24	0·228	0·22297	0·24	0·225	0·22223
0·375	0·35	0·323	0·31548	0·34	0·322	0·31512	0·33	0·318	0·31412
0·5	0·44	0·406	0·39682	0·43	0·405	0·39638	0·42	0·400	0·39517
0·75	0·58	0·543	0·53176	0·57	0·541	0·53122	0·56	0·536	0·52973
1	0·69	0·649	0·63662	0·68	0·647	0·63604	0·66	0·641	0·63442
1·25	0·78	0·731	0·71808	0·76	0·729	0·71749	0·75	0·723	0·71584
1·5	0·84	0·794	0·78135	0·83	0·792	0·78077	0·81	0·786	0·77916
2	0·92	0·879	0·86860	0·91	0·877	0·86810	0·89	0·873	0·86669
2·5	0·96	0·930	0·92113	0·95	0·928	0·92072	0·94	0·924	0·91957
3	0·98	0·959	0·95272	0·97	0·958	0·95240	0·96	0·955	0·95151
4	1·00	0·987	0·98307	0·99	0·986	0·98290	0·99	0·984	0·98241

TABLE 10—*continued*

Velocity distributions within the boundary layer

Values of u/U for varying ξ, ζ and K

ζ	ξ = 90 deg			ξ = 120 deg			ξ = 150 deg		
	K = 5	10	20	K = 5	10	20	K = 5	10	20
0	0	0	0	0	0	0	0	0	0
0·125	0·12	0·118	0·11751	0·11	0·115	0·11693	0·09	0·113	0·11650
0·25	0·23	0·221	0·22121	0·21	0·217	0·22016	0·18	0·214	0·21938
0·375	0·32	0·313	0·31272	0·29	0·307	0·31130	0·26	0·302	0·31023
0·5	0·40	0·394	0·39349	0·37	0·387	0·39176	0·33	0·381	0·39047
0·75	0·53	0·528	0·52766	0·50	0·519	0·52552	0·45	0·512	0·52393
1	0·64	0·633	0·63215	0·60	0·623	0·62982	0·55	0·615	0·62807
1·25	0·72	0·714	0·71352	0·68	0·704	0·71114	0·62	0·696	0·70935
1·5	0·79	0·777	0·77690	0·74	0·768	0·77456	0·69	0·759	0·77280
2	0·87	0·865	0·86470	0·83	0·856	0·86263	0·78	0·849	0·86106
2·5	0·93	0·918	0·91794	0·89	0·911	0·91624	0·85	0·905	0·91495
3	0·96	0·951	0·95024	0·93	0·945	0·94890	0·89	0·940	0·94787
4	0·99	0·982	0·98170	0·97	0·979	0·98095	0·95	0·976	0·98036

TABLE 11

ξ (deg)		$\tau_0/\rho U_0 v_0$			$v_0\delta^*/v$			$v_0\theta/v$			H		
	U/U_0	K = 5	10	20	K = 5	10	20	K = 5	10	20	K = 5	10	20
0	0	0	0	0	0·779	0·947	0·98553	0·366	0·469	0·49159	2·08	2·017	2·00477
10	0·34730	0·389	0·360	0·35067	0·786	0·948	0·98573	0·371	0·470	0·49171	2·08	2·017	2·00470
20	0·68404	0·767	0·708	0·69038	0·805	0·950	0·98636	0·383	0·471	0·49207	2·07	2·016	2·00450
30	1	1·118	1·033	1·00855	0·828	0·953	0·98738	0·401	0·473	0·49266	2·05	2·015	2·00418
40	1·28558	1·427	1·323	1·29533	0·861	0·958	0·98878	0·421	0·476	0·49347	2·041	2·014	2·00373
50	1·53209	1·680	1·570	1·54187	0·889	0·964	0·99052	0·438	0·479	0·49447	2·029	2·012	2·00317
60	1·73205	1·865	1·766	1·74069	0·912	0·971	0·99255	0·452	0·483	0·49565	2·023	2·010	2·00251
70	1·87939	1·981	1·906	1·88585	0·930	0·979	0·99483	0·461	0·488	0·49698	2·022	2·007	2·00176
80	1·96962	2·028	1·984	1·97311	0·947	0·988	0·99729	0·468	0·493	0·49841	2·024	2·005	2·00095
90	2	2·011	2·001	2·00009	0·968	0·998	0·99988	0·478	0·499	0·49991	2·025	2·002	2·00010
100	1·96962	1·934	1·957	1·96628	1·000	1·009	1·00250	0·494	0·505	0·50144	2·021	1·998	1·99922
110	1·87939	1·799	1·854	1·87300	1·047	1·019	1·00508	0·522	0·511	0·50295	2·008	1·994	1·99835
120	1·73205	1·609	1·697	1·72338	1·111	1·030	1·00754	0·561	0·518	0·50440	1·984	1·991	1·99752
130	1·53209	1·363	1·492	1·52218	1·191	1·041	1·00980	0·612	0·524	0·50572	1·949	1·987	1·99675
140	1·28558	1·101	1·245	1·27563	1·279	1·051	1·01177	0·669	0·530	0·50688	1·906	1·983	1·99606
150	1	0·817	0·963	0·99123	1·369	1·059	1·01339	0·726	0·535	0·50784	1·86	1·980	1·99550
160	0·68404	0·535	0·657	0·67751	1·44	1·065	1·01459	0·775	0·539	0·50855	1·82	1·977	1·99508
170	0·34730	0·263	0·333	0·34382	1·49	1·069	1·01534	0·81	0·541	0·50899	1·8	1·976	1·99483

TABLE 12

$\pi - \theta$ (deg)	F	F'	F''	$F''' \times 10^{-6}$	$F'''' \times 10^{-8}$	C_1	$C_2 \times 10^{-4}$	$C_3 \times 10^{-6}$	$C_4 \times 10^{-8}$
4	5·38977	−134·476	−23526	15·8201		−268·952	+10·436	2234	
6	5·03506	−157·504	+1698·13	3·1795	−15·2	−315·008	−17·621	+21·32	+33100
8	4·63896	−139·158	8383·77	+0·5085	−3·35	−278·316	−19·393	−339·90	−6500
12	3·93528	−85·278	5648·88	−0·5396	−0·302	−170·556	−8·617	−144·99	−3720
18	3·20480	−38·992	1718·97	−0·1300	+0·187	−77·984	−1·952	−17·22	−285

Part III. Flow with Variation of Suction Velocity

Summary.—The asymptotic theory is here extended to cover general two-dimensional flow with arbitrary distributions both of the main stream velocity and of the velocity of suction. The method follows closely the treatment used in Part II, except that the suction velocity is now considered as a function of position on the surface. This increases considerably the number of functions which occur in the results, and hence renders their application more complicated. The similar velocity distributions examined in Part I are special cases of this general theory, and it has been verified that the results do reduce to those of Part I for this case. No further applications are made here, but the theory will probably be needed in considering the actual flow into a constant pressure suction chamber for an aerofoil, since the variation in external pressure causes a corresponding variation in the velocity of suction. This will be particularly important for suction over the leading edge of an aerofoil to obtain high lifts.

1. *The Transformation.*—As in Part II, we take a representative length c and a representative velocity U_0 and write R for the Reynolds number

$$R = \frac{U_0 c}{\nu} . \qquad (1)$$

The boundary-layer equation is reduced to non-dimensional form by writing

$$\xi = x/c , \qquad (2)$$

$$\eta = \left(\frac{U_0}{c\nu}\right)^{1/2} y, \qquad (3)$$

$$\psi = (U_0 c\nu)^{1/2} f(\xi, \eta) , \qquad (4)$$

$$U = U_0 F(\xi) , \qquad (5)$$

and becomes

$$\frac{\partial f}{\partial \eta}\frac{\partial^2 f}{\partial \xi \partial \eta} - \frac{\partial f}{\partial \xi}\frac{\partial^2 f}{\partial \eta^2} = F(\xi)F'(\xi) + \frac{\partial^3 f}{\partial \eta^3} , \qquad (6)$$

with the boundary conditions

$$\left.\begin{aligned} \frac{\partial f}{\partial \eta}(\xi, 0) &= 0 \\ \frac{\partial f}{\partial \eta}(\xi, \infty) &= F(\xi) . \end{aligned}\right\} \qquad (7)$$

The velocity of suction is

$$v_0(\xi) = -v(\xi, 0) = \left(\frac{U_0 \nu}{c}\right)^{1/2} \frac{\partial f}{\partial \xi}(\xi, 0) . \qquad (8)$$

We therefore put

$$f(\xi, 0) = KG(\xi) \qquad (9)$$

as an additional boundary condition, and then have

$$v_0 = K\left(\frac{U_0 \nu}{c}\right)^{1/2} G'(\xi) . \qquad (10)$$

Now write

$$\zeta = KG'\eta = \frac{v_0 y}{\nu} , \qquad (11)$$

and put

$$f(\xi, \eta) = KG + \frac{1}{KG'}\phi(\xi, \zeta). \qquad (12)$$

This transformation gives

$$\frac{\partial^3 \phi}{\partial \zeta^3} + \frac{\partial^2 \phi}{\partial \zeta^2} + \frac{1}{K^2 G'^2}\left[\frac{\partial \phi}{\partial \xi}\frac{\partial^2 \phi}{\partial \zeta^2} - \frac{\partial^2 \phi}{\partial \xi \partial \zeta}\frac{\partial \phi}{\partial \zeta} - \frac{G''}{G'}\phi\frac{\partial^2 \phi}{\partial \zeta^2} + FF'\right] = 0, \qquad (13)$$

and hence

$$\left.\begin{aligned}\phi_0'' &= e^{-\zeta},\\ \phi_0' &= 1 - e^{-\zeta},\\ \phi_0 &= \zeta - 1 + e^{-\zeta}.\end{aligned}\right\} \qquad (26)$$

