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Theoretical Determination of Subsonic Oscillatory Airforce Coefficients

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1. Introduction

Methods for calculating generalised airforce coefficients on a harmonically oscillating flat plate wing in subsonic flow based on linearised theory, have been in the course of development for many years. The linearised theory is used to set up an integral equation relating the unknown loading distribution to the known upwash distribution on the wing. Basically two methods have been in the course of development for solving the integral equation numerically. In the one the loading distribution is replaced by a distribution of concentrated loads on certain lines and is known as the doublet lattice method, whereas in the other the loading distribution is replaced by an approximation which is continuous over the wing except in the neighbourhood of its leading edge and is known as the lifting surface method. There are also methods which are not based on the above-mentioned integral equation, for example the vortex lattice method. In this paper we concern ourselves exclusively with the lifting surface method.

The lifting surface theory of Multhopp[1] was for steady flow and required the loading to be approximated to by a polynomial in the wing coordinates ξ and η multiplied by a function of ξ and η which took into account the known singular behaviour of the loading at the edges of the wing. This approximation to the loading was substituted into the integral equation to get an approximation to the upwash, and this approximation to the upwash was equated to the known upwash at a set of points on the wing equal in number to the number of unknown coefficients in the expression for the approximation to the loading. A set of linear equations for the unknown coefficients was thus obtained and this set could be solved. The approximation to the loading was then known and an approximation to any required generalised airforce coefficients could be obtained from it.

Multhopp's method was extended to low-frequency harmonic oscillations in Ref. 2 and to general-frequency harmonic oscillations by, among others, Acum[3], Richardson[4] and Davies[5]. It was found, however, that in all these methods, in which the chordwise integration was carried out first, the spanwise integral was evaluated numerically by too coarse a method and this resulted in inaccurate estimation of the approximation to the upwash at points near to the leading and trailing edges of the wing with consequent loss of accuracy in the results for generalised airforce coefficients. Garner and Fox[6] refined the method of numerical integration of the spanwise integral and applied their refinement to the case of low-frequency harmonic oscillations. Long[7] applied this same refinement to the case of general-frequency harmonic oscillations. Zandbergen, Labrujere and Wouters[8] also refined the method of numerical integration of the spanwise integral in a manner somewhat different from that of Ref. 6 and for the particular case of steady flow. Lehrian and Garner[9] then extended the refinement of Ref. 8 to the case of general-frequency harmonic oscillations. A method in which the spanwise integral is evaluated first is that of Hewitt and Kellaway[10]. The numerical integrations in Ref. 10 are all carried out accurately so that no refinements are necessary.

All the above methods require that the approximation to the upwash be equated to the known upwash at a set of points on the wing, their number being the same as that of the unknown coefficients in the expression for the loading, and then these unknown coefficients may be determined. There are other methods of determining these unknown coefficients and one of these methods is discussed by Davies[11]. In this method the coefficients are determined by equating integrals involving the approximation to the upwash to corresponding integrals involving the known upwash. This process, theoretically, leads to the generalised airforce coefficients being obtained with the highest possible precision for a loading approximation of a particular form. The said integrals are evaluated numerically for the present paper, but the number of integration points may exceed the number of unknown coefficients in the expression for the loading. If the number of integration points equals the number of unknown coefficients in the expression for the loading then the method of equating the approximation to the upwash to the given upwash at a set of points on the wing is retrieved. Furthermore the refinement of Garner and Fox[6] is used in evaluating the spanwise integral in the integral equation, although this is modified to some extent in that the parameter q which determines the number of spanwise integration points may depend on the location of the upwash point concerned.

The process is described in detail in the present paper. A program has been written in ICL 1900 FORTRAN to calculate, by using this process, the generalised airforce coefficients and the loading distribution for a wing oscillating harmonically at general frequencies in subsonic flow. The procedure for using the program is described in Ref. 17. Calculations, using the program, have been carried out for swept tapered and rectangular wings. The results obtained are given here and their convergence is studied.

2. Theoretical Considerations

2.1. Preliminary Formulae

We refer points of space to an inertial right-handed Cartesian coordinate system $Oxyz$ where O is the origin of coordinates and Ox, Oy, Oz are the axes of x, y, z coordinates, positive z being upwards. We introduce a thin wing W into the space and consider it to be vibrating in such a way that the position of any material point on the surfaces of W is near to a fixed point which is the mean position of that surface point. We consider a flow of fluid in the space about the wing W, which at large distances ahead of the wing is uniform and horizontal with speed V in the direction of the positive x-axis. We then have the problem of determining the flow about the vibrating impervious surfaces of the wing W and we take this to be potential flow except across the wake surface which extends as a sheet of vorticity downstream from the trailing edge of W. Once the flow is known the pressure forces acting on the wing can be determined.

When the wing W is very thin and is vibrating in such a manner that all the points on its surfaces are always very near to the plane $z=0$, the governing equations of motion of the fluid may be linearised. Let the orthogonal projection of the mean positions of the material points on the surface of W on to the plane $z=0$ define the area S, which we shall call the wing planform. Then, as far as the aerodynamic problem is concerned, the wing is replaced by the area S in the plane $z=0$ and the boundary condition of the fluid not penetrating the surfaces of W is replaced by given fluid speed distributions normal to the top and the bottom surfaces of the planform area S. The wake becomes a flat surface in the plane $z=0$ extending from the trailing edge of S to infinity.

There are two material points on the surfaces of W, one on the top and one on the bottom surface, whose mean positions have an orthogonal projection onto the point $(x, y, 0)$ of the planform area S. Let the components, in the direction of the positive z-axis, of the displacement of these points at time t from the point $(x, y, 0)$ on the planform area S be denoted by $Z_+(x, y, t)$ and $Z_-(x, y, t)$ respectively for the point on the top surface and for the point on the bottom surface of W. According to linearised theory, the fluid speed $W_+(x, y, t)$ normal to the top of the planform area S and measured positive upwards is given by

$$W_+(x, y, t) = V\frac{\partial Z_+(x, y, t)}{\partial x} + \frac{\partial Z_+(x, y, t)}{\partial t}. \tag{1}$$

Similarly the fluid speed $W_-(x, y, t)$ normal to the bottom of the planform area S and measured positive upwards is given by

$$W_-(x, y, t) = V\frac{\partial Z_-(x, y, t)}{\partial x} + \frac{\partial Z_-(x, y, t)}{\partial t}. \tag{2}$$

The perturbation velocity potential in the fluid can be split up into the sum of two constituents, one of which is symmetric about the plane $z=0$ and the other of which is antisymmetric about the plane $z=0$. The speed distribution normal to the planform area S, corresponding to the symmetric velocity potential constituent is the same in magnitude but opposite in sign on the top and bottom of the planform area S, whereas the speed distribution normal to the planform area S, corresponding to the antisymmetric velocity potential constituent is the same in magnitude and sign on the top and bottom of the planform area S. Corresponding to the symmetric velocity potential constituent there is no net pressure loading across the planform area S but corresponding to the antisymmetric velocity potential constituent there is a net pressure loading across the planform area S, and, by the principle of superposition, this is the total pressure loading across the planform area S. This pressure loading will give rise to generalised airforces on S and these can be taken to be the linearised values of the corresponding generalised airforces acting on the wing W. Accordingly, to determine these generalised airforces we need deal only with the antisymmetric velocity potential constituent.

Let us now write

$$Z_+(x, y, t) = Z(x, y, t) + Z_1(x, y, t) \tag{3}$$

and

$$Z_-(x, y, t) = Z(x, y, t) - Z_1(x, y, t). \tag{4}$$

The function $Z_1(x, y, t)$ describes the wing thickness distribution, which will not normally be changing with time, whereas the function $Z(x, y, t)$ describes the position of the camber surface, which will be changing with time. The antisymmetric velocity potential constituent depends exclusively on $Z(x, y, t)$ whereas the symmetric velocity potential constituent depends exclusively on $Z_1(x, y, t)$. Since, as stated above, we need deal only with the antisymmetric velocity potential constituent we may disregard the thickness function $Z_1(x, y, t)$

henceforth, and consider only the displacement constituent function $Z(x, y, t)$. The corresponding upward component of the fluid velocity normal to the planform area S on its top and bottom is $W(x, y, t)$ where

$$W(x, y, t) = V\frac{\partial Z(x, y, t)}{\partial x} + \frac{\partial Z(x, y, t)}{\partial t}. \tag{5}$$

The quantity $W(x, y, t)$ is called the upwash on S.

For a vibrating wing the displacement function $Z(x, y, t)$ can be given as a linear combination of independent modes of oscillation. Thus we can write

$$Z(x, y, t) = l \sum_{k=1}^{\infty} \zeta_k(x, y) b_k(t) \tag{6}$$

where l is some typical length of the planform S, $\zeta_k(x, y)$ is the modal function and $b_k(t)$ is the generalised coordinate for the mode number k, both of which are non-dimensional. The modal functions $\zeta_k(x, y)$, $k = 1, 2, \ldots$, need to be a complete set of functions for (6) to be valid, in general, and to be of practical value the summation in (6) must be truncated to a finite number of terms. This truncation may entail an error but if the number of terms retained is sufficiently large the resulting error is negligibly small.

The generalised coordinate $b_k(t)$ is a real function of time t, which we shall assume to consist of a linear superposition of harmonic constituents

$$\bar{b}_k(\omega)\, e^{i\omega t} + \bar{b}_k^*(\omega)\, e^{-i\omega t} \tag{7}$$

over a range of values of circular frequency ω, where $\bar{b}_k^*(\omega)$ is the complex conjugate of the complex number $\bar{b}_k(\omega)$. Since our aerodynamic problem has been linearised we may take

$$b_k(t) = \bar{b}_k(\omega)\, e^{i\omega t} \tag{8}$$

to carry out the determination of the generalised airforces at the circular frequency ω. The generalised airforces for the problem when

$$b_k(t) = \bar{b}_k^*(\omega)\, e^{-i\omega t} \tag{9}$$

are the complex conjugates of those for $b_k(t)$ given by (8). The generalised airforces corresponding to the real function $b_k(t)$ given by the expression (7) are then the real quantities obtained by adding the two complex conjugate generalised airforces corresponding to (8) and (9).

If we substitute for $Z(x, y, t)$ from (6) into (5) and use the expression (8) for $b_k(t)$ we get

$$W(x, y, t) = V \sum_{k=1}^{P} \alpha_k(x, y; \nu) \bar{b}_k(\omega)\, e^{i\omega t} \tag{10}$$

where P is the number of terms to be retained in (6),

$$\alpha_k(x, y; \nu) = l\frac{\partial \zeta_k(x, y)}{\partial x} + i\nu\zeta_k(x, y) \tag{11}$$

and

$$\nu = \frac{\omega l}{V} \tag{12}$$

is the frequency parameter corresponding to the circular frequency ω.

Corresponding to the upwash

$$V\alpha_k(x, y; \nu)\bar{b}_k(\omega)\, e^{i\omega t} \tag{13}$$

in the mode number k, there is, across the planform area S, a normal pressure force per unit area in the direction of the positive z-axis, called the aerodynamic loading at the point $(x, y, 0)$ of the planform S at time t. We can write this loading in the form

$$\rho V^2 l_k(x, y; \nu, M_\infty)\bar{b}_k(\omega)\, e^{i\omega t} \tag{14}$$

where ρ is the density of the fluid in the uniform flow far upstream of the wing and M_∞ is the Mach number

$$M_\infty = \frac{V}{a} \tag{15}$$

where a is the speed of sound in the uniform flow far upstream of the wing.

On using the governing linearised partial differential equation for the perturbation velocity potential, the boundary condition of prescribed upwash on S, and the condition of no loading on the wake, we can set up an integral equation relating the upwash on S to the loading on S (see Ref. 5). For the mode number k this takes the form

$$\alpha_k(x, y; \nu) = \frac{1}{4\pi l^2}\iint_S l_k(x_0, y_0; \nu, M_\infty) K\left(\frac{x-x_0}{l}, \frac{y-y_0}{l}; \nu, M_\infty\right) \exp\left\{\frac{-i\nu(x-x_0)}{l}\right\} dx_0\, dy_0 \tag{16}$$

where, for subsonic flow,

$$K\left(\frac{x}{l}, \frac{y}{l}; \nu, M_\infty\right) = l^2 \int_{(-x+M_\infty R)/(1-M_\infty^2)}^{\infty} e^{-i\nu u/l} \frac{du}{(u^2+y^2)^{\frac{3}{2}}} + l^2 \frac{M_\infty(M_\infty x + R)}{R(x^2+y^2)} \exp\left\{-\frac{i\nu}{l}\left(\frac{-x+M_\infty R}{1-M_\infty^2}\right)\right\} \tag{17}$$

and

$$R = \sqrt{\{x^2 + (1-M_\infty^2)y^2\}}. \tag{18}$$

For dynamical analyses of oscillating wings in an airstream we need to know the generalised airforce coefficients $Q_{jk}(\nu, M)$ which are given by the expressions

$$Q_{jk}(\nu, M_\infty) = \frac{1}{l^2}\iint_S \zeta_j(x_0, y_0) l_k(x_0, y_0; \nu, M_\infty)\, dx_0\, dy_0 \tag{19}$$

in the linearised approximation. It is the main purpose of this paper to discuss the numerical evaluation of $Q_{jk}(\nu, M_\infty)$.

2.2. Approximate Solution of the Integral Equation

The integral equation (16) does not have a unique solution, but if we impose the condition that the loading at the trailing edge of S vanishes then, generally, the solution becomes unique. As a consequence of imposing this condition the loading acquires a certain behaviour near the edges of the planform S and this behaviour is known. We shall seek an approximate solution of the integral equation (16) which has this known behaviour near the edges of the planform S.

We introduce parametric coordinates on the planform S by means of the formulae

$$\left.\begin{aligned} \xi &= \frac{1}{c(y)}\{x - x_L(y)\} \\ \eta &= \frac{y}{s} \\ \xi_0 &= \frac{1}{c(y_0)}\{x_o - x_L(y_0)\} \\ \eta_0 &= \frac{y_0}{s} \end{aligned}\right\} \tag{20}$$

where s is the semi-span of the planform S, $c(y)$ is the chord length and $x_L(y)$ is the x coordinate of the leading edge at the spanwise position y of the planform S, as shown in Fig. 1.

The integral equation (16) may now be written as

$$\alpha_k(x, y; \nu) = \frac{1}{4\pi}\frac{s}{l}\int_{-1}^{+1}\frac{c(y_0)}{l}\,d\eta_0\int_0^1 l_k(x_0, y_0; \nu, M)K\left(\frac{x-x_0}{l}, \frac{y-y_0}{l}; \nu, M_\infty\right)\exp\left\{\frac{-i\nu}{l}(x-x_0)\right\}d\xi_0. \tag{21}$$

We now take an approximation $\hat{l}_k(x_0, y_0)$ to $l_k(x_0, y_0; \nu, M_\infty)$ which is given by the formula

$$\hat{l}_k(x_0, y_0) = \frac{l}{c(y_0)}\exp\left(\frac{-i\nu x_0}{l}\right)\sum_{i=1}^{n}\sum_{p=1}^{m} A_{k;i,p}\xi_0^{i-1}\eta_0^{p-1}\sqrt{\frac{1-\xi_0}{\xi_0}}\sqrt{1-\eta_0^2}. \tag{22}$$

The approximation $\hat{l}_k(x_0, y_0)$ to $l_k(x_0, y_0; \nu, M_\infty)$ vanishes at the trailing edge of S and has the required behaviour near the edges of S. The values of the coefficients $A_{k;i,p}$ are such that the simultaneous linear equations (33) are satisfied.

The approximation $\hat{l}_k(x_0, y_0)$ in formula (22) has been expressed in terms of n chordline base functions ξ_0^{i-1}, $i = 1, 2, \ldots, n$, and of m spanwise base functions η_0^{p-1}, $p = 1, 2, \ldots, m$. The formula (22) will be put into a different form, which is more convenient for numerical evaluation than is the formula (22) which involves merely simple monomials $\xi_0^{i-1}\eta_0^{p-1}$.

The points

$$\xi_i^{(n)} = \frac{1}{2}\left[1-\cos\left(\frac{2i-1}{2n+1}\pi\right)\right], \qquad i = 1, 2, \ldots, n \tag{23}$$

are a set of n distinct points in (0, 1). We form interpolation polynomials $h_r^{(n)}(\xi_0)$, $r = 1, 2, \ldots, n$, based on these points. These interpolation polynomials are given by the expressions

$$h_r^{(n)}(\xi_0) = \prod_{\substack{i=1\\ i\neq r}}^{n}\left(\frac{\xi_0 - \xi_i^{(n)}}{\xi_r^{(n)} - \xi_i^{(n)}}\right), \qquad r = 1, 2, \ldots, n \tag{24}$$

and have the property

$$h_r^{(n)}(\xi_i^{(n)}) = \delta_{ri} \tag{25}$$

where δ_{ri} is Kronecker's delta

$$\delta_{ri} = \begin{cases} 1 & r = i \\ 0 & r \neq i. \end{cases} \tag{26}$$

The function ξ_0^{i-1}, $i = 1, 2, \ldots, n$, can be expressed as a linear combination of the $h_r^{(n)}(\xi_0)$, $r = 1, 2, \ldots, n$, because these are a set of n linearly independent functions each of degree $(n-1)$ in ξ_0.

The points

$$\eta_p^{(m)} = \cos\left(\frac{p\pi}{m+1}\right), \qquad p = 1, 2, \ldots, m \tag{27}$$

are a set of m distinct points in $(-1, 1)$. We form interpolation polynomials $g_s^{(m)}(\eta_0)$, $s = 1, 2, \ldots, m$, based on these points and given by the expressions

$$g_s^{(m)}(\eta_0) = \prod_{\substack{p=1\\ p\neq s}}^{m}\left(\frac{\eta_0 - \eta_p^{(m)}}{\eta_s^{(m)} - \eta_p^{(m)}}\right), \qquad s = 1, 2, \ldots, m \tag{28}$$

which have the property

$$g_s^{(m)}(\eta_p^{(m)}) = \delta_{sp} \tag{29}$$

where δ_{sp} is Kronecker's delta. The function η_0^{p-1}, $p = 1, 2, \ldots, m$, can be expressed as a linear combination of the $g_s^{(m)}(\eta_0)$, $s = 1, 2, \ldots, m$, because these are a set of m linearly independent functions each of degree $(m-1)$ in η_0.

Therefore $\hat{l}_k(x_0, y_0)$ of formula (23) can be expressed in the different, but equivalent, form

$$\hat{l}_k(x_0, y_0) = \frac{l}{c(y_0)} \exp\left(\frac{-i\nu x_0}{l}\right) \sum_{r=1}^{n} \sum_{s=1}^{m} B_{k;r,s} h_r^{(n)}(\xi_0) g_s^{(m)}(\eta_0) \sqrt{\frac{1-\xi_0}{\xi_0}} \sqrt{1-\eta_0^2} \tag{30}$$

where the unknown coefficients $A_{k;i,p}$ of formula (22) have now been replaced by the new unknown coefficients $B_{k;r,s}$, which are linear combinations of the $A_{k;i,p}$. We need determine only the coefficients $B_{k;r,s}$ in order to know the function $\hat{l}_k(x_0, y_0)$.

The n distinct points $\xi_i^{(n)}$, $i = 1, 2, \ldots, n$, in (0, 1) and the m distinct points $\eta_p^{(m)}$, $p = 1, 2, \ldots, m$, in $(-1, 1)$ have the advantage over other choices in that simple expressions, concerning integration, which have been developed in Appendix A, may be used in the ensuing part of the Report. Other choices of points may be just as good as far as the numerical accuracy of the final results is concerned, but it is possible to have an unfortunate choice of the points ξ_0 in (0, 1) and η_0 in $(-1, 1)$ resulting in ill-conditioning of sets of simultaneous equations and the numerical accuracy of the final results is poor when only a moderately small number of significant figures is used in the calculations.

If we use the approximation $\hat{l}_k(x_0, y_0)$ from (30) for $l_k(x_0, y_0; \nu, M_\infty)$ in the integral equation (21) we shall get a corresponding approximation $\hat{\alpha}_k(x, y)$ to the upwash function $\alpha_k(x, y; \nu)$ which is given by

$$\hat{\alpha}_k(x, y) = \frac{s}{l} \sum_{r=1}^{n} \sum_{s=1}^{m} B_{k;r,s} U_{r,s}(x, y; \nu, M_\infty) \exp\left(\frac{-i\nu x}{l}\right) \tag{31}$$

where

$$U_{r,s}(x, y; \nu, M_\infty) = \frac{1}{4\pi} \int_{-1}^{+1} g_s^{(m)}(\eta_0) \sqrt{1-\eta_0^2}\, d\eta_0 \int_0^1 h_r^{(n)}(\xi_0) \sqrt{\frac{1-\xi_0}{\xi_0}} K\left(\frac{x-x_0}{l}, \frac{y-y_0}{l}; \nu, M_\infty\right) d\xi_0,$$
$$r = 1, 2, \ldots, n;\ s = 1, 2, \ldots, m. \tag{32}$$

Following Ref. 11 we determine the unknown coefficients $B_{k;r,s}$, $r = 1, 2, \ldots, n$; $s = 1, 2, \ldots, m$, from the set of mn linear simultaneous equations

$$\int_{-1}^{+1} g_p^{(m)}(\eta) \sqrt{1-\eta^2}\, d\eta \int_0^1 h_i^{(n)}(1-\xi) \sqrt{\frac{\xi}{1-\xi}} \{\alpha_k(x, y; \nu) - \hat{\alpha}_k(x, y)\} \exp\left(\frac{i\nu x}{l}\right) d\xi = 0,$$
$$i = 1, 2, \ldots, n;\ p = 1, 2, \ldots, m. \tag{33}$$

The corresponding approximation $\hat{Q}_{jk}$ to the generalised airforce coefficient Q_{jk} is then obtained from formula (19) on replacing $l_k(x_0, y_0; \nu, M_\infty)$ by $\hat{l}_k(x_0, y_0)$ and is

$$\hat{Q}_{jk} = \frac{1}{l^2} \iint_S \zeta_j(x_0, y_0) \hat{l}_k(x_0, y_0)\, dx_0\, dy_0. \tag{34}$$

If we substitute for $\hat{\alpha}_k(x, y)$ from (31) into (33) then we can write the set of mn linear simultaneous equations (33) in the alternative form

$$\frac{s}{l} \sum_{r=1}^{n} \sum_{s=1}^{m} \psi_{i,p;r,s} B_{k;r,s} = \theta_{k;i,p}, \qquad i = 1, 2, \ldots, n;\ p = 1, 2, \ldots, m, \tag{35}$$

where

$$\theta_{k;i,p}=\int_{-1}^{+1} g_p^{(m)}(\eta)\sqrt{1-\eta^2}\,d\eta \int_0^1 h_i^{(n)}(1-\xi)\sqrt{\frac{\xi}{1-\xi}}\,\alpha_k(x,y;\nu)\exp\left(\frac{i\nu x}{l}\right)d\xi,$$

$$i=1,2,\ldots,n;\,p=1,2,\ldots,m, \qquad (36)$$

and

$$\psi_{i,p;r,s}=\int_{-1}^{+1} g_p^{(m)}(\eta)\sqrt{1-\eta^2}\,d\eta \int_0^1 h_i^{(n)}(1-\xi)\sqrt{\frac{\xi}{1-\xi}}\,U_{r,s}(x,y;\nu,M_\infty)\,d\xi,$$

$$i=1,2,\ldots,n;\,p=1,2,\ldots,m,\,r=1,2,\ldots,n;\,s=1,2,\ldots,m. \qquad (37)$$

We assume that the wing planform S is symmetric about the x–z coordinate plane and that the modes of oscillation are either all symmetric modes or all antisymmetric. Then we must have

$$c(-y)=c(y) \qquad (38)$$

$$x_L(-y)=x_L(y) \qquad (39)$$

and

$$\zeta_k(x,-y)=\kappa\zeta_k(x,y) \qquad (40)$$

where $\kappa=+1$ for the case in which all the modes are symmetric and $\kappa=-1$ for the case in which all the modes are antisymmetric.

We note, from formula (27), that

$$\eta_{m-p+1}^{(m)}=-\eta_p^{(m)}, \qquad p=1,2,\ldots,m. \qquad (41)$$

Then, it follows from (28) that

$$g_{m-s+1}^{(m)}(\eta)=g_s^{(m)}(-\eta), \qquad s=1,2,\ldots,m, \qquad (42)$$

from (32) that

$$U_{r,m-s+1}(x,y;\nu,M_\infty)=U_{r,s}(x,-y;\nu,M_\infty), \qquad r=1,2,\ldots,n;\,s=1,2,\ldots,m, \qquad (43)$$

because the kernel function $K(x,y;\nu,M)$ is symmetric in y, and from (37) that

$$\psi_{i,m-p+1;r,m-s+1}=\psi_{i,p;r,s}, \qquad i=1,2,\ldots,n;\,p=1,2,\ldots,m,\,r=1,2,\ldots,n;\,s=1,2,\ldots,m. \qquad (44)$$

From (40) and (11) we get that the reduced upwash function $\alpha_k(x,y;\nu)$ satisfies

$$\alpha_k(x,-y;\nu)=\kappa\alpha_k(x,y;\nu), \qquad (45)$$

and then, from (36), we get that

$$\theta_{k;i,m-p+1}=\kappa\theta_{k;i,p}, \qquad i=1,2,\ldots,n;\,p=1,2,\ldots,m. \qquad (46)$$

Because of (44) and (46), the coefficients $B_{k;r,s}$, $r=1,2,\ldots,n$; $s=1,2,\ldots,m$, which satisfy the set of mn linear simultaneous equations (35), must satisfy the relations

$$B_{k;r,m-s+1}=\kappa B_{k;r,s}, \qquad r=1,2,\ldots,n;\,s=1,2,\ldots,m. \qquad (47)$$

We now define the integer m_H by means of the formula

$$m_H = \tfrac{1}{4}\{2m + \kappa - \kappa(-1)^m\}. \tag{48}$$

We note, from (46) and (47), that when m is an odd integer and $\kappa = -1$ we must have

$$\theta_{k;i,m_H+1} = 0, \tag{49}$$

and

$$B_{k;r,m_H+1} = 0. \tag{50}$$

For all values of m and κ, only m_H values of the $B_{k;r,s}$ are therefore unknown and consequently the set of mn linear simultaneous equations (35) may be replaced by the set of m_Hn linear simultaneous equations

$$\frac{s}{l}\sum_{r=1}^{n}\sum_{s=1}^{m_H} \Psi_{i,p;r,s}C_{k;r,s} = \theta_{k:i,p}, \qquad i = 1, 2, \ldots, n; p = 1, 2, \ldots, m_H, \tag{51}$$

where

$$C_{k;r,s} = 2B_{k;r,s}, \qquad r = 1, 2, \ldots, n; s = 1, 2, \ldots, m_H - 1, \tag{52}$$

$$C_{k;r,m_H} = \begin{cases} 2B_{k;r,m_H} & m \text{ even} \\ 2B_{k;r,m_H} & m \text{ odd}, \kappa = -1, \\ B_{k;r,m_H} & m \text{ odd}, \kappa = +1, \qquad r = 1, 2, \ldots, n, \end{cases} \tag{53}$$

and

$$\Psi_{i,p;r,s} = \tfrac{1}{2}(\psi_{i,p;r,s} + \kappa\psi_{i,p;r,m-s+1}), \qquad i = 1, 2, \ldots, n; p = 1, 2, \ldots, m_H, r = 1, 2, \ldots, n; s = 1, 2, \ldots, m_H. \tag{54}$$

Furthermore, the set of m_Hn linear simultaneous equations (51) may be written as the matrix equation

$$\frac{s}{l}[\Psi][C_k] = [\theta_k] \tag{55}$$

where $[\Psi]$ is a square matrix of order $m_Hn \times m_Hn$ consisting of elements which are the quantities $\Psi_{i,p;r,s}$, $i = 1, 2, \ldots, n$; $p = 1, 2, \ldots, m_H$; $r = 1, 2, \ldots, n$; $s = 1, 2, \ldots, m_H$, $[C_k]$ is a column matrix consisting of the m_Hn elements $C_{k;r,s}$, $r = 1, 2, \ldots, n$; $s = 1, 2, \ldots, m_H$, and $[\theta_k]$ is a column matrix consisting of the m_Hn elements $\theta_{k;i,p}$, $i = 1, 2, \ldots, n; p = 1, 2, \ldots, m_H$. The arrangement of the elements $C_{k;r,s}$ in the column matrix $[C_k]$ is immaterial and so is the arrangement of the elements $\theta_{k;i,p}$ in the column matrix $[\theta_k]$, but once these two arrangements have been specified the arrangement of the elements $\Psi_{i,p;r,s}$ in the square matrix $[\Psi]$ is determined.

To be definite we may specify the following arrangement for the elements in the matrices $[C_k]$, $[\theta_k]$ and $[\Psi]$. The column matrix $[C_k]$ has the element

$$C_{k;r,s}, \qquad r = 1, 2, \ldots, n; s = 1, 2, \ldots, m_H, \tag{56}$$

in the $n(m_H - s) + r$th row. The column matrix $[\theta_k]$ has the element

$$\theta_{k;i,p}, \qquad i = 1, 2, \ldots, n; p = 1, 2, \ldots, m_H, \tag{57}$$

in the $n(m_H - p) + n - i + 1$st row. Consequently the square matrix $[\Psi]$ has the element

$$\Psi_{i,p;r,s}, \qquad i = 1, 2, \ldots, n; p = 1, 2, \ldots, m_H, r = 1, 2, \ldots, n; s = 1, 2, \ldots, m_H, \tag{58}$$

in the $n(m_H - p) + n - i + 1$st row, and $n(m_H - s) + r$th column.

2.3. Evaluation of the Generalised Forces

If we substitute for $\hat{l}_k(x_0, y_0)$ from equation (30) into formula (34) we get

$$\hat{Q}_{jk} = \frac{s}{l}\sum_{r=1}^{n}\sum_{s=1}^{m} \chi_{j;r,s}B_{k;r,s} \tag{59}$$

where

$$\chi_{j;r,s} = \int_{-1}^{+1} g_s^{(m)}(\eta_0)\sqrt{1-\eta_0^2}\,d\eta_0 \int_0^1 \zeta_j(x_0, y_0)\exp\left(\frac{-i\nu x_0}{l}\right)h_r^{(n)}(\xi_0)\sqrt{\frac{1-\xi_0}{\xi_0}}\,d\xi_0,$$

$$r = 1, 2, \ldots, n;\ s = 1, 2, \ldots, m. \tag{60}$$

On using (40) and (42) in formula (60) we get the relations

$$\chi_{j;r,m-s+1} = \kappa\chi_{j;r,s}, \qquad r = 1, 2, \ldots, n;\ s = 1, 2, \ldots, m. \tag{61}$$

Therefore, from (59), on using (47), (52), (53) and (61), and noting that $\kappa^2 = 1$ we get

$$\hat{Q}_{jk} = \frac{s}{l}\sum_{r=1}^{n}\sum_{s=1}^{m_H} \chi_{j;r,s}C_{k;r,s}. \tag{62}$$

The formula (62) may be written as the matrix formula

$$[\hat{Q}_{jk}] = \frac{s}{l}[\chi_j][C_k] \tag{63}$$

where $[\chi_j]$ is a row matrix of $m_H n$ elements, with the element

$$\chi_{j;r,s} \qquad r = 1, 2, \ldots, n;\ s = 1, 2, \ldots, m_H, \tag{64}$$

in the $n(m_H - s) + r$th column and $[\hat{Q}_{jk}]$ is the matrix of order 1×1 consisting of the one element $\hat{Q}_{jk}$.

If we solve the matrix equation (55) for the column matrix $[C_k]$ and insert the result into formula (63) we get

$$[\hat{Q}_{jk}] = [\chi_j][\Psi]^{-1}[\theta_k]. \tag{65}$$

Let us now write

$$\zeta_{j;r,s} = \frac{1}{H_r^{(n)}G_s^{(m)}}\exp\left(\frac{i\nu}{l}x_{r,s}^{(n,m)}\right)\chi_{j;r,s}, \qquad r = 1, 2, \ldots, n;\ s = 1, 2, \ldots, m, \tag{66}$$

and

$$\alpha_{k;i,p} = \frac{1}{H_i^{(n)}G_p^{(m)}}\exp\left(\frac{-i\nu}{l}\bar{x}_{i,p}^{(n,m)}\right)\theta_{k;i,p}, \qquad i = 1, 2, \ldots, n;\ p = 1, 2, \ldots, m, \tag{67}$$

where

$$y_s^{(m)} = s\eta_s^{(m)}, \qquad s = 1, 2, \ldots, m, \tag{68}$$

$$x_{r,s}^{(n,m)} = c(y_s^{(m)})\xi_r^{(n)} + x_L(y_s^{(m)}), \qquad r = 1, 2, \ldots, n;\ s = 1, 2, \ldots, m, \tag{69}$$

$$\bar{x}_{i,p}^{(n,m)} = c(y_p^{(m)})(1-\xi_i^{(n)}) + x_L(y_p^{(m)}), \qquad i = 1, 2, \ldots, n;\ p = 1, 2, \ldots, m, \tag{70}$$

and (see Appendix A, formulae (A-13) and (A-32))

$$H_i^{(n)} = \int_0^1 h_i^{(n)}(\xi)\sqrt{\frac{1-\xi}{\xi}}\,d\xi$$

$$= \frac{2\pi}{(2n+1)}(1-\xi_i^{(n)}), \qquad i = 1, 2, \ldots, n, \tag{71}$$

$$G_p^{(m)} = \int_{-1}^{+1} g_p^{(m)}(\eta)\sqrt{1-\eta^2}\,d\eta$$

$$= \frac{\pi}{(m+1)}[1-(\eta_p^{(m)})^2], \qquad p = 1, 2, \ldots, m. \tag{72}$$

If we use the numerical integration formulae (A-16) and (A-35) with n integration points chordwise and m integration points spanwise to evaluate $\chi_{j;r,s}$ and $\theta_{k;i,p}$ from formulae (60) and (61) respectively we get

$$\chi_{j;r,s} \simeq \sum_{i=1}^{n}\sum_{p=1}^{m} H_i^{(n)}G_p^{(m)}h_r^{(n)}(\xi_i^{(n)})g_s^{(m)}(\eta_p^{(m)})\zeta_j(x_{i,p}^{(n,m)}, y_p^{(m)})\exp\left(\frac{-i\nu}{l}x_{i,p}^{(n,m)}\right)$$

$$= H_r^{(n)}G_s^{(m)}\zeta_j(x_{r,s}^{(n,m)}, y_s^{(m)})\exp\left(\frac{-i\nu}{l}x_{r,s}^{(n,m)}\right), \qquad r = 1, 2, \ldots, n;\ s = 1, 2, \ldots, m, \tag{73}$$

and

$$\theta_{k;i,p} \simeq \sum_{r=1}^{n}\sum_{s=1}^{m} H_r^{(n)}G_s^{(m)}h_i^{(n)}(\xi_r^{(n)})g_p^{(m)}(\eta_s^{(m)})\alpha_k(\bar{x}_{r,s}^{(m,n)}, y_s^{(m)};\ \nu)\exp\left(\frac{i\nu}{l}\bar{x}_{r,s}^{(n,m)}\right)$$

$$= H_i^{(n)}G_p^{(m)}\alpha_k(\bar{x}_{i,p}^{(n,m)}, y_p^{(m)};\ \nu)\exp\left(\frac{i\nu}{l}\bar{x}_{i,p}^{(n,m)}\right), \qquad i = 1, 2, \ldots, n;\ p = 1, 2, \ldots, m. \tag{74}$$

On substituting from (73) and (74) respectively into (66) and (67) respectively we then get

$$\zeta_{j;r,s} \simeq \zeta_j(x_{r,s}^{(n,m)}, y_s^{(m)}), \qquad r = 1, 2, \ldots, n;\ s = 1, 2, \ldots, m, \tag{75}$$

and

$$\alpha_{k;i,p} \simeq \alpha_k(\bar{x}_{i,p}^{(n,m)}, y_p^{(m)};\ \nu), \qquad i = 1, 2, \ldots, n;\ p = 1, 2, \ldots, m. \tag{76}$$

The formulae (73) and (74) are only approximate formulae, but they may be very good approximations to the correct values at high values of n and m if $c(y)$, $\alpha_k(x, y;\ \nu)$ and $\zeta_j(x, y)$ are continuous functions with only a few undulations over the planform S. There are instances, e.g. control surface rotation, in which their accuracy is not as good as may be desired unless n and m are unpractically high. If it is considered that the approximations (73) and (74) are sufficiently accurate then formulae (75) and (76) may be used for $\zeta_{j;r,s}$ and $\alpha_{k;i,p}$ respectively instead of the expressions (66) and (67) with $\chi_{j;r,s}$ and $\theta_{k;i,p}$ being obtained accurately by numerical integration of the integral relations (60) and (36) using formulae which are much more accurate than formulae (73) and (74), or, indeed, by carrying out the integrations analytically if this is possible.

Let $[\zeta_j]$ be the row matrix of $m_H n$ elements with the element

$$\zeta_{j;r,s}, \qquad r = 1, 2, \ldots, n;\ s = 1, 2, \ldots, m_H, \tag{77}$$

in the $n(m_H - s) + r$th column. Let $[\alpha_k]$ be the column matrix of $m_H n$ elements with the element

$$\alpha_{k;i,p}, \qquad i = 1, 2, \ldots, n;\ p = 1, 2, \ldots, m_H, \tag{78}$$

in the $n(m_H - p) + n - i + 1$st row. Let $[E]$ be the diagonal matrix of order $m_H n \times m_H n$ with the element

$$H_r^{(n)}G_s^{(m)}\exp\left(\frac{-i\nu}{l}x_{r,s}^{(n,m)}\right), \qquad r = 1, 2, \ldots, n;\ s = 1, 2, \ldots, m_H, \tag{79}$$

in the $n(m_H-s)+r$th row and column. Let $[D]$ be the diagonal matrix of order $m_Hn \times m_Hn$ with the element

$$H_i^{(n)}G_p^{(m)}\exp\left(\frac{i\nu}{l}\bar{x}_{i,p}^{(n,m)}\right), \qquad i=1,2,\ldots,n;\, p=1,2,\ldots,m_H, \tag{80}$$

in the $n(m_H-p)+n-i+1$st row and column. Then we can write formula (66) and (67) as the equivalent matrix formulae

$$[\chi_i]=[\zeta_j][E] \tag{81}$$

and

$$[\theta_k]=[D][\alpha_k] \tag{82}$$

respectively. On substituting from (81) and (82) into (65) we get

$$[\hat{Q}_{jk}]=[\zeta_j][E][\Psi]^{-1}[D][\alpha_k] \tag{83}$$

which is the final formulation for the approximation $\hat{Q}_{jk}$ to the generalised airforce coefficient Q_{jk}.

Suppose that there are P modes of oscillation corresponding to the index k in the summation on the right hand side of (6) taking the values $k=1,2,\ldots,P$, only, just as in formula (10). Let $[\zeta]$ be the matrix of order $P\times m_Hn$ obtained by arranging the row matrices $[\zeta_j]$, $j=1,2,\ldots,P$, sequentially one below another. Let $[\alpha]$ be the matrix of order $m_Hn\times P$ obtained by arranging the column matrices $[\alpha_k]$, $k=1,2,\ldots,P$, sequentially one alongside another. Let $[\hat{Q}]$ be the square matrix of order $P\times P$ which has the element $[\hat{Q}_{jk}]$ in the jth row and kth column, $j=1,2,\ldots,P$; $k=1,2,\ldots,P$. Then we get immediately from formula (83)

$$[\hat{Q}]=[\zeta][E][\Psi]^{-1}[D][\alpha]. \tag{84}$$

2.4. Evaluation of the Loading

We can obtain the approximations $\hat{l}_k(x_0, y_0)$, $k=1,2,\ldots,P$, to the loading distribution $l_k(x_0, y_0; \nu, M)$ in the mode k of oscillation directly from formula (30). By using the definitions (52) and (53) in formula (30) in order to replace the coefficients $B_{k;r,s}$ by $C_{k;r,s}$ we get

$$\hat{l}_k(x_0, y_0)=\frac{l}{2c(y_0)}\exp\left(\frac{-i\nu x_0}{l}\right)\sum_{r=1}^{n}\sum_{s=1}^{m_H} C_{k;r,s}h_r^{(n)}(\xi_0)k_s^{(m_H)}(\eta_0)\sqrt{\frac{1-\xi}{\xi_0}}\sqrt{1-\eta_0^2} \tag{85}$$

where

$$k_s^{(m_H)}(\eta_0)=g_s^{(m)}(\eta_0)+\kappa g_s^{(m)}(-\eta_0), \qquad s=1,2,\ldots,m_H. \tag{86}$$

The relations (42) and (47) have also been used in obtaining the formula (85).

Let $[F(x_0, y_0)]$ be the row matrix of m_Hn elements with the element

$$\frac{l^2}{sc(y_0)}\exp\left(\frac{-i\nu x_0}{l}\right)h_r^{(n)}(\xi_0)k_s^{(m_H)}(\eta_o)\sqrt{\frac{1-\xi_0}{\xi_0}}\sqrt{1-\eta_0^2}, \qquad r=1,2,\ldots,n;\, s=1,2,\ldots,m_H, \tag{87}$$

in the $n(m_H-s)+r$th column and let $[\hat{l}_k(x_0, y_0)]$ be the matrix consisting of the single element $\hat{l}_k(x_0, y_0)$. Then we can replace the formula (85) by the matrix formula

$$[\hat{l}_k(x_0, y_0)]=\frac{1}{2}\frac{s}{l}[F(x_0, y_0)][C_k]. \tag{88}$$

If we obtain $[C_k]$ from formula (55) and express $[\theta_k]$ in terms of $[\alpha_k]$ by means of formula (82), then we can write, instead of (88)

$$[\hat{l}_k(x_0, y_0)]=\tfrac{1}{2}[F(x_0, y_o)][\Psi]^{-1}[D][\alpha_k] \tag{89}$$

which is a formula for determining $\hat{l}_k(x_0, y_0)$.

In particular, if we put in formula (89)

$$\left.\begin{aligned} x_0 &= x_{i,p}^{(n,m)} \\ y_0 &= y_p^{(m)} \end{aligned}\right\} \tag{90}$$

where $x_{i,p}^{(n,m)}$, $y_p^{(m)}$ are defined in formulae (68) and (69), with i and p replaced by r and s, we get

$$[\hat{l}_k(x_{i,p}^{(n,m)}, y_p^{(m)})] = \tfrac{1}{2}[F(x_{i,p}^{(n,m)}, y_p^{(m)})][\Psi]^{-1}[D][\alpha_k]. \tag{91}$$

We note that the row matrix $[F(x_{i,p}^{(n,m)}, y_p^{(m)})]$ has only one non-zero element, namely the one in the $n(m_H - p) + i$th column. This fact comes from applying the properties (25) and (29) of the functions $h_r^{(n)}(\xi_0)$ and $g_s^{(m)}(\eta_0)$ to the functions $k_s^{(m_H)}(\eta_0)$ of (86) and the elements (87) of $[F(x_0, y_0)]$.

Now let $[\hat{l}_k]$ be the column matrix of $m_H n$ elements with the element

$$\hat{l}_k(x_{i,p}^{(n,m)}, y_p^{(m)}), \qquad i = 1, 2, \ldots, n; p = 1, 2, \ldots, m_H, \tag{92}$$

in the $n(m_H - p) + i$th row and let $[F]$ be the matrix of order $m_H n \times m_H n$ whose $n(m_H - p) + i$th row is the row matrix $F(x_{i,p}^{(n,m)}, y_p^{(m)})$. Actually, according to the above, the square matrix $[F]$ is a diagonal matrix. It can be written as the product of diagonal matrices

$$[F] = [H][E] \tag{93}$$

where the diagonal matrix $[E]$ has been defined immediately before expression (79) and $[H]$ is the diagonal matrix of order $m_H n \times m_H n$ with the element

$$\frac{l^2}{sc(y_p^{(m)})} \frac{1}{H_i^{(n)} G_p^{(m)}} k_p^{(m_H)}(\eta_p^{(m)}) \sqrt{\frac{1 - \xi_i^{(n)}}{\xi_i^{(n)}}} \sqrt{1 - [\eta_p^{(m)}]^2}, \qquad i = 1, 2, \ldots, n; p = 1, 2, \ldots, m_H, \tag{94}$$

in the $n(m_H - p) + i$th row and column.

From equation (91) for $i = 1, 2, \ldots, n$; $p = 1, 2, \ldots, m_H$, we then deduce the matrix equation

$$[\hat{l}_k] = \tfrac{1}{2}[H][E][\Psi]^{-1}[D][\alpha_k]. \tag{95}$$

If we solve equation (95) for $[\Psi]^{-1}[D][\alpha_k]$ and substitute the result into formula (89) we get

$$[\hat{l}_k(x_0, y_0)] = [F(x_0, y_0)][E]^{-1}[H]^{-1}[\hat{l}_k] \tag{96}$$

which is an alternative formula to formula (89) for determining $[\hat{l}_k(x_0, y_0)]$. Formula (96) is useful if one computer program is constructed to evaluate the $m_H n$ values $\hat{l}_k(x_{i,p}^{(n,m)}, y_p^{(m)})$, $i = 1, 2, \ldots, n$; $p = 1, 2, \ldots, m_H$, for each value of k from equation (95) and a second computer program is constructed to evaluate $\hat{l}_k(x_0, y_0)$ at given values of x_0 and y_0 using these $m_H n$ values $\hat{l}_k(x_{i,p}^{(n,m)}, y_p^{(m)})$, $i = 1, 2, \ldots, n$; $p = 1, 2, \ldots, m_H$, for each value of k in formula (96).

We notice that the matrix product $[E][\Psi]^{-1}[D]$ occurs both in formula (84) for the generalised airforce coefficients and in formula (95) for the loading at the loading points. It was to achieve this that the product formula (93) for $[F]$ was introduced.

If $[\hat{l}]$ is the matrix of order $m_H n \times P$ obtained by arranging the column matrices $[\hat{l}_k]$, $k = 1, 2, \ldots, P$, sequentially one alongside another, and if $[\hat{l}(x_0, y_0)]$ is the row matrix of P elements obtained by arranging the elements $\hat{l}_k(x_0, y_0)$, $k = 1, 2, \ldots, P$, sequentially one alongside another, then we get immediately from formulae (95) and (96) the formulae

$$[\hat{l}] = \tfrac{1}{2}[H][E][\Psi]^{-1}[D][\alpha] \tag{97}$$

and

$$[\hat{l}(x_0, y_0)] = [F(x_0, y_0)][E]^{-1}[H]^{-1}[\hat{l}]. \tag{98}$$

3. Numerical Integration

The quantities $\psi_{i,p;r,s}$ defined in formulae (37) as double integrals, are to be evaluated numerically, and we do this by evaluating the double integrals using the Gaussian numerical integration formula (A-16) over N chordwise points ξ in (0, 1) and the Gaussian numerical integration formula (A-35) over M spanwise points η in $(-1, 1)$. This process leads to the formula

$$\psi_{i,p;r,s} = \sum_{I=1}^{N} \sum_{J=1}^{M} H_I^{(N)} G_J^{(M)} h_i^{(n)}(\xi_I^{(N)}) g_p^{(m)}(\eta_J^{(M)}) U_{r,s}(\bar{x}_{I,J}^{(N,M)}, y_J^{(M)}; \nu, M_\infty),$$

$$i=1,2,\ldots,n;\ p=1,2,\ldots,m,\ r=1,2,\ldots,n;\ s=1,2,\ldots,m, \qquad (99)$$

where (see Appendix A)

$$\xi_I^{(N)} = \frac{1}{2}\left[1-\cos\left(\frac{2I-1}{2N+1}\pi\right)\right], \qquad I=1,2,\ldots,N, \qquad (100)$$

$$\eta_J^{(M)} = \cos\left(\frac{J\pi}{M+1}\right), \qquad J=1,2,\ldots,M, \qquad (101)$$

$$H_I^{(N)} = \int_0^1 h_I^{(N)}(\xi)\sqrt{\frac{1-\xi}{\xi}}\,d\xi$$

$$= \frac{2\pi}{2N+1}(1-\xi_I^{(N)}), \qquad I=1,2,\ldots,N, \qquad (102)$$

$$G_J^{(M)} = \int_{-1}^{+1} g_J^{(M)}(\eta)\sqrt{1-\eta^2}\,d\eta$$

$$= \frac{\pi}{M+1}[1-(\eta_J^{(M)})^2], \qquad J=1,2,\ldots,M, \qquad (103)$$

$$\bar{x}_{I,J}^{(N,M)} = c(y_J^{(M)})(1-\xi_I^{(N)}) + x_L(y_J^{(M)}), \qquad I=1,2,\ldots,N;\ J=1,2,\ldots,M, \qquad (104)$$

and

$$y_J^{(M)} = s\eta_J^{(M)}. \qquad (105)$$

We define the integer M_H by means of the formula

$$M_H = \tfrac{1}{4}\{2M+\kappa-(-1)^M\kappa\}. \qquad (106)$$

Then, if we use the relations (41), (42) and (43) in (99), and use the definitions (54) we get

$$\Psi_{i,p;r,s} = \tfrac{1}{2}\sum_{I=1}^{N}\sum_{J=1}^{M_H} H_I^{(N)} G_J^{(M)} h_i^{(n)}(\xi_I^{(N)}) L_{p,J} \times$$

$$\times\{U_{r,s}(\bar{x}_{I,J}^{(N,M)}, y_J^{(M)}; \nu, M_\infty) + \kappa U_{r,m-s+1}(\bar{x}_{I,J}^{(N,M)}, y_J^{(M)}; \nu, M_\infty)\},$$

$$i=1,2,\ldots,n;\ p=1,2,\ldots,m_H,\ r=1,2,\ldots,n;\ s=1,2,\ldots,m_H, \qquad (107)$$

where

$$L_{p,J} = g_p^{(m)}(\eta_J^{(M)}) + \kappa g_p^{(m)}(-\eta_J^{(M)}) \qquad J<M_H \qquad (108)$$

$$L_{p,M_H} = \begin{cases} g_p^{(m)}(\eta_{M_H}^{(M)}) + \kappa g_p^{(m)}(-\eta_{M_H}^{(M)}) & M \text{ even} \\ g_p^{(m)}(\eta_{M_H}^{(M)}) - g_p^{(m)}(-\eta_{M_H}^{(M)}) & M \text{ odd}, \kappa=-1 \\ g_p^{(m)}(0) & M \text{ odd}, \kappa=+1. \end{cases} \qquad (109)$$

We may now write for the matrix $[\Psi]$, appearing in formula (55),

$$[\Psi]=[L][U] \tag{110}$$

where $[L]$ is the matrix of order $m_H n \times M_H N$ with the element

$$\tfrac{1}{2}H_I^{(N)}G_J^{(M)}h_i^{(n)}(\xi_I^{(N)})L_{p,j}, \qquad i=1,2,\dots,n;\, p=1,2,\dots,m_N,\, I=1,2,\dots,N;\, J=1,2,\dots,M_H, \tag{111}$$

in the $n(m_H-p)+n-i+1$st row and $N(M_H-J)+N-I+1$st column, and $[U]$ is the matrix of order $M_HN\times m_Hn$ with the element

$$U_{r,s}(\bar{x}_{I,J}^{(N,M)}, y_J^{(M)}; \nu, M_\infty)+\kappa U_{r,m-s+1}(\bar{x}_{I,J}^{(N,M)}, y_J^{(M)}; \nu, M_\infty), \qquad I=1,2,\dots,N;\, J=1,2,\dots,M_H,\; r=1,2,\dots,n;\, s=1,2,\dots,m_H, \tag{112}$$

in the $N(M_H-J)+N-I+1$st row and $n(m_H-s)+r$th column.

