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By

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Ill-conditioned flutter equations and their improvement for simulator use

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SUMMARY

If simple arbitrary modes of the form η^a are used as coordinates in a wing flutter calculation, the equations are ill-conditioned and cannot be solved satisfactorily on a simulator. This ill-conditioning can be avoided by transforming the flutter matrix so as to reduce the inertia couplings between like modes to zero. This transformation is described with numerical examples, and some observations are made on the general problem of choice of coordinates in a flutter calculation.

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1 Introduction

The calculation of flutter speeds is now commonly carried out on electronic simulators, of which the R.A.E. flutter simulator in six degrees of freedom is a notable example. The simulators have brought about a great reduction in the time required to solve the flutter equations, and this has led to the adoption of techniques of calculation which were previously impractical. The most obvious example of this change is that when the solution had to be obtained longhand (i.e. by use of desk calculating machines) it was essential to restrict the number of degrees of freedom to the minimum that could be used without introducing large errors. With a simulator this is no longer so important, and it is often more convenient, particularly in the design stage of an aircraft, to use a relatively large number of simple arbitrary modes in a calculation, rather than first to calculate the normal modes of the structure and then to use a smaller selection of these in the flutter calculations.

Unfortunately, the behaviour of the simulator depends on the type of modes chosen, and its behaviour is best when the modes are calculated normal modes. The reason for this is that the modes are then well separated from each other in frequency, and are completely uncoupled from each other at zero airspeed. This means that the equations are well conditioned and the numerical accuracy good; that there is no tendency for prolonged beating to occur; that there is no tendency for a disturbance in one degree of freedom to cause overloading in another (as happens when the coupling terms are large); and there is no tendency for the amplifiers to give an undamped response to a high frequency.

The way to avoid these difficulties with arbitrary modes is to choose modes which possess increasing numbers of nodes (one for the first overtone, two for the second and so on), which, being of the same general character as normal modes, likewise lead to well conditioned equations. Modes of this sort can be chosen by experience, but it is often more convenient to choose simple arbitrary modes and then to transform the flutter matrix just before scaling for the simulator. Details of a suitable transformation are given in the present paper with numerical examples. It is shown that although several figures must be retained in calculating the flutter matrix for the simple arbitrary modes, this does not represent any real hardship.

2 The use of simple arbitrary modes in flutter calculations

Simple arbitrary modes have often been used in flutter calculations. They will not generally give so accurate a solution as an equal number of normal modes, but in the project stage of an aircraft it is more expedient to calculate the wing flutter speed from six arbitrary modes, than first to work out the normal modes and then to use four or five of these in a flutter calculation. That it is more expedient is a result of having a flutter simulator available for the solution. Moreover, six simple arbitrary modes may be expected to give at least as good an answer as four or five calculated normal modes.

Calculations using six arbitrary modes have recently been made to investigate the wing flutter of aircraft which have already flown in the clean condition but which have now to be modified to carry heavy external stores on the wing, such as fuel tanks or bombs. In such cases as these, the ground resonance tests on the clean aircraft have been completed and it may seem simpler to use the normal modes of the clean aircraft for the new flutter calculations with allowance for the extra masses in the inertia coefficients. The modes would no longer be normal, but the elastic coefficients would be unchanged (the cross-stiffnesses would be zero and the direct stiffnesses obtained from the inertia coefficients and frequencies of the clean aircraft) so that one of the big advantages of using measured normal modes would be preserved. There are no objections in principle to this method of calculation, but experience shows that at least four wing modes would be needed to give an adequate prediction of the flutter speed and it is unlikely that so many modes will have been obtained from the ground resonance tests of the clean aircraft, especially for a fighter aircraft. Even if sufficient modes were available, the overtones would probably not be accurate enough.

The method of using simple arbitrary modes has the disadvantage that the elastic coefficients have to be calculated by integration of the strain energy, but some guidance is available to the distribution of GJ and EI from the stiffness test results. Perhaps the simplest set of modes to choose is the following, involving three bending modes and three torsion modes.

$$\left.\begin{array}{lll} (1) & f_1(\eta) = \eta^2 \\ (2) & f_2(\eta) = \eta^3 \\ (3) & f_3(\eta) = \eta^4 \end{array}\right\} \text{bending modes}$$

$$\left.\begin{array}{lll} (4) & f_4(\eta) = \eta \\ (5) & f_5(\eta) = \eta^2 \\ (6) & f_6(\eta) = \eta^3 \end{array}\right\} \text{torsion modes} \qquad (1)$$

where $f(\eta)$ represents the displacement at a fraction η of the semi-span, divided by the tip displacement, and for modes (1), (2) and (3) the displacement is linear and vertical, for modes (4), (5) and (6) the displacement is a rotation.

It is clear that these modes will lead to ill-conditioned equations, because modes 1, 2 and 3 are superficially very similar, and so are modes 4, 5 and 6. As an example, consider an inertia coefficient of the bending modes. This will have the form

$$A_{rs} = \int_0^1 m\, f_r\, f_s\, d\eta \qquad (2)$$

where m represents the mass distribution. If, for simplicity, m is assumed to be constant the inertia matrix for the first three modes is

$$A = m \begin{bmatrix} 1/5 & 1/6 & 1/7 \\ 1/6 & 1/7 & 1/8 \\ 1/7 & 1/8 & 1/9 \end{bmatrix} \qquad (3)$$

and the near equality of the coefficients indicates that trouble may be experienced because of the introduction of small differences. Another way of looking at it is that the inertia determinant is very small. When this happens the highest natural coupled frequency appropriate to the degrees of freedom concerned becomes very large. The relation for the natural frequency ω, is

$$-\omega^2 A q + Eq = 0 \qquad (4)$$

where E is the square matrix of elastic coefficients and q the column of generalised coordinates. Equation (4) can be written

$$A^{-1} E q = \omega^2 q \qquad (5)$$

and since the elements of the matrix A^{-1} are inversely proportional to the determinant of A, it follows that the elements of $A^{-1}E$ are large if (A) is small, and hence at least one solution for ω will be large. This result will be expected for the modes given above because the third coupled bending mode will contain two nodes, and will appear as a typical high frequency mode. If three bending modes are to be chosen they must inevitably contain a high frequency mode, either explicitly as a high frequency in one of the degrees of freedom or through large inertia coupling terms. The difference in representation between the explicit high frequency and the large inertia coupling is only important when the digital accuracy is restricted; but as low digital accuracy is inherent in electronic simulators it is essential in simulator calculations that consideration be given to the coordinates chosen. It is suggested in the present paper that simple arbitrary modes can be used with success in simulator flutter calculations, but the coordinates should be transformed before scaling the coefficients. The transformation is designed to improve the conditioning of the equations.

3 Transformation to improve conditioning

3.1 General rules for changing coordinates

The flutter equations can be derived, in Lagrangian form, from the expression

$$X = \lfloor \delta q \rfloor \, [u] \, \{q\} \tag{6}$$

where the brackets $\lfloor \; \rfloor$ denote a row matrix, and the brackets $\{ \; \}$ a column matrix, as in Ref.1: $[u]$ is the (complex) square matrix of flutter coefficients, and δq_r represents a small displacement of the coordinate q_r. The r^{th} equation is obtained by equating the coefficient of δq_r to zero in expression (6). If the matrix u is of order n, then there will be n simultaneous equations.

Suppose that the n modal functions appropriate to expression (6) were such that the required displacement, z, is given by

$$z = f_1 q_1 + \dots f_r q_r + \dots f_n q_n = f' q \tag{7}$$

where f and q are columns, and f' denotes the transposed of f.

We may wish to write the flutter equations in terms of a new set of functions $F_1 \dots F_N$ where these new functions are linear combinations of the original functions f. Thus

$$\begin{bmatrix} F_1 \\ \vdots \\ F_N \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \dots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{N1} & & \alpha_{Nn} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = \alpha f \tag{8}$$

It should be noted that N must be less than, or equal to, n.

Let the quantities Q_i be defined by

$$q = \alpha' Q \tag{9}$$

Then

$$z = f' q = f' \alpha' Q \tag{10}$$

i.e.

$$z = F' Q \tag{11}$$

and Q_1 are thus the new coordinates corresponding to F .

Finally

$$X = \delta q' u q = \delta Q' \alpha u \alpha' Q = \delta Q' U Q \qquad (12)$$

where

$$U = \alpha u \alpha' \qquad (13)$$

is the new matrix.

Equation (13) represents the standard transformation used, for example, when a flutter calculation of large order is reduced to a binary or ternary by gearing some of the modes together. In the present paper the transformation is used to improve the conditioning of the flutter determinant without, of course, reducing the order.