The first approximation to the velocity distribution is therefore the familiar asymptotic suction profile

$$\frac{u}{U} = 1 - e^{-v_0 y/\nu}. \qquad (27)$$

Substitution in equation (19) gives

$$\frac{\partial^3 \Phi_1}{\partial \zeta^3} + \frac{\partial^2 \Phi_1}{\partial \zeta^2} + F'\phi_0 F\phi_0'' - F'\phi_0' F\phi_0' - g'F\phi_0 F\phi_0'' + FF' = 0,$$

so that

$$\Phi_1 = \tfrac{1}{2}(FF'\phi_1 + F^2 g'\chi_1), \qquad (28)$$

where

$$\phi_1''' + \phi_1'' + 2(\phi_0\phi_0'' - \phi_0'^2 + 1) = 0, \qquad (29)$$

$$\chi_1''' + \chi_1'' - 2\phi_0\phi_0'' = 0. \qquad (30)$$

Similarly we find that

$$\Phi_2 = \tfrac{1}{4}(FF'^2\phi_{21} + F^2F''\phi_{22} + F^2F'g'\chi_{21} + F^3g'^2\chi_{22} + F^3g''\chi_{23}), \qquad (31)$$

where

$$\phi_{21}''' + \phi_{21}'' + 2(\phi_0\phi_1'' + \phi_1\phi_0'' - 2\phi_0'\phi_1') = 0, \qquad (32)$$

$$\phi_{22}''' + \phi_{22}'' + 2(\phi_1\phi_0'' - \phi_0'\phi_1') = 0, \qquad (33)$$

$$\chi_{21}''' + \chi_{21}'' + 2(\phi_0\chi_1'' - 3\phi_1\phi_0'' + 2\chi_1\phi_0'' - \phi_0\phi_1'' - 3\phi_0'\chi_1' + 2\phi_0'\phi_1') = 0, \qquad (34)$$

$$\chi_{22}''' + \chi_{22}'' - 2(3\chi_1\phi_0'' + \phi_0\chi_1'' - 2\phi_0'\chi_1') = 0, \qquad (35)$$

$$\chi_{23}''' + \chi_{23}'' + 2(\chi_1\phi_0'' - \phi_0'\chi_1') = 0; \qquad (36)$$

and

$$\begin{aligned}\Phi_3 = \tfrac{1}{8}(&FF'^3\phi_{31} + F^2F'F''\phi_{32} + F^3F'''\phi_{33} + F^2F'^2g'\chi_{31} + F^3F''g'\chi_{32}\\ &+ F^3F'g'^2\chi_{33} + F^4g'^3\chi_{34} + F^3F'g''\chi_{35} + F^4g'g''\chi_{36} + F^4g'''\chi_{37}),\end{aligned} \qquad (37)$$

where

$$\phi_{31}''' + \phi_{31}'' + 2(\phi_0\phi_{21}'' + \phi_1\phi_1'' + \phi_{21}\phi_0'' - 2\phi_0'\phi_{21}' - \phi_1'^2) = 0, \qquad (38)$$

$$\phi_{32}''' + \phi_{32}'' + 2(\phi_0\phi_{22}'' + \phi_1\phi_1'' + 2\phi_{21}\phi_0'' + 2\phi_{22}\phi_0'' - 3\phi_0'\phi_{22}' - \phi_1'^2 - 2\phi_0'\phi_{21}') = 0, \qquad (39)$$

$$\phi_{33}''' + \phi_{33}'' + 2(\phi_{22}\phi_0'' - \phi_0'\phi_{22}') = 0, \qquad (40)$$

$$\begin{aligned}\chi_{31}''' + \chi_{31}'' + 2(&\phi_0\chi_{21}'' - 3\phi_1\phi_1'' + 2\chi_1\phi_1'' + \phi_1\chi_1'' - 5\phi_{21}\phi_0'' + 2\chi_{21}\phi_0'' - \phi_0\phi_{21}''\\ &- 3\phi_0'\chi_{21}' + 2\phi_1'^2 - 3\phi_1'\chi_1' + 4\phi_0'\phi_{21}') = 0,\end{aligned} \qquad (41)$$

$$\chi_{32}''' + \chi_{32}'' + 2(\phi_1\chi_1'' - 5\phi_{22}\phi_0'' + \chi_{21}\phi_0'' - \phi_0\phi_{22}'' - \phi_1'\chi_1' + 4\phi_0'\phi_{22}' - \phi_0'\chi_{21}') = 0, \qquad (42)$$

$$\begin{aligned}\chi_{33}''' + \chi_{33}'' + 2(&\phi_0\chi_{22}'' - 3\chi_1\phi_1'' - 3\phi_1\chi_1'' + 2\chi_1\chi_1'' - 5\chi_{21}\phi_0'' + 3\chi_{22}\phi_0'' - \phi_0\chi_{21}''\\ &- 4\phi_0\chi_{22}' + 4\phi_1'\chi_1' - 2\chi_1'^2 + 4\phi_0'\chi_{21}') = 0,\end{aligned} \qquad (43)$$

$$\chi_{34}''' + \chi_{34}'' - 2(3\chi_1\chi_1'' + 5\chi_{22}\phi_0'' + \phi_0\chi_{22}'' - 2\chi_1'^2 - 4\phi_0'\chi_{22}') = 0, \qquad (44)$$

$$\chi_{35}''' + \chi_{35}'' + 2(\phi_0\chi_{23}'' + \chi_1\phi_1'' + \chi_{21}\phi_0'' + 3\chi_{23}\phi_0'' - 4\phi_0'\chi_{23}' - \phi_1\chi_1' - \phi_0'\chi_{21}') = 0, \qquad (45)$$

$$\chi_{36}''' + \chi_{36}'' + 2(\chi_1\chi_1'' + 2\chi_{22}\phi_0'' - 5\chi_{23}\phi_0'' - \phi_0\chi_{23}'' - \chi_1'^2 - 2\phi_0'\chi_{22}' + 4\phi_0'\chi_{23}') = 0, \qquad (46)$$

$$\chi_{37}''' + \chi_{37}'' + 2(\chi_{23}\phi_0'' - \phi_0'\chi_{23}') = 0. \qquad (47)$$

The functions ϕ are the same as those occurring in Part II, and values of their derivatives ϕ' are given there. The new functions χ which occur in the terms involving g are calculated by precisely the same process as that employed for the ϕ functions, and the following results are obtained:

$$\chi_1' = -(\zeta^2 + 1)e^{-\zeta} + e^{-2\zeta} \quad \ldots \quad (48)$$

$$\chi_{21}' = (\zeta^4 + 6\zeta^3 + 23\zeta^2 + 12\zeta + 42)e^{-\zeta} - (2\zeta^2 + 18\zeta + 42)e^{-2\zeta}, \quad \ldots \quad (49)$$

$$\chi_{22}' = -(\tfrac{1}{2}\zeta^4 + 1\tfrac{1}{3}\zeta^3 + 7\zeta^2 + \zeta + 17\tfrac{1}{2})e^{-\zeta} + (2\zeta^2 + 8\zeta + 18)e^{-2\zeta} - \tfrac{1}{2}e^{-3\zeta}, \quad \ldots \quad (50)$$

$$\chi_{23}' = (\tfrac{2}{3}\zeta^3 + 2\zeta^2 + \zeta + 4\tfrac{1}{6})e^{-\zeta} - (2\zeta + 4)e^{-2\zeta} - \tfrac{1}{6}e^{-3\zeta}, \quad \ldots \quad (51)$$

$$\begin{aligned}\chi_{31}' = &-(\tfrac{1}{2}\zeta^6 + 6\zeta^5 + 48\tfrac{1}{2}\zeta^4 + 228\zeta^3 + 828\zeta^2 + 654\zeta + 1825\tfrac{2}{3})e^{-\zeta} \\ &+ (2\zeta^4 + 36\zeta^3 + 272\zeta^2 + 1064\zeta + 1826)e^{-2\zeta} - \tfrac{1}{3}e^{-3\zeta}, \quad \ldots \quad (52)\end{aligned}$$

$$\begin{aligned}\chi_{32}' = &-(\tfrac{2}{3}\zeta^5 + 9\tfrac{1}{3}\zeta^4 + 54\zeta^3 + 207\zeta^2 + 72\zeta + 553\tfrac{7}{18})e^{-\zeta} \\ &+ (5\tfrac{1}{3}\zeta^3 + 62\zeta^2 + 296\zeta + 554)e^{-2\zeta} - (\tfrac{1}{3}\zeta + \tfrac{11}{18})e^{-3\zeta}, \quad \ldots \quad (53)\end{aligned}$$

$$\begin{aligned}\chi_{33}' = &(\tfrac{1}{2}\zeta^6 + 5\tfrac{1}{3}\zeta^5 + 44\tfrac{2}{3}\zeta^4 + 201\tfrac{2}{3}\zeta^3 + 781\tfrac{1}{2}\zeta^2 + 373\zeta + 2242\tfrac{7}{18})e^{-\zeta} \\ &- (4\zeta^4 + 56\zeta^3 + 366\zeta^2 + 1324\zeta + 2272)e^{-2\zeta} + (1\tfrac{1}{2}\zeta^2 + 13\tfrac{1}{3}\zeta + 29\tfrac{11}{18})e^{-3\zeta}, \quad \ldots \quad (54)\end{aligned}$$

$$\begin{aligned}\chi_{34}' = &-(\tfrac{1}{6}\zeta^6 + 1\tfrac{1}{3}\zeta^5 + 11\tfrac{1}{6}\zeta^4 + 47\tfrac{2}{3}\zeta^3 + 191\tfrac{1}{2}\zeta^2 + 63\tfrac{1}{3}\zeta + 627\tfrac{2}{3})e^{-\zeta} \\ &+ (2\zeta^4 + 18\tfrac{2}{3}\zeta^3 + 112\zeta^2 + 378\zeta + 642)e^{-2\zeta} - (1\tfrac{1}{2}\zeta^2 + 6\tfrac{2}{3}\zeta + 14\tfrac{5}{9})e^{-3\zeta} + \tfrac{2}{9}e^{-4\zeta}, \quad (55)\end{aligned}$$

$$\begin{aligned}\chi_{35}' = &-(\tfrac{2}{3}\zeta^5 + 7\tfrac{1}{3}\zeta^4 + 46\tfrac{1}{3}\zeta^3 + 156\tfrac{1}{6}\zeta^2 + 111\tfrac{2}{3}\zeta + 378\tfrac{1}{2})e^{-\zeta} \\ &+ (4\zeta^3 + 48\zeta^2 + 224\zeta + 368)e^{-2\zeta} + (\tfrac{1}{2}\zeta^2 + 4\tfrac{2}{3}\zeta + 10\tfrac{1}{2})e^{-3\zeta}, \quad \ldots \quad (56)\end{aligned}$$

$$\begin{aligned}\chi_{36}' = &(\tfrac{2}{3}\zeta^5 + 5\tfrac{2}{3}\zeta^4 + 33\tfrac{2}{3}\zeta^3 + 118\tfrac{1}{6}\zeta^2 + 57\tfrac{5}{9}\zeta + 337\tfrac{23}{54})e^{-\zeta} - (5\tfrac{1}{3}\zeta^3 + 46\zeta^2 + 198\zeta + 334\tfrac{1}{3})e^{-2\zeta} \\ &- (\tfrac{1}{2}\zeta^2 + 1\tfrac{1}{3}\zeta + 3\tfrac{2}{9})e^{-3\zeta} + \tfrac{7}{54}e^{-4\zeta}, \quad \ldots \quad (57)\end{aligned}$$

$$\begin{aligned}\chi_{37}' = &-(\tfrac{1}{3}\zeta^4 + 2\tfrac{2}{3}\zeta^3 + 9\zeta^2 + 5\tfrac{1}{9}\zeta + 21\tfrac{20}{27})e^{-\zeta} + (2\zeta^2 + 12\zeta + 21)e^{-2\zeta} \\ &+ (\tfrac{1}{3}\zeta + \tfrac{13}{18})e^{-3\zeta} + \tfrac{1}{54}e^{-4\zeta}. \quad \ldots \quad (58)\end{aligned}$$