We note that the matrix $[\Psi]$ given by (110) is non-singular only if $M_HN\geqslant m_Hn$. We shall take $M_H\geqslant m_H$ and $N\geqslant n$ so that this condition is satisfied and then $[\Psi]$ can be inverted to give $[\Psi]^{-1}$ required in formula (65).

Now, from (32), we get

$$U_{r,s}(x, y; \nu, M_\infty)=\frac{l^2}{s^2}\int_{-1}^{+1}\frac{g_s^{(m)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta-\eta_0)^2}I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)\,d\eta_0 \tag{113}$$

where

$$I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)=\frac{1}{4\pi}\left(\frac{y-y_0}{l}\right)^2\int_0^1 h_r^{(n)}(\xi_0)\sqrt{\frac{1-\xi_0}{\xi_0}}K\left(\frac{x-x_0}{l}, \frac{y-y_0}{l}; \nu, M_\infty\right)d\xi_0. \tag{114}$$

To evaluate $U_{r,s}(x, y; \nu, M_\infty)$ from (113) we write, following Ref. 12,

$$\begin{aligned} U_{r,s}(x, y; \nu, M_\infty)=&\frac{l^2}{s^2}\int_{-1}^{+1}\frac{g_s^{(m)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta-\eta_0)^2}\times\\ &\times\left\{I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)-I_r^{(n)}(\xi, \eta, \eta; \nu, M_\infty)-(\eta_0-\eta)\left[\frac{\partial}{\partial\eta_0}I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)\right]_{\eta_0=\eta}\right\}d\eta_0+\\ &+\frac{l^2}{s^2}I_r^{(n)}(\xi, \eta, \eta; \nu, M_\infty)\int_{-1}^{+1}\frac{g_s^{(m)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta-\eta_0)^2}d\eta_0-\\ &-\frac{l^2}{s^2}\left[\frac{\partial}{\partial\eta_0}I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)\right]_{\eta_0=\eta}\int_{-1}^{+1}\frac{g_s^{(m)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta-\eta_0)}d\eta_0. \end{aligned} \tag{115}$$

Now

$$\frac{1}{(\eta-\eta_0)^2}\left\{I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)-I_r^{(n)}(\xi, \eta, \eta; \nu, M_\infty)-(\eta_0-\eta)\left[\frac{\partial}{\partial\eta_0}I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)\right]_{\eta_0=\eta}\right\} \tag{116}$$

becomes logarithmically infinite when $\eta_0=\eta$, so that a numerical integration process for evaluating directly the first integral on the right hand side of formula (115) must not have an integration point at $\eta_0=\eta$. Also it is not easy to estimate the accuracy of such a numerical integration. We can replace the function $I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)$ in the first integral on the right hand side of formula (115) by a function $\hat{I}_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)$ so that numerical integration of the resulting integral is straightforward using a Gaussian numerical integration process and analytical integration of the difference between the two integrals can be carried out. This is achieved by using the known analytical behaviour of $I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)$ for η_0 near to η (see Ref. 5, Appendix 4).

We introduce the functions $\hat{I}_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)$ by means of the formulae

$$I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty) = \hat{I}_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty) + F_r^{(n)}(\xi, \eta; \nu, M_\infty)(\eta-\eta_0)^2 \log|\eta-\eta_0|, \qquad r=1, 2, \ldots, n, \quad (117)$$

where

$$F_r^{(n)}(\xi, \eta; \nu, M_\infty) = \frac{1}{4\pi}\left(\frac{s}{c(y)}\right)^2\left\{-(1-M_\infty^2)\frac{d}{d\xi}\left(h_r^{(n)}(\xi)\sqrt{\frac{1-\xi}{\xi}}\right) + 2i\nu\frac{c(y)}{l}h_r^{(n)}(\xi)\sqrt{\frac{1-\xi}{\xi}} - \right.$$
$$\left. -(i\nu)^2\frac{c^2(y)}{l^2}\int_0^\xi h_r^{(n)}(\xi_0)\sqrt{\frac{1-\xi_0}{\xi_0}}\, d\xi_0\right\}, \qquad r=1, 2, \ldots, n. \quad (118)$$

The lowest order logarithmic contribution to $I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)$ is then completely separated as the second term on the right hand side of (117). It follows that the function

$$\frac{1}{(\eta-\eta_0)^2}\left\{\hat{I}_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty) - I_r^{(n)}(\xi, \eta, \eta; \nu, M_\infty) - (\eta_0-\eta)\left[\frac{\partial}{\partial\eta_0}I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)\right]_{\eta_0=\eta}\right\},$$
$$r=1, 2, \ldots, n \quad (119)$$

is finite when $\eta_0=\eta$ if $x_L''(y)$ and $c''(y)$ are finite. The function (119) is then finite for all η_0 in $(-1, 1)$ and therefore is straightforward to deal with as part of an integrand of an integral which is to be evaluated using a Gaussian numerical integration process.

If we substitute for $I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)$ from (117) into the first integral on the right hand side of (115) we obtain

$$U_{r,s}(x, y; \nu, M_\infty) = \frac{l^2}{s^2}\int_{-1}^{+1}\frac{g_s^{(m)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta-\eta_0)^2}\left\{\hat{I}_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty) - I_r^{(n)}(\xi, \eta, \eta; \nu, M_\infty) - \right.$$
$$\left. -(\eta_0-\eta)\left[\frac{\partial}{\partial\eta_0}I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)\right]_{\eta_0=\eta}\right\} d\eta_0 +$$
$$+\frac{l^2}{s^2}F_r^{(n)}(\xi, \eta; \nu, M_\infty)\int_{-1}^{+1} g_s^{(m)}(\eta_0)\log|\eta-\eta_0|\sqrt{1-\eta_0^2}\, d\eta_0 +$$
$$+\frac{l^2}{s^2}I_r^{(n)}(\xi, \eta, \eta; \nu, M_\infty)\int_{-1}^{+1}\frac{g_s^{(m)}(\eta_0)}{(\eta-\eta_0)}\sqrt{1-\eta_0^2}\, d\eta_0 -$$
$$-\frac{l^2}{s^2}\left[\frac{\partial}{\partial\eta_0}I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)\right]_{\eta_0=\eta}\int_{-1}^{+1}\frac{g_s^{(m)}(\eta_0)}{(\eta-\eta_0)}\sqrt{1-\eta_0^2}\, d\eta_0. \quad (120)$$

It is observed that the third and fourth integrals on the right hand side of (120) are principal value singular integrals.

The first integral on the right hand side of (120) may be evaluated numerically by means of a Gaussian numerical integration. The obvious formula is the one with $\sqrt{1-\eta_0^2}$ as weight function, but we shall find it necessary to use also other weight functions in order to justify the final form (156) for $U_{r,s}(x, y_J^{(M)}; \nu, M_\infty)$ in all circumstances. These weight functions are $\sqrt{(1-\eta_0)/(1+\eta_0)}$, $\sqrt{(1+\eta_0)/(1-\eta_0)}$ and $1/\sqrt{1-\eta_0^2}$. If we use Λ integration points, the application of the Gaussian numerical integration corresponding to each of these weight functions gives the result

$$\int_{-1}^{+1}\frac{g_s^{(m)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta-\eta_0)^2}\left\{\hat{I}_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty) - I_r^{(n)}(\xi, \eta, \eta; \nu, M_\infty) - (\eta_o-\eta)\left[\frac{\partial}{\partial\eta_0}I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)\right]_{\eta_0=\eta}\right\} d\eta_0$$
$$= \frac{\pi}{P_\Lambda}\sum_{p=1}^{\Lambda}\frac{g_s^{(m)}(\zeta_p^{(\Lambda)})\{1-(\zeta_p^{(\Lambda)})^2\}}{(\eta-\zeta_p^{(\Lambda)})^2}\left\{\hat{I}_r^{(n)}(\xi, \eta, \zeta_p^{(\Lambda)}; \nu, M_\infty) - \right.$$
$$\left. -I_r^{(n)}(\xi, \eta, \eta; \nu, M_\infty) - (\zeta_p^{(\Lambda)}-\eta)\left[\frac{\partial}{\partial\eta_0}I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)\right]_{\eta_0=\eta}\right\}. \quad (121)$$

The values of P_Λ and $\zeta_p^{(\Lambda)}$, $p=1,2,\ldots,\Lambda$, for the different weight functions may be obtained from equations (A-86), (A-88), (A-91) and (A-94) of Appendix A.

For the weight function $\sqrt{1-\eta_0^2}$ these values are

$$P_\Lambda=(\Lambda+1) \tag{122}$$

and

$$\begin{aligned}\zeta_p^{(\Lambda)}&=\eta_p^{(\Lambda)}\\ &=\cos\left(\frac{p\pi}{\Lambda+1}\right), \qquad p=1,2,\ldots,\Lambda.\end{aligned} \tag{123}$$

For the weight function $\sqrt{(1-\eta_0)/(1+\eta_0)}$ these values are

$$P_\Lambda=(\Lambda+\tfrac{1}{2}) \tag{124}$$

and

$$\begin{aligned}\zeta_p^{(\Lambda)}&=\eta_p^{(2\Lambda)}\\ &=\cos\left(\frac{2p\pi}{2\Lambda+1}\right), \qquad p=1,2,\ldots,\Lambda.\end{aligned} \tag{125}$$

For the weight function $\sqrt{(1+\eta_0)/(1-\eta_0)}$ these values are

$$P_\Lambda=(\Lambda+\tfrac{1}{2}) \tag{126}$$

and

$$\begin{aligned}\zeta_p^{(\Lambda)}&=\eta_{2p-1}^{(2\Lambda)}\\ &=\cos\left(\frac{2p-1}{2\Lambda+1}\pi\right), \qquad p=1,2,\ldots,\Lambda.\end{aligned} \tag{127}$$

For the weight function $1/\sqrt{1-\eta_0^2}$ these values are

$$P_\Lambda=\Lambda \tag{128}$$

and

$$\begin{aligned}\zeta_p^{(\Lambda)}&=\eta_{2p-1}^{(2\Lambda-1)}\\ &=\cos\left(\frac{2p-1}{2\Lambda}\pi\right), \qquad p=1,2,\ldots,\Lambda.\end{aligned} \tag{129}$$

Formula (121) is, in general, only an approximate formula but for the weight function $\sqrt{1-\eta_0^2}$ it would be exact if the function (119) were a polynomial of degree $\leqslant 2\Lambda-m$ in η_0. For the weight functions $\sqrt{(1-\eta_0)/(1+\eta_0)}$ and $\sqrt{(1+\eta_0)/(1-\eta_0)}$ it would be exact if the function (119) were a polynomial of degree $\leqslant 2\Lambda-m-1$ in η_0 and for the weight function $1/\sqrt{1-\eta_0^2}$ it would be exact if the function (119) were a polynomial of degree $\leqslant 2\Lambda-m-2$ in η_0. However, since the integration weights are positive numbers and the function (119) is finite for all η_0 in $(-1,1)$, the right hand side of (121) will converge to the exact value of the integral on the left hand side of (121) as Λ tends to infinity. We can take Λ to be large enough for formula (121) to be sufficiently accurate for all practical purposes.

We note that the form of the right hand side of (121) can be used only if

$$\eta\neq\zeta_p^{(\Lambda)}, \qquad p=1,2,\ldots,\Lambda, \tag{130}$$

but if this is not the case the right hand side of (121) is easily modified and the resulting formula involves the second derivative $(\partial^2/\partial\eta_0^2)\hat{I}_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)$ for $\eta_0=\zeta_p^{(\Lambda)}$ for the appropriate value of p.

We can further write

$$\int_{-1}^{+1} g_s^{(m)}(\eta_0)\log|\eta-\eta_0|\sqrt{1-\eta_0^2}\,d\eta_0$$

$$=\int_{-1}^{+1}\{g_s^{(m)}(\eta_0)-g_s^{(m)}(\eta)\}\log|\eta-\eta_0|\sqrt{1-\eta_0^2}\,d\eta_0+g_s^{(m)}(\eta)\int_{-1}^{+1}\log|\eta-\eta_0|\sqrt{1-\eta_0^2}\,d\eta_0$$

$$=\frac{\pi}{P_\Lambda}\sum_{p=1}^{\Lambda}\{g_s^{(m)}(\zeta_p^{(\Lambda)})-g_s^{(m)}(\eta)\}\log|\eta-\zeta_p^{(\Lambda)}|\{1-(\zeta_p^{(\Lambda)})^2\}+\frac{\pi}{2}g_s^{(m)}(\eta)\{\eta^2-\tfrac{1}{2}-\log 2\} \quad (131)$$

where one of the above Gaussian integration formulae with Λ integration points has been used to give

$$\int_{-1}^{+1}\{g_s^{(m)}(\eta_0)-g_s^{(m)}(\eta)\}\log|\eta-\eta_0|\sqrt{1-\eta_0^2}\,d\eta_0$$

$$=\frac{\pi}{P_\Lambda}\sum_{p=1}^{\Lambda}\{g_s^{(m)}(\zeta_p^{(\Lambda)})-g_s^{(m)}(\eta)\}\log|\eta-\zeta_p^{(\Lambda)}|\{1-(\zeta_p^{(\Lambda)})^2\}. \quad (132)$$

The formula (132) is only approximate but since the integration weights are positive numbers and the function $\{g_s^{(m)}(\eta_0)-g_s^{(m)}(\eta)\}\log|\eta-\eta_0|$ is finite for all η_0 in $(-1, 1)$ the right hand side of (132) will converge to the exact value of the integral on the left hand side of (132) as Λ tends to infinity. We can take Λ to be large enough for the formula (132) to be sufficiently accurate for all practical purposes. Actually, the integral on the left hand side of (131) can be evaluated analytically, but it is simpler to get its value from the expression on the right hand side of (131), and then the accuracy is consistent with the accuracy of formula (121).

On substituting from (121) and (131) into (120) we get

$$U_{r,s}(x, y; \nu, M_\infty)=\frac{\pi}{P_\Lambda}\frac{l^2}{s^2}\sum_{p=1}^{\Lambda}\frac{g_s^{(m)}(\zeta_p^{(\Lambda)})\{1-(\zeta_p^{(\Lambda)})^2\}}{(\eta-\zeta_p^{(\Lambda)})^2}\times$$

$$\times[\hat{I}_r^{(n)}(\xi, \eta, \zeta_p^{(\Lambda)}; \nu, M_\infty)+F_r^{(n)}(\xi, \eta; \nu, M_\infty)(\eta-\zeta_p^{(\Lambda)})^2\log|\eta-\zeta_p^{(\Lambda)}|]+$$

$$+\frac{l^2}{s^2}I_r^{(n)}(\xi, \eta, \eta; \nu, M_\infty)\left[\int_{-1}^{+1}\frac{g_s^{(m)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta-\eta_0)^2}d\eta_0-\frac{\pi}{P_\Lambda}\sum_{p=1}^{\Lambda}\frac{g_s^{(m)}(\zeta_p^{(\Lambda)})\{1-(\zeta_p^{(\Lambda)})^2\}}{(\eta-\zeta_p^{(\Lambda)})^2}\right]-$$

$$-\frac{l^2}{s^2}\left[\frac{\partial}{\partial\eta_0}I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)\right]_{\eta_0=\eta}\times$$

$$\times\left[\int_{-1}^{+1}\frac{g_s^{(m)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta-\eta_0)}d\eta_0-\frac{\pi}{P_\Lambda}\sum_{p=1}^{\Lambda}\frac{g_s^{(m)}(\zeta_p^{(\Lambda)})\{1-(\zeta_p^{(\Lambda)})^2\}}{(\eta-\zeta_p^{(\Lambda)})}\right]+$$

$$+\frac{l^2}{s^2}F_r^{(n)}(\xi, \eta; \nu, M_\infty)g_s^{(m)}(\eta)\left[\frac{\pi}{2}(\eta^2-\tfrac{1}{2}-\log 2)-\frac{\pi}{P_\Lambda}\sum_{p=1}^{\Lambda}\log|\eta-\zeta_p^{(\Lambda)}|\{1-(\zeta_p^{(\Lambda)})^2\}\right]$$

$$=\frac{\pi}{P_\Lambda}\frac{l^2}{s^2}\sum_{p=1}^{\Lambda}\frac{g_s^{(m)}(\zeta_p^{(\Lambda)})\{1-(\zeta_p^{(\Lambda)})^2\}}{(\eta-\zeta_p^{(\Lambda)})^2}I_r^{(n)}(\xi, \eta, \zeta_p^{(\Lambda)}; \nu, M_\infty)+$$

$$+\frac{l^2}{s^2}I_r^{(n)}(\xi, \eta, \eta; \nu, M_\infty)\left[\int_{-1}^{+1}\frac{g_s^{(m)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta-\eta_0)^2}d\eta_0-\frac{\pi}{P_\Lambda}\sum_{p=1}^{\Lambda}\frac{g_s^{(m)}(\zeta_p^{(\Lambda)})\{1-(\zeta_p^{(\Lambda)})^2\}}{(\eta-\zeta_p^{(\Lambda)})^2}\right]-$$

$$-\frac{l^2}{s^2}\left[\frac{\partial}{\partial\eta_0}I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)\right]_{\eta_0=\eta}\left[\int_{-1}^{+1}\frac{g_s^{(m)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta-\eta_0)}d\eta_0-\frac{\pi}{P_\Lambda}\sum_{p=1}^{\Lambda}\frac{g_s^{(m)}(\zeta_p^{(\Lambda)})\{1-(\zeta_p^{(\Lambda)})^2\}}{(\eta-\zeta_p^{(\Lambda)})}\right]$$

$$+\frac{l^2}{s^2}F_r^{(n)}(\xi, \eta; \nu, M_\infty)g_s^{(m)}(\eta)\left[\frac{\pi}{2}(\eta^2-\tfrac{1}{2}-\log 2)-\frac{\pi}{P_\Lambda}\sum_{p=1}^{\Lambda}\log|\eta-\zeta_p^{(\Lambda)}|\{1-(\zeta_p^{(\Lambda)})^2\}\right]. \quad (133)$$

Again,

$$\int_{-1}^{+1} \frac{g_s^{(m)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta-\eta_0)^2}\,d\eta_0$$

$$= \int_{-1}^{+1} \frac{\{g_s^{(m)}(\eta_0)-g_s^{(m)}(\eta)-(\eta_0-\eta)g_s^{(m)\prime}(\eta)\}}{(\eta-\eta_0)^2}\sqrt{1-\eta_0^2}\,d\eta_0+$$

$$+g_s^{(m)}(\eta)\int_{-1}^{+1}\frac{\sqrt{1-\eta_0^2}}{(\eta-\eta_0)^2}\,d\eta_0-g_s^{(m)\prime}(\eta)\int_{-1}^{+1}\frac{\sqrt{1-\eta_0^2}}{(\eta-\eta_0)}\,d\eta_0$$

$$=\frac{\pi}{P_\Lambda}\sum_{p=1}^{\Lambda}\frac{\{g_s^{(m)}(\zeta_p^{(\Lambda)})-g_s^{(m)}(\eta)-(\zeta_p^{(\Lambda)}-\eta)g_s^{(m)\prime}(\eta)\}}{(\eta-\zeta_p^{(\Lambda)})^2}\{1-(\zeta_p^{(\Lambda)})^2\}-$$

$$-\pi g_s^{(m)}(\eta)-\pi\eta g_s^{(m)\prime}(\eta) \tag{134}$$

where the principal value singular integrals $\int_{-1}^{+1}\sqrt{(1-\eta_0^2)}/(\eta-\eta_0)^2\,d\eta_0$ and $\int_{-1}^{+1}\sqrt{(1-\eta_0^2)}/(\eta-\eta_0)\,d\eta_0$ have been replaced by their respective values $-\pi$ and $\pi\eta$. Similarly

$$\int_{-1}^{+1}\frac{g_s^{(m)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta-\eta_0)}\,d\eta_0$$

$$=\int_{-1}^{+1}\frac{\{g_s^{(m)}(\eta_0)-g_s^{(m)}(\eta)\}}{(\eta-\eta_0)}\sqrt{1-\eta_0^2}\,d\eta_0+g_s^{(m)}(\eta)\int_{-1}^{+1}\frac{\sqrt{1-\eta_0^2}}{(\eta-\eta_0)}\,d\eta_0$$

$$=\frac{\pi}{P_\Lambda}\sum_{p=1}^{\Lambda}\frac{\{g_s^{(m)}(\zeta_p^{(\Lambda)})-g_s^{(m)}(\eta)\}}{(\eta-\zeta_p^{(\Lambda)})}\{1-(\zeta_p^{(\Lambda)})^2\}+\pi\eta g_s^{(m)}(\eta). \tag{135}$$

In equations (134) and (135) one of the above Gaussian integration formulae with Λ integration points has been used to give

$$\int_{-1}^{+1}\frac{\{g_s^{(m)}(\eta_0)-g_s^{(m)}(\eta)-(\eta_0-\eta)g_s^{(m)\prime}(\eta)\}}{(\eta-\eta_0)^2}\sqrt{1-\eta_0^2}\,d\eta_0$$

$$=\frac{\pi}{P_\Lambda}\sum_{p=1}^{\Lambda}\frac{\{g_s^{(m)}(\zeta_p^{(\Lambda)})-g_s^{(m)}(\eta)-(\zeta_p^{(\Lambda)}-\eta)g_s^{(m)\prime}(\eta)\}}{(\eta-\zeta_p^{(\Lambda)})^2}\{1-(\zeta_p^{(\Lambda)})^2\} \tag{136}$$

and

$$\int_{-1}^{+1}\frac{\{g_s^{(m)}(\eta_0)-g_s^{(m)}(\eta)\}}{(\eta-\eta_0)}\sqrt{1-\eta_0^2}\,d\eta_0$$

$$=\frac{\pi}{P_\Lambda}\sum_{p=1}^{\Lambda}\frac{\{g_s^{(m)}(\zeta_p^{(\Lambda)})-g_s^{(m)}(\eta)\}}{\eta-\zeta_p^{(\Lambda)}}\{1-(\zeta_p^{(\Lambda)})^2\}. \tag{137}$$

The function $g_s^{(m)\prime}(\eta)$ is the derivative with respect to η of $g_s^{(m)}(\eta)$. The formula (136) is exact if

$$\Lambda\geqslant\tfrac{1}{2}m-1 \tag{138}$$

for weight function $\sqrt{1-\eta_0^2}$, if

$$\Lambda\geqslant\tfrac{1}{2}m-\tfrac{1}{2} \tag{139}$$

for weight functions $\sqrt{(1-\eta_0)/(1+\eta_0)}$ and $\sqrt{(1+\eta_0)/(1-\eta_0)}$, and if

$$\Lambda\geqslant\tfrac{1}{2}m \tag{140}$$

for weight function $1/\sqrt{1-\eta_0^2}$.

The formula (137) is exact if

$$\Lambda \geq \tfrac{1}{2}m - \tfrac{1}{2} \tag{141}$$

for weight function $\sqrt{1-\eta_0^2}$, if

$$\Lambda \geq \tfrac{1}{2}m \tag{142}$$

for weight functions $\sqrt{(1-\eta_0)/(1+\eta_0)}$ and $\sqrt{(1+\eta_0)/(1-\eta_0)}$, and if

$$\Lambda \geq \tfrac{1}{2}m + \tfrac{1}{2} \tag{143}$$

for weight function $1/\sqrt{1-\eta_0^2}$.

Hence, if one of the conditions (141), (142), (143) holds, as appropriate to the weight function used, then both formulae (134) and (135) are exact and we can substitute them into the right hand side of (133) to get an alternative formula which is precisely equivalent to (133),

$$\begin{aligned}
U_{r,s}(x, y; \nu, M_\infty) = {} & \frac{\pi}{P_\Lambda}\frac{l^2}{s^2}\sum_{p=1}^{\Lambda} \frac{g_s^{(m)}(\zeta_p^{(\Lambda)})\{1-(\zeta_p^{(\Lambda)})^2\}}{(\eta-\zeta_p^{(\Lambda)})^2} I_r^{(n)}(\xi, \eta, \zeta_p^{(\Lambda)}; \nu, M_\infty) - \\
& - \pi\frac{l^2}{s^2} I_r^{(n)}(\zeta, \eta, \eta; \nu, M_\infty) g_s^{(m)}(\eta)\left[1 + \frac{1}{P_\Lambda}\sum_{p=1}^{\Lambda} \frac{\{1-(\zeta_p^{(\Lambda)})^2\}}{(\eta-\zeta_p^{(\Lambda)})^2}\right] - \\
& - \pi\frac{l^2}{s^2}\left[\frac{\partial}{\partial\eta_0}\{I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty) g_s^{(m)}(\eta_0)\}\right]_{\eta_0=\eta} \times \left[\eta - \frac{1}{P_\Lambda}\sum_{p=1}^{\Lambda} \frac{\{1-(\zeta_p^{(\Lambda)})^2\}}{(\eta-\zeta_p^{(\Lambda)})}\right] + \\
& + \pi\frac{l^2}{s^2} F_r^{(n)}(\xi, \eta; \nu, M_\infty) g_s^{(m)}(\eta) \times \\
& \times\left[\tfrac{1}{2}(\eta^2 - \tfrac{1}{2} - \log 2) - \frac{1}{P_\Lambda}\sum_{p=1}^{\Lambda} \log|\eta - \zeta_p^{(\Lambda)}| \times \{1-(\zeta_p^{(\Lambda)})^2\}\right]
\end{aligned} \tag{144}$$

and this is the numerical integration formula that we use for evaluating $U_{r,s}(x, y; \nu, M)$. However, we need the values $U_{r,s}(x, y; \nu, M)$ only for

$$y = y_J^{(M)}, \qquad J = 1, 2, \ldots, M, \tag{145}$$

in order to evaluate the elements (112) of the matrix $[U]$, and these values of y correspond to

$$\eta = \eta_J^{(M)}, \qquad J = 1, 2, \ldots, M. \tag{146}$$

With the values of η given by (146) a simplification of the expression (144) occurs with certain special values of Λ, these special values of Λ being specific to the weight function $\sqrt{1-\eta_0^2}$, $\sqrt{(1-\eta_0)/(1+\eta_0)}$, $\sqrt{(1+\eta_0)/(1-\eta_0)}$ or $1/\sqrt{1-\eta_0^2}$ being used. In fact we have (see Appendix B)

$$\eta_J^{(M)} - \frac{1}{P_\Lambda}\sum_{p=1}^{\Lambda} \frac{\{1-(\zeta_p^{(\Lambda)})^2\}}{(\eta_J^{(M)}-\zeta_p^{(\Lambda)})} = 0 \tag{147}$$

$$1 + \frac{1}{P_\Lambda}\sum_{p=1}^{\Lambda} \frac{\{1-(\zeta_p^{(\Lambda)})^2\}}{(\eta_J^{(M)}-\zeta_p^{(\Lambda)})^2} = P_\Lambda \tag{148}$$

n the following five cases:

Case (i) Any positive integer M, any positive integer J, $1 \leq J \leq M$, and weight function $1/\sqrt{1-\eta_0^2}$.

Values of Λ are given by

$$\Lambda = a(M+1) \tag{149}$$

where a is any positive integer.

Case (ii) Any odd positive integer M, any even positive integer J, $2 \leqslant J \leqslant M-1$, and weight function $1/\sqrt{1-\eta_0^2}$.
Values of Λ are given by

$$\Lambda = \tfrac{1}{2}(2a-1)(M+1) \tag{150}$$

where a is any positive integer.

Case (iii) Any even positive integer M, any even positive integer J, $2 \leqslant J \leqslant M$, and weight function $\sqrt{1+\eta_0/1-\eta_0}$.
Values of Λ are given by

$$\Lambda = \tfrac{1}{2}(2a-1)(M+1) - \tfrac{1}{2} \tag{151}$$

where a is any positive integer.

Case (iv) Any even positive integer M, any odd positive integer J, $1 \leqslant J \leqslant M-1$, and weight function $\sqrt{1-\eta_0/1+\eta_0}$.
Values of Λ are given by

$$\Lambda = \tfrac{1}{2}(2a-1)(M+1) - \tfrac{1}{2} \tag{152}$$

where a is any positive integer.

Case (v) Any odd positive integer M, any odd positive integer J, $1 \leqslant J \leqslant M$, and weight function $\sqrt{1-\eta_0^2}$.
Values of Λ are given by

$$\Lambda = \tfrac{1}{2}(2a-1)(M+1) - 1 \tag{153}$$

where a is any positive integer.

If we substitute from (147) and (148) into (144) for the values of y given by (145) we get

$$\begin{aligned} U_{r,s}(x, y_J^{(M)}; \nu, M_\infty) &= \frac{\pi}{P_\Lambda} \frac{l^2}{s^2} \sum_{p=1}^{\Lambda} \frac{g_s^{(m)}(\zeta_p^{(\Lambda)})\{1-(\zeta_p^{(\Lambda)})^2\}}{(\eta_J^{(M)} - \zeta_p^{(\Lambda)})^2} I_r^{(n)}(\xi, \eta_J^{(M)}, \zeta_p^{(\Lambda)}; \nu, M_\infty) - \\ &\quad -\pi P_\Lambda \frac{l^2}{s^2} I_r^{(n)}(\xi, \eta_J^{(M)}, \eta_J^{(M)}; \nu, M_\infty) g_s^{(m)}(\eta_J^{(M)}) + \pi \frac{l^2}{s^2} F_r^{(n)}(\xi, \eta_J^{(M)}; \nu, M_\infty) g_s^{(m)}(\eta_J^{(M)}) \times \\ &\quad \times \left[\tfrac{1}{2}((\eta_J^{(M)})^2 - \tfrac{1}{2} - \log 2) - \frac{1}{P_\Lambda} \sum_{p=1}^{\Lambda} \log |\eta_J^{(M)} - \zeta_p^{(\Lambda)}| \times \{1-(\zeta_p^{(\Lambda)})^2\} \right]. \end{aligned} \tag{154}$$

It is to be noted that the quantity $[(\partial/\partial\eta_0)\{I_r^{(n)}(\xi, \eta_J^{(M)}, \eta_0; \nu, M_\infty) g_s^{(m)}(\eta_0)\}]_{\eta_0=\eta_J^{(M)}}$ does not occur in formula (154).

Formula (154) is valid for the following cases:

Case (i) Any positive integer M and any positive integer J, $1 \leqslant J \leqslant M$, when Λ is given by (149), P_Λ is given by (128) and $\zeta_p^{(\Lambda)}$, $p = 1, 2, \ldots, \Lambda$, are given by (129).

Case (ii) Any odd positive integer M and any even positive integer J, $2 \leqslant J \leqslant M-1$, when Λ is given by (150), P_Λ is given by (128) and $\zeta_p^{(\Lambda)}$, $p = 1, 2, \ldots, \Lambda$, are given by (129).

Case (iii) Any even positive integer M and any even positive integer J, $2 \leqslant J \leqslant M$, when Λ is given by (151), P_Λ is given by (126) and $\zeta_p^{(\Lambda)}$, $p = 1, 2, \ldots, \Lambda$, are given by (127).

Case (iv) Any even positive integer M and any odd positive integer J, $1 \leqslant J \leqslant M-1$, when Λ is given by (152), P_Λ is given by (124) and $\zeta_p^{(\Lambda)}$, $p = 1, 2, \ldots, \Lambda$, are given by (125).

Case (v) Any odd positive integer M and any odd positive integer J, $1 \leqslant J \leqslant M$, when Λ is given by (153), P_Λ is given by (122) and $\zeta_p^{(\Lambda)}$, $p = 1, 2, \ldots, \Lambda$, are given by (123).

By using formula (A-55) we can put the result (154) into yet a different form, in which we do not have to distinguish between the cases (i), (ii), (iii), (iv) and (v). If we put

$$\bar{M} = q(M+1) - 1 \tag{155}$$

then the result (154) can be expressed in the form

$$\begin{aligned} U_{r,s}(x, y_J^{(M)}; \nu, M_\infty) &= \frac{l^2}{s^2} \sum_{p=1}^{\bar{M}} I_r^{(n)}(\xi, \eta_J^{(M)}, \eta_p^{(\bar{M})}; \nu, M_\infty) g_s^{(m)}(\eta_p^{(\bar{M})}) \int_{-1}^{+1} \frac{g_p^{(\bar{M})}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta_0 - \eta_J^{(M)})^2} d\eta_0 + \\ &+ \frac{l^2}{s^2} F_r^{(n)}(\xi, \eta_J^{(M)}; \nu, M_\infty) g_s^{(m)}(\eta_J^{(M)}) \Big[\frac{\pi}{2}((\eta_J^{(M)})^2 - \tfrac{1}{2} - \log 2) - \\ &- \sum_{p=1}^{\bar{M}} (\eta_J^{(M)} - \eta_p^{(\bar{M})})^2 \log |\eta_J^{(M)} - \eta_p^{(\bar{M})}| \int_{-1}^{+1} \frac{g_p^{(\bar{M})}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta_0 - \eta_J^{(M)})^2} d\eta_0 \Big]. \end{aligned} \tag{156}$$

If q is a positive even number then we put

$$q = 2a, \tag{157}$$

where a is a positive number, and formula (156) follows directly from case (i) above.

If q is a positive odd number then we put

$$q = 2a - 1, \tag{158}$$

where a is a positive number. Formula (156) then follows immediately from case (ii) above when M is any odd positive integer and J, $2 \leqslant J \leqslant M-1$ is any even positive integer, from case (iii) above when M is any even positive integer and J, $2 \leqslant J \leqslant M$ is any even positive integer, from case (iv) above when M is any even positive integer and J, $1 \leqslant J \leqslant M-1$ is any odd positive integer and from case (v) above when M is any odd positive integer and J, $1 \leqslant J \leqslant M$, is any odd positive integer.

The formula (156) is precisely the formula derived by Garner and Fox[6] and now it has been demonstrated that the numerical procedure converges and provides a value for $U_{r,s}(x, y_J^{(M)}; \nu, M_\infty)$ which is correct when q becomes indefinitely large. Garner and Fox[6] considered quasi steady flow only but their formula is valid for general frequencies of harmonic oscillation and was applied to this case by Long[7]. For a given value of q the corresponding value of a is obtained from (157) when q is an even positive integer and from (158) when q is an odd positive integer. The number of integration points Λ is determined then from (149) when q is an even positive integer and from (150), (151), (152) or (153) when q is an odd positive integer.

The value of q used in evaluating $U_{r,s}(\bar{x}_{I,J}^{(N,M)}, y_J^{(M)}; \nu, M_\infty)$ from formula (156) can be taken individually for each combination (r, s, I, J), that occurs, as a value which gives adequate accuracy for $U_{r,s}(\bar{x}_{I,J}^{(N,M)}, y_J^{(M)}; \nu, M_\infty)$. It is to be observed that a higher value of q is needed for a given accuracy in $U_{r,s}(\bar{x}_{I,J}^{(N,M)}, y_J^{(M)}; \nu, M_\infty)$ as the point $\bar{x}_{I,J}^{(N,M)}$ approaches the leading edge or trailing edge of the planform than is otherwise needed. In our work that follows we shall take q to depend only on I and denote it by q_I. This will enable us to take higher values of q for points $\bar{x}_{I,J}^{(N,M)}$ near the leading or trailing edges of the planform than for others further away from the leading or trailing edges, if we so desire, without unduly complicating the arrangement of the calculation, but we note that a more general variation of q could be taken.

We would like to mention here that Zandbergen, Labrujere and Wouters in Ref. 8 take

$$\Lambda = a(M+1) - 1 \tag{159}$$

where a is any positive integer, and use the weight function $\sqrt{1-\eta_0^2}$ for the numerical integration in formula (121). Provided that condition (130) holds, their final formula is our formula (133), but, as we have indicated, this is precisely equivalent to (144) if condition (141) holds.

In this case

$$\zeta_p^{(\Lambda)} = \eta_p^{(\Lambda)} = \cos\left(\frac{p\pi}{\Lambda+1}\right) \tag{160}$$

and

$$\eta_J^{(M)} = \cos\left(\frac{J\pi}{M+1}\right) = \cos\left(\frac{aJ\pi}{\Lambda+1}\right) = \zeta_{aJ}^{(\Lambda)}. \tag{161}$$

It therefore follows that the condition (130) is not satisfied when

$$\eta = \eta_J^{(M)} \tag{162}$$

for one value of $p = 1, 2, \ldots, \Lambda$. The formula (121) must then be modified to read

$$\begin{aligned}
&\int_{-1}^{+1} \frac{g_s^{(m)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta_J^{(M)}-\eta_0)^2}\Big\{\hat{I}_r^{(n)}(\xi, \eta_J^{(M)}, \eta_0; \nu, M_\infty) - I_r^{(n)}(\xi, \eta_J^{(M)}, \eta_J^{(M)}; \nu, M_\infty) - \\
&\quad -(\eta_0-\eta_J^{(M)})\Big[\frac{\partial}{\partial\eta_0} I_r^{(n)}(\xi, \eta_J^{(M)}, \eta_0; \nu, M_\infty)\Big]_{\eta_0=\eta_J^{(M)}}\Big\} d\eta_0 \\
&= \frac{\pi}{P_\Lambda}\sum_{\substack{p=1\\ p\neq aJ}}^{\Lambda} \frac{g_s^{(m)}(\zeta_p^{(\Lambda)})\{1-(\zeta_p^{(\Lambda)})^2\}}{(\eta_J^{(M)}-\zeta_p^{(\Lambda)})^2}\Big\{\hat{I}_r^{(n)}(\xi, \eta_J^{(M)}, \zeta_p^{(\Lambda)}; \nu, M_\infty) - \\
&\quad - I_r^{(n)}(\xi, \eta_J^{(M)}, \eta_J^{(M)}; \nu, M_\infty) - \\
&\quad -(\zeta_p^{(\Lambda)}-\eta_J^{(M)})\Big[\frac{\partial}{\partial\eta_0} I_r^{(n)}(\xi, \eta_J^{(M)}, \eta_0; \nu, M_\infty)\Big]_{\eta_0=\eta_J^{(M)}}\Big\} + \\
&\quad + \frac{\pi}{2P_\Lambda} g_s^{(m)}(\zeta_{aJ}^{(\Lambda)})\{1-(\zeta_{aJ}^{(\Lambda)})^2\}\Big[\frac{\partial^2}{\partial\eta_0^2}\hat{I}_r^{(n)}(\xi, \eta_J^{(M)}, \eta_0; \nu, M_\infty)\Big]_{\eta_0=\eta_J^{(M)}}
\end{aligned} \tag{163}$$

and correspondingly the formula (133) must likewise be modified. The second derivative $[(\partial^2/\partial\eta_0^2)\hat{I}_r^{(n)}(\xi, \eta_J^{(M)}, \eta_0; \nu, M_\infty)]_{\eta_0=\eta_J^{(M)}}$ as well as the first derivative $[(\partial/\partial\eta_0)I_r^{(n)}(\xi, \eta_J^{(M)}; \eta_0; \nu, M_\infty)]_{\eta_0=\eta_J^{(M)}}$ appear in these modified formulae, in contrast to formula (156), where neither derivative appears. That the second derivative appears for all $J = 1, 2, \ldots, M$, is a direct result of Λ having been given the formula (159), and it is only when Λ has this formula that the second derivative appears for all J.

The modified formula (133) would be exact with the Λ integration points given by formula (159) if the function (119) were a polynomial of degree $2a(M+1)-m-2$ in η_0. The formula requires the values of $\hat{I}_r^{(n)}(\xi, \eta_J^{(M)}, \eta_0; \nu, M_\infty)$ at the $a(M+1)-1$ integration points $\eta_0 = \zeta_p^{(\Lambda)}$, $p = 1, 2, \ldots, \Lambda$, together with the values $[(\partial^2/\partial\eta_0^2)\hat{I}_r^{(n)}(\xi, \eta_J^{(M)}, \eta_0; \nu, M_\infty)]_{\eta_0=\eta_J^{(M)}}$ and $[(\partial/\partial\eta_0)I_r^{(n)}(\xi, \eta_J^{(M)}, \eta_0; \nu, M_\infty)]_{\eta_0=\eta_J^{(M)}}$, i.e. $a(M+1)+1$ values in all.

On the other hand, if we take Λ to be given by (149) and use the weight function $1/\sqrt{1-\eta_0^2}$ for the numerical integration in formula (121), then the formula (154) is valid and would be exact if the function (119) were a polynomial of degree $\leqslant 2a(M+1)-m-2$ in η_0. The formula requires the values of $I_r^{(n)}(\xi, \eta_J^{(M)}, \zeta_p^{(\Lambda)}; \nu, M_\infty)$, $p = 1, 2, \ldots, \Lambda$, where $\zeta_p^{(\Lambda)}$ is given in formula (129), and $I_r^{(n)}(\xi, \eta_J^{(M)}, \eta_J^{(M)}; \nu, M_\infty)$, i.e. $a(M+1)+1$ values in all. We conclude that formula (154) for case (i) above is of comparable accuracy with the modified formula (133) for the same value of a, although less work is required in the evaluation of (154) than in the evaluation of the modified formula (133). All the same, the modified formula (133) is somewhat better conditioned than formula (154) because the absolute values of the quantities

$$\frac{1-(\zeta_p^{(\Lambda)})^2}{(\eta_J^{(M)}-\zeta_p^{(\Lambda)})^2} \tag{164}$$

attain greater values in (154) for a given value of a than they do in the modified formula (133).

We note further that formula (154) for case (i) above with $a = 1$ is the formula used by Hsu[13]. Also the modified formula (133) with weight function $\sqrt{1-\eta_0^2}$ and with $a = 1$ in the formula (159) for determining Λ is the formula suggested by Multhopp in Appendix V of Ref. 1.

If $M=m$ and $q=1$ so that $\bar{M}=M=m$, then formula (156) is the formula for $U_{r,s}(x, y_J^{(M)}; \nu, M_\infty)$ which is normally used in lifting surface theory. The corresponding value of Λ and the weight function for the integration formula (121) are those appropriate to case (ii), (iii), (iv) or (v) above, whichever has to be used, and this depends on whether the numbers M and J are even or odd. The numerical integration formulae (121) and (131) are not of good accuracy in this case because the value of Λ given by any one of the formulae (150), (151), (152) or (153) is not large enough. The numerical integration formulae (134) and (135), on the other hand, are precise. Consequently the formula (156) may not give good accuracy for $U_{r,s}(x, y_J^{(M)}; \nu, M_\infty)$ in this case. A significant improvement in accuracy for $U_{r,s}(x, y_J^{(M)}; \nu, M_\infty)$ should be obtained if q is increased to 2, and this is equivalent to using Hsu's formula[13]. By taking the value of q high enough we can obtain $U_{r,s}(x, y_J^{(M)}; \nu, M_\infty)$ to any accuracy that we like.

If now we take

$$q=q_I, \qquad I=1,2,\ldots,N, \tag{165}$$

and

$$M_I=q_I(M+1)-1, \qquad I=1,2,\ldots,N, \tag{166}$$

then, analogously to formula (156) we have the formula

$$\begin{aligned} &U_{r,s}(\bar{x}_{I,J}^{(N,M)}, y_J^{(M)}; \nu, M_\infty) \\ &\quad =\frac{l^2}{s^2}\sum_{p=1}^{M_I} I_n^{(n)}(\bar{\xi}_I^{(N)}, \eta_J^{(M)}, \eta_p^{(M_I)}; \nu, M_\infty) g_s^{(m)}(\eta_p^{(M_I)})\times \\ &\quad \times\int_{-1}^{+1}\frac{g_p^{(M_I)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta_0-\eta_J^{(M)})^2}\,d\eta_0+\frac{l^2}{s^2}F_r^{(n)}(\bar{\xi}_I^{(N)}, \eta_J^{(M)}; \nu, M_\infty) g_s^{(m)}(\eta_J^{(M)})\times \\ &\quad \times\left[\frac{\pi}{2}((\eta_J^{(M)})^2-\tfrac{1}{2}-\log 2)-\sum_{p=1}^{M_I}(\eta_J^{(M)}-\eta_p^{(M_I)})^2\log|\eta_J^{(M)}-\eta_p^{(M_I)}|\times\right. \\ &\quad \left.\times\int_{-1}^{+1}\frac{g_p^{(M_I)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta_0-\eta_J^{(M)})^2}\,d\eta_0\right], \end{aligned} \tag{167}$$

where

$$\bar{\xi}_I^{(N)}=1-\xi_I^{(N)}. \tag{168}$$

Quite analogously to the derivation of formula (167) we derive the numerical formula

$$\begin{aligned} &U_{r,m-s+1}(\bar{x}_{I,J}^{(N,M)}, y_J^{(M)}; \nu, M_\infty) \\ &\quad =\frac{l^2}{s^2}\sum_{p=1}^{M_I} I_r^{(n)}(\bar{\xi}_I^{(N)}, \eta_J^{(M)}, \eta_p^{(M_I)}; \nu, M_\infty) g_{m-s+1}^{(m)}(\eta_p^{(M_I)})\times \\ &\quad \times\int_{-1}^{+1}\frac{g_p^{(M_I)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta_0-\eta_J^{(M)})^2}\,d\eta_0+ \\ &\quad +\frac{l^2}{s^2}F_r^{(n)}(\bar{\xi}_I^{(N)}, \eta_J^{(M)}; \nu, M_\infty) g_{m-s+1}^{(m)}(\eta_J^{(M)})\times \\ &\quad \times\left[\frac{\pi}{2}((\eta_J^{(M)})^2-\tfrac{1}{2}-\log 2)-\sum_{p=1}^{M_I}(\eta_J^{(M)}-\eta_p^{(M_I)})^2\log|\eta_J^{(M)}-\eta_p^{(M_I)}|\times\right. \\ &\quad \left.\times\int_{-1}^{+1}\frac{g_p^{(M_I)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta_0-\eta_J^{(M)})^2}\,d\eta_0\right]. \end{aligned} \tag{169}$$

Then, on making use of (42), we get from (167) and (169)

$$U_{r,s}(\bar{x}_{I,J}^{(N,M)}, y_J^{(M)}; \nu, M_\infty)+\kappa U_{r,m-s+1}(\bar{x}_{I,J}^{(N,M)}, y_J^{(M)}; \nu, M_\infty)$$

$$=\frac{l^2}{s^2}\sum_{p=1}^{M_I}(g_s^{(m)}(\eta_p^{(M_I)})+\kappa g_s^{(m)}(-\eta_p^{(M_I)}))I_r^{(n)}(\bar{\xi}_I^{(N)}, \eta_J^{(M)}, \eta_p^{(M_I)}; \nu, M_\infty)\times$$

$$\times\int_{-1}^{+1}\frac{g_p^{(M_I)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta_0-\eta_J^{(M)})^2}d\eta_0+\frac{l^2}{s^2}(g_s^{(m)}(\eta_J^{(M)})+\kappa g_s^{(m)}(-\eta_J^{(M)}))F_r^{(n)}(\bar{\xi}_I^{(N)}, \eta_J^{(M)}; \nu, M_\infty)\times$$

$$\times\Bigg[\frac{\pi}{2}((\eta_J^{(M)})^2-\tfrac{1}{2}-\log 2)-\sum_{p=1}^{M_I}(\eta_J^{(M)}-\eta_p^{(M_I)})^2\log|\eta_J^{(M)}-\eta_p^{(M_I)}|\times$$

$$\times\int_{-1}^{+1}\frac{g_p^{(M_I)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta_0-\eta_J^{(M)})^2}d\eta_0\Bigg], \tag{170}$$

which is the formula we use in getting the elements (112) of the matrix $[U]$ appearing in equation (110).

We note that

$$\frac{\pi}{2}((\eta_J^{(M)})^2-\tfrac{1}{2}-\log 2)-\sum_{p=1}^{M_I}(\eta_J^{(M)}-\eta_p^{(M_I)})^2\log|\eta_J^{(M)}-\eta_p^{(M_I)}|\times$$

$$\times\int_{-1}^{+1}\frac{g_p^{(M_I)}(\eta_0)\sqrt{1-\eta_0^2}}{(\eta_0-\eta_J^{(M)})^2}d\eta_0, \tag{171}$$

appears to converge to zero as $q_I\to\infty$, as far as one can judge from numerical results, but the present author has not succeeded in proving this analytically. If this is true then we could miss out the coefficient of $F_r^{(n)}(\bar{\xi}_I^{(N)}, \eta_J^{(M)}; \nu, M_\infty)$ in formula (170) and still get convergence to the correct value as $q_I\to\infty$, but we must expect the convergence to be slower than that of formula (170).

To apply formula (170) we must still evaluate $I_r^{(n)}(\bar{\xi}_I^{(N)}, \eta_J^{(M)}, \eta_p^{(M_I)}; \nu, M_\infty)$ numerically from (114) and the process for doing this is described in Appendix C, where the use of Chebyshev polynomials is recommended and illustrated.

