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Second-Order Small-Perturbation Theory for Finite Wings in Incompressible Flow

By J. WEBER

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Summary

The incompressible second-order theory for two-dimensional aerofoils is extended to finite swept wings. The flow field is represented by distributions of sources and 'lifting singularities' on the 'chord surface' which contains the chord at each spanwise station.

The strength of the source distribution is obtained as the sum of the distribution from first-order theory and a correction which is derived from the second-order boundary condition. This involves the computation of the velocity which planar singularity distributions induce on and off the plane; the computation can be done by computer programs developed at the R.A.E.

It is suggested that the determination of the strength of the lifting singularities aims from the start at the solution for the wing of finite thickness.

First a generally applicable solution is derived. By means of Taylor series expansions, this solution is simplified for the part of the wing away from centre and tip.

The problem of designing a wing, of given thickness distribution, which has a prescribed pressure distribution on the upper surface is also treated.

The Report describes only the calculation procedures, but does not give actual sample calculations. One of the procedures suggested for determining the pressure distribution has been applied successfully by C. C. L. Sells to untwisted uncambered wings[17] and to wings with camber and twist.[18]

*Replaces R.A.E. Technical Report 72171—A.R.C. 34 469.

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1. Introduction

The present Report is concerned with methods for calculating the pressure distribution over the surface of three-dimensional swept wings of given geometry. The aim is also to provide a method which can be applied to design wings with given properties, for example, a prescribed pressure distribution on the upper surface.

The Report deals only with inviscid, incompressible flow; as a consequence the problems can be solved by means of distributions of singularities: sources, doublets or vortices.

There exist several methods for determining the pressure distribution over wings of given shape, which use singularity distributions on the surface of the wing, as for example those by A. M. O. Smith[1] and A. Roberts.[2] (The method of Ref. 1, which deals only with non-lifting wings, has been extended by several authors to lifting configurations.)

With the method of A. M. O. Smith, the surface of the wing is approximated by a large number of plane quadrilateral 'facets', on each of which is placed a source distribution of constant density. The boundary condition of zero normal velocity is satisfied at one particular point on each facet and the tangential velocity is computed at the same point. The method cannot compute sensible values near the edges of the facets, since the slope of the panels and the strength of the sources vary generally in a discontinuous manner. The pressure distribution at the centre section of a swept wing can only be derived by extrapolation, and it is not possible to check the accuracy of the solution.

These difficulties are avoided by the method of A. Roberts. He uses curved panels and ensures that all quantities involved (the surface ordinates and the strengths of the sources and doublets used to represent the flow) are as smooth as possible, by approximating all the variables by a double cubic spline system. As a result, relatively accurate results can be obtained and the accuracy can be examined, but the computing time can be large. There arises, therefore, the requirement for a simpler and faster method.

There also exist cruder methods, based on first-order theory, where the flow is represented by singularity distributions in a plane, which satisfy an approximate boundary condition with respect to the velocity normal to the plane, and the pressure distribution is derived from the velocity tangential to this plane. The methods have the advantage that they can use smooth functions for the strength of the singularity distributions, are much faster than the panel methods and can also provide a solution to the design problem.

However, it is known for two-dimensional aerofoils that the accuracy of the results from first-order theory can be insufficient. (For example, the lift slope for a 10 per cent thick symmetrical aerofoil can be too low by 8 per cent.)

The R.A.E. standard method,[3,4] which is based on first-order theory, therefore includes some second-order terms; but they are derived from a second-order theory for two-dimensional aerofoils and hence may not be applicable in three-dimensional situations as they can occur near the centre section of swept wings.

The aim of the present Report is therefore to consider methods which are more accurate than those from first-order theory (and the R.A.E. standard method) but less time consuming than the Roberts method. It is also intended to derive a method for the design of wings, which the Roberts program, in its present form, does not deal with.

It is of course true that the effects of viscosity and compressibility, which we ignore here, can be of a similar magnitude as the difference between an exact inviscid, incompressible pressure distribution and the pressure distribution from first-order theory. But it is clear that to determine the pressure distribution in viscous flow, we need to have an inviscid solution of sufficient accuracy.

Our aim is a 'second-order theory' by which we mean that we intend to compute the perturbation velocity at the surface of the wing to the accuracy $0(\varepsilon^2)$ where ε is a geometric scale parameter which is related to the thickness-to-chord ratio, the camber-to-chord ratio, the angle of incidence and the angle of twist. (We shall see that, in contrast to the second-order theory for two-dimensional aerofoils,[5–7] we cannot express everywhere on the wing all components of the perturbation velocity in powers of ε.) We require the accuracy $0(\varepsilon^2)$ only where the components of the perturbation velocity are of $0(\varepsilon)$, which means away from the leading and trailing edges.

The methods to be considered use singularity distributions which are situated in a single surface inside the wing. For the two-dimensional aerofoil Lighthill[5] has shown that, to obtain a valid solution from a distribution of singularities on a straight line, it is necessary that the line passes through leading and trailing edges. We extend Lighthill's approach to the three-dimensional wing and choose therefore singularity distributions on the 'chord surface' which contains at each spanwise station the chord of the section.

To derive a fairly simple method we intend to make use of the computer programs derived by Ledger[8] and Sells[9] for computing the velocity field induced by given singularity distributions. These programs apply only to planar distributions. For the general wing shape the leading and trailing edges are not coplanar so that the chord surface is not a plane. We discuss in Section 2.1 how we can obtain an approximate solution by means of planar singularity distributions.

The condition for the strength of the planar singularity distributions is derived in Section 2.2 from the boundary condition that the flow is tangential to the wing surface. With the assumption that the perturbation velocity is small, the boundary condition is approximated to second-order accuracy by an equation in terms of the velocity components induced by planar singularity distributions. Such an approximation is of course not necessary when the spanwise derivative of the ordinate of the mean surface of the wing is negligible, as for an untwisted, uncambered wing.