These functions are tabulated at the end of the report, and are needed for calculating the velocity distribution in the boundary layer, since

$$\begin{aligned}\frac{u}{U} = \phi_0' &+ \frac{1}{2K^2G'^2}(F'\phi_1' + F'g\chi_1') \\ &+ \frac{1}{4K^4G'^4}(F'^2\phi_{21}' + FF''\phi_{22}' + FF'g'\chi_{21}' + F^2g'^2\chi_{22}' + F^2g''\chi_{23}') \\ &+ \frac{1}{8K^6G'^6}(F'^3\phi_{31}' + FF'F''\phi_{32}' + F^2F'''\phi_{33}' + FF'^2g'\chi_{31}' + F^2F''g'\chi_{32}' \\ &+ F^2F'g'^2\chi_{33}' + F^3g'^3\chi_{34}' + F^2F'g''\chi_{35}' + F^3g'g''\chi_{36}' + F^3g'''\chi_{37}') + \ldots \quad \ldots \quad (59)\end{aligned}$$

It may be noted that this is a power series in

$$\frac{1}{K^2G'^2} = \frac{U_0\nu}{v_0^2c} \quad \ldots \quad (60)$$

and that when v_0 varies with x, so does $\zeta = v_0y/\nu$ for fixed y.

3. *Properties of the Boundary Layer.*—As in Part II, we find that the skin friction is given by

$$\frac{\tau_0}{\rho U v_0} = \frac{1}{F}\left(\frac{\partial^2 \phi}{\partial \zeta^2}\right)_{\zeta = 0}$$

$$= 1 + \frac{1}{2K^2G'^2}\left(4F' - Fg'\right) - \frac{1}{4K^4G'^4}\left(26F'^2 + 7FF'' - 36FF'g' + 10F^2g'^2 - 3\tfrac{1}{3}F^2g''\right)$$
$$+ \frac{1}{8K^6G'^6}\left(409\tfrac{1}{3}F'^3 + 355FF'F'' + 27\tfrac{8}{9}F^2F''' - 1415\tfrac{1}{3}FF'^2g' - 329\tfrac{1}{9}F^2F''g'\right.$$
$$\left. + 1275\tfrac{1}{9}F^2F'g'^2 - 305\tfrac{5}{9}F^3g'^3 - 272F^2F'g'' + 198\tfrac{11}{18}F^3g'g'' - 15\tfrac{5}{18}F^3g'''\right) - \ldots \qquad (61)$$

For the displacement thickness δ^* we have

$$\frac{v_0\delta^*}{\nu} = \int_0^\infty \left(1 - \frac{u}{U}\right) d\zeta$$

$$= 1 - \frac{1}{2K^2G'^2}\left(6F' - 2\tfrac{1}{2}Fg'\right) + \frac{1}{4K^4G'^4}\left(89F'^2 + 20FF'' - 134FF'g' + 41\tfrac{1}{6}F^2g'^2 - 10\tfrac{11}{18}F^2g''\right)$$
$$- \frac{1}{8K^6G'^6}\left(1922\tfrac{7}{9}F'^3 + 1386\tfrac{1}{6}FF'F'' + 95\tfrac{1}{27}F^2F''' - 6485\tfrac{7}{9}FF'^2g' - 1299\tfrac{7}{54}F^2F''g'\right.$$
$$\left. + 5889\tfrac{19}{54}F^2F'g'^2 - 1461\tfrac{35}{54}F^3g'^3 - 1078\tfrac{17}{18}F^2F'g'' + 817\tfrac{199}{216}F^3g'g'' - 54\tfrac{41}{72}F^3g'''\right) + \ldots \qquad (62)$$

Similarly for the momentum thickness θ,

$$\frac{v_0\theta}{\nu} = \int_0^\infty \frac{u}{U}\left(1 - \frac{u}{U}\right) d\zeta$$

$$= \tfrac{1}{2} - \frac{1}{2K^2G'^2}\left(3\tfrac{1}{2}F' - 1\tfrac{2}{3}Fg'\right)$$
$$+ \frac{1}{4K^4G'^4}\left(57\tfrac{5}{6}F'^2 + 13\tfrac{17}{18}FF'' - 94\tfrac{5}{6}FF'g' + 30\tfrac{5}{9}F^2g'^2 - 7\tfrac{23}{36}F^2g''\right)$$
$$- \frac{1}{8K^6G'^6}\left(1350\tfrac{2}{9}F'^3 + 1012\tfrac{19}{27}FF'F'' + 69\tfrac{61}{72}F^2F''' - 4803\tfrac{7}{9}FF'^2g' - 975\tfrac{19}{24}F^2F''g'\right.$$
$$\left. + 4492\tfrac{23}{27}F^2F'g'^2 - 1138\tfrac{13}{45}F^3g'^3 - 804\tfrac{1}{72}F^2F'g'' + 623\tfrac{83}{270}F^3g'g'' - 40\tfrac{349}{540}F^3g'''\right) + \ldots \qquad (63)$$

Taking the quotient of the series (62) and (63),

$$H = \frac{\delta^*}{\theta} = 2 + \frac{1}{2K^2G'^2}\left(2F' - 1\tfrac{2}{3}Fg'\right)$$
$$- \frac{1}{4K^4G'^4}\left(39\tfrac{1}{3}F'^2 + 15\tfrac{7}{9}FF'' - 93FF'g' + 34\tfrac{1}{3}F^2g'^2 - 9\tfrac{1}{3}F^2g''\right)$$
$$+ \frac{1}{8K^6G'^6}\left(1048\tfrac{2}{3}F'^3 + 1112\tfrac{7}{27}FF'F'' + 89\tfrac{17}{54}F^2F''' - 4889\tfrac{1}{3}FF'^2g' - 1205\tfrac{5}{6}F^2F''g'\right.$$
$$+ 5204\tfrac{1}{27}F^2F'g'^2 - 1413\tfrac{76}{135}F^3g'^3 - 962\tfrac{5}{18}F^2F'g'' + 800\tfrac{439}{540}F^3g'g''$$
$$\left. - 53\tfrac{241}{540}F^3g'''\right) - \ldots \qquad (64)$$

4. *Case of Similar Velocity Profiles.*—In Part 1 the flows were considered for which

$$U = kx^m, \qquad (65)$$

$$v_0 = K\left(\frac{m+1}{2}\right)^{1/2}\left(\frac{U\nu}{x}\right)^{1/2}. \qquad (66)$$

We therefore have

$$F(\xi) = \xi^m, \qquad (67)$$

$$G'(\xi) = \left(\frac{m+1}{2}\right)^{1/2} \xi^{(m-1)/2}, \qquad (68)$$

$$g'(\xi) = \frac{m-1}{2\xi}, \qquad (69)$$

and it follows that the relation between $\phi(\xi, \zeta)$ and the function $\phi(\zeta)$ of Part I is

$$\phi(\xi, \zeta) = \xi^m \phi(\zeta). \qquad (70)$$

Putting this in equation (13) we see that the equation satisfied by $\phi(\zeta)$ is

$$\phi''' + \phi'' + \frac{1}{K^2}[\phi\phi'' + \beta(1 - \phi'^2)] = 0, \qquad (71)$$

where

$$\beta = \frac{2m}{m+1}, \qquad (72)$$

thus agreeing with equation (31) of Part I. The solution of this equation was obtained in Part I in the form

$$\phi(\zeta) = \phi_0 + \frac{1}{4K^2}(\phi_{10}^* + \beta\phi_{11}^*) + \frac{1}{8K^4}(\phi_{20}^* + \beta\phi_{21}^* + \beta^2\phi_{22}^*) + \ldots, \qquad (73)$$

where stars have been added to distinguish the ϕ_{10} etc. of Part I from the functions bearing the same suffices of Parts II and III. If we substitute in the solution (15) of Part III we find that

$$\phi(\zeta) = \phi_0 + \frac{1}{2K^2\left(\frac{m+1}{2}\right)\xi^{m-1}}\left[m\xi^{m-1}\phi_1 + \xi^m\left(\frac{m-1}{2\xi}\right)\chi_1\right] + \ldots, \qquad (74)$$

and by comparing equations (73) and (74) we see that

$$\phi_{10}^* + \beta\phi_{11}^* = 2\,\frac{m\phi_1 + \frac{1}{2}(m-1)\chi_1}{\frac{1}{2}(m+1)}$$

etc.

and by expressing the right hand side in terms of β we obtain the following relations giving the functions of Part I in terms of those of Part III.

$$\phi_{10}^* = -2\chi_1, \qquad (75)$$

$$\phi_{11}^* = 2\phi_1 + 2\chi_1, \qquad (76)$$

$$\phi_{20}^* = 2\chi_{22} + 4\chi_{23}, \qquad (77)$$

$$\phi_{21}^* = -4\phi_{22} - 2\chi_{21} - 4\chi_{22} - 6\chi_{23}, \qquad (78)$$

$$\phi_{22}^* = 2\phi_{21} + 4\phi_{22} + 2\chi_{21} + 2\chi_{22} + 2\chi_{23}, \qquad (79)$$

$$\phi_{30}^* = -2\chi_{34} - 4\chi_{36} - 16\chi_{37}, \qquad (80)$$

$$\phi_{31}^* = 16\phi_{33} + 4\chi_{32} + 2\chi_{33} + 6\chi_{34} + 4\chi_{35} + 10\chi_{36} + 32\chi_{37}, \qquad (81)$$

$$\phi_{32}^* = -4\phi_{32} - 28\phi_{33} - 2\chi_{31} - 8\chi_{32} - 4\chi_{33} - 6\chi_{34} - 6\chi_{35} - 8\chi_{36} - 20\chi_{37}, \qquad (82)$$

$$\phi_{33}^* = 2\phi_{31} + 4\phi_{32} + 12\phi_{33} + 2\chi_{31} + 4\chi_{32} + 2\chi_{33} + 2\chi_{34} + 2\chi_{35} + 2\chi_{36} + 4\chi_{37} \qquad (83)$$

These relations provide a valuable check on the differential equations for the functions χ as well as for the functions themselves. Similar relations enable the expressions (61) to (64) for τ_0, δ^*, θ and H to be checked. All the formulae have in fact been checked in this manner.