Examples

We give, in this section, a selection of results of calculations carried out on four planforms. The approximations $\hat{Q}_{jk}$ to the generalised airforce coefficients $Q_{jk}(\nu, M_\infty)$ are obtained from formula (84) and the approximations $\hat{l}_k(x_0, y_0)$ to the loading functions $l_k(x_0, y_0; \nu, M_\infty)$ in the mode k of oscillation at points (x_0, y_0) on the wing planform are obtained from formula (98).

The four planforms considered are a tapered swept wing of aspect ratio 2, a tapered swept wing of aspect ratio 6, and rectangular wings of aspect ratios 2 and 8. Diagrams of these planforms are given in Figs. 2, 3, 4a and 4b.

The tapered swept wings have trailing-edge motivators as shown and each is taken to oscillate in rotation about the hinge line at its leading edge. The loading in this mode of oscillation has a logarithmic singularity at the hinge line, but, despite this, an approximation to the loading in the form (22) is permissible, except in the immediate neighbourhood of the hinge line, and the corresponding approximations to the generalised airforce coefficients obtained from formula (34) should be acceptable. The approximations (75) and (76) are not valid for the motivator mode so that we must use the exact expressions (66) and (67) together with the associated formulae (60) and (36).

4.1. Tapered Swept Wing of Aspect Ratio 2

The planform of this wing is illustrated in Fig. 2. The x coordinate $x_L(y)$ of the leading edge at spanwise position y is given by the formula

$$x_L(y)=\sqrt{3}|y| \qquad -s\leq y\leq s, \tag{172}$$

and the chord length $c(y)$ at spanwise position y is given by the formula

$$c(y)=\frac{s}{4}(2\sqrt{3}+3)-\frac{|y|}{4}(4\sqrt{3}-2) \qquad -s\leqslant y\leqslant s. \tag{173}$$

The leading edge and trailing edge of the wing have a discontinuity of slope at the centre line of the wing and, in order to enhance the convergence of the numerical results, it is necessary to change the shape of the leading and trailing edges of the wing so that there are no discontinuities of slope. This change of shape is made in the region $-y_R\leqslant y\leqslant y_R$, where $0<y_R<s$, and is known as rounding of the leading and trailing edges in the neighbourhood of their central portions. It is desirable to arrange that $x_L''(y)$ and $c''(y)$ are continuous, as well as $x_L(y)$, $x_L'(y)$, $c(y)$, $c'(y)$, because a more accurate numerical estimate for $U_{r,s}(x, y; \nu, M_\infty)$ of formula (113) is then obtained. The change of shape is achieved by taking

$$x_L(y)=\begin{cases}\sqrt{3}y_R f(\lambda) & -y_R\leqslant y\leqslant y_R\\ \sqrt{3}y_R|\lambda| & y_R\leqslant |y|\leqslant s\end{cases} \tag{174}$$

and

$$c(y)=\begin{cases}\frac{s}{4}(2\sqrt{3}+3)-\frac{y_R}{4}(4\sqrt{3}-2)f(\lambda) & -y_R\leqslant y\leqslant y_R\\ \frac{s}{4}(2\sqrt{3}+3)-\frac{y_R}{4}(4\sqrt{3}-2)|\lambda| & y_R\leqslant |y|\leqslant s\end{cases} \tag{175}$$

where

$$\lambda=\frac{y}{y_R} \tag{176}$$

and $f(\lambda)$ is an arbitrary even function of λ with continuous second derivative $f''(\lambda)$ in $-1\leqslant\lambda\leqslant 1$ which is such that

$$f(1)=1, \tag{177}$$

$$f'(1)=1, \tag{178}$$

$$f''(1)=0. \tag{179}$$

The origin of coordinates at the apex of the tapered wing is, in general, not on the rounded leading edge of the modified wing.

Here we shall follow Hewitt[10] and take

$$f(\lambda)=\tfrac{5}{16}+\tfrac{15}{16}\lambda^2-\tfrac{5}{16}\lambda^4+\tfrac{1}{16}\lambda^6. \tag{180}$$

Further we shall take

$$\frac{y_R}{s}=\sin\left(\frac{\pi}{16}\right)=0{\cdot}1950903 \tag{181}$$

and we shall take the typical length l to be the geometric mean chord $\bar{c}$ of the planform, i.e.,

$$l=\bar{c}=s. \tag{182}$$

The motivators shown in Fig. 2 are known when the coordinates (X_1, Y_1) and (X_2, Y_2) of the inboard and outboard extremities respectively on the leading edge of the motivator on the starboard side of the wing are known. These coordinates are given by

$$X_1 = \tfrac{1}{4}(3+2\sqrt{3})\bar{c}, \tag{183}$$

$$Y_1 = \tfrac{1}{2}\bar{c}, \tag{184}$$

$$X_2 = \tfrac{1}{8}(9+4\sqrt{3})\bar{c}, \tag{185}$$

$$Y_2 = \bar{c}. \tag{186}$$

All the edges of the motivators are straight and the port motivator is the mirror image of the starboard motivator in the plane of symmetry of the wing planform.

The leading edge of the starboard motivator has the equation

$$x = x_H(y) \qquad Y_1 \leq y \leq Y_2 \tag{187}$$

and the leading edge of the port motivator has the equation

$$x = x_H(-y) \qquad -Y_2 \leq y \leq -Y_1. \tag{188}$$

The explicit form of the function $x_H(y)$ may be obtained from the information given above.

Three modes of oscillation are considered. These are specified by giving the functions $\zeta_k(x, y)$, introduced in formula (6), for $k = 1, 2$ and 3. These functions are taken in this example to be

$$\zeta_1(x, y) = 1, \tag{189}$$

$$\zeta_2(x, y) = \frac{x}{\bar{c}}, \tag{190}$$

and

$$\zeta_3(x, y) = \left(\frac{x - x_H(y)}{\bar{c}}\right) H(x - x_H(y))[H(y - Y_1) + H(-y - Y_1) - H(y - Y_2) - H(-y - Y_2)] \tag{191}$$

where $H(x)$ is Heaviside's unit function

$$H(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0. \end{cases} \tag{192}$$

Approximations $\hat{Q}_{ij}$ to the generalised airforce coefficients $Q_{ij}(\nu, M)$ have been evaluated for $i = 1, 2, 3;$ $j = 1, 2, 3$, when

$$\nu = 0{\cdot}32506 \quad \text{and} \quad M_\infty = 0{\cdot}78060.$$

We write $\hat{Q}_{ij}$ in the form

$$\hat{Q}_{ij} = \hat{Q}'_{ij} + i\nu\hat{Q}''_{ij} \tag{193}$$

where $\hat{Q}'_{ij}$ and $\hat{Q}''_{ij}$ are real quantities.

The numerical values of $\hat{Q}'_{ij}$, $\hat{Q}''_{ij}$, $i = 1, 2, 3$; $j = 1, 2, 3$, obtained with $m = M$ and

$$q_I = q, \qquad I = 1, 2, \ldots, N, \tag{194}$$

for a selection of values of the parameters m, n, N and q are given in Table 1. Examination of Table 1 reveals that the $\hat{Q}'_{ij}$ and $\hat{Q}''_{ij}$ need a fairly high value of q in their evaluation for them to be reliable for a given set of parameters m, n and N. As n and N are increased with $m = M = 15$ the values $\hat{Q}'_{ij}$ and $\hat{Q}''_{ij}$ seem to converge. This behaviour is illustrated graphically in Figs. 5a to 5r. The convergence with increasing q and increasing n and N appears to be best for the motivator mode of oscillation, mode 3.

Approximations $\hat{l}_k(x, y)$ to the loading also have been evaluated for $k = 3$ for the above values of ν and M_∞. We write $\hat{l}_k(x, y)$ in the form

$$\hat{l}_k(x, y) = \hat{l}'_k(x, y) + i\hat{l}''_k(x, y) \tag{195}$$

where $\hat{l}'_k(x, y)$ and $\hat{l}''_k(x, y)$ are real quantities.

Numerical values of $\hat{l}'_3(x, y)$ and $\hat{l}''_3(x, y)$ are given in Table 2 for $m = 15$, $n = 10$, $M = 15$, $N = 10$, $q_I = 8$, $I = 1, 2, \ldots, 10$. The coordinates (x, y) have been transformed to the coordinates (ξ, η) by means of the transformation (20). The loadings $\hat{l}'_3(x, y)$ and $\hat{l}''_3(x, y)$ are given at the set of chordwise lines

$$\eta = \eta_j = \cos\left\{\frac{(9-j)\pi}{16}\right\} \qquad j = 1, 2, \ldots, 8. \tag{196}$$

The locations of the hinge $\xi = \xi_H(\eta)$ at the chordwise lines within the motivator span are given by the following set of numbers:

$j =$ 4	5	6	7	8
$\eta_j = 0{\cdot}55557$	0·70711	0·83147	0·92388	0·98079
$\xi_H(\eta_j) = 0{\cdot}74654$	0·73387	0·71750	0·69853	0·68158

Curves of $\hat{l}'_3(x, y)$ and $\hat{l}''_3(x, y)$ for a range of values of ξ along each chordwise line $\eta = \eta_j$, $j = 1, 2, \ldots, 8$, are given in Figs. 6a and 6b. The curve for each value of η has its own origin, which is marked.

It is to be noticed that $\hat{l}'_3(x, y)$ along a chordwise line within the motivator span has a deep minimum near to the hinge line. The actual theoretical real part of the loading becomes infinite like $\log|\xi - \xi_H(\eta)|$ at the hinge line. Curves for both $\hat{l}'_3(x, y)$ and $\hat{l}''_3(x, y)$ have a number of undulations along them and this is due to the truncation of an infinite series, implied in formula (22), for a function which has a logarithmic singularity at the hinge line.

4.2. Tapered Swept Wing of Aspect Ratio 6

The planform of this wing is illustrated in Fig. 3. The x coordinate $x_L(y)$ of the leading edge at spanwise position y is given by the formula

$$x_L(y) = \tfrac{1}{3}(\sqrt{3} + \tfrac{1}{2})|y| \qquad -s \leq y \leq s, \tag{197}$$

and the chord length $c(y)$ at spanwise position y is given by the formula

$$c(y) = -\tfrac{1}{3}|y| + \tfrac{1}{2}s \qquad -s \leq y \leq s. \tag{198}$$

Here again, as with the tapered swept wing of aspect ratio 2 we round the leading and trailing edges in the neighbourhood of their central portions. The functions $x_L(y)$ and $c(y)$ above are thereby modified to the forms

$$x_L(y) = \begin{cases} \tfrac{1}{3}(\sqrt{3} + \tfrac{1}{2})y_R f(\lambda) & -y_R \leq y \leq y_R \\ \tfrac{1}{3}(\sqrt{3} + \tfrac{1}{2})y_R|\lambda| & y_R \leq |y| \leq s, \end{cases} \tag{199}$$

and

$$c(y) = \begin{cases} \tfrac{1}{2}s - \tfrac{1}{3}y_R f(\lambda) & -y_R \leq y \leq y_R \\ \tfrac{1}{2}s - \tfrac{1}{3}y_R|\lambda| & y_R \leq |y| \leq s, \end{cases} \tag{200}$$

where λ is given by formula (176).

In this example we shall follow Zandbergen, Labrujere and Wouters[8] and take

$$f(\lambda)=\tfrac{1}{3}+\lambda^2-\tfrac{1}{3}|\lambda|^3. \tag{201}$$

We shall again take y_R to be given by formula (181) and the typical length l to be the geometric chord $\bar{c}$ of the planform, i.e.,

$$l=\bar{c}=\tfrac{1}{3}s. \tag{202}$$

The motivators shown in Fig. 2 are known when the coordinates (X_1, Y_1) and (X_2, Y_2) of the inboard and outboard extremities respectively on the leading edge of the motivator on the starboard side of the wing are known. These coordinates are given by

$$X_1=\left(\frac{2\sqrt{3}}{5}+\frac{97}{100}\right)\bar{c}, \tag{203}$$

$$Y_1=\frac{6\bar{c}}{5}, \tag{204}$$

$$X_2=\left(\frac{7\sqrt{3}}{10}+\frac{91}{100}\right)\bar{c}, \tag{205}$$

$$Y_2=\frac{21\bar{c}}{10}. \tag{206}$$

All the edges of the motivators are straight and the port motivator is the mirror image of the starboard motivator in the plane of symmetry of the wing planform. The ratio of motivator chord to wing chord, in this example, is independent of the spanwise location of the chord and has the value 0·3.

Three modes of oscillation are considered and these are again defined by equations (189), (190) and (191).

Approximations $\hat{Q}_{ij}$ to the generalised airforce coefficients $Q_{ij}(\nu, M)$ have been evaluated for $i=1, 2, 3$; $j=1, 2, 3$, when

$$\nu=3{\cdot}1569 \quad \text{and} \quad M_\infty=0{\cdot}4.$$

We write $\hat{Q}_{ij}$ in the form (193). The numerical values of $\hat{Q}'_{ij}$, $\hat{Q}''_{ij}$, $i=1, 2, 3; j=1, 2, 3$, obtained for a selection of values of the parameters m, n, M, N and q_I, $I=1, 2, \ldots, N$, are given in Table 3. For all evaluations we have again taken all the q_I equal as in formula (194).

Results for $\hat{Q}_{ij}$, $i=1, 2; j=1, 2$, for this same tapered swept wing of aspect ratio 6 with the same rounding of the leading and trailing edges and the same frequency parameter and Mach number have been presented by Lehrian and Garner[9]. Lehrian and Garner's results have to be multiplied by -6 to make them compatible with the present results because of their different non-dimensionalising factor in the definition of Q_{ij}. Their results, after multiplication by -6, are presented in Table 4.

Lehrian and Garner[9] use an extension to oscillatory flow of the steady flow method of Zandbergen, Labrujere and Wouters[8]. In their development of this method, the number of upwash points is equal to the number of basic loading functions, so that $M=m$ and $N=n$. The parameter a determines the number of spanwise integration stations to be used in the numerical evaluation of the spanwise integral in the integral equation much as the parameter q determines the number of spanwise integration stations to be used in the present method for the numerical evaluation of, effectively, the same spanwise integral. The amount of computation necessary to get results with $q=2a$ in the present method is comparable with that required in the method of Lehrian and Garner and the results also may be expected to be of comparable accuracy, and this is borne out by examination of the respective results obtained. Lehrian and Garner used effectively the same values (66) and (60) to obtain $\zeta_{j;r,s}$ as were used to get the present results but they used values of $\alpha_{k;i,p}$ corresponding to the approximations (76) rather than those obtained from (67) and (36). The differences in results arising from this cause are expected to be small.

In the present method we may take $M \geq m$ and $N \geq n$. The effect of taking $M>m$, $N=n$ can be seen by examining Table 3. The comparatively large differences in the values of $\hat{Q}_{ij}$ for $m=14$ and $m=15$ when

$M = m$ are considerably reduced when $M > m$. The differences in values of $\hat{Q}_{ij}$ for $m = 22$ and $m = 23$ when $M = m$ are not so large as those for $m = 14$ and $m = 15$, but these differences also are reduced when $M > m$. The differences in values of $\hat{Q}_{ij}$ for $m = 30$ and $m = 31$ when $M = m$ are again smaller than are those for $m = 22$ and $m = 23$. This is an indication that the results are converging as m increases, and that the convergence is more rapid when $M > m$.

4.3. Rectangular Wings of Aspect Ratios 2 and 8

The planforms of these two wings are illustrated in Figs. 4a and 4b. The origin of coordinates is taken to be at the middle point of the leading edge. The typical length l is taken to be the wing chord c. Two modes of oscillation, heave and pitch about the leading edge, are considered. These are specified by the functions $\zeta_k(x, y)$ introduced in formula (6) and defined for $k = 1$ and 2 respectively in equations (189) and (190), with $\bar{c} = c$.

Approximations $\hat{Q}_{ij}$ to the generalised airforce coefficients $Q_{ij}(\nu, M_\infty)$ have been evaluated for $i = 1, 2$; $j = 1, 2$ when

$$\nu = 1{\cdot}0, \qquad M_\infty = 0{\cdot}8 \quad \text{and} \quad q_I = 32, \qquad I = 1, 2, \ldots, N,$$

for a selection of values of the parameters m, n, M, N. We write $\hat{Q}_{ij}$ in the form (193). The numerical values obtained for $\hat{Q}'_{ij}$, $\hat{Q}''_{ij}$, $i = 1, 2$; $j = 1, 2$, are given in Table 5 for the rectangular wing of aspect ratio 2 and in Table 6 for the rectangular wing of aspect ratio 8. For both sets of results there is evidence of convergence as m, n, M and N are increased, although it would appear that the convergence is more rapid with odd values of m than with even values of m. We shall consider the values obtained for $m = 19$, $n = 8$, $M = 19$, $N = 8$ to be the best estimates of the values of Q'_{ij}, Q''_{ij}, $i = 1, 2$; $J = 1, 2$. Values of $\hat{Q}'_{ij}$, $\hat{Q}''_{ij}$ are in close agreement with these best estimates for smaller values of m, n, M and N. To quantify the closeness of agreement of $\hat{Q}_{ij}$ with Q_{ij} we introduce a measure ε_{ij} of percentage difference by means of the formula

$$\varepsilon_{ij} = 100 \times \left[\frac{(\hat{Q}'_{ij} - Q'_{ij})^2 + \nu^2(\hat{Q}''_{ij} - Q''_{ij})^2}{(Q'_{ij})^2 + \nu^2(Q''_{ij})^2} \right]^{\frac{1}{2}} \tag{207}$$

and the arithmetic mean ε of ε_{ij} over $i = 1, 2$; $j = 1, 2$,

$$\varepsilon = \tfrac{1}{4}(\varepsilon_{11} + \varepsilon_{12} + \varepsilon_{21} + \epsilon_{22}). \tag{208}$$

A selection of values ε_{ij} for the wings of aspect ratios 2 and 8 are given in Tables 7 and 8 respectively.

The values of $\hat{Q}_{ij}$ are all in poor agreement with the best estimates Q_{ij} when $n = 2$. For $m = 4$, $n = 2$, $M = 4$, $N = 2$, the mean ε is 10·9 for the wing of aspect ratio 2 and 10·1 for the wing of aspect ratio 8. The percentage differences do not change greatly when m, M and N are changed with n kept at 2, even when m, M and N are all increased substantially.

There is a distinct improvement in the values of $\hat{Q}_{ij}$ when n is increased to 4. For $m = 4$, $n = 4$, $M = 4$, $N = 4$, the mean ε is 0·042 for the wing of aspect ratio 2 and 0·18 for the wing of aspect ratio 8. For $m = 4$, $n = 4$, $M = 19$, $N = 8$ the mean ε is 0·032 for the wing of aspect ratio 2 and 0·27 for the wing of aspect ratio 8. Thus changing (M, N) from (4, 4) to (19, 8) with (m, n) kept at (4, 4) has caused a small improvement in the values of $\hat{Q}_{ij}$ for the wing aspect ratio 2 and a deterioration in these values for the wing of aspect ratio 8. This latter suggests that $m = 4$ is not large enough for the wing of aspect ratio 8.

For $m = 9$, $n = 4$, $M = 9$, $N = 4$ the mean ε is 0·022 for the wing of aspect ratio 2 and 0·10 for the wing of aspect ratio 8. For $m = 9$, $n = 4$, $M = 19$, $N = 8$ the mean ε is 0·018 for the wing of aspect ratio 2 and 0·08 for the wing of aspect ratio 8. Thus changing (M, N) from (9, 4) to (19, 8) with (m, n) kept at (9, 4) has caused merely a marginal improvement in the values of $\hat{Q}_{ij}$. There are no further distinct improvements in increasing m to 14 or to 19 while keeping $n = 4$.

Further improvements in the values of $\hat{Q}_{ij}$ are obtained when n is increased to 6. For $m = 4$ the percentage differences for $n = 4$ and $n = 6$ are of the same order, but for higher values of m the percentage differences are lower for $n = 6$ than they are for $n = 4$ when $M = 19$. For $m = 9$, $n = 6$, $M = 9$, $N = 6$ the mean ε is 0·032 for the wing of aspect ratio 2 and 0·12 for the wing of aspect ratio 8. For $m = 9$, $n = 6$, $M = 19$, $N = 8$ the mean ε is 0·001 for the wing of aspect ratio 2 and 0·000 for the wing of aspect ratio 8. Thus changing (M, N) from (9, 6) to (19, 8) with (m, n) kept at (9, 6) has produced almost complete agreement of the $\hat{Q}_{ij}$ with the best estimates.

For $m = 14$, $n = 6$, $M = 14$, $N = 6$ the mean ε is 0·015 for the wing of aspect ratio 2 and 0·038 for the wing of aspect ratio 8. For $m = 14$, $n = 6$, $M = 19$, $N = 18$ the mean ε is 0·014 for the wing of aspect ratio 2 and 0·037 for the wing of aspect ratio 8. Thus changing (M, N) from (14, 6) to (19, 8) with (m, n) kept at (14, 6) has practically no effect on the values of $\hat{Q}_{ij}$. We may note that almost complete agreement of the $\hat{Q}_{ij}$ with the best estimates has not been achieved with (m, n) kept at (14, 6), but, all the same, the percentage differences ε_{ij} are very small.

It is as well to remark here that our results have been obtained with $q_I = 32$, $I = 1, 2, \ldots, N$, whereas results from earlier theories, such as those of Refs. 3, 4 and 5, are comparable with the results we would get by taking $q_I = 1$, $I = 1, 2, \ldots, N$; $M = m$ and $N = n$. For $m = M = 4$, $N = n = 4$ and $q_I = 1$, $I = 1, 2, \ldots, N$, we get $\hat{Q}_{11} = 0{\cdot}84678 - i3{\cdot}2052$, $\hat{Q}_{12} = -3{\cdot}2858 - i3{\cdot}1810$, $\hat{Q}_{21} = 0{\cdot}90492 - i0{\cdot}83073$, $\hat{Q}_{22} = -0{\cdot}51381 - i2{\cdot}0731$ for the wing of aspect ratio 2 and $\hat{Q}_{11} = -1{\cdot}1040 - i13{\cdot}627$, $\hat{Q}_{12} = -16{\cdot}484 - i7{\cdot}6979$, $\hat{Q}_{21} = 1{\cdot}7608 - i4{\cdot}5769$, $\hat{Q}_{22} = -4{\cdot}5283 - i6{\cdot}2760$ for the wing of aspect ratio 8. The corresponding percentage differences ε_{ij} are $\varepsilon_{11} = 2{\cdot}5$, $\varepsilon_{12} = 3{\cdot}1$, $\varepsilon_{21} = 5{\cdot}0$, $\varepsilon_{22} = 5{\cdot}4$ for the wing of aspect ratio 2 and $\varepsilon_{11} = 16{\cdot}6$, $\varepsilon_{12} = 17{\cdot}7$, $\varepsilon_{21} = 21{\cdot}7$, $\varepsilon_{22} = 26{\cdot}3$ for the wing of aspect ratio 8. The magnitudes of the ε_{ij} are of the order one hundred times the corresponding values in Tables 7 and 8 where the q_I are changed to $q_I = 32$, $I = 1, 2, \ldots, N$. To get ε_{ij} of the same order as those in Tables 7 and 8 with $q_I = 1$, $I = 1, 2, \ldots, N$, $M = m$ and $N = n$ an undesirably large value of m would be needed when $n = 4$. The values of the $\hat{Q}_{ij}$ with $q_I = 1$, $I = 1, 2, \ldots, N$, $m = M = 4$, $N = n = 4$ are quite unacceptable for the aspect ratio 8 wing.

To appraise the convergence of the results as M and N are increased we examine in Tables 5 and 6 the values of $\hat{Q}'_{ij}$, $\hat{Q}''_{ij}$ when $m = 4$, $n = 4$. For the wing of aspect ratio 2 the convergence of all the $\hat{Q}'_{ij}$, $\hat{Q}''_{ij}$ is rapid for N fixed and M increasing through 4, 9, 14, 19. Then with M held at 19 the convergence is rapid for N increasing through 4, 6, 8. For the wing of aspect ratio 8 convergence of these quantities is on the whole slower. Thus, while $\hat{Q}'_{11}$, $\hat{Q}''_{11}$ continue to converge quite rapidly, the convergence of $\hat{Q}'_{22}$, $\hat{Q}''_{22}$ for N fixed and M increasing through 4, 9, 14, 19 is less rapid, and to achieve convergence to five significant figures the value of M would have to be increased beyond 19 when $N = 4$, 6 and 8. Convergence to four figures, however, has practically been achieved at $M = 19$, and also at $N = 8$ when M is held at 19. The behaviour of the other $\hat{Q}'_{ij}$, $\hat{Q}''_{ij}$ is intermediate between the behaviours of $\hat{Q}'_{11}$, $\hat{Q}''_{11}$ and $\hat{Q}'_{22}$, $\hat{Q}''_{22}$.

The same pattern of convergence may be observed with other values of m and n kept fixed while M and N are increased although there are more numbers available in any convergence sequence for the lower values of m and n. With M and N being increased while m and n are fixed there are converged values of $\hat{Q}'_{ij}$, $\hat{Q}''_{ij}$ appropriate to m spanwise loading functions and n chordwise loading functions. Convergence with respect to m and n being increased then leads to the best estimates of these values.

We may conclude, for both rectangular wings, that increasing n from $n = 2$ effects a substantial improvement in the results for $\hat{Q}'_{ij}$, $\hat{Q}''_{ij}$, $i = 1, 2$; $j = 1, 2$, even while keeping m fixed at $m = 4$, whereas increasing m from $m = 4$ and keeping n fixed at $n = 2$ hardly effects any improvement at all. For practical purposes values of Q'_{ij}, Q''_{ij}, $i = 1, 2$; $j = 1, 2$, can be obtained to sufficient accuracy with $n = 4$ for both wings and respectively with $m = 4$ and $m = 9$ for the wings of aspect ratios 2 and 8.

4.4. Discussion

Results for the generalised airforce coefficients on a tapered swept wing of aspect ratio 2 show that a more refined numerical integration of the spanwise integral than the one used by Multhopp[1] must be used in order to obtain acceptable results. Values of M and N equal to m and n respectively were used for the results on the tapered swept wing of aspect ratio 2, but for a tapered swept wing of aspect ratio 6 values of $M > m$ were considered. For the tapered swept wing of aspect ratio 6 values obtained with $M \simeq 2m$ showed better convergence on the whole than did those with $M = m$. In particular, the values of the generalised airforce coefficients for $m = 15$, $n = 6$, $M = 30$, $N = 6$ are, on the whole, closer to the values for $m = 30$, $n = 6$, $M = 30$, $N = 6$ than are those for $m = 15$, $n = 6$, $M = 15$, $N = 6$.

A comprehensive set of results for rectangular wings, of aspect ratios 2 and 8 respectively, illustrate the nature of convergence of values of the generalised airforce coefficients when m, n, M and N are increased. Generally, the effect of increasing the values of M and N is to give convergence of results for given m and n, and then the effect of increasing the values of m and n is to give the final converged results.

5. Concluding Remarks

In the lifting surface theory developed in this paper the loading is represented approximately as a linear superposition of a finite number of known linearly independent elementary functions, there being a combination of n chordwise by m spanwise of them, as in a number of other theories. For given n and m, there are $N \geqslant n$ chordwise by $M \geqslant m$ spanwise integration points at which the upwash is evaluated. The mathematical analysis leads to the same kind of refined process of numerical evaluation of the spanwise integral in the integral equation as was introduced by Garner and Fox in Ref. 6.

The results obtained for generalised airforce coefficients, using the Garner and Fox[6] refined process of numerical evaluation of the spanwise integral in a lifting surface theory are a considerable improvement over those obtained by a simple extension of Multhopp's steady flow method to oscillatory flow. The results obtained with $N = n$ and $M = m$ are so good, particularly for rectangular wings, that it is not easy to detect any further improvement obtained by taking $N > n$ and $M > m$. Nethertheless results for the example of a tapered swept wing aspect ratio 6 do show evidence of a further improvement being obtained when $M > m$. Thus, for a given M, if one allows $m < M$ it may be possible to use a lower value of m than if one insists on having $m = M$, and get results of comparable accuracy, with, incidentally, less storage space and execution time on the computing machine.

LIST OF SYMBOLS

a	Speed of sound in undisturbed main flow. Also a positive integer which determines Λ from one of (149), (150), (151), (152) or (153)
a_n	Defined by formula (A-19)
A	The integral (A-75). Also a point within $(0, \infty)$
$A^{(r)}$	Approximation given by formula (A-79) to the integral A
$A_r(A)$	Coefficients appearing in formula (C-46)
$A_{k;i,p}$	Coefficients appearing in formula (22)
$b_k(t)$	Generalised coordinate for mode number k (see formula (6))
$\bar{b}_k(\omega)$	Quantity defining amount of harmonic constituent in $b_k(t)$ (see formula (7))
$\bar{b}_k^*(\omega)$	Complex conjugate of $\bar{b}_k(\omega)$
B	The integral (A-76)
$B^{(r)}$	Approximation given by formula (A-83) to the integral B
$B_r(A)$	Coefficients appearing in formula (C-47)
$B_{k;r,s}$	Coefficients appearing in formula (30)
$c(y)$	Chord length of planform S at spanwise position y (see Fig. 1)
$C_r(A)$	Coefficients appearing in formulae (C-44) and (C-45)
$[C_k]$	Matrix appearing in formula (55) and defined immediately afterwards
$C_{k;r,s}$	Coefficients defined in formulae (52) and (53)
d_m	Defined by formula (A-38)
$[D]$	Diagonal matrix with (80) as general diagonal element
$D_r(A)$	Coefficients appearing in formula (C-44)
$[E]$	Diagonal matrix with (79) as general diagonal element
$E_r(\alpha)$	The functions defined by formula (C-62)
$E_r(A)$	Coefficients appearing in formula (C-45)
$f(\alpha)$	The function defined by equation (C-23)
f_m	Defined by formula (A-73)
$[F]$	Matrix appearing in equation (93) and defined immediately before
$F(\alpha)$	The function defined by equation (C-30)
$F_n(\alpha)$	The functions defined by equations (C-34) and (C-35)
$F_r(A)$	Coefficients appearing in equation (C-75)
$[F(x_0, y_0)]$	Row matrix with (87) as general element
$F_r^{(n)}(\xi, \eta; \nu, M_\infty)$	Quantities defined by formula (118)
$g(\alpha)$	The function defined by equation (C-24)
$g_s^{(m)}(\eta_0)$	Set of interpolation polynomials defined by equations (28)
$g_j^{(r)}(\eta)$	Set of interpolation polynomials defined by equations (A-27)
$\bar{g}_j^{(r)}(\eta)$	Set of interpolation polynomials defined by equations (A-62)
$G(\alpha)$	The function defined by equation (C-31)

$G_n(\alpha)$	The functions defined by equations (C-36)
$G_r(A)$	Coefficients appearing in equation (C-75)
$G_p^{(m)}$	Quantities defined in formulae (72)
$G_J^{(M)}$	Quantities defined in formulae (103)
$G_i^{(r)}$	Quantities defined in formulae (A-32)
$\bar{G}_i^{(r)}$	Quantities defined in formulae (A-67)
$h(\alpha)$	The function defined in formula (C-67)
$h_n(\alpha)$	The function defined by equation (C-70)
$h_r^{(n)}(\xi_0)$	Set of interpolation polynomials defined by equations (24)
$h_i^{(r)}(\xi)$	Set of interpolation polynomials defined by equations (A-8)
$[H]$	Diagonal matrix with (94) as general diagonal element
$H(\alpha)$	The function defined by equation (C-65)
$H_i^{(n)}$	Quantities defined in formulae (71)
$H_I^{(N)}$	Quantities defined in formulae (102)
$H_i^{(r)}$	Quantities defined in formulae (A-13)
I	Integral introduced in formula (A-14)
$I^{(r)}$	Approximation to I
$I_1(\alpha)$	Modified Bessel function of the first kind and first order
$I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)$	Quantities defined in formula (114)
$\hat{I}_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)$	Quantities defined in formula (117)
$j(\alpha)$	The function defined in equation (C-22)
J	Integral introduced in formula (A-33)
$J^{(r)}$	Approximation to J
$J^{(r,1)}$	Approximation to J
$J^{(r,2)}$	Approximation to J
$J^{(r,3)}$	Approximation to J
$J\left(\frac{x}{l}, \frac{y}{l}; \nu, M_\infty\right)$	Quantity defined in formula (C-9)
$k(\alpha)$	The function defined in formula (C-67)
$k_n(\alpha)$	The function defined by equation (C-71)
$k_s^{(m_H)}(\eta_0)$	The function defined by equation (86)
K	Integral introduced in formula (A-68)
$K(\alpha)$	The function defined by equation (C-66)
$K_1(\alpha)$	Modified Bessel function of the second kind and first order
$K\left(\frac{x}{l}, \frac{y}{l}; \nu, M_\infty\right)$	Subsonic kernel function defined by equation (17)
l	Typical length of the planform S
$l_k(x, y; \nu, M_\infty)$	Loading function introduced in expression (14)
$\hat{l}_k(x_0, y_0)$	Approximation (22) to the loading function $l_k(x_0, y_0; \nu, M)$
$[\hat{l}_k]$	Column matrix with (92) as general element

$[\hat{l}_k(x_0, y_0)]$	Matrix consisting of the single element $\hat{l}_k(x_0, y_0)$
$l_r(\xi)$	Orthogonal polynomials of degree r in ξ satisfying equation (A-1)
$[L]$	Matrix with (111) as general element
$L_1(\alpha)$	Modified Struve function which is related to the Struve function $\lvert H\rvert_{-1}(i\alpha)$ by means of formula (C-18). The function is defined by equation (C-21)
$L_{p,J}$	Quantities defined in formulae (108) and (109)
m	Number of spanwise loading functions (see formula (22))
m_H	Integer defined by formula (48)
M_∞	Mach number defined by formula (15)
$\bar{M}$	Integer defined by formula (155)
M	Number of integration points spanwise for evaluation of $\psi_{i,p;r,s}$ from the numerical formula (99)
M_H	Integer defined by formula (106)
M_I	Set of N positive integers defined by formulae (166)
$M_r(A)$	Coefficients appearing in equation (C-74)
n	Number of chordwise loading functions (see formula (22))
N	Number of integration points chordwise for evaluation of $\psi_{i,p;r,s}$ from the numerical formula (99)
$N_r(A)$	Coefficients appearing in equation (C-74)
$P(\alpha)$	Function defined in formula (C-15)
P_Λ	Quantities defined in equations (122), (124), (126) or (128)
q	A positive integer for determining $\bar{M}$ from formula (155)
q_I	A set of N positive integers introduced in formula (165)
$Q_{jk}(\nu, M_\infty)$	Generalised airforce coefficients, defined by formula (19)
$\hat{Q}_{jk}$	Approximation (34) to the generalised airforce coefficient $Q_{jk}(\nu, M_\infty)$
$[\hat{Q}]$	Matrix appearing in formula (84) and defined immediately before
$Q_\Lambda(\eta)$	Function defined by formula (B-46)
R	Quantity defined by formula (18)
$R_\Lambda(\eta)$	Function defined by formula (B-39)
s	Semi-span on the planform S
S	Wing planform
$S(\alpha)$	Function defined by formula (C-12)
$S_\Lambda(\eta)$	Function defined by formula (B-22)
t	Time
$T_r(\eta)$	Orthogonal polynomials of degree r in η satisfying equation (A-56). Also Chebyshev polynomial defined in formula (C-43)
$[U]$	Matrix with (112) as general element
$U_{r,s}(x, y; \nu, M)$	Quantity defined by formula (32)
V	Speed of main flow. It is in the positive x direction
$W(x, y, t)$	Quantity defined by equation (5)

$W_+(x, y, t)$	Quantity defined by equation (1)
$W_-(x, y, t)$	Quantity defined by equation (2)
x, y, z	Rectangular cartesian coordinates of a point relative to a frame fixed with respect to the mean position of the wing
$x_L(y)$	x coordinate of the leading edge of the planform S at spanwise position y (see Fig. 1)
$x_{r,s}^{(n,m)}$	Quantities defined by formulae (69)
$\bar{x}_{i,p}^{(n,m)}$	Quantities defined by formulae (70)
$\bar{x}_{I,J}^{(N,M)}$	Quantities defined by formulae (104)
X	Quantity defined by formula (C-10)
$y_s^{(m)}$	Quantities defined by formulae (68)
$y_J^{(M)}$	Quantities defined by formulae (105)
$Z(x, y, t)$	Quantity obtained from equations (3) and (4)
$Z_1(x, y, t)$	Quantity obtained from equations (3) and (4)
$Z_+(x, y, t)$	Displacement from the plane of S of a point on the wing top surface
$Z_-(x, y, t)$	Displacement from the plane of S of a point on the wing bottom surface
$\alpha_k(x, y; \nu)$	Reduced upwash function, defined by formula (11)
$\hat{\alpha}_k(x, y)$	Approximation (31) to the upwash function $\alpha_k(x, y)$ corresponding to the approximation $\hat{l}_k(x_0, y_0)$ to the loading function $l_k(x_0, y_0; \nu, M)$
$[\alpha]$	Matrix appearing in formula (84) and defined immediately before
$\alpha_j^{(r)}$	Quantities defined by formulae (A-80)
$\beta_j^{(r)}$	Quantities defined by formulae (A-84)
$\gamma = 0{\cdot}5772156649$	Euler's constant
$\gamma_r(\eta)$	Orthogonal polynomials of degree r in η satisfying equation (A-21)
δ_{ri}	Kronecker's delta, defined in formula (26)
$\delta_n(\alpha)$	Function defined by equation (C-32)
ε	Quantity defined by formula (208)
$\varepsilon_n(\alpha)$	Function defined by equation (C-33)
ε_{ij}	Quantities defined by formula (207)
$\zeta_k(x, y)$	Modal function for the mode number k as it appears in equation (6)
$\zeta_p^{(\Lambda)}$	Set of Λ points defined in equations (123), (125), (127) or (129)
η	Parametric coordinate defined by formula (20)
η_0	Parametric coordinate defined by formula (20)
$\eta_p^{(m)}$	Set of m points in $(-1, 1)$ defined by formulae (27)
$\eta_J^{(M)}$	Set of M points in $(-1, 1)$ defined by formulae (101)
$\eta_j^{(r)}$	Set of r points in $(-1, 1)$ defined by formulae (A-25)
$[\theta_k]$	Matrix appearing in formula (55) and defined immediately afterwards
$\theta_i^{(r)}$	Quantities defined by formulae (A-7)
$\theta_{k;i,p}$	Quantities defined by formulae (36)

$\kappa = \begin{cases} +1 \\ -1 \end{cases}$	For symmetric modes of oscillation For anti-symmetric modes of oscillation
Λ	Number of integration points used in the numerical evaluation of the integral (121)
$\mu_n(\alpha)$	Function defined by equation (C-68)
ν	Frequency parameter defined by formula (12)
$\nu_n(\alpha)$	Function defined by equation (C-69)
ξ	Parametric coordinate defined by formula (20)
ξ_0	Parametric coordinate defined by formula (20)
$\xi_i^{(n)}$	Set of n points in (0, 1) defined by formula (23)
$\xi_I^{(N)}$	Set of N points in (0, 1) defined by formula ((100)
$\bar{\xi}_I^{(N)}$	Set of N points in (0, 1) defined by formula (168)
$\xi_i^{(r)}$	Set of r points in (0, 1) defined by formula (A-6)
ρ	Density of the fluid in the main flow
ϕ	Variable defined in equation (B-1)
$\phi_j^{(r)}$	Quantities defined by formulae (A-26)
$\bar{\phi}_j^{(r)}$	Quantities defined by formulae (A-61)
$[\chi_i]$	A row matrix appearing in formula (63) and defined immediately afterwards
$\chi_i^{(r)}$	Quantities defined by formulae (A-60)
$\chi_{j;r,s}$	Quantities defined by formulae (60)
$\psi_{i,p;r,s}$	Quantities defined by formulae (37)
$\Psi_{i,p;r,s}$	Quantities defined by formulae (54)
$[\Psi]$	Matrix appearing in formula (55) and defined immediately afterwards
ω	Circular frequency of harmonic oscillation
$'$	Indicates differentiation
$''$	Indicates differentiation twice

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APPENDIX A

Integration Formulae

Integrand with Weight Function $\sqrt{(1-\xi)/\xi}$

Let $l_r(\xi)$, $r = 0, 1, 2, \ldots$, be the set of polynomials of degree r in ξ which satisfy the orthogonality relations

$$\int_0^1 l_r(\xi) l_s(\xi) \sqrt{\frac{1-\xi}{\xi}}\, d\xi = \delta_{rs} \tag{A-1}$$

where δ_{rs} is Kronecker's delta

$$\delta_{rs} = \begin{cases} 1 & r = s \\ 0 & r \neq s. \end{cases} \tag{A-2}$$

If we make the change of variables

$$\xi = \tfrac{1}{2}(1-\cos\theta) \tag{A-3}$$

in the integrand of (A-1) we get

$$\int_0^\pi l_r\!\left(\frac{1-\cos\theta}{2}\right) l_s\!\left(\frac{1-\cos\theta}{2}\right)(\cos\tfrac{1}{2}\theta)^2\, d\theta = \delta_{rs} \tag{A-4}$$

from which it follows that

$$l_r\!\left(\frac{1-\cos\theta}{2}\right) = \sqrt{\frac{2}{\pi}}\,\frac{\cos(r+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}, \tag{A-5}$$

for this is a polynomial of degree r in the original variable ξ and both (A-1) and (A-4) are satisfied. Since $\sqrt{(1-\xi)/\xi}$ is positive for ξ in (0, 1) the polynomial $l_r(\xi)$ will have r zeros in (0, 1). Let these zeros be denoted by $\xi_i^{(r)}$, $i = 1, 2, \ldots, r$. The locations of these zeros are obtained directly from (A-5) and (A-3) and are given by

$$\xi_i^{(r)} = \tfrac{1}{2}(1-\cos\theta_i^{(r)}), \qquad i = 1, 2, \ldots, r, \tag{A-6}$$

where

$$\theta_i^{(r)} = \frac{2i-1}{2r+1}\pi, \qquad i = 1, 2, \ldots, r. \tag{A-7}$$

We define the interpolation polynomials $h_i^{(r)}(\xi)$ by means of the formulae

$$h_i^{(r)}(\xi) = \prod_{\substack{k=1 \\ k\neq i}}^{r} \left(\frac{\xi - \xi_k^{(r)}}{\xi_i^{(r)} - \xi_k^{(r)}}\right), \qquad i = 1, 2, \ldots, r. \tag{A-8}$$

These polynomials are of degree $(r-1)$ in ξ and have the property

$$h_i^{(r)}(\xi_k^{(r)}) = \delta_{ik}. \tag{A-9}$$

They are given also by the alternative formula

$$h_i^{(r)}(\xi) = \frac{l_r(\xi)}{l_r'(\xi_i^{(r)})(\xi - \xi_i^{(r)})} \qquad i = 1, 2, \ldots, r. \tag{A-10}$$

where the dash represents differentiation with respect to the argument of the appropriate function. By differentiating formula (A-5) with respect to θ we easily establish that

$$l_r'(\xi_i^{(r)}) = (-1)^i \sqrt{\frac{2}{\pi}} \frac{(2r+1)}{\sin \theta_i^{(r)} \cos \frac{1}{2}\theta_i^{(r)}}. \tag{A-11}$$

Then, from formula (A-10), we get

$$h_i^{(r)}(\xi) = \frac{2(-1)^{i+1}}{(2r+1)} \sin \theta_i^{(r)} \cos \tfrac{1}{2}\theta_i^{(r)} \frac{\cos (r+\frac{1}{2})\theta}{\cos \frac{1}{2}\theta(\cos \theta - \cos \theta_i^{(r)})}. \tag{A-12}$$

Therefore

$$\begin{aligned}
H_i^{(r)} &= \int_0^1 h_i^{(r)}(\xi)\sqrt{\frac{1-\xi}{\xi}}\, d\xi \\
&= \int_0^{\pi} h_i^{(r)}(\xi)(\cos \tfrac{1}{2}\theta)^2\, d\theta \\
&= \frac{(-1)^{i+1}}{(2r+1)} \sin \theta_i^{(r)} \cos \tfrac{1}{2}\theta_i^{(r)} \int_0^{\pi} \frac{[\cos (r+1)\theta + \cos r\theta]}{(\cos \theta - \cos \theta_i^{(r)})}\, d\theta \\
&= \frac{\pi(-1)^{i+1}}{(2r+1)} \sin \theta_i^{(r)} \cos \tfrac{1}{2}\theta_i^{(r)} \left[\frac{\sin (r+1)\theta_i^{(r)}}{\sin \theta_i^{(r)}} + \frac{\sin r\theta_i^{(r)}}{\sin \theta_i^{(r)}}\right] \\
&= \frac{2\pi(-1)^{i+1}}{(2r+1)} (\cos \tfrac{1}{2}\theta_i^{(r)})^2 \sin (r+\tfrac{1}{2})\theta_i^{(r)} \\
&= \frac{2\pi}{(2r+1)} (\cos \tfrac{1}{2}\theta_i^{(r)})^2 \\
&= \frac{2\pi}{(2r+1)} (1-\xi_i^{(r)}).
\end{aligned} \tag{A-13}$$

We are interested in the numerical evaluation of the integral

$$I = \int_0^1 f(\xi)\sqrt{\frac{1-\xi}{\xi}}\, d\xi \tag{A-14}$$

where $f(\xi)$ is an arbitrary continuous function of ξ in (0, 1). If we take an approximation $f^{(r)}(\xi)$ to $f(\xi)$ which is the interpolation polynomial of degree $(r-1)$ in ξ

$$f^{(r)}(\xi) = \sum_{i=1}^{r} f(\xi_i^{(r)}) h_i^{(r)}(\xi), \tag{A-15}$$

we get a corresponding approximation $I^{(r)}$ to I which is given by

$$I^{(r)} = \sum_{i=1}^{r} H_i^{(r)} f(\xi_i^{(r)}) \tag{A-16}$$

by replacing $f(\xi)$ on the right of (A-14) by $f^{(r)}(\xi)$. Formula (A-16) is the Gaussian numerical integration formula for a weight function $\sqrt{(1-\xi)/\xi}$ when r integration points are used. Because of the orthogonality relationship (A-1) we have that the approximation $I^{(r)}$ is exactly equal to I whenever $f(\xi)$ is a polynomial in ξ of degree $\leqslant 2r-1$. If $f(\xi)$ is not such a polynomial then we can give an estimate for the error $I - I^{(r)}$ by using the following procedure.

We can show that the approximation $I^{(r)}$ to I given by formula (A-16) is also given by the formula

$$I^{(r)}=\frac{1}{2p}\lim_{p\to\infty}\int_0^{\pi} f\left(\frac{1-\cos\theta}{2}\right)\frac{\{1+(-1)^{p+1}\cos(2r+1)p\theta\}}{\{\cos(r+\frac{1}{2})\theta\}^2}(\cos\tfrac{1}{2}\theta)^2\,d\theta, \tag{A-17}$$

where p runs through the positive integers. To show that (A-16) and (A-17) are equivalent formulae for continuous functions $f(\xi)$, the range $(0, \pi)$ of integration in (A-17) is divided into r subranges with each of the subranges containing one of the zeros $\theta=\theta_i^{(r)}$, $i=1,2,\ldots,r$ of $\cos(r+\frac{1}{2})\theta$ internal to it. For each of the subranges a new variable of integration is introduced by translation of θ so that the zero of the new variable is at $\theta=\theta_i^{(r)}$ in the ith subrange and Fejér's integral (see Ref. 14) is thereby obtained for that subrange. Then, by applying Fejér's theorem (see Ref. 14) to each of the subranges the result that (A-16) and (A-17) are equivalent formulae is obtained.

From (A-17) we get

$$\begin{aligned}
I^{(r)}&=\lim_{p\to\infty}\int_0^{\pi} f\left(\frac{1-\cos\theta}{2}\right)\left\{1+2\sum_{s=1}^{p}(-1)^s\left(1-\frac{s}{p}\right)\cos(2r+1)s\theta\right\}(\cos\tfrac{1}{2}\theta)^2\,d\theta\\
&=\int_0^{\pi} f\left(\frac{1-\cos\theta}{2}\right)(\cos\tfrac{1}{2}\theta)^2\,d\theta+\lim_{p\to\infty}\sum_{s=1}^{r}(-1)^s\left(1-\frac{s}{p}\right)\int_0^{\pi} f\left(\frac{1-\cos\theta}{2}\right)\times\\
&\quad\times[\cos\{(2r+1)s-\tfrac{1}{2}\}\theta+\cos\{(2r+1)s+\tfrac{1}{2}\}\theta]\cos\tfrac{1}{2}\theta\,d\theta\\
&=\int_0^{1} f(\xi)\sqrt{\frac{1-\xi}{\xi}}\,d\xi+\lim_{p\to\infty}\sqrt{\frac{\pi}{2}}\sum_{s=1}^{p}(-1)^s\left(1-\frac{s}{p}\right)\int_0^{1} f(\xi)\times\\
&\quad\times[l_{(2r+1)s-1}(\xi)+l_{(2r+1)s}(\xi)]\sqrt{\frac{1-\xi}{\xi}}\,d\xi\\
&=I+\lim_{p\to\infty}\sqrt{\frac{\pi}{2}}\sum_{s=1}^{p}(-1)^s\left(1-\frac{s}{p}\right)[a_{(2r+1)s-1}+a_{(2r+1)s}]
\end{aligned} \tag{A-18}$$

where

$$a_n=\int_0^{1} f(\xi)l_n(\xi)\sqrt{\frac{1-\xi}{\xi}}\,d\xi. \tag{A-19}$$

The formula

$$I^{(r)}=I+\lim_{p\to\infty}\sqrt{\frac{\pi}{2}}\sum_{s=1}^{p}(-1)^s[a_{(2r+1)s-1}+a_{(2r+1)s}] \tag{A-20}$$

is equivalent to the formula (A-18) provided that the limit on the right hand side of (A-20) exists. The formula (A-18) is valid whether the limit on the right hand side of (A-20) exists or not. Taking the limit as $p\to\infty$ in (A-20) corresponds to summing an infinite series directly whereas taking the limit as $p\to\infty$ in (A-18) corresponds to summing the same series by arithmetic means or by Cesàro's means of first order.

