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The Laminar Boundary Layer Associated with the Retarded Flow of a Compressible Fluid

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Summary.—Two aspects of the solution of the equations governing steady gas flow in a laminar boundary layer, when the main stream velocity is non-uniform, are considered. In the first place it is shown that the equations can be reduced to ordinary differential equations, whose solution implies the similarity* of the distributions of velocity and temperature in planes perpendicular to the boundary, only in the case when the main stream velocity is uniform. In the second part, an extension of Pohlhausen's method is used to determine the point of separation of the boundary layer in an air flow in which the pressure increases with a uniform gradient.

PART I

1.1. *Introduction.*—Exact solutions of the boundary layer equations for a compressible fluid flowing with uniform velocity in the main stream have been given by Busemann, and by Kármán and Tsien, for the special case in which the Prandtl number is equal to 1. Much more extensive sets of solutions have been obtained by Brainerd and Emmons for a series of Prandtl numbers, originally with constant values for the coefficients of viscosity and thermal conductivity [1](1941) and later with these coefficients varying with temperature [2](1942). In both these cases conditions depend only on one independent variable $y/x^{1/2}$, and consequently the solutions simply involve the integration of ordinary differential equations.

For an *incompressible* fluid with a uniform velocity in the main stream the solution is also a function of $y/x^{1/2}$ only. Otherwise, when the main stream velocity U is not uniform, it has been shown by Goldstein [3](1939) that there are no solutions in which the velocity distributions are similar for different values of x, apart from the cases when $U = cx^m$ or $U = ce^{ax}$.

The corresponding question for a *compressible* fluid will be examined:—What special forms of the variations of U with x allow solutions with similar velocity and temperature distributions for different values of x?

1.2. *Boundary Layer Equations.*—The steady flow of a compressible fluid along a fixed boundary will be considered. With the co-ordinate axes of x and y along and at right-angles respectively to the wall the usual approximations of boundary layer theory yield the equations

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \qquad (1)$$

* In the sense that each distribution function is a function of a single variable of the form y times a function of x, where x, y are distances measured along and perpendicular to the boundary.

$$0 = -\frac{\partial p}{\partial y}\,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (2)$$

$$\rho u \frac{\partial}{\partial x}(c_p T) + \rho v \frac{\partial}{\partial y}(c_p T) = u \frac{\partial p}{\partial x} + \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{\partial}{\partial y}\left(\varkappa \frac{\partial T}{\partial y}\right). \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (3)$$

The equation of continuity is

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0\,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (4)$$

It is assumed that the fluid is a perfect gas so the relation between pressure, density and temperature is

$$p = R\rho T\,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (5)$$

In the main stream, where the effects of viscosity and heat condition are neglected, the pressure, density, temperature and velocity are all functions of x only. They satisfy the following equations, in which the suffix 1 refers to the main stream,

$$\rho_1 U \frac{dU}{dx} = -\frac{dp_1}{dx}\,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (6)$$

$$\rho_1 \frac{d}{dx}(c_p T_1) = \frac{dp_1}{dx}\,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (7)$$

$$p_1 = R\rho_1 T_1\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (8)$$

From equations (6) and (7) it follows that

$$c_p T_1 + \tfrac{1}{2}U^2 = \text{const.} = c_p T_0.$$

It follows from equation (2) that the pressure is constant, for a given value of x, throughout the boundary layer, and therefore that it is equal to the main stream value for the same value of x. Consequently, for all values of y,

$$\rho(x, y)\, T(x, y) = \rho_1(x)\, T_1(x)\,.$$

By using this relation, ρ can be eliminated from equations (1) to (5) which then give

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{T}{T_1} U \frac{dU}{dx} + \frac{RT}{p_1} \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (9)$$

$$u \frac{\partial}{\partial x}(c_p T) + v \frac{\partial}{\partial y}(c_p T) = -\frac{uT}{T_1} U \frac{dU}{dx} + \frac{RT}{p_1} \mu \left(\frac{\partial u}{\partial y}\right)^2 + \frac{RT}{p_1} \frac{\partial}{\partial y}\left(\varkappa \frac{\partial T}{\partial y}\right), \quad \ldots \quad \ldots \qquad (10)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{u}{RT_1} U \frac{dU}{dx} + \frac{1}{T}\left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y}\right). \qquad \ldots \quad \ldots \quad \ldots \qquad (11)$$

When the main stream velocity is uniform these equations can be reduced to three ordinary differential equations involving the three unknowns, u, v and T, which are functions of the independent variable $y/x^{1/2}$. Brainerd and Emmons have integrated these ordinary differential equations in several cases, for different main stream velocities with different laws of dependence of the viscosity coefficient μ on the temperature T.

1.3. *Reducibility to Three Ordinary Differential Equations.*—The possibility of obtaining a reduction from partial to ordinary differential equations when the main stream velocity is non-uniform will now be examined. Suppose that by employing the substitutions

$$u(x, y) = F_1(x, y)\, u_2(\eta)\,,$$
$$v(x, y) = F_2(x, y)\, v_2(\eta)\,,$$
$$T(x, y) = F_3(x, y)\, T_2(\eta)\,,$$

where η is a function of x and y, the resulting equations involve only one independent variable, η. These substitutions are not completely general, but at first sight they appear to be less restrictive than those required by the assumption that the velocity and temperature distributions are similar for different values of x. When these forms for u, v and T are used, the equations become

$$\begin{aligned} &F_1 F_{1x} u_2^2 + F_1^2 \eta_x u_2 u_2' + F_{1y} F_2 u_2 v_2 + F_1 F_2 \eta_y u_2' v_2 - U(dU/dx) T_1^{-1} F_3 T_2 \\ &- Rp_1^{-1} \mu [F_{1yy} F_3 u_2 T_2 + (2F_{1y} F_3 \eta_y + F_1 F_3 \eta_{yy}) u_2' T_2 + F_1 F_3 \eta_y^2 u_2'' T_2] \\ &- Rp_1^{-1} (d\mu/dT) [F_{1y} F_3 F_{3y} u_2 T_2^2 + F_{1y} F_3^2 \eta_y u_2 T_2 T_2' + F_1 F_3 F_{3y} \eta_y u_2' T_2^2 \\ &\qquad + F_1 F_3^2 \eta_y^2 u_2' T_2 T_2'] = 0 , \qquad \ldots \quad \ldots \quad (12) \end{aligned}$$

$$\begin{aligned} &[c_p F_1 F_{3x} + U(dU/dx) T_1^{-1} F_1 F_3] u_2 T_2 + c_p F_2 F_{3y} v_2 T_2 + c_p F_1 F_3 \eta_x u_2 T_2' + c_p F_2 F_3 \eta_y v_2 T_2' \\ &- Rp_1^{-1} \mu [F_{1y}^2 F_3 u_2^2 T_2 + 2F_1 F_{1y} F_3 \eta_y u_2 u_2' T_2 + F_1^2 F_3 \eta_y^2 u_2'^2 T_2] \\ &- Rp_1^{-1} \varkappa [F_3 F_{3yy} T_2^2 + (2F_3 F_{3y} \eta_y + F_3^2 \eta_{yy}) T_2 T_2' + F_3^2 \eta_y^2 T_2 T_2''] \\ &- Rp_1^{-1} (d\varkappa/dT) [F_3 F_{3y}^2 T_2^3 + 2F_3^2 F_{3y} \eta_y T_2^2 T_2' + F_3^2 \eta_y^2 T_2 T_2'^2] = 0 , \qquad \ldots \quad \ldots \quad (13) \end{aligned}$$

$$\begin{aligned} &[F_{1x} - U(dU/dx) (RT_1)^{-1} F_1 - F_1 F_3^{-1} F_{3x}] u_2 + [F_{2y} - F_2 F_3^{-1} F_{3y}] v_2 \\ &\qquad + F_1 \eta_x u_2' + F_2 \eta_y v_2' - F_1 \eta_x u_2 T_2^{-1} T_2' - F_2 \eta_y v_2 T_2^{-1} T_2' = 0 . \qquad \ldots \quad \ldots \quad (14) \end{aligned}$$

In these equations, the suffixes x, y denote $\partial/\partial x$, $\partial/\partial y$, and the dash denotes $\partial/\partial\eta$. The specific heat c_p has been assumed to be constant.

Apart from the possibility of a common factor in each equation, the part of each term not involving u_2, v_2, or T_2 must be a function of η only. Thus comparison of the coefficients of $u_2 v_2$ and $u_2' v_2$ in equation (12) leads to the equation $F_1^{-1} F_{1y} = G_1(\eta)\eta_y$, where G_1 is an undetermined function of η. Consequently $\log F_1 = \int G_1(\eta) d\eta + H_1(x)$, where H_1 is an undetermined function of x. This equation can be written as $F_1(x, y) = J_1(\eta) K_1(x)$, where $J_1 = \exp [\int G_1(\eta) d\eta]$ and $K_1 = \exp H_1(x)$. But $J_1(\eta)$ must be unity since it is supposed that all factors of u which are functions of η have already been absorbed into $u_2(\eta)$. It therefore follows that $F_1 (x, y)$ is a function of x only. Similarly comparison of the coefficients of $v_2 T_2$ and $v_2 T_2'$ in equation (13) gives the result $F_3(x, y) = K_3(x)$. Finally, from the coefficients of v_2 and v_2' in equation (14) $F_2(x, y) = K_2(x)$. Now the coefficients of u_2^2 and $u_2 u_2'$ in equation (12) give the equation $F_1^{-1} F_{1x} = K_1^{-1} (dK_1/dx) = L_1(\eta) \eta_x$, where L_1 is an undetermined function of η. Integration of this equation gives $\log K_1 = \int L_1(\eta) d\eta + M_1(y)$, where M_1 is an undetermined function of y. This is equivalent to $N_1(\eta) = K_1(x) P_1(y)$, where $N_1 = \exp [\int L_1(\eta) d\eta]$ and $P_1 = \exp [-M_1(y)]$. There is no restriction in taking $N_1(\eta)$ equal to η. For, if this is not the case, a new independent variable ζ can be defined by the equation $\zeta = N_1(\eta)$ and each of u, v and T can be written in the form $f(x)g(\zeta)$ where $\zeta = K_1(x) P_1(y)$. Thus the independent variable in the ordinary differential equations is expressible as the product of a function of x and a function of y.