3.2 Partial orthogonalisation of the inertia matrix

Equation (13) can be used to reduce all the cross inertia coefficients to zero, when α takes the form of a triangular matrix, h say.

$$[\alpha] = \begin{bmatrix} 1 & 0 & \cdots & & 0 \\ h_{12} & 1 & & & \\ \vdots & & 1 & & \\ \vdots & & & 1 & 0 \\ h_{1n} & & & h_{n-1,n} & 1 \end{bmatrix} \qquad (14)$$

If we now carry out the transformation (13) on the inertia matrix [a] and equate the new cross inertias to zero we obtain the equations

$$\left.\begin{array}{l} a_{12} + h_{12}a_{11} = 0 \\ \left.\begin{array}{l} a_{13} + h_{13}a_{11} + h_{23}a_{12} = 0 \\ a_{23} + h_{13}a_{12} + h_{23}a_{22} = 0 \end{array}\right\} \end{array}\right\} \qquad (15)$$

etc.

which provides sufficient equations to solve for the elements of [h].

Solution of the equations (15) would be a laborious task for a large number of coordinates*, but fortunately on most current flutter simulators which provide up to six degrees of freedom it is unnecessary to use more than the first three equations. The reason for this is that the whole of the inertia matrix does not need to be treated. Suppose the modes used are those given in equation (1). The three torsion modes will be referred to the flexural axis, if one exists, which will be close to the locus of centres of gravity so that a_{rs} (r = 1,2,3; s = 4,5,6) will be small. If there is no

---

* The process of reducing the cross-inertias to zero is synonymous with that of reducing the inertia matrix or sub-matrix to canonical form, for which there are several methods available (see, for example, Ref.1). The examples considered here, however, are too simple for any formal process to be necessary.

flexural axis the reference axis should be chosen either as the c.g. axis, or as a fixed axis near to this, the half chord for example. Because of this it is only necessary to reduce to zero the cross inertias a_{rs} $(r,s = 1,2,3;\ r \neq s)$ and a_{rs} $(r,s = 4,5,6;\ r \neq s)$. The matrix [h] now becomes

$$h = \begin{bmatrix} 1 & 0 & & & & \\ h_{12} & 1 & 0 & & & \\ h_{13} & h_{23} & 1 & 0 & & \\ 0 & 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & h_{45} & 1 & 0 \\ 0 & 0 & 0 & h_{46} & h_{56} & 1 \end{bmatrix} \qquad (16)$$

and the equations for the elements of [h] are the first three equations of equation (15) together with a corresponding set for the coefficients h_{45}, h_{46} and h_{56}.

The effect of the matrix (16) used in the transformation is to transform the modal functions f into functions F, where

$$\left.\begin{aligned} F_1 &= f_1 \\ F_2 &= h_{12}f_1 + f_2 \\ F_3 &= h_{13}f_1 + h_{23}f_2 + f_3 \end{aligned}\right\} \qquad (17)$$

and it is to be expected that if the original functions f are those given in equation (1), then F_2 will give one node within the wing span and F_3 will give two nodes within the wing span.

3.3 Numerical examples of the transformation

A typical inertia matrix for modes of the type given in equation (1) is given below in expression (18). The wing is unswept and carries tip fuel tanks full of fuel, and with forward c.g. which explains why the cross inertias a_{rs} $(r = 1,2,3;\ s = 4,5,6)$ are relatively large.

$$a = \begin{bmatrix} 6.75807 & 6.41834 & 6.12402 & -1.53257 & -1.47300 & -1.41490 \\ 6.41834 & 6.12402 & 5.85636 & -1.47300 & -1.41490 & -1.35873 \\ 6.12402 & 5.85636 & 5.60715 & -1.41490 & -1.35873 & -1.30461 \\ -1.53257 & -1.47300 & -1.41490 & 0.736961 & 0.704833 & 0.675363 \\ -1.47300 & -1.41490 & -1.35873 & 0.704833 & 0.675363 & 0.647635 \\ -1.41490 & -1.35873 & -1.30461 & 0.675363 & 0.647635 & 0.621292 \end{bmatrix} \qquad (18)$$

The matrix is symmetric (in the further examples only the upper triangle will be printed) and in addition $a_{22} = a_{13}$, by equations (1) and (2). The ill-conditioning between modes 1, 2 and 3, and between modes 4, 5 and 6 is quite obvious and is of the same order as that shown by equation (3), but is in fact rather worse because of the taper.

It is to be expected that the aerodynamic coefficients will be ill-conditioned in the same way as the inertia coefficients, because they also depend upon the displacements of the modes. The matrices are given, for interest, in equations (19) and (20), where the aerodynamic dampings are denoted by b and the stiffnesses by c.

$$
b = \left[\begin{array}{cccccc}
0.296018 & 0.238036 & 0.197535 & -0.0642220 & -0.0505741 & -0.0411556 \\
0.238036 & 0.197535 & 0.167632 & -0.0505741 & -0.0411556 & -0.0343055 \\
0.197535 & 0.167632 & 0.144619 & -0.0411556 & -0.0343055 & -0.0291243 \\
-0.028870 & -0.022198 & -0.0177957 & 0.139419 & 0.0855822 & 0.0583567 \\
-0.0221980 & -0.0177957 & -0.0146942 & 0.0855822 & 0.0583567 & 0.0426254 \\
-0.0177957 & -0.0146942 & -0.0124009 & 0.0583567 & 0.0436254 & 0.0326867
\end{array}\right] \qquad (19)
$$

$$
c = \left[\begin{array}{cccccc}
0.0944859 & 0.0755716 & 0.0624528 & 0.381631 & 0.290883 & 0.232744 \\
0.0755716 & 0.0624528 & 0.0528250 & 0.290883 & 0.232744 & 0.192427 \\
0.0624528 & 0.0528250 & 0.0454534 & 0.232744 & 0.192427 & 0.162838 \\
-0.0241943 & -0.0173972 & -0.0132898 & -0.0382347 & -0.0276215 & -0.0212347 \\
-0.0173972 & -0.0132898 & -0.0105861 & -0.0276215 & -0.0212347 & -0.0170210 \\
-0.0132898 & -0.0105861 & -0.00869208 & -0.0212347 & -0.0170210 & -0.0140554
\end{array}\right] \qquad (20)
$$

The matrices b and c have been partitioned to illustrate the fact that although the matrices as a whole are not symmetric, each partitioned segment is symmetric. The conditioning is not as bad as in the inertia matrix because the presence of any large concentrated mass tends to worsen the conditioning; a point mass always has a zero inertia determinant. The elastic coefficients, e, depend on the curvature of the bending modes and on the slope of the torsion modes and again are not as badly conditioned as the inertia coefficients. The flexural axis was used as a reference axis, so that the elastic couplings between the bending and torsion modes are absent; the matrix is symmetric.

$$
e = \left[\begin{array}{cccccc}
0.294954 & 0.229401 & 0.211839 & 0 & 0 & 0 \\
 & 0.317759 & 0.382835 & 0 & 0 & 0 \\
 & & 0.518615 & 0 & 0 & 0 \\
 & & & 0.242633 & 0.162189 & 0.119759 \\
 & & & & 0.159679 & 0.144218 \\
 & & & & & 0.146743
\end{array}\right] \qquad (21)
$$

If the coefficients are scaled for the simulator as they stand, the results could mean very little even if the simulator responded in a satisfactory manner, because only at most three significant figures could be retained. Apart from this, however, if each degree of freedom is scaled to a frequency appropriate to its own direct inertia and stiffness coefficients (which is the normal practice) those frequencies are, in non-dimensional form,

$$
0.209, \quad 0.228, \quad 0.304, \quad 0.575, \quad 0.487, \quad 0.486 \qquad (22)
$$

whereas the natural coupled frequencies, at which the simulator will respond when all the degrees of freedom are being used, cover a much wider range. The result is that unstable oscillations occur at the highest natural frequency and the simulator is unusable.

The transformation matrix, h, is of the form (16) and is found very quickly, requiring the solution of only two pairs of simultaneous equations.

$$
h = \left[\begin{array}{cccccc}
1 & & & & & \\
-0.9497 & 1 & & & & \\
0.4413 & -1.4188 & 1 & & & \\
 & & & 1 & & \\
 & & & -0.9564 & 1 & \\
 & & & 0.3878 & -1.364 & 1
\end{array}\right] \qquad (23)
$$

The transformed inertia matrix, a_t, is now given in equation (24); it is still symmetric, but the top right hand partition is no longer symmetric.

$$a_t = \begin{bmatrix} 6.75807 & 0 & 0 & -1.53257 & -0.00724277 & -0.000551385 \\ & 0.0283317 & 0 & -0.0174727 & 0.000762873 & 0.0000145729 \\ & & 0.000663624 & -0.00132775 & -0.0000320136 & 0.0000249775 \\ & & & 0.736961 & 0 & 0 \\ & & & & 0.00125737 & 0 \\ & & & & & 0.0000402576 \end{bmatrix} \quad (24)$$

The transformation acts in a similar way on the aerodynamic coefficients, except that none of them is reduced to zero. For example, the transformed damping matrix for degrees of freedom 1, 2 and 3 becomes

$$b_t = \begin{bmatrix} 0.296018 & -0.0431011 & -0.00955971 \\ & 0.0123995 & 0.00149236 \\ & & 0.000499346 \end{bmatrix} \quad (25)$$

and is still symmetric, although the symmetry of cross partitioned segments is destroyed and, of course, the whole matrix is unsymmetric.