The error $I-I^{(r)}$ may now be gauged from either formula (A-18) or formula (A-20). The coefficients a_n tend to zero as n tends to infinity, as is shown by an application of the Riemann–Lebesgue theorem (see Ref. 14) to formula (A-19) after changing the integration variable from ξ to θ. The error $I-I^{(r)}$ converges to zero as r increases indefinitely because the multipliers $H_i^{(r)}$ of formula (A-16) are, according to formula (A-13), all positive.

The formula (A-18) is also valid for more general $f(\xi)$ than continuous functions, e.g. for $f(\xi)$ having a finite number of jump discontinuities for ξ in (0, 1), but in this paper we are interested only in continuous functions $f(\xi)$.

Integrand with Weight Function $\sqrt{1-\eta^2}$

Let $\gamma_r(\eta)$, $r=0, 1, 2, \ldots$ be the set of polynomials of degree r in η which satisfy the orthogonality relations

$$\int_{-1}^{+1} \gamma_r(\eta)\gamma_s(\eta)\sqrt{1-\eta^2}\, d\eta = \delta_{rs}. \tag{A-21}$$

If we make the change of variables

$$\eta = \cos\phi \tag{A-22}$$

in the integrand of (A-20) we get

$$\int_0^{\pi} \gamma_r(\cos\phi)\gamma_s(\cos\phi)(\sin\phi)^2\, d\phi = \delta_{rs} \tag{A-23}$$

from which it follows that

$$\gamma_r(\cos\phi) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(r+1)\phi}{\sin\phi}, \tag{A-24}$$

for this is a polynomial of degree r in the original variable η and both (A-21) and (A-23) are satisfied. Since $\sqrt{1-\eta^2}$ is positive for η in $(-1, 1)$ the polynomial $\gamma_r(\eta)$ will have r zeros in $(-1, 1)$. Let these zeros be denoted by $\eta_j^{(r)}$, $j=1, 2, \ldots, r$. The locations of these zeros are obtained directly from (A-24) and (A-22) and are given by

$$\eta_j^{(r)} = \cos\phi_j^{(r)}, \qquad j=1, 2, \ldots, r, \tag{A-25}$$

where

$$\phi_j^{(r)} = \frac{j\pi}{(r+1)}, \qquad j=1, 2, \ldots, r. \tag{A-26}$$

We define the interpolation polynomials $g_j^{(r)}(\eta)$ by means of the formulae

$$g_j^{(r)}(\eta) = \prod_{\substack{k=1\\k\neq j}}^{r}\left(\frac{\eta-\eta_k^{(r)}}{\eta_j^{(r)}-\eta_k^{(r)}}\right) \qquad j=1, 2, \ldots, r. \tag{A-27}$$

These polynomials are of degree $(r-1)$ in η and have the property

$$g_j^{(r)}(\eta_k^{(r)}) = \delta_{jk}. \tag{A-28}$$

They are given also by the alternative formula

$$g_j^{(r)}(\eta) = \frac{\gamma_r(\eta)}{\gamma_r'(\eta_j^{(r)})(\eta-\eta_j^{(r)})} \tag{A-29}$$

where the dash represents differentiation with respect to the argument of the appropriate function. By differentiating formula (A-24) with respect to ϕ and putting $\phi=\phi_j^{(r)}$ we easily establish that

$$\gamma_r'(\eta_j^{(r)}) = (-1)^{j+1}\sqrt{\frac{2}{\pi}}\,\frac{(r+1)}{(\sin\phi_j^{(r)})^2}. \tag{A-30}$$

Then, from formula (A-29) we get

$$g_j^{(r)}(\eta) = \frac{(-1)^{j+1}}{(r+1)}(\sin\phi_j^{(r)})^2\,\frac{\sin(r+1)\phi}{\sin\phi(\cos\phi-\cos\phi_j^{(r)})}. \tag{A-31}$$

Therefore

$$
\begin{aligned}
G_j^{(r)} &= \int_{-1}^{+1} g_j^{(r)}(\eta)\sqrt{1-\eta^2}\, d\eta \\
&= \int_0^{\pi} g_j^{(r)}(\cos\phi)(\sin\phi)^2\, d\phi \\
&= \frac{(-1)^{j+1}}{2(r+1)}(\sin\phi_j^{(r)})^2 \int_0^{\pi} \frac{(\cos r\phi - \cos(r+2)\phi)}{(\cos\phi - \cos\phi_j^{(r)})}\, d\phi \\
&= \frac{\pi(-1)^{j+1}}{2(r+1)}(\sin\phi_j^{(r)})^2\left[\frac{\sin r\phi_j^{(r)}}{\sin\phi_j^{(r)}} - \frac{\sin(r+2)\phi_j^{(r)}}{\sin\phi_j^{(r)}}\right] \\
&= \frac{\pi(-1)^{j}}{(r+1)}(\sin\phi_j^{(r)})^2 \cos(r+1)\phi_j^{(r)} \\
&= \frac{\pi}{(r+1)}(\sin\phi_j^{(r)})^2 \\
&= \frac{\pi}{(r+1)}[1-(\eta_j^{(r)})^2].
\end{aligned}
\tag{A-32}
$$

We are interested in the numerical evaluation of the integral

$$
J = \int_{-1}^{+1} f(\eta)\sqrt{1-\eta^2}\, d\eta \tag{A-33}
$$

where $f(\eta)$ is an arbitrary continuous function of η in $(-1, 1)$. If we take an approximation $f^{(r)}(\eta)$ to $f(\eta)$ which is the interpolation polynomial of degree $(r-1)$ in η

$$
f^{(r)}(\eta) = \sum_{j=1}^{r} f(\eta_j^{(r)})g_j^{(r)}(\eta), \tag{A-34}
$$

we get a corresponding approximation $J^{(r)}$ to J which is given by

$$
J^{(r)} = \sum_{j=1}^{r} G_j^{(r)} f(\eta_j^{(r)}) \tag{A-35}
$$

by replacing $f(\eta)$ on the right of (A-33) by $f^{(r)}(\eta)$.

Formula (A-35) is the Gaussian numerical integration formula for a weight function $\sqrt{1-\eta^2}$ when r integration points are used. Because of the orthogonality relationship (A-21) we have that the approximation $J^{(r)}$ is exactly equal to J whenever $f(\eta)$ is a polynomial in η of degree $\leqslant 2r-1$. If $f(\eta)$ is not such a polynomial then we can give an estimate for the error $J-J^{(r)}$.

Instead of formula (A-35) we can write

$$
J^{(r)} = \lim_{p\to\infty} \frac{1}{p} \int_0^{\pi} f(\cos\phi)\left(\frac{\sin(r+1)p\phi}{\sin(r+1)\phi}\right)^2 (\sin\phi)^2\, d\phi \tag{A-36}
$$

where p runs through the positive integers. That (A-35) and (A-36) are equivalent formulae for continuous functions $f(\eta)$ is demonstrated if we divide the range $(0, \pi)$ of integration in (A-36) into r subranges with each of the subranges containing one of the zeros $\phi = \phi_j^{(r)}$, $j = 1, 2, \ldots, r$, of $\sin(r+1)\phi$ internal to it, and then apply Fejér's integral formula (see Ref. 14) to each of the subranges.

From (A-36) we get

$$J^{(r)} = \lim_{p\to\infty} \int_0^{\pi} f(\cos\phi)\left\{1+2\sum_{s=1}^{p}\left(1-\frac{s}{p}\right)\cos 2(r+1)s\phi\right\}(\sin\phi)^2\, d\phi$$

$$= \int_0^{\pi} f(\cos\phi)(\sin\phi)^2\, d\phi + \lim_{p\to\infty}\sum_{s=1}^{p}\left(1-\frac{s}{p}\right)\int_0^{\pi} f(\cos\phi)\times$$

$$\times[\sin\{2(r+1)s+1\}\phi - \sin\{2(r+1)s-1\}\phi]\sin\phi\, d\phi$$

$$= \int_{-1}^{+1} f(\eta)\sqrt{1-\eta^2}\, d\eta +$$

$$+ \lim_{p\to\infty}\sqrt{\frac{\pi}{2}}\sum_{s=1}^{p}\left(1-\frac{s}{p}\right)\int_{-1}^{1} f(\eta)[\gamma_{2(r+1)s}(\eta)-\gamma_{2(r+1)s-2}]\sqrt{1-\eta^2}\, d\eta$$

$$= J + \lim_{p\to\infty}\sqrt{\frac{\pi}{2}}\sum_{s=1}^{p}\left(1-\frac{s}{p}\right)[d_{2(r+1)s}-d_{2(r+1)s-2}] \qquad \text{(A-37)}$$

where

$$d_m = \int_{-1}^{+1} f(\eta)\gamma_m(\eta)\sqrt{1-\eta^2}\, d\eta. \qquad \text{(A-38)}$$

We can write, instead of (A-37), the formula

$$J^{(r)} = J + \sqrt{\frac{\pi}{2}}\lim_{p\to\infty}\sum_{s=1}^{p}[d_{2(r+1)s}-d_{2(r+1)s-2}] \qquad \text{(A-39)}$$

provided that the limit on the right hand side of (A-39) exists. The formula (A-37) is valid whether the right hand side of (A-39) exists or not.

The error $J - J^{(r)}$ may now be gauged from either formula (A-37) or (A-39). The coefficients d_m tend to zero as m tends to infinity, as is shown by an application of the Riemann–Lebesgue theorem (see Ref. 14) to formula (A-38) after changing the integration variable from η to ϕ. The error $J - J^{(r)}$ converges to zero as r increases indefinitely because the multipliers $G_j^{(r)}$ of formula (A-35) are, according to formula (A-32), all positive.

The formula (A-37) is also valid for more general $f(\eta)$, e.g. $f(\eta)$ having a finite number of jump discontinuities for η in $(-1, 1)$, but in this paper we are interested only in continuous functions $f(\eta)$.

We are also interested in the analytical evaluation of the integral

$$\int_{-1}^{+1} \frac{g_j^{(r)}(\eta)\sqrt{1-\eta^2}}{(\eta-\eta_k^{(r)})^2}\, d\eta. \qquad \text{(A-40)}$$

Let us write

$$\int_{-1}^{+1} \frac{g_j^{(r)}(\eta)\sqrt{1-\eta^2}}{(\eta-\zeta)}\, d\eta = \frac{1}{\gamma_r'(\eta_j^{(r)})}\int_{-1}^{+1}\frac{\gamma_r(\eta)\sqrt{1-\eta^2}}{(\eta-\eta_j^{(r)})(\eta-\zeta)}\, d\eta$$

$$= \frac{1}{\gamma_r'(\eta_j^{(r)})(\zeta-\eta_j^{(r)})}\int_{-1}^{+1}\left\{\frac{1}{(\eta-\zeta)}-\frac{1}{(\eta-\eta_j^{(r)})}\right\}\gamma_r(\eta)\sqrt{1-\eta^2}\, d\eta. \qquad \text{(A-41)}$$

Now, we have

$$\int_{-1}^{+1}\frac{\gamma_r(\eta)}{(\eta-\zeta)}\sqrt{1-\eta^2}\, d\eta = \frac{1}{\sqrt{2\pi}}\int_0^{\pi}\frac{(\cos r\phi - \cos(r+2)\phi)}{(\cos\phi-\cos\psi)}\, d\phi$$

$$= \sqrt{\frac{\pi}{2}}\frac{(\sin r\psi - \sin(r+2)\psi)}{\sin\psi}$$

$$= -\sqrt{2\pi}\cos(r+1)\psi \qquad \text{(A-42)}$$

where

$$\zeta = \cos\psi. \tag{A-43}$$

On changing ζ to $\eta_j^{(r)}$ in (A-42), and consequently, on changing ψ to $\phi_j^{(r)}$ we get immediately

$$\int_{-1}^{+1} \frac{\gamma_r(\eta)}{(\eta-\eta_j^{(r)})}\sqrt{1-\eta^2}\,d\eta = -\sqrt{2}\pi\cos(r+1)\phi_j^{(r)}. \tag{A-44}$$

If we substitute from (A-42) and (A-44) into (A-41) and use (A-30) we get

$$\int_{-1}^{+1} \frac{g_j^{(r)}(\eta)\sqrt{1-\eta^2}}{(\eta-\zeta)}\,d\eta = \frac{\pi(-1)^j}{(r+1)}[1-(\eta_j^{(r)})^2]\frac{(\cos(r+1)\psi-\cos(r+1)\phi_j^{(r)})}{(\zeta-\eta_j^{(r)})}. \tag{A-45}$$

Hence, if

$$\zeta = \eta_k^{(r)} \qquad k \neq j \tag{A-46}$$

we get from (A-45)

$$\int_{-1}^{+1} \frac{g_j^{(r)}(\eta)\sqrt{1-\eta^2}}{(\eta-\eta_k^{(r)})}\,d\eta = \frac{\pi[1-(\eta_j^{(r)})^2]}{(r+1)(\eta_j^{(r)}-\eta_k^{(r)})}[1-(-1)^{j+k}], \tag{A-47}$$

and if

$$\zeta = \eta_j^{(r)} \tag{A-48}$$

we get from (A-45)

$$\begin{aligned}\int_{-1}^{+1} \frac{g_j^{(r)}(\eta)\sqrt{1-\eta^2}}{(\eta-\eta_j^{(r)})}\,d\eta &= \frac{\pi(-1)^j}{(r+1)}[1-(\eta_j^{(r)})^2]\left[\frac{d}{d\zeta}\cos(r+1)\psi\right]_{\zeta=\eta_j^{(r)}}\\ &= \frac{\pi(-1)j}{(r+1)}[1-(\eta_j^{(r)})^2]\left[\frac{d/d\psi\cos(r+1)\psi}{(d\cos\psi)/d\psi}\right]_{\psi=\phi_j^{(r)}}\\ &= 0.\end{aligned} \tag{A-49}$$

Also from (A-45), on differentiating with respect to ζ, we get

$$\begin{aligned}\int_{-1}^{+1} \frac{g_j^{(r)}(\eta)\sqrt{1-\eta^2}}{(\eta-\zeta)^2}\,d\eta &= \frac{d}{d\zeta}\int_{-1}^{+1} \frac{g_j^{(r)}(\eta)\sqrt{1-\eta^2}}{(\eta-\zeta)}\,d\eta\\ &= \frac{\pi(-1)^{j+1}}{(r+1)}[1-(\eta_j^{(r)})^2]\frac{(\cos(r+1)\psi-\cos(r+1)\phi_j^{(r)})}{(\zeta-\eta_j^{(r)})^2}+\\ &\quad +\pi(-1)^j[1-(\eta_j^{(r)})^2]\frac{\sin(r+1)\psi}{\sin\psi}\frac{1}{(\zeta-\eta_j^{(r)})}.\end{aligned} \tag{A-50}$$

Hence, if

$$\zeta = \eta_k^{(r)} \qquad k \neq j \tag{A-51}$$

we get from (A-50)

$$\int_{-1}^{+1} \frac{g_j^{(r)}(\eta)\sqrt{1-\eta^2}}{(\eta-\eta_k^{(r)})^2}\,d\eta = \frac{\pi[1-(\eta_j^{(r)})^2]}{(r+1)(\eta_j^{(r)}-\eta_k^{(r)})^2}[1-(-1)^{j+k}], \tag{A-52}$$

whereas if

$$\zeta = \eta_j^{(r)} \tag{A-53}$$

we get from (A-50)

$$\begin{aligned}\int_{-1}^{+1} \frac{g_j^{(r)}(\eta)\sqrt{1-\eta^2}}{(\eta-\eta_j^{(r)})^2}\,d\eta
&= \frac{\pi(-1)^j}{2(r+1)}[1-(\eta_j^{(r)})^2]\left[\frac{d^2}{d\zeta^2}\cos(r+1)\psi\right]_{\zeta=\eta_j^{(r)}} \\
&= \frac{\pi(-1)^j}{2(r+1)}[1-(\eta_j^{(r)})^2]\left[\frac{1}{\sin\psi}\frac{d}{d\psi}\frac{1}{\sin\psi}\frac{d}{d\psi}\cos(r+1)\psi\right]_{\psi=\eta_j^{(r)}} \\
&= \frac{\pi}{2}(-1)^{j+1}[1-(\eta_j^{(r)})^2]\left[(r+1)\frac{\cos(r+1)\psi}{(\sin\psi)^2}-\frac{\sin(r+1)\psi}{(\sin\psi)^3}\right]_{\psi=\eta_j^{(r)}} \\
&= -\frac{\pi}{2}(r+1).\end{aligned} \tag{A-54}$$

Collecting together the results (A-52) and (A-54) into one formula we get

$$\int_{-1}^{+1} \frac{g_j^{(r)}(\eta)\sqrt{1-\eta^2}}{(\eta-\eta_k^{(r)})^2}\,d\eta = \begin{cases} -\dfrac{\pi}{2}(r+1) & k=j \\[2ex] \dfrac{\pi[1-(\eta_j^{(r)})^2]}{(r+1)(\eta_j^{(r)}-\eta_k^{(r)})^2}[1-(-1)^{j+k}] & k\neq j \end{cases} \tag{A-55}$$

Integrand with Weight Function $1/\sqrt{1-\eta^2}$

Let $T_r(\eta)$, $r=0, 1, 2, \ldots$, be the set of polynomials of degree r in η which satisfy the orthogonality relations

$$\int_{-1}^{+1} T_r(\eta)T_s(\eta)\frac{d\eta}{\sqrt{1-\eta^2}} = \delta_{rs}. \tag{A-56}$$

If we make the change of variables

$$\eta = \cos\phi \tag{A-57}$$

we get

$$\int_0^{\pi} T_r(\cos\phi)T_s(\cos\phi)\,d\phi = \delta_{rs}, \tag{A-58}$$

from which it follows that

$$T_r(\cos\phi) = \begin{cases} \dfrac{1}{\sqrt{\pi}} & r=0 \\[2ex] \sqrt{\dfrac{2}{\pi}}\cos r\phi & r\neq 0 \end{cases} \tag{A-59}$$

for this is a polynomial of degree r in the original variable η and both (A-56) and (A-58) are satisfied. Since $1/\sqrt{1-\eta^2}$ is positive for η in $(-1, 1)$ the polynomial $T_r(\eta)$ will have r zeros in $(-1, 1)$. Let these zeros be denoted by $\chi_j^{(r)}, j=1,2,\ldots,r$. The locations of these zeros are obtained directly from (A-59) and (A-57) and are given by

$$\chi_j^{(r)} = \cos\bar{\phi}_j^{(r)} \qquad j=1,2,\ldots,r, \tag{A-60}$$

where

$$\bar{\phi}_j^{(r)} = \frac{2j-1}{2r}\pi \qquad j=1,2,\ldots,r. \tag{A-61}$$

We define interpolation polynomials $\bar{g}_j^{(r)}(\eta)$ by means of the formulae

$$\bar{g}_j^{(r)}(\eta) = \prod_{\substack{k=1\\k\neq j}}^{r}\left(\frac{\eta-\chi_k^{(r)}}{\chi_j^{(r)}-\chi_k^{(r)}}\right) \qquad j=1,2,\ldots,r. \tag{A-62}$$

These polynomials are of degree $(r-1)$ in η and have the property

$$\bar{g}_j^{(r)}(\eta_k^{(r)}) = \delta_{jk}. \tag{A-63}$$

They are also given by the alternative formula

$$\bar{g}_j^{(r)}(\eta) = \frac{T_r(\eta)}{T_r'(\chi_j^{(r)})(\eta-\chi_j^{(r)})} \tag{A-64}$$

where the dash represents differentiation with respect to the argument of the appropriate function. By differentiating formula (A-59) with respect to ϕ we easily establish that

$$T_r'(\chi_j^{(r)}) = (-1)^{j+1}\sqrt{\frac{2}{\pi}}\,\frac{r}{\sin\bar{\phi}_j^{(r)}}. \tag{A-65}$$

Then, from formula (A-64) we get

$$\bar{g}_j^{(r)}(\eta) = \frac{(-1)^{j+1}}{r}\sin\bar{\phi}_j^{(r)}\frac{\cos r\phi}{(\cos\phi-\cos\bar{\phi}_j^{(r)})}. \tag{A-66}$$

Therefore

$$\begin{aligned}
\bar{G}_j^{(r)} &= \int_{-1}^{+1}\bar{g}_j^{(r)}(\eta)\frac{d\eta}{\sqrt{1-\eta^2}}\\
&= \int_0^{\pi}\bar{g}_j^{(r)}(\cos\phi)\,d\phi\\
&= \frac{(-1)^{j+1}}{r}\sin\bar{\phi}_j^{(r)}\int_0^{\pi}\frac{\cos r\phi}{(\cos\phi-\cos\bar{\phi}_j^{(r)})}\,d\phi\\
&= \frac{\pi(-1)^{j+1}}{r}\sin r\bar{\phi}_j^{(r)}\\
&= \frac{\pi}{r}.
\end{aligned} \tag{A-67}$$

We are interested in the numerical evaluation of the integral

$$K = \int_{-1}^{+1} f(\eta)\frac{d\eta}{\sqrt{1-\eta^2}} \tag{A-68}$$

where $f(\eta)$ is an arbitrary continuous function of η in $(-1, 1)$. If we take an approximation $f^{(r)}(\eta)$ to $f(\eta)$ which is the interpolation polynomial of degree $(r-1)$ in η

$$f^{(r)}(\eta)=\sum_{j=1}^{r} f(\chi_j^{(r)})\bar{g}_j^{(r)}(\eta), \tag{A-69}$$

we get a corresponding approximation $K^{(r)}$ to K which is given by

$$K^{(r)}=\sum_{j=1}^{r} \bar{G}_j^{(r)} f(\chi_j^{(r)}) \tag{A-70}$$

by replacing $f(\eta)$ on the right of (A-68) by $f^{(r)}(\eta)$. Formula (A-70) is the Gaussian numerical integration formula for a weight function $1/\sqrt{1-\eta^2}$ when r integration points are used. Because of the orthogonality relationship (A-56) we have that the approximation $K^{(r)}$ is exactly equal to K whenever $f(\eta)$ is a polynomial in η of degree $\leqslant 2r-1$. If $f(\eta)$ is not such a polynomial then we can give an estimate for the error $K-K^{(r)}$.

Instead of formula (A-70) we can write

$$K^{(r)}=\lim_{p\to\infty}\frac{1}{2p}\int_0^{\pi} f(\cos\phi)\frac{\{1+(-1)^p\cos 2rp\phi\}}{(\cos r\phi)^2}\,d\phi \tag{A-71}$$

where p runs through the positive integers. That (A-70) and (A-71) are equivalent formulae for continuous functions $f(\eta)$ is demonstrated if we divide the range $(0, \pi)$ of integration in (A-71) into r subranges with each of the subranges containing one of the zeros $\phi=\bar{\phi}_j^{(r)}$, $j=1, 2, \ldots, r$, of $\cos r\phi$ internal to it, and then apply Fejér's integral formula (see Ref. 14) to each of the subranges.

From (A-71) we get

$$\begin{aligned}K^{(r)}&=\lim_{p\to\infty}\int_0^{\pi} f(\cos\phi)\left\{1+2\sum_{s=1}^{p}(-1)^s\left(1-\frac{s}{p}\right)\cos 2rs\phi\right\}d\phi\\
&=\int_{-1}^{+1} f(\eta)\frac{d\eta}{\sqrt{1-\eta^2}}+\lim_{p\to\infty}\sqrt{2\pi}\sum_{s=1}^{p}(-1)^s\left(1-\frac{s}{p}\right)\int_{-1}^{+1} f(\eta)T_{2rs}(\eta)\frac{d\eta}{\sqrt{1-\eta^2}}\\
&=K+\lim_{p\to\infty}\sqrt{2\pi}\sum_{s=1}^{p}(-1)^s\left(1-\frac{s}{p}\right)f_{2rs}\end{aligned} \tag{A-72}$$

where

$$f_m=\int_{-1}^{+1} f(\eta)T_m(\eta)\frac{d\eta}{\sqrt{1-\eta^2}}. \tag{A-73}$$

We can write, instead of (A-72), the formula

$$K^{(r)}=K+\sqrt{2\pi}\lim_{p\to\infty}\sum_{s=1}^{p}(-1)^s f_{2rs} \tag{A-74}$$

provided that the limit on the right hand side of (A-74) exists. The formula (A-72) is valid whether the right hand side of (A-74) exists or not.

The error $K-K^{(r)}$ may now be gauged from either formula (A-72) or (A-74). The coefficients f_m tend to zero as m tends to infinity, as is shown by an application of the Riemann–Lebesgue theorem (see Ref. 14) to formula (A-73) after changing the integration variable from η to ϕ. The error $K-K^{(r)}$ converges to zero as r increases indefinitely because the multipliers $\bar{G}_j^{(r)}$ of formula (A-70) are, according to formula (A-67), all positive.

The formula (A-72) is also valid for more general $f(\eta)$, e.g. $f(\eta)$ having a finite number of jump discontinuities for η in $(-1, 1)$, but in this paper we are interested only in continuous functions $f(\eta)$.

Integrands with Weight Functions $\sqrt{(1+\eta)/(1-\eta)}$ and $\sqrt{(1-\eta)/(1+\eta)}$

We obtain numerical formulae of integration for the integrals

$$A=\int_{-1}^{+1} f(\eta)\sqrt{\frac{1+\eta}{1-\eta}}\,d\eta \tag{A-75}$$

and

$$B=\int_{-1}^{+1} f(\eta)\sqrt{\frac{1-\eta}{1+\eta}}\,d\eta \tag{A-76}$$

by application of the numerical formula (A-16). If we make the transformation of variables

$$\xi=\tfrac{1}{2}(1-\eta) \tag{A-77}$$

in the integrand in (A-75) we get

$$A=2\int_{0}^{1} f(1-2\xi)\sqrt{\frac{1-\xi}{\xi}}\,d\xi. \tag{A-78}$$

Then, on applying the numerical integration formula (A-16), for weight function $\sqrt{(1-\xi)/\xi}$ with r integration points, to (A-78) we get the approximation $A^{(r)}$ to A which is given by

$$A^{(r)}=\frac{2\pi}{(2r+1)}\sum_{j=1}^{r}(1+\alpha_j^{(r)})f(\alpha_j^{(r)}) \tag{A-79}$$

where

$$\begin{aligned}\alpha_j^{(r)}&=1-2\xi_j^{(r)}\\ &=\cos\left(\frac{2j-1}{2r+1}\pi\right) \qquad j=1,2,\ldots,r.\end{aligned} \tag{A-80}$$

If we make the transformation of variables

$$\xi=\tfrac{1}{2}(1+\eta) \tag{A-81}$$

in the integrand in (A-76) we get

$$B=2\int_{0}^{1} f(2\xi-1)\sqrt{\frac{1-\xi}{\xi}}\,d\xi. \tag{A-82}$$

Then, on applying the numerical integration formula (A-16), for weight function $\sqrt{(1-\xi)/\xi}$ with r integration points, to (A-82) we get the approximation $B^{(r)}$ to B which is given by

$$B^{(r)}=\frac{2\pi}{(2r+1)}\sum_{j=1}^{r}(1-\beta_j^{(r)})f(\beta_j^{(r)}) \tag{A-83}$$

where

$$\begin{aligned}\beta_j^{(r)}&=2\xi_{r-j+1}^{(r)}-1\\ &=\cos\left(\frac{2j}{2r+1}\pi\right), \qquad j=1,2,\ldots,r.\end{aligned} \tag{A-84}$$

The approximation $A^{(r)}$ is exactly equal to A and the approximation $B^{(r)}$ is exactly equal to B whenever $f(\eta)$ is a polynomial in η of degree $\leqslant 2r-1$. If $f(\eta)$ is not such a polynomial then we can give estimates for the errors $A-A^{(r)}$ and $B-B^{(r)}$, which are based on formula (A-18).

Spanwise Integration Formulae

Finally, in this Appendix, let us consider several different evaluations of the integral

$$J=\int_{-1}^{+1} f(\eta)\sqrt{1-\eta^2}\,d\eta \tag{A-85}$$

which are based on the numerical integration formulae (A-35), (A-70), (A-79) and (A-83). These integration formulae are required for justifying formula (121) of the main text.

Straightforwardly we have the numerical estimate $J^{(r)}$ from formula (A-35),

$$J^{(r)}=\frac{\pi}{(r+1)}\sum_{j=1}^{r} f(\eta_j^{(r)})[1-(\eta_j^{(r)})^2]. \tag{A-86}$$

If we write

$$J=\int_{-1}^{+1} f(\eta)(1+\eta)\sqrt{\frac{1-\eta}{1+\eta}}\,d\eta \tag{A-87}$$

and apply the formula of integration (A-83) we get the estimate $J^{(r,1)}$ for J which is given by

$$J^{(r,1)}=\frac{\pi}{(r+\frac{1}{2})}\sum_{j=1}^{r} f(\eta_{2j+1}^{(2r)})[1-(\eta_{2j+1}^{(2r)})^2] \tag{A-88}$$

since

$$\beta_j^{(r)}=\eta_{2j+1}^{(2r)} \qquad j=1,2,\ldots,r. \tag{A-89}$$

If we write

$$J=\int_{-1}^{+1} f(\eta)(1-\eta)\sqrt{\frac{1+\eta}{1-\eta}}\,d\eta \tag{A-90}$$

and apply the formula of integration (A-79) we get the estimate $J^{(r,2)}$ for J which is given by

$$J^{(r,2)}=\frac{\pi}{(r+\frac{1}{2})}\sum_{j=1}^{r} f(\eta_{2j-1}^{(2r)})[1-(\eta_{2j-1}^{(2r)})^2] \tag{A-91}$$

since

$$\alpha_j^{(r)}=\eta_{2j-1}^{(2r)} \qquad j=1,2,\ldots,r. \tag{A-92}$$

If we write

$$J=\int_{-1}^{+1} f(\eta)(1-\eta^2)\frac{d\eta}{\sqrt{1-\eta^2}} \tag{A-93}$$

and apply the formula of integration (A-70) we get the estimate $J^{(r,3)}$ for J which is given by

$$J^{(r,3)}=\frac{\pi}{r}\sum_{j=1}^{r} f(\eta_{2j-1}^{2r-1})[1-(\eta_{2j-1}^{(2r-1)})^2] \tag{A-94}$$

since

$$\chi_j^{(r)}=\eta_{2j-1}^{(2r-1)} \qquad j=1,2,\ldots,r. \tag{A-95}$$

Formula (121) of the main text is obtained by applying one of the estimates (A-86), (A-88), (A-91) and (A-94) for the integral appearing on its left hand side.

APPENDIX B

Derivation of Some Identities used in the Main Text

In this Appendix we derive the formulae (147) and (148) of the main text. In the derivation we shall use the transformation of variables

$$\eta = \cos\phi \tag{B-1}$$

and the numbers

$$\eta_J^{(M)} = \cos\left(\frac{J\pi}{M+1}\right) \qquad J = 1, 2, \ldots, M, \tag{B-2}$$

where M and J are positive integers. There are five cases to be considered. The method of derivation is basically that of Williams[12].

Cases (i) and (ii)

The function $T_\Lambda(\eta)$ is defined by the formula

$$T_\Lambda(\eta) = \sqrt{\frac{2}{\pi}}\cos\Lambda\phi \qquad \Lambda \geq 1 \tag{B-3}$$

in conformity with formula (A-59). The function $T_\Lambda(\eta)$ is a polynomial of degree Λ in η and its zeros $\chi_p^{(\Lambda)}$, $p = 1, 2, \ldots, \Lambda$, are given by

$$\chi_p^{(\Lambda)} = \cos\left(\frac{2p-1}{2\Lambda}\right) \qquad p = 1, 2, \ldots, \Lambda, \tag{B-4}$$

just as in equation (A-60). We note that

$$\sum_{p=1}^{\Lambda} \chi_p^{(\Lambda)} = 0. \tag{B-5}$$

We can write for $T_\Lambda(\eta)$ the alternative formula

$$T_\Lambda(\eta) = T_\Lambda \prod_{p=1}^{\Lambda} (\eta - \chi_p^{(\Lambda)}) \tag{B-6}$$

where T_Λ is a constant for a given integer Λ. From (B-6) and the equation obtained from (B-6) after differentiation with respect to η we get

$$\frac{T'_\Lambda(\eta)}{T_\Lambda(\eta)} = \sum_{p=1}^{\Lambda} \frac{1}{(\eta - \chi_p^{(\Lambda)})}, \tag{B-7}$$

and on differentiating (B-7) with respect to η we get

$$\frac{T''_\Lambda(\eta)}{T_\Lambda(\eta)} - \left(\frac{T'_\Lambda(\eta)}{T_\Lambda(\eta)}\right)^2 = -\sum_{p=1}^{\Lambda} \frac{1}{(\eta - \chi_p^{(\Lambda)})^2}. \tag{B-8}$$

Therefore

$$\begin{aligned}\sum_{p=1}^{\Lambda} \frac{\{1 - (\chi_p^{(\Lambda)})^2\}}{(\eta - \chi_p^{(\Lambda)})} &= \sum_{p=1}^{\Lambda} \left\{\frac{(1-\eta^2)}{(\eta - \chi_p^{(\Lambda)})} + \eta + \chi_p^{(\Lambda)}\right\} \\ &= (1-\eta^2)\frac{T'_\Lambda(\eta)}{T_\Lambda(\eta)} + \Lambda\eta,\end{aligned} \tag{B-9}$$

and

$$\sum_{p=1}^{\Lambda} \frac{\{1-(\chi_p^{(\Lambda)})^2\}}{(\eta-\chi_p^{(\Lambda)})^2} = \sum_{p=1}^{\Lambda} \left\{ \frac{(1-\eta^2)}{(\eta-\chi_p^{(\Lambda)})^2} + \frac{2\eta}{(\eta-\chi_p^{(\Lambda)})} - 1 \right\}$$

$$= -(1-\eta^2)\left\{ \frac{T''_\Lambda(\eta)}{T_\Lambda(\eta)} - \left(\frac{T'_\Lambda(\eta)}{T_\Lambda(\eta)}\right)^2 \right\} + 2\eta \frac{T'_\Lambda(\eta)}{T_\Lambda(\eta)} - \Lambda. \tag{B-10}$$

By making use of the formula (B-3) for $T_\Lambda(\eta)$ we establish that

$$\frac{T'_\Lambda(\eta)}{T_\Lambda(\eta)} = \Lambda \frac{\sin \Lambda\phi}{\cos \Lambda\phi \sin \phi} \tag{B-11}$$

and

$$\frac{T''_\Lambda(\eta)}{T_\Lambda(\eta)} = -\frac{1}{(1-\eta^2)}\left\{ \Lambda^2 - \frac{T'_\Lambda(\eta)}{T_\Lambda(\eta)} \cos \phi \right\}. \tag{B-12}$$

In case (i) Λ is given by (see formula (149))

$$\Lambda = a(M+1) \tag{B-13}$$

where a and M are any positive integers. Then, from (B-11), we get

$$\frac{T'_\Lambda(\eta_J^{(M)})}{T_\Lambda(\eta_J^{(M)})} = \Lambda \frac{\sin [J\Lambda\pi/(M+1)]}{\cos [J\Lambda\pi/(M+1)] \sin [J\pi/(M+1)]}$$

$$= \Lambda \frac{\sin (aJ\pi)}{\cos (aJ\pi) \sin [J\pi/(M+1)]}$$

$$= 0 \tag{B-14}$$

for all positive integers J from 1 to M inclusive.

In case (ii) Λ is given by (see formula (150))

$$\Lambda = \tfrac{1}{2}(2a-1)(M+1) \tag{B-15}$$

where a is any positive integer and M is any odd positive integer. Then, from (B-11), we get

$$\frac{T'_\Lambda(\eta_J^{(M)})}{T_\Lambda(\eta_J^{(M)})} = \Lambda \frac{\sin (\frac{1}{2}(2a-1)J\pi)}{\cos (\frac{1}{2}(2a-1)J\pi) \sin [J\pi/(M+1)]}$$

$$= 0 \tag{B-16}$$

for all even positive integers J from 2 to $(M-1)$ inclusive. From (B-12) we then get, in cases (i) and (ii)

$$\frac{T''_\Lambda(\eta_J^{(M)})}{T_\Lambda(\eta_J^{(M)})} = -\frac{\Lambda^2}{\{1-(\eta_J^{(M)})^2\}}. \tag{B-17}$$

Finally, on putting

$$\eta = \eta_J^{(M)} \tag{B-18}$$

into formulae (B-9) and (B-10), using results (B-14), (B-16) and (B-17), and rearranging the resulting expressions, we get

$$\eta_J^{(M)} - \frac{1}{\Lambda} \sum_{p=1}^{\Lambda} \frac{\{1-(\chi_p^{(\Lambda)})^2\}}{(\eta_J^{(M)} - \chi_p^{(\Lambda)})} = 0 \tag{B-19}$$

and

$$1+\frac{1}{\Lambda}\sum_{p=1}^{\Lambda}\frac{\{1-(\chi_p^{(\Lambda)})^2\}}{(\eta_J^{(M)}-\chi_p^{(\Lambda)})^2}=\Lambda \tag{B-20}$$

in both cases (i) and (ii). It must be remembered however that in case (i) the quantity M is any positive integer and J is any positive integer from 1 to M inclusive, whereas in case (ii) the quantity M is any odd positive integer and J is any even positive integer from 2 to $(M-1)$ inclusive.

Formulae (B-19) and (B-20) agree with formulae (147) and (148) of the main text for cases (i) and (ii) if we take

$$P_\Lambda=\Lambda \tag{B-21}$$

because the $\chi_p^{(\Lambda)}$, $p=1,2,\ldots,\Lambda$, are the Gaussian integration points corresponding to the weight function $1/\sqrt{1-\eta_0^2}$.

Case (iii)

We define the function $S_\Lambda(\eta)$ by means of the formula

$$S_\Lambda(\eta)=\cos(\Lambda+\tfrac{1}{2})\phi. \tag{B-22}$$

For any positive integer Λ it is the product of $\sqrt{1+\eta}$ with a polynomial in η of degree Λ and its zeros $\alpha_p^{(\Lambda)}$, $p=1,2,\ldots,\Lambda$, are given by

$$\alpha_p^{(\Lambda)}=\cos\left(\frac{2p-1}{2\Lambda+1}\pi\right)\qquad p=1,2,\ldots,\Lambda, \tag{B-23}$$

which is in conformity with (A-80). We note that

$$\sum_{p=1}^{\Lambda}\alpha_p^{(\Lambda)}=\tfrac{1}{2}. \tag{B-24}$$

We can write for $S_\Lambda(\eta)$ the alternative formula

$$S_\Lambda(\eta)=S_\Lambda\prod_{p=1}^{\Lambda}(\eta-\alpha_p^{(\Lambda)})\sqrt{1+\eta} \tag{B-25}$$

where S_Λ is a constant for a given integer Λ. From (B-25) and the equation obtained from (B-25) after differentiation with respect to η we get

$$\frac{S'_\Lambda(\eta)}{S_\Lambda(\eta)}=\sum_{p=1}^{\Lambda}\frac{1}{(\eta-\alpha_p^{(\Lambda)})}+\frac{1}{2(1+\eta)}, \tag{B-26}$$

and on differentiating (B-26) with respect to η we get

$$\frac{S''_\Lambda(\eta)}{S_\Lambda(\eta)}-\left(\frac{S'_\Lambda(\eta)}{S_\Lambda(\eta)}\right)^2=-\sum_{p=1}^{\Lambda}\frac{1}{(\eta-\alpha_p^{(\Lambda)})^2}-\frac{1}{2(1+\eta)^2}. \tag{B-27}$$

Therefore

$$\sum_{p=1}^{\Lambda}\frac{\{1-(\alpha_p^{(\Lambda)})^2\}}{(\eta-\alpha_p^{(\Lambda)})}=(1-\eta^2)\frac{S'_\Lambda(\eta)}{S_\Lambda(\nu)}+(\Lambda+\tfrac{1}{2})\eta \tag{B-28}$$

and

$$\sum_{p=1}^{\Lambda} \frac{\{1-(\alpha_p^{(\Lambda)})^2\}}{(\eta-\alpha_p^{(\Lambda)})^2} = -(1-\eta^2)\left\{\frac{S''_\Lambda(\eta)}{S_\Lambda(\eta)} - \left(\frac{S'_\Lambda(\eta)}{S_\Lambda(\eta)}\right)^2\right\} + 2\eta\frac{S'_\Lambda(\eta)}{S_\Lambda(\eta)} - (\Lambda+\tfrac{1}{2}). \tag{B-29}$$

By making use of the formula (B-22) for $S_\Lambda(\eta)$ we establish that

$$\frac{S'_\Lambda(\eta)}{S_\Lambda(\eta)} = (\Lambda+\tfrac{1}{2})\frac{\sin(\Lambda+\tfrac{1}{2})\phi}{\cos(\Lambda+\tfrac{1}{2})\phi \sin\phi} \tag{B-30}$$

and

$$\frac{S''_\Lambda(\eta)}{S_\Lambda(\eta)} = -\frac{1}{(1-\eta^2)}\left\{(\Lambda+\tfrac{1}{2})^2 - \frac{S'_\Lambda(\eta)}{S_\Lambda(\eta)}\cos\phi\right\}. \tag{B-31}$$

In case (iii) Λ is given by (see formula (151))

$$\Lambda = \tfrac{1}{2}(2a-1)(M+1) - \tfrac{1}{2} \tag{B-32}$$

where a is any positive integer and M is any even positive integer. Then, from (B-30) and (B-31), we get

$$\frac{S'_\Lambda(\eta_J^{(M)})}{S_\Lambda(\eta_J^{(M)})} = 0 \tag{B-33}$$

and

$$\frac{S''_\Lambda(\eta_J^{(M)})}{S_\Lambda(\eta_J^{(M)})} = -\frac{(\Lambda+\tfrac{1}{2})^2}{\{1-(\eta_J^{(M)})^2\}} \tag{B-34}$$

for all even positive integers J from 2 to M inclusive. Finally, on putting

$$\eta = \eta_J^{(M)} \tag{B-35}$$

into formulae (B-28) and (B-29), using results (B-33) and (B-34), and rearranging the resulting expressions, we get

$$\eta_J^{(M)} - \frac{1}{(\Lambda+\tfrac{1}{2})}\sum_{p=1}^{\Lambda}\frac{\{1-(\alpha_p^{(\Lambda)})^2\}}{(\eta_J^{(M)}-\alpha_p^{(\Lambda)})} = 0 \tag{B-36}$$

and

$$1 + \frac{1}{(\Lambda+\tfrac{1}{2})}\sum_{p=1}^{\Lambda}\frac{\{1-(\alpha_p^{(\Lambda)})^2\}}{(\eta_J^{(M)}-\alpha_p^{(\Lambda)})^2} = \Lambda+\tfrac{1}{2}, \tag{B-37}$$

provided that M is any even positive integer and J is any even positive integer from 2 to M inclusive.

Formulae (B-36) and (B-37) agree with formulae (147) and (148) of the main text for case (iii) if we take

$$P_\Lambda = \Lambda+\tfrac{1}{2}, \tag{B-38}$$

because the $\alpha_p^{(\Lambda)}$, $p=1, 2, \ldots, \Lambda$, are the Gaussian integration points corresponding to the weight function $\sqrt{(1+\eta_0)/(1-\eta_0)}$.

Case (iv)

We define the function $R_\Lambda(\eta)$ by means of the formula

$$R_\Lambda(\eta) = \sin(\Lambda+\tfrac{1}{2})\phi. \tag{B-39}$$

For any positive integer Λ it is the product of $\sqrt{1-\eta}$ with a polynomial in η of degree Λ and its zeros $\beta_p^{(\Lambda)}$, $p=1, 2, \ldots, \Lambda$, are given by

$$\beta_p^{(\Lambda)}=\cos\left(\frac{2p\pi}{2\Lambda+1}\right) \qquad p=1, 2, \ldots, \Lambda, \tag{B-40}$$

which is in conformity with (A-48). We note that

$$\sum_{p=1}^{\Lambda} \beta_p^{(\Lambda)}=-\tfrac{1}{2}. \tag{B-41}$$

We can write for $R_\Lambda(\eta)$ the alternative formula

$$R_\Lambda(\eta)=R_\Lambda \prod_{p=1}^{\Lambda} (\eta-\beta_p^{(\Lambda)})\sqrt{1-\eta} \tag{B-42}$$

where R_Λ is a constant for a given integer Λ.

By proceeding in exactly the same manner as in case (iii) from formula (B-25) onwards we derive the formulae

$$\eta_J^{(M)}-\frac{1}{(\Lambda+\frac{1}{2})}\sum_{p=1}^{\Lambda}\frac{\{1-(\beta_p^{(\Lambda)})^2\}}{(\eta_J^{(M)}-\beta_p^{(\Lambda)})}=0 \tag{B-43}$$

and

$$1+\frac{1}{(\Lambda+\frac{1}{2})}\sum_{p=1}^{\Lambda}\frac{\{1-(\beta_p^{(\Lambda)})^2\}}{(\eta_J^{(M)}-\beta_p^{(\Lambda)})^2}=\Lambda+\tfrac{1}{2} \tag{B-44}$$

provided that M is any even positive integer and J is any odd positive integer from 1 to $(M-1)$ inclusive.

Formulae (B-43) and (B-44) agree with formulae (147) and (148) of the main text for case (iv) if we take

$$P_\Lambda=\Lambda+\tfrac{1}{2}, \tag{B-45}$$

because the $\beta_p^{(\Lambda)}$, $p=1, 2, \ldots, \Lambda$, are the Gaussian integration points corresponding to the weight function $\sqrt{(1-\eta_0)/(1+\eta_0)}$.

Case (v)

We define the function $Q_\Lambda(\eta)$ by means of the formula

$$Q_\Lambda(\eta)=\sin(\Lambda+1)\phi. \tag{B-46}$$

For any positive integer Λ it is the product of $\sqrt{1-\eta^2}$ with a polynomial in η of degree Λ, the polynomial being a multiple of $\gamma_\Lambda(\eta)$ of formula (A-24). The zeros $\eta_p^{(\Lambda)}$, $p=1, 2, \ldots, \Lambda$, of $Q_\Lambda(\eta)$ are given by

$$\eta_p^{(\Lambda)}=\cos\left(\frac{p\pi}{\Lambda+1}\right) \qquad p=1, 2, \ldots, \Lambda, \tag{B-47}$$

just as in equation (A-25). We note that

$$\sum_{p=1}^{\Lambda} \eta_p^{(\Lambda)}=0. \tag{B-48}$$

We can write for $Q_\Lambda(\eta)$ the alternative formula

$$Q_\Lambda(\eta)=Q_\Lambda \prod_{p=1}^{\Lambda} (\eta-\eta_p^{(\Lambda)})\sqrt{1-\eta^2} \tag{B-49}$$

where Q_Λ is a constant for a given integer Λ. From (B-49) and the equation obtained from (B-49) after differentiation with respect to η we get

$$\frac{Q'_\Lambda(\eta)}{Q_\Lambda(\eta)}=\sum_{p=1}^{\Lambda}\frac{1}{(\eta-\eta_p^{(\Lambda)})}-\frac{\eta}{(1-\eta^2)}, \tag{B-50}$$

and on differentiating (B-50) with respect to η we get

$$\frac{Q''_\Lambda(\eta)}{Q_\Lambda(\eta)}-\left(\frac{Q'_\Lambda(\eta)}{Q_\Lambda(\eta)}\right)^2=-\sum_{p=1}^{\Lambda}\frac{1}{(\eta-\eta_p^{(\Lambda)})^2}-\frac{(1+\eta^2)}{(1-\eta^2)^2}. \tag{B-51}$$

Therefore

$$\sum_{p=1}^{\Lambda}\frac{\{1-(\eta_p^{(\Lambda)})^2\}}{(\eta-\eta_p^{(\Lambda)})}=(1-\eta^2)\frac{Q'_\Lambda(\eta)}{Q_\Lambda(\eta)}+(\Lambda+1)\eta \tag{B-52}$$

and

$$\sum_{p=1}^{\Lambda}\frac{\{1-(\eta_p^{(\Lambda)})^2\}}{(\eta-\eta_p^{(\Lambda)})^2}=-(1-\eta^2)\left\{\frac{Q''_\Lambda(\eta)}{Q_\Lambda(\eta)}-\left(\frac{Q'_\Lambda(\eta)}{Q_\Lambda(\eta)}\right)^2\right\}+2\eta\frac{Q'_\Lambda(\eta)}{Q_\Lambda(\eta)}-(\Lambda+1). \tag{B-53}$$

By making use of the formula (B-46) for $Q_\Lambda(\eta)$ we establish that

$$\frac{Q'_\Lambda(\eta)}{Q_\Lambda(\eta)}=-(\Lambda+1)\frac{\cos(\Lambda+1)\phi}{\sin(\Lambda+1)\phi\sin\phi} \tag{B-54}$$

and

$$\frac{Q''_\Lambda(\eta)}{Q_\Lambda(\eta)}=-\frac{1}{(1-\eta^2)}\left\{(\Lambda+1)^2-\frac{Q'_\Lambda(\eta)}{Q_\Lambda(\eta)}\cos\phi\right\}. \tag{B-55}$$

In case (v) Λ is given by (see formula (153))

$$\Lambda=\tfrac{1}{2}(2a-1)(M+1)-1 \tag{B-56}$$

where a is any positive integer and M is any odd positive integer. Then, from (B-54) and (B-55), we get

$$\frac{Q'_\Lambda(\eta_J^{(M)})}{Q_\Lambda(\eta_J^{(M)})}=0 \tag{B-57}$$

and

$$\frac{Q''_\Lambda(\eta_J^{(M)})}{Q_\Lambda(\eta_J^{(M)})}=-\frac{(\Lambda+1)^2}{\{1-(\eta_J^{(M)})^2\}} \tag{B-58}$$

for all odd positive integers J from 1 to M inclusive. Finally, on putting

$$\eta=\eta_J^{(M)} \tag{B-59}$$

into formulae (B-52) and (B-53), using results (B-57) and (B-58), and rearranging the resulting expressions, we get

$$\eta_J^{(M)}-\frac{1}{(\Lambda+1)}\sum_{p=1}^{\Lambda}\frac{\{1-(\eta_p^{(\Lambda)})^2\}}{(\eta_J^{(M)}-\eta_p^{(\Lambda)})}=0 \tag{B-60}$$

and

$$1+\frac{1}{(\Lambda+1)}\sum_{p=1}^{\Lambda}\frac{\{1-(\eta_p^{(\Lambda)})^2\}}{(\eta_J^{(M)}-\eta_p^{(\Lambda)})^2}=\Lambda+1, \tag{B-61}$$

provided that M is any odd positive integer and J is any odd positive integer from 1 to M inclusive.