5. *Note on Practical Applications.*—When suction is applied through a porous aerofoil surface, the pressure at the inner side of the porous material will be nearly constant, but that at the outer surface will vary with the external velocity, by Bernoulli's theorem. Since the velocity of suction is proportional to the pressure difference if the thickness and porosity of the surface are constant, the greatest value of the suction velocity will coincide with the least value of the external velocity. In fact v_0 will be of the form

$$v_0 = a - bU^2,$$

where a and b depend on the porosity of the surface and the internal pressure. To calculate the boundary layer then involves $F = U/U_0$,

$$\frac{1}{K^2G'^2} = \frac{U_0\nu}{{v_0}^2c}, \text{ and } g'(\xi) = \frac{c}{v_0}\frac{dv_0}{dx} = \frac{2FF'\,b{U_0}^2}{F^2b{U_0}^2 - a}.$$

The amount of suction needed to prevent separation can be found by the method indicated in Part II, section 8. It is evident that the decision whether or not to take into account this variation of suction velocity depends on the magnitude of $(p_0 - p_1)/\frac{1}{2}\rho {U_0}^2$, where $(p_0 - p_1)$ is the pressure difference between the flow at infinity and the interior of the aerofoil, and U_0 is the main stream velocity. If this quantity is large the suction will be effectively uniform, but if it becomes comparable with $(U/U_0)^2$ then the variation will have to be taken into consideration. In particular we see that for a case such as suction at the nose of an aerofoil at high incidence, where large local velocities occur, this criterion may be of importance.

TABLE 13

Table of the functions χ'

ζ	χ_1'	χ_{21}'	χ_{22}'	χ_{23}'	χ_{31}'	χ_{32}'	χ_{33}'	χ_{34}'	χ_{35}'	χ_{36}'	χ_{37}'
0	0	0	0	0	0	0	0	0	0	0	0
0·125	−0·11749	4·23005	−1·17503	0·39166	−166·303	−38·671	149·827	−35·903	−31·960	23·337	−1·795
0·25	−0·22095	7·96212	−2·21194	0·73705	−313·032	−72·786	282·023	−67·583	−60·153	43·925	−3·378
0·375	−0·31157	11·25223	−3·12674	1·04105	−442·408	−102·853	398·595	−95·522	−84·997	62·073	−4·773
0·5	−0·39028	14·14804	−3·93330	1·30770	−556·351	−129·306	501·277	−120·140	−106·848	78·043	−5·998
0·75	−0·51494	18·91233	−5·26762	1·74195	−744·357	−172·800	670·780	−160·817	−142·746	104·329	−8·005
1	−0·60042	22·51108	−6·29087	2·06141	−887·876	−205·662	800·306	−191·980	−169·825	124·252	−9·503
1·25	−0·65208	25·13299	−7·05832	2·28281	−995·143	−229·700	897·270	−215·423	−189·592	138·924	−10·575
1·5	−0·67539	26·92343	−7·61056	2·42017	−1072·474	−246·320	967·341	−232·513	−203·231	149·210	−11·285
2	−0·65836	28·46930	−8·18667	2·49210	−1156·306	−261·691	1043·831	−251·713	−215·818	159·273	−11·833
2·5	−0·58838	27·94133	−8·20626	2·36761	−1168·919	−259·037	1056·260	−255·855	−213·679	158·734	−11·511
3	−0·49539	26·00501	−7·81727	2·12433	−1131·087	−244·039	1023·239	−249·260	−201·502	150·833	−10·621
4	−0·31103	20·06155	−6·32495	1·51311	−964·261	−194·606	874·730	−216·102	−161·310	122·823	−8·067

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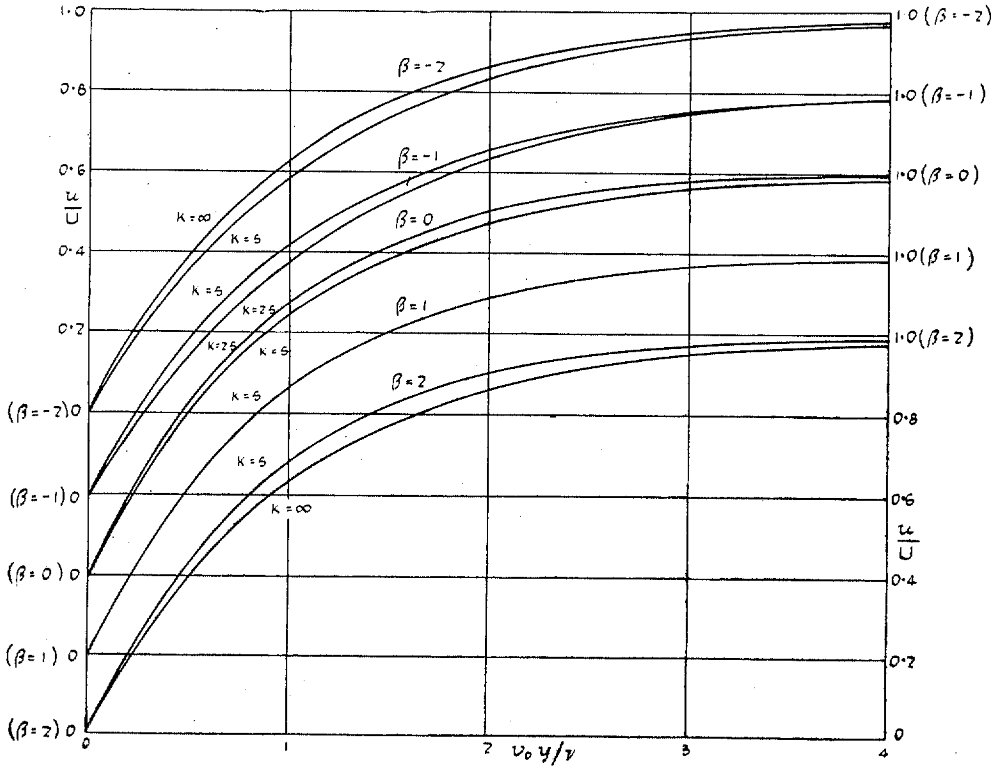

FIG. 1. Velocity distributions in the boundary layer.

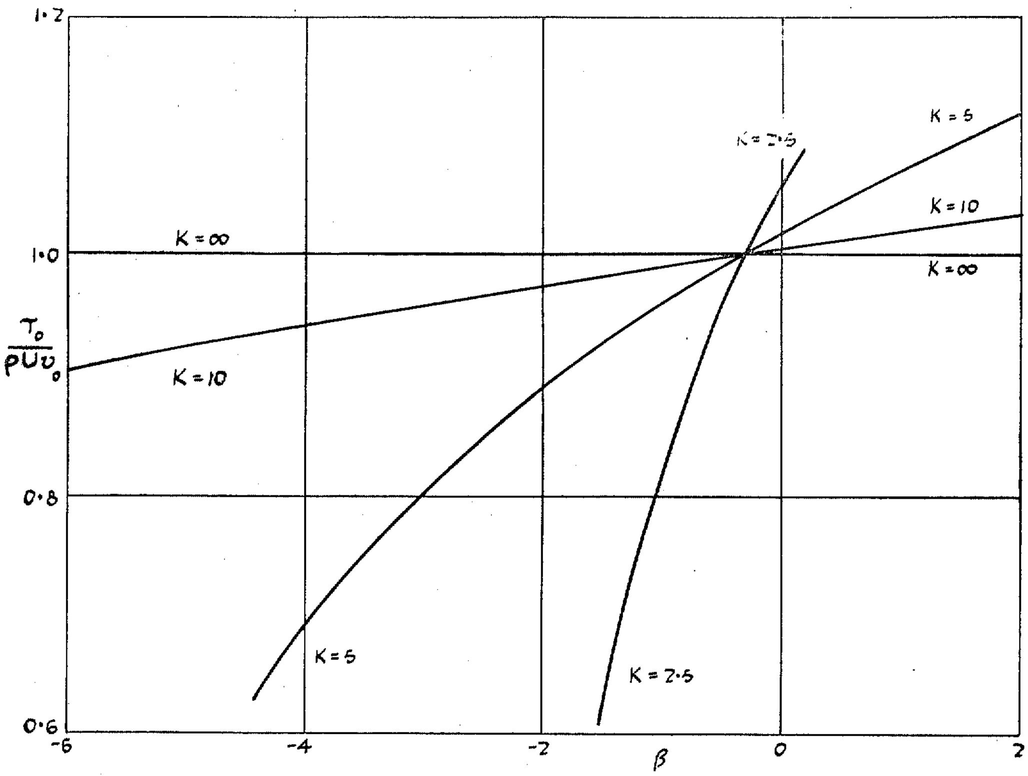

FIG. 2. Variation of skin friction.

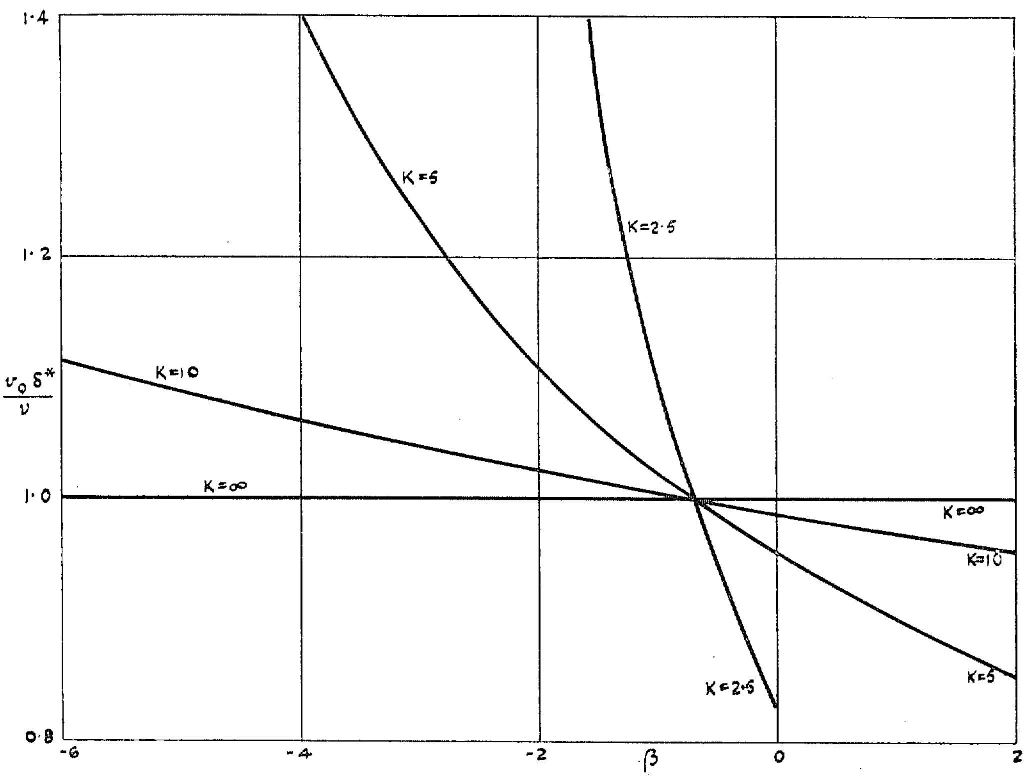

FIG. 3. Variation of displacement thickness.