The strengths of the singularity distributions are determined by an iterative procedure. The further discussion is restricted to wing shapes for which the local dihedral angle is of second-order magnitude. The trailing vortices are assumed to lie along the extension of the wing chord. As a consequence the velocity field can be expressed in terms of a source and a load distribution, $q(x, y)$, $l(x, y)$, in the plane $z = 0$.

In Section 2.3 we first ignore the fact that the small-perturbation assumption breaks down near the leading and trailing edges and derive a formal solution. We express the unknown $q(x, y)$, $l(x, y)$ as the sums of the solutions from a first-order theory, $q^{(1)}(x, y)$, $l^{(1)}(x, y)$, and second-order correction terms $\Delta q(x, y)$, $\Delta l(x, y)$ and derive the equations which determine Δq, Δl. The source distribution Δq and the velocity component $\Delta v_{zl}(x, y, z = 0)$ normal to the plane have a strong singular behaviour near the leading edge. It is therefore suggested to shift the singularity distributions rearwards (similar to the Lighthill technique for two-dimensional aerofoils); to compute all velocity components, at least near the leading edge, at the surface of the wing; to examine whether the solution satisfies the boundary condition with sufficient accuracy; and when necessary to modify the singularities.

It is not essential to begin with an exact solution of first-order theory, and it may prove beneficial to use as input for the first step of the iteration approximate source and load distributions which would already take account of some of the second-order effects, as for example given by the R.A.E. standard method. In Section 2.4 such an iterative procedure is discussed for the simple example of an uncambered untwisted wing since this is a suitable example to examine the convergence of the iterative procedure. For uncambered untwisted wings C. C. L. Sells[17] has shown that the iteration cycle converges to acceptable accuracy after two steps.

For the two-dimensional aerofoil the method has been simplified by expressing the values of the velocity components off the plane $z = 0$ in terms of the values in the plane by means of Taylor series expansions. We do not introduce Taylor series expansions from the start, because they are not permissible at the centre section of a swept wing for the velocity component v_z, induced by the source distribution $q^{(1)}(x, y)$. But since the use of Taylor series expansions can reduce the computing effort, we discuss in Section 2.5 how Taylor series expansions might be used at least for the part of the wing away from centre and tips.

The solutions derived by Taylor series expansions are not valid near the leading and trailing edges. By comparing the exact solution for a parabola or an ellipse with the formal solution, van Dyke[6] and Gretler[7] have derived rules for modifying the formal solution for two-dimensional aerofoils. Following a suggestion by R. C. Lock,[10] we make use of direct results for ellipsoids to derive a formula for three-dimensional wings which produces valid results near the leading edge, except close to the centre section. The accuracy of such a procedure can be examined by a comparison with the results obtained from the previous method where we compute the velocity components at the surface of the wing and determine the error left in the boundary condition. We do not attempt for inviscid flow, to satisfy in detail the conditions very near the trailing edge or the tips since the real viscous flow departs there a great deal from the inviscid flow field. The same is true for the methods of A. M. O. Smith and Roberts, (though Roberts method deals properly with the trailing edge and can incorporate a curved wake of given shape).

Finally, the problem of designing a wing, for which the thickness distribution and the pressure distribution on the upper surface are prescribed, is discussed in Section 3. Part of the solution requires the design of a wing for which the load distribution, $l(x, y)$, in $z = 0$ is given. This task is therefore discussed first, in Section 3.1.

2. Wings with Given Geometry

2.1 Approximations Involved in the use of Planar Singularity Distributions

We consider in this Report only wing shapes for which the mean surface does not depart much from a plane, i.e. the angle of twist, the maximum camber and the angle of dihedral are assumed to be small.

We consider a rectangular coordinate system x^*, y^*, z^*, where the plane $z^* = 0$ is a reference plane close to the wing. The wing shape is specified by the ordinate $z_w^*(x^*, y^*)$, the leading edge by the coordinates $x_L^*(y^*)$, $z_L^*(y^*)$. The 'chord surface' is then given by

$$z_c^*(x^*, y^*) = z_L^*(y^*) - \tan\alpha_T(y^*)[x^* - x_L^*(y^*)], \tag{1}$$

where $\alpha_T(y^*)$ is the spanwise twist distribution. Wing thickness and camber are given for each spanwise station, $y^* = \text{const.}$, in the form used with two-dimensional sections, i.e. as ordinates z_w normal to the chord

$$z_w(x; y^*) = \pm z_t(x; y^*) + z_s(x; y^*), \tag{2}$$

where the x-axis (i.e. $z(x; y^* = \text{const.}) = 0$) contains the wing chord. $z_t(x; y^*)$ is the thickness distribution and $z_s(x; y^*)$ the camber distribution. Both z_t and z_s vanish at the leading edge and at the trailing edge. The relations between x^*, z_w^* and x, z_w read

$$x^*(x, y^*) = x_L^*(y^*) + [x - x_L(y^*)] \cos \alpha_T(y^*) + z_w(x, y^*) \sin \alpha_T(y^*),$$

$$z_w^*(x, y^*) = z_L^*(y^*) - [x - x_L(y^*)] \sin \alpha_T(y^*) + z_w(x, y^*) \cos \alpha_T(y^*).$$

We choose

$$x_L(y^*) = \frac{x_L^*(y^*)}{\cos \alpha_T(y^*)}, \tag{3}$$

so that

$$x^*(x, y^*) = x \cos \alpha_T(y^*) + z_w(x, y^*) \sin \alpha_T(y^*), \tag{4}$$

$$z_w^*(x, y^*) = z_L^*(y^*) + x_L^*(y^*) \tan \alpha_T(y^*) - x \sin \alpha_T(y^*) + z_w(x, y^*) \cos \alpha_T(y^*). \tag{5}$$

We assume that the maximum thickness-to-chord ratio as well as maximum camber and twist are small. We restrict the discussion further to configurations where the spanwise gradients of the angle of twist, $\alpha_T(y^*)$, and of the ordinates of the leading and trailing edges $z_L^*(y^*)$, $z_T^*(y^*)$, are small. We consider only such wings for which the spanwise derivatives $\partial z_w(x, y^*)/\partial y^*$, $d\alpha_T/dy^*$, dx_L^*/dy^*, dz_L^*/dy^* are continuous except at the centre section, $y^* = 0$.