Formulae (B-60) and (B-61) agree with formulae (147) and (148) of the main text for case (v) if we take

$$P_\Lambda=\Lambda+1, \tag{B-62}$$

because the $\eta_p^{(\Lambda)}$, $p=1, 2, \ldots, \Lambda$, are the Gaussian integration points corresponding to the weight function $\sqrt{1-\eta_0^2}$.

APPENDIX C

Numerical Evaluation of the Function $I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)$

Integration with respect to ξ_0

In this Appendix we discuss the process for the numerical evaluation of the function $I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty)$ from the formula

$$I_r^{(n)}(\xi, \eta, \eta_0; \nu, M_\infty) = \frac{1}{4\pi}\left(\frac{y-y_0}{l}\right)^2 \int_0^1 h_r^{(n)}(\xi_0)\sqrt{\frac{1-\xi_0}{\xi_0}}K\left(\frac{x-x_0}{l}, \frac{y-y_0}{l}; \nu, M_\infty\right) d\xi_0 \qquad \text{(C-1)}$$

where

$$\left.\begin{aligned} \xi &= \frac{1}{c(y)}[x - x_L(y)] \\ \eta &= \frac{y}{s} \\ \xi_0 &= \frac{1}{c(y_0)}[x_0 - x_L(y_0)] \\ \eta_0 &= \frac{y_0}{s} \end{aligned}\right\} \qquad \text{(C-2)}$$

$$K\left(\frac{x}{l}, \frac{y}{l}; \nu, M_\infty\right) = l^2 \int_{(-x+M_\infty R)/(1-M_\infty^2)}^{\infty} e^{-i\nu u/l}\frac{du}{(u^2+y^2)^{\frac{3}{2}}} + l^2\frac{M_\infty(M_\infty x + R)}{R(x^2+y^2)}\exp\left\{\frac{-i\nu}{l}\left(\frac{-x+M_\infty R}{1-M_\infty^2}\right)\right\} \qquad \text{(C-3)}$$

and

$$R = \sqrt{\{x^2 + (1-M_\infty^2)y^2\}}. \qquad \text{(C-4)}$$

Formula (C-1) is identical with formula (114) of the main text.

If $\eta_0 = \eta$ the evaluation of the integral on the right hand side of (C-1) is straightforward because, from (C-3) we have

$$\lim_{y\to 0}\frac{y^2}{l^2}K\left(\frac{x}{l}, \frac{y}{l}; \nu, M_\infty\right) = \begin{cases} 0 & x<0, \\ 2 & x>0, \end{cases} \qquad \text{(C-5)}$$

and therefore,

$$I_r^{(n)}(\xi, \eta, \eta; \nu, M_\infty) = \frac{1}{2\pi}\int_0^{\xi} h_r^{(n)}(\xi_0)\sqrt{\frac{1-\xi_0}{\xi_0}}\,d\xi_0. \qquad \text{(C-6)}$$

The integral on the right hand side of (C-6) can be evaluated analytically using formulae of Appendix A or it can be evaluated numerically. To evaluate it numerically, divide the range $(0, \xi)$ of ξ_0 into a number of intervals of equal length and apply Gaussian integration formulae, using a small number of integration points, to the integral over each of these intervals. In the interval abutting on $\xi_0 = 0$ the weight function $1/\sqrt{\xi_0}$ is used whereas in all the other intervals the weight function 1 is used for the Gaussian integration. The number of intervals of integration which it is necessary to take to obtain a given accuracy will depend on the number of integration points in each interval and on the value of n, and will need to be increased if n is increased because of the resulting increase in the number of undulations of the function $h_r^{(n)}(\xi_0)$. The actual number of intervals required can be assessed by experience.

If $\eta_0 \neq \eta$ the range (0, 1) of ξ_0 in the integral on the right hand side of (C-1) is divided into a number of intervals, not necessarily of equal length, and Gaussian integration formulae using a small number of integration points are applied to the integral over each of these intervals. In the interval abutting on $\xi_0 = 0$ the

Then formula (C-11) can be written as

$$J\left(\frac{x}{l}, \frac{y}{l}; \nu, M_\infty\right) = S\left(\frac{\nu|y|}{l}\right) - \int_0^{X/|y|} e^{-i\nu|y|u/l} \frac{du}{(u^2+1)^{\frac{3}{2}}}. \tag{C-13}$$

We can show that $S(\alpha)$ satisfies the differential equation

$$\alpha\left\{\frac{d^2S(\alpha)}{d\alpha^2} - S(\alpha)\right\} - \frac{dS(\alpha)}{d\alpha} = i \tag{C-14}$$

by inserting $S(\alpha)$ from (C-12) into (C-14). If we put

$$S(\alpha) = \alpha P(\alpha), \tag{C-15}$$

substitute for $S(\alpha)$ from (C-15) into (C-14) and divide the resulting equation through by α^2 we get the differential equation

$$P''(\alpha) + \frac{1}{\alpha}P'(\alpha) - \left(1 + \frac{1}{\alpha^2}\right)P(\alpha) = \frac{i}{\alpha^2} \tag{C-16}$$

for $P(\alpha)$, which we recognise as a modified form of Bessel's differential equation with a non-zero right hand side. This differential equation has the general solution (see e.g. Ref. 15)

$$P(\alpha) = \frac{\pi}{2}\{iL_1(\alpha) + CI_1(\alpha) + DK_1(\alpha)\} \tag{C-17}$$

where C and D are integration constants, $L_1(\alpha)$ and $K_1(\alpha)$ are modified Bessel functions of order 1 and of the first and second kinds respectively and $L_1(\alpha)$ is a modified Struve function which is related to the Struve function $H_{-1}(i\alpha)$ by means of the formula (see Ref. 15)

$$L_1(\alpha) = -H_{-1}(i\alpha). \tag{C-18}$$

For small values of $|\alpha|$ we may write (see Ref. 15)

$$L_1(\alpha) = \alpha j(\alpha) \tag{C-19}$$

$$K_1(\alpha) = \frac{1}{\alpha} + \alpha\left[\log\left(\frac{\alpha}{2}\right) + \gamma\right] j(\alpha) - \alpha f(\alpha) \tag{C-20}$$

$$L_1(\alpha) = -\frac{2}{\pi} g(\alpha), \tag{C-21}$$

where $j(\alpha)$, $f(\alpha)$ and $g(\alpha)$ are even integral functions of α which have the power series expansions:

$$j(\alpha) = \frac{1}{2}\sum_{r=0}^{\infty} \frac{1}{r!(r+1)!}\left(\frac{\alpha}{2}\right)^{2r} \tag{C-22}$$

$$f(\alpha) = \frac{1}{2}\sum_{r=0}^{\infty} \frac{1}{r!(r+1)!}\left\{\sum_{s=0}^{r} \frac{1}{(s+1)} - \frac{1}{2(r+1)}\right\}\left(\frac{\alpha}{2}\right)^{2r} \tag{C-23}$$

and

$$g(\alpha) = \sum_{r=0}^{\infty} \frac{2^{2r} r! r!}{(2r)!(2r+1)!} \alpha^{2r}. \tag{C-24}$$

For real positive α the branch of $\log(\alpha/2)$ which is real is taken in formula (C-20). Therefore, on using the above expansions, we get, for small values of α

$$S(\alpha)=\tfrac{1}{2}\pi D-(i\alpha)+\tfrac{1}{4}\pi C\alpha^2+\tfrac{1}{4}\pi D\alpha^2\left[\gamma+\log\left(\frac{\alpha}{2}\right)-\frac{1}{2}\right]+0(\alpha^3). \tag{C-25}$$

However, we can obtain an expansion of $S(\alpha)$, for small values of α, directly from the integral representation (C-12). In fact, for small real positive α, we can show that

$$S(\alpha)=1-(i\alpha)+\tfrac{1}{2}\alpha^2\left[\gamma+\log\left(\frac{\alpha}{2}\right)-\frac{1}{2}+\frac{i\pi}{2}\right]+0(\alpha^3). \tag{C-26}$$

On comparing the expansions (C-25) and (C-26) we get immediately

$$C=i \tag{C-27}$$

and

$$D=\frac{2}{\pi} \tag{C-28}$$

for the values of the integration constants C and D introduced in formula (C-17).

Having obtained the values of the integration constants C and D we can substitute for $P(\alpha)$ from (C-17) into (C-15) to get $S(\alpha)$ in the form

$$S(\alpha)=F(\alpha)+iG(\alpha) \tag{C-29}$$

where

$$F(\alpha)=1+\left[\gamma+\log\left(\frac{\alpha}{2}\right)\right]\alpha^2 j(\alpha)-\alpha^2 f(\alpha) \tag{C-30}$$

$$G(\alpha)=\frac{\pi}{2}\alpha^2 j(\alpha)-\alpha g(\alpha) \tag{C-31}$$

are real functions of α for real positive α.

We may also deduce from (C-12) the following asymptotic expansions for $F(\alpha)$ and $G(\alpha)$ of formula (C-29) for large real positive α. With the integer $n\geqslant 0$ arbitrary we get

$$F(\alpha)=\sqrt{\frac{\pi\alpha}{2}}\,e^{-\alpha}\left\{F_n(\alpha)+\frac{1}{\alpha^n}\delta_n(\alpha)\right\} \tag{C-32}$$

and

$$G(\alpha)=G_n(\alpha)+\frac{1}{\alpha^{2n+1}}\varepsilon_n(\alpha) \tag{C-33}$$

where

$$F_0(\alpha)=1 \tag{C-34}$$

$$F_n(\alpha)=1+\sum_{r=1}^{n}\frac{1}{r!(8\alpha)^r}\prod_{s=1}^{r}(4-(2s-1)^2) \qquad n\geqslant 1, \tag{C-35}$$

$$G_n(\alpha)=-\frac{1}{\alpha}\sum_{r=0}^{n}\frac{(2r)!(2r+1)!}{r!r!(2\alpha)^{2r}} \qquad n\geqslant 0. \tag{C-36}$$

The remainder functions $\delta_n(\alpha)$, $\varepsilon_n(\alpha)$, for any $n \geqslant 0$ have the behaviour

$$\delta_n(\alpha) = 0(1) \tag{C-37}$$

$$\varepsilon_n(\alpha) = 0(1) \tag{C-38}$$

for $\alpha \to +\infty$.

The same asymptotic expansions (C-32) and (C-33) may be deduced by using the asymptotic expansions of $I_1(\alpha)$, $K_1(\alpha)$ and $L_1(\alpha)$, as given for example in Ref. 15, in formula (C-17) for $P(\alpha)$ with C and D given by (C-27) and (C-28) and substituting the resulting asymptotic expansion into (C-15). It is, however, quite easy to get the required expansions directly from (C-12).

The power series (C-22), (C-23) and (C-24) have to be truncated at finite values of r in order to be able to evaluate from them the values of the functions $j(\alpha)$, $f(\alpha)$ and $g(\alpha)$, and these finite values of r will depend upon the accuracy to which these functions are required and on the value of α under consideration. If we work numerically to a given number of significant figures the accuracy with which we can evaluate the sums of the truncated series will decrease as α increases. Thus, even though these power series expansions are convergent for any finite value of α, they cannot be used to give accurate values of the functions $j(\alpha)$, $f(\alpha)$ and $g(\alpha)$ when α becomes indefinitely large, if we are limited in the number of significant figures used in the arithmetical operations. By working with a given number of significant figures there is a maximum value of α for which formula (C-30), with $j(\alpha)$ and $f(\alpha)$ obtained from the power series expansions (C-22) and (C-23) respectively, can be used to obtain $F(\alpha)$ to within some prescribed accuracy $\varepsilon > 0$. Similarly there is a maximum value of α for which formula (C-31), with $j(\alpha)$ and $g(\alpha)$ obtained from the power series expansions (C-22) and (C-24) respectively, can be used to obtain $G(\alpha)$ with the accuracy ε. The smaller α is, the fewer terms, in general, will be needed in the truncation of the power series expansions (C-22), (C-23), (C-24) to obtain $F(\alpha)$ and $G(\alpha)$ to within the accuracy ε.

For very large values of α we can use the asymptotic formulae (C-23) and (C-33) to evaluate the values of $F(\alpha)$ and $G(\alpha)$. Because of formula (C-32) we can, for given $\varepsilon > 0$, find $\alpha_1(n, \varepsilon) > 0$ such that

$$\left| F(\alpha) - \sqrt{\frac{\pi\alpha}{2}}\, e^{-\alpha} F_n(\alpha) \right| < \varepsilon \tag{C-39}$$

whenever

$$\alpha > \alpha_1(n, \varepsilon). \tag{C-40}$$

If we take $\alpha_1(n, \varepsilon)$ to be the minimum quantity for which (C-39) is true under the condition (C-40) we find that, for fixed ε, $\alpha_1(n, \varepsilon)$ decreases in general as n is increased from zero up to a certain value of n and then increases as n is increased beyond this certain value. The minimum value of $\alpha_1(n, \varepsilon)$, for all the values of n, is then the minimum value of α for which $F(\alpha)$ may be obtained to within accuracy ε from formula (C-32). This accuracy may be somewhat reduced if we work numerically to a given number of significant figures. The higher α is, beyond the minimum value, the smaller will the value of n need to be, in general, for (C-39) to be true. Likewise, because of formula (C-33) we can, for given $\varepsilon > 0$, find $\alpha_2(n, \varepsilon) > 0$ such that

$$|G(\alpha) - G_n(\alpha)| < \varepsilon \tag{C-41}$$

whenever

$$\alpha > \alpha_2(n, \varepsilon). \tag{C-42}$$

Again there is a minimum value of α for which $G(\alpha)$ may be obtained to within accuracy ε from formula (C-33).

The maximum values of α for which $F(\alpha)$ and $G(\alpha)$ can be evaluated to the given accuracy from formulae (C-30) and (C-31) with $j(\alpha)$, $f(\alpha)$ and $g(\alpha)$ obtained from the power series expansions (C-22), (C-23) and (C-24) respectively, depends strongly on the number of significant figures used in the arithmetic, whereas the minimum values of α for which $F(\alpha)$ and $G(\alpha)$ can be evaluated to the given accuracy from formulae (C-32) and (C-31) is hardly dependent on the number of significant figures used in the arithmetic, provided that this number is greater than the number of significant figures required in the values of the functions. The functions

$F(\alpha)$ and $G(\alpha)$ cannot be evaluated to the given accuracy ε, for all α in $(0, \infty)$, using formulae (C-30), (C-31), (C-32) and (C-33) as described above, if the number of significant figures used in the arithmetic is not sufficiently high. In other words, if we work to a given number of significant figures, then ε must be greater than a certain lower bound in order that $F(\alpha)$ and $G(\alpha)$ may be evaluated in the above manner to the given accuracy ε for all α in $(0, \infty)$. If ε is less than this bound then some other means of evaluating the functions $F(\alpha)$ and $G(\alpha)$ must be used, at least over the ranges of α for which the above method does not yield the required accuracy ε. We shall expand the functions in series of Chebyshev polynomials rather than power series. Although ε will still have to be greater than a certain lower bound in order that $F(\alpha)$ and $G(\alpha)$ may be evaluated to the given accuracy ε, this lower bound should be less than the former lower bound, thus rendering the procedure involving expansion of fuctions in series of Chebyshev polynomials of wider application than that involving expansion of functions in power series. It may be true that with the number of significant figures available on a particular computing machine, functions may be evaluated to a sufficient accuracy for some applications using expansions in power series, but it would seem to be good practice to use another procedure which is capable of giving superior accuracy and is no more difficult to apply.

Use of Chebyshev Polynomials

The formulae (C-30), (C-31), (C-32) and (C-33) are valid for complex values of α provided the branch line of $\log(\alpha/2)$ lies in the half-plane $\mathrm{Im}(\alpha) \geq 0$, but we are here concerned only with real positive values of α. There are convergent expansions for $j(\alpha)$, $f(\alpha)$ and $g(\alpha)$, which are valid only for real α but which are more suitable for numerical computation than are (C-22), (C-23) and (C-24) when α is in some restricted interval $(0, A)$ where A is some positive finite number. We can also use expansions for $F(\alpha)$ and $G(\alpha)$ which are valid only for real α but which are convergent when α is in the restricted range (A, ∞) as opposed to the expansions (C-32) and (C-33) which are asymptotic expansions. These convergent expansions in series of orthogonal polynomials are again more suitable for numerical computation than are the asymptotic expansions. Because of their simple properties we shall use the Chebyshev polynomials rather than other orthogonal polynomials, but it is well to remember that other orthogonal polynomials may be more appropriate to use in some circumstances. The Chebyshev polynomial $T_n(x)$ defined by

$$T_n(x) = \cos(n \cos^{-1} x) \qquad n = 0, 1, 2, \ldots, \tag{C-43}$$

is a polynomial of degree n in x. Because $j(\alpha)$, $f(\alpha)$ and $g(\alpha)$ in formulae (C-30) and (C-31) are integral functions of α^2 we can write, for $0 \leq \alpha \leq A$,

$$F(\alpha) = \sum_{r=0}^{\infty}{}' D_r(A) T_r\left(\frac{2\alpha^2}{A^2} - 1\right) + \frac{1}{2}\alpha^2 \log\left(\frac{\alpha}{A}\right) \sum_{r=0}^{\infty}{}' C_r(A) T_r\left(\frac{2\alpha^2}{A^2} - 1\right) \tag{C-44}$$

and

$$G(\alpha) = -\alpha \sum_{r=0}^{\infty}{}' E_r(A) T_r\left(\frac{2\alpha^2}{A^2} - 1\right) + \frac{\pi}{4}\alpha^2 \sum_{r=0}^{\infty}{}' C_r(A) T_r\left(\frac{2\alpha^2}{A^2} - 1\right) \tag{C-45}$$

where the dash $'$ on the summation sign $\sum$ indicates that the quantity under the summation sign for $r = 0$ is to be multiplied by $\frac{1}{2}$. The coefficients $C_r(A)$, $D_r(A)$ and $E_r(A)$ in the formulae (C-44) and (C-45) may be determined numerically for $r = 0, 1, 2, \ldots$, for a given value of A by applying the method of Clenshaw[16].

It is apparent from equations (C-32) and (C-33) that the function $e^{\alpha} F(\alpha)/\sqrt{\alpha}$ is a function of bounded variation in $1/\alpha$ and the function $\alpha G(\alpha)$ is a function of bounded variation $1/\alpha^2$ for large α. Hence we can write, for $A \leq \alpha \leq \infty$

$$F(\alpha) = \sqrt{\alpha}\, e^{-\alpha} \sum_{r=0}^{\infty}{}' A_r(A) T_r\left(\frac{2A}{\alpha} - 1\right) \tag{C-46}$$

and

$$G(\alpha) = -\frac{1}{\alpha} \sum_{r=0}^{\infty}{}' B_r(A) T_r\left(\frac{2A^2}{\alpha^2} - 1\right), \tag{C-47}$$

where, again, the dash ′ on the summation sign $\sum$ indicates that the quantity under the summation sign for $r = 0$ is to be multiplied by $\frac{1}{2}$. The coefficients $A_r(A)$ and $B_r(A)$ in the formulae (C-46) and (C-47) may be determined numerically for $r = 0, 1, 2, \ldots$, for a given value of A by applying the method of Clenshaw[16].

If we determine the coefficients $A_r(A)$, $B_r(A)$, $C_r(A)$, $D_r(A)$ and $E_r(A)$ numerically for $A = 7$ we get the following results:

$$\left.\begin{aligned} A_0(7) &= 2{\cdot}571\,72 \\ A_1(7) &= 0{\cdot}032\,21 \\ A_2(7) &= -0{\cdot}000\,32 \\ A_3(7) &= 0{\cdot}000\,01 \end{aligned}\right\} \qquad \text{(C-48)}$$

$$\left.\begin{aligned} B_0(7) &= 2{\cdot}080\,503\,0 \\ B_1(7) &= 0{\cdot}043\,102\,0 \\ B_2(7) &= 0{\cdot}002\,271\,0 \\ B_3(7) &= -0{\cdot}000\,790\,2 \\ B_4(7) &= -0.000\,144\,5 \\ B_5(7) &= 0{\cdot}000\,075\,0 \\ B_6(7) &= -0{\cdot}000\,002\,4 \\ B_7(7) &= -0{\cdot}000\,008\,1 \\ B_8(7) &= 0{\cdot}000\,003\,5 \\ B_9(7) &= -0{\cdot}000\,000\,4 \\ B_{10}(7) &= -0{\cdot}000\,000\,4 \\ B_{11}(7) &= 0{\cdot}000\,000\,3 \\ B_{12}(7) &= -0{\cdot}000\,000\,1 \end{aligned}\right\} \qquad \text{(C-49)}$$

$$\left.\begin{aligned} C_0(7) &= 31{\cdot}850\,997\,610 \\ C_1(7) &= 20{\cdot}478\,045\,274 \\ C_2(7) &= 6{\cdot}700\,050\,944 \\ C_3(7) &= 1{\cdot}298\,530\,059 \\ C_4(7) &= 0{\cdot}164\,782\,643 \\ C_5(7) &= 0{\cdot}014\,674\,957 \\ C_6(7) &= 0{\cdot}000\,964\,651 \\ C_7(7) &= 0{\cdot}000\,048\,629 \\ C_8(7) &= 0{\cdot}000\,001\,937 \\ C_9(7) &= 0{\cdot}000\,000\,062 \\ C_{10}(7) &= 0{\cdot}000\,000\,002 \end{aligned}\right\} \qquad \text{(C-50)}$$

$$\left.\begin{aligned}
D_0(7) &= 48{\cdot}347\,809\,077 \\
D_1(7) &= 14{\cdot}168\,616\,622 \\
D_2(7) &= -19{\cdot}622\,120\,487 \\
D_3(7) &= -14{\cdot}015\,111\,195 \\
D_4(7) &= -3{\cdot}981\,038\,753 \\
D_5(7) &= -0{\cdot}646\,707\,438 \\
D_6(7) &= -0{\cdot}068\,862\,606 \\
D_7(7) &= -0{\cdot}005\,195\,562 \\
D_8(7) &= -0{\cdot}000\,292\,608 \\
D_9(7) &= -0{\cdot}000\,012\,776 \\
D_{10}(7) &= -0{\cdot}000\,000\,445 \\
D_{11}(7) &= -0{\cdot}000\,000\,013
\end{aligned}\right\} \qquad \text{(C-51)}$$

$$\left.\begin{aligned}
E_0(7) &= 156{\cdot}661\,749\,51 \\
E_1(7) &= 111{\cdot}982\,807\,86 \\
E_2(7) &= 43{\cdot}246\,690\,85 \\
E_3(7) &= 9{\cdot}928\,919\,83 \\
E_4(7) &= 1{\cdot}474\,963\,03 \\
E_5(7) &= 0{\cdot}151\,515\,57 \\
E_6(7) &= 0{\cdot}011\,328\,83 \\
E_7(7) &= 0{\cdot}000\,641\,65 \\
E_8(7) &= 0{\cdot}000\,028\,41 \\
E_9(7) &= 0{\cdot}000\,001\,01 \\
E_{10}(7) &= 0{\cdot}000\,000\,03
\end{aligned}\right\} \qquad \text{(C-52)}$$

If we now use the values of the coefficients $A_r(7)$, $B_r(7)$, $C_r(7)$, $D_r(7)$ and $E_r(7)$ from (C-48), (C-49), (C-50), (C-51) and (C-52) in formulae (C-44), (C-45), (C-46), (C-47), and neglect the remaining higher order coefficients, we can evaluate $F(\alpha)$ and $G(\alpha)$ for any given value of α. By doing this for all the integers α from 0 to 25, using 11 significant figures in the arithmetic, we find the values $F(\alpha)$ and $G(\alpha)$ of Table C-1 (overleaf).

The function values tabulated in Table C-1 should all be correct to seven decimal places. To achieve this accuracy the coefficients $A_r(7)$ needed to be given only to five decimal places because of the factor $\sqrt{\alpha}\,e^{-\alpha}$ in formula (C-46), and the coefficients $B_r(7)$ needed to be given only to seven decimal places because of the factor $1/\alpha$ in formula (C-47). The coefficients $C_r(7)$, $D_r(7)$ and $E_r(7)$ could have been given to one fewer decimal place each, but there is not much gain in this because these coefficients reduce so rapidly as r is increased.

It is also possible to evaluate $F(\alpha)$ to seven places of decimals for $\alpha = 7$ from the asymptotic approximation $\sqrt{\pi\alpha/2}\,e^{-\alpha}F_n(\alpha)$ with $n = 5$. For higher values of α the value of n required to get this accuracy may be less than 5. However it is possible to evaluate $G(\alpha)$ accurate to only three places of decimals for $\alpha = 7$ from the asymptotic approximation $G_n(\alpha)$ and to get the highest accuracy for $\alpha = 7$ we must take $n = 2$. Thus with the demarcation value $A = 7$ we must turn to some other formula, such as (C-47), in order to be able to evaluate $G(\alpha)$ to seven places of decimals for $\alpha \geqslant 7$. The asymptotic approximation $G_n(\alpha)$ cannot be used to evaluate $G(\alpha)$ to seven decimal places unless $\alpha > 17$.

If $F(\alpha)$ and $G(\alpha)$ were required to higher accuracy than seven decimal places then the demarcation value A in formulae (C-44), (C-45), (C-46) and (C-47) would have to be taken to be some value less than 7, unless the

TABLE C-1

α	$F(\alpha)$	$G(\alpha)$
0	1·000 000 0	0·000 000 0
1	0·601 907 2	−0·468 450 8
2	0·279 731 8	−0·467 289 0
3	0·120 469 3	−0·376 343 2
4	0·049 934 0	−0·291 743 6
5	0·020 223 1	−0·229 284 5
6	0·008 063 5	−0·185 609 1
7	0·003 179 3	−0·154 965 3
8	0·001 243 0	−0·132 881 5
9	0·000 482 7	−0·116 408 0
10	0·000 186 5	−0·103 692 7
11	0·000 071 7	−0·093 577 4
12	0·000 027 5	−0·085 324 8
13	0·000 010 5	−0·078 451 3
14	0·000 004 0	−0·072 629 2
15	0·000 001 5	−0·067 628 7
16	0·000 000 6	−0·063 283 7
17	0·000 000 2	−0·059 471 1
18	0·000 000 0	−0·056 097 1
19	0·000 000 0	−0·053 089 4
20	0·000 000 0	−0·050 390 6
21	0·000 000 0	−0·047 955 1
22	0·000 000 0	−0·045 745 8
23	0·000 000 0	−0·043 732 4
24	0·000 000 0	−0·041 889 7
25	0·000 000 0	−0·040 196 9

number of significant figures used in the arithmetic were increased beyond 11. The coefficients $A_r(A)$ and $B_r(A)$ would decrease more slowly as r increased and the coefficients $C_r(A)$, $D_r(A)$ and $E_r(A)$ would decrease more rapidly as r is increased with this lower value of A than was the case with $A=7$. For high enough required accuracy it will no longer be possible to obtain $F(\alpha)$ from the asymptotic approximation $\sqrt{\pi\alpha/2}\,e^{-\alpha}F_n(\alpha)$ at the demarcation value $\alpha=A$.

The numbers of terms which need to be retained in the infinite series in (C-44), (C-45), (C-46) and (C-47) depend only on A and not on the value of α. On the other hand, the numbers of terms which need to be retained in the infinite series (C-22), (C-23) and (C-24) do depend on α and are very small when α is very small. Thus for very small α it is less work numerically to evaluate $F(\alpha)$ and $G(\alpha)$ from (C-30) and (C-31) using the series (C-22), (C-23) and (C-24) for $j(\alpha)$, $f(\alpha)$ and $g(\alpha)$, than it is to evaluate $F(\alpha)$ and $G(\alpha)$ from (C-44) and (C-45), but over the whole range $(0, A)$ the formulae (C-44) and (C-45) are more economical.

The numerical evaluation of the summations in formulae (C-44), (C-45), (C-46) and (C-47) is easily carried out by using the scheme described by Clenshaw[16].

From (C-43) we get, on using elementary properties of the cosine function,

$$T_0(x)=1 \tag{C-53}$$

$$T_1(x)=x \tag{C-54}$$

and the reduction formula

$$T_{n+1}(x)-2xT_n(x)+T_{n-1}(x)=0. \tag{C-55}$$

Suppose now that we wish to evaluate $k(x)$ where $k(x)$ is given by the formula

$$k(x)=\sum_{r=0}^{n}{}' a_r T_r(x). \tag{C-56}$$

Clenshaw's scheme is to put

$$b_{n+2}=0 \tag{C-57}$$

$$b_{n+1}=0 \tag{C-58}$$

$$b_{n-r}=2xb_{n-r+1}-b_{n-r+2}+a_{n-r}, \qquad r=0,1,2,\ldots,n. \tag{C-59}$$

Then

$$k(x)=\tfrac{1}{2}(b_0-b_2) \tag{C-60}$$

as may be shown by application of (C-53), (C-54) and (C-55).

The function $S(\alpha)$ is now obtained from (C-29) and the result used in (C-13) to obtain $J(x/l, y/l; \nu, M_\infty)$ when $|X|/|y|$ is not large.

Procedures when $|X|/|y|$ is Large

To evaluate $J(x/l, y/l; \nu, M_\infty)$ when $X/|y|$ is large and positive we replace $1/(u^2+1)^{\frac{3}{2}}$ in the integrand on the right hand side of (C-9) by its expansion as a power series in $1/u$ and integrate term by term to get

$$J\left(\frac{x}{l},\frac{y}{l};\nu,M_\infty\right)=\sum_{r=0}^{\infty}(-1)^r\frac{(2r+1)!}{2^{2r}r!r!}\left(\frac{|y|}{X}\right)^{2r+2}E_{2r+3}\left(\frac{\nu X}{l}\right) \tag{C-61}$$

where

$$E_r(\alpha)=\alpha^{r-1}\int_\alpha^\infty e^{-iv}\frac{dv}{v^r}, \qquad r=3,5,7,\ldots. \tag{C-62}$$

The expansion on the right hand side of (C-61) is convergent for $X/|y|\geqslant 1$ and, the larger $X/|y|$ is, the faster does the expansion (C-61) converge, i.e. the fewer terms in a finite truncation of it are necessary to get $J(x/l, y/l; \nu, M_\infty)$ to a given accuracy. We may arbitrarily choose to use the expansion on the right hand side of (C-61) for obtaining $J(x/l, y/l; \nu, M_\infty)$ when $X/|y|\geqslant 2$.

To evaluate $E_{2r+3}(\alpha)$ we first use the recurrence relationships

$$E_{2p+3}(\alpha)=\left[\frac{1}{(2p+2)}-\frac{i\alpha}{(2p+2)(2p+1)}\right]e^{-i\alpha}-\frac{\alpha^2}{(2p+2)(2p+1)}E_{2p+1}(\alpha) \qquad p=1,2,\ldots,r, \tag{C-63}$$

to express $E_{2r+3}(\alpha)$ in terms of $E_3(\alpha)$.

We may express $E_3(\alpha)$ by means of the formula

$$E_3(\alpha)=\tfrac{1}{2}\alpha^2\left(\log\alpha+\frac{i\pi}{2}\right)+H(\alpha)-i\alpha K(\alpha), \tag{C-64}$$

where the branch of the function $\log\alpha$ which is real for α real positive is taken and $H(\alpha)$ and $K(\alpha)$ are even integral functions of α. We may deduce the power series expansions

$$H(\alpha)=\tfrac{1}{2}-\tfrac{1}{2}(\tfrac{3}{2}-\gamma)\alpha^2-\tfrac{1}{2}\alpha^4\sum_{s=0}^{\infty}\frac{(-1)^s\alpha^{2s}}{(s+1)(2s+4)!} \tag{C-65}$$

and

$$K(\alpha)=1+\alpha^2\sum_{s=0}^{\alpha}\frac{(-1)^s\alpha^{2s}}{(2s+1)(2s+3)!} \tag{C-66}$$

from the integral representation (C-62) with $r=3$ for $E_3(\alpha)$.

For large values of α the power series expansions (C-65) and (C-66) are of little value. It is then better to represent $E_3(\alpha)$ by means of the formula

$$E_3(\alpha)=\mathrm{e}^{-i\alpha}\{h(\alpha)+ik(\alpha)\}. \tag{C-67}$$

We may then deduce from (C-62) with $r=3$ the following asymptotic expansions for $h(\alpha)$ and $k(\alpha)$ for real positive α. With the integer $n\geqslant 0$ arbitrary we get

$$h(\alpha)=h_n(\alpha)+\frac{1}{\alpha^{2n+2}}\mu_n(\alpha) \tag{C-68}$$

and

$$k(\alpha)=k_n(\alpha)+\frac{1}{\alpha^{2n+1}}\nu_n(\alpha) \tag{C-69}$$

where

$$h_n(\alpha)=\frac{1}{2\alpha^2}\sum_{r=0}^{n}\frac{(-1)^r(2r+3)!}{\alpha^{2r}} \tag{C-70}$$

and

$$k_n(\alpha)=-\frac{1}{2\alpha}\sum_{r=0}^{n}\frac{(-1)^r(2r+2)!}{\alpha^{2r}}. \tag{C-71}$$

The remainder functions $\mu_n(\alpha)$, $\nu_n(\alpha)$, for any $n\geqslant 0$ have the behaviour

$$\mu_n(\alpha)=0(1) \tag{C-72}$$

$$\nu_n(\alpha)=0(1) \tag{C-73}$$

for $\alpha\to+\infty$.

Because $H(\alpha)$ and $K(\alpha)$ in (C-64) are integral functions of α^2 we can write, for $0\leqslant\alpha\leqslant A$,

$$E_3(\alpha)=\tfrac{1}{2}\alpha^2\left[\log\left(\frac{\alpha}{A}\right)+\frac{i\pi}{2}\right]+\sum_{r=0}^{\infty}{}' M_r(A)T_r\left(\frac{2\alpha^2}{A^2}-1\right)-(i\alpha)\sum_{r=0}^{\infty}{}' N_r(A)T_r\left(\frac{2\alpha^2}{A^2}-1\right), \tag{C-74}$$

where the dash ′ on the summation sign $\sum$ indicates that the quantity under the summation sign for $r=0$ is to be multiplied by $\frac{1}{2}$. The coefficients $M_r(A)$ and $N_r(A)$ in the formula (C-74) may be determined numerically for $r=0, 1, 2, \ldots$, for a given value of A by applying the method of Clenshaw[16].

The functions $\alpha^2h(\alpha)$ and $\alpha k(\alpha)$, where $h(\alpha)$ and $k(\alpha)$ are defined in formula (C-67), are functions of bounded variation in $1/\alpha^2$ for large α. Hence we can write, for $A\leqslant\alpha\leqslant\infty$

$$E_3(\alpha)=\frac{\mathrm{e}^{-i\alpha}}{\alpha^2}\sum_{r=0}^{\infty}{}' F_r(A)T_r\left(\frac{2A^2}{\alpha^2}-1\right)+\frac{i\,\mathrm{e}^{-i\alpha}}{\alpha}\sum_{r=0}^{\infty}{}' G_r(A)T_r\left(\frac{2A^2}{\alpha^2}-1\right) \tag{C-75}$$

where, again, the dash ′ on the summation sign $\sum$ indicates that the quantity under the summation sign for $r=0$ is to be multiplied by $\frac{1}{2}$. The coefficients $F_r(A)$ and $G_r(A)$ in the formulae (C-75) may be determined numerically for $r=0, 1, 2, \ldots$, for a given value of A by applying the method of Clenshaw[16].

If we determine the coefficients $F_r(A)$, $G_r(A)$, $M_r(A)$ and $N_r(A)$ numerically for $A = 7$ we get the following results:

$$\left.\begin{aligned}
F_0(7) &= +5{\cdot}180\,814\\
F_1(7) &= -0{\cdot}360\,529\\
F_2(7) &= +0{\cdot}041\,133\\
F_3(7) &= -0{\cdot}006\,359\\
F_4(7) &= +0{\cdot}001\,210\\
F_5(7) &= -0{\cdot}000\,268\\
F_6(7) &= +0{\cdot}000\,067\\
F_7(7) &= -0{\cdot}000\,018\\
F_8(7) &= +0{\cdot}000\,005\\
F_9(7) &= -0{\cdot}000\,002\\
F_{10}(7) &= +0{\cdot}000\,001
\end{aligned}\right\} \qquad \text{(C-76)}$$

$$\left.\begin{aligned}
G_0(7) &= -1{\cdot}819\,858\,0\\
G_1(7) &= +0{\cdot}081\,768\,8\\
G_2(7) &= -0{\cdot}007\,181\,8\\
G_3(7) &= +0{\cdot}000\,926\,5\\
G_4(7) &= -0{\cdot}000\,153\,8\\
G_5(7) &= +0{\cdot}000\,030\,6\\
G_6(7) &= -0{\cdot}000\,007\,0\\
G_7(7) &= +0{\cdot}000\,001\,7\\
G_8(7) &= -0{\cdot}000\,000\,5\\
G_9(7) &= +0{\cdot}000\,000\,1
\end{aligned}\right\} \qquad \text{(C-77)}$$

$$\left.\begin{aligned}
M_0(7) &= +4{\cdot}768\,526\,27\\
M_1(7) &= -0{\cdot}677\,567\,89\\
M_2(7) &= -2{\cdot}089\,539\,69\\
M_3(7) &= +0{\cdot}398\,136\,24\\
M_4(7) &= -0{\cdot}065\,821\,07\\
M_5(7) &= +0{\cdot}007\,658\,89\\
M_6(7) &= -0{\cdot}000\,634\,37\\
M_7(7) &= +0{\cdot}000\,038\,88\\
M_8(7) &= -0{\cdot}000\,001\,83\\
M_9(7) &= +0{\cdot}000\,000\,07
\end{aligned}\right\} \qquad \text{(C-78)}$$

$$\left.\begin{aligned}N_0(7)&=+7{\cdot}151\,718\,43\\N_1(7)&=+2{\cdot}201\,177\,93\\N_2(7)&=-0{\cdot}310\,914\,27\\N_3(7)&=+0{\cdot}055\,598\,29\\N_4(7)&=-0{\cdot}007\,412\,33\\N_5(7)&=+0{\cdot}000\,704\,63\\N_6(7)&=-0{\cdot}000\,049\,07\\N_7(7)&=+0{\cdot}000\,002\,59\\N_8(7)&=-0{\cdot}000\,000\,11\end{aligned}\right\}\qquad\text{(C-79)}$$

If now we use the values of the coefficients $F_r(7)$, $G_r(7)$, $M_r(7)$ and $N_r(7)$ from (C-76), (C-77), (C-78), (C-79) in formulae (C-74) and (C-75), and neglect the remaining higher order coefficients, we can evaluate $E_3(\alpha)$ for any given value of α. By doing this for all the integers α from 0 to 25, using nine significant figures in the arithmetic, we find the values $E_3(\alpha)$ of Table C-2.

TABLE C-2

α	Re $E_3(\alpha)$	Im $E_3(\alpha)$
0	+0·500 000 0	+0·000 000 0
1	+0·018 117 6	−0·378 530 2
2	−0·271 409 2	−0·107 735 2
3	−0·168 342 2	+0·164 075 8
4	+0·058 929 6	+0·186 434 0
5	+0·163 769 9	+0·031 120 2
6	+0·093 301 2	−0·110 845 2
7	−0·043 467 6	−0·120 258 2
8	−0·112 298 8	−0·021 174 8
9	−0·068 523 3	+0·077 155 1
10	+0·027 748 1	+0·089 804 8
11	+0·083 589 3	+0·021 269 7
12	+0·057 204 0	−0·055 431 2
13	−0·015 793 7	−0·072 263 6
14	−0·065 040 0	−0·023 107 1
15	−0·050 651 5	+0·040 227 3
16	+0·006 772 4	+0·060 467 0
17	+0·051 729 4	+0·024 944 2
18	+0·046 076 9	−0·028 863 5
19	+0·000 160 8	−0·051 616 2
20	−0·041 447 3	−0·026 368 1
21	−0·042 393 6	+0·019 961 0
22	−0·005 685 7	+0·044 441 6
23	+0·033 085 1	+0·027 298 5
24	+0·039 121 4	−0·012 760 0
25	+0·009 811 7	−0·038 304 3

The function values tabulated in Table C-2 should all be correct to seven decimal places. To achieve this accuracy the coefficients $F_r(7)$ needed to be given only to six decimal places because of the factor $e^{-i\alpha}/\alpha^2$ multiplying the first series on the right of (C-75) and the coefficients $G_r(7)$ needed to be given only to seven decimal places because of the factor $e^{-i\alpha}/\alpha$ multiplying the second series on the right of (C-75).

It is possible to evaluate $E_3(\alpha)$ only to three places of decimals for $\alpha = 7$ using the asymptotic approximation $h_n(\alpha)$ to $h(\alpha)$ and $k_n(\alpha)$ to $k(\alpha)$ and to get the highest accuracy at $\alpha = 7$ we must take $n = 2$ for evaluating both

$h(\alpha)$ and $k(\alpha)$. Thus with the demarcation value $A = 7$ we must turn to a formula, such as (C-75), in order to be able to evaluate $E_3(\alpha)$ to seven places of decimals for $\alpha \geq 7$. The asymptotic approximations $h_n(\alpha)$ and $k_n(\alpha)$ cannot be used to get $E_3(\alpha)$ to seven places of decimals unless $\alpha > 17$.

We may now evaluate $J(x/l, y/l; \nu, M_\infty)$ from the infinite series (C-61) when $X/|y| \geq 2$ if we use the relations (C-63) to express the $E_{2r+3}(\nu X/l)$, occurring in (C-61), in terms of $E_3(\nu X/l)$ and then evaluate $E_3(\nu X/l)$ from either formula (C-74) or (C-75) depending on whether $\nu X/l < A$ or $\nu X/l > A$.

To evaluate $J(x/l, y/l; \nu, M_\infty)$ when $X/|y|$ is large and negative, we write, from (C-9),

$$J\left(\frac{x}{l}, \frac{y}{l}; \nu, M_\infty\right) = \int_{-\infty}^{\infty} e^{-i\nu|y|u/l} \frac{du}{(u^2+1)^{\frac{3}{2}}} - \int_{-\infty}^{X/|y|} e^{-i\nu|y|u/l} \frac{du}{(u^2+1)^{\frac{3}{2}}}$$

$$= 2\frac{\nu|y|}{l} K_1\left(\frac{\nu|y|}{l}\right) - \int_{-X/|y|}^{\infty} e^{i\nu|y|u/l} \frac{du}{(u^2+1)^{\frac{3}{2}}}. \qquad \text{(C-80)}$$

The evaluation of $\alpha K_1(\alpha)$ for $0 \leq \alpha \leq \infty$ has already been considered. The evaluation of

$$\int_{-X/|y|}^{\infty} e^{i\nu|y|u/l} \frac{du}{(u^2+1)^{\frac{3}{2}}}$$

for $-X/|y| > 2$ is carried out in exactly the same manner as the evaluation of $J(x/l, y/l; \nu, M_\infty)$ for $X/|y| > 2$ was carried out, apart from a change in the sign of the imaginary part.