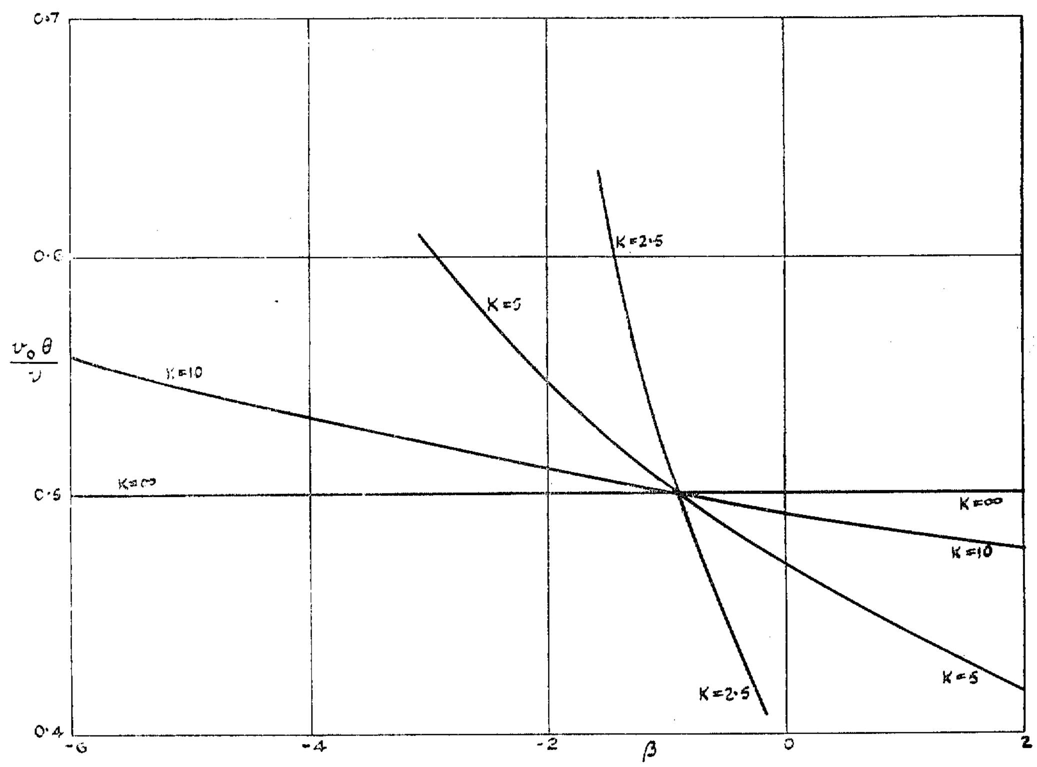

FIG. 4. Variation of momentum thickness.

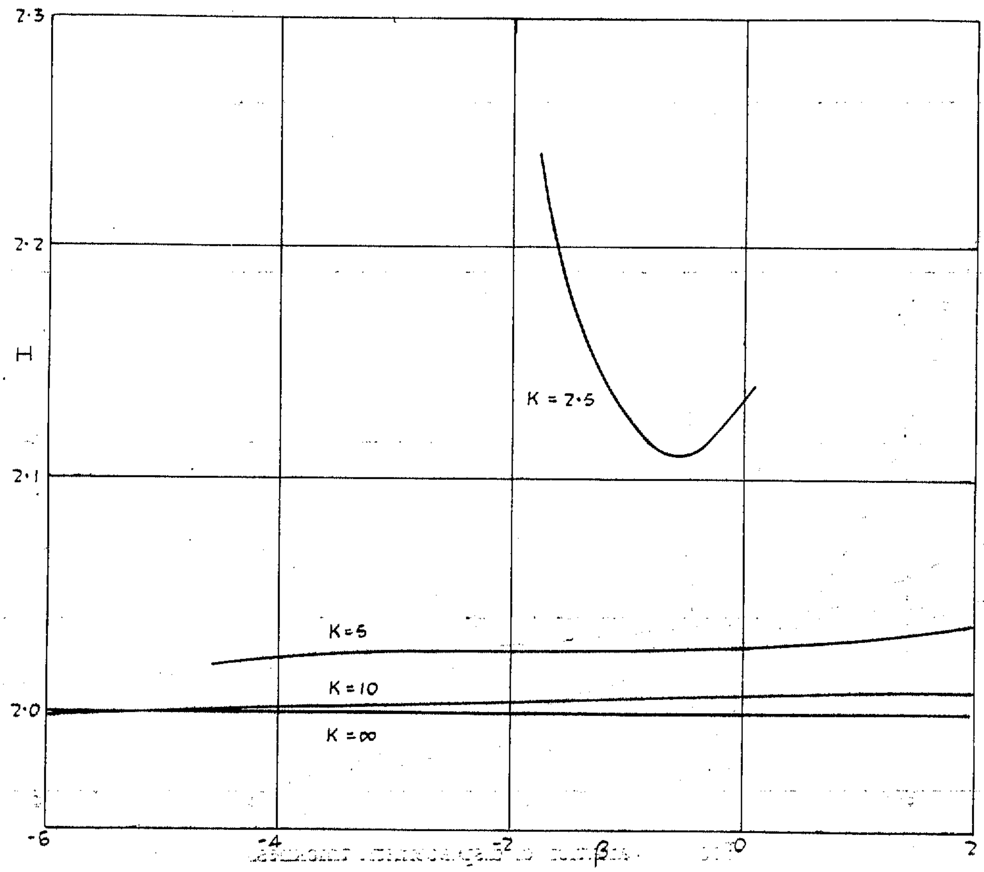

FIG. 5. Variation of H.

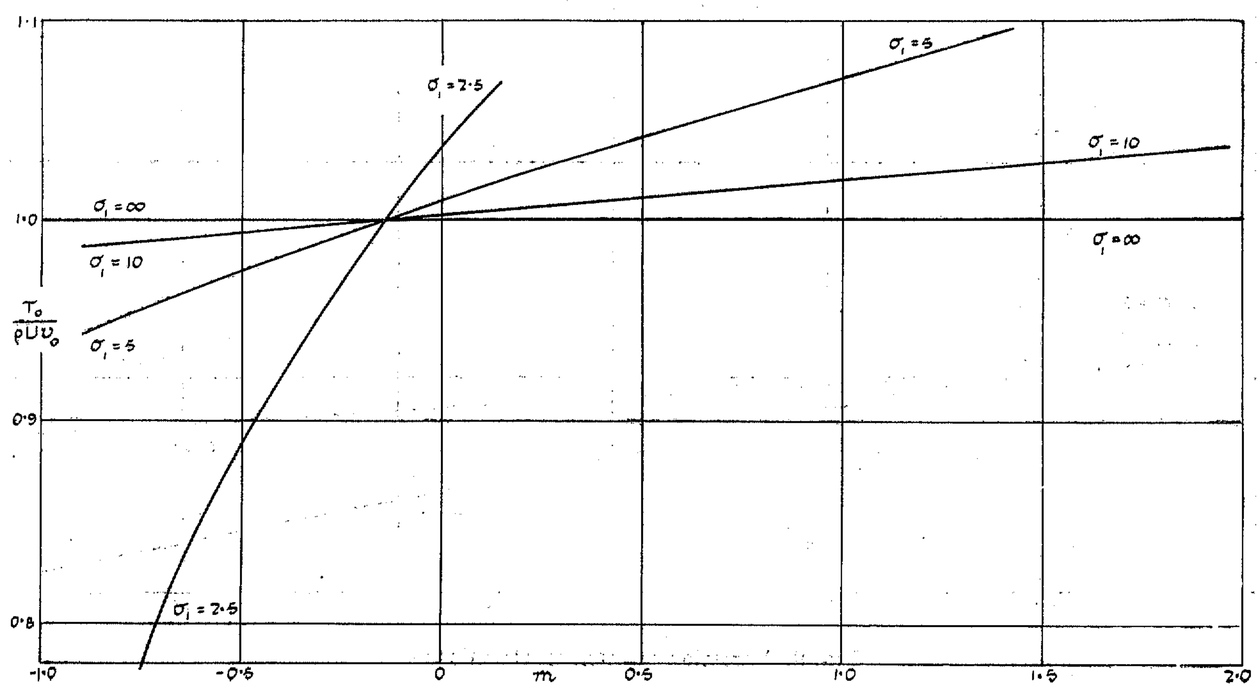

FIG. 6. Variation of skin friction.