The elastic matrix is not affected so much as the others, and becomes (still symmetric)

$$e_t = \begin{bmatrix} 0.294954 & -0.0507256 & 0.0165259 & 0 & 0 & 0 \\ & 0.148066 & 0.0175406 & 0 & 0 & 0 \\ & & 0.0290768 & 0 & 0 & 0 \\ & & & 0.242633 & -0.0698664 & -0.00732030 \\ & & & & 0.0713812 & -0.00363263 \\ & & & & & 0.00818287 \end{bmatrix} \quad (26)$$

and it can be seen that the reduction in the coefficients of the third and sixth degrees of freedom is much less than in the inertia coefficients. This has the effect of spreading out the frequencies of the individual degrees of freedom, so that the new values (comparable with (22)) are

$$0.209,\ 2.28,\ 6.63,\ 0.575,\ 7.53,\ 14.2 \quad (27)$$

The frequency range here is enormous compared with that of the unmodified coefficients. There are no longer any high frequencies masked by ill-conditioning, and the coefficients can be scaled directly for the R.A.E. flutter simulator, with appropriate choice of time constants. In fact this was done and the solution on the simulator presented no difficulty at all. In all, five different mass conditions were covered in the problem, but only one complete rescaling was made; the same transformation was used for the first three cases, and a new transformation for the other two.

It is of interest to determine the new modal shapes appropriate to the transformed coefficients. These are given by the expressions for F_1 etc. in (17) in terms of f_1 etc. which are the functions given in (1). The new modes are simple combinations of the old, and can best be described in terms of the nodal points, viz:

mode 1 is unchanged

mode 2 has a node at $\eta = 0.95$ span

mode 3 has nodes at $\eta = 0.46$ and 0.96

mode 4 is unchanged

mode 5 has a node at $\eta = 0.96$ span

mode 6 has nodes at $\eta = 0.40$ and 0.96 span.

The modes are also compared in Figs.1 to 4.

The effect of the transformation is now apparent: it has yielded an approximation to the first three uncoupled normal modes in bending and torsion. This illustrates the reason for the wide range of frequencies given in (27); in particular the large gap between the fundamental frequencies of bending and torsion and their respective overtones is caused by the dominating effect of the heavy tip mass. The approximation to the uncoupled normal modes is, of course, quite crude since only the inertia matrix is orthogonal in this respect, but what matters is that the desired effect has been achieved.

In the second example the original choice of modes was quite different and the need for the transformation less certain. In this case the example is for illustrative purposes only and is not taken from an actual aircraft. The wing is divided into three equal spanwise lengths and each length allowed two degree of freedom (see Ref.2 for example): parabolic bending and linear torsion. This choice of arbitrary modes is particularly valuable when a parameter has to be varied which only affects part of the wing span. If, for example, there was reason to suppose that the most economical way of increasing the flutter speed would be to stiffen a particular section of the wing, then that section would be given its own degrees of freedom and the variations in stiffness would only affect one binary in the whole matrix.

The modes are

$$\left.\begin{aligned} f_1 &= \eta_1^2 \quad \text{(over inner wing)} \\ f_2 &= \eta_2^2 \quad \text{(over middle wing)} \\ f_3 &= \eta_3^2 \quad \text{(over outer wing)} \end{aligned}\right\} \text{Bending modes}$$

$$\left.\begin{aligned} f_4 &= \eta_1 \quad \text{(over inner wing)} \\ f_5 &= \eta_2 \quad \text{(over middle wing)} \\ f_6 &= \eta_3 \quad \text{(over outer wing)} \end{aligned}\right\} \text{torsion modes} \qquad (28)$$

so that over the outer wing, for example, the deflection is

$$z = \ell\left\{(3q_1 + q_2) + 2(q_1 + q_2)\,\eta_3 + \eta_3^2 q_3\right\} \qquad (29)$$

and

$$\theta = q_4 + q_5 + \eta_3 q_6 \qquad (30)$$

where ℓ is the length of a wing section and η_3 is the distance along the outer section divided by ℓ. With these coordinates the inertia matrix is

$$a = \begin{bmatrix} 53.4051 & 21.5333 & 2.85 & -0.488 & -1 & -0.5 \\ & 10.3 & 1.51667 & -0.5 & -0.5 & -0.25 \\ & & 0.395833 & -0.0625 & -0.0625 & -0.03125 \\ & & & 0.6010196 & 0.226567 & 0.0792725 \\ & & & & 0.202087 & 0.0792725 \\ & & & & & 0.0411377 \end{bmatrix} \qquad (31)$$

where those numbers which are given to less than six significant figures are exact, as a result of the simplicity of the assumed wing properties.

The elastic matrix is diagonal

$$e = \begin{bmatrix} 53.4051 & & & & & \\ & 20.6 & & & & \\ & & 1.18333 & & & \\ & & & 1.202039 & & \\ & & & & 0.808348 & \\ & & & & & 0.329102 \end{bmatrix} \qquad (32)$$

In fact the elastic coefficients were chosen to be in a simple ratio to the direct inertia coefficients. The frequencies appropriate to the six individual degrees of freedom are, therefore

$$1.0, \ \sqrt{2}, \ 2.0, \ \sqrt{2}, \ 2.0, \ 2\sqrt{2} \qquad (33)$$

To provide a check of the behaviour of the flutter simulator with the coefficients of equations (31) and (32), they were scaled as they stand and combined with 1% critical damping in the direct terms and zero aerodynamic coefficients. The simulator was not violently unstable at a basic frequency of 10 rad/sec as it had been in the previous example, but it did not settle down well and the equilibrium condition was easily disturbed. As there were no aerodynamic coefficients available the flutter speed could not be determined and instead it was decided to try to evaluate the six natural frequencies. On the present R.A.E. flutter simulator, this is a more severe test of the conditioning of a problem since it is usual to work at a mean frequency of 100 rad/sec when the amplifiers overload much more easily. In this case to prevent overloading the structural damping had to be increased to such a degree that the conditions of resonance were quite unobtainable, so that no solution to the problem was possible.

The transformation (13) was again used to see if the conditioning would again be improved to a satisfactory level. It was possible, in this case, that the improvement in the inertia matrix would be balanced by a worsening of the elastic matrix. As the determinant of the elastic coefficients is reduced to zero the effect is not to cause violent overloading at a high frequency, but to cause static instability. This also renders the simulator unusable, but it can be used at smaller values of the stiffness determinant than of the inertia determinant so that the effect is not quite so serious. The transformation is

$$h = \begin{bmatrix} 1 & & & & & \\ -0.403208 & 1 & & & & \\ 0.0382451 & -0.227205 & 1 & & & \\ & & & 1 & & \\ & & & -0.376971 & 1 & \\ & & & 0.0276724 & -0.423293 & 1 \end{bmatrix} \qquad (34)$$

giving

$$a_t = \begin{bmatrix} 53.4051 & 0 & 0 & -0.488 & -0.816038 & -0.0902111 \\ & 1.61759 & 0 & -0.303235 & 0.0175185 & -0.0158159 \\ & & 0.0602372 & 0.0324389 & 0.000628897 & 0.00188393 \\ & & & 0.6010196 & 0 & 0 \\ & & & & 0.116678 & 0 \\ & & & & & 0.00977587 \end{bmatrix} \qquad (35)$$

$$e_t = \begin{bmatrix} 53.4051 & -21.5333 & 2.04248 & 0 & 0 & 0 \\ & 29.2824 & -5.50397 & 0 & 0 & 0 \\ & & 2.32486 & 0 & 0 & 0 \\ & & & 1.20204 & -0.453134 & 0.0332633 \\ & & & & 0.979168 & -0.354708 \\ & & & & & 0.474860 \end{bmatrix} \qquad (36)$$

The frequencies of the individual degrees of freedom now have the values

$$1.0, \quad 4.35, \quad 6.21, \quad 1.41, \quad 2.90, \quad 6.97 \qquad (37)$$

which again show a wider spread than those of (33). In the form of (31) and (32) the coefficients were scaled for the simulator which was then used at a mean frequency of 100 rad/sec to find the natural frequencies with no trouble. The results were

$$0.882, \quad 4.25, \quad 6.87, \quad 1.31, \quad 2.65, \quad 7.26 \qquad (38)$$

where the first three numbers represent the frequencies in ascending order of the predominantly bending modes, and the last three of the predominantly torsional modes, the arrangement being for ease of comparison with (37). The modes obtained by the transformation are shown in Figs.6 and 8 for comparison with the original modes shown in Figs.5 and 7. It will be seen that the respective overtone modes contain the same number of nodes as the corresponding normal modes.