The undisturbed flow is parallel to the planes $y^* = \text{const.}$ and at small angle α^* to the plane $z^* = 0$; the velocity V_0 of the undisturbed flow is taken as unity.

We intend to determine the pressure distribution on the wing by means of distributions of singularities inside the wing and intend to extend to three dimensions the simple approach to the two-dimensional problem which uses a singularity distribution on the chord extending from the leading edge to the trailing edge.[5 to 7] A summary of the two-dimensional theory is given in Appendix A.

In the three-dimensional case any surface which joins the leading and trailing edges is generally non-planar. The projection of the wing onto the plane $z^* = 0$ produces a planar surface which may be partly outside the wing, so that we cannot use this surface directly as the location for the singularities. For this we take the warped surface through the leading and trailing edges generated by straight lines joining the leading and trailing edge of any spanwise station y^*.

We make then the assumption that a singularity distribution on this warped surface produces at a small normal distance h (including $h = 0$) from the surface at any point x^*, y^* the same velocity—to second-order accuracy—as the same singularity distribution in a plane $z = 0$ produces at x^*, y^*, $z = h$. Such an assumption has to be made here, since we intend to make use of the computer programs of Refs. 8 and 9, which apply only to singularity distributions in a plane. (If one were to use non-planar distributions, then it becomes questionable whether any gain in computing effort can be obtained compared to solving the problem by means of surface singularities.) We have however not yet examined the errors involved in the assumption; the error will of course depend on the values of the first and higher spanwise derivatives of the functions $z_L^*(y^*)$, $z_T^*(y^*)$, $\alpha_T(y^*)$.

The assumption need not hold at a station where the spanwise derivative of $z_c^*(x^*, y^*)$ changes discontinuously, as for a wing with dihedral or for the centre section of a wing for which the twist varies near the centre section. It may then be necessary to use singularity distributions in more than one plane to approximate to the chord surface. The computer programs of Refs. 8 and 9 can be used with several planar singularity distributions, but in the present Report we shall consider mainly wings for which the local dihedral angle is of second-order magnitude so that the approximation by one planar singularity distribution is sufficient.

2.2 Conditions for the Strength of the Singularity Distributions

The strength of the singularities has to be determined such that the boundary condition of zero normal velocity at the wing surface is satisfied, at least approximately.

When the wing surface is defined by $F(x^*, y^*, z^*) = 0$, then the condition that the component of the total velocity V normal to the wing surface vanishes can be written in the form

$$V_{x^*}\frac{\partial F}{\partial x^*} + V_{y^*}\frac{\partial F}{\partial y^*} + V_{z^*}\frac{\partial F}{\partial z^*} = 0.$$

With $F(x^*, y^*, z^*) = z^* - z_w^*(x^*, y^*)$, this condition reads

$$\frac{\partial z_w^*(x^*, y^*)}{\partial x^*}[\cos\alpha^* + v_{x^*}(x^*, y^*, z_w^*)] + \frac{\partial z_w^*(x^*, y^*)}{\partial y^*}v_{y^*}(x^*, y^*, z_w^*) = \sin\alpha^* + v_{z^*}(x^*, y^*, z_w^*), \tag{6}$$

where $v_{x^*}, v_{y^*}, v_{z^*}$ are the components of the perturbation velocity along the various axes, computed at the surface of the wing.

We have stated above that we approximate the velocity induced by the singularity distributions in the chord surface by the velocity derived from the same singularity distribution in a plane which is tangential to the chord surface at the point $x^*, y^*, z_c^*(x^*, y^*)$ under consideration.

We resolve the velocity induced by the singularity distributions in the tangent plane into the component normal to the chord surface, of strength v_n, and a component in the tangent plane of the chord surface; the latter is resolved into a component parallel to the chord, of strength v_x, and a component normal to the chord, of strength v_t.

At the point $x^*, y^*, z_c^*(x^*, y^*)$ the unit vector normal to the chord surface, $\mathbf{n}$, can be resolved in its three components parallel to the x^*, y^*, z^*-axes:

$$\mathbf{n} = \frac{\tan\alpha_T\mathbf{i}^* - \tan\psi\,\mathbf{j}^* + \mathbf{k}^*}{\sqrt{1 + \tan^2\alpha_T + \tan^2\psi}},$$

where $\mathbf{i}^*, \mathbf{j}^*, \mathbf{k}^*$ are unit vectors parallel to the x^*, y^*, z^*-axes and

$$\tan\psi(x^*, y^*) = \frac{\partial z_c^*(x^*, y^*)}{\partial y^*}. \tag{7}$$

The unit vector parallel to the chord, $\mathbf{i}$, can be written as

$$\mathbf{i} = \cos\alpha_T\mathbf{i}^* - \sin\alpha_T\mathbf{k}^*$$

and the unit vector which is normal to the chord and lies in the tangent plane, $\mathbf{j}$, can be obtained from

$$\mathbf{j} = \mathbf{n} \times \mathbf{i}.$$

With the notation

$$\kappa = \frac{1}{\sqrt{1 + \tan^2\alpha_T + \tan^2\psi}}, \tag{8}$$

$$\mathbf{j} = \kappa\sin\alpha_T\tan\psi\mathbf{i}^* + \frac{\kappa}{\cos\alpha_T}\mathbf{j}^* + \kappa\cos\alpha_T\tan\psi\mathbf{k}^*.$$