This does not mean that the solution of the differential equations necessarily has the property of similarity, whereby the distributions of the velocity component and temperature are similar in the planes of constant x. Compare, for example, the distribution in the plane through the point $(x, 0)$ with that in a standard plane $x = x_0$. The ordinate y of the point in the arbitrary plane for which conditions correspond to those at the point (x_0, y_0) in the standard plane is given by the equation $P_1(y)/P_1(y_0) = K_1(x_0)/K_1(x)$. The condition for similarity is satisfied only if this equation implies that y/y_0 is independent of y and y_0 individually. This is certainly the case if $P_1(y)$ is a power of y.

Now by comparison of the coefficients of $u_2'T_2$ and $u_2''T_2$ in equation (12) it is seen that the expression $(2F_{1y}F_3\eta_y + F_1F_3\eta_{yy})/(F_1F_3\eta_y^2) = (d^2P_1/dy^2)/[K_1(dP_1/dy)^2]$ must be a function of η. It follows that $P_1(d^2P_1/dy^2)/(dP_1/dy)^2$ must be a constant, and therefore that either $P_1(y) = y^s$ or $P_1(y) = e^{ay}$. That the second alternative is not permissible is seen by considering the behaviour of $u(x, y)$ at the wall $y = 0$. If $P_1(y) = e^{ay}$, then $u = K_1(x)u_2(K_1 e^{ay})$, and at the wall $u = K_1(x)u_2(K_1(x))$. It is not possible for the last expression to vanish as required for all values of x, unless either $K_1(x)$ is a constant or $u_2(\eta) = 0$ everywhere, and neither of these conditions can be satisfied. The conclusion is that $P_1(y)$ is of the form y^s and that a solution of the ordinary differential equations must have the property of similarity.

By considering the coefficients of u_2T_2 and u_2T_2' in equation (13) it follows that the expression

$$[c_pF_1F_{3x} + U(dU/dx)\, T_1^{-1}\, F_1F_3]/[c_p\, F_1F_3\, \eta_x] = [K_3^{-1}(dK_3/dx) + U(dU/dx)\, (c_pT_1)^{-1}]/\eta_x$$
$$= [K_3^{-1}(dK_3/dx) - T_1^{-1}(dT_1/dx)]/[K_1^{-1}(dK_1/dx)\,\eta]$$

must be a function of η. Consequently $[K_3^{-1}(dK_3/dx) - T_1^{-1}(dT_1/dx)]/[K_1^{-1}(dK_1/dx)] = \alpha$, where α is a constant. Hence, $K_3/T_1 = AK_1^{\alpha}$ where A is also a constant.

Comparison of the coefficients of u_2 and u_2' in equation (14) shows that the expression

$$[F_{1x} - U(dU/dx)\,(RT_1)^{-1}\, F_1 - F_1\, F_3^{-1}\, F_{3x}]/[F_1\,\eta_x]$$
$$= [K_1^{-1}(dK_1/dx) + \gamma(\gamma - 1)^{-1}\, T_1^{-1}(dT_1/dx) - K_3^{-1}\,(dK_3/dx)]/[K_1^{-1}\,(dK_1/dx)\,\eta]$$

must be a function of η. This requires that

$$K_1^{-1}(dK_1/dx) + \gamma(\gamma - 1)^{-1}\, T_1^{-1}(dT_1/dx) - K_3^{-1}(dK_3/dx) = \beta K_1^{-1}\,(dK_1/dx)\,,$$

where β is a constant; and this equation in turn requires that $T_1^{\gamma/(\gamma-1)}/K_3 = BK_1^{\beta-1}$, where B is also a constant. Hence $ABK_1^{\alpha+\beta-1} = T_1^{1/(\gamma-1)}$, so $K_1 = T_1^{n}$, where the constant factor has been taken equal to unity, without loss of generality. Also $K_3(= AK_1^{\alpha}\, T_1) = T_1^{m}$, where the constant factor has again been taken equal to unity.

Finally from the coefficients of u_2^2 and T_2 in equation (12) it is necessary that

$$[U(dU/dx)\; T_1^{-1}\; F_3]/[F_1\; F_{1x}] = -\,[c_p\; T_1^{-1}(dT_1/dx)\; K_3]/[K_1(dK_1/dx)] = -\,c_p\; T_1^{m-2n}/n$$

is a function of η. But this expression is a function of x only and must therefore be a constant, and so $m = 2n$.

Thus the expressions for u and T are

$$u(x, y) = T_1^{n}(x)u_2(\eta), \qquad T(x, y) = T_1^{2n}(x)\,T_2(\eta),$$

where $\eta = y^sT_1^{n}(x)$.

As $y \to \infty$, $\eta \to 0$ or ∞ according as $s \lesseqgtr 0$, and u_2 and T_2 must in either case become constant. But $T(x, \infty) = T_1$ so it is necessary to have $2n = 1$. Consequently $u(x, \infty) = U(x) = CT_1^{1/2}$, where C is a constant. In order to satisfy the equation $c_pT_1 + \frac{1}{2}\,U^2 = c_pT_0$ it is necessary that $c_pT_1 + \frac{1}{2}\,C^2T_1 = c_pT_0$, so T_1, and p_1, ρ_1 and U are all constants.

If the boundary layer equations are to be reducible to three ordinary differential equations by substitutions of the type used above (which have been shown to imply solutions with the property of similarity), then conditions in the main stream must be uniform.

PART II

2.1. In a recent paper Cope and Hartree[4] (1945) have assessed the problem of solving the boundary layer equations for compressible fluids when the main stream velocity is non-uniform. Cope has examined the possibility of extending the approximate method of solution for incompressible fluids given by Pohlhausen. He has found that the use of the equation of energy, $c_p T + \frac{1}{2}u^2 = \text{const.}$, to express the density and temperature in terms of the velocity considerably complicates the method. The different extension of Pohlhausen's method used here was suggested by the treatment of the problem of forced convection in a laminar boundary layer at a flat plate, given in *Modern Developments in Fluid Dynamics*[5]. At a later stage it was found that a similar method had been given previously in a paper by Frankl[6] (1934). The accuracy of the method is measured against an exact solution in the case of a uniform main stream velocity.* The method provides a forecast of the position of separation in retarded flow.

2.2. *Momentum and Energy Equations.*—By considering the change of linear momentum in a semi-infinite section of the fluid, extending from the boundary in the direction normal to the boundary, the momentum equation is obtained. The same result is reached by integrating equation (1) with respect to y from 0 to ∞. Similarly the energy equation is obtained by integrating equation (3) in this way. Infinite integrals are replaced by finite integrals by introducing the so-called thicknesses of the velocity and temperature boundary-layers, denoted by δ and δ' respectively[5]. At points further from the boundary than δ the velocity u is indistinguishable from the main stream velocity U to the extent that $\int_\delta^\infty (U - u)dy$ can be neglected. Similarly δ' is such that $\int_{\delta'}^\infty (T - T_1)dy$ can be neglected.

It is not assumed that $\delta' = \delta$, necessarily. In the first place suppose that $\delta' < \delta$. At a position $y = \delta''$ such that $\delta' < \delta'' < \delta$, $\partial T/\partial y = 0$ and $\partial u/\partial y = 0$ since δ'' is outside the temperature boundary layer but inside the velocity boundary layer. Hence equation (3) reads, at $y = \delta''$, as

$$\rho_1 u (d/dx)(c_p T_1) = u\, dp_1/dx + \mu(\partial u/\partial y)^2 = u\rho_1 (d/dx)(c_p T_1) + \mu(\partial u/\partial y)^2,$$

using equation (7), or

$$0 = \mu(\partial u/\partial y)^2.$$

This involves a contradiction, and therefore it is necessary to have $\delta' \geqslant \delta$. So long as there is dissipation in the flow the temperature boundary layer is at least as thick as the velocity boundary layer.

In what follows, it is assumed that the gas is perfect, obeying equation (5), and that c_p and c_v are constant.

Momentum Equation

By using equation (4) and the facts that $u = U$ at $y = \infty$ and $v = 0$ at $y = 0$, it follows, after integration by parts, that

$$\int_0^\infty \rho v \frac{\partial}{\partial y}(u - U)dy = \int_0^\infty (u - U)\frac{\partial}{\partial x}(\rho u)dy.$$

Equation (1) can be written in the form

$$\rho u\, \partial u/\partial x - \rho U\, dU/dx + \rho U\, dU/dx - \rho_1 U\, dU/dx + \rho v\,(\partial/\partial y)(u - U) = (\partial/\partial y)(\mu\, \partial u/\partial y).$$

When this equation is integrated with respect to y from 0 to ∞ it gives

$$\int_0^\delta \rho\left(u\frac{\partial u}{\partial x} - U\frac{dU}{dx}\right)dy + U\frac{dU}{dx}\int_0^{\delta'}(\rho - \rho_1)dy + \int_0^\delta (u - U)\frac{\partial}{\partial x}(\rho u)dy = -\left(\mu\frac{\partial u}{\partial y}\right)_{y=0},$$

* When this was written, in 1946, no solutions other than that for a uniform main stream were available, so far as the author was aware.

since

$$\int_{\delta}^{\infty} \rho \frac{\partial}{\partial x}(u^2 - U^2)dy = 0 \quad , \quad \int_{\delta}^{\infty} (u - U) \frac{\partial}{\partial x}(\rho u)dy = 0$$

and

$$\int_{\delta'}^{\infty} (\rho - \rho_1)dy = \frac{p_1}{R}\int_{\delta'}^{\infty} \frac{T_1 - T}{TT_1}\, dy = 0 .$$

This equation can be rearranged as

$$\int_0^{\delta} \frac{\partial}{\partial x}(\rho u^2)dy - U\int_0^{\delta} \frac{\partial}{\partial x}(\rho u)dy + U\frac{dU}{dx}\left[\int_0^{\delta}(\rho - \rho_1)dy - \int_0^{\delta}\rho dy\right] = -\left(\mu\frac{\partial u}{\partial y}\right)_{y=0},$$

which is the same as

$$\frac{d}{dx}\int_0^{\delta}\rho u^2 dy - U\frac{d}{dx}\int_0^{\delta}\rho u\, dy + U\frac{dU}{dx}\left[\int_{\delta}^{\delta}\rho dy - \rho_1\delta'\right] = -\left(\mu\frac{\partial u}{\partial y}\right)_{y=0} . \qquad (15)$$