At the beginning of this example it was remarked that the type of arbitrary modes used here had particular merit where the effect of local stiffening was to be investigated. Much of the advantage of this will be lost if the transformation is such that all the new coefficients depend on the local stiffness concerned. It follows that the variable parameter should be contained in the (3,6) binary terms if the transformation is to be used in the form given by equation (16). Thus the arrangement used in the example would be suitable for investigating changes in a parameter confined to the tip section; if the variable were in the middle section then the original matrix would be rearranged to make the degrees of freedom 3 and 6 contain the distortions of this section.

3.4 Accuracy of the basic coefficients

It will have been noticed that the examples of section 3.3 are worked to six significant figures, and the reason is simply that figures are lost very rapidly in the transformation, as always when dealing with ill-conditioned equations. In fact seven significant figures should be kept throughout the work where simple arbitrary modes are used. It may seem that this high degree of accuracy is pointless since the physical data from which the coefficients are derived are never known to more than two or three significant figures. The reason it is not pointless is that all the coefficients are functions of the same physical data and a small change in the physical data would produce small changes in all the coefficients. Moreover when the conditioning is poor the functions are all similar to each other and hence all the small changes are similar. Conversely a small isolated change in one coefficient only would represent a very great physical change and in many cases would turn the problem into an unreal one. The physical properties are shown in the fine balance of the coefficients and a small change in the physical properties does not upset that balance, although it may give rise to a small change in the overall level of the coefficients.

The coefficients must therefore be worked out very accurately, and also the physical data must be used in exactly the same way in each coefficient. Thus approximations to the physical data are permissible, but numerical approximations in evaluating the coefficients are not. It is satisfactory, for example, to approximate to the true mass distribution by a mathematical function in terms of the span, so that the integrations can be carried out analytically, but it is not satisfactory to use the true mass distribution and evaluate the integrals graphically or by any method that might introduce random errors. The standard engineering method of evaluating the integrals by dividing the wing into say ten strips parallel to the line of flight, expressing the function to be integrated by its values at the middle of the strips, and then summing these values multiplied by the widths of the strips, is again satisfactory since the true mass distribution (or wing chord etc) is represented by a series of rectangular steps for which an exact summation is carried out. The high degree of numerical accuracy must be retained until after the transformation has been used.

4 The use of more complicated arbitrary modes

An alternative to using the transformation is to try to choose modes in the first place which will not lead to ill-conditioned equations, as Minhinnick recommends[3] in the choice of modes for normal mode calculations. In practice this is not always easy to do on an aircraft wing which may be carrying large concentrated masses, and cases have occurred where it has been attempted only to find that the final coefficients were still too ill-conditioned for satisfactory solution on the flutter simulator. But even if the choice is satisfactory there is no saving in numerical work, which is in fact usually longer. As an example, consider the three torsion modes of equations (1) applied to a uniform beam

i.e.
$$\theta = \eta q_1 + \eta^2 q_2 + \eta^3 q_3 \tag{39}$$

The kinetic energy, T, is given by

$$2T = \int_0^1 I\dot{\theta}^2 \, d\eta = \int_0^1 I(\eta\dot{q}_1 + \eta^2\dot{q}_2 + \eta^3\dot{q}_3)^2 d\eta \tag{40}$$

where I is the torsional inertia per unit η . The inertia matrix, by inspection, is (in non-dimensional form)

$$a = \begin{bmatrix} 1/3 & 1/4 & 1/5 \\ & 1/5 & 1/6 \\ & & 1/7 \end{bmatrix} \tag{41}$$

and similarly the elastic matrix is

$$e = \begin{bmatrix} 1 & 1 & 1 \\ & 4/3 & 6/4 \\ & & 9/5 \end{bmatrix} \tag{42}$$

The transformation is

$$h = \begin{bmatrix} 1 & & \\ -3/4 & 1 & \\ 2/5 & -4/3 & 1 \end{bmatrix} \tag{43}$$

and the transformed matrices are

$$a_t = \begin{bmatrix} 1/3 & 0 & 0 \\ & 1/80 & 0 \\ & & 1/1575 \end{bmatrix} \quad e_t = \begin{bmatrix} 1 & 1/4 & 1/15 \\ & 19/48 & 13/180 \\ & & 43/675 \end{bmatrix} \tag{44}$$

The transformed modes are

$$\left.\begin{aligned} F_1 &= f_1 = \eta \\ F_2 &= f_2 - \tfrac{3}{4}f_1 = \eta^2 - \tfrac{3}{4}\eta \text{ (node at 0.75)} \\ F_3 &= f_3 - {}^4/_3\, f_2 + {}^2/_5 f_1 = \eta^3 - {}^4/_3\eta^2 + {}^2/_5\eta \text{ (nodes at 0.45 and 0.89)} \end{aligned}\right\} \tag{45}$$

In the alternative method modes are chosen which have the correct number of nodes in roughly the correct positions, e.g.

$$\left.\begin{aligned} &\text{mode 1} \quad \theta = \eta q_1 \\ &\text{mode 2} \quad \theta = (\eta^2 - 0.7\eta)q_2 \text{ (node at 0.7)} \\ &\text{mode 3} \quad \theta = (\eta^3 - 1.3\eta^2 + 0.4\eta)q_3 \text{ (nodes at 0.5 and 0.8)} \end{aligned}\right\} \tag{46}$$

In this case the expressions for the inertia coefficients are more complicated and have to be evaluated very carefully because they result in small differences of large quantities. For example

$$a_{23} = \int_0^1 (\eta^2 - 0.7\eta)\,(\eta^3 - 1.3\eta^2 + 0.4\eta)\,d\eta \tag{47}$$

$$= {}^1/_6 - {}^2/_5 + \frac{1.31}{4} - \frac{0.28}{3} = 0.00093\dot{3} \tag{48}$$

where it can be seen that three significant figures are lost in the summation of equation (48). In fact it appears that if polynomial functions are used at all then small differences cannot be avoided in the calculation and the integrals must all be evaluated with great numerical accuracy. For comparison with (44) the inertia and elastic matrices are

$$a = \begin{bmatrix} 0.3\dot{3} & 0.016\dot{6} & 0.0083\dot{3} \\ & 0.013\dot{3} & 0.000933 \\ & & 0.0008571 \end{bmatrix} \quad e = \begin{bmatrix} 1.0 & 0.3 & 0.1 \\ & 0.4233 & 0.096\dot{6} \\ & & 0.073\dot{3} \end{bmatrix} \tag{49}$$

and it can be seen that they are similar. With a more complicated structure, however, it is more difficult to choose modes as well conditioned as those given in (46).

One refinement suggested by Minhinnick and not so far considered here is to use Duncan functions rather than simple algebraic modes (see Ref.3). The Duncan functions are polynomials arranged to satisfy the conditions of zero moments at the wing tip. This refinement is not, however, thought to be of great value in flutter work when as many as six modes are chosen. Collar has shown[4] that in semi-rigid work a failure to satisfy the tip condition is not important.

5 General remarks on choice of coordinates

The availability of flutter simulators of up to six degrees of freedom has considerably widened the choice of coordinates in flutter problems. Without a simulator, or other suitable electronic computor, the overriding factor is that the required accuracy of representation should be achieved with as few coordinates as possible. With a simulator the important factors are

(i) the time taken to evaluate the flutter coefficients

(ii) The time taken to change the coefficients in accordance with the variation that may be desired in any parameters.

(iii) The suitability of the coordinates for solution on a simulator.

(iv) The requirement not to exceed the number of equations which the simulator can solve.

The time taken in (i) will not always be shortest for the scheme which has the fewest coordinates; the form of the modal functions is also important. The time taken in (ii) can vary very greatly depending on whether much recalculation is necessary when the variable is changed. As regards (iii) the coefficients put on the simulator should be such as to avoid serious overloading both at high and low frequencies if possible; a transformation as suggested in the present paper may be used. Finally, the process of simplifying the assumed modes should not be carried so far that the capacity of the simulator becomes inadequate for satisfactory representation of the problem.