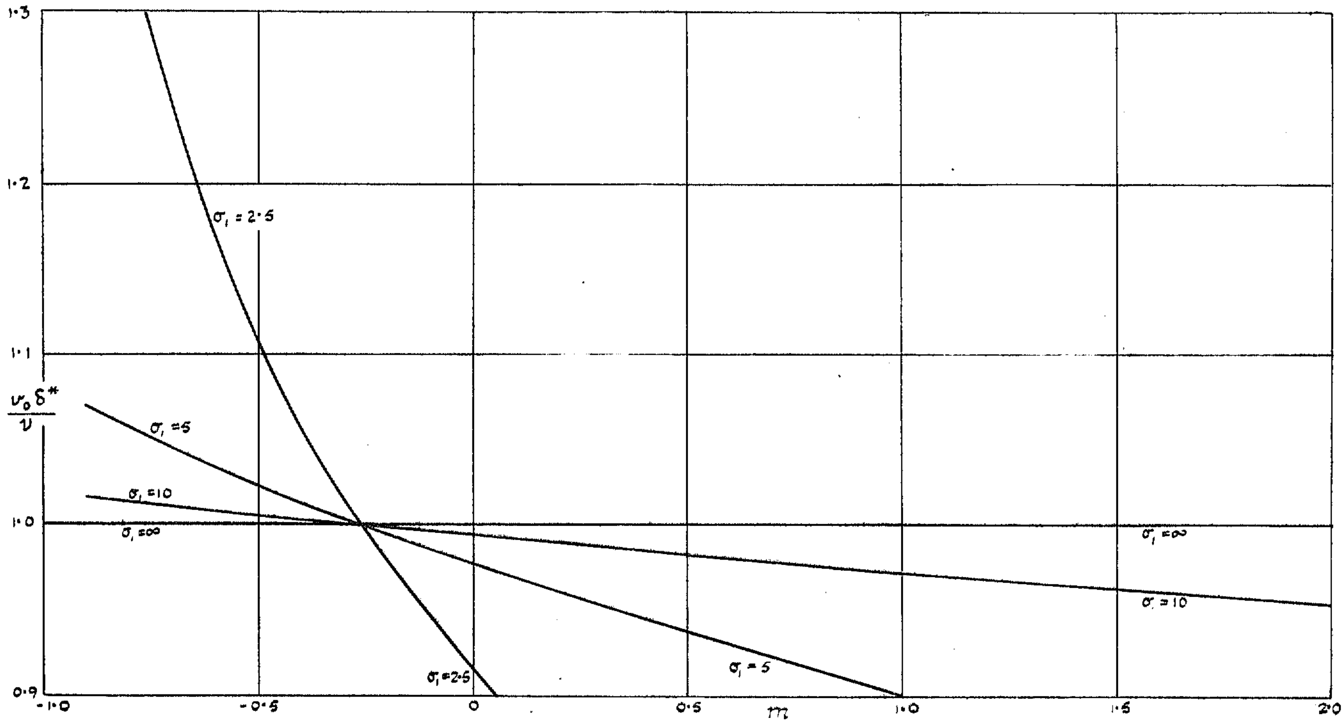

FIG. 7. Variation of displacement thickness.

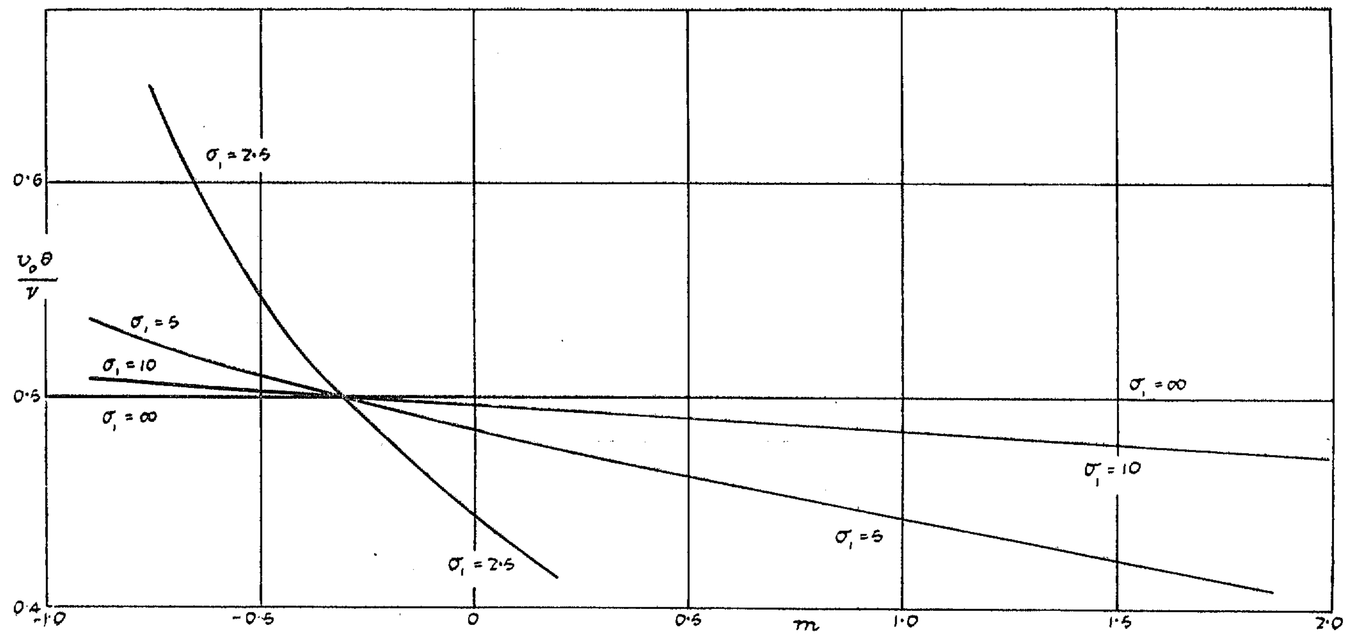

FIG. 8. Variation of momentum thickness.

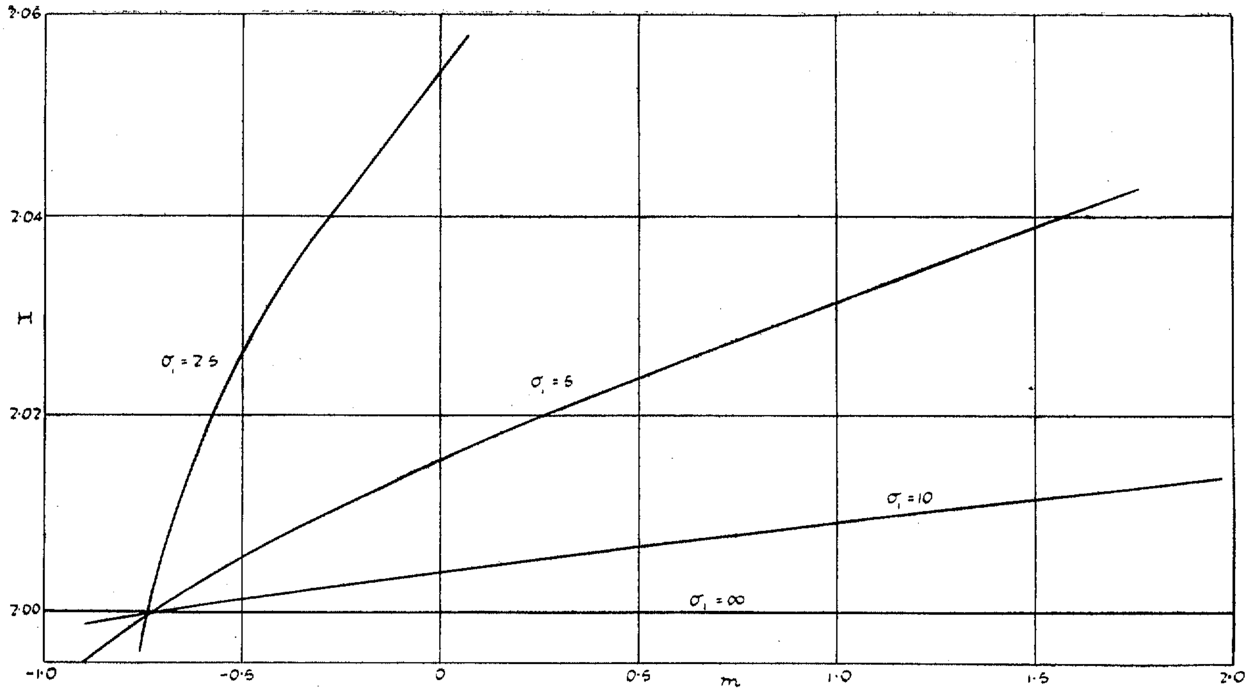

FIG. 9. Variation of H.

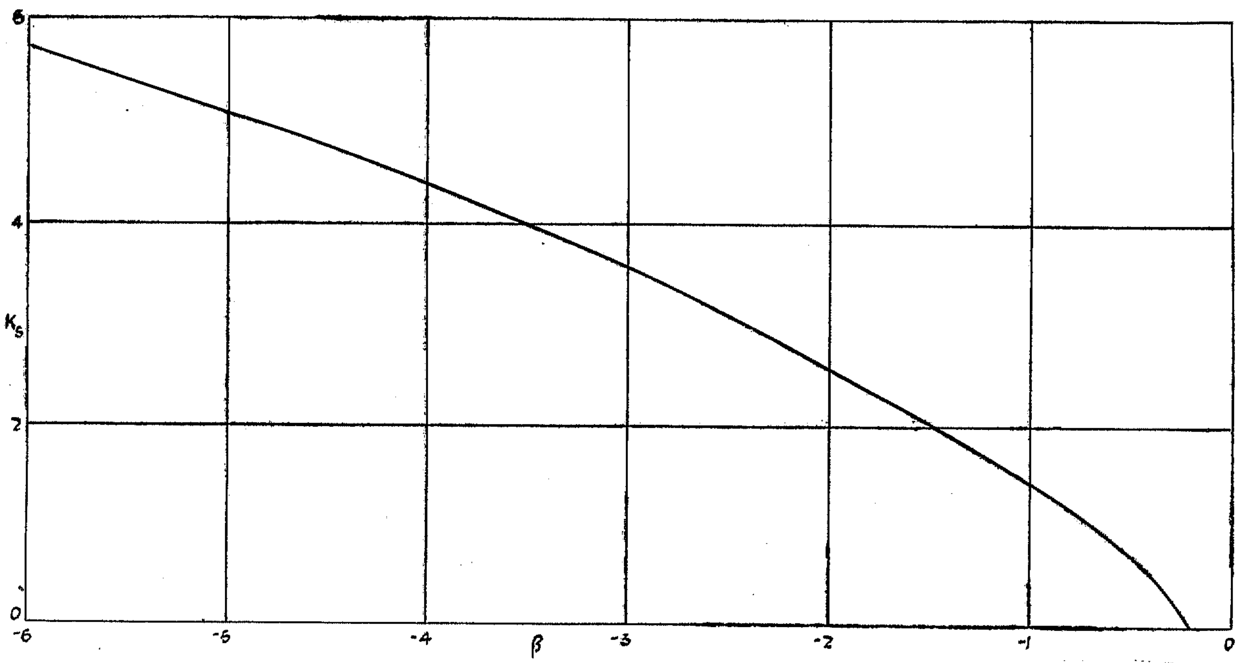

FIG. 10. Variation of the amount of suction for separation profiles.