Energy Equation

Again by use of equation (4), after integration by parts,

$$\int_0^{\infty}\rho v\frac{\partial}{\partial y}[c_p(T - T_1)]dy = \int_0^{\infty} c_p(T - T_1)\frac{\partial}{\partial x}(\rho u)dy .$$

The energy equation (3) can be written as

$$\rho u\,(\partial/\partial x)\,[c_p(T - T_1)] + (\rho - \rho_1)\,u\,(d/dx)\,(c_pT_1) + \rho v\,(\partial/\partial y)\,[c_p(T - T_1)] = \mu(\partial u/\partial y)^2 + (\partial/\partial y)\,(\varkappa\,\partial T/\partial y) .$$

Integration of this equation gives

$$\int_0^{\delta'}\rho u\frac{\partial}{\partial x}[c_p(T - T_1)]dy + \int_0^{\delta'}(\rho - \rho_1)\,u\frac{d}{dx}(c_pT_1)dy + \int_0^{\delta'} c_p(T - T_1)\frac{\partial}{\partial x}(\rho u)dy = \int_0^{\delta}\mu\left(\frac{\partial u}{\partial y}\right)^2 dy - \left(\varkappa\frac{\partial T}{\partial y}\right)_{y=0} .$$

This can be written as

$$\int_0^{\delta'}\frac{\partial}{\partial x}(\rho u c_p T)dy - c_pT_1\int_0^{\delta'}\frac{\partial}{\partial x}(\rho u)dy - \rho_1\frac{d}{dx}(c_pT_1)\int_0^{\delta'} u\, dy = \int_0^{\delta}\mu\left(\frac{\partial u}{\partial y}\right)^2 dy - \left(\varkappa\frac{\partial T}{\partial y}\right)_{y=0},$$

and finally in the form

$$\frac{d}{dx}\left(\frac{\gamma}{\gamma - 1}p_1\int_0^{\delta'} u dy\right) - c_pT_1\frac{d}{dx}\int_0^{\delta'}\rho u dy + \rho_1 U\frac{dU}{dx}\int_0^{\delta'} u dy = \int_0^{\delta}\mu\left(\frac{\partial u}{\partial y}\right)^2 dy - \varkappa\left(\frac{\partial T}{\partial y}\right)_{y=0} . \qquad (16)$$

It is required to solve equations (15) and (16), together with (5), subject to the boundary conditions which must be satisfied at the boundary $y = 0$ and at the lines of transition into the conditions of the main stream, *viz.* $y = \delta$, for the velocity, and $y = \delta'$, for the density and temperature.

In formulating the boundary conditions, two alternative ways of stating the thermal condition at the solid boundary $y = 0$ present themselves. Either the temperature of the boundary is known, and then the problem is to determine the rate of heat transfer between the flowing gas and the wall, or vice versa the heat transfer is prescribed and it is required to find the temperature of the wall. In both cases, of course, the gas in contact with the wall is at rest. In passing from the boundary layer into the main stream there are no discontinuous changes in either the velocity or the temperature.

Boundary Conditions

At $y = 0$, $u = 0$, $v = 0$ and either T or $\partial T/\partial y$ is prescribed. Accordingly, from equation (1),

$$0 = -\, dp_1/dx + [(\partial/\partial y)\,(\mu\, \partial u/\partial y)]_{y=0}\,,$$

and from equation (3)

$$0 = [\mu\,(\partial u/\partial y)^2]_{y=0} + [(\partial/\partial y)\,(\varkappa\, \partial T/\partial y)]_{y=0}\,.$$

At $y = \delta$, $u = U$, $\partial^n u/\partial y^n = 0 \quad (n = 1, 2, \ldots)$.

At $y = \delta'$, $T = T_1$, $\partial^n T/\partial y^n = 0 \quad (n = 1, 2 \ldots)$.

2.3. *Application of Pohlhausen's Method.*

The solution of equations (15) and (16) is carried out by writing

$$\eta = y/\delta, \quad \chi = \delta'/\delta, \quad \theta = \rho_{y=0}/\rho_1, \quad u/U = f(\eta) \text{ and } \rho/\rho_1 = \theta + (1-\theta)\, g\,(\eta/\chi)\,,$$

and assuming suitable forms for the functions $f(\eta)$ and $g(\eta/\chi)$. Henceforward it is assumed that μ and $\varkappa$ are constants. The boundary conditions in terms of f and g are

$$f(0) = 0,\ g(0) = 0, \text{ and either } \theta \text{ or } g'(0) \text{ is given,}$$

$$f''(0) = \lambda,$$

$$[f'(0)]^2 + \frac{1-\theta}{\sigma(\gamma-1)M^2\chi^2\theta^2}\left\{\frac{2(1-\theta)}{\theta}\,[g'(0)]^2 - g''(0)\right\} = 0\,,$$

$$f(1) = 1,\ f^{(n)}(1) = 0\ (n = 1, 2 \ldots)\,,$$

$$g(1) = 1,\ g^{(n)}(1) = 0\ (n = 1, 2 \ldots)\,,$$

in which $\lambda = \delta^2 \rho_1 (dU/dx)/\mu$, M is the Mach number of the flow in the main stream and $\sigma\ (= \mu c_p/\varkappa)$ is the Prandtl number.

For incompressible fluids, Pohlhausen took f as a polynomial in η, of either first, second, third or fourth degree, and Lamb suggested the form $f = \sin \tfrac{1}{2}\pi\eta$. For the present consideration of compressible fluids, some of these forms for f are also used and in each case g has the same form as the corresponding f. Of course, the number of boundary conditions which can be satisfied is determined by the degree of the polynomials adopted.

The particular case considered is that of the flat plate thermometer, which records the temperature of the wall when thermal equilibrium has been attained. In this case there is no heat transfer and accordingly $g'(0) = 0$.

2.3.1. *Main stream velocity uniform.*—When conditions in the main stream are uniform, θ and χ are constants and equations (15) and (16) take the following simpler forms

$$\rho_1 U^2 \left\{\int_0^1 [\theta + (1-\theta)g](f^2 - f)\,d\eta\right\} \frac{d\delta}{dx} = -\,\frac{\mu U}{\delta}\, f'(0)\,, \qquad \ldots \quad \ldots \quad (17)$$

$$\frac{\gamma}{\gamma - 1} p_1 U\left\{(1 - \theta)\left[\int_0^1 f(1 - g)d\eta + (\chi - 1) - \int_1^\chi g\, d\eta\right]\right\} \frac{d\delta}{dx} = \frac{\mu U^2}{\delta} \int_0^1 f'^2 d\eta . \quad \cdots \quad (18)$$

If

$$I = \int_0^1 (f - f^2)\, d\eta, \quad J = \int_0^1 (1 - g)(f - f^2)\, d\eta ,$$

$$K = \int_0^1 f(1 - g)\, d\eta + (\chi - 1) - \int_1^\chi g\, d\eta \text{ and } L = \int_0^1 f'^2\, d\eta ,$$

then

$$\gamma p_1 M^2[(1 - \theta)J - I] \frac{d}{dx} (\tfrac{1}{2}\delta^2) = - \mu U f'(0)$$

and

$$\gamma p_1 M^2[(1 - \theta)K] \frac{d}{dx} (\tfrac{1}{2}\delta^2) = (\gamma - 1)M^2 \mu U L ,$$

so

$$(1 - \theta)\left[J + \frac{f'(0)}{(\gamma - 1)M^2} \frac{K}{L}\right] = I . \quad \cdots \quad (19)$$

This is an equation between the two unknowns θ and χ. A second equation for θ and χ is provided by the boundary condition

$$\frac{1 - \theta}{\sigma(\gamma - 1)M^2\chi^2\theta^2} g''(0) = [f'(0)]^2 . \quad \cdots \quad (20)$$

It should be noted in reference to equation (20) that the expression for g is chosen so that $g''(0) \neq 0$. Otherwise the implication is that $f'(0) = 0$ and $(\partial u/\partial y)_{y=0} = 0$ everywhere along the plate.

The values of θ and χ are obtained from equations (19) and (20) most easily in practice by obtaining θ for a series of values of χ from equation (19), and then using equation (20) to give the corresponding values of σ. It is found that values of χ between 1 and 2 give most of the required range of values of σ. When the values of θ and χ have been obtained, the skin frictional drag on the plate and the temperature of the plate are easily derived.

Since

$$\gamma p_1 M^2[I - (1 - \theta)J]\, (d/dx)(\tfrac{1}{2}\delta^2) = \mu U f'(0) ,$$

it follows that

$$\delta^2 = \frac{2\mu f'(0) x}{\rho_1 U[I - (1 - \theta) J]} ,$$

and

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = \left\{\frac{\rho_1 U^3[I - (1 - \theta)J] f'(0)}{2\mu x}\right\}^{1/2} .$$

The drag coefficient (for both sides of the plate) is given by

$$C_D = \frac{2\int_0^x \mu(\partial u/\partial y)_{y=0}\, dx}{\rho_1 U^2 x} = 4R_1^{1/2} C_U ,$$

where R_1 is the Reynolds number of the main stream flow, and

$$C_U = \{\tfrac{1}{2} f'(0)\, [I - (1 - \theta) J]\}^{1/2} .$$

The temperature of the plate ($T_{y=0}$) can be expressed in terms of the actual temperature (T_1) and the stagnation temperature (T_0) of the main stream flow by means of the coefficient

$$C_T = \frac{T_{y=0} - T_1}{T_0 - T_1} = \frac{\theta^{-1} - 1}{U^2/(2c_p T_1)} = \frac{2(1 - \theta)}{\theta(\gamma - 1)M^2} = \frac{2\sigma\theta\chi^2[f'(0)]^2}{g''(0)} .$$