The perturbation velocity vector $\mathbf{v}$ can therefore be written as

$$\begin{aligned}\mathbf{v} &= v_n\mathbf{n} + v_x\mathbf{i} + v_t\mathbf{j}\\ &= [v_n\kappa\tan\alpha_T + v_x\cos\alpha_T + v_t\kappa\sin\alpha_T\tan\psi]\mathbf{i}^* + \\ &\quad + \left[-v_n\kappa\tan\psi + v_t\frac{\kappa}{\cos\alpha_T}\right]\mathbf{j}^* + [v_n\kappa - v_x\sin\alpha_T + v_t\kappa\cos\alpha_T\tan\psi]\mathbf{k}^*.\end{aligned} \tag{9}$$

The derivatives $\partial z_w^*(x^*, y^*)/\partial x^*$ and $\partial z_w^*(x^*, y^*)/\partial y^*$, which occur in equation (6), can be obtained from the following relations

$$\frac{\partial z_w^*(x^*, y^*)}{\partial x^*} = \frac{\partial z_w^*(x, y^*)/\partial x}{\partial x^*(x, y^*)/\partial x},$$

$$\frac{\partial z_w^*(x^*, y^*)}{\partial y^*} = -\frac{\partial z_w^*(x^*, y^*)}{\partial x^*}\frac{\partial x^*(x, y^*)}{\partial y^*} + \frac{\partial z_w^*(x, y^*)}{\partial y^*}. \tag{10}$$

By using equations (4) and (5), we obtain

$$\frac{\partial z_w^*(x^*, y^*)}{\partial x^*} = \frac{-\sin\alpha_T + \cos\alpha_T\dfrac{\partial z_w(x, y^*)}{\partial x}}{\cos\alpha_T + \sin\alpha_T\dfrac{\partial z_w(x, y^*)}{\partial x}}. \tag{11}$$

By using equations (1), (5) and (7), we obtain

$$\frac{\partial z_w^*(x, y^*)}{\partial y^*} = \frac{\partial}{\partial y^*}[z_L^* + x_L^*\tan\alpha_T] - x\cos\alpha_T\frac{d\alpha_T}{dy^*} + \frac{\partial}{\partial y^*}[\cos\alpha_T z_w(x, y^*)]$$

$$= \tan\psi + \cos\alpha_T\frac{\partial z_w(x, y^*)}{\partial y^*} + [x\cos\alpha_T + \sin\alpha_T z_w(x, y^*)]\tan^2\alpha_T\frac{d\alpha_T}{dy^*}. \tag{12}$$

Using equation (10), we can write the boundary condition in the form

$$\frac{\partial z_w^*(x^*, y^*)}{\partial x^*}\left[\cos\alpha^* + v_{x^*} - v_{y^*}\frac{\partial x^*(x, y^*)}{\partial y^*}\right] + \frac{\partial z_w^*(x, y^*)}{\partial y^*}v_{y^*} = \sin\alpha^* + v_{z^*}. \tag{13}$$

By inserting the velocity components given by equation (9) and using equations (4), (11), (12), we obtain

$$\begin{aligned}
&\left(\frac{-\sin\alpha_T + \cos\alpha_T\dfrac{\partial z_w}{\partial x}}{\cos\alpha_T + \sin\alpha_T\dfrac{\partial z_w}{\partial x}}\right)\Bigg\{\cos\alpha^* + v_n\kappa\tan\alpha_T + v_x\cos\alpha_T + v_t\kappa\sin\alpha_T\tan\psi + \\
&\quad + \left[v_t\frac{\kappa}{\cos\alpha_T} - v_n\kappa\tan\psi\right]\left[(x\sin\alpha_T - z_w\cos\alpha_T)\frac{d\alpha_T}{dy^*} - \sin\alpha_T\frac{\partial z_w}{\partial y^*}\right]\Bigg\} + \\
&\quad + \left\{\tan\psi + \cos\alpha_T\frac{\partial z_w}{\partial y^*} + [x\cos\alpha_T + z_w\sin\alpha_T]\tan^2\alpha_T\frac{d\alpha_T}{dy^*}\right\}\left\{v_t\frac{\kappa}{\cos\alpha_T} - v_n\kappa\tan\psi\right\} \\
&= \sin\alpha^* + v_n\kappa - v_x\sin\alpha_T + v_t\kappa\cos\alpha_T\tan\psi.
\end{aligned}$$

This equation can be written in the form

$$\begin{aligned}
&\frac{\partial z_w}{\partial x}\left\{\cos(\alpha^* + \alpha_T) + v_x + \left[v_t\frac{\kappa}{\cos\alpha_T} - v_n\kappa\tan\psi\right]\left[\sin\alpha_T\tan\psi + \left(x\tan\alpha_T - z_w\frac{\cos 2\alpha_T}{\cos^2\alpha_T}\right)\frac{d\alpha_T}{dy^*}\right]\right\} + \\
&\quad + \left[\frac{\partial z_w}{\partial y^*} + z_w\tan\alpha_T\frac{d\alpha_T}{dy^*}\right]\left[v_t\frac{\kappa}{\cos\alpha_T} - v_n\kappa\tan\psi\right] \\
&= \sin(\alpha^* + \alpha_T) + v_n\sqrt{\frac{1 + \tan^2\alpha_T + \tan^2\psi}{1 + \tan^2\alpha_T}}.
\end{aligned} \tag{14}$$

To elucidate equation (14), we consider the special case where locally α_T is zero, which means that $x = x^*$ at the station considered. The perturbation velocity has the components

$$v_{x^*} = v_x,$$

$$v_{y^*} = v_t \cos\psi - v_n \sin\psi,$$

$$v_{z^*} = v_t \sin\psi + v_n \cos\psi.$$

Further

$$\frac{\partial z_w^*(x^*, y^*)}{\partial x^*} = \frac{\partial z_w(x, y^*)}{\partial x},$$

$$\frac{\partial z_w^*(x, y^*)}{\partial y^*} = \tan\psi + \frac{\partial z_w(x, y^*)}{\partial y^*},$$

$$\frac{\partial x^*(x, y^*)}{\partial y^*} = z_w(x, y^*)\frac{d\alpha_T}{dy^*}.$$