TABLE 1

Numerical Values of Approximations $\hat{Q}_{ij}$, $i = 1, 2, 3$; $j = 1, 2, 3$, to the Generalised Airforces on a Tapered Swept Wing of Aspect Ratio 2, Oscillating in Heave, Pitch and Control Surface Rotation with $\nu = 0{\cdot}32560$, $M_\infty = 0{\cdot}7806$

m	n	M	N	q	$\hat{Q}'_{11}$	$\hat{Q}''_{11}$	$\hat{Q}'_{12}$	$\hat{Q}''_{12}$	$\hat{Q}'_{13}$	$\hat{Q}''_{13}$
15	2	15	2	1	0·056 532	−2·4867	−2·4586	−4·5856	−0·59672	0·047 442
15	2	15	2	2	0·059 524	−2·5112	−2·4806	−4·6469	−0·59274	0·058 493
15	2	15	2	4	0·059 303	−2·5094	−2·4791	−4·6423	−0·59301	0·058 082
15	2	15	2	8	0·059 245	−2·5092	−2·4790	−4·6412	−0·59296	0·058 138
15	3	15	3	1	0·063 709	−2·5395	−2·5066	−4·7566	−0·59599	0·092 102
15	3	15	3	2	0·060 337	−2·5124	−2·4836	−4·6788	−0·58650	0·084 073
15	3	15	3	4	0·061 863	−2·5223	−2·4921	−4·7058	−0·58572	0·083 562
15	3	15	3	8	0·061 750	−2·5214	−2·4913	−4·7032	−0·58585	0·083 567
15	4	15	4	1	0·072 016	−2·5627	−2·5224	−4·8710	−0·59746	0·081 139
15	4	15	4	2	0·061 005	−2·5217	−2·4921	−4·7037	−0·58622	0·087 618
15	4	15	4	4	0·061 854	−2·5220	−2·4921	−4·7079	−0·58420	0·084 590
15	4	15	4	8	0·061 849	−2·5227	−2·4927	−4·7084	−0·58426	0·084 585
15	5	15	5	1	0·069 369	−2·5493	−2·5125	−4·8268	−0·58719	0·085 382
15	5	15	5	2	0·065 804	−2·5421	−2·5080	−4·7790	−0·58680	0·082 546
15	5	15	5	4	0·060 595	−2·5179	−2·4891	−4·6924	−0·58435	0·082 547
15	5	15	5	8	0·062 152	−2·5243	−2·4940	−4·7140	−0·58371	0·085 162
15	6	15	6	1	0·068 997	−2·5496	−2·5130	−4·8208	−0·58459	0·083 682
15	6	15	6	2	0·067 449	−2·5440	−2·5086	−4·7980	−0·58530	0·082 405
15	6	15	6	4	0·061 730	−2·5239	−2·4938	−4·7125	−0·58410	0·084 300
15	6	15	6	8	0·061 857	−2·5225	−2·4926	−4·7093	−0·58364	0·084 845
15	7	15	7	1	0·068 974	−2·5507	−2·5142	−4·8176	−0·58518	0·085 373
15	7	15	7	2	0·065 972	−2·5380	−2·5042	−4·7755	−0·58626	0·082 964
15	7	15	7	4	0·063 669	−2·5315	−2·4996	−4·7419	−0·58488	0·084 483
15	7	15	7	8	0·061 305	−2·5207	−2·4912	−4·7021	−0·58346	0·085 272
15	8	15	8	1	0·067 845	−2·5448	−2·5095	−4·7959	−0·58257	0·087 852
15	8	15	8	2	0·065 255	−2·5356	−2·5024	−4·7649	−0·58684	0·085 725
15	8	15	8	4	0·064 781	−2·5348	−2·5019	−4·7571	−0·58619	0·085 461
15	8	15	8	8	0·061 416	−2·5218	−2·4921	−4·7052	−0·58416	0·084 262
15	9	15	9	1	0·067 288	−2·5435	−2·5087	−4·7863	−0·57992	0·086 495
15	9	15	9	2	0·065 524	−2·5388	−2·5053	−4·7699	−0·58603	0·086 130
15	9	15	9	4	0·064 664	−2·5336	−2·5010	−4·7544	−0·58664	0·085 478
15	9	15	9	8	0·062 050	−2·5245	−2·4943	−4·7152	−0·58462	0·084 989
15	10	15	10	1	0·066 940	−2·5415	−2·5070	−4·7783	−0·57914	0·085 793
15	10	15	10	2	0·065 865	−2·5382	−2·5045	−4·7711	−0·58455	0·085 210
15	10	15	10	4	0·064 066	−2·5307	−2·4987	−4·7450	−0·58584	0·084 590
15	10	15	10	8	0·062 761	−2·5272	−2·4963	−4·7258	−0·58466	0·084 965
15	2	15	3	1	0·058 199	−2·5305	−2·5003	−4·6620	−0·59146	0·049 011
15	2	15	3	2	0·056 910	−2·4996	−2·4713	−4·6048	−0·58893	0·043 323
15	2	15	3	4	0·058 384	−2·5112	−2·4814	−4·6337	−0·58732	0·049 199
15	2	15	3	8	0·058 298	−2·5103	−2·4806	−4·6313	−0·58745	0·048 993

TABLE 1—(continued)

m	n	M	N	q	$\hat{Q}'_{11}$	$\hat{Q}''_{11}$	$\hat{Q}'_{12}$	$\hat{Q}''_{12}$	$\hat{Q}'_{13}$	$\hat{Q}''_{13}$
15	2	15	4	1	0·066 883	−2·5526	−2·5159	−4·7788	−0·59040	0·072 441
15	2	15	4	2	0·056 460	−2·5081	−2·4799	−4·6138	−0·58954	0·044 034
15	2	15	4	4	0·058 345	−2·5092	−2·4796	−4·6308	−0·58752	0·048 557
15	2	15	4	8	0·058 262	−2·5101	−2·4806	−4·6310	−0·58735	0·048 878
15	3	15	4	1	0·071 518	−2·5608	−2·5219	−4·8634	−0·59576	0·083 508
15	3	15	4	2	0·060 772	−2·5210	−2·4917	−4·6987	−0·58872	0·083 986
15	3	15	4	4	0·061 712	−2·5211	−2·4914	−4·7036	−0·58510	0·081 262
15	3	15	4	8	0·061 692	−2·5219	−2·4922	−4·7042	−0·58510	0·081 363
15	3	15	5	1	0·067 870	−2·5428	−2·5078	−4·8026	−0·59178	0·083 598
15	3	15	5	2	0·065 649	−2·5406	−2·5070	−4·7738	−0·59044	0·082 491
15	3	15	5	4	0·060 261	−2·5166	−2·4882	−4·6850	−0·58587	0·082 290
15	3	15	5	8	0·061 963	−2·5231	−2·4932	−4·7086	−0·58504	0·081 277
15	3	15	6	1	0·067 323	−2·5448	−2·5100	−4·7963	−0·58833	0·083 298
15	3	15	6	2	0·066 517	−2·5394	−2·5053	−4·7812	−0·59007	0·081 535
15	3	15	6	4	0·061 439	−2·5229	−2·4932	−4·7058	−0·58691	0·082 380
15	3	15	6	8	0·061 634	−2·5213	−2·4917	−4·7031	−0·58509	0·081 406
15	4	15	5	1	0·069 064	−2·5449	−2·5081	−4·8175	−0·59207	0·082 452
15	4	15	5	2	0·065 842	−2·5414	−2·5072	−4·7783	−0·59036	0·082 924
15	4	15	5	4	0·060 581	−2·5176	−2·4888	−4·6917	−0·58422	0·086 010
15	4	15	5	8	0·062 137	−2·5240	−2·4938	−4·7134	−0·58434	0·084 148

m	n	M	N	q	$\hat{Q}'_{21}$	$\hat{Q}''_{21}$	$\hat{Q}'_{22}$	$\hat{Q}''_{22}$	$\hat{Q}'_{23}$	$\hat{Q}''_{23}$
15	2	15	2	1	−0·083 903	2·7077	2·6375	5·7238	0·95694	0·050 595
15	2	15	2	2	−0·088 217	2·7098	2·6344	5·7720	0·95471	0·044 836
15	2	15	2	4	−0·087 931	2·7090	2·6340	5·7680	0·95500	0·045 119
15	2	15	2	8	−0·087 884	2·7086	2·6337	5·7667	0·95494	0·045 098
15	3	15	3	1	−0·100 36	2·7431	2·6537	5·9967	0·98120	−0·028 454
15	3	15	3	2	−0·094 451	2·7286	2·6470	5·9099	0·96384	−0·027 201
15	3	15	3	4	−0·096 970	2·7265	2·6419	5·9322	0·96210	−0·026 717
15	3	15	3	8	−0·096 806	2·7259	2·6415	5·9295	0·96234	−0·026 787
15	4	15	4	1	−0·111 53	2·7054	2·6033	6·0884	0·97587	−0·019 808
15	4	15	4	2	−0·096 315	2·7395	2·6555	5·9499	0·96552	−0·033 136
15	4	15	4	4	−0·097 049	2·7281	2·6436	5·9408	0·95943	−0·030 097
15	4	15	4	8	−0·097 155	2·7285	2·6439	5·9415	0·95952	−0·029 990
15	5	15	5	1	−0·106 49	2·7071	2·6116	6·0428	0·96472	−0·024 649
15	5	15	5	2	−0·103 00	2·7259	2·6341	6·0109	0·96199	−0·025 359
15	5	15	5	4	−0·095 435	2·7354	2·6528	5·9339	0·95856	−0·029 318
15	5	15	5	8	−0·097 602	2·7283	2·6431	5·9469	0·95864	−0·030 893
15	6	15	6	1	−0·106 64	2·7106	2·6148	6·0408	0·95871	−0·025 191
15	6	15	6	2	−0·104 82	2·7152	2·6215	6·0219	0·95983	−0·024 189
15	6	15	6	4	−0·097 225	2·7344	2·6495	5·9530	0·95930	−0·029 144
15	6	15	6	8	−0·097 122	2·7291	2·6446	5·9437	0·95848	−0·030 833

TABLE 1—(continued)

m	n	M	N	q	$\hat{Q}'_{21}$	$\hat{Q}''_{21}$	$\hat{Q}'_{22}$	$\hat{Q}''_{22}$	$\hat{Q}'_{23}$	$\hat{Q}''_{23}$
15	7	15	7	1	−0·106 55	2·7060	2·6104	6·0281	0·95920	−0·028 807
15	7	15	7	2	−0·102 44	2·7191	2·6283	6·0024	0·96111	−0·026 344
15	7	15	7	4	−0·099 863	2·7280	2·6400	5·9760	0·96023	−0·028 960
15	7	15	7	8	−0·096 415	2·7320	2·6483	5·9398	0·95872	−0·031 418
15	8	15	8	1	−0·104 87	2·7059	2·6125	6·0057	0·95623	−0·032 553
15	8	15	8	2	−0·101 72	2·7223	2·6323	5·9956	0·96328	−0·030 357
15	8	15	8	4	−0·101 25	2·7229	2·6334	5·9867	0·96242	−0·029 838
15	8	15	8	8	−0·096 685	2·7328	2·6488	5·9438	0·95905	−0·030 459
15	9	15	9	1	−0·104 17	2·7066	2·6140	5·9944	0·95113	−0·031 572
15	9	15	9	2	−0·102 14	2·7215	2·6310	5·9965	0·96176	−0·030 934
15	9	15	9	4	−0·100 92	2·7222	2·6332	5·9836	0·96308	−0·030 275
15	9	15	9	8	−0·097 600	2·7312	2·6460	5·9522	0·96010	−0·030 638
15	10	15	10	1	−0·103 63	2·7053	2·6134	5·9838	0·94942	−0·031 183
15	10	15	10	2	−0·102 47	2·7172	2·6265	5·9936	0·95880	−0·029 849
15	10	15	10	4	−0·100 07	2·7240	2·6360	5·9764	0·96142	−0·029 413
15	10	15	10	8	−0·098 555	2·7286	2·6423	5·9602	0·96007	−0·030 257
15	2	15	3	1	−0·086 112	2·7368	2·6638	5·7803	0·94719	0·049 598
15	2	15	3	2	−0·083 509	2·7138	2·6440	5·7246	0·94308	0·056 959
15	2	15	3	4	−0·086 132	2·7129	2·6397	5·7488	0·94305	0·053 779
15	2	15	3	8	−0·086 013	2·7122	2·6392	5·7466	0·94319	0·053 965
15	2	15	4	1	−0·101 25	2·6985	2·6083	5·9073	0·95868	0·041 854
15	2	15	4	2	−0·082 895	2·7257	2·6564	5·7324	0·94343	0·054 703
15	2	15	4	4	−0·085 998	2·7116	2·6387	5·7467	0·94308	0·053 934
15	2	15	4	8	−0·085 955	2·7125	2·6396	5·7466	0·94290	0·053 571
15	3	15	4	1	−0·110 63	2·7075	2·6079	6·0811	0·97631	−0·026 072
15	3	15	4	2	−0·095 684	2·7389	2·6561	5·9385	0·96706	−0·023 964
15	3	15	4	4	−0·096 643	2·7268	2·6432	5·9319	0·96006	−0·023 619
15	3	15	4	8	−0·096 725	2·7274	2·6437	5·9325	0·96012	−0·023 528
15	3	15	5	1	−0·104 36	2·7114	2·6190	6·0227	0·97059	−0·028 846
15	3	15	5	2	−0·102 56	2·7245	2·6338	6·0014	0·96833	−0·024 178
15	3	15	5	4	−0·094 744	2·7343	2·6529	5·9208	0·96194	−0·023 842
15	3	15	5	8	−0·097 121	2·7267	2·6426	5·9362	0·95990	−0·023 570
15	3	15	6	1	−0·104 06	2·7148	2·6224	6·0145	0·96499	−0·027 106
15	3	15	6	2	−0·103 23	2·7163	2·6251	6·0039	0·96726	−0·025 185
15	3	15	6	4	−0·096 636	2·7335	2·6497	5·9409	0·96343	−0·023 819
15	3	15	6	8	−0·096 583	2·7274	2·6439	5·9318	0·96011	−0·023 699
15	4	15	5	1	−0·106 38	2·7056	2·6100	6·0371	0·96894	−0·024 067
15	4	15	5	2	−0·103 05	2·7249	2·6329	6·0097	0·96737	−0·025 570
15	4	15	5	4	−0·095 427	2·7352	2·6526	5·9332	0·96108	−0·031 900
15	4	15	5	8	−0·097 576	2·7280	2·6429	5·9459	0·95930	−0·029 343

TABLE 1—(continued)

m	n	M	N	q	$\hat{Q}'_{31}$	$\hat{Q}''_{31}$	$\hat{Q}'_{32}$	$\hat{Q}''_{32}$	$\hat{Q}'_{33}$	$\hat{Q}''_{33}$
15	2	15	2	1	−0·000 532 58	0·004 866 3	0·004 244 7	0·018 135	0·009 041 4	0·001 303 0
15	2	15	2	2	−0·000 592 28	0·004 193 7	0·003 478 1	0·017 756	0·009 115 8	0·001 474 1
15	2	15	2	4	−0·000 587 63	0·004 253 2	0·003 545 4	0·017 800	0·009 112 6	0·001 462 0
15	2	15	2	8	−0·000 587 55	0·004 253 6	0·003 546 0	0·017 800	0·009 112 6	0·001 462 0
15	3	15	3	1	−0·000 747 41	0·003 061 3	0·001 754 3	0·017 798	0·014 066	0·004 315 3
15	3	15	3	2	−0·000 749 42	0·003 109 8	0·001 810 9	0·017 860	0·013 810	0·004 252 3
15	3	15	3	4	−0·000 740 01	0·003 105 1	0·001 817 0	0·017 738	0·013 788	0·004 251 2
15	3	15	3	8	−0·000 741 04	0·003 107 0	0·001 817 5	0·017 756	0·013 795	0·004 252 5
15	4	15	4	1	−0·000 769 11	0·002 632 7	0·001 274 2	0·017 394	0·012 822	0·006 553 8
15	4	15	4	2	−0·000 708 77	0·003 065 0	0·001 794 3	0·017 260	0·012 665	0·006 316 3
15	4	15	4	4	−0·000 707 38	0·003 016 3	0·001 750 8	0·017 127	0·012 543	0·006 240 2
15	4	15	4	8	−0·000 708 01	0·003 010 7	0·001 744 0	0·017 130	0·012 548	0·006 242 9
15	5	15	5	1	−0·000 716 63	0·002 941 1	0·001 653 3	0·017 273	0·010 740	0·007 433 1
15	5	15	5	2	−0·000 718 13	0·002 934 6	0·001 650 4	0·017 209	0·010 931	0·007 201 0
15	5	15	5	4	−0·000 702 06	0·003 080 4	0·001 826 8	0·017 184	0·010 933	0·007 019 9
15	5	15	5	8	−0·000 702 45	0·003 014 8	0·001 760 3	0·017 076	0·010 929	0·006 996 0
15	6	15	6	1	−0·000 715 48	0·002 983 1	0·001 705 9	0·017 247	0·010 436	0·007 430 0
15	6	15	6	2	−0·000 717 90	0·002 984 1	0·001 700 7	0·017 290	0·010 667	0·007 376 3
15	6	15	6	4	−0·000 706 13	0·003 048 7	0·001 787 4	0·017 204	0·010 761	0·007 220 3
15	6	15	6	8	−0·000 702 07	0·003 028 9	0·001 775 5	0·017 097	0·010 784	0·007 146 8
15	7	15	7	1	−0·000 710 60	0·002 839 6	0·001 573 9	0·016 903	0·010 363	0·007 203 2
15	7	15	7	2	−0·000 719 90	0·002 968 9	0·001 684 5	0·017 291	0·010 565	0·007 331 6
15	7	15	7	4	−0·000 711 16	0·003 001 2	0·001 731 4	0·017 206	0·010 594	0·007 272 3
15	7	15	7	8	−0·000 701 44	0·003 055 0	0·001 802 3	0·017 136	0·010 590	0·007 197 7
15	8	15	8	1	−0·000 697 83	0·002 835 8	0·001 591 7	0·016 684	0·009 960 5	0·007 092 5
15	8	15	8	2	−0·000 717 94	0·002 938 8	0·001 658 3	0·017 191	0·010 416	0·007 258 9
15	8	15	8	4	−0·000 713 29	0·002 978 8	0·001 705 2	0·017 200	0·010 406	0·007 282 0
15	8	15	8	8	−0·000 703 39	0·003 051 0	0·001 795 2	0·017 162	0·010 327	0·007 232 7
15	9	15	9	1	−0·000 694 17	0·002 788 0	0·001 551 2	0·016 526	0·009 701 3	0·007 078 1
15	9	15	9	2	−0·000 704 23	0·002 989 3	0·001 730 4	0·017 086	0·010 332	0·007 218 9
15	9	15	9	4	−0·000 709 97	0·003 013 0	0·001 744 5	0·017 211	0·010 412	0·007 279 8
15	9	15	9	8	−0·000 705 05	0·003 037 1	0·001 778 2	0·017 166	0·010 320	0·007 251 4
15	10	15	10	1	−0·000 685 53	0·002 811 6	0·001 588 9	0·016 437	0·009 577 7	0·007 043 7
15	10	15	10	2	−0·000 707 57	0·002 915 6	0·001 653 8	0·016 982	0·010 125	0·007 221 9
15	10	15	10	4	−0·000 713 28	0·002 986 4	0·001 714 2	0·017 204	0·010 265	0·007 299 4
15	10	15	10	8	−0·000 706 86	0·003 018 8	0·001 756 8	0·017 164	0·010 241	0·007 272 8
15	2	15	3	1	−0·000 638 71	0·003 708 3	0·002 922 5	0·017 632	0·009 239 5	0·001 600 1
15	2	15	3	2	−0·000 530 33	0·004 772 9	0·004 151 8	0·017 972	0·009 043 1	0·001 396 2
15	2	15	3	4	−0·000 582 01	0·004 271 3	0·003 570 9	0·017 786	0·009 117 8	0·001 505 0
15	2	15	3	8	−0·000 578 53	0·004 309 8	0·003 615 0	0·017 807	0·009 114 4	0·001 497 1

TABLE 1—(concluded)

m	n	M	N	q	$\hat{Q}'_{31}$	$\hat{Q}''_{31}$	$\hat{Q}'_{32}$	$\hat{Q}''_{32}$	$\hat{Q}'_{33}$	$\hat{Q}''_{33}$
15	2	15	4	1	−0·000 787 12	0·002 694 9	0·001 694 3	0·017 780	0·009 4936	0·001 801 0
15	2	15	4	2	−0·000 557 74	0·004 461 1	0·003 796 5	0·017 825	0·009 1015	0·001 468 8
15	2	15	4	4	−0·000 570 95	0·004 390 8	0·003 707 7	0·017 842	0·009 1025	0·001 479 5
15	2	15	4	8	−0·000 577 16	0·004 322 8	0·003 630 0	0·017 811	0·009 1134	0·001 495 1
15	3	15	4	1	−0·000 578 90	0·004 402 9	0·003 317 0	0·017 892	0·014 239	0·004 327 4
15	3	15	4	2	−0·000 759 44	0·003 095 1	0·001 783 2	0·017 999	0·013 917	0·004 280 5
15	3	15	4	4	−0·000 744 61	0·003 102 0	0·001 809 2	0·017 805	0·013 817	0·004 266 8
15	3	15	4	8	−0·000 741 43	0·003 128 9	0·001 840 5	0·017 810	0·013 821	0·004 266 5
15	3	15	5	1	−0·000 517 96	0·004 790 6	0·003 785 4	0·017 784	0·014 207	0·004 330 8
15	3	15	5	2	−0·000 667 89	0·003 656 4	0·002 457 8	0·017 782	0·013 999	0·004 299 3
15	3	15	5	4	−0·000 785 41	0·002 879 6	0·001 534 3	0·017 955	0·013 837	0·004 274 4
15	3	15	5	8	−0·000 734 17	0·003 169 4	0·001 890 4	0·017 784	0·013 821	0·004 267 4
15	3	15	6	1	−0·000 559 15	0·004 372 9	0·003 311 6	0·017 567	0·014 050	0·004 305 9
15	3	15	6	2	−0·000 614 17	0·004 030 5	0·002 900 6	0·017 729	0·014 025	0·004 301 4
15	3	15	6	4	−0·000 749 38	0·003 127 3	0·001 829 2	0·017 922	0·013 878	0·004 279 6
15	3	15	6	8	−0·000 747 98	0·003 083 9	+0·001 786 4	0·017 817	0·013 822	0·004 269 3
15	4	15	5	1	−0·000 925 61	0·001 522 9	−0·000 047 544	0·017 482	0·012 581	0·006 693 6
15	4	15	5	2	−0·000 761 52	0·002 674 8	+0·001 330 0	0·017 315	0·012 733	0·006 394 5
15	4	15	5	4	−0·000 689 51	0·003 183 7	0·001 939 6	0·017 185	0·012 642	0·006 235 5
15	4	15	5	8	−0·000 712 48	0·002 980 5	0·001 707 2	0·017 134	0·012 578	0·006 235 6

TABLE 2

Numerical Values of Approximations $\hat{l}_3(x, y)$ to the Loading on a Tapered Swept Wing of Aspect Ratio 2 when the Control Surface is Oscillating about its Hinge.
$m = 15, n = 10, M = 15, N = 10, q_I = 8, I = 1, 2, \ldots, 10, \nu = 0{\cdot}32560, M_\infty = 0{\cdot}78060$

$\hat{l}_3'(x, y)$

ξ \ η	0·00000	0·19509	0·38268	0·55557	0·70711	0·83147	0·92388	0·98079
0·01	−0·00074	−0·00189	−0·04349	+0·03650	−0·07581	−0·43460	−0·95712	−1·39046
0·03	−0·00755	−0·00257	−0·03097	+0·11061	+0·06418	−0·14523	−0·51628	−0·78938
0·06	−0·00203	−0·00715	−0·01899	−0·10096	−0·21939	−0·39160	−0·68185	−0·63627
0·10	+0·00095	−0·01028	−0·02427	−0·19141	−0·30860	−0·47131	−0·67715	−0·45242
0·15	−0·00389	−0·01220	−0·04030	−0·07286	−0·12525	−0·29648	−0·44353	−0·24766
0·20	−0·01010	−0·01618	−0·04694	+0·01337	−0·04112	−0·23565	−0·36681	−0·20765
0·25	−0·01365	−0·02281	−0·04667	−0·03830	−0·16006	−0·37652	−0·48970	−0·28164
0·30	−0·01568	−0·03045	−0·05097	−0·15836	−0·34213	−0·56731	−0·62918	−0·34285
0·35	−0·01914	−0·03783	−0·06600	−0·23743	−0·43422	−0·65711	−0·64111	−0·32812
0·40	−0·02594	−0·04521	−0·08841	−0·22306	−0·40129	−0·62260	−0·53781	−0·27671
0·45	−0·03603	−0·05410	−0·10946	−0·14912	−0·33515	−0·56491	−0·45755	−0·28526
0·50	−0·04787	−0·06625	−0·12268	−0·10915	−0·38132	−0·62569	−0·55382	−0·42883
0·55	−0·05963	−0·08251	−0·12996	−0·19997	−0·64113	−0·88676	−0·88252	−0·69828
0·60	−0·07051	−0·10232	−0·14260	−0·46456	−1·10029	−1·30704	−1·34765	−0·98741
0·65	−0·08130	−0·12396	−0·17616	−0·85816	−1·61686	−1·72711	−1·73382	−1·13935
0·68	−0·08857	−0·13699	−0·21109	−1·10536	−1·86525	−1·89247	−1·82991	−1·11053
0·70	−0·09407	−0·14550	−0·24070	−1·25281	−1·97644	−1·94227	−1·81539	−1·03474
0·72	−0·10024	−0·15377	−0·27458	−1·37358	−2·03157	−1·93528	−1·73395	−0.91715
0·75	−0·11093	−0·16571	−0·33037	−1·48342	−1·99595	−1·81480	−1·49554	−0·68112
0·80	−0·13218	−0·18417	−0·41834	−1·42519	−1·63370	−1·37169	−0·90088	−0·24695
0·85	−0·15446	−0·19936	−0·46281	−1·07807	−1·04704	−0·82157	−0·34836	+0·01419
0·90	−0·16898	−0·20407	−0·42714	−0·61656	−0·53606	−0·43508	−0·12911	−0·02855
0·94	−0·16326	−0·18677	−0·33586	−0·36251	−0·34633	−0·30926	−0·15724	−0·13087
0·97	−0·13517	−0·14597	−0·23535	−0·29872	−0·30318	−0·23479	−0·12230	−0·06467
0·99	−0·08648	−0·08813	−0·13772	−0·24236	−0·22438	−0·12760	−0·01873	+0·05462

$\hat{l}_3''(x, y)$

ξ \ η	0·00000	0·19509	0·38268	0·55557	0·70711	0·83147	0·92388	0·98079
0·01	0·01419	0·03523	0·07638	0·17930	0·27935	0·37386	0·43515	0·40115
0·03	0·00714	0·02440	0·04585	0·12139	0·17180	0·21293	0·24798	0·21555
0·06	0·00999	0·01971	0·04192	0·05792	0·08133	0·11276	0·13914	0·10881
0·10	0·01318	0·01935	0·03879	0·03238	0·05149	0·07843	0·09330	0·04887
0·15	0·01402	0·02197	0·03377	0·04896	0·07269	0·08848	0·08465	0·02136
0·20	0·01488	0·02428	0·03326	0·06320	0·08265	0·08761	0·06718	+0·00401
0·25	0·01732	0·02573	0·03654	0·05840	0·06797	0·06633	0·03517	−0·01132
0·30	0·02075	0·02720	0·03966	0·04474	0·04758	0·04179	+0·00530	−0·01997
0·35	0·02408	0·02933	0·04023	0·03569	0·03774	0·02668	−0·01161	−0·02024
0·40	0·02671	0·03194	0·03867	0·03661	0·03998	0·02041	−0·01733	−0·01658
0·45	0·02868	0·03434	0·03698	0·04414	0·04510	0·01428	−0·02057	−0·01569
0·50	0·03043	0·03582	0·03681	0·05087	0·04240	+0·00019	−0·02878	−0·02172
0·55	0·03230	0·03608	0·03825	0·05021	+0·02626	−0·02396	−0·04388	−0·03413

TABLE 2—(concluded)

	$\hat{l}_3''(x, y)$							
η / ξ	0·00000	0·19509	0·38268	0·55557	0·70711	0·83147	0·92388	0·98079
0·60	0·03428	0·03532	0·03982	0·03884	−0·00278	−0·05457	−0·06323	−0·04890
0·65	0·03598	0·03391	0·03922	+0·01652	−0·04098	−0·08622	−0·08347	−0·06159
0·68	0·03660	0·03285	0·03697	−0·00162	−0·06677	−0·10417	−0·09543	−0·06712
0·70	0·03674	0·03203	0·03451	−0·01546	−0·08478	−0·11552	−0·10336	−0·06985
0·72	0·03662	0·03110	0·03123	−0·03056	−0·10325	−0·12636	−0·11126	−0·07186
0·75	0·03587	0·02935	0·02478	−0·05515	−0·13119	−0·14154	−0·12283	−0·07357
0·80	0·03282	0·02503	+0·01028	−0·09878	−0·17424	−0·16223	−0·13852	−0·07270
0·85	0·02731	0·01820	−0·00764	−0·13810	−0·20113	−0·17022	−0·14048	−0·06537
0·90	0·01949	0·00883	−0·02550	−0·15820	−0·19454	−0·15493	−0·11781	−0·05046
0·94	0·01210	+0·00077	−0·03415	−0·14672	−0·15865	−0·12400	−0·08496	−0·03775
0·97	0·00646	−0·00378	−0·03140	−0·11278	−0·11470	−0·09209	−0·06022	−0·03212
0·99	0·00281	−0·00418	−0·02012	−0·06734	−0·06932	−0·05866	−0·04003	−0·02534

TABLE 3

Numerical Values of Approximations $\hat{Q}_{ij}$, $i = 1, 2, 3$; $j = 1, 2, 3$, to the Generalised Airforces on a Tapered Wing of Aspect Ratio 6, Oscillating in Heave, Pitch and Control Surface Rotation with $\nu = 3{\cdot}1569$, $M_\infty = 0{\cdot}4$

m	*n*	*M*	*N*	*q*	$\hat{Q}'_{11}$	$\hat{Q}''_{11}$	$\hat{Q}'_{12}$	$\hat{Q}''_{12}$	$\hat{Q}'_{21}$	$\hat{Q}''_{21}$	$\hat{Q}'_{22}$	$\hat{Q}''_{22}$
14	6	14	6	12	37·780	−13·417	32·235	−25·150	55·368	−16·452	60·928	−35·208
14	6	27	6	12	36·990	−13·526	31·155	−25·022	54·831	−16·602	59·937	−35·190
14	6	28	6	12	37·148	−13·514	31·330	−25·046	54·959	−16·582	60·104	−35·201
15	6	15	6	12	36·604	−13·628	30·576	−24·988	54·702	−16·741	59·529	−35·275
15	6	29	6	12	37·108	−13·606	31·040	−25·121	55·015	−16·653	59·968	−35·287
15	6	30	6	12	37·156	−13·589	31·107	−25·118	55·035	−16·633	60·012	−35·273
22	6	22	6	8	37·456	−13·633	31·312	−25·232	55·391	−16·652	60·377	−35·380
22	6	43	6	8	37·134	−13·645	30·919	−25·166	55·074	−16·676	59·926	−35·327
22	6	44	6	8	37·173	−13·642	30·962	−25·171	55·105	−16·671	59·966	−35·329
23	6	23	6	8	36·983	−13·684	30·708	−25·177	54·984	−16·737	59·762	−35·381
23	6	45	6	8	37·168	−13·653	30·936	−25·180	55·111	−16·682	59·953	−35·342
23	6	46	6	8	37·167	−13·645	30·945	−25·173	55·101	−16·673	59·952	−35·331
30	6	30	6	8	37·186	−13·646	30·969	−25·183	55·128	−16·677	59·994	−35·347
31	6	31	6	8	37·189	−13·663	30·952	−25·198	55·155	−16·694	60·002	−35·369

m	*n*	*M*	*N*	*q*	$\hat{Q}'_{13}$	$\hat{Q}''_{13}$	$\hat{Q}'_{31}$	$\hat{Q}''_{31}$	$\hat{Q}'_{23}$	$\hat{Q}''_{23}$	$\hat{Q}'_{32}$	$\hat{Q}''_{32}$
14	6	14	6	12	−1·9962	−0·36897	0·39310	−0·041884	−3·4483	−0·84922	0·67051	−0·14343
14	6	27	6	12	−1·9856	−0·37948	0·38768	−0·040999	−3·4323	−0·85459	0·66562	−0·14118
14	6	28	6	12	−1·9867	−0·37862	0·38801	−0·041066	−3·4334	−0·85371	0·66577	−0·14132
15	6	15	6	12	−1·9823	−0·39329	0·38081	−0·042441	−3·4224	−0·86763	0·65651	−0·14093
15	6	29	6	12	−1·9693	−0·38541	0·38485	−0·042582	−3·4107	−0·85736	0·65844	−0·14201
15	6	30	6	12	−1·9634	−0·38517	0·38512	−0·042490	−3·4100	−0·85704	0·65884	−0·14194
22	6	22	6	8	−1·9833	−0·39212	0·39046	−0·043107	−3·4242	−0·86718	0·66427	−0·14396
22	6	43	6	8	−1·9788	−0·38920	0·38894	−0·042988	−3·4211	−0·86169	0·66146	−0·14347
22	6	44	6	8	−1·9792	−0·38908	0·38894	−0·042987	−3·4215	−0·86154	0·66140	−0·14346
23	6	23	6	8	−1·9763	−0·39093	0·39053	−0·043204	−3·4188	−0·86521	0·66399	−0·14407
23	6	45	6	8	−1·9791	−0·38918	0·38901	−0·042994	−3·4215	−0·86153	0·66144	−0·14348
23	6	46	6	8	−1·9790	−0·38916	0·38894	−0·042996	−3·4213	−0·86148	0·66129	−0·14347
30	6	30	6	8	−1·9790	−0·38933	0·38959	−0·042961	−3·4217	−0·86316	0·66308	−0·14360
31	6	31	6	8	−1·9785	−0·38957	0·38934	−0·042982	−3·4212	−0·86319	0·66275	−0·14356

m	*N*	*M*	*N*	*q*	$\hat{Q}'_{33}$	$\hat{Q}''_{33}$
14	6	14	6	12	−0·023448	−0·035733
14	6	27	6	12	−0·023851	−0·035654
14	6	28	6	12	−0·023865	−0·035641
15	6	15	6	12	−0·024078	−0·036449
15	6	29	6	12	−0·023986	−0·036189
15	6	30	6	12	−0·024001	−0·036180

TABLE 3—(concluded)

m	N	M	N	q	$\hat{Q}'_{33}$	$\hat{Q}''_{33}$
22	6	22	6	8	−0·023692	−0·037150
22	6	43	6	8	−0·023994	−0·036940
22	6	44	6	8	−0·024002	−0·036934
23	6	23	6	8	−0·023757	−0·037180
23	6	45	6	8	−0·024011	−0·036952
23	6	46	6	8	−0·024017	−0·036946
30	6	30	6	8	−0·023652	−0·037585
31	6	31	6	8	−0·023689	−0·037647

TABLE 4

Numerical Values, Evaluated by Lehrian and Garner[9], of Approximations $\hat{Q}_{ij}$, $i = 1, 2$; $j = 1, 2$, to the Generalised Airforces on a Tapered Swept Wing of Aspect Ratio 6, Oscillating in Heave and Pitch with $\nu = 3{\cdot}1569$, $M_\infty = 0{\cdot}4$

m	n	a	$\hat{Q}'_{11}$	$\hat{Q}''_{11}$	$\hat{Q}'_{12}$	$\hat{Q}''_{12}$	$\hat{Q}'_{21}$	$\hat{Q}''_{21}$	$\hat{Q}'_{22}$	$\hat{Q}''_{22}$
14	6	6	37·745	−13·391	32·129	−25·110	55·284	−16·399	60·733	−35·123
15	6	6	36·413	−13·586	30·391	−24·871	54·398	−16·664	59·203	−35·087
22	6	4	37·370	−13·610	31·180	−25·169	55·214	−16·610	60·094	−35·270
23	6	4	36·841	−13·646	30·545	−25·084	54·737	−16·675	59·438	−35·230

TABLE 5

Numerical Values of Approximations $\hat{Q}_{ij}$, $i=1, 2$; $j=1, 2$, to the Generalised Airforces on a Rectangular Wing of Aspect Ratio 2 Oscillating in Heave and Pitch with $\nu=1\cdot0$, $M_\infty=0\cdot8$, $q_I=q=32$, $I=1, 2, \ldots, N$

m	n	M	N	$\hat{Q}'_{11}$	$\hat{Q}''_{11}$	$\hat{Q}'_{12}$	$\hat{Q}''_{12}$	$\hat{Q}'_{21}$	$\hat{Q}''_{21}$	$\hat{Q}'_{22}$	$\hat{Q}''_{22}$
4	2	4	2	0·90594	−3·0921	−2·9891	−3·2292	0·79937	−0·77193	−0·43632	−1·8191
4	2	9	2	0·90593	−3·0920	−2·9890	−3·2291	0·79937	−0·77192	−0·43631	−1·8190
4	2	14	2	0·90593	−3·0920	−2·9890	−3·2292	0·79937	−0·77192	−0·43631	−1·8190
4	2	19	2	0·90593	−3·0920	−2·9890	−3·2292	0·79937	−0·77192	−0·43630	−1·8190
4	2	4	4	0·91725	−3·1079	−3·0054	−3·2545	0·81469	−0·77069	−0·42610	−1·8405
4	2	9	4	0·91730	−3·1079	−3·0054	−3·2546	0·81475	−0·77069	−0·42608	−1·8406
4	2	14	4	0·91730	−3·1079	−3·0054	−3·2546	0·81475	−0·77069	−0·42608	−1·8406
4	2	19	4	0·91730	−3·1079	−3·0054	−3·2546	0·81474	−0·77069	−0·42608	−1·8406
4	2	4	6	0·91731	−3·1081	−3·0056	−3·2547	0·81480	−0·77077	−0·42615	−1·8407
4	2	9	6	0·91738	−3·1081	−3·0055	−3·2547	0·81486	−0·77068	−0·42603	−1·8407
4	2	14	6	0·91735	−3·1080	−3·0055	−3·2547	0·81483	−0·77066	−0·42602	−1·8407
4	2	19	6	0·91734	−3·1080	−3·0055	−3·2546	0·81481	−0·77065	−0·42601	−1·8406
4	2	4	8	0·91742	−3·1083	−3·0059	−3·2550	0·81494	−0·77085	−0·42619	−1·8409
4	2	9	8	0·91735	−3·1080	−3·0055	−3·2547	0·81483	−0·77065	−0·42600	−1·8406
4	2	14	8	0·91728	−3·1079	−3·0053	−3·2545	0·81475	−0·77060	−0·42597	−1·8405
4	2	19	8	0·91723	−3·1078	−3·0052	−3·2543	0·81470	−0·77056	−0·42595	−1·8404
4	4	4	4	0·90950	−3·2618	−3·3188	−3·3228	0·96652	−0·84864	−0·49919	−2·1919
4	4	9	4	0·90957	−3·2618	−3·3188	−3·3229	0·96660	−0·84865	−0·49915	−2·1920
4	4	14	4	0·90956	−3·2618	−3·3188	−3·3229	0·96660	−0·84865	−0·49915	−2·1920
4	4	19	4	0·90956	−3·2618	−3·3188	−3·3229	0·96660	−0·84865	−0·49915	−2·1920
4	4	4	6	0·90972	−3·2620	−3·3190	−3·3232	0·96675	−0·84866	−0·49911	−2·1923
4	4	9	6	0·90983	−3·2620	−3·3190	−3·3233	0·96681	−0·84858	−0·49900	−2·1924
4	4	14	6	0·90981	−3·2620	−3·3190	−3·3233	0·96680	−0·84858	−0·49901	−2·1923
4	4	19	6	0·90981	−3·2620	−3·3190	−3·3232	0·96679	−0·84858	−0·49901	−2·1923
4	4	4	8	0·90991	−3·2622	−3·3191	−3·3236	0·96694	−0·84873	−0·49909	−2·1927
4	4	9	8	0·90987	−3·2620	−3·3190	−3·3234	0·96685	−0·84858	−0·49899	−2·1925
4	4	14	8	0·90983	−3·2620	−3·3190	−3·3233	0·96680	−0·84857	−0·49899	−2·1924
4	4	19	8	0·90981	−3·2620	−3·3190	−3·3232	0·96679	−0·84856	−0·49900	−2·1923
4	6	4	6	0·90947	−3·2621	−3·3194	−3·3228	0·96684	−0·84890	−0·49963	−2·1928
4	6	9	6	0·90958	−3·2621	−3·3195	−3·3229	0·96689	−0·84881	−0·49950	−2·1929
4	6	14	6	0·90957	−3·2621	−3·3194	−3·3228	0·96688	−0·84881	−0·49950	−2·1929
4	6	19	6	0·90956	−3·2621	−3·3194	−3·3228	0·96687	−0·84881	−0·49950	−2·1929
4	6	4	8	0·90966	−3·2622	−3·3195	−3·3232	0·96703	−0·84898	−0·49961	−2·1932
4	6	9	8	0·90963	−3·2621	−3·3194	−3·3230	0·96694	−0·84882	−0·49949	−2·1930
4	6	14	8	0·90959	−3·2621	−3·3194	−3·3229	0·96690	−0·84881	−0·49949	−2·1929
4	6	19	8	0·90959	−3·2621	−3·3194	−3·3229	0·96689	−0·84881	−0·49949	−2·1929
4	8	4	8	0·90966	−3·2622	−3·3195	−3·3232	0·96703	−0·84898	−0·49961	−2·1932
4	8	9	8	0·90963	−3·2621	−3·3194	−3·3230	0·96694	−0·84882	−0·49949	−2·1930
4	8	14	8	0·90960	−3·2621	−3·3194	−3·3229	0·96690	−0·84881	−0·49949	−2·1929
4	8	19	8	0·90959	−3·2621	−3·3194	−3·3229	0·96689	−0·84880	−0·49949	−2·1929

TABLE 5—(continued)

m	n	M	N	$\hat{Q}'_{11}$	$\hat{Q}''_{11}$	$\hat{Q}'_{12}$	$\hat{Q}''_{12}$	$\hat{Q}'_{21}$	$\hat{Q}''_{21}$	$\hat{Q}'_{22}$	$\hat{Q}''_{22}$
9	2	9	2	0·90653	−3·0924	−2·9891	−3·2302	0·79973	−0·77198	−0·43620	−1·8197
9	2	14	2	0·90593	−3·0920	−2·9890	−3·2292	0·79938	−0·77192	−0·43631	−1·8191
9	2	19	2	0·90608	−3·0921	−2·9890	−3·2295	0·79946	−0·77194	−0·43628	−1·8192
9	2	9	4	0·91817	−3·1088	−3·0060	−3·2562	0·81539	−0·77081	−0·42596	−1·8417
9	2	14	4	0·91732	−3·1079	−3·0054	−3·2546	0·81477	−0·77068	−0·42606	−1·8406
9	2	19	4	0·91753	−3·1081	−3·0056	−3·2550	0·81492	−0·77071	−0·42604	−1·8409
9	2	9	6	0·91838	−3·1091	−3·0064	−3·2567	0·81567	−0·77085	−0·42590	−1·8421
9	2	14	6	0·91738	−3·1080	−3·0055	−3·2547	0·81486	−0·77065	−0·42600	−1·8407
9	2	19	6	0·91760	−3·1083	−3·0057	−3·2552	0·81504	−0·77068	−0·42597	−1·8410
9	2	9	8	0·91845	−3·1093	−3·0066	−3·2569	0·81577	−0·77086	−0·42588	−1·8423
9	2	14	8	0·91730	−3·1079	−3·0053	−3·2545	0·81478	−0·77058	−0·42595	−1·8405
9	2	19	8	0·91752	−3·1081	−3·0055	−3·2549	0·81495	−0·77060	−0·42590	−1·8408
9	4	9	4	0·91062	−3·2623	−3·3187	−3·3246	0·96719	−0·84853	−0·49870	−2·1931
9	4	14	4	0·90966	−3·2619	−3·3188	−3·3230	0·96667	−0·84864	−0·49911	−2·1922
9	4	19	4	0·90990	−3·2620	−3·3188	−3·3234	0·96680	−0·84861	−0·49900	−2·1924
9	4	9	6	0·91112	−3·2626	−3·3190	−3·3254	0·96760	−0·84844	−0·49846	−2·1938
9	4	14	6	0·90997	−3·2620	−3·3190	−3·3235	0·96692	−0·84854	−0·49892	−2·1926
9	4	19	6	0·91025	−3·2622	−3·3190	−3·3239	0·96709	−0·84851	−0·49881	−2·1928
9	4	9	8	0·91129	−3·2628	−3·3192	−3·3258	0·96776	−0·84844	−0·49840	−2·1941
9	4	14	8	0·90999	−3·2620	−3·3190	−3·3235	0·96693	−0·84853	−0·49891	−2·1926
9	4	19	8	0·91029	−3·2622	−3·3190	−3·3240	0·96711	−0·84849	−0·49878	−2·1929
9	6	9	6	0·91086	−3·2627	−3·3195	−3·3250	0·96766	−0·84869	−0·49896	−2·1943
9	6	14	6	0·90972	−3·2621	−3·3194	−3·3231	0·96699	−0·84878	−0·49941	−2·1931
9	6	19	6	0·91000	−3·2623	−3·3194	−3·3236	0·96715	−0·84876	−0·49930	−2·1934
9	6	9	8	0·91106	−3·2629	−3·3196	−3·3254	0·96784	−0·84870	−0·49891	−2·1946
9	6	14	8	0·90977	−3·2622	−3·3194	−3·3232	0·96703	−0·84878	−0·49940	−2·1932
9	6	19	8	0·91007	−3·2623	−3·3195	−3·3237	0·96722	−0·84875	−0·49928	−2·1935
9	8	9	8	0·91106	−3·2629	−3·3196	−3·3254	0·96784	−0·84869	−0·49890	−2·1946
9	8	14	8	0·90977	−3·2622	−3·3194	−3·3232	0·96703	−0·84877	−0·49939	−2·1932
9	8	19	8	0·91007	−3·2623	−3·3194	−3·3237	0·96721	−0·84875	−0·49926	−2·1935
14	2	14	2	0·90593	−3·0920	−2·9890	−3·2292	0·79938	−0·77192	−0·43631	−1·8191
14	2	19	2	0·90593	−3·0920	−2·9890	−3·2292	0·79938	−0·77192	−0·43631	−1·8191
14	2	14	4	0·91732	−3·1079	−3·0054	−3·2546	0·81477	−0·77068	−0·42606	−1·8406
14	2	19	4	0·91732	−3·1079	−3·0054	−3·2546	0·81477	−0·77068	−0·42606	−1·8406
14	2	14	6	0·91738	−3·1080	−3·0055	−3·2547	0·81486	−0·77065	−0·42600	−1·8407
14	2	19	6	0·91736	−3·1080	−3·0055	−3·2547	0·81484	−0·77064	−0·42599	−1·8407
14	2	14	8	0·91730	−3·1079	−3·0053	−3·2545	0·81478	−0·77058	−0·42595	−1·8405
14	2	19	8	0·91725	−3·1078	−3·0053	−3·2544	0·81473	−0·77054	−0·42593	−1·8404

TABLE 5—(concluded)

m	n	M	N	$\hat{Q}'_{11}$	$\hat{Q}''_{11}$	$\hat{Q}'_{12}$	$\hat{Q}''_{12}$	$\hat{Q}'_{21}$	$\hat{Q}''_{21}$	$\hat{Q}'_{22}$	$\hat{Q}''_{22}$
14	4	14	4	0·90966	−3·2619	−3·3188	−3·3230	0·96667	−0·84864	−0·49911	−2·1922
14	4	19	4	0·90966	−3·2619	−3·3188	−3·3230	0·96667	−0·84864	−0·49911	−2·1922
14	4	14	6	0·90997	−3·2620	−3·3190	−3·3235	0·96692	−0·84854	−0·49892	−2·1926
14	4	19	6	0·90996	−3·2620	−3·3190	−3·3235	0·96692	−0·84854	−0·49892	−2·1925
14	4	14	8	0·90999	−3·2620	−3·3190	−3·3235	0·96693	−0·84853	−0·49891	−2·1926
14	4	19	8	0·90997	−3·2620	−3·3190	−3·3235	0·96692	−0·84852	−0·49891	−2·1925
14	6	14	6	0·90972	−3·2621	−3·3194	−3·3231	0·96699	−0·84878	−0·49941	−2·1931
14	6	19	6	0·90972	−3·2621	−3·3194	−3·3231	0·96699	−0·84878	−0·49941	−2·1931
14	6	14	8	0·90977	−3·2622	−3·3194	−3·3232	0·96703	−0·84878	−0·49940	−2·1932
14	6	19	8	0·90976	−3·2622	−3·3194	−3·3231	0·96703	−0·84878	−0·49940	−2·1931
14	8	14	8	0·90977	−3·2622	−3·3194	−3·3232	0·96703	−0·84877	−0·49939	−2·1932
14	8	19	8	0·90976	−3·2621	−3·3194	−3·3231	0·96702	−0·84877	−0·49939	−2·1931
19	2	19	2	0·90608	−3·0921	−2·9890	−3·2295	0·79946	−0·77194	−0·43628	−1·8192
19	2	19	4	0·91753	−3·1081	−3·0056	−3·2550	0·81492	−0·77071	−0·42604	−1·8409
19	2	19	6	0·91760	−3·1083	−3·0057	−3·2552	0·81504	−0·77068	−0·42597	−1·8410
19	2	19	8	0·91752	−3·1081	−3·0055	−3·2549	0·81495	−0·77060	−0·42590	−1·8408
19	4	19	4	0·90990	−3·2620	−3·3188	−3·3234	0·96680	−0·84861	−0·49900	−2·1924
19	4	19	6	0·91025	−3·2622	−3·3190	−3·3239	0·96709	−0·84851	−0·49881	−2·1928
19	4	19	8	0·91025	−3·2622	−3·3190	−3·3240	0·96710	−0·84851	−0·49882	−2·1929
19	6	19	6	0·91000	−3·2623	−3·3194	−3·3236	0·96715	−0·84876	−0·49930	−2·1934
19	6	19	8	0·91007	−3·2623	−3·3195	−3·3237	0·96722	−0·84875	−0·49928	−2·1935
19	8	19	8	0·91007	−3·2623	−3·3194	−3·3237	0·96721	−0·84875	−0·49926	−2·1935

TABLE 6

Numerical Values of Approximations $\hat{Q}_{ij}$, $i = 1, 2$; $j = 1, 2$, to the Generalised Airforces on a Rectangular Wing of Aspect Ratio 8 Oscillating in Heave and Pitch with $\nu = 1{\cdot}0$, $M_\infty = 0{\cdot}8$, $q_I = q = 32$, $I = 1, 2, \ldots, N$

m	n	M	N	$\hat{Q}'_{11}$	$\hat{Q}''_{11}$	$\hat{Q}'_{12}$	$\hat{Q}''_{12}$	$\hat{Q}'_{21}$	$\hat{Q}''_{21}$	$\hat{Q}'_{22}$	$\hat{Q}''_{22}$
4	2	4	2	−1·2372	−15·939	−19·123	−9·5648	1·9318	−5·1566	−5·0567	−7·3567
4	2	9	2	−1·2323	−15·956	−19·143	−9·5824	1·9402	−5·1633	−5·0615	−7·3746
4	2	14	2	−1·2323	−15·956	−19·143	−9·5824	1·9402	−5·1632	−5·0614	−7·3746
4	2	19	2	−1·2322	−15·956	−19·142	−9·5824	1·9402	−5·1632	−5·0614	−7·3746
4	2	4	4	−1·1413	−15·986	−19·113	−9·7481	1·9823	−5·0979	−4·9262	−7·3833
4	2	9	4	−1·1294	−16·001	−19·125	−9·7725	1·9939	−5·0996	−4·9226	−7·4018
4	2	14	4	−1·1289	−16·000	−19·123	−9·7724	1·9938	−5·0989	−4·9215	−7·4011
4	2	19	4	−1·1288	−16·000	−19·123	−9·7724	1·9938	−5·0987	−4·9213	−7·4009
4	2	4	6	−1·1416	−15·991	−19·120	−9·7490	1·9859	−5·0979	−4·9246	−7·3876
4	2	9	6	−1·1308	−16·005	−19·132	−9·7733	1·9946	−5·1018	−4·9255	−7·4048
4	2	14	6	−1·1295	−16·002	−19·126	−9·7729	1·9941	−5·0999	−4·9228	−7·4024
4	2	19	6	−1·1290	−16·001	−19·124	−9·7727	1·9940	−5·0991	−4·9218	−7·4016
4	2	4	8	−1·1363	−15·982	−19·100	−9·7464	1·9870	−5·0885	−4·9105	−7·3798
4	2	9	8	−1·1312	−16·007	−19·135	−9·7735	1·9958	−5·1022	−4·9255	−7·4065
4	2	14	8	−1·1301	−16·003	−19·129	−9·7732	1·9946	−5·1009	−4·9240	−7·4039
4	2	19	8	−1·1294	−16·002	−19·126	−9·7728	1·9942	−5·0998	−4·9226	−7·4024
4	4	4	4	−1·9903	−16·192	−20·312	−8·3273	2·1285	−5·8804	−6·2842	−8·3902
4	4	9	4	−1·9848	−16·204	−20·325	−8·3398	2·1356	−5·8812	−6·2822	−8·4013
4	4	14	4	−1·9849	−16·203	−20·324	−8·3387	2·1352	−5·8804	−6·2812	−8·3998
4	4	19	4	−1·9849	−16·203	−20·323	−8·3384	2·1351	−5·8802	−6·2809	−8·3395
4	4	4	6	−1·9886	−16·196	−20·316	−8·3284	2·1317	−5·8776	−6·2799	−8·3924
4	4	9	6	−1·9838	−16·208	−20·331	−8·3452	2·1377	−5·8828	−6·2828	−8·4067
4	4	14	6	−1·9840	−16·206	−20·327	−8·3426	2·1367	−5·8808	−6·2804	−8·4029
4	4	19	6	−1·9841	−16·205	−20·326	−8·3415	2·1363	−5·8800	−6·2795	−8·4015
4	4	4	8	−1·9886	−16·187	−20·303	−8·3155	2·1295	−5·8654	−6·2657	−8·3761
4	4	9	8	−1·9834	−16·210	−20·332	−8·3453	2·1387	−5·8824	−6·2823	−8·4079
4	4	14	8	−1·9838	−16·207	−20·329	−8·3440	2·1374	−5·8818	−6·2815	−8·4051
4	4	19	8	−1·9840	−16·206	−20·327	−8·3426	2·1368	−5·8808	−6·2804	−8·4031
4	6	4	6	−1·9935	−16·189	−20·310	−8·3134	2·1268	−5·8806	−6·2905	−8·3868
4	6	9	6	−1·9884	−16·202	−20·325	−8·3306	2·1330	−5·8863	−6·2941	−8·4019
4	6	14	6	−1·9885	−16·200	−20·321	−8·3284	2·1322	−5·8845	−6·2919	−8·3986
4	6	19	6	−1·9855	−16·199	−20·320	−8·3275	2·1319	−5·8838	−6·2910	−8·3974
4	6	4	8	−1·9929	−16·182	−20·298	−8·3024	2·1254	−5·8696	−6·2772	−8·3728
4	6	9	8	−1·9880	−16·204	−20·326	−8·3309	2·1340	−5·8859	−6·2936	−8·4031
4	6	14	8	−1·9883	−16·202	−20·323	−8·3297	2·1328	−5·8854	−6·2930	−8·4006
4	6	19	8	−1·9884	−16·200	−20·321	−8·3285	2·1323	−5·8846	−6·2920	−8·3988
4	8	4	8	−1·9929	−16·182	−20·298	−8·3024	2·1254	−5·8696	−6·2772	−8·3728
4	8	9	8	−1·9880	−16·204	−20·326	−8·3309	2·1340	−5·8860	−6·2937	−8·4032
4	8	14	8	−1·9883	−16·202	−20·323	−8·3297	2·1328	−5·8854	−6·2930	−8·4007
4	8	19	8	−1·9884	−16·200	−20·321	−8·3285	2·1323	−5·8846	−6·2920	−8·3989