When the main stream flow is very slow, $M \to 0$ and $\theta \to 1$, so from equation (19)

$$\lim_{M\to 0} (1-\theta)/M^2 = (\gamma-1)IL/[Kf'(0)]$$

and from equation (20)

$$\lim_{M\to 0} (1-\theta)/M^2 = \sigma(\gamma-1)\chi^2[f'(0)]^2/g''(0) .$$

χ is then determined from the equation connecting these two expressions for the same limit, *viz.*

$$IL\, g''(0) - \sigma K \chi^2 [f'(0)]^3 = 0.$$

In this case

$$C_D = 4R_1^{-1/2}\,[\tfrac{1}{2}If'(0)]^{1/2} ,$$

and

$$\lim_{M\to 0} (T_{y=0} - T_1)/(T_0 - T_1) = 2\sigma\chi^2[f'(0)]^2/g''(0) .$$

Forms for f and g

1. *Quadratic*

It is possible in this case to satisfy the boundary conditions

$$f(0) = 0,\ f(1) = 1,\ f'(1) = 0$$

and

$$g(0) = 0,\ g(1) = 1,\ g'(0) = 0 .$$

Hence

$$f = 2\eta - \eta^2, \quad g = (\eta/\chi)^2 ,$$

so

$$I = \frac{2}{15}, \quad J = \frac{2}{15} - \frac{1}{42}\frac{1}{\chi^2}, \quad K = \frac{2}{3}\chi - \frac{1}{3} + \frac{1}{30}\frac{1}{\chi^2} \text{ and } L = \frac{4}{3} .$$

2. *Cubic*

In addition to the boundary conditions given in 1, there are

$$f''(0) = 0 \text{ and } g'(1) = 0 .$$

In this case

$$f = \tfrac{3}{2}\eta - \tfrac{1}{2}\eta^3, \quad g = 3(\eta/\chi)^2 - 2(\eta/\chi)^3 ,$$

from which

$$I = \frac{39}{280}, \quad J = \frac{39}{280} - \frac{71}{840}\frac{1}{\chi^2} + \frac{9}{280}\frac{1}{\chi^3}, \quad K = \tfrac{1}{2}\chi - \frac{3}{8} + \frac{1}{8}\frac{1}{\chi^2} - \frac{3}{70}\frac{1}{\chi^3} \text{ and } L = \frac{6}{5} .$$

3. *Quartic*

The boundary conditions are

$$f(0) = 0, f''(0) = 0, f(1) = 1, f'(1) = 0, f''(1) = 0$$

and

$$g(0) = 0, g'(0) = 0, g(1) = 1, g'(1) = 0, g''(1) = 0 .$$

These give

$$f = 2\eta - 2\eta^3 + \eta^4, \quad g = 6(\eta/\chi)^2 - 8\,(\eta/\chi)^3 + 3(\eta/\chi)^4 ,$$

and so

$$I = \frac{37}{315}, \quad J = \frac{37}{315} - \frac{118}{1155}\frac{1}{\chi^2} + \frac{47}{693}\frac{1}{\chi^3} - \frac{281}{20020}\frac{1}{\chi^4} ,$$

$$K = \frac{2}{5}\chi - \frac{3}{10} + \frac{1}{7}\frac{1}{\chi^2} - \frac{3}{35}\frac{1}{\chi^3} + \frac{1}{60}\frac{1}{\chi^4} \text{ and } L = \frac{52}{35} .$$

4. *Series of odd powers for f, even powers for g*

With the same boundary conditions as in the case of the quartic,

$$f = \frac{1}{8}(15\eta - 10\eta^3 + 3\eta^5), \qquad g = 3(\eta/\chi)^2 - 3(\eta/\chi)^4 + (\eta/\chi)^6,$$

$$I = \frac{445}{3696}, \quad J = \frac{445}{3696} - \frac{10585}{192192}\frac{1}{\chi^2} + \frac{2501}{160160}\frac{1}{\chi^4} - \frac{311}{155584}\frac{1}{\chi^6},$$

$$K = \frac{16}{35}\chi - \frac{5}{16} + \frac{5}{64}\frac{1}{\chi^2} - \frac{3}{160}\frac{1}{\chi^4} + \frac{1}{448}\frac{1}{\chi^6} \text{ and } L = \frac{45}{64}.$$

5. $f = \sin\frac{1}{2}\pi\eta$, $g = \sin^2\frac{1}{2}\pi\eta/\chi$

$$I = \frac{2}{\pi} - \frac{1}{2},$$

$$J = \frac{1}{\pi} - \frac{1}{4} + \frac{1 + \sin\pi/\chi}{2\pi(1 + 2/\chi)} + \frac{1 - \sin\pi/\chi}{2\pi(1 - 2/\chi)} - \frac{\chi\sin\pi/\chi}{4\pi} - \frac{\sin\pi/\chi}{8\pi(1 + 1/\chi)} + \frac{\sin\pi/\chi}{8\pi(1 - 1/\chi)},$$

$$K = \frac{1}{\pi} + \frac{\chi - 1}{2} - \frac{\chi}{2\pi}\sin\frac{\pi}{\chi} + \frac{1 + \sin\pi/\chi}{2\pi(1 + 2/\chi)} + \frac{1 - \sin\pi/\chi}{2\pi(1 - 2/\chi)} \text{ and } L = \frac{\pi^2}{8}.$$

Reference to Brainerd and Emmons's exact solution[4] for the flow along a flat-plate thermometer shows which of the above approximations is the most satisfactory. In their paper, the drag coefficient and the coefficient giving the temperature of the plate are tabulated for $\sigma = 0$, $0 \cdot 25$, $0 \cdot 733$, $1 \cdot 0$ and $1 \cdot 20$ at Mach numbers of 0, 1, $1 \cdot 5$, 2 and $\sqrt{10}$. For the purpose of making the comparison the present approximations have been worked out only for Mach numbers 0 and $\sqrt{10}$. The results are given in Table 1. In some cases, the approximation does not cover the whole range of interest without violating the condition that $\chi \nless 1$. However, if an approximation is adequate at $M = 0$ then it is adequate to cover the range at all Mach numbers.

Inspection of Table 1 shows that the approximation $f = \sin\frac{1}{2}\pi\eta$ and $g = \sin^2\frac{1}{2}\pi\eta/\chi$ gives the best results, although it does not quite cover the range of Prandtl numbers. In all the approximations the drag coefficient is given much more accurately than the temperature coefficient, which is generally underestimated. Even with the trigonometric form for f and g the temperature coefficient is as much as 26 per cent too low at $M = \sqrt{10}$ for a Prandtl number of $1 \cdot 20$. The discrepancy becomes smaller as the Mach number is reduced, and it is also less for smaller Prandtl numbers.

For atmospheric air, the results of Frankl's generalisation of Pohlhausen's method are quoted in Table 2. Frankl assumes that $\chi = 1$, takes the quartic forms for f and g, and obtains more accurate results than the present method gives.

Although the choice of f and g is to some extent a question of 'trial and error' with the exact solutions available as a guide, one may say that it is the form of g rather than f which is critical. In the case of an incompressible fluid, the drag coefficient depends on the integral $\int_0^1 f(1 - f)\,d\eta$ and $f'(0)$, since $C_D = 4R_1^{-1/2}[\frac{1}{2}If'(0)]^{1/2}$. The integral does not vary rapidly with f, and any form of f for which $f'(0) \doteqdot 3/2$ will give a good approximation to the drag coefficient. In the compressible case, it seems to be $g''(0)$, occurring in equation (20), which is particularly influential in determining θ and the density and temperature at the plate. The simple polynomials tried have given values of $g''(0)$ rather too small.

2.3.2. *Main stream velocity non-uniform*.—In this section it is no longer assumed that the main stream flow is uniform. In the case when there is no pressure gradient in the main stream, apart from the trigonometric form for f and g which does not cover the whole range of interest, the cubic form gives the best results. However it is not suitable for use when there is a pressure gradient.* Accordingly, the quartic form is used.

* See the note at the end of the paper.

The boundary conditions which can be satisfied are:

$$f(0) = 0, f''(0) = -\lambda, f(1) = 1, f'(1) = 0, f''(1) = 0$$

and

$$g(0) = 0, \quad (1-\theta)g''(0)/[\sigma(\gamma-1)M^2\chi^2\theta^2] = [f'(0)]^2, \quad g(1) = 1, g'(1) = 0, g''(1) = 0.$$

Then

$$f = 2\eta - 2\eta^3 + \eta^4 + \tfrac{1}{6}\lambda\eta(1-\eta)^3, \quad g = 6(\eta/\chi)^2 - 8(\eta/\chi)^3 + 3(\eta/\chi)^4,$$

where

$$12(1-\theta)/[\sigma(\gamma-1)M^2\chi^2\theta^2] = (2 + \tfrac{1}{6}\lambda)^2. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (21)$$

In this case,

$$\int_0^\delta \rho u^2 dy = \rho_1 U^2 \delta \left\{\theta\left(\frac{367}{630} + \lambda\frac{71}{7560} + \lambda^2\frac{1}{9072}\right) + (1-\theta)\left[\left(\frac{2027}{1155}\frac{1}{\chi^2} - \frac{914}{495}\frac{1}{\chi^3} + \frac{8548}{15015}\frac{1}{\chi^4}\right)\right.\right.$$
$$\left.\left. + \lambda\left(\frac{17}{1540}\frac{1}{\chi^2} - \frac{82}{10395}\frac{1}{\chi^3} + \frac{31}{18018}\frac{1}{\chi^4}\right) + \lambda^2\left(\frac{1}{13860}\frac{1}{\chi^2} - \frac{1}{24948}\frac{1}{\chi^3} + \frac{1}{144144}\frac{1}{\chi^4}\right)\right]\right\}$$
$$= \rho_1 U^2 \delta F_1, \text{ say,}$$

$$\int_0^\delta \rho u dy = \rho_1 U\delta\left\{\theta\left(\frac{7}{10} + \lambda\frac{1}{120}\right)\right.$$
$$\left. + (1-\theta)\left[\left(\frac{13}{7}\frac{1}{\chi^2} - \frac{67}{35}\frac{1}{\chi^3} + \frac{7}{12}\frac{1}{\chi^4}\right) + \lambda\left(\frac{1}{140}\frac{1}{\chi^2} - \frac{1}{210}\frac{1}{\chi^3} + \frac{1}{1008}\frac{1}{\chi^4}\right)\right]\right\}$$
$$= \rho_1 U\delta F_2,$$

$$\int_\delta^{\delta'} \rho dy - \rho_1\delta' = \rho_1\delta\left[-1 + (1-\theta)\left(-\frac{\chi}{2} + 1 - \frac{1}{\chi^2} + \frac{1}{2}\frac{1}{\chi^3}\right)\right] = \rho_1\delta F_3,$$

$$\int_0^{\delta'} u dy = U\delta\left[\chi - \frac{3}{10} + \lambda\frac{1}{120}\right] = U\delta F_4,$$

$$\int_0^{\delta'} \rho u dy = \rho_1 U\delta\left\{\theta\left(\chi - \frac{3}{10} + \lambda\frac{1}{120}\right)\right.$$
$$\left. + (1-\theta)\left[\left(\frac{3}{5}\chi - \frac{1}{7}\frac{1}{\chi^2} + \frac{3}{35}\frac{1}{\chi^3} - \frac{1}{60}\frac{1}{\chi^4}\right) + \lambda\left(\frac{1}{140}\frac{1}{\chi^2} - \frac{1}{210}\frac{1}{\chi^3} + \frac{1}{1008}\frac{1}{\chi^4}\right)\right]\right\}$$
$$= \rho_1 U\delta F_5,$$

$$\int_0^\delta \left(\frac{\partial u}{\partial y}\right)^2 dy = \frac{U^2}{\delta}\left[\frac{52}{35} + \lambda\frac{4}{105} + \lambda^2\frac{1}{420}\right] = \frac{U^2}{\delta}F_6.$$