When we insert these relations into equation (13), then the boundary condition can be written in the form

$$\frac{\partial z_w}{\partial x}\left[\cos\alpha^* + v_x - z_w\frac{d\alpha_T}{dy^*}(v_t \cos\psi - v_n \sin\psi)\right] + \frac{\partial z_w}{\partial y^*}(v_t \cos\psi - v_n \sin\psi) = \sin\alpha^* + \frac{v_n}{\cos\psi}.$$

We consider only such wing shapes for which α_T, $d\alpha_T/d(y^*/c(y^*))$ and ψ are at most of first-order magnitude. The velocity component v_n is everywhere of a magnitude comparable to ε; it does not behave like $\partial z_w/\partial x$ which is a term of formal order ε but tends to infinity at the leading edge. The velocity component v_t is close to the leading edge of a magnitude comparable to one and away from the leading edge of a magnitude comparable to ε. We may therefore write equation (14) in the form

$$\frac{\partial z_w}{\partial x}\left[\cos(\alpha^* + \alpha_T) + v_x + 0(v_t\varepsilon\psi) + 0\left(v_t\varepsilon\frac{d\alpha_T}{d(y^*/c(y^*))}\right)\right] + \frac{\partial z_w}{\partial y^*}[v_t + 0(\varepsilon\psi) + 0(v_t\psi^2)]$$

$$= \sin(\alpha^* + \alpha_t) + v_n + 0(\varepsilon\psi^2) + 0\left(v_t\varepsilon^2\frac{d\alpha_T}{d(y^*/c(y^*))}\right) \qquad (15)$$

where all the terms $0(\chi)$ are of a magnitude comparable to χ even near the leading edge. Equations (14) and (15) can, within second-order accuracy, be approximated by the equation

$$\frac{\partial z_w}{\partial x}[\cos(\alpha^* + \alpha_T) + v_x] + \frac{\partial z_w}{\partial y^*}\cdot v_t = \sin(\alpha^* + \alpha_T) + v_n. \qquad (16)$$

We note that, away from the leading edge, the term $d\alpha_T/d(y^*/c(y^*))$ occurs in equation (14) only in terms which are of fourth order, whilst ψ occurs in terms which are of third order. This means that for wings with large dihedral, where ψ is of zeroth order, we must not reduce equation (14) into equation (16) if we intend to satisfy the boundary condition to second order. When ψ and $d\alpha_T/d(y^*/c(y^*))$ are $0(\varepsilon^2)$ then equation (16) is correct to $0(\varepsilon^3)$.

The computer programs of Refs. 8 and 9 are written for singularity distributions in the plane $z = 0$, for which v_n corresponds to v_z and v_t to v_y.

When we compute the perturbation velocity for given values of x and y^*, then (in the context of a second-order theory and for small values of ψ) we need not differentiate between the point on the wing surface which lies on the normal to the chord surface and the point on the wing which lies in the plane $y^* =$ const.

Within second-order accuracy, we may also ignore the difference between y^* and y, the spanwise coordinate in the tangent plane for the chord surface.

The task is thus to determine a planar singularity distribution which satisfies, to second-order accuracy, the boundary condition

$$\frac{\partial z_w(x, y)}{\partial x}[\cos(\alpha^* + \alpha_T) + v_x(x, y, z_w)] + \frac{\partial z_w(x, y)}{\partial y} v_y(x, y, z_w) = \sin(\alpha^* + \alpha_T(y)) + v_z(x, y, z_w) \tag{17a}$$

or

$$\frac{\partial z_w(x, y)}{\partial x}[1 + v_x(x, y, z_w)] + \frac{\partial z_w(x, y)}{\partial y} v_y(x, y, z_w) = \alpha^* + \alpha_T(y) + v_z(x, y, z_w). \tag{17b}$$

The centre section requires some further consideration since, for some wing shapes used in practice, the function $\psi(x^*, y^*)$, equation (8), is not continuous at $y = 0$. We consider only symmetrical flow conditions (zero sideslip) and symmetrical wing shapes. For these configurations, $v_{y^*}(x^*, y^* = 0, z^*) = 0$, so that the boundary condition for $y^* = 0$ reads

$$\frac{\partial z_w^*(x^*, y = 0)}{\partial x^*}[\cos \alpha^* + v_{x^*}(x^*, 0, z_w^*)] = \sin \alpha^* + v_{z^*}(x^*, 0, z_w^*). \tag{18}$$

Brebner and Wyatt[11] have determined the velocity induced by two semi-infinite source- and vortex-distributions in two half-planes $z = |y| \tan \psi$ (with constant $\psi(x, y)$) and have shown that the velocity component v_z induced at $y = 0$, $z = 0$ contains besides the term for $\psi = 0$ also terms of order ψ. This means that, when the values of $[\partial z_c^*(x^*, y^*)/\partial y^*]_{y^*=0}$ are of first-order magnitude, we ought to approximate the perturbation velocity at the centre section by the velocities induced by two planar distributions (instead of one planar distribution), if we require the pressure distribution to be correct to second order.

At $y^* = 0$ the velocity components v_{x^*}, v_{z^*} are related to the components parallel and normal to the chord line, v_x, v_z by the relations, similar to equations (4) and (5):

$$v_{x^*}(x^*, 0, z_w^*) = \cos \alpha_T v_x(x, 0, z_w) + \sin \alpha_T v_z(x, 0, z_w), \tag{19}$$

$$v_{z^*}(x^*, 0, z_w^*) = -\sin \alpha_T v_x(x, 0, z_w) + \cos \alpha_T v_z(x, 0, z_w). \tag{20}$$

When equations (14), (19), (20) are introduced into equation (18), then we obtain for $y = 0$ the equation

$$\frac{\partial z_w}{\partial x}[\cos(\alpha^* + \alpha_T) + v_x(x, 0, z_w)] = \sin(\alpha^* + \alpha_T) + v_z(x, 0, z_w). \tag{21}$$

We note that equation (21) agrees with equation (17a).