TABLE 6—(continued)

m	n	M	N	$\hat{Q}'_{11}$	$\hat{Q}''_{11}$	$\hat{Q}'_{12}$	$\hat{Q}''_{12}$	$\hat{Q}'_{21}$	$\hat{Q}''_{21}$	$\hat{Q}'_{22}$	$\hat{Q}''_{22}$
9	2	9	2	−1·2419	−15·950	−19·138	−9·5696	1·9332	−5·1624	−5·0659	−7·3735
9	2	14	2	−1·2429	−15·938	−19·126	−9·5601	1·9311	−5·1573	−5·0601	−7·3650
9	2	19	2	−1·2426	−15·941	−19·129	−9·5625	1·9317	−5·1585	−5·0615	−7·3671
9	2	9	4	−1·1379	−15·998	−19·128	−9·7635	1·9904	−5·0986	−4·9264	−7·4075
9	2	14	4	−1·1355	−15·980	−19·103	−9·7545	1·9869	−5·0901	−4·9160	−7·3939
9	2	19	4	−1·1359	−15·984	−19·108	−9·7567	1·9878	−5·0918	−4·9181	−7·3969
9	2	9	6	−1·1413	−16·006	−19·141	−9·7637	1·9920	−5·1028	−4·9320	−7·4133
9	2	14	6	−1·1360	−15·982	−19·106	−9·7550	1·9873	−5·0911	−4·9173	−7·3953
9	2	19	6	−1·1365	−15·986	−19·111	−9·7568	1·9882	−5·0927	−4·9192	−7·3983
9	2	9	8	−1·1433	−16·010	−19·149	−9·7634	1·9938	−5·1046	−4·9339	−7·4170
9	2	14	8	−1·1367	−15·983	−19·109	−9·7552	1·9877	−5·0921	−4·9185	−7·3967
9	2	19	8	−1·1373	−15·987	−19·114	−9·7568	1·9885	−5·0937	−4·9205	−7·3996
9	4	9	4	−2·0062	−16·200	−20·327	−8·3118	2·1204	−5·8855	−6·2962	−8·3944
9	4	14	4	−2·0091	−16·184	−20·310	−8·2965	2·1165	−5·8790	−6·2889	−8·3802
9	4	19	4	−2·0084	−16·188	−20·313	−8·2998	2·1173	−5·8802	−6·2902	−8·3831
9	4	9	6	−2·0050	−16·207	−20·337	−8·3187	2·1236	−5·8886	−6·2991	−8·4031
9	4	14	6	−2·0080	−16·187	−20·313	−8·3006	2·1180	−5·8793	−6·2880	−8·3835
9	4	19	6	−2·0074	−16·190	−20·316	−8·3035	2·1188	−5·8803	−6·2892	−8·3860
9	4	9	8	−2·0047	−16·211	−20·341	−8·3198	2·1253	−5·8894	−6·3003	−8·4068
9	4	14	8	−2·0078	−16·188	−20·315	−8·3020	2·1187	−5·8803	−6·2892	−8·3856
9	4	19	8	−2·0073	−16·192	−20·319	−8·3048	2·1194	−5·8814	−6·2905	−8·3882
9	6	9	6	−2·0099	−16·201	−20·331	−8·3038	2·1187	−5·8921	−6·3103	−8·3980
9	6	14	6	−2·0126	−16·181	−20·307	−8·2863	2·1135	−5·8831	−6·2997	−8·3793
9	6	19	6	−2·0120	−16·184	−20·311	−8·2893	2·1143	−5·8842	−6·3008	−8·3819
9	6	9	8	−2·0097	−16·205	−20·336	−8·3050	2·1205	−5·8928	−6·3113	−8·4016
9	6	14	8	−2·0125	−16·182	−20·309	−8·2876	2·1142	−5·8840	−6·3007	−8·3813
9	6	19	8	−2·0119	−16·186	−20·313	−8·2905	2·1149	−5·8852	−6·3021	−8·3839
9	8	9	8	−2·0097	−16·205	−20·336	−8·3051	2·1205	−5·8929	−6·3114	−8·4017
9	8	14	8	−2·0125	−16·182	−20·309	−8·2876	2·1142	−5·8841	−6·3008	−8·3813
9	8	19	8	−2·0119	−16·186	−20·313	−8·2905	2·1149	−5·8852	−6·3021	−8·3840
14	2	14	2	−1·2429	−15·938	−19·126	−9·5601	1·9311	−5·1573	−5·0601	−7·3650
14	2	19	2	−1·2429	−15·938	−19·126	−9·5601	1·9312	−5·1573	−5·0601	−7·3650
14	2	14	4	−1·1355	−15·980	−19·103	−9·7545	1·9869	−5·0901	−4·9160	−7·3939
14	2	19	4	−1·1354	−15·980	−19·102	−9·7544	1·9869	−5·0899	−4·9157	−7·3937
14	2	14	6	−1·1360	−15·982	−19·106	−9·7550	1·9873	−5·0911	−4·9173	−7·3953
14	2	19	6	−1·1356	−15·980	−19·103	−9·7547	1·9871	−5·0903	−4·9162	−7·3944
14	2	14	8	−1·1367	−15·983	−19·109	−9·7552	1·9877	−5·0921	−4·9185	−7·3967
14	2	19	8	−1·1360	−15·981	−19·105	−9·7549	1·9873	−5·0910	−4·9171	−7·3952

TABLE 6—(concluded)

m	n	M	N	$\hat{Q}'_{11}$	$\hat{Q}''_{11}$	$\hat{Q}'_{12}$	$\hat{Q}''_{12}$	$\hat{Q}'_{21}$	$\hat{Q}''_{21}$	$\hat{Q}'_{22}$	$\hat{Q}''_{22}$
14	4	14	4	−2·0091	−16·184	−20·310	−8·2965	2·1165	−5·8789	−6·2889	−8·3803
14	4	19	4	−2·0091	−16·184	−20·309	−8·2963	2·1164	−5·8788	−6·2887	−8·3799
14	4	14	6	−2·0080	−16·187	−20·313	−8·3007	2·1181	−5·8793	−6·2880	−8·3835
14	4	19	6	−2·0081	−16·186	−20·312	−8·2997	2·1177	−5·8785	−6·2871	−8·3821
14	4	14	8	−2·0078	−16·188	−20·315	−8·3021	2·1188	−5·8803	−6·2892	−8·3857
14	4	19	8	−2·0079	−16·187	−20·313	−8·3008	2·1181	−5·8793	−6·2880	−8·3837
14	6	14	6	−2·0126	−16·181	−20·307	−8·2864	2·1135	−5·8831	−6·2997	−8·3794
14	6	19	6	−2·0126	−16·180	−20·306	−8·2855	2·1133	−5·8824	−6·2988	−8·3781
14	6	14	8	−2·0124	−16·182	−20·309	−8·2877	2·1142	−5·8840	−6·3007	−8·3814
14	6	19	8	−2·0125	−16·181	−20·307	−8·2865	2·1137	−5·8832	−6·2997	−8·3796
14	8	14	8	−2·0124	−16·182	−20·309	−8·2877	2·1142	−5·8840	−6·3008	−8·3814
14	8	19	8	−2·0125	−16·181	−20·307	−8·2865	2·1137	−5·8832	−6·2997	−8·3796
19	2	19	2	−1·2426	−15·941	−19·129	−9·5625	1·9317	−5·1585	−5·0615	−7·3671
19	2	19	4	−1·1358	−15·984	−19·108	−9·7567	1·9878	−5·0918	−4·9181	−7·3969
19	2	19	6	−1·1365	−15·986	−19·111	−9·7568	1·9882	−5·0927	−4·9192	−7·3938
19	2	19	8	−1·1373	−15·987	−19·114	−9·7569	1·9885	−5·0937	−4·9205	−7·3996
19	4	19	4	−2·0084	−16·188	−20·313	−8·2998	2·1173	−5·8802	−6·2902	−8·3831
19	4	19	6	−2·0074	−16·190	−20·316	−8·3035	2·1189	−5·8803	−6·2892	−8·3860
19	4	19	8	−2·0072	−16·192	−20·319	−8·3049	2·1195	−5·8814	−6·2905	−8·3882
19	6	19	6	−2·0119	−16·184	−20·311	−8·2893	2·1144	−5·8814	−6·3008	−8·3820
19	6	19	8	−2·0118	−16·186	−20·313	−8·2906	2·1149	−5·8852	−6·3021	−8·3840
19	8	19	8	−2·0118	−16·186	−20·313	−8·2906	2·1149	−5·8852	−6·3021	−8·3840

TABLE 7

Numerical Values of Percentage Differences ε_{ij}, $i = 1, 2$; $j = 1, 2$, for a Rectangular Wing of Aspect Ratio 2

m	n	M	N	ε_{11}	ε_{12}	ε_{21}	ε_{22}
4	2	4	2	5·0	7·3	14·3	16·9
19	2	19	8	4·6	6·8	13·3	16·0
4	4	4	4	0·022	0·023	0·054	0·071
4	4	19	8	0·012	0·014	0·036	0·055
9	4	9	4	0·016	0·024	0·017	0·031
9	4	19	8	0·007	0·011	0·022	0·034
14	4	14	4	0·017	0·020	0·043	0·058
14	4	19	8	0·009	0·010	0·029	0·047
19	4	19	4	0·010	0·014	0·034	0·050
19	4	19	8	0·006	0·010	0·020	0·033
4	6	4	6	0·019	0·019	0·031	0·035
4	6	19	8	0·015	0·017	0·025	0·029
9	6	9	6	0·026	0·028	0·035	0·038
9	6	19	8	0·000	0·002	0·000	0·000
14	6	14	6	0·012	0·013	0·017	0·019
14	6	19	8	0·010	0·013	0·014	0·019
19	6	19	6	0·002	0·002	0·004	0·005
19	6	19	8	0·000	0·002	0·000	0·000
4	8	4	8	0·013	0·011	0·023	0·021
4	8	19	8	0·015	0·017	0·025	0·–29
9	8	9	8	0·034	0·036	0·049	0·051
9	8	19	8	0·000	0·000	0·000	0·000
14	8	14	8	0·009	0·011	0·014	0·015
14	8	19	8	0·011	0·013	0·015	0·019

TABLE 8

Numerical Values of Percentage Differences ε_{ij}, $i = 1, 2$; $j = 1, 2$, for a Rectangular Wing of Aspect Ratio 8

m	n	M	N	ε_{11}	ε_{12}	ε_{21}	ε_{22}
4	2	4	2	5·0	8·0	12·0	15·4
19	2	19	8	5·5	8·6	12·8	16·1
4	4	4	4	0·14	0·17	0·23	0·18
4	4	19	8	0·21	0·25	0·36	0·28
9	4	9	4	0·09	0·12	0·09	0·11
9	4	19	8	0·05	0·07	0·09	0·12
14	4	14	4	0·02	0·03	0·10	0·13
14	4	19	8	0·03	0·05	0·11	0·14
19	4	19	4	0·02	0·04	0·09	0·11
19	4	19	8	0·05	0·07	0·09	0·12
4	6	4	6	0·11	0·11	0·20	0·11
4	6	19	8	0·17	0·18	0·28	0·17
9	6	9	6	0·09	0·10	0·13	0·16
9	6	19	8	0·001	0·001	0·000	0·001
14	6	14	6	0·031	0·033	0·040	0·049
14	6	19	8	0·031	0·033	0·037	0·048
19	6	19	6	0·012	0·011	0·061	0·023
19	6	19	8	0·000	0·000	0·000	0·000
4	8	4	8	0·12	0·09	0·30	0·26
4	8	19	8	0·17	0·18	0·28	0·17
9	8	9	8	0·12	0·12	0·15	0·19
9	8	19	8	0·001	0·001	0·000	0·000
14	8	14	8	0·024	0·023	0·022	0·028
14	8	19	8	0·031	0·033	0·037	0·048

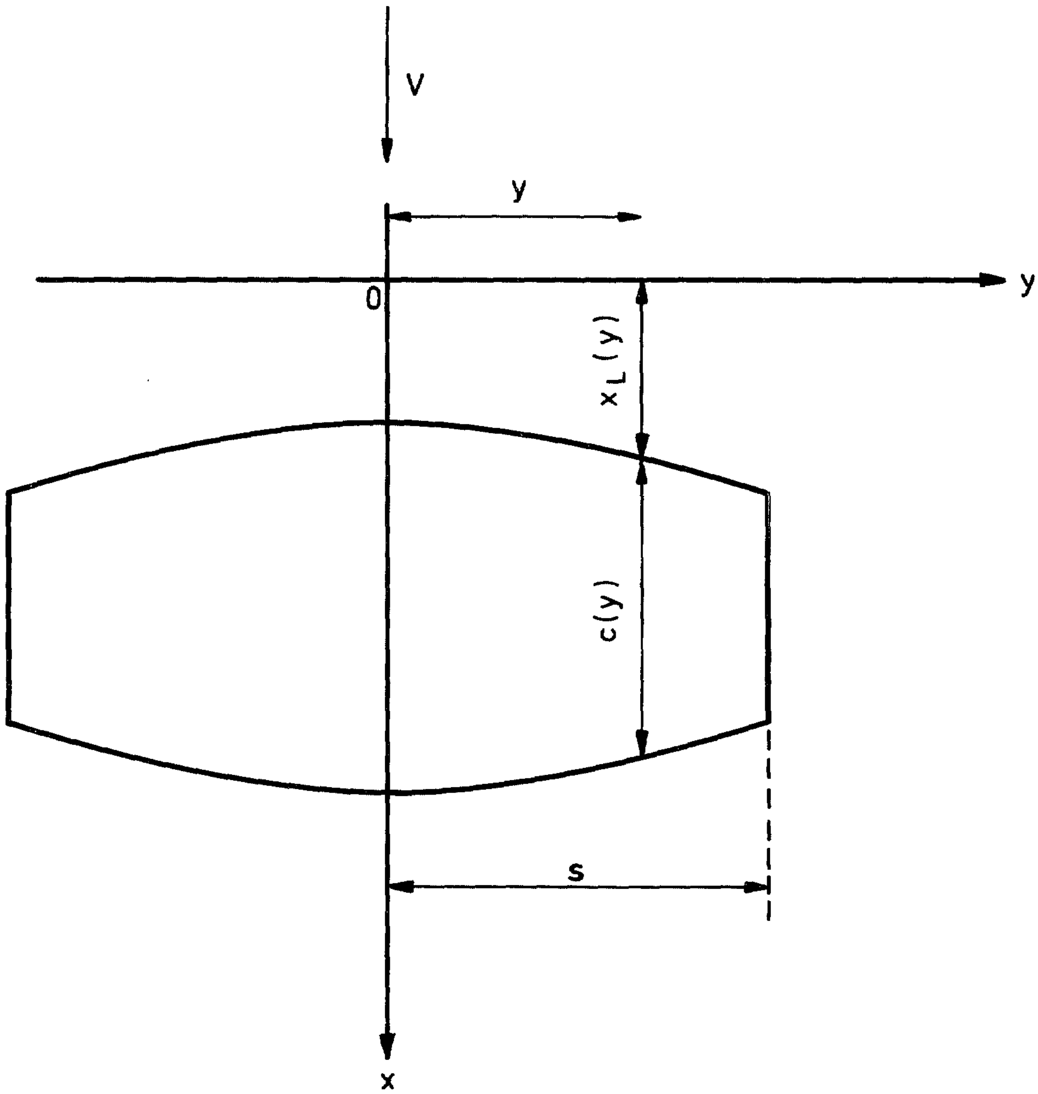
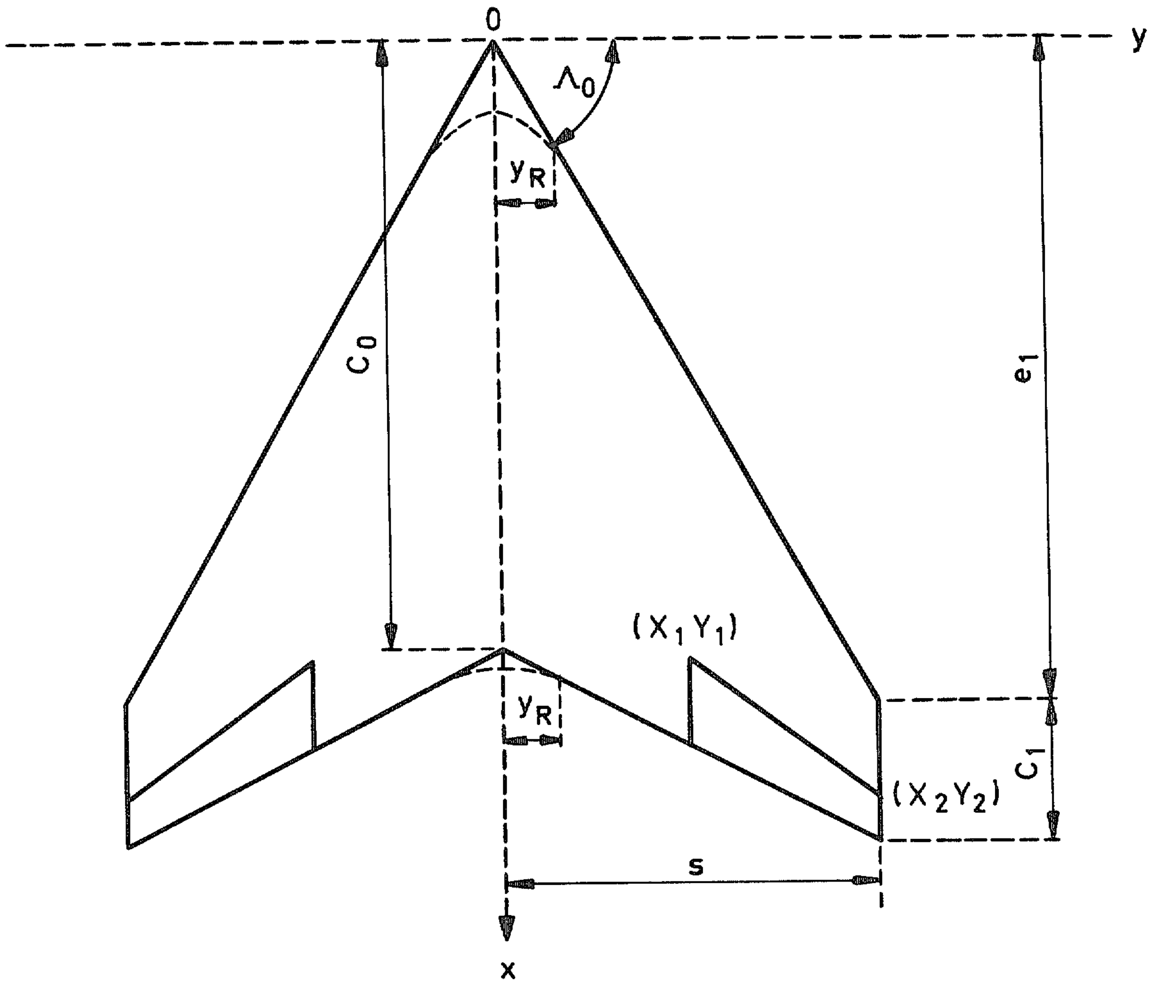


FIG. 1. General wing planform *S*.



$$\Lambda_0 = \pi/3^c$$

$$s = \bar{c}$$

$$C_0 = \frac{1}{4}\left(3+2\sqrt{3}\right)\bar{c} = 1{\cdot}6160254\ \bar{c}$$

$$C_1 = \frac{1}{4}\left(5-2\sqrt{3}\right)\bar{c} = 0{\cdot}3839746\ \bar{c}$$

$$e_1 = \sqrt{3}\ \bar{c} = 1{\cdot}7320508\ \bar{c}$$

$$X_1 = \frac{1}{4}\left(3+2\sqrt{3}\right)\bar{c} = 1{\cdot}6160254\ \bar{c}$$

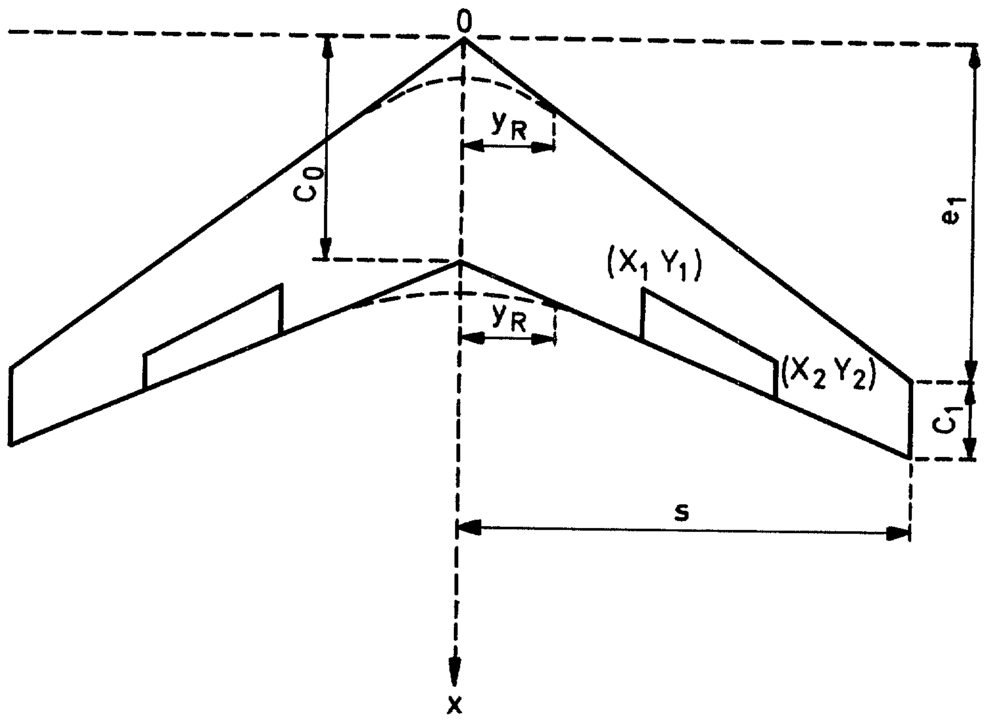
$$Y_1 = \frac{1}{2}\bar{c}$$

$$X_2 = \frac{1}{8}\left(9+4\sqrt{3}\right)\bar{c} = 1{\cdot}9910254\ \bar{c}$$

$$Y_2 = \bar{c}$$

$$y_R = \sin(\pi/16)\ \bar{c} = 0{\cdot}1950903\ \bar{c}$$

FIG. 2. Tapered swept wing of aspect ratio 2.



$$s = 3\bar{c}$$

$$C_0 = 3\bar{c}/2$$

$$C_1 = \bar{c}/2$$

$$e_1 = \left(\sqrt{3} + 1/2\right)\bar{c} = 2{\cdot}232051\,\bar{c}$$

$$X_1 = \left(\frac{2\sqrt{3}}{5} + \frac{97}{100}\right)\bar{c} = 1{\cdot}662820\,\bar{c}$$

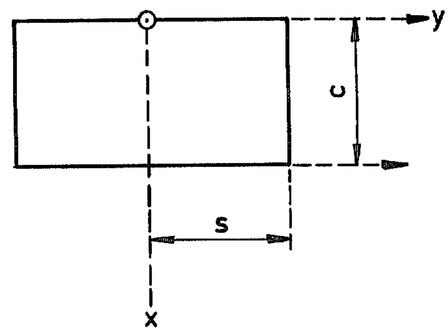
$$Y_1 = 6\bar{c}/5$$

$$X_2 = \left(\frac{7\sqrt{3}}{10} + \frac{91}{100}\right)\bar{c} = 2{\cdot}122436\,\bar{c}$$

$$Y_2 = 21\bar{c}/10$$

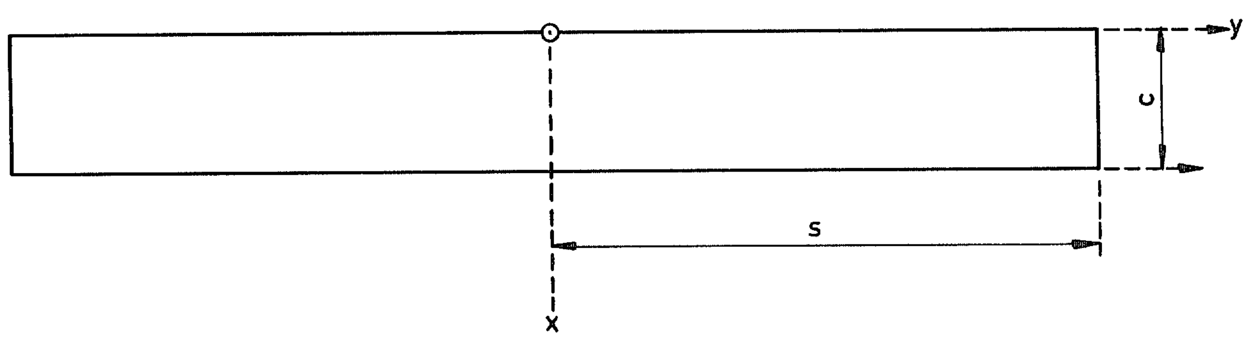
$$y_R = s\sin\frac{\pi}{16} = 0{\cdot}1950903\,s$$

FIG. 3. Tapered swept wing of aspect ratio 6.



s = c

FIG. 4a. Rectangular wing of aspect ratio 2.



s = 4 c

FIG. 4b. Rectangular wing of aspect ratio 8.