The two differential equations (15) and (16) are

$$(d/dx)(\rho_1 U^2\delta F_1) - U(d/dx)(\rho_1 U\delta F_2) + U(dU/dx)\,\rho_1\delta F_3 = -\mu U\,\delta^{-1}(2 + \tfrac{1}{6}\lambda),$$
$$(d/dx)[\gamma(\gamma-1)^{-1}p_1 U\delta F_4] - [\gamma(\gamma-1)^{-1}p_1\rho_1^{-1}](d/dx)(\rho_1 U\delta F_5) + \rho_1 U^2(dU/dx)\delta F_4 = \mu U^2\delta^{-1}F_6,$$

or

$$\lambda(d/dx)\,(F_1 - F_2) + \tfrac{1}{2}(F_1 - F_2)\,d\lambda/dx = -\,[2 + \tfrac{1}{6}\lambda + \lambda(2F_1 - F_2 + F_3)]\,U^{-1}(dU/dx)$$
$$-\,\tfrac{1}{2}\lambda(F_1 - F_2)\,\rho_1^{-1}(d\rho_1/dx) + \tfrac{1}{2}\lambda(F_1 - F_2)\,(dU/dx)^{-1}\,(d^2U/dx^2)\,. \qquad (22)$$

and

$$\lambda(d/dx)\,(F_4 - F_5) + \tfrac{1}{2}(F_4 - F_5)\,d\lambda/dx = -\lambda(F_4 - F_5)\,U^{-1}(dU/dx) + \tfrac{1}{2}\lambda(F_4 + F_5)\,\rho_1^{-1}(d\rho_1/dx)$$
$$+\,[F_6 + (\gamma - 1)^{-1}\,\lambda F_4]\,\gamma^{-1}(\gamma - 1)\,p_1^{-1}\,\rho_1 U(dU/dx) + \tfrac{1}{2}\lambda(F_4 - F_5)\,(dU/dx)^{-1}\,(d^2U/dx^2)\,. \qquad (23)$$

Although the functions F_1, F_2, F_3 and F_4 involve the three dependent variables λ, θ and χ, the equations can be written as first-order differential equations in the two dependent variables λ and θ by introducing χ as a function of θ and λ according to equation (21).

Before proceeding to the solution of these equations for the flow past a flat plate, it is worth noticing a similarity with the incompressible case which appears in the application to the flow round a bluff-nosed body. The integration of the equations is started at the nose where $U = 0$, $dU/dx \neq 0$, and is continued until the point of separation is reached at $\lambda = -12$. Equation (21) shows that $\theta = 1$ at the nose and, further, that $(1 - \theta) \propto U^2$ in the neighbourhood of the nose. The initial values of λ and χ can be obtained from equations (22) and (23). For equation (22) cannot be satisfied when $U = 0$ unless, using the fact that $\theta = 1$,

$$2 + \frac{\lambda}{6} + \lambda(2F_1 - F_2 + F_3) = 2 - \frac{116}{315}\,\lambda + \frac{79}{7560}\,\lambda^2 + \frac{1}{4536}\,\lambda^3 = 0\,.$$

This is the equation for the value of λ at the forward stagnation point in the flow of an incompressible fluid; the appropriate root is $\lambda = 7{\cdot}052$. Also, equation (23) can be written as $C_1U + C_2U^2 = 0$ where C_1 and C_2 are certain coefficients. When $U = 0$, it is necessary that $C_1 = 0$ and this gives, with $\theta = 1$,

$$\frac{\sigma\chi^2(2 + \lambda/6)^2}{4}\,\lambda\,\left[\left(\frac{2}{5}\,\chi - \frac{3}{10} + \frac{1}{7}\,\frac{1}{\chi^2} - \frac{3}{35}\,\frac{1}{\chi^3} + \frac{1}{60}\,\frac{1}{\chi^4}\right)\right.$$
$$\left. + \lambda\left(\frac{1}{120} - \frac{1}{140}\,\frac{1}{\chi^2} + \frac{1}{210}\,\frac{1}{\chi^3} - \frac{1}{1008}\,\frac{1}{\chi^4}\right)\right] = \frac{52}{35} + \frac{4}{105}\,\lambda + \frac{1}{420}\,\lambda^2\,.$$

This gives, when the value $\lambda = 7{\cdot}052$ is used, the following equation for the initial value of χ

$$\sigma(0{\cdot}4\chi^3 - 0{\cdot}24123\chi^2 + 0{\cdot}09249 - 0{\cdot}05213\chi^{-1} + 0{\cdot}009671\chi^{-2} = 0{\cdot}10535\,.$$

The range of application of the method in this case is limited to Prandtl numbers less than $0{\cdot}5$. The equation has the root $\chi = 1$ for $\sigma = 0{\cdot}505$, but for all higher Prandtl numbers the roots are less than 1.

2.3.3. *Flat plate thermometer. Uniform pressure gradient.*—Hartree[4] has examined the possibility of integrating the boundary layer equations on the differential analyser in the case when the pressure in the main stream increases at a uniform rate. This example will be considered here.

Let the suffix i refer to conditions at the leading edge of the plate which is taken as the origin of co-ordinates. If $\beta(> 0)$ is the uniform pressure gradient, the pressure in the plane at distance x downstream from the leading edge is given by $p_1 = p_{1i} + \beta x$. It is convenient to use a new variable ξ, defined by $\xi = 1 + \beta x/p_{1i}$, and conditions in the main stream can then be expressed in terms of ξ as follows

$$p_1/p_{1i} = \xi\,, \quad \rho_1/\rho_{1i} = \xi^{1/\gamma}\,, \quad U^2 = U_i^2 + 2c_pT_{1i}\,(1 - \xi^{(\gamma-1)/\gamma})$$

and

$$M^2 = [M_i^2 + 2/(\gamma - 1)]\,\xi^{-(\gamma-1)/\gamma} - 2/(\gamma - 1)\,.$$

If differentiation with respect to the variable ξ is denoted by dashes, then

$$\frac{U'}{U} = -\frac{(\gamma-1)\,\xi^{-1/\gamma}}{\gamma[(\gamma-1)M_i^2 + 2(1-\xi^{(\gamma-1)/\gamma})]}\,,\quad \frac{U''}{U'} = -\frac{1}{\gamma\xi} - \frac{U'}{U}\,.$$

Equations (22) and (23), become in terms of the independent variable ξ,

$$\lambda(d/d\xi)(F_1 - F_2) + \tfrac{1}{2}(F_1 - F_2)\,d\lambda/d\xi = -(U'/U)[2 + \tfrac{1}{6}\lambda + \tfrac{1}{2}\lambda(5F_1 - 3F_2 + 2F_3)] - [\lambda(F_1 - F_2)]/(\gamma\xi),$$

$$\lambda(d/d\xi)(F_4 - F_5) + \tfrac{1}{2}(F_4 - F_5)\,d\lambda/d\xi = -(U'/U)[\tfrac{3}{2}\lambda(F_4 - F_5)] - [(\gamma-1)F_6 + \lambda(F_4 - F_5)]/(\gamma\xi)\,.$$

From these equations θ and λ can be found in terms of ξ. The solution will refer the boundary layer along the plate fıom the leading edge, where $\delta = 0$, $\lambda = 0$, up to the point of separation where $\lambda = -12$. Since the range of values of λ is known previously, and especially since it is the same range whatever the Mach number and Prandtl number of the main stream flow, it is convenient now to regard λ as the independent variable and θ and ξ as dependent variables. The differential equations are then

$$a_1\,d\theta/d\lambda + b_1\,d\xi/d\lambda + c_1 = 0\,, \qquad (24)$$

$$a_2\,d\theta/d\lambda + b_2\,d\xi/d\lambda + c_2 = 0\,, \qquad (25)$$

where

$$a_1 = \lambda(\partial/\partial\theta)(F_1 - F_2)\,,$$

$$b_1 = \lambda(\partial/\partial M)(F_1 - F_2)\,dM/d\xi + (U'/U)\,[2 + \tfrac{1}{6}\lambda + \tfrac{1}{2}\lambda(5F_1 - 3F_2 + 2F_3)] + [\lambda(F_1 - F_2)]/(\gamma\xi),$$

$$c_1 = \lambda(\partial/\partial\lambda)(F_1 - F_2) + \tfrac{1}{2}(F_1 - F_2)\,,$$

$$a_2 = \lambda(\partial/\partial\theta)(F_4 - F_5)\,,$$

$$b_2 = \lambda(\partial/\partial M)(F_4 - F_5)\,dM/d\xi + (U'/U)\,[\tfrac{3}{2}\lambda(F_4 - F_5)] + [(\gamma-1)F_6 + \lambda(F_4 - F_5)]/(\gamma\xi)\,,$$

$$c_2 = \lambda(\partial/\partial\lambda)(F_4 - F_5) + \tfrac{1}{2}(F_4 - F_5)\,.$$