The general problem is thus to find the strength of planar singularity distributions which satisfy equation (17). When $\psi(x, y = 0)$ is of first-order magnitude, then $v_x(x, 0, z_w)$ and $v_z(x, 0, z_w)$ are to be derived from two planar singularity distributions in $z = |y| \tan \psi$.

The values of the velocity components induced at $y \neq 0$ by a singularity distribution in the two half-planes $z = |y| \tan \psi$ differ from those induced by the same singularity distribution in $z = 0$; the differences decrease of course with increasing $|y|$. With a source distribution or a vortex distribution of constant strength along the span, the differences between values of $v_x(x, y, z = |y| \tan \psi)$ and $v_z(x, y, z = |y| \tan \psi)$ computed for $\psi \neq 0$ and $\psi = 0$ are approximately proportional to the strength of the singularity at x. This means that we can expect to improve the accuracy of the approximate values for $v_x(x, y, z_w)$ and $v_z(x, y, z_w)$ by computing them also for $y \neq 0$, at least for small values of y, from singularity distributions in $z = |y| \tan \psi(x, |y|)$ instead of in $z = 0$. We must of course expect relatively large inaccuracies when the spanwise and chordwise gradients of $\psi(x, y)$ are fairly large.

We may note that it is sufficient to derive values for the spanwise velocity component, v_y or v_t, from singularity distributions in $z = 0$, since, in a second-order theory, we require to know v_y only to first-order accuracy, both with respect to the boundary condition and to derive the pressure distribution on the wing. We require to know v_x to second order when we determine the total velocity and v_z to second order to satisfy the boundary condition.

In the following Sections, 2.3, 2.4 and most of 2.5, we restrict the discussion to wing shapes for which $\psi(x, y = 0)$ is of second-order magnitude, so that we have to deal only with singularity distributions in $z = 0$. We shall compute the velocities induced by the singularities in the plane $z = 0$, both in that plane and at a distance $z_w = \pm z_t + z_s$. The latter are to approximate the velocities at the surface of the wing, whilst those at $z = 0$ are to approximate those at the chord line of each spanwise station.

The assumption that any chord line joining the leading and trailing edge lies wholly inside the wing implies that we restrict the discussion to wing shapes for which $z_t(x, y) + z_s(x, y)$ is everywhere positive and $-z_t + z_s$ is everywhere negative. However, for two-dimensional flow, the derived formulae give meaningful results also for sections with large camber, where part of the chord line lies outside of the section. A similar interpretation of the formulae seems possible in three dimensions.

2.3 Solution derived by Using Results from a First-Order Theory

We represent the wing by a source distribution and a vortex distribution in the 'chord surface'. This means that the shape of the wake is not taken into account. Cross-sections y = const. through the stream surface which leaves the trailing edge are actually curved, because the vorticity distribution induces behind the wing velocity components v_z which vary with distance from the trailing edge. The present approach is thus based on the assumption that the pressure distribution on the wing, for small values of α and $|z_s|$, depends less on the curvature of the wake than on the differences between the values of the velocity components on the wing surface and the values in the chord surface.

We make the further assumption that the trailing vortices are lines y = const. This implies an approximation because the source distribution, which represents a three-dimensional wing, induces velocity components v_y which are in general non-zero. The streamlines leaving the trailing edge are therefore curved in plan view. The vortex distribution induces also a non-zero spanwise velocity component at the trailing edge. Since the contribution to v_y from the vortices is of opposite sign on the upper and the lower surface of the wing, the magnitude of v_y at the trailing edge has different values on the upper and lower surface. To satisfy the condition of zero pressure difference at the trailing edge, the chordwise velocity components at the trailing edge on the upper and lower surface must not both vanish (*see* Mangler and Smith[12]). We do not intend to represent these features of the inviscid solution. With an exact solution, the spanwise velocity induced by the source distribution is small near the trailing edge except for a very narrow region. This exception is connected with the fact that the chordwise velocity varies rapidly over a very short chordwise distance near the trailing edge. This behaviour of v_x and v_y is completely altered by viscosity. We therefore do not intend to solve the inviscid thickness problem or the lifting problem correctly near the trailing edge.

The velocity potential related to a planar distribution of bound vortices and straight trailing vortices can be written[9] as an integral, extending only over the wing area S (i.e. the projection of the wing into the plane $z^* = 0$), where the integrand is a kernel function multiplied by the load distribution† $l(x, y)$ in the plane $z = 0$

$$\phi_l(x, y, z) = \frac{z}{8\pi} \iint_S \frac{l(x', y')}{(y - y')^2 + z^2} \left[1 + \frac{x - x'}{\sqrt{(x - x')^2 + (y - y')^2 + z^2}} \right] dx'\, dy',$$

where (22)

$$l(x, y) = 4\left[\frac{\partial \phi_l(x, y, +0)}{\partial x} - \frac{\partial \phi_l(x, y, -0)}{\partial x} \right],$$

($z = +0$ refers to the upper surface and $z = -0$ to the lower surface of the plane $z = 0$).

The task is thus to determine the strength of the source distribution, $q(x, y)$, and of the load distribution, $l(x, y)$.

We intend to solve the problem to second-order accuracy, i.e. to determine the velocity at the surface of the wing to the accuracy $O(\varepsilon^2)$, where ε is the maximum value of the various functions $z_t(x, y)/c(y)$, $z_s(x, y)/(c(y)$, $\alpha_T(y)$, α^*. We require the accuracy $O(\varepsilon^2)$ only away from the leading and trailing edges, where the components of the perturbation velocity are of $O(\varepsilon)$. Near the leading and trailing edges (and also near the tip of a wing with finite tip chord), we allow a reduced accuracy.