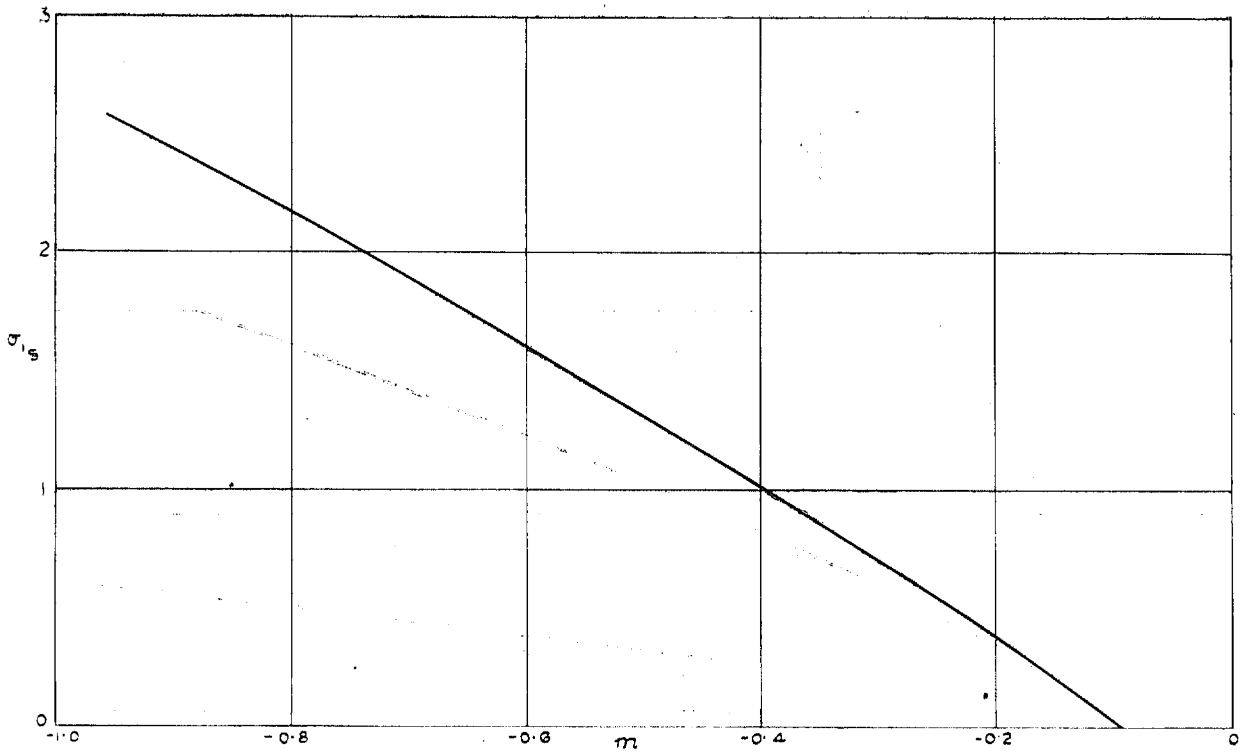

FIG. 11. Variation of the amount of suction for separation profiles.

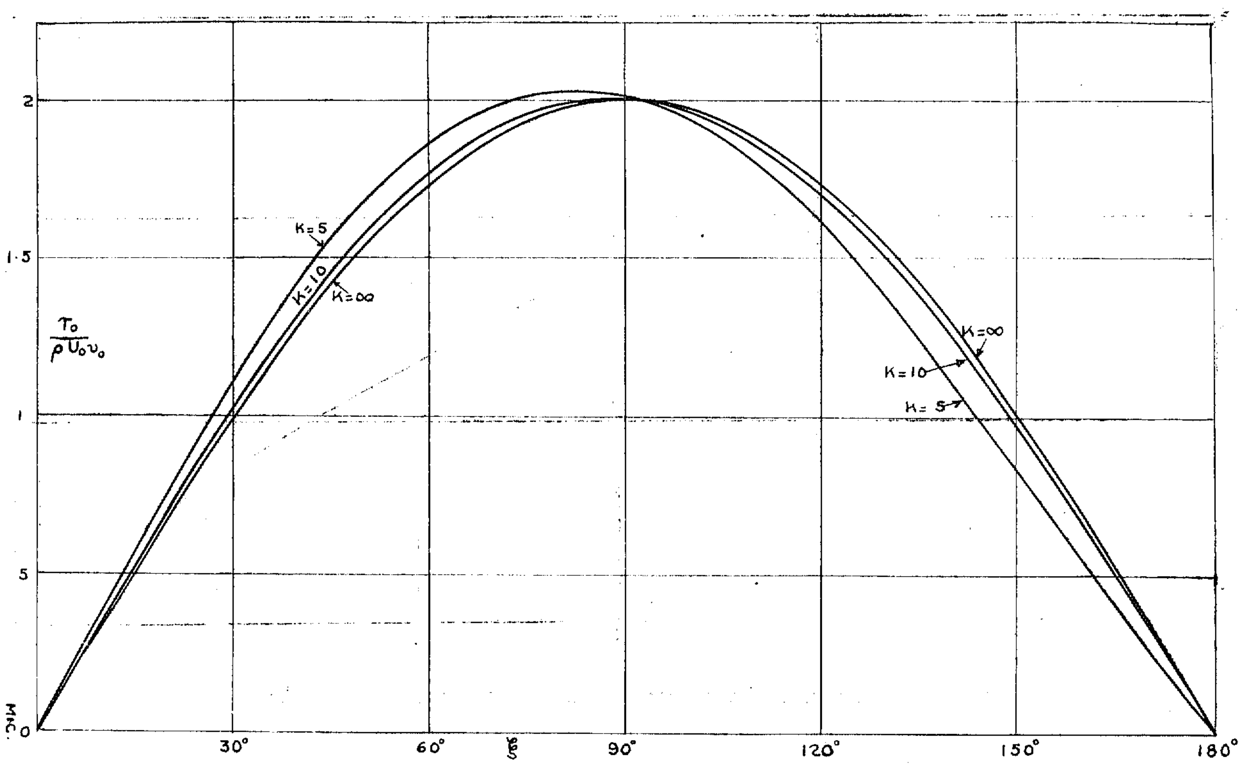

FIG. 12.

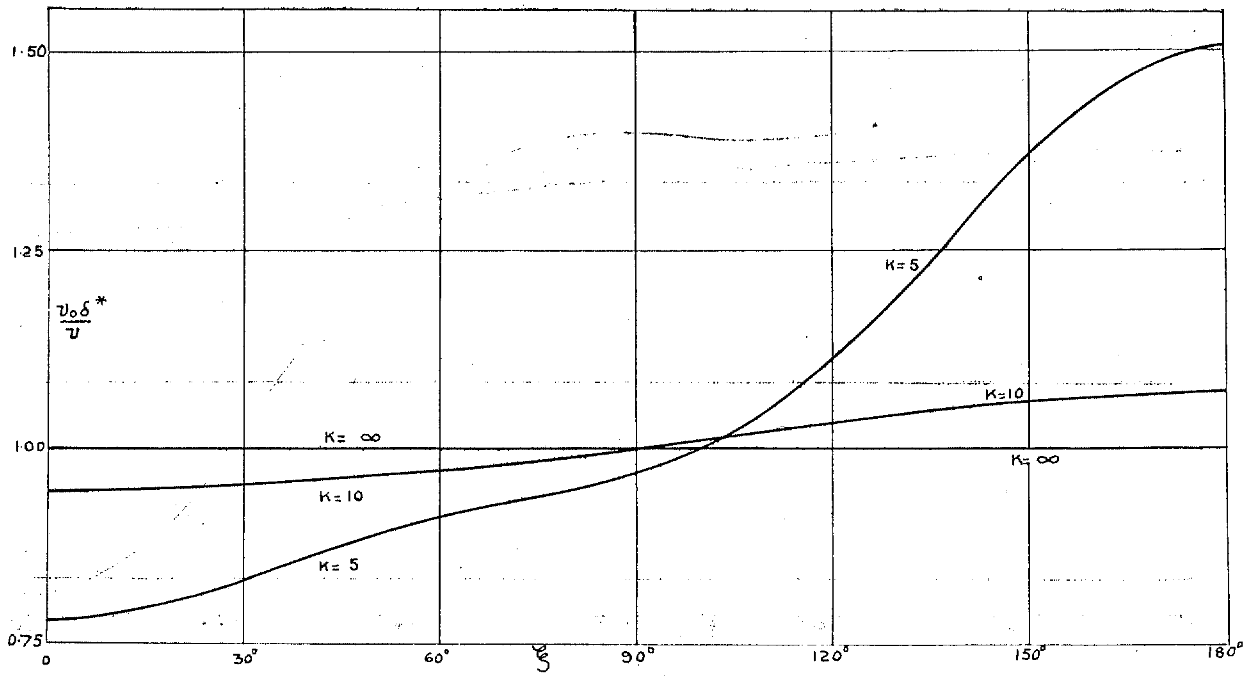
1·50
1·25
1·00
0·75
$\frac{v_0 \delta^*}{v}$
K= 5
K=10
K= ∞
K= 10
K= ∞
K= 5
0
30°
60°
ξ
90°
120°
150°
180°

FIG. 13.

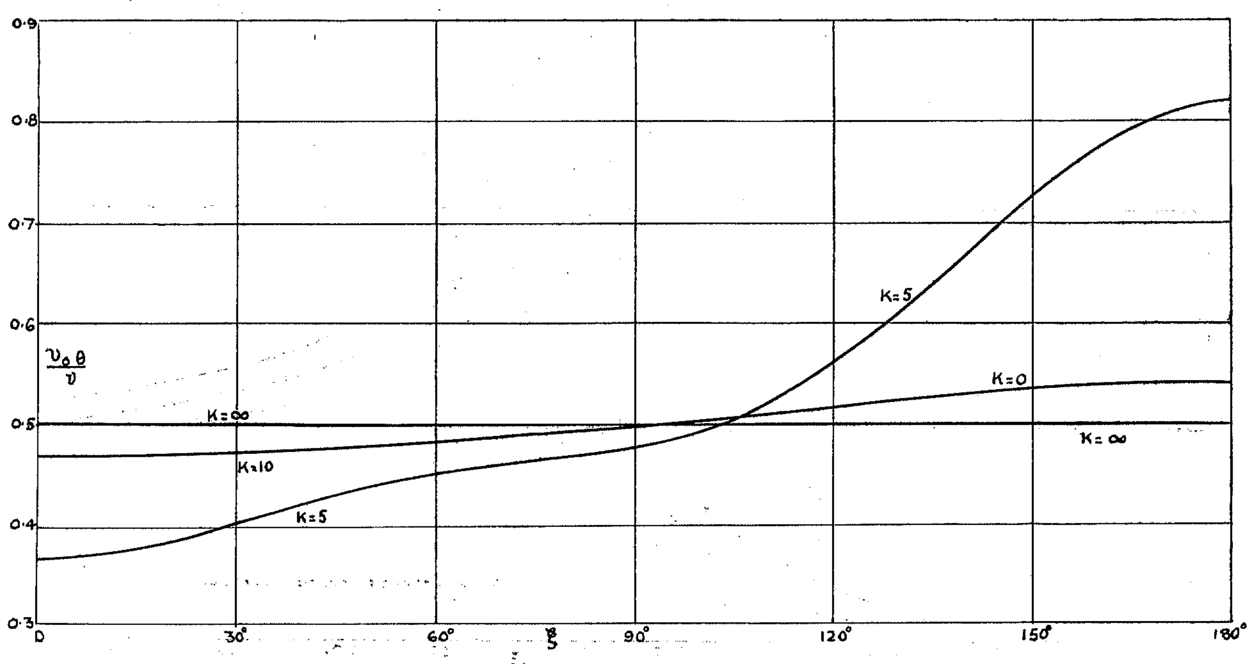
0·9
0·8
0·7
0·6
0·5
0·4
0·3
$\frac{v_0 \theta}{v}$
K=5
K=0
K=∞
K=10
K=5
K=∞
0
30°
60°
ξ
90°
120°
150°
180°

FIG. 14.