With

$$Z = \left[\frac{\sigma(\gamma-1)}{432}\right]^{1/2} \frac{M\theta}{(1-\theta)^{1/2}}\,,$$

the quantities a_1, b_1, etc. may be written as

$$a_1 = a_{10} + \sum_{n=1}^{3} a_{1n}Z^{n+1} + \theta^{-1}\sum_{n=4}^{6} a_{1n}Z^{n-2}\,.$$

$$b_1 = (U'/U)\,[b_{10} + (1-\theta)\,(b_{11}Z^{-1} + b_{12} + \sum_{n=3}^{5} b_{1n}Z^{n-2})] + (\gamma\xi)^{-1}\,[\beta_{10} + (1-\theta)(\beta_{11} + \sum_{n=2}^{4} \beta_{1n}Z^n)]\,,$$

$$c_1 = c_{10} + (1-\theta)\,(c_{11} + \sum_{n=2}^{4} c_{1n}Z^n)\,,$$

$$a_2 = a_{20}Z^{-1} + a_{21} + \sum_{n=2}^{4} a_{2n}Z^n + \theta^{-1}(a_{25}Z^{-1} + \sum_{n=6}^{8} a_{2n}Z^{n-4})\,,$$

$$b_2 = (U'/U)\,[(1-\theta)\,(b_{20}Z^{-1} + b_{21} + \sum_{n=2}^{4} b_{2n}Z^n)] + (\gamma\xi)^{-1}[\beta_{20} + (1-\theta)\,(\beta_{21}Z^{-1} + \beta_{22} + \sum_{n=3}^{5} \beta_{2n}Z^{n-1})]\,,$$

$$c_2 = (1-\theta)\,(c_{20}Z^{-1} + c_{21} + \sum_{n=2}^{4} c_{2n}Z^n)\,,$$

where the coefficients a_{rs}, etc. are functions of λ only, and are given in Table 3.

When $\lambda = 0$, $\xi = 1$ and $a_1 = a_2 = 0$ so equations (24) and (25) do not contain $d\theta/d\lambda$, and they can be solved to give the initial values of $d\xi/d\lambda$ and θ. By differentiating these equations with respect to λ and substituting the values just determined, one obtains two equations which can be solved to give the initial values of $d\theta/d\lambda$ and $d^2\xi/d\lambda^2$.

On the other hand at $\lambda = -12$, $f'(0) = 0$ and therefore $g''(0) = 0$, so the density ρ satisfies the boundary conditions

$$\partial\rho/\partial y = \partial^2\rho/\partial y^2 = 0 \text{ at } y = 0\,,$$

and

$$\rho = \rho_1,\ \partial\rho/\partial y = \partial^2\rho/\partial y^2 = 0 \text{ at } y = \delta'\,.$$

Since ρ has been taken as a quartic in (η/χ) the coefficients of all four powers of (n/χ) are zero, and $\rho = \rho_1$ for all values of y at this position along the plate. It follows that $\theta = 1$, and the implication is that $\chi = 0$ at $\lambda = -12$. A solution which is continuous cannot approach the value $\chi = 0$ as $\lambda \to -12$, since the equations were formulated on the assumption that $\chi \geqslant 1$. It should be emphasised, however, that this mathematical difficulty arises from the restrictions involved by assuming that ρ is a quartic in η. Assuming that $1 - \theta \sim (12 + \lambda)^n$ in the neighbourhood of $\lambda = -12$, it is necessary that $n \leqslant 1/2$, otherwise $\chi \to 0$ as $\lambda \to -12$. Then $d\theta/d\lambda \to \infty$ as $\lambda \to -12$ and from equation (24) it follows that $d\xi/d\lambda \to \infty$. The difficulty of completing the numerical integration as far as $\lambda = -12$ has been skirted, in this approximate treatment of the boundary layer equations, by extrapolating for the value of ξ over the interval from $\lambda = -11$ to $\lambda = -12$.

The integration has been carried out in three cases of air flow ($\sigma = 0{\cdot}715$) past a flat plate, when the incident stream has specific speeds of $0{\cdot}1$, $1{\cdot}0$ and $\sqrt{10}$. In the first case separation occurs when the pressure has risen by one-fifth of one per cent of its value at the front edge; in the second case the corresponding fractional increase is twenty per cent and at the highest speed the pressure rises to nine times its initial value before separation occurs. The details of the integration are given in Table 4.

The behaviour at a specific speed of $0{\cdot}1$ should be much the same as in the incompressible case. It is, therefore, of interest to compare the present result with the estimation given by Howarth[7] (1938) of the point of separation for an incompressible fluid whose main stream *velocity* is decreasing linearly. Howarth uses a variable which in the present notation is given by $(U_i - U)/U_i$ and finds that separation occurs at the place where this variable has the value $0{\cdot}120$. At the point of separation estimated here, $\xi = 1{\cdot}0019$ and $(U_i - U)/U_i = 0{\cdot}149$ which is close to the value $0{\cdot}156$ obtained by applying Pohlhausen's method to Howarth's case.

From the results for the other two specific speeds, it appears that, with a given linearly increasing *pressure*, separation is delayed by increasing the Mach number of the incident stream. For the same pressure gradient (β) and the same pressure (p_{1i}) of the incident stream the distances down the plate of the points of separation are in the ratio 1: 118: 4216 for Mach numbers of $0{\cdot}1$, $1{\cdot}0$ and $\sqrt{10}$ respectively.

Note added in revision for R. & M.

Howarth[8] (1948) has shown that the choice of a uniform pressure gradient that is *independent of the Mach number* of the incident stream is not a convenient basis for presenting the effect of Mach number on separation. To obtain an alternative basis, write $\beta = \rho_{1i}U_i^2/l$, where l is a length independent of x; then, $p_1 = p_{1i}(1 + \gamma M_i^2 x/l)$, and the values of x/l at separation are in the ratios $1:1{\cdot}18:4{\cdot}22$ for $M_i = 0{\cdot}1$, 1 and $\sqrt{10}$. With the new assumption that l is independent of M_i, these ratios are the ratios of the distances of the points of separation from the leading edge of the plate.

Howarth considers a given main stream velocity, of the form $U = U_i(1 - x/l')$, where l' is a length independent of x. He finds that the values of x/l' at separation are in the ratios $1:0{\cdot}95:0{\cdot}69$ for $M_i = 0$, 1 and $\sqrt{10}$, and shows that these ratios are qualitatively consistent with the previous ones, although at first sight they appear to be contradictory.

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Note on the choice of polynomial for f and g

The choice between the cubic and quartic forms for f and g when considering the flow with a linear pressure gradient past a boundary is easily made on the following considerations.

For incompressible fluids, when the quartic form is used

$$u/U = (2\eta - 2\eta^3 + \eta^4) - \tfrac{1}{6}\lambda\eta(1-\eta)^3$$

and

$$\frac{d\lambda}{d\xi} = \frac{1}{U}\frac{dU}{d\xi}\left[\frac{7257\cdot6 - 1549\cdot44\lambda + 39\cdot84\lambda^2 + \lambda^3}{213\cdot12 - 5\cdot76\lambda - \lambda^2}\right], \quad \ldots \ldots \ldots \quad \text{(A)}$$

where $\xi = p_1/p_{1i}$ and p_{1i} is the pressure at the front edge of the plate.

When the cubic form is used

$$u/U = (\tfrac{3}{2}\eta - \tfrac{1}{2}\eta^3) + \tfrac{1}{4}\lambda\eta(1-\eta)^2$$

and

$$\frac{d\lambda}{d\xi} = \frac{1}{U}\frac{dU}{d\xi}\left[\frac{1008 + 354\lambda + 17\lambda^2 + \lambda^3}{46\cdot8 - 1\cdot8\lambda - \lambda^2}\right]. \quad \ldots \ldots \ldots \ldots \ldots \quad \text{(B)}$$

Also $U\,dU/d\xi = -p_{1i}/\rho_1$ so that $dU/d\xi$ is negative.

Equation (A) must be integrated from the front edge of the plate where $\lambda = 0$ to the point of separation where $\lambda = -12$. There is no difficulty in this, since the polynomials in λ in the numerator and in the denominator are both positive in the range $0 \geqslant \lambda \geqslant -12$. On the other hand, equation (B) must be integrated from $\lambda = 0$ to $\lambda = -6$, and although the denominator remains positive in this range the polynomial in the numerator changes sign from positive to negative in passing from $\lambda = -3$ to $\lambda = -4$, and the value $\lambda = -6$ will not be reached. This difficulty must also arise in the case of compressible fluids, at least for low enough values of the main stream velocity.

TABLE 1

Comparison of Drag and Temperature Coefficients with the Exact Values

Approximation I Quadratic
II Cubic
III Quartic
IV Odd powers
V $f = \sin\frac{1}{2}\pi\eta$

$M = \sqrt{10}$

Approximation		I	II	III	IV	V	Exact solution
σ							
1·2	C_U	*	0·234	0·255	0·262	0·236	0·218
	C_T	*	0·749	0·711	0·62	0·82	1·105
1·0	C_U	*	0·236	0·257	0·266	0·238	0·222
	C_T	*	0·693	0·658	0·57	0·76	1·000
0·733	C_U	*	0·240	0·260	0·270	0·241	0·228
	C_T	*	0·608	0·578	0·49	0·67	0·838
0·25	C_U	0·276	0·254	0·274	0·281	0·254	0·248
	C_T	0·52	0·393	0·373	0·32	0·43	0·247

$M = 0$

Approximation		I	II	III	IV	V	Exact solution
σ							
1·2	C_U	0·351	0·323	0·343	0·336	0·328	0·331
	C_T	*	0·988	0·904	*	*	—
1·0	C_U	0·351	0·323	0·343	0·336	0·328	0·331
	C_T	*	0·907	0·831	*	1·00	1·000
0·733	C_U	0·351	0·323	0·343	0·336	0·328	0·331
	C_T	*	0·783	0·720	*	0·87	0·852
0·25	C_U	0·351	0·323	0·343	0·336	0·328	0·331
	C_T	*	0·480	0·437	0·34	0·53	—

* Denotes that the result is outside the range $\chi \geqslant 1$.