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† In the present Report, the term 'load distribution' is used to denote a distribution of lifting singularities, as in equation (22); it should be realised that this is in general only the same as the physical load distribution $-\Delta C_p$ in first-order theory.

We consider first the derivation of a formal solution of equation (17), where we ignore the fact that the perturbation velocity is not small near the leading and trailing edges; we discuss later how one may derive a solution which is valid also near the leading edge.

In this section, we intend to determine $q(x, y)$ and $l(x, y)$ in two steps, by determining first a solution $q^{(1)}(x, y)$, $l^{(1)}(x, y)$ which is accurate to first order and then a solution $q^{(2)}(x, y)$, $l^{(2)}(x, y)$ which is correct to second order.

One possible choice for $q^{(1)}(x, y)$ and $l^{(1)}(x, y)$ would be given by the singularity distributions which satisfy the first-order boundary condition

$$\pm\frac{\partial z_t(x, y)}{\partial x} + \frac{\partial z_s(x, y)}{\partial x} = \alpha^* + \alpha_T(y) + v_z^{(1)}(x, y, \pm 0). \tag{23}$$

The upper sign applies to the upper surface of the wing and the lower sign to the lower surface. We denote by the superscript (1) the velocity components induced by $q^{(1)}(x, y)$ and $l^{(1)}(x, y)$. The superscript (2) denotes the velocity components induced by $q^{(2)}(x, y)$, $l^{(2)}(x, y)$.

$v_z^{(1)}(x, y, \pm 0)$ can be expressed as the sum of two terms where one term is symmetrical with respect to the plane $z = 0$, $v_{zl}^{(1)}(x, y, 0)$, and the other term is antisymmetrical, $\pm v_{zt}^{(1)}(x, y, 0)$. The antisymmetrical distribution

$$v_{zt}^{(1)}(x, y, \pm 0) = \pm\frac{\partial z_t(x, y)}{\partial x} \tag{24}$$

can be represented by the source distribution

$$q^{(1)}(x, y) = 2\frac{\partial z_t(x, y)}{\partial x}. \tag{25}$$

The symmetrical upwash distribution

$$v_{zl}^{(1)}(x, y, 0) = \frac{\partial z_s(x, y)}{\partial x} - \alpha^* - \alpha_T(y) \tag{26}$$

is represented by the load distribution $l^{(1)}(x, y)$, which satisfies the equation

$$\frac{1}{8\pi}\iint\limits_S \frac{l^{(1)}(x', y')}{(y - y')^2}\left[1 + \frac{x - x'}{\sqrt{(x - x')^2 + (y - y')^2}}\right] dx'\, dy' = v_{zl}^{(1)}(x, y, 0). \tag{27}$$

It is not yet certain what conditions must be satisfied with respect to the shape of the mean surface to ensure the existence of a physically meaningful solution, $l^{(1)}(x, y)$, of equation (27). For example, we do not yet know whether $l^{(1)}(x, y)$ exists for a swept wing (with straight leading edges up to the centre section) with the same chordwise camber shape along the span. For such wings, $\partial z_s(x, y)/\partial y$ is discontinuous at $y = 0$, which means we seek a load distribution for which $\partial v_z/\partial y$ is discontinuous at $y = 0$. We shall discuss in the next section how we might overcome this difficulty. In this section, we assume that $l^{(1)}(x, y)$ exists, which is certainly the case for regular planforms, and that we have a method for computing $l^{(1)}(x, y)$.

We express the second-order singularity distributions in the form

$$q^{(2)}(x, y) = q^{(1)}(x, y) + \Delta q(x, y), \tag{28}$$

$$l^{(2)}(x, y) = l^{(1)}(x, y) + \Delta l(x, y). \tag{29}$$

The velocity components induced by $\Delta q(x, y)$ and $\Delta l(x, y)$ are denoted by Δv_x, Δv_y and Δv_z. The boundary condition, equation (17),

$$\frac{\partial z_w}{\partial x}[1 + v_x^{(1)}(x, y, z_w) + \Delta v_x(x, y, z_w)] + \frac{\partial z_w}{\partial y}[v_y^{(1)}(x, y, z_w) + \Delta v_y(x, y, z_w)]$$
$$= \alpha^* + \alpha_T + v_z^{(1)}(x, y, z_w) + \Delta v_z(x, y, z_w) \tag{30}$$

can be approximated to second-order accuracy by

$$\frac{\partial z_w}{\partial x}[1 + v_x^{(1)}(x, y, z_w)] + \frac{\partial z_w}{\partial y} v_y^{(1)}(x, y, z_w) = \alpha^* + \alpha_T + v_z^{(1)}(x, y, z_w) + \Delta v_z(x, y, \pm 0). \qquad (31)$$

For the two-dimensional aerofoil, the task has been simplified by expressing $v_z^{(1)}(x, z_w)$ as a Taylor series expansion in terms of the velocity components induced in $z = 0$; but such a Taylor series expansion is not always possible on a three-dimensional wing, for the following reason. Let us consider a swept wing which has the same symmetrical chordwise section along the span. At the centre section, the values of $v_{zt}^{(1)}(x, 0, z_t)$ induced by the source distribution $q^{(1)}(x, y)$ contain the terms (*see* for example Ref. 8):

$$q^{(1)}(x)(\tfrac{1}{2} + 0(z_t)) + \frac{\tan\varphi}{\pi}\frac{dq^{(1)}(x)}{dx} z_t \log z_t + 0(z_t^2).$$

This shows that for $v_{zt}^{(1)}(x, 0, z)$ an expansion in powers of ε is not possible.