As a brief illustration of the rapidity with which the calculated flutter speed converges with increasing numbers of degrees of freedom, a simple flutter calculation has been carried out on a rectangular wing using strip theory and constant derivatives. The calculation was divided into two parts, of which the first was a binary using what might be termed standard arbitrary modes of parabolic bending and linear torsion about the flexural axis, which was assumed to be at the half chord and coincident with the inertia axis. In the second part of the calculation the modes chosen were parabolic bending and parabolic and cubic torsion about the leading edge. The intention was to compare the result of the standard calculation with that of a calculation in which the torsion mode was replaced by two admittedly unrealistic torsion modes. It could be, for example, that in a particular case the assumption of a linear torsion mode might lead to complications either aerodynamically or structurally, so that the two modes with zero rate of twist at the root would be more convenient to use than the single linear mode. The modes were taken as twisting about the leading edge so as to remove still further their association with the structure (again the leading edge could be a more convenient reference axis in some circumstances).

In the limiting condition as the bending stiffness tends to zero, the exact solution is available for this wing[2], and can be compared with the solutions obtained on the semi-rigid theory for the different modes. This comparison is made in Table I below, where

s is the distance from root to tip

c is the chord

V is the flutter speed

ρ is the air density (assumed small compared with the wing mass)

GJ is the torsional rigidity

z is the vertical displacement of a point

$s\eta$ is the distance of the point from the root

and x is the distance of the point aft of the leading edge.

/Table I

TABLE I

	Exact solution	standard binary $z = cq_1\eta^2 + (x - \frac{c}{2})q_2\eta$	quadratic binary $z = cq_1\eta^2 + xq_2\eta^2$	ternary $z = cq_1\eta^2 + xq_2\eta^2 + xq_3\eta^3$
$scV\sqrt{\frac{\rho}{GJ}}$	2.48	2.27	3.35	2.39
error	-	8.5%	35.1%	3.6%

Points to note are

(i) The standard binary gives a satisfactory answer, and the error is on the safe side;

(ii) The quadratic binary gives an unsatisfactory answer and errs on the unsafe side. In this particular binary the position of the reference axis is immaterial since both spanwise functions have the same form;

(iii) The ternary gives a better answer than the standard binary in spite of the fact that both its torsion modes are unrealistic by themselves.

In Fig. 9 the exact flutter torsion mode (a sine curve) is compared with the linear mode of the standard binary, the quadratic mode and the torsion mode obtained from the ternary solution.

This simple example illustrates the importance of giving consideration in advance to the most economical choice of degrees of freedom, when the solution is to be found on a simulator. It is easy to imagine circumstances in which the numerical work required to carry out the ternary would be less than that for the standard binary (see Table I) and for which the ternary would therefore represent the better choice of coordinates.

REFERENCES

No.	Author	Title, etc.
1	R.A. Frazer, W.J. Duncan and A.R. Collar	Elementary matrices Cambridge University Press, 1938
2	W.J. Duncan, A.R. Collar and H.M. Lyon	Oscillations of elastic blades and wings in an airstream R & M 1716, January 1936
3	I.T. Minhinnick	The theoretical determination of normal modes and frequencies of vibration R.A.E. Report No. Structures 197, January 1956, A.R.C.18414 (to be published in the R & M series)
4	A.R. Collar	Semi-rigid representation of elastic cantilevers for torsion problems in aerodynamics Phil. Mag. Ser. 7, Vol.XXXIX p.97, February 1948

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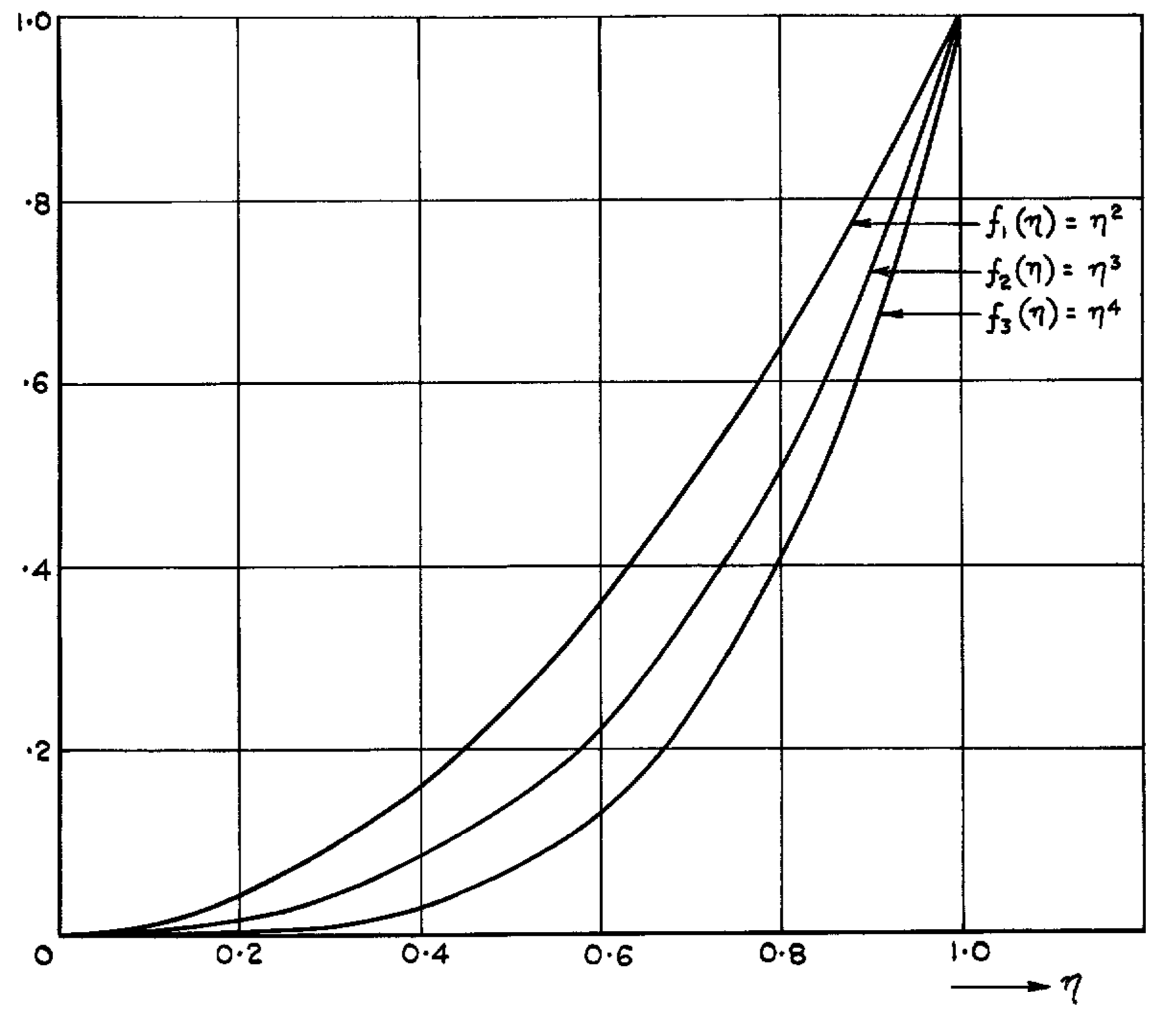

FIG. 1. THREE SIMPLE ARBITRARY MODES FOR BENDING.