TABLE 2

Coefficients for Drag and Temperature (see Figs. 1, 2) using the Quartic Forms for f and g

M / σ		0	0·5	1·0	1·5	2·0	$\sqrt{10}$
1·2	C_U	0·343	0·337	0·325	0·310	0·292	0·255
	C_T	0·904	0·890	0·865	0·830	0·792	0·711
1·0	C_U	0·343	0·337	0·327	0·311	0·294	0·257
	C_T	0·831	0·819	0·797	0·765	0·732	0·658
0·8	C_U	0·343	0·338	0·328	0·312	0·296	0·259
	C_T	0·749	0·740	0·721	0·694	0·665	0·599
0·6	C_U	0·343	0·338	0·328	0·314	0·298	0·262
	C_T	0·657	0·650	0·635	0·613	0·589	0·532
0·4	C_U	0·343	0·339	0·330	0·317	0·301	0·267
	C_T	0·549	0·543	0·531	0·515	0·496	0·451
0.2	C_U	0·343	0·339	0·332	0·321	0·308	0·276
	C_T	0·406	0·402	0·395	0·385	0·373	0·343
0	C_U	0·343	0·343	0·343	0·343	0·343	0·343
	C_T	0	0	0	0	0	0

$\sigma = 0{\cdot}733$

M		Exact solution	Frankl's solution	Present solution
0·5	C_U	0·326	0·333	0·338
	C_T	0·864	0·78	0·71
1·0	C_U	0·312	0·326	0·328
	C_T	0·855	0·76	0·69
1·5	C_U	0·294	0·310	0·313
	C_T	0·850	0·76	0·67
2·0	C_U	0·274	0·290	0·297
	C_T	0·852	0·77	0·64
$\sqrt{10}$	C_U	0·228	0·245	0·260
	C_T	0·838	0·82	0·58

TABLE 3

λ	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}
0	0	0	0	0	0	0	0
−1	0·1184	−25·65	235·9	−648	25·65	−283·1	864
−2	0·2383	−43·87	369·6	−928	43·87	−443·5	1237
−3	0·3589	−55·02	420·1	−954	55·02	−504·1	1272·
−4	0·4797	−59·70	408·0	−828	59·70	−489·6	1104
−5	0·6000	−58·72	353·5	−630	58·72	−424·2	841
−6	0·7191	−53·11	275·8	−424	53·11	−331·0	565
−7	0·8363	−44·06	191·9	−247	44·06	−230·3	329
−8	0·9510	−32·95	115·5	−119	32·95	−138·6	159
−9	1·0625	−21·28	56·32	− 43·8	21·28	− 67·59	58·4
−10	1·1702	−10·71	19·01	− 9·9	10·71	− 22·82	13·2
−11	1·2734	− 2·998	2·68	− 0·70	2·998	− 3·21	0·93
−12	1·3715	0	0	0	0	0	0

λ	b_{10}	b_{11}	b_{12}	b_{13}	b_{14}	$\frac{1}{10}b_{15}$
0	2·0000	0	0	0	0	0
−1	2·4377	0·03636	−0·6043	75·86	− 640	166
−2	2·8957	0·08000	−1·229	130·1	−1004	238
−3	3·3723	0·1300	−1·872	163·7	−1143	245
−4	3·8660	0·2000	−2·533	178·2	−1112	213
−5	4·3750	0·2857	−3·208	175·9	− 966	163
−6	4·8976	0·4000	−3·898	159·6	− 755	109
−7	5·4324	0·5600	−4·599	132·9	− 526	64
−8	5·9775	0·8000	−5·311	99·7	− 318	31
−9	6·5313	1·200	−6·031	64·7	− 155	11
−10	7·0922	2·000	−6·759	32·7	− 53	3
−11	7·6586	4·400	−7·492	9·2	− 7	0
−12	8·2287	∞	−8·229	0	0	0

λ	β_{10}	β_{11}	β_{12}	β_{13}	β_{14}
0	0	0	0	0	0
−1	0·11841	−0·1184	7·631	−37·0	41
−2	0·23827	−0·2383	13·05	−58·0	59
−3	0·35893	−0·3589	16·37	−66·0	60
−4	0·47972	−0·4797	17·76	−64·1	52
−5	0·59998	−0·6000	17·47	−55·5	40
−6	0·71906	−0·7191	15·80	−43·3	27
−7	0·83628	−0·8363	13·11	−30·1	16
−8	0·95099	−0·9510	9·80	−18·1	8
−9	1·0625	−1·062	6·33	− 8·8	3
−10	1·1702	−1·170	3·19	− 3·0	1
−11	1·2734	−1·273	0·89	− 0·4	0
−12	1·3715	−1·371	0	0	0

TABLE 3—*continued*

λ	c_{10}	c_{11}	c_{12}	c_{13}	c_{14}
0	−0·058730	0·058730	−7·356	58·60	−145·5
− 1	−0·060042	0·060042	−4·534	+25·50	− 39·9
− 2	−0·060803	0·060803	−1·818	− 1·46	+ 32·4
− 3	−0·061013	0·061013	+0·687	−21·70	74·8
− 4	−0·060672	0·060672	2·88	−35·05	92·4
− 5	−0·059780	0·059780	4·67	−41·79	91·2
− 6	−0·058337	0·058337	5·98	−42·54	77·3
− 7	−0·056342	0·056342	6·73	−38·33	57·1
− 8	−0·053797	0·053797	6·86	−30·50	36·0
− 9	−0·05070	0·05070	6·29	−20·69	18·2
−10	−0·04705	0·04705	4·99	−10·84	6·3
−11	−0·04286	0·04286	2·90	− 3·14	0·92
−12	−0·03811	0·03811	0	0	0

λ	a_{20}	a_{21}	a_{22}	a_{23}	a_{24}
0	0	0	0	0	0
− 1	0·01818	−0·3083	36·30	−301·1	776
− 2	0·04000	−0·6333	62·86	−476·2	1119
− 3	0·06667	−0·9750	79·84	−546·7	1160
− 4	0·1000	−1·333	87·77	−536·4	1014
− 5	0·1429	−1·708	87·50	−469·6	779
− 6	0·2000	−2·100	80·23	−370·3	528
− 7	0·2800	−2·508	67·50	−260·4	310
− 8	0·4000	−2·933	51·20	−158·5	151
− 9	0·6000	−3·375	33·56	− 78·1	56
−10	1·000	−3·833	17·14	− 26·7	13
−11	2·200	−4·308	4·87	− 3·8	1
−12	∞	−4·800	0	0	0

λ	a_{25}	a_{26}	a_{27}	a_{28}
0	0	0	0	0
− 1	0·03636	−36·30	361·3	−1034
− 2	0·08000	−62·86	571·4	−1492
− 3	0·1333	−79·84	656·1	−1547
− 4	0·2000	−87·77	643·7	−1352
− 5	0·2857	−87·50	563·5	−1039
− 6	0·4000	−80·23	444·3	− 704
− 7	0·5600	−67·50	312·5	− 413
− 8	0·8000	−51·20	190·2	− 202
− 9	1·200	−33·56	93·7	− 75
−10	2·000	−17·14	32·0	− 17
−11	4·400	− 4·87	4·6	− 1
−12	∞	0	0	0

TABLE 3—*continued*

λ	b_{20}	b_{21}	b_{22}	b_{23}	b_{24}
0	0	0	0	0	0
− 1	−0·01818	0·4625	− 63·5	541·9	−1422
− 2	−0·04000	0·9500	−110·0	857·1	−2052
− 3	−0·06667	1·462	−139·7	984·1	−2126
− 4	−0·1000	2·000	−153·6	965·5	−1859
− 5	−0·1429	2·562	−153·1	845·2	−1428
− 6	−0·2000	3·150	−140·4	666·5	− 967
− 7	−0·2800	3·762	−118·1	468·7	− 568
− 8	−0·4000	4·400	− 89·6	285·3	− 277
− 9	−0·6000	5·062	− 58·7	140·6	− 103
−10	−1·000	5·750	− 30·0	48·0	− 23
−11	−2·200	6·462	− 8·5	6·8	− 2
−12	− ∞	7·200	0	0	0

λ	β_{20}	β_{21}	β_{22}	β_{23}	β_{24}	β_{25}
0	0·60171	0	0	0	0	0
− 1	0·58725	−0·04372	0·3083	−10·80	47·3	−49
− 2	0·57471	−0·09620	0·6333	−18·70	74·8	−71
− 3	0·56411	−0·1603	0·9750	−23·75	85·8	−73
− 4	0·55543	−0·2405	1·333	−26·11	84·2	−64
− 5	0·54868	−0·3436	1·708	−26·03	73·7	−49
− 6	0·54386	−0·4810	2·100	−23·87	58·1	−33
− 7	0·54096	−0·6734	2·508	−20·08	40·9	−20
− 8	0·54000	−0·9620	2·933	−15·23	24·9	−10
− 9	0·54096	−1·443	3·375	− 9·98	12·3	− 4
−10	0·54386	−2·405	3·833	− 5·10	4·2	− 1
−11	0·54868	−5·291	4·308	− 1·4	0·6	− 0
−12	0·55543	− ∞	4·800	0	0	0

λ	c_{20}	c_{21}	c_{22}	c_{23}	c_{24}
0	0·016667	−0·1500	10·286	−74·06	172·8
− 1	0·021488	−0·1625	6·639	−33·7	+ 49·8
− 2	0·028000	−0·1750	+ 2·998	0	− 36·1
− 3	0·037037	−0·1875	− 0·482	+26·0	− 87·9
− 4	0·050000	−0·2000	− 3·66	43·9	−111
− 5	0·069388	−0·2125	− 6·37	53·6	−110
− 6	0·10000	−0·2250	− 8·49	55·5	− 95
− 7	0·15200	−0·2375	− 9·84	50·9	− 71
− 8	0·25000	−0·2500	−10·29	41·1	− 45
− 9	0·46667	−0·2625	− 9·68	28·3	− 23
−10	1·1000	−0·2750	− 7·86	15·1	− 8
−11	4·6000	−0·2875	− 4·68	4·4	− 1
−12	∞	−0·3000	0	0	0