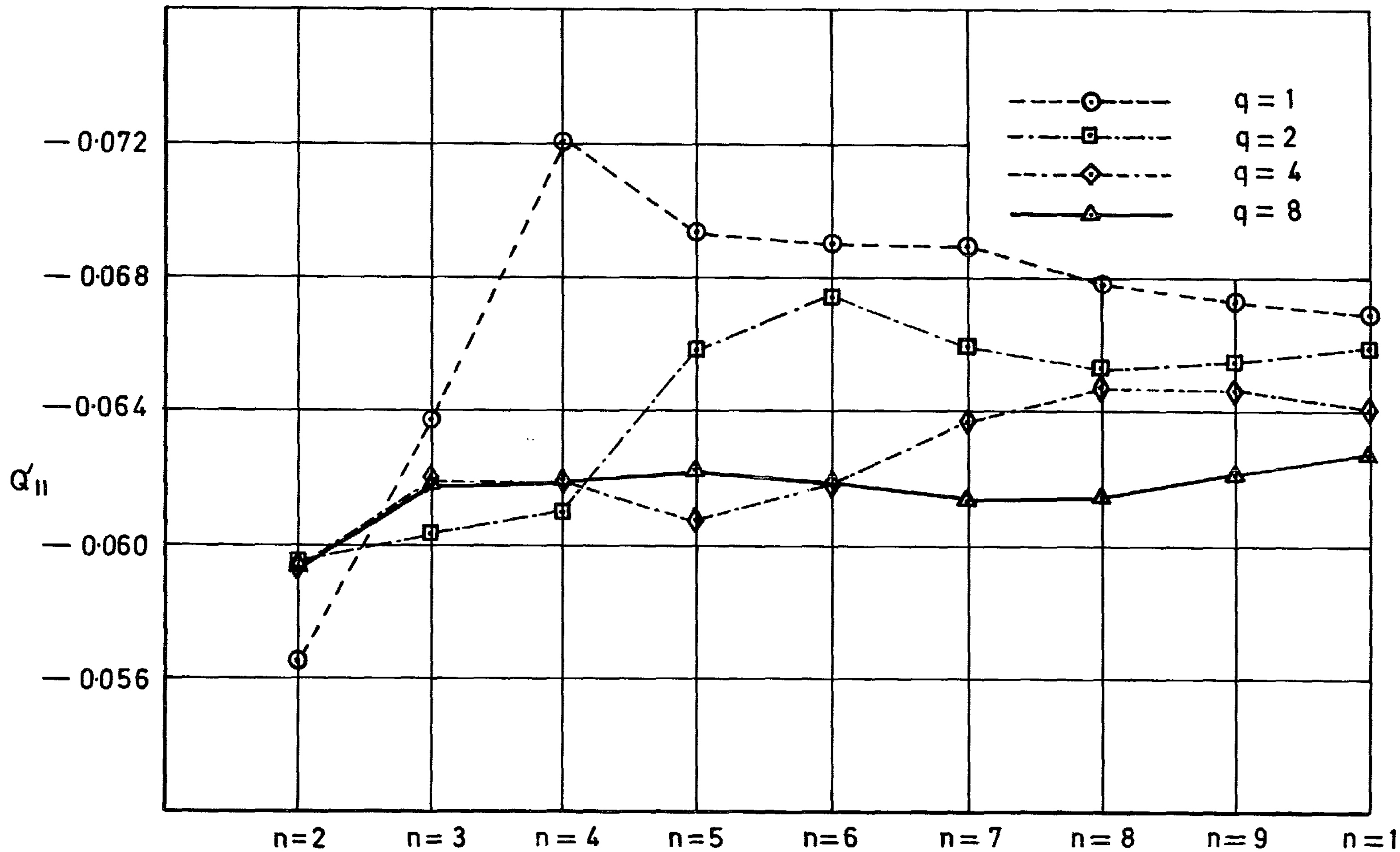


FIG. 5a. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

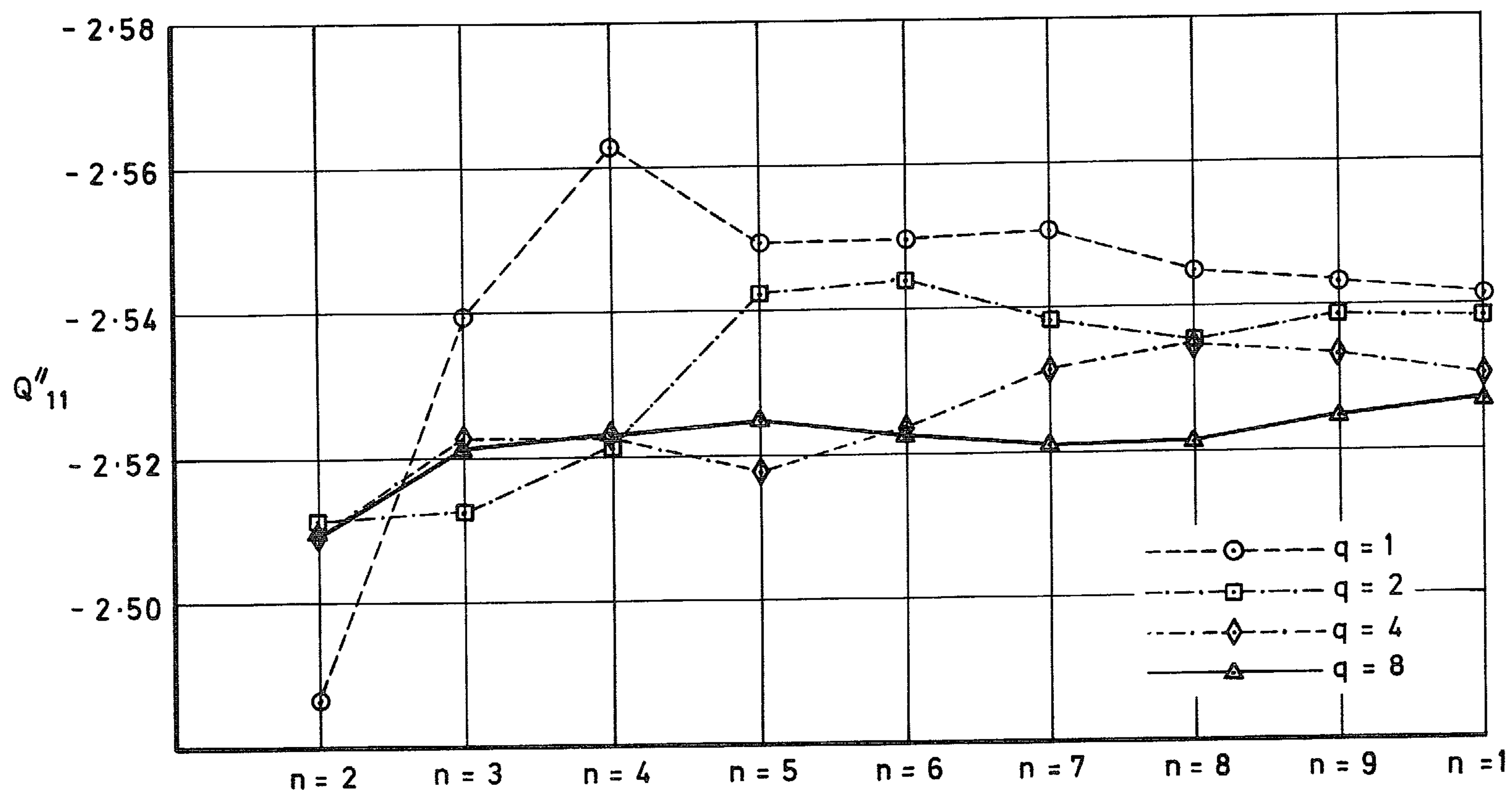


FIG. 5b. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

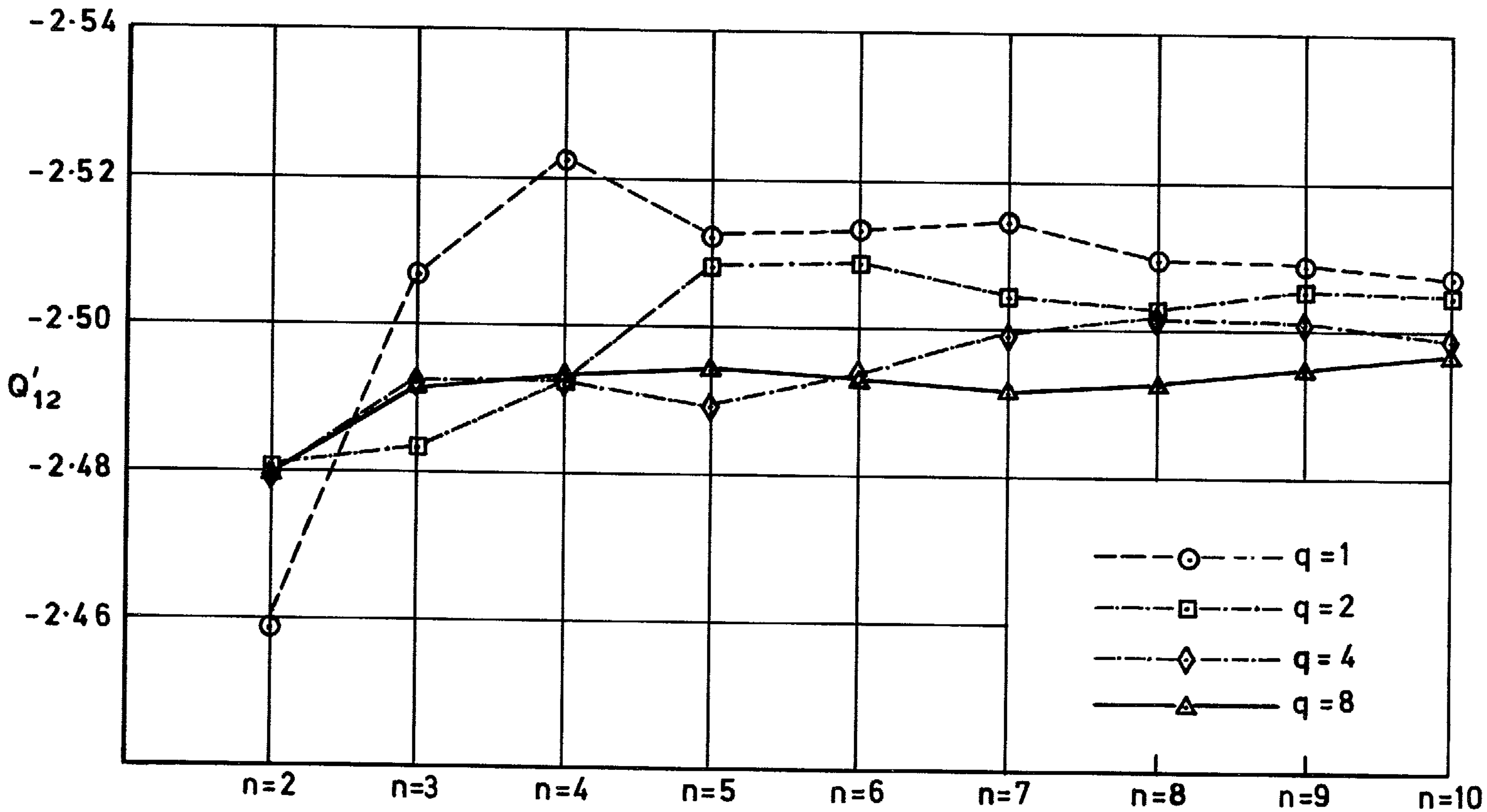


FIG. 5c. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

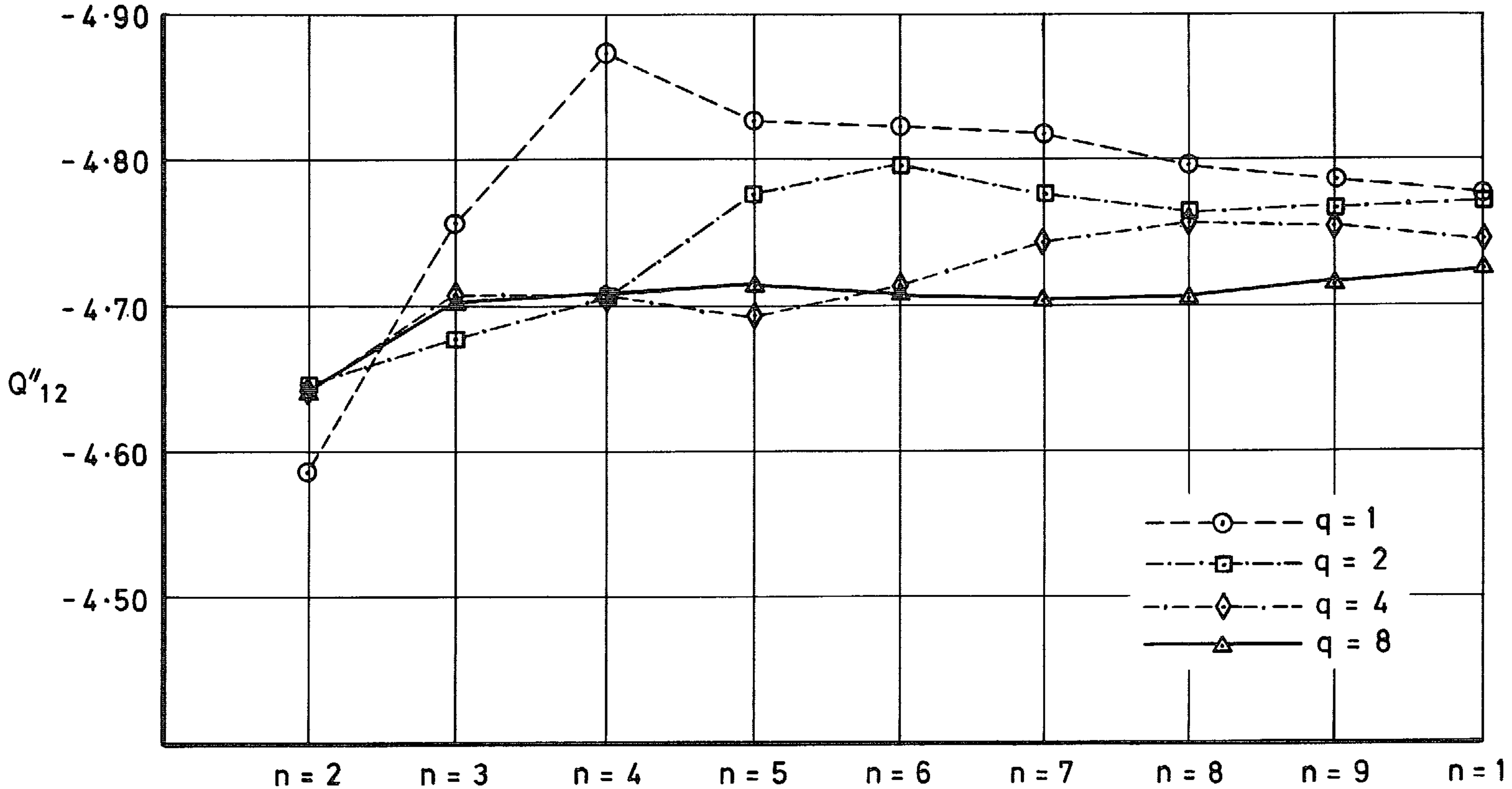


FIG. 5d. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

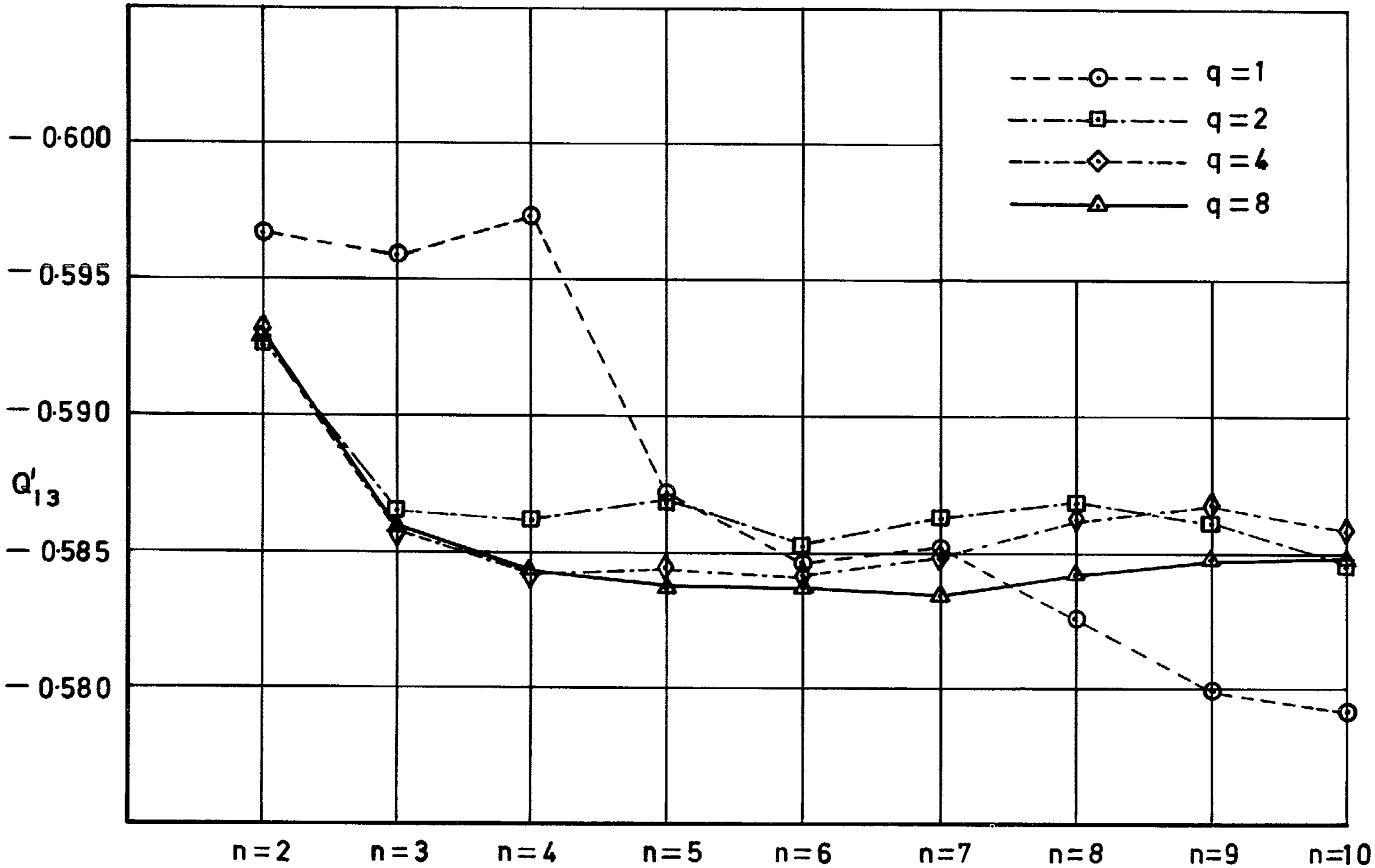


FIG. 5e. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

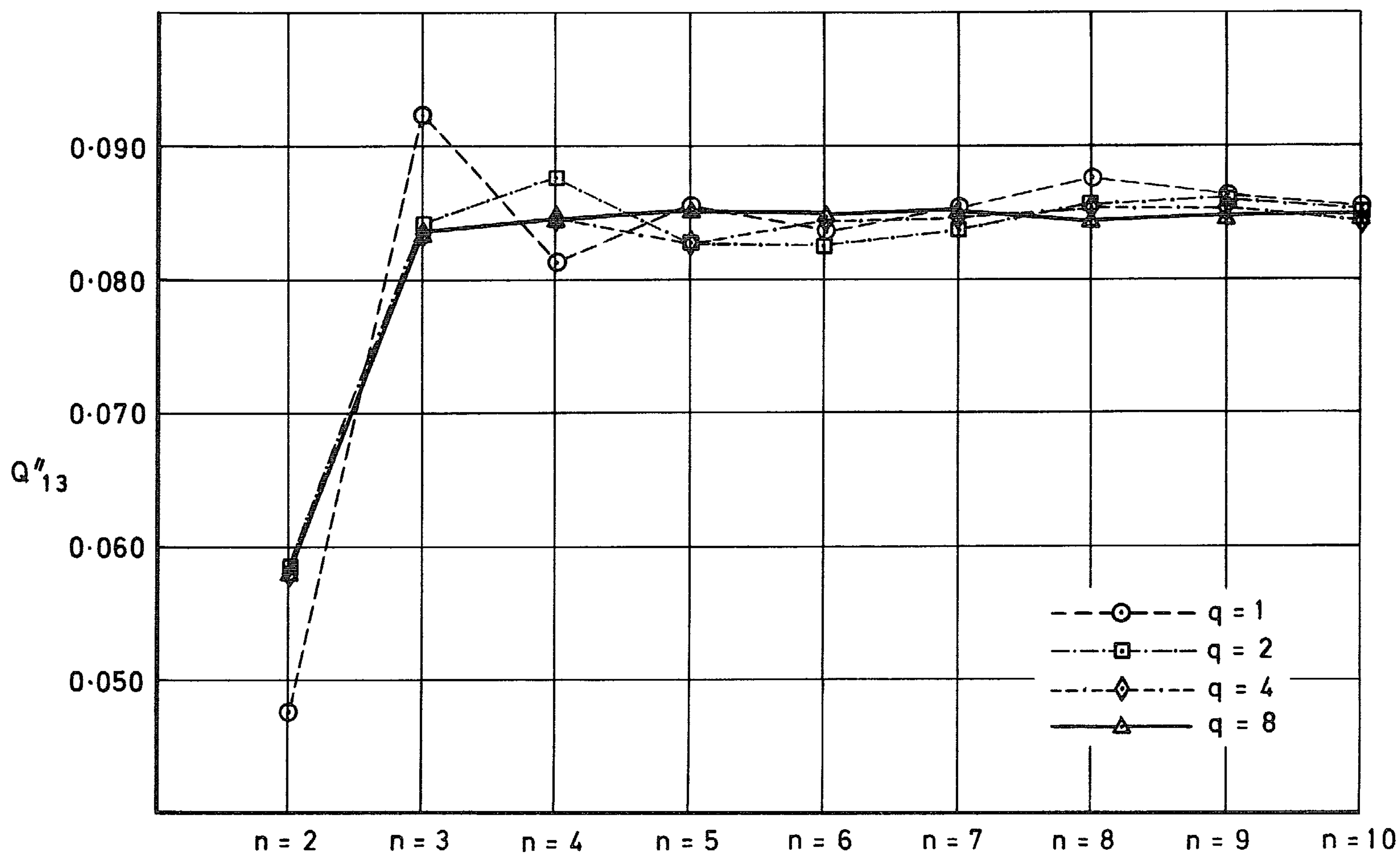


FIG. 5f. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

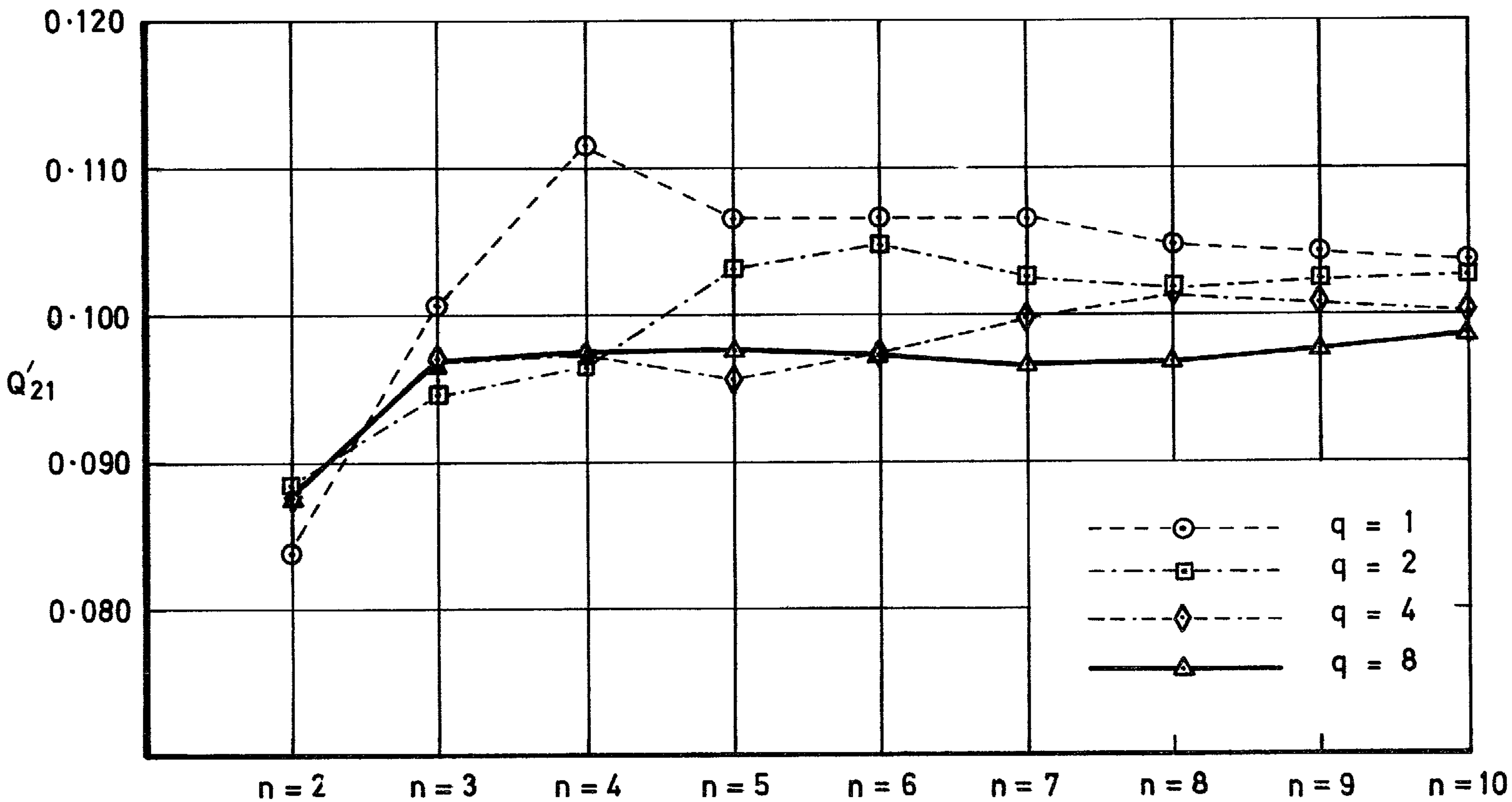


FIG. 5g. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

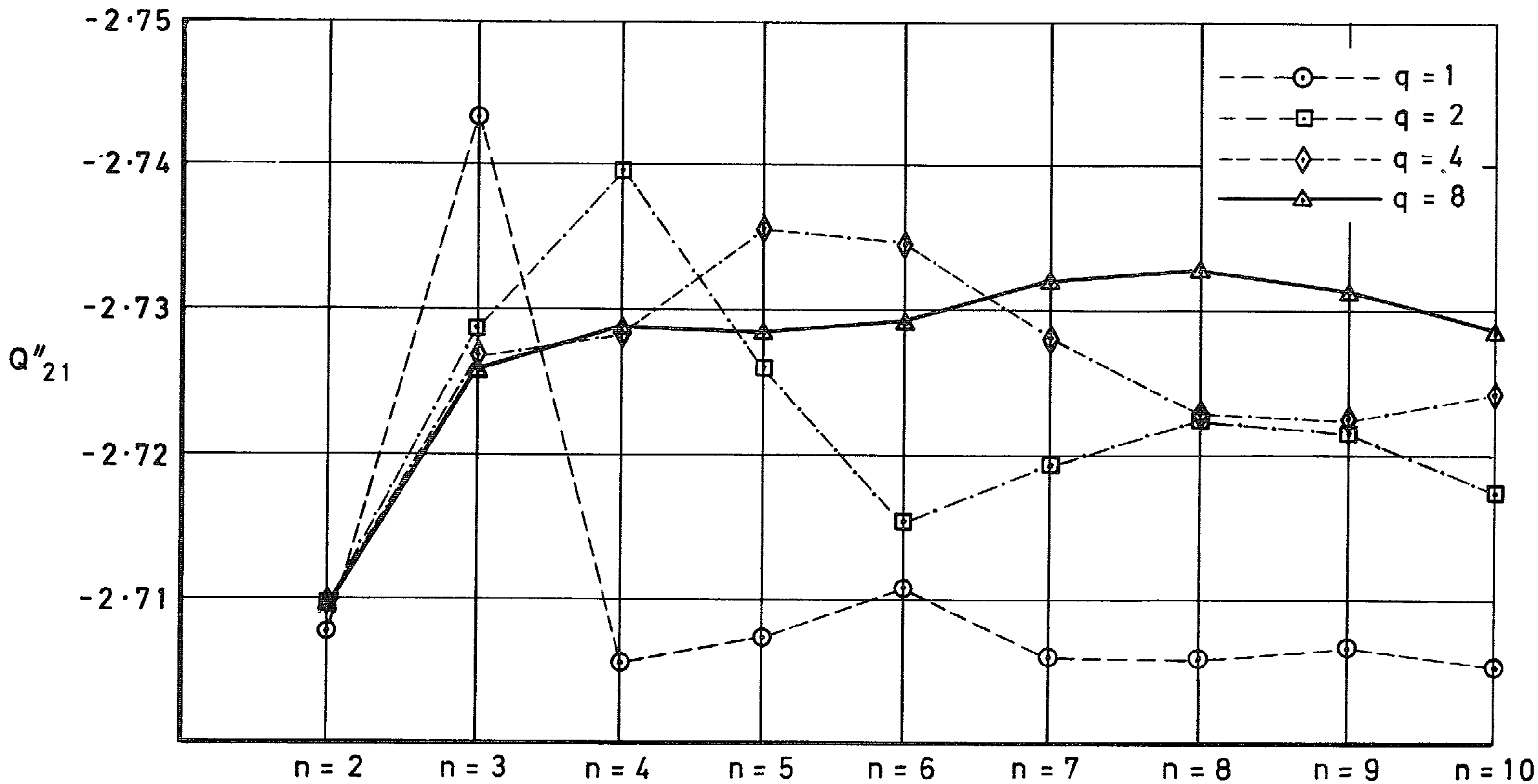


FIG. 5h. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

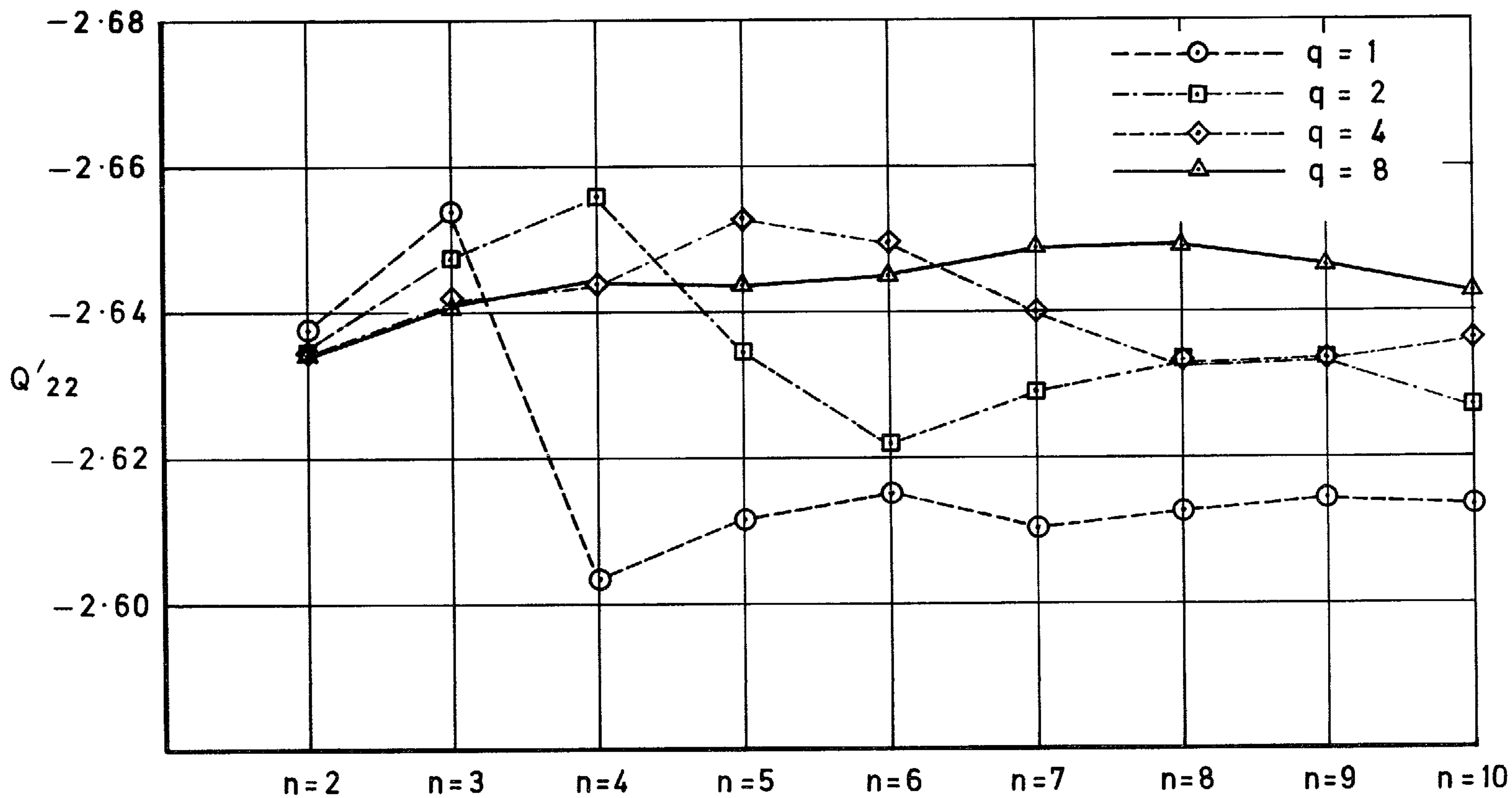


FIG. 5i. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

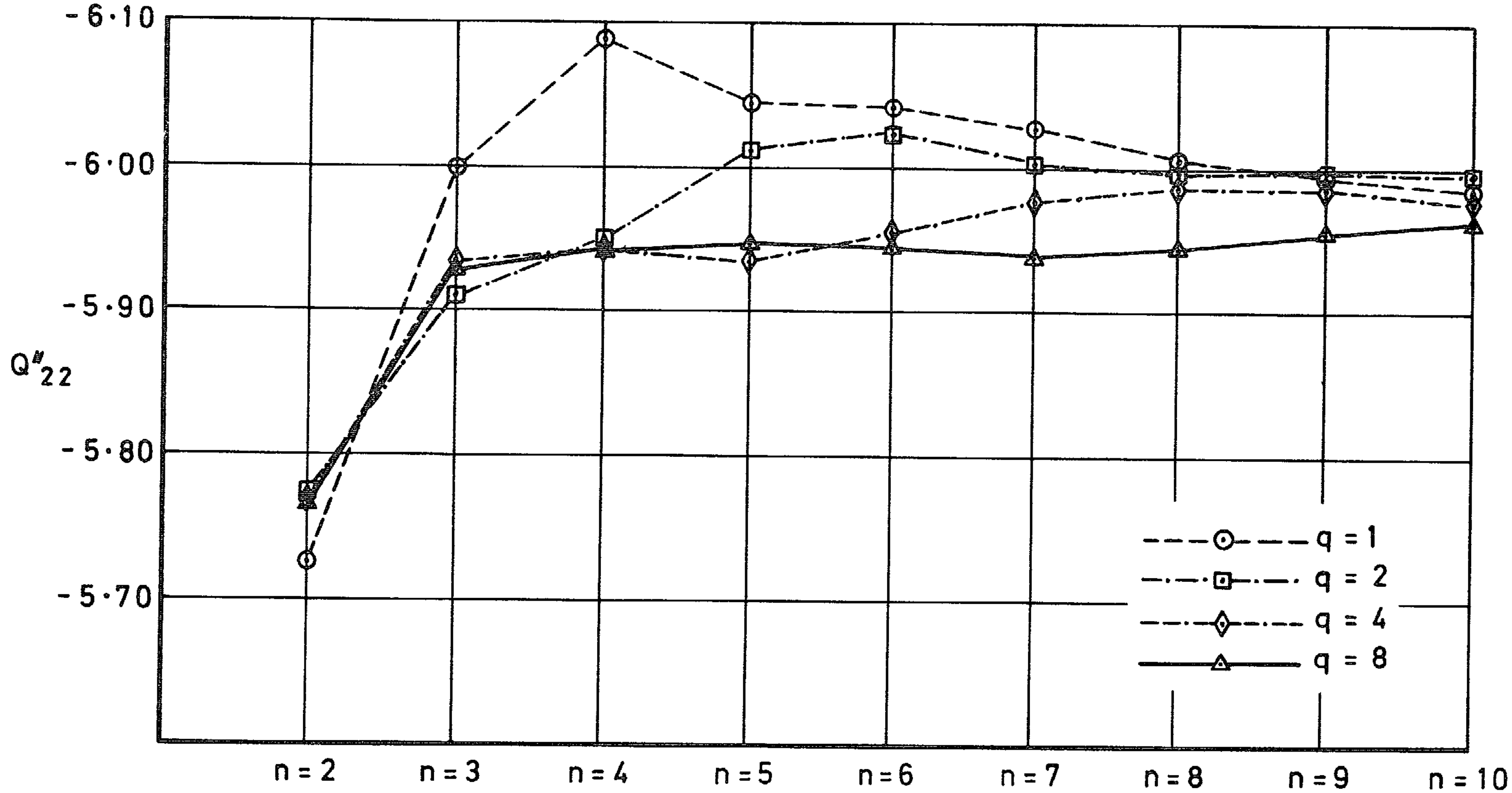


FIG. 5j. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

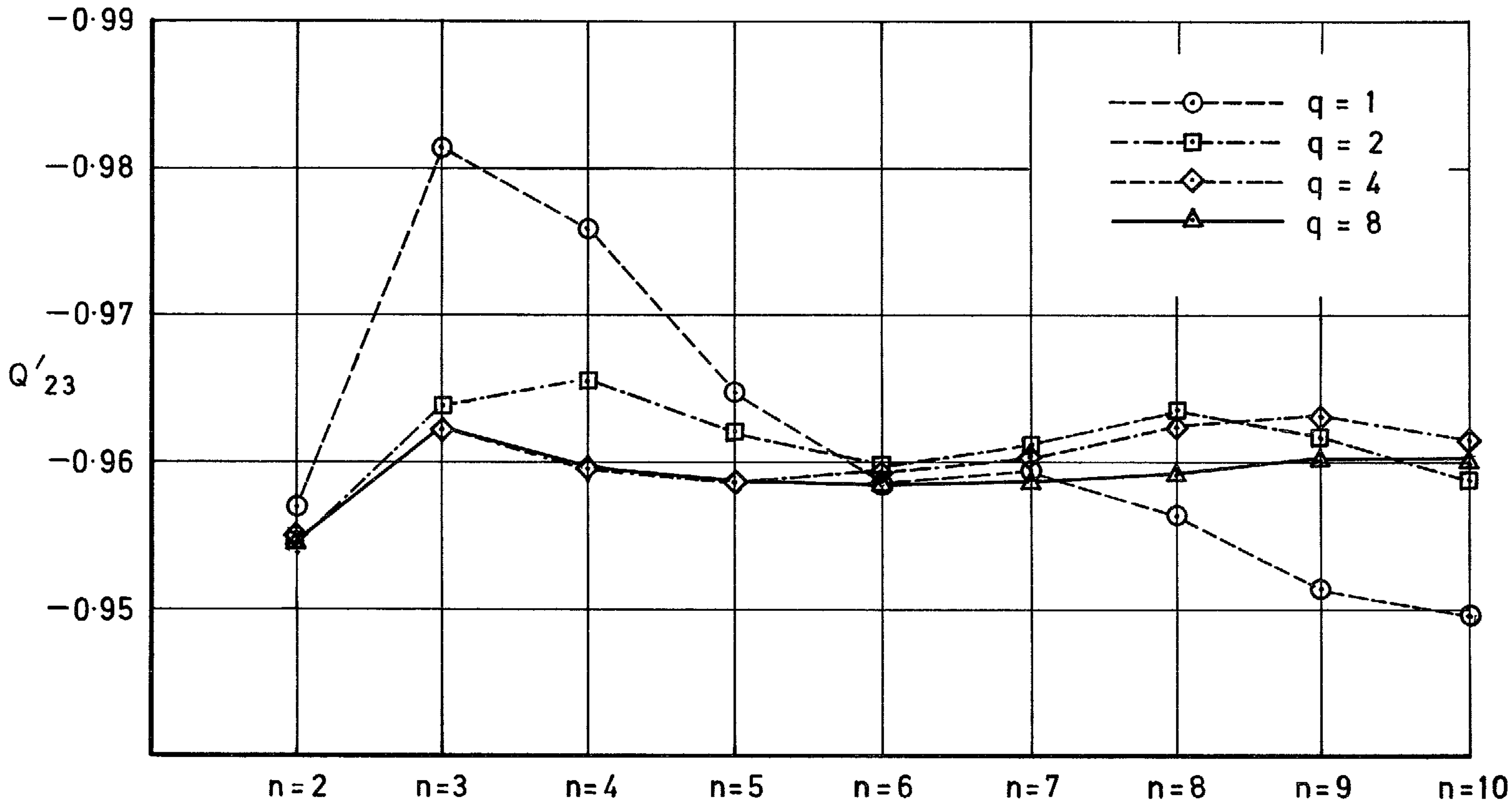


FIG. 5k. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

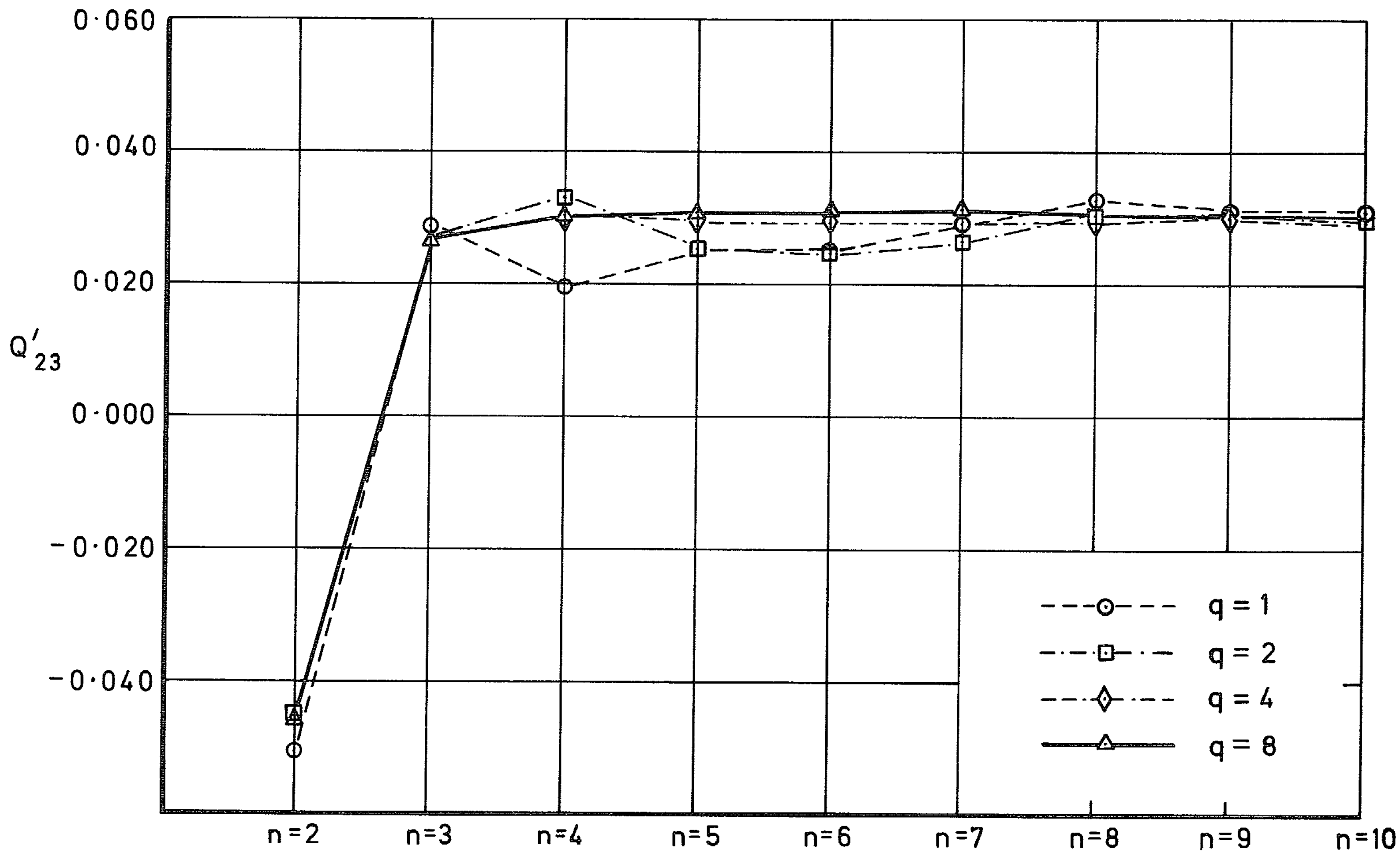


FIG. 51. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

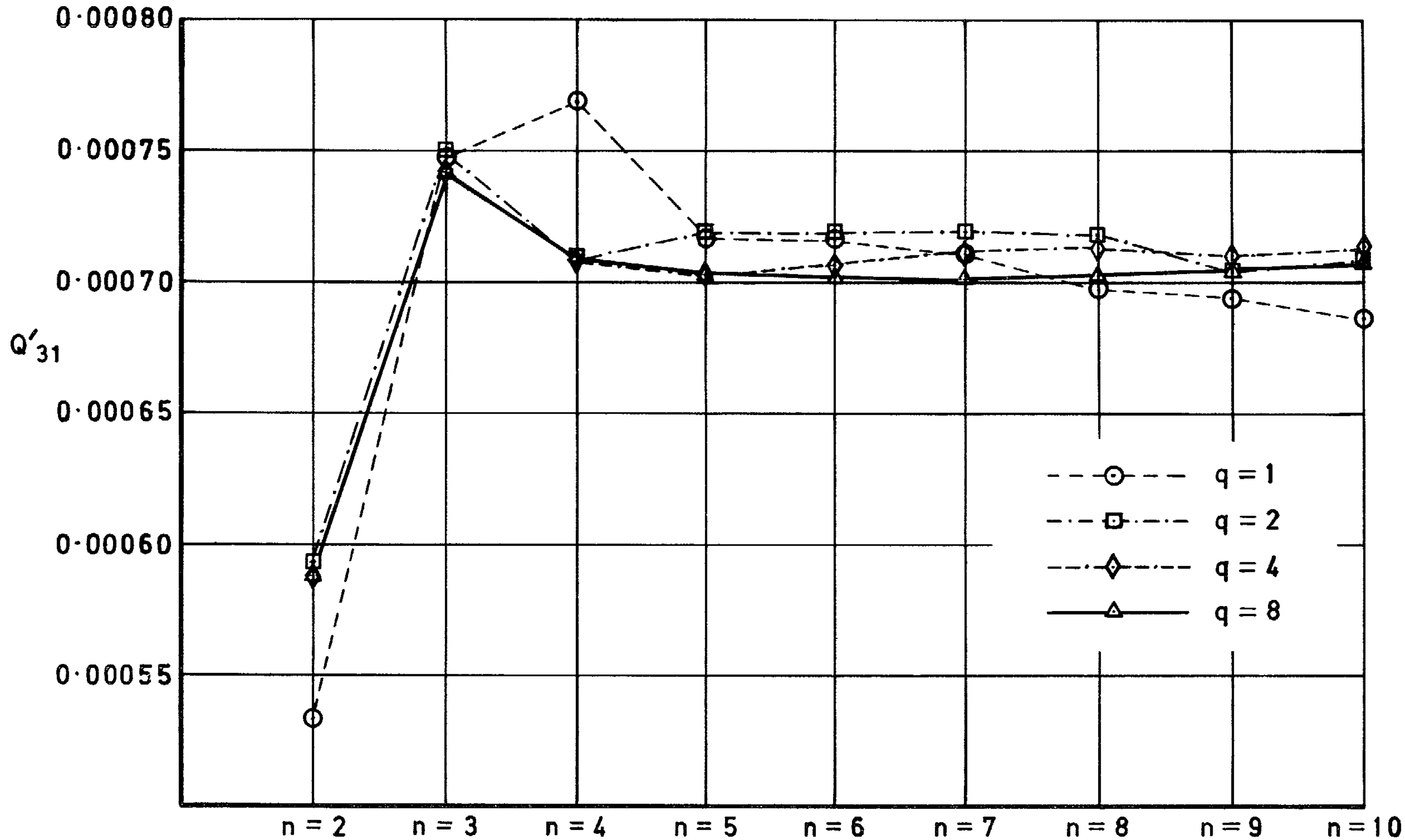


FIG. 5m. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

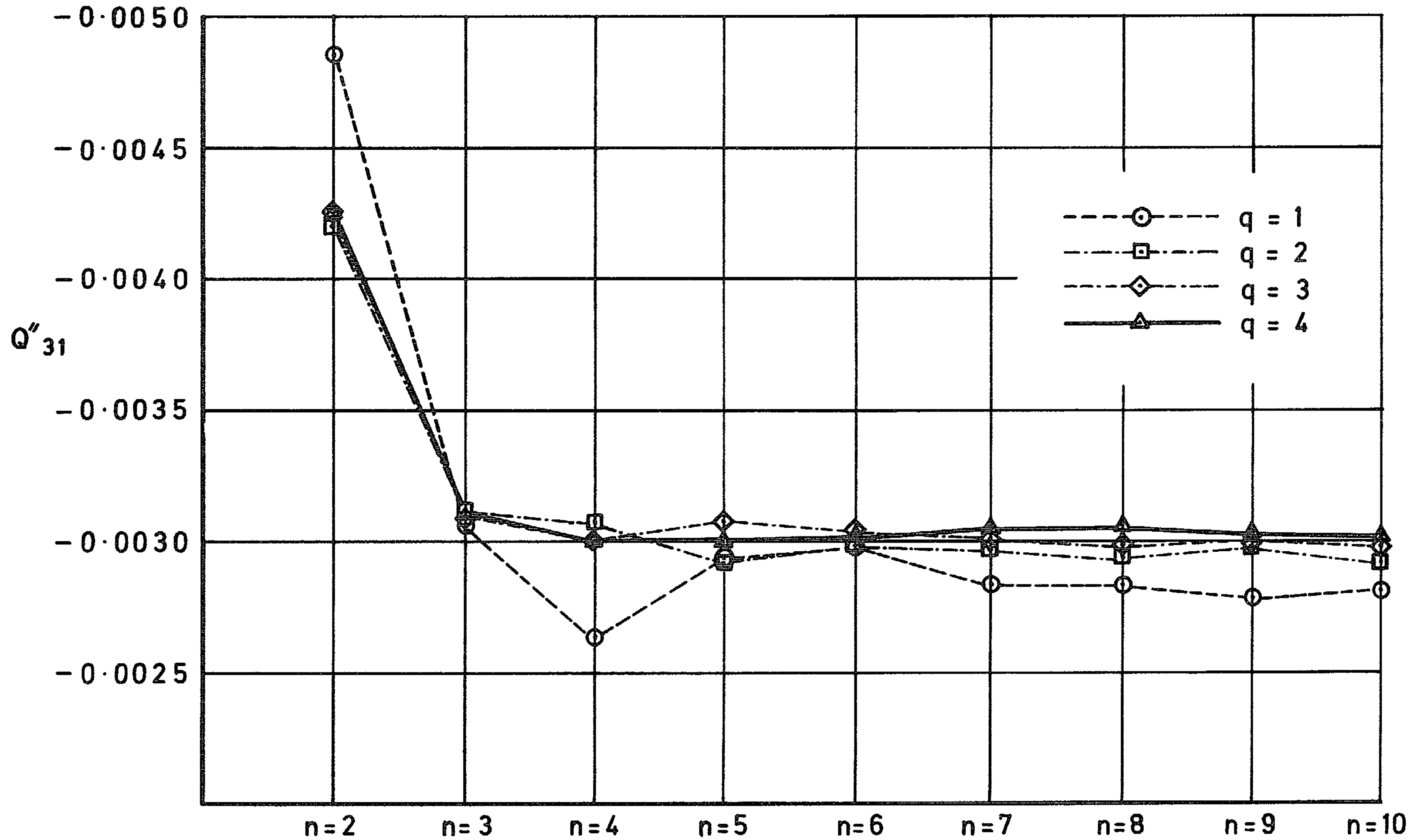


FIG. 5n. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

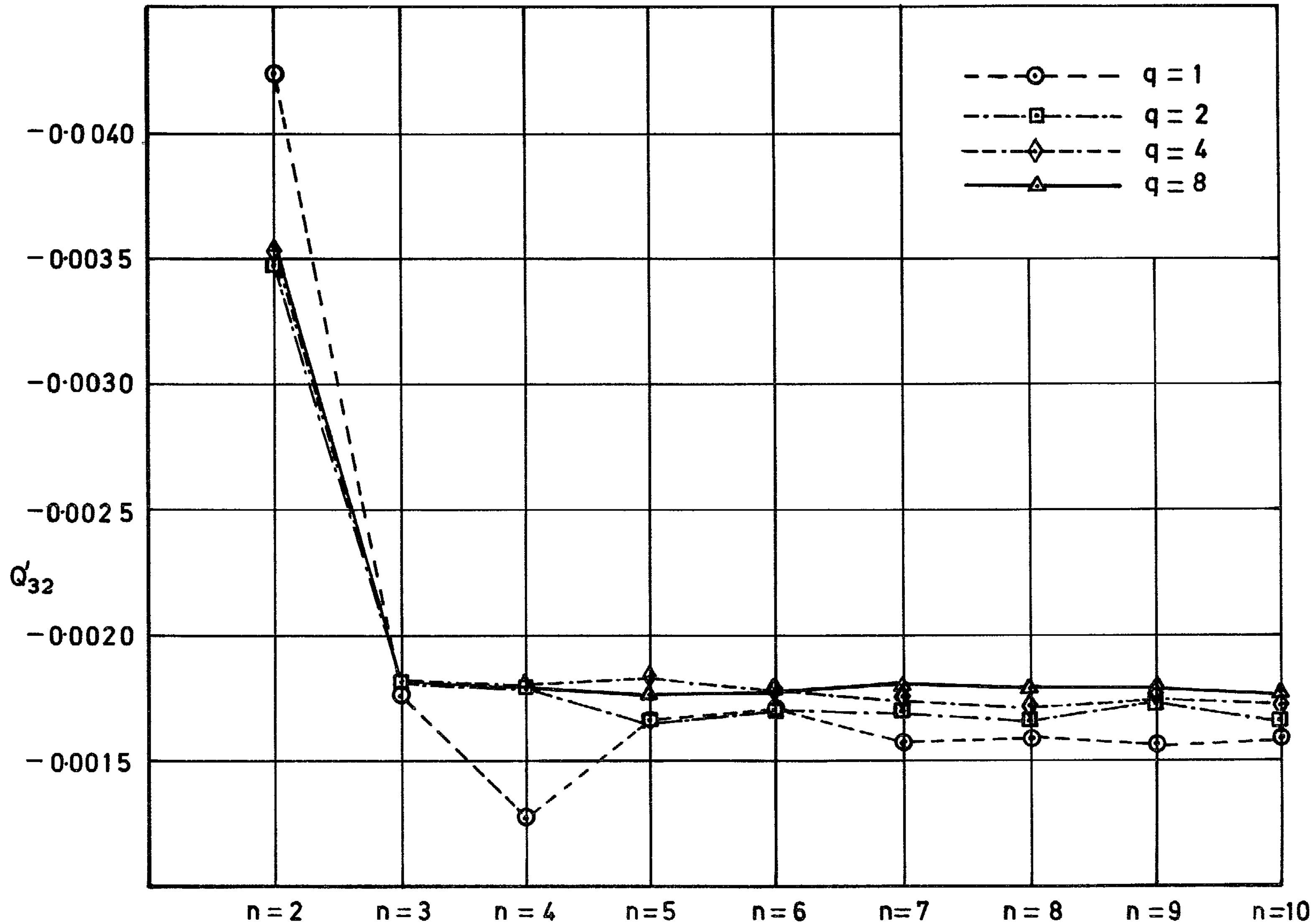


FIG. 5o. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

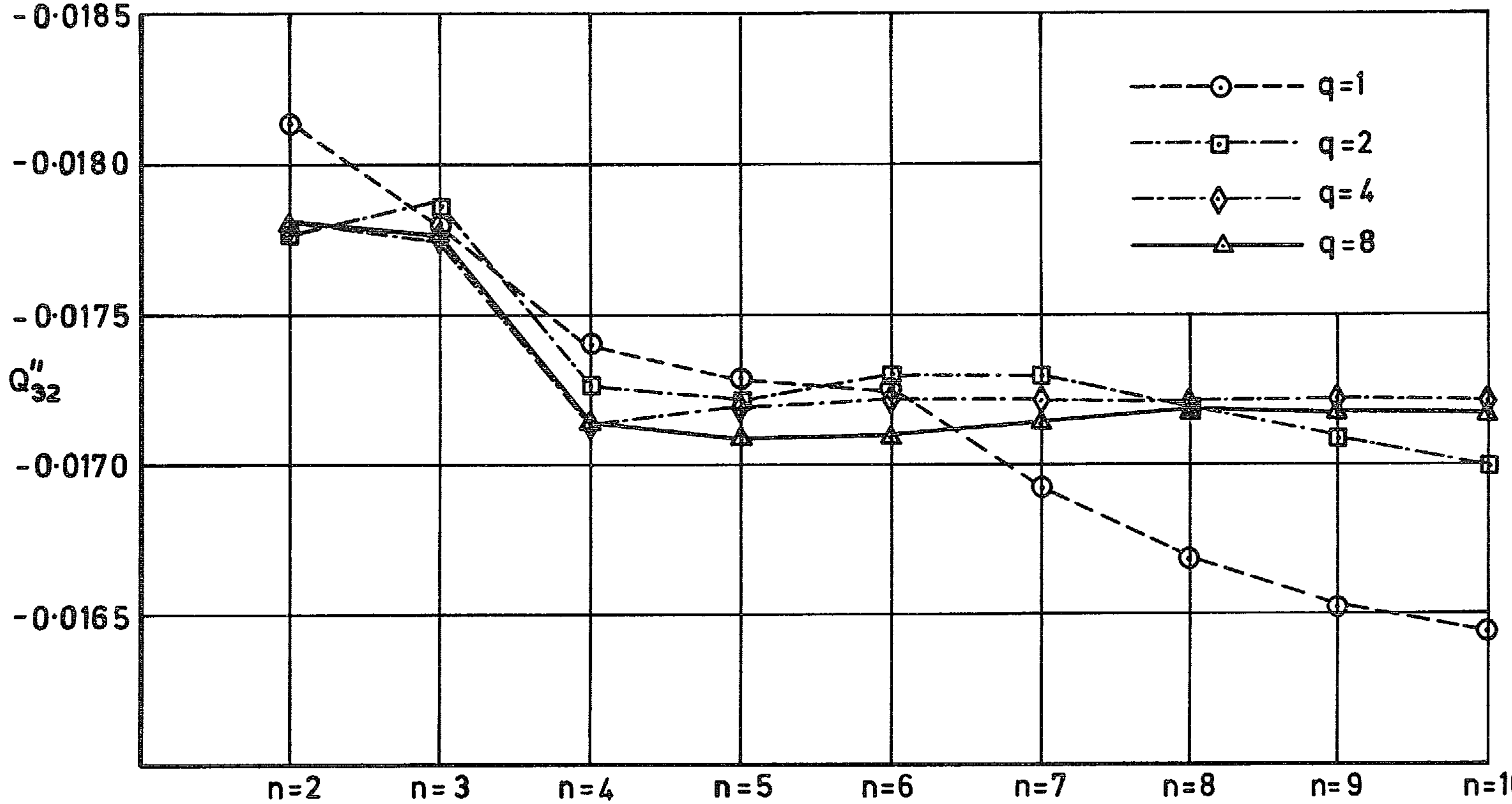


FIG. 5p. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu=0{\cdot}32560$, $M=0{\cdot}78060$.

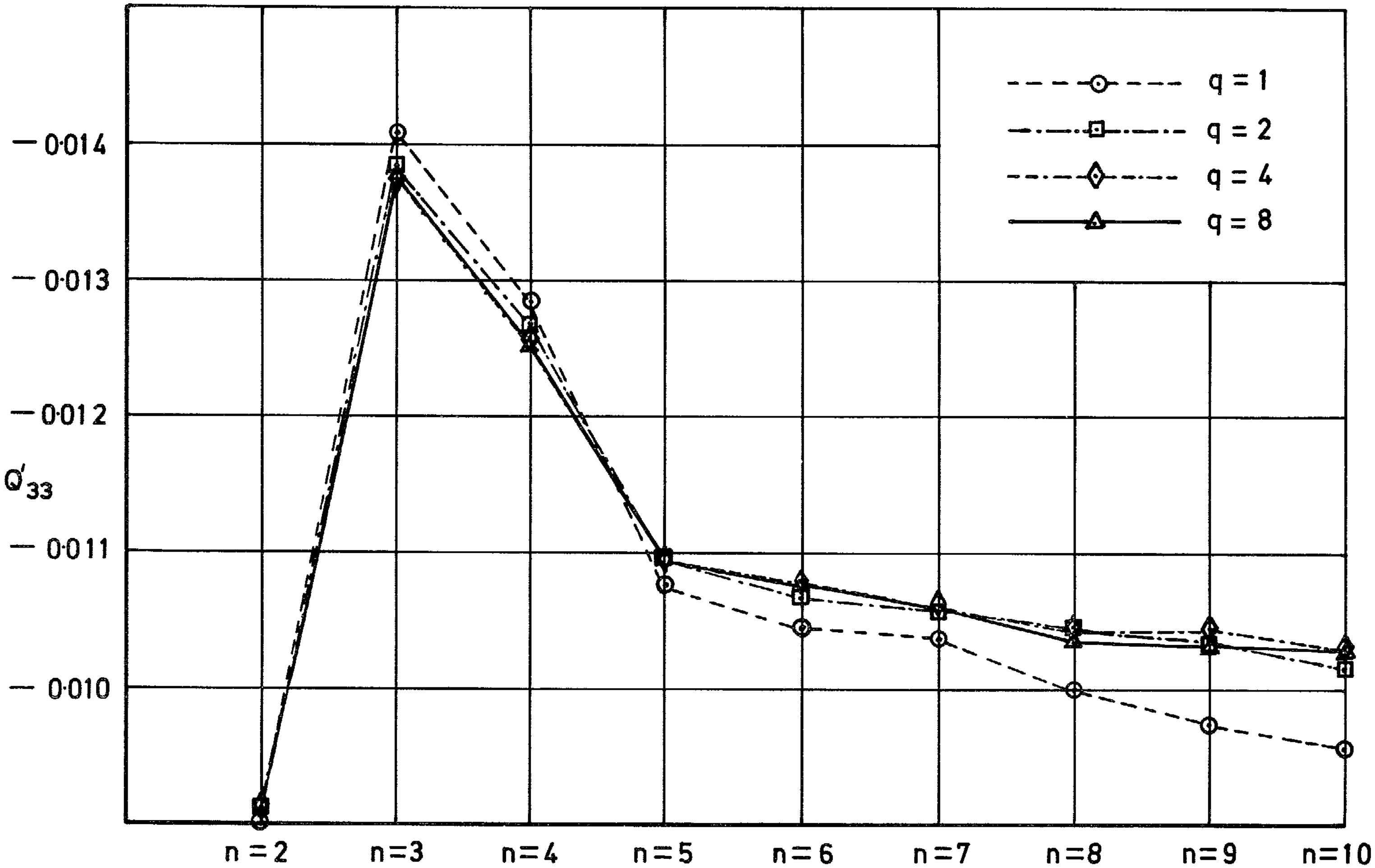


FIG. 5q. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

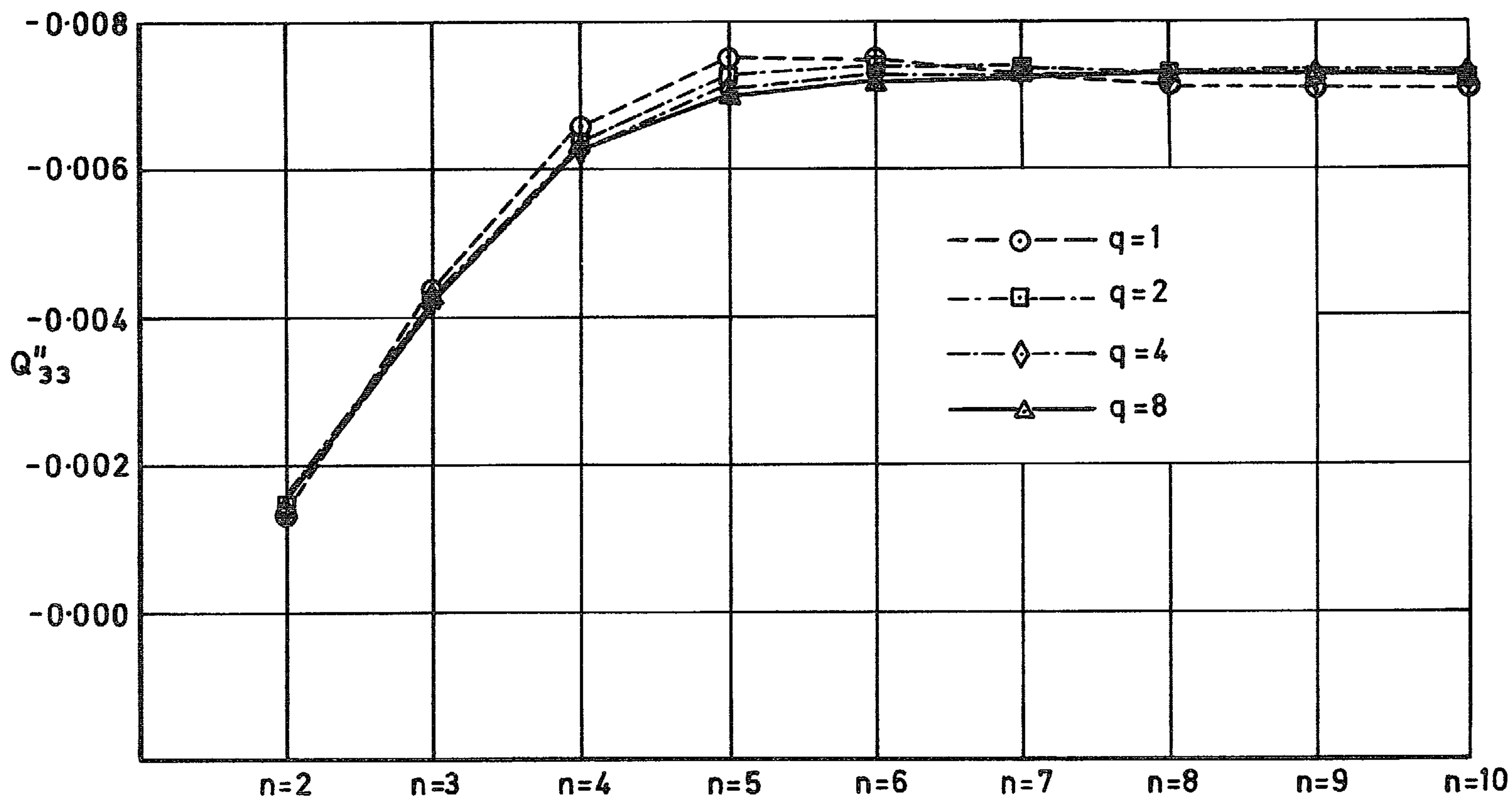


FIG. 5r. Generalised airforce coefficients on a tapered swept wing of aspect ratio 2. $\nu = 0{\cdot}32560$, $M = 0{\cdot}78060$.

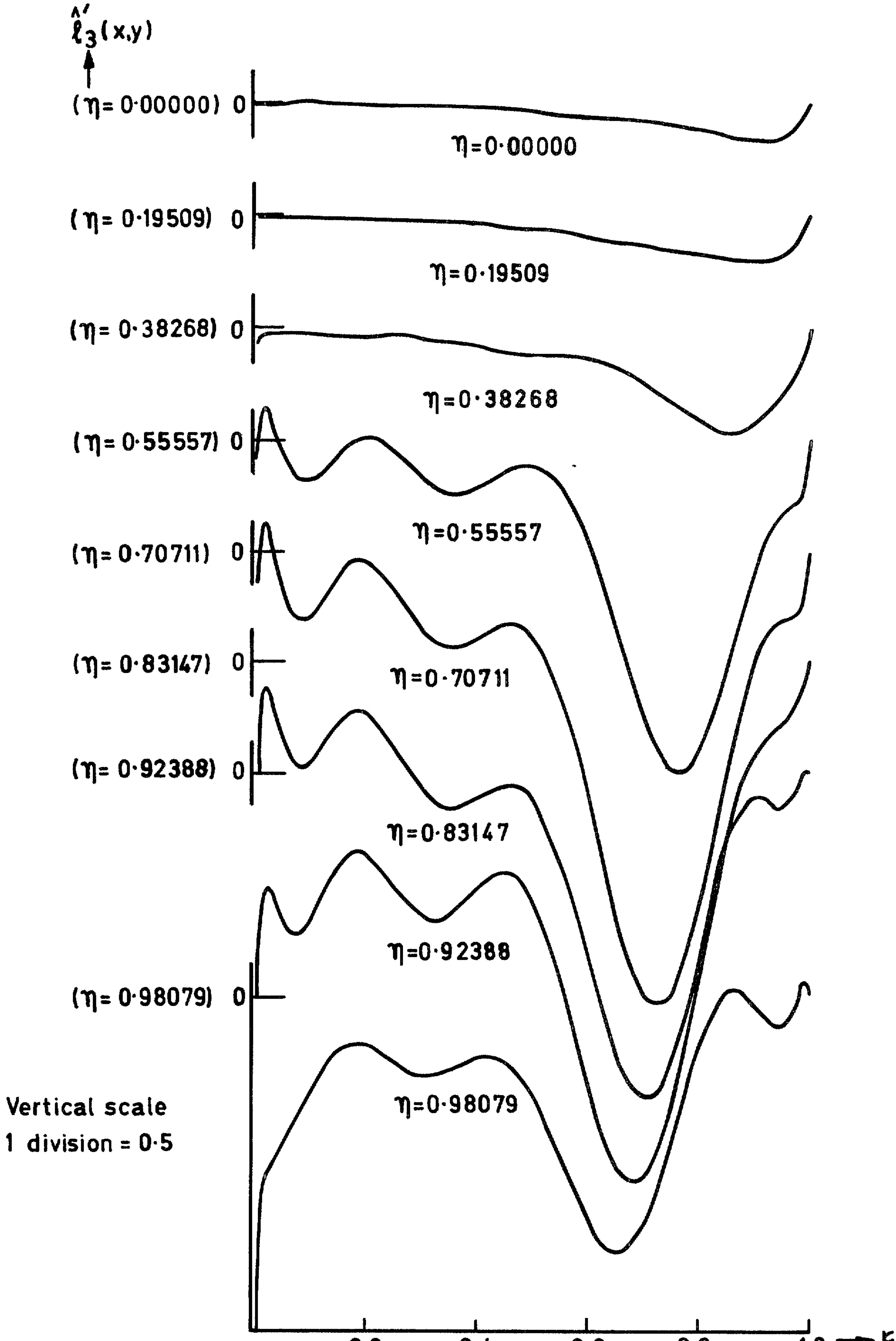


FIG. 6a. Real part of loading on a tapered swept wing of aspect ratio 2. $\nu = 1 \cdot 0$, $M = 0 \cdot 8$.

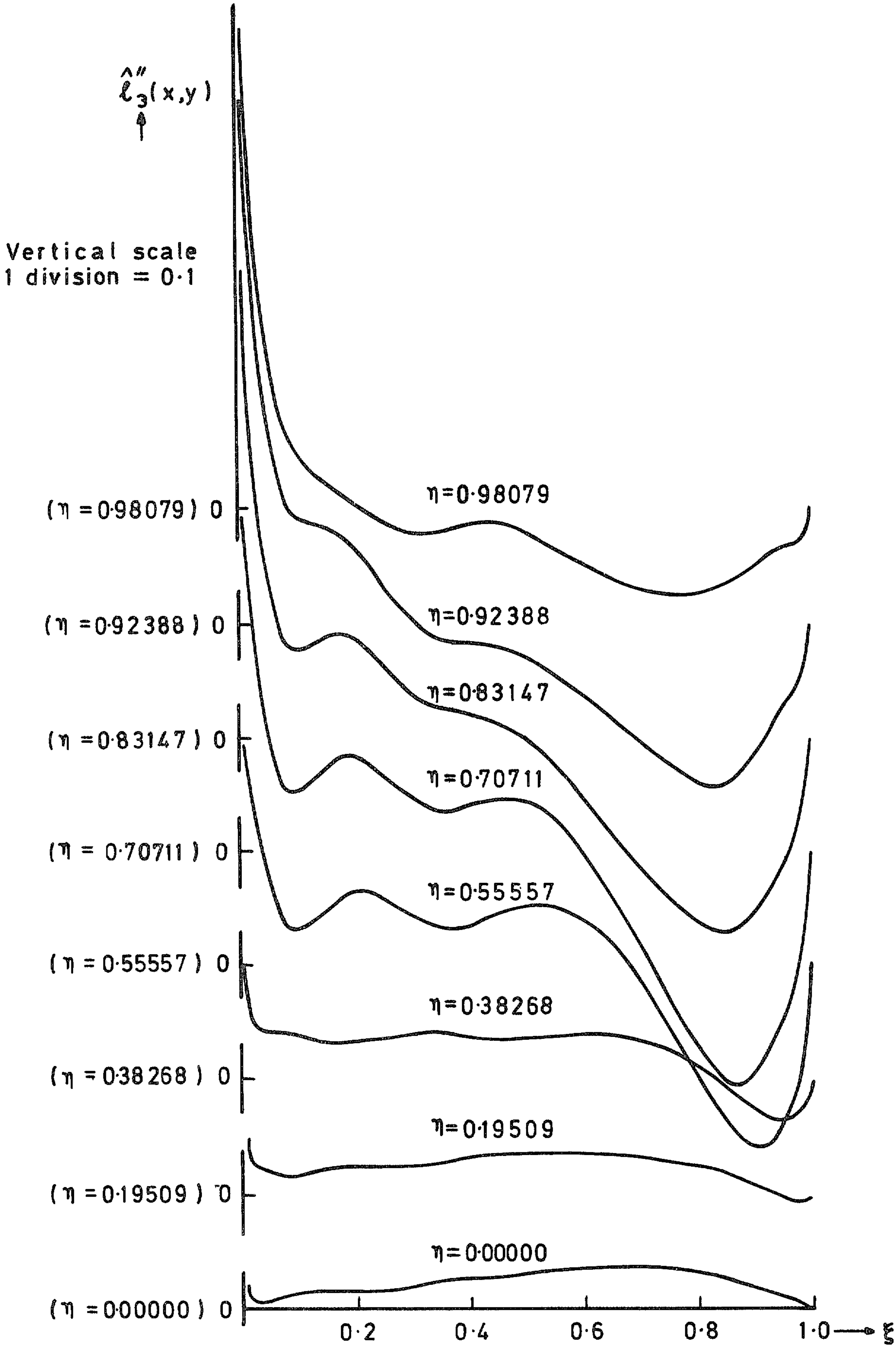


FIG. 6b. Imaginary part of loading on a tapered swept wing of aspect ratio 2. $\nu = 1{\cdot}0$, $M = 0{\cdot}8$.

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