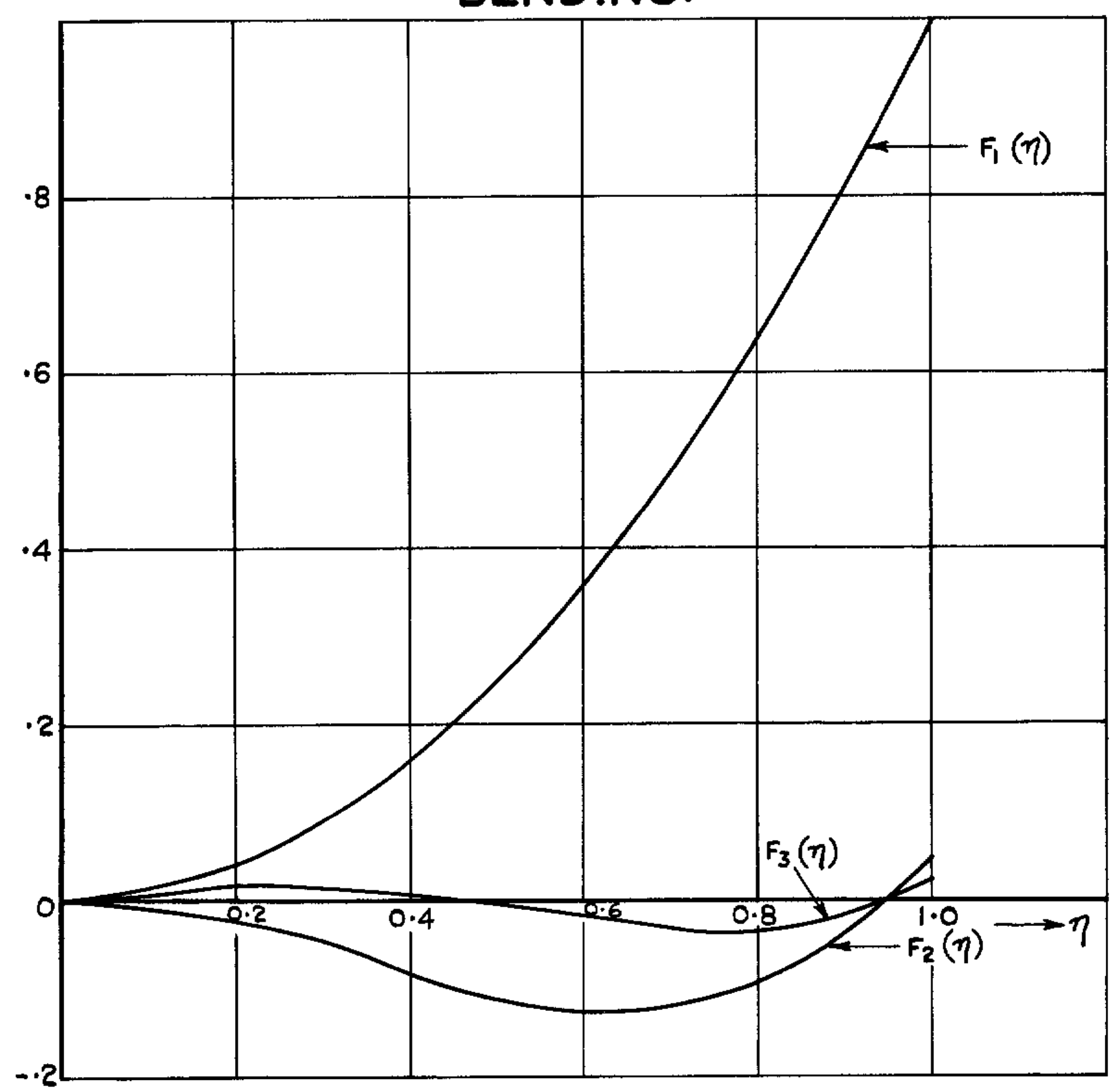

FIG. 2. THE MODES OF FIG. 1 AFTER TRANSFORMATION.

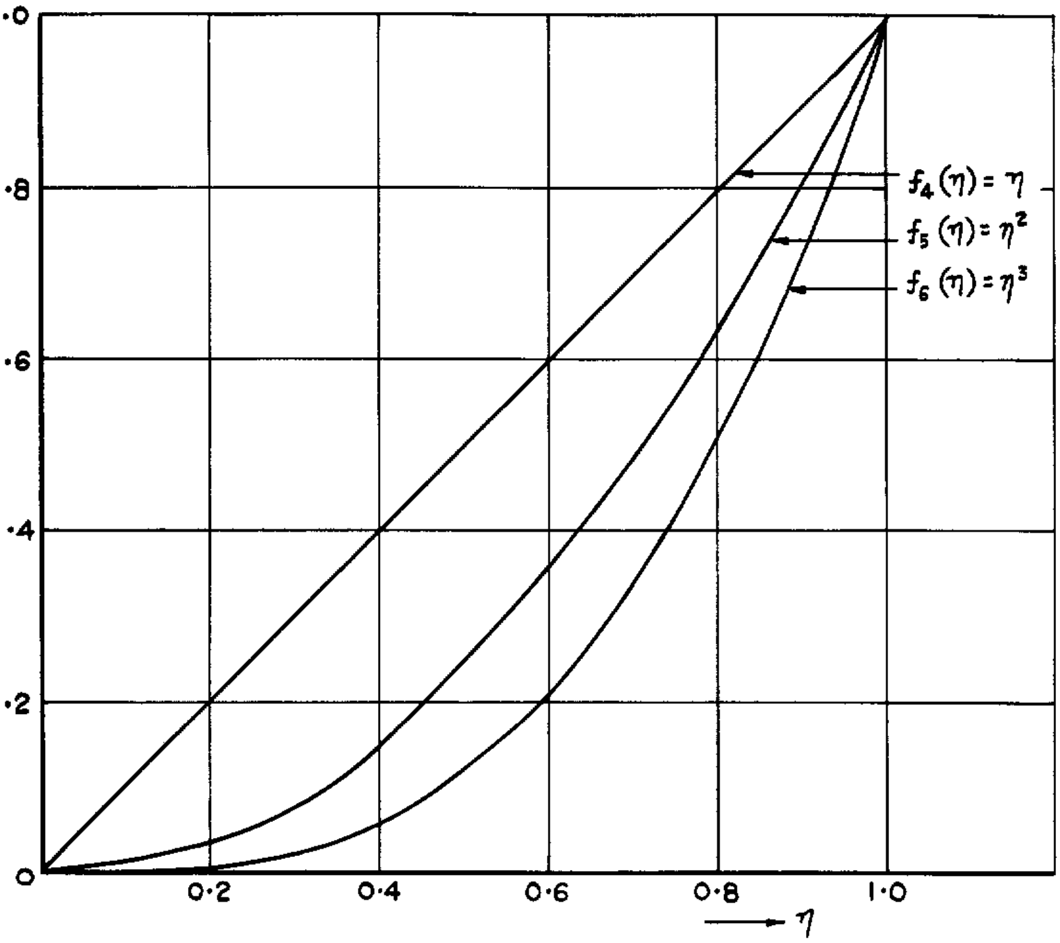

FIG.3. THREE SIMPLE ARBITRARY MODES OF TORSION

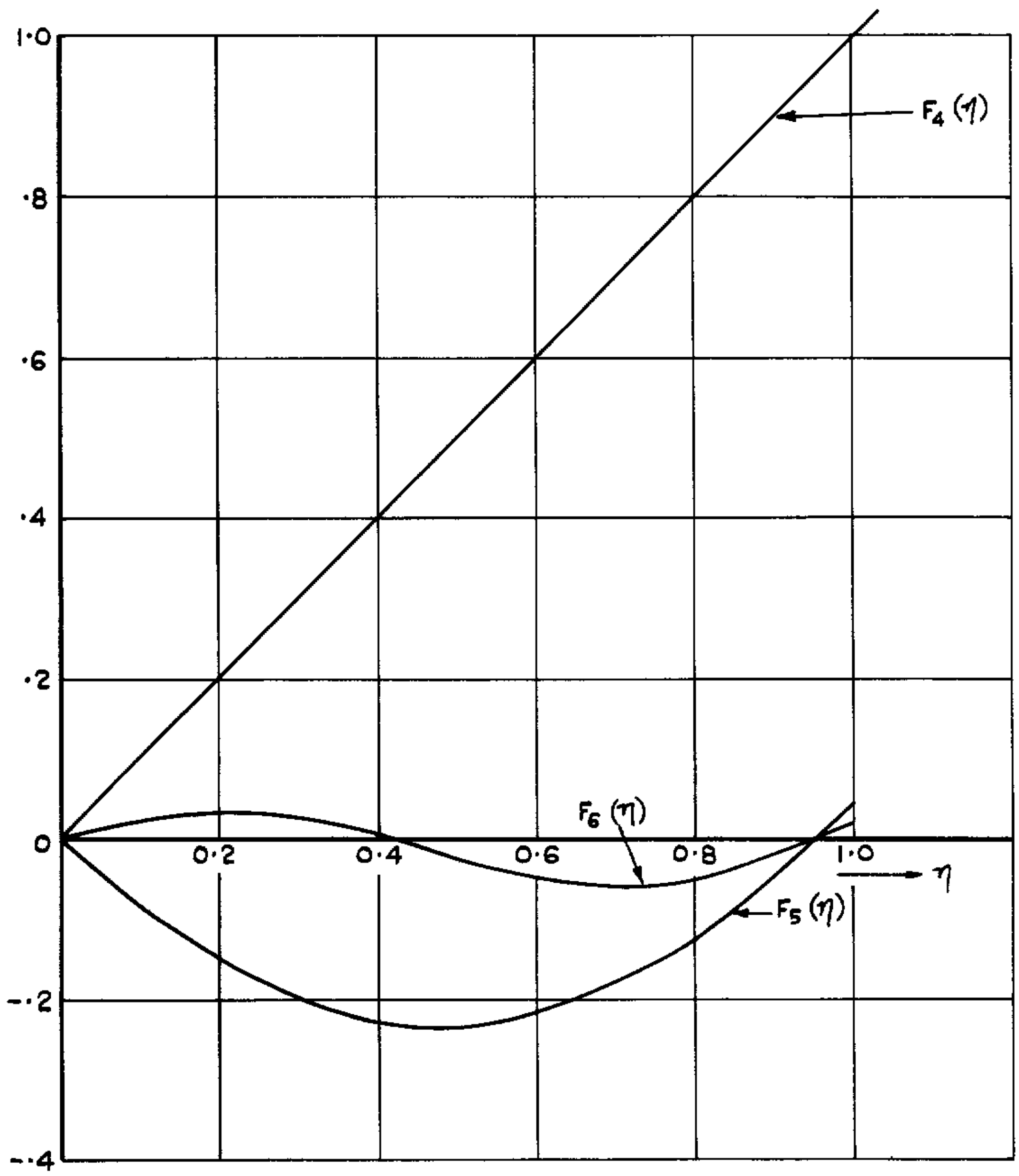

FIG.4. THE MODES OF FIG.3 AFTER TRANSFORMATION.