TABLE 4

Flow Along Flat-plate Thermometer. Linear Pressure Gradient

$\sigma = 0\cdot715$

Mach number of incident flow = 0·1

λ	ξ	M	θ	χ
0	1·0000	0·1000	0·9986	1·22
− 1	1·0004	0·0973	0·9987	1·29
− 2	1·0007	0·0952	0·9989	1·36
− 3	1·0009	0·0933	0·9990	1·45
− 4	1·0011	0·0917	0·9991	1·56
− 5	1·0013	0·0904	0·9992	1·68
− 6	1·0014	0·0893	0·9994	1·83
− 7	1·0016	0·0882	0·9995	2·03
− 8	1·0017	0·0874	0·9996	2·31
− 9	1·0017	0·0867	0·9997	2·72
−10	1·0018	0·0861	0·9998	3·46
−11	1·0019	0·0855	0·9999	5·23
−12	1·0019	0·0851	1·0000	

Mach number of incident flow = 1

λ	ξ	M	θ	χ
0	1·000	1·000	0·878	1·28
− 1	1·035	0·970	0·891	1·34
− 2	1·066	0·944	0·902	1·41
− 3	1·093	0·922	0·913	1·50
− 4	1·116	0·903	0·924	1·60
− 5	1·136	0·886	0·934	1·71
− 6	1·154	0·872	0·943	1·87
− 7	1·170	0·859	0·952	2·06
− 8	1·183	0·848	0·961	2·34
− 9	1·196	0·838	0·970	2·74
−10	1·206	0·830	0·978	3·51
−11	1·215	0·822	0·988	5·14
−12	1·224	0·815	1·000	

Mach number of incident flow = $\sqrt{10}$

λ	ξ	M	θ	χ
0	1·00	3·16	0·463	1·61
− 1	1·29	2·99	0·503	1·64
− 2	1·71	2·80	0·549	1·68
− 3	2·28	2·62	0·599	1·73
− 4	2·97	2·44	0·650	1·80
− 5	3·75	2·30	0·699	1·89
− 6	4·56	2·17	0·744	2·01
− 7	5·36	2·07	0·788	2·18
− 8	6·13	1·98	0·829	2·44
− 9	6·87	1·91	0·868	2·83
−10	7·59	1·84	0·907	3·53
−11	8·28	1·78	0·947	5·29
−12	9·01	1·73	1·000	

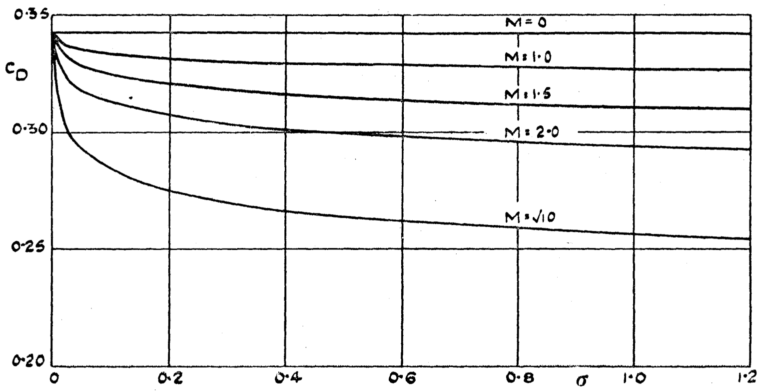

FIG. 1. Drag coefficient C_D as a function of σ.

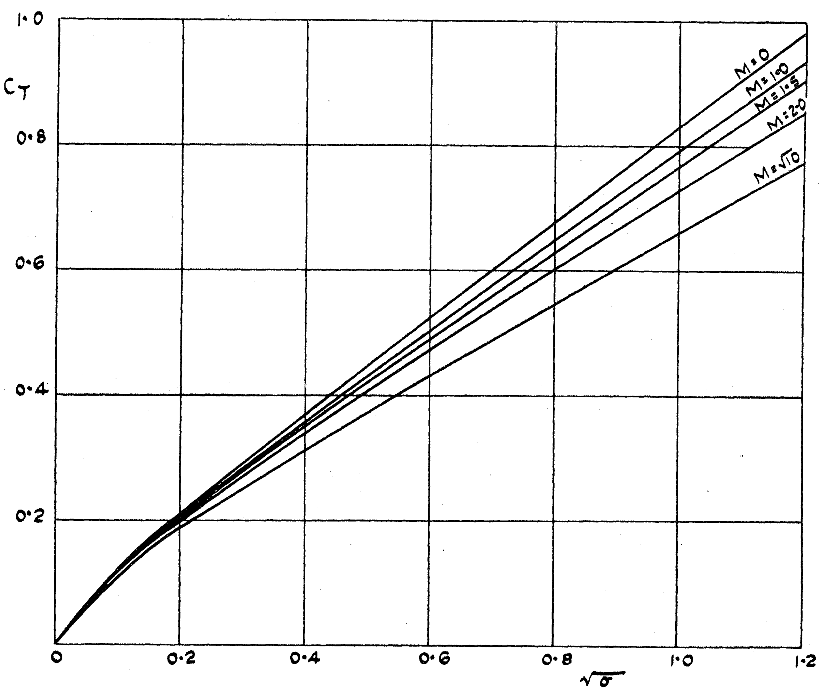

FIG. 2. Temperature coefficient C_T as a function of $\sqrt{\sigma}$.

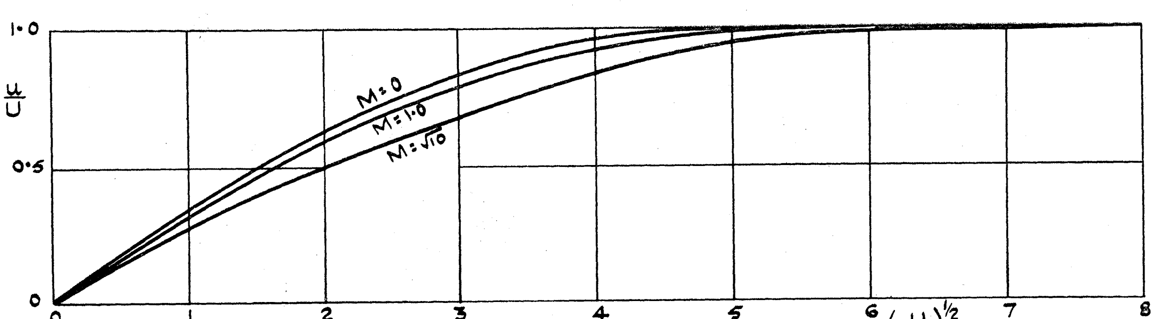

FIG. 3. Variation, for $\sigma = 0 \cdot 733$, of u/U in the boundary layer.

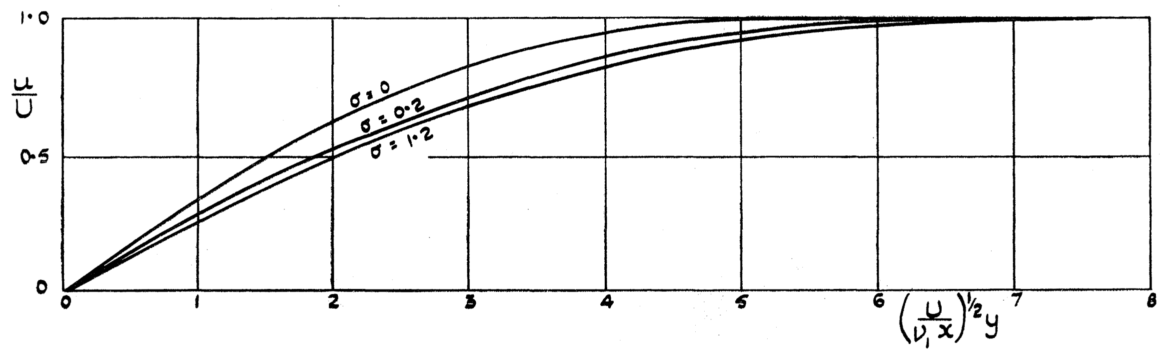

FIG. 4. Variation, at a Mach number of $\sqrt{10}$, of u/U in the boundary layer.

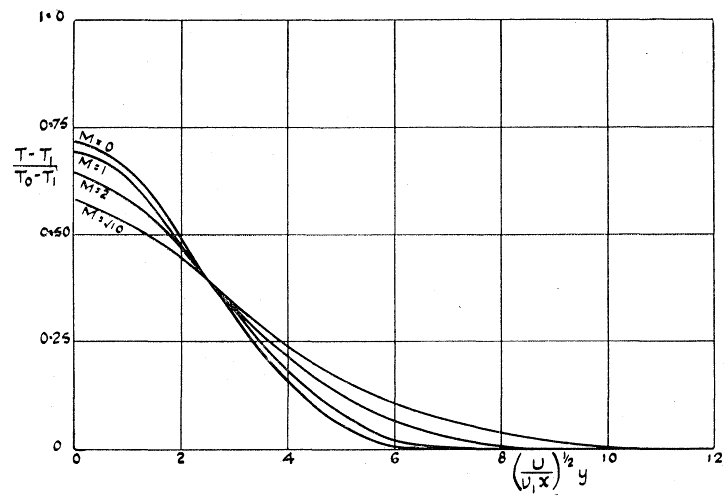

FIG. 5. Variation of temperature in the boundary layer for $\sigma = 0\cdot733$.

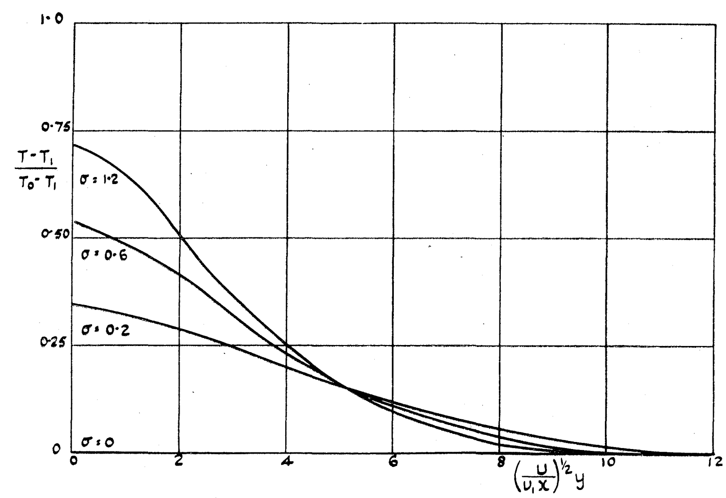

FIG. 6. Variation of temperature in the boundary layer for $M = \sqrt{10}$.

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