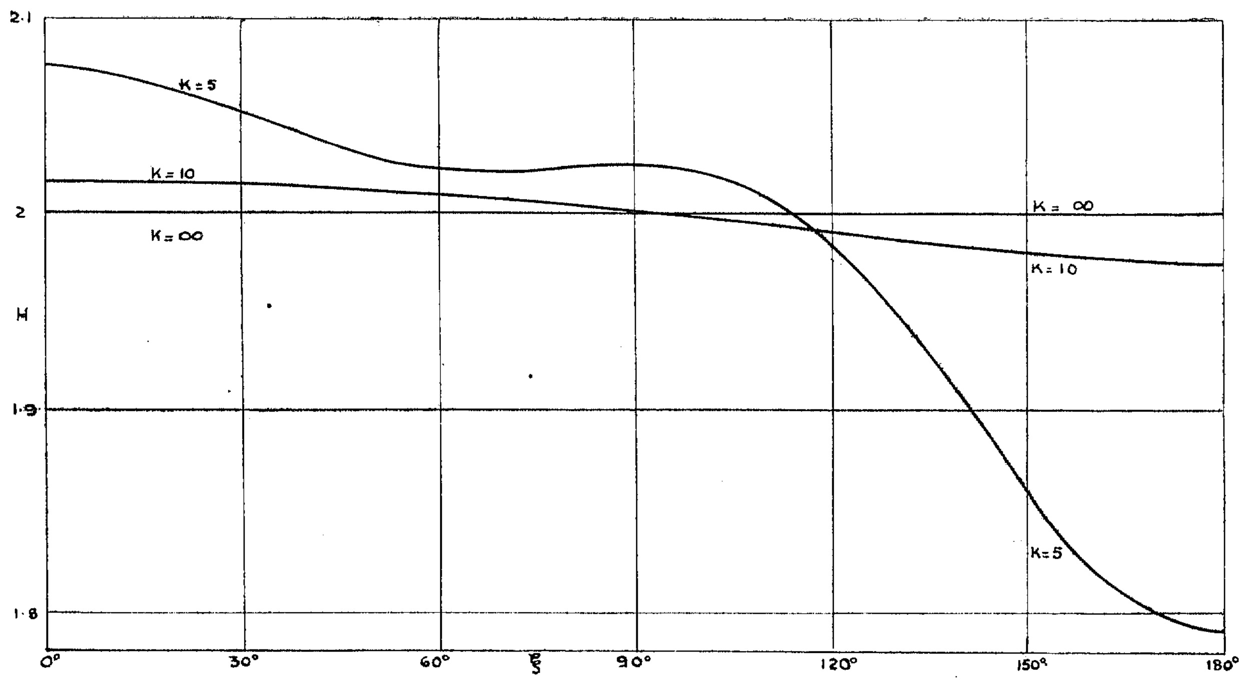

FIG. 15.

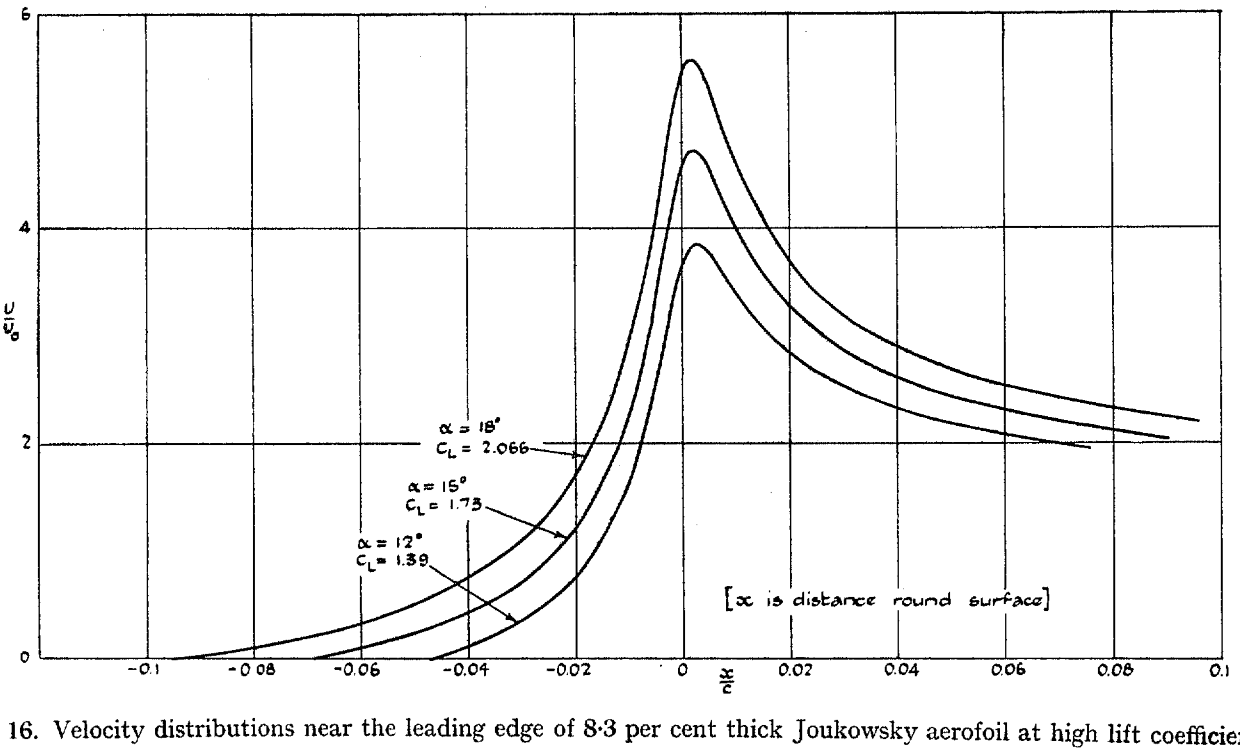

FIG. 16. Velocity distributions near the leading edge of 8·3 per cent thick Joukowsky aerofoil at high lift coefficients.

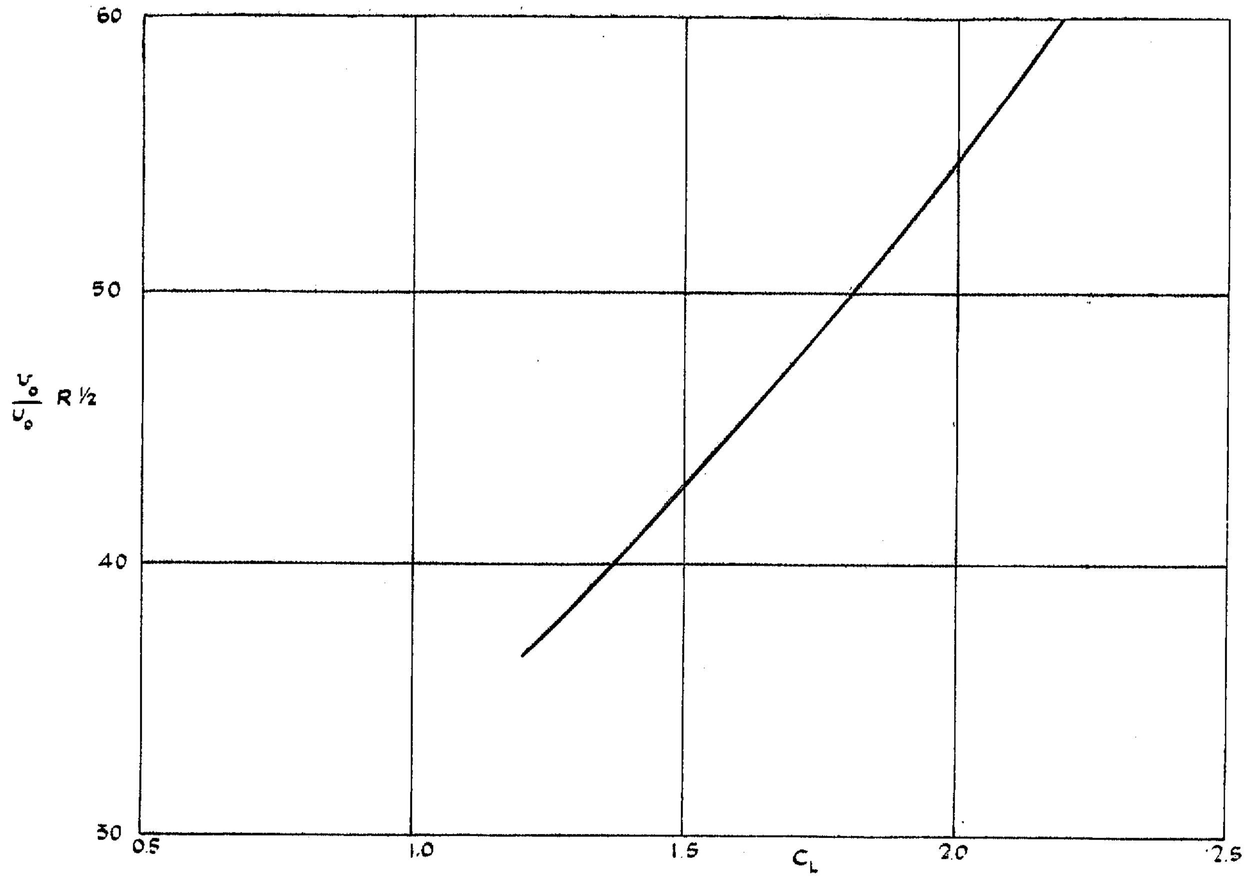

FIG. 17. Variation of suction velocity with lift coefficient.

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