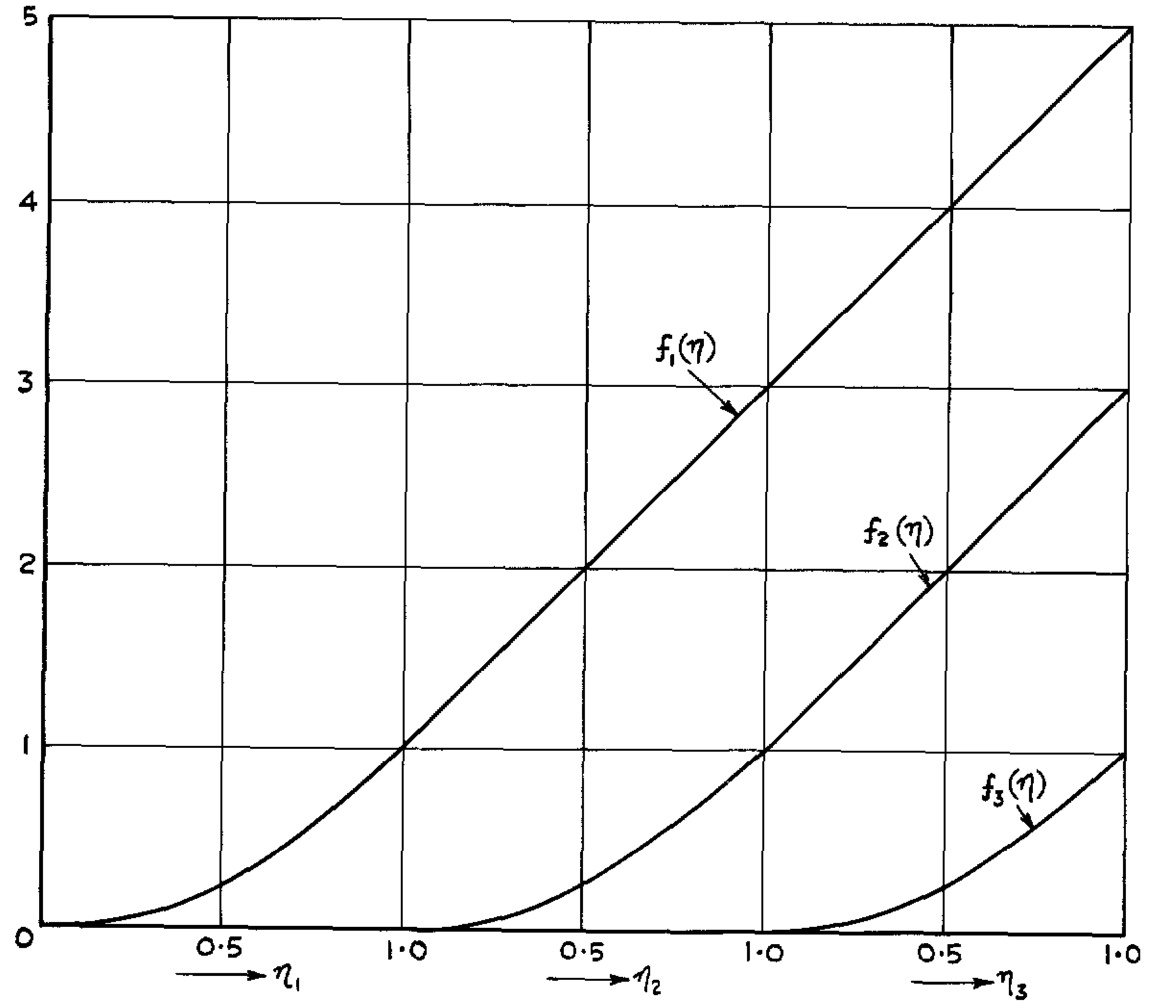

FIG. 5. THREE SIMPLE BENDING MODES FOR A WING DIVIDED SPANWISE.

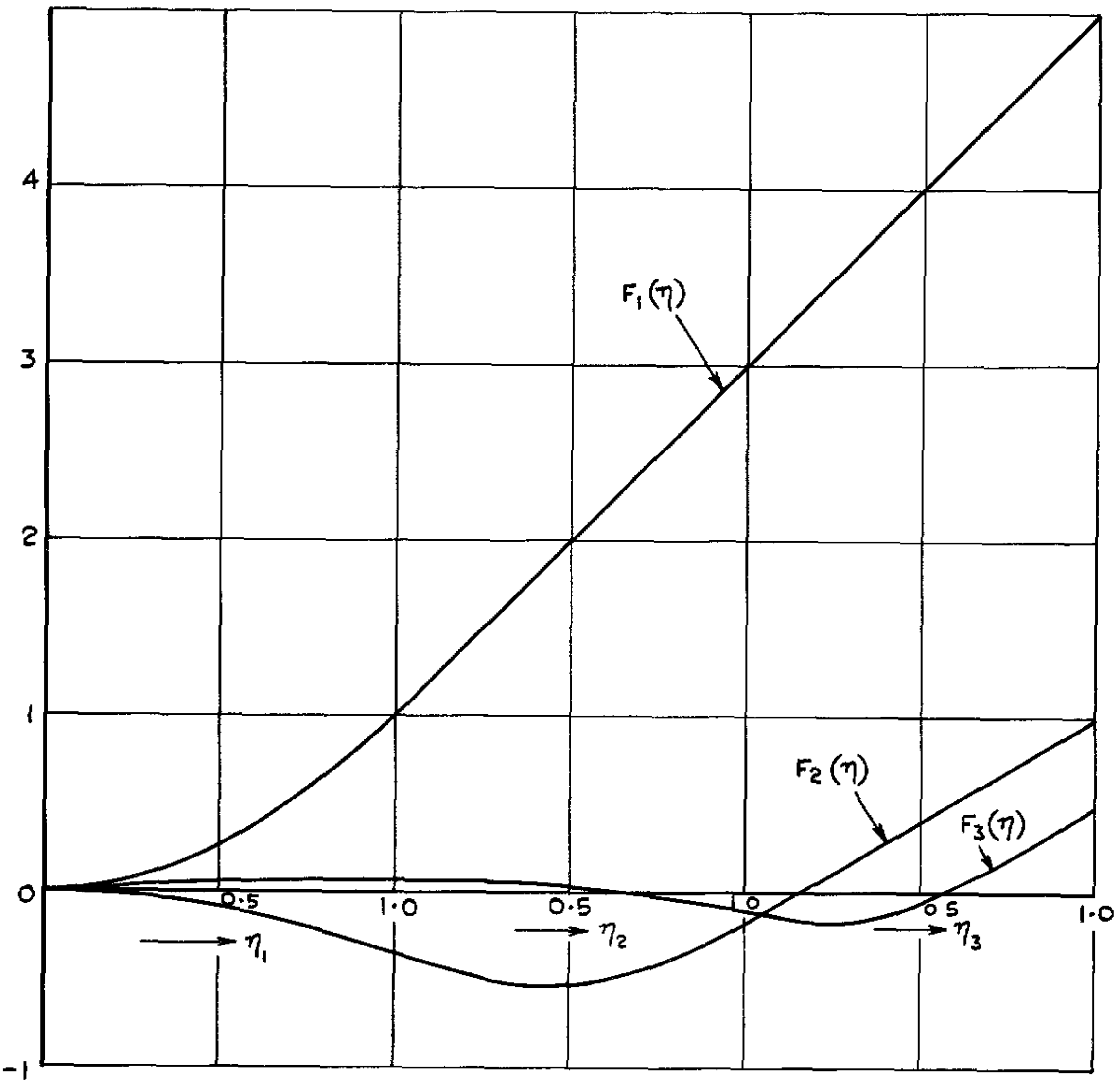

FIG. 6. THE MODES OF FIG. 5 AFTER TRANSFORMATION.

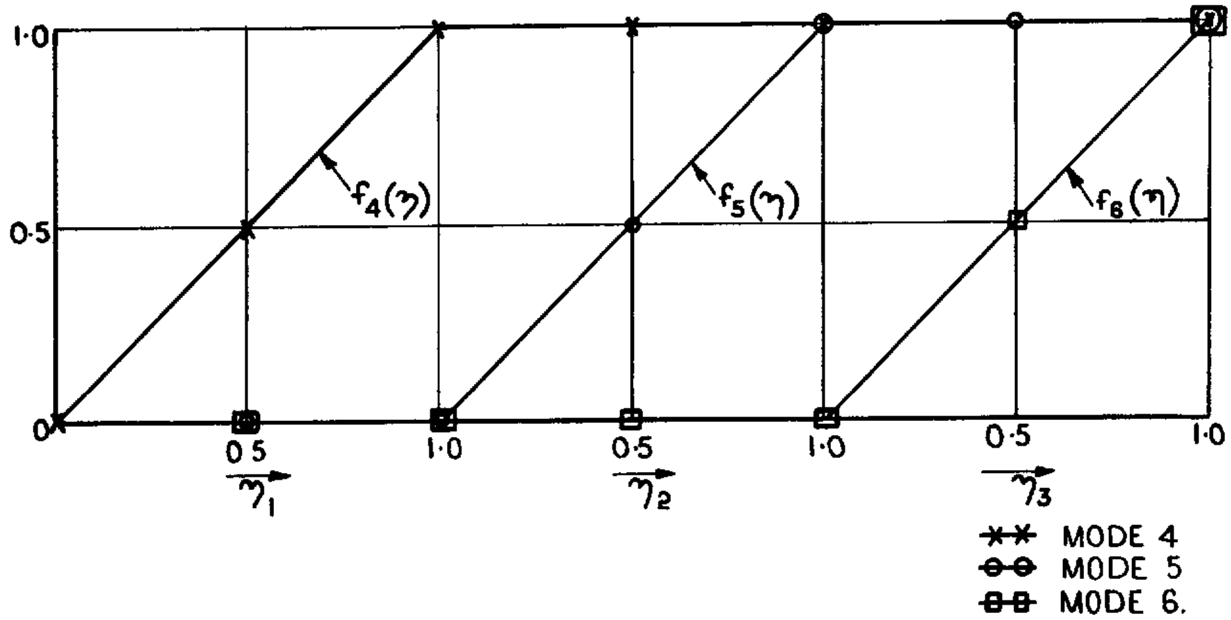

FIG. 7. THREE SIMPLE TORSION MODES FOR A WING DIVIDED SPANWISE.

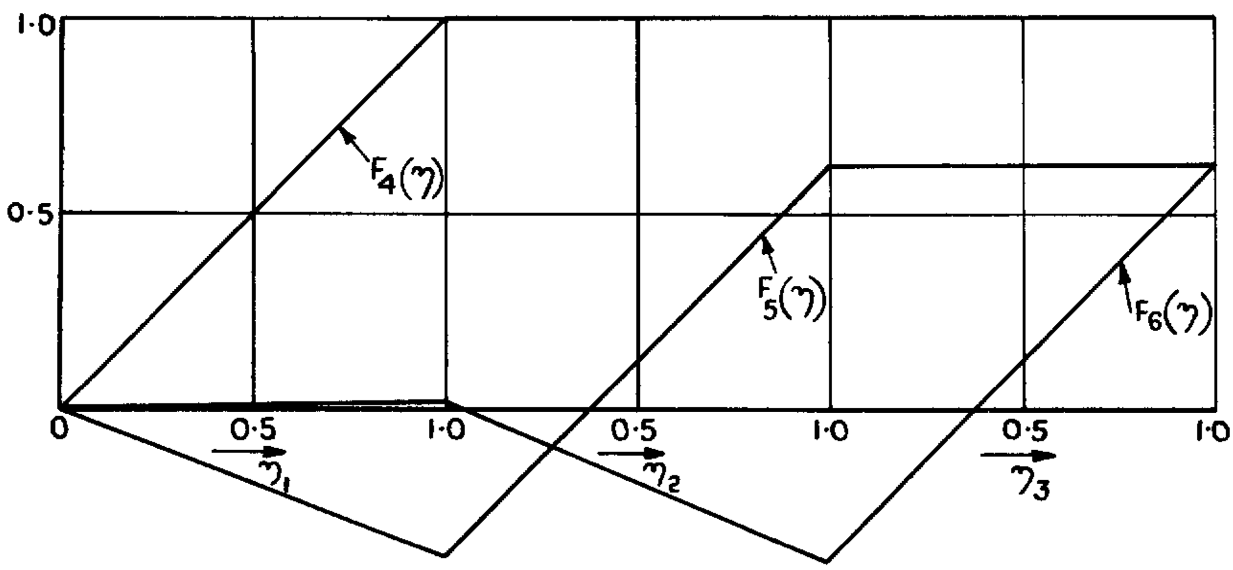

FIG. 8. THE MODES OF FIG. 7. AFTER TRANSFORMATION.

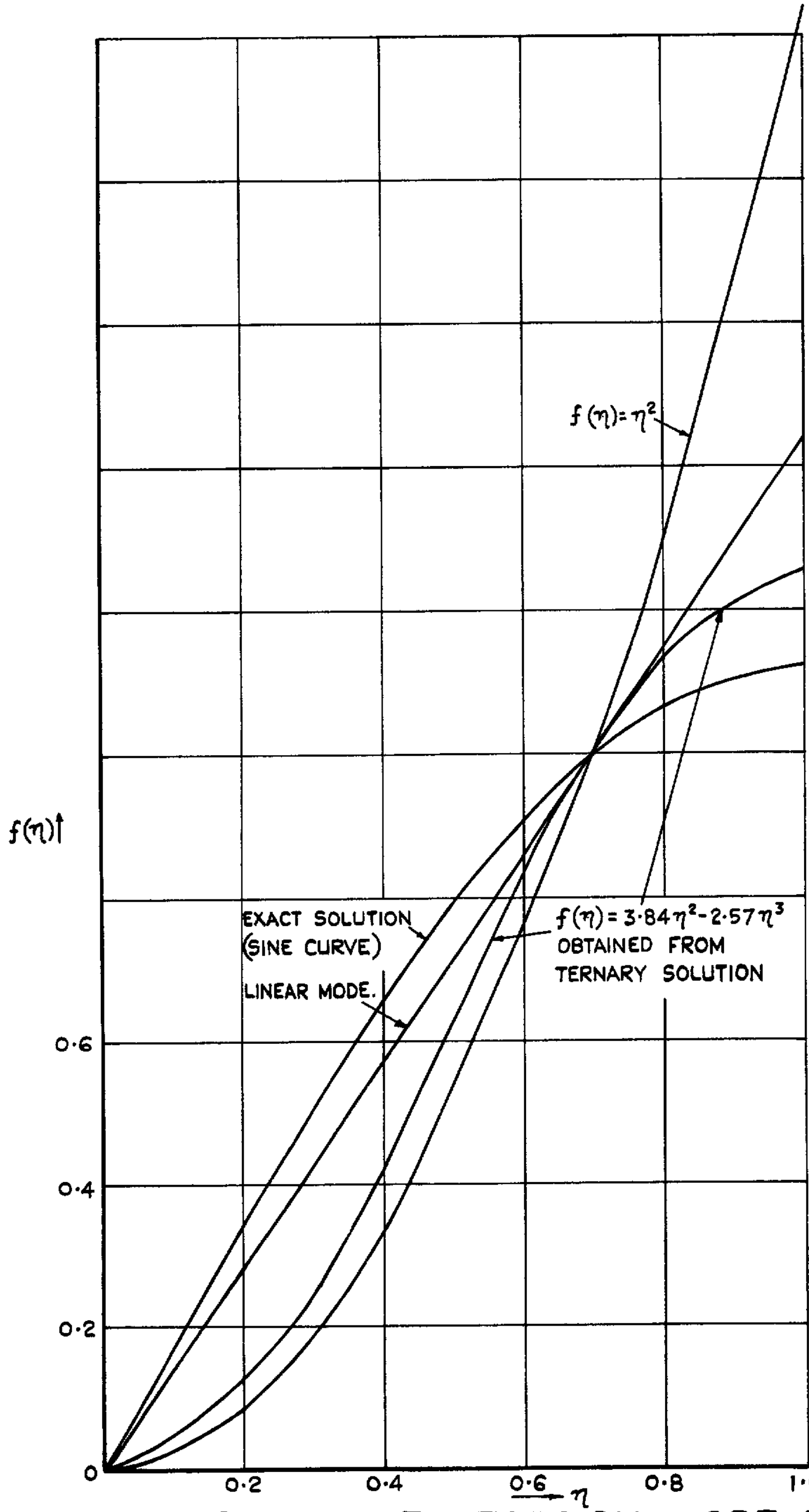

FIG.9. THE EXACT FLUTTER TORSION MODE FOR AN IDEALISED WING COMPARED WITH LINEAR AND PARABOLIC TORSION AS ASSUMED IN TWO BINARY CALCULATIONS, AND WITH A COMBINED MODE OBTAINED FROM A TERNARY CALCULATION.

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