We therefore discuss first how we may proceed when a Taylor series expansion of v_z is not everywhere possible.

Within the second-order theory for two-dimensional aerofoils the term $v_x^{(1)}(x, z_w)$ in equation (31) has been replaced by its value at $z = 0$, $v_x^{(1)}(x, 0)$. One may be inclined to replace both $v_x^{(1)}(x, y, z_w)$ and $v_y^{(1)}(x, y, z_w)$ in equation (31) by their values at $z = 0$. However, in Appendix B it is shown, for elliptic sections, that the accuracy of the resulting pressure distribution is noticeably improved if one retains $v_x^{(1)}(x, z_w)$ in equation (31) when computed values for $v_z^{(1)}(x, z_w)$ are used, and that one uses $v_x^{(1)}(x, 0)$ together with values for $v_z^{(1)}(x, z_w)$ derived by Taylor series expansion. Until we have checked the procedure by numerical calculations for three-dimensional wings (and non-elliptic aerofoils), we suggest that the same rule should be applied for three-dimensional wings. Thus, when using equation (31) with values of $v_z^{(1)}(x, y, z_w)$ computed on the wing surface, we take for the terms $v_x^{(1)}(x, y, z_w)$ and $v_y^{(1)}(x, y, z_w)$ also the values on the wing surface.

At least near the leading edge, the values $v_x^{(1)}(x, y, z_w)$, $v_y^{(1)}(x, y, z_w)$ have to be computed at $z = z_w(x, y)$. Away from the leading edge, i.e. for $\xi = (x - x_L(y))/c(y) > 0{\cdot}1$ say, one may use values derived by Taylor series expansions. These Taylor series expansions will be discussed further in Section 2.5.

In the following, we denote by the suffix t the velocity components induced by a source distribution and by the suffix l the velocity components induced by a load distribution. The velocity components v_{xt}, v_{yt}, v_{zl} are symmetrical functions of z with respect to the plane $z = 0$ and v_{zt}, v_{xl}, v_{yl} are antisymmetrical functions. When we apply equation (31) to the upper and the lower surface, respectively, then we obtain the two equations:

$$\left(\frac{\partial z_t}{\partial x} + \frac{\partial z_s}{\partial x}\right)[1 + v_{xt}^{(1)}(x, y, z_t + z_s) + v_{xl}^{(1)}(x, y, z_t + z_s)] + \left(\frac{\partial z_t}{\partial y} + \frac{\partial z_s}{\partial y}\right)[v_{yt}^{(1)}(x, y, z_t + z_s) + v_{yl}^{(1)}(x, y, z_t + z_s)]$$
$$= \alpha^* + \alpha_T(y) + v_{zt}^{(1)}(x, y, z_t + z_s) + v_{zl}^{(1)}(x, y, z_t + z_s) + \Delta v_{zt}(x, y, 0) + \Delta v_{zl}(x, y, 0) \qquad (32)$$

and

$$\left(-\frac{\partial z_t}{\partial x} + \frac{\partial z_s}{\partial x}\right)[1 + v_{xt}^{(1)}(x, y, z_t - z_s) - v_{xl}^{(1)}(x, y, z_t - z_s)] + \left(-\frac{\partial z_t}{\partial y} + \frac{\partial z_s}{\partial y}\right)[v_{yt}^{(1)}(x, y, z_t - z_s) - v_{yl}^{(1)}(x, y, z_s - z_s)]$$
$$= \alpha^* + \alpha_T(y) - v_{zt}^{(1)}(x, y, z_t - z_s) + v_{zl}^{(1)}(x, y, z_t - z_s) - \Delta v_{zt}(x, y, 0) + \Delta v_{zl}(x, y, 0). \qquad (33)$$

We may remind the reader that we have stated in Section 2.1 that we restrict the present discussion to wing shapes for which $z_t(x, y) + z_s(x, y)$ and $z_t(x, y) - z_s(x, y)$ are everywhere positive.

By subtracting equation (33) from equation (32) we obtain for the strength of the additional source distribution $\Delta q(x, y) = 2\Delta v_{zt}(x, y, 0)$ the equation:

$$\Delta q(x, y) = \frac{\partial z_t}{\partial x}[2 + v_{xt}^{(1)}(x, y, z_t + z_s) + v_{xt}^{(1)}(x, y, z_t - z_s) + v_{xl}^{(1)}(x, y, z_t + z_s) - v_{xl}^{(1)}(x, y, z_t - z_s)] +$$
$$+ \frac{\partial z_s}{\partial x}[v_{xt}^{(1)}(x, y, z_t + z_s) - v_{xt}^{(1)}(x, y, z_t - z_s) + v_{xl}^{(1)}(x, y, z_t + z_s) + v_{xl}^{(1)}(x, y, z_t - z_s)] +$$

$$+ \frac{\partial z_t}{\partial y}[v_{yt}^{(1)}(x, y, z_t + z_s) + v_{yt}^{(1)}(x, y, z_t - z_s) + v_{yl}^{(1)}(x, y, z_t + z_s) - v_{yl}^{(1)}(x, y, z_t - z_s)] +$$

$$+ \frac{\partial z_s}{\partial y}[v_{yt}^{(1)}(x, y, z_t + z_s) - v_{yt}^{(1)}(x, y, z_t - z_s) + v_{yl}^{(1)}(x, y, z_t + z_s) + v_{yl}^{(1)}(x, y, z_t - z_s)] -$$

$$- [v_{zt}^{(1)}(x, y, z_t + z_s) + v_{zt}^{(1)}(x, y, z_t - z_s)] - [v_{zl}^{(1)}(x, y, z_t + z_s) - v_{zl}^{(1)}(x, y, z_t - z_s)]. \quad (34)$$

Adding equations (32) and (33), we obtain for the upwash $\Delta v_{zl}(x, y, 0)$, which has to be produced by the load distribution $\Delta l(x, y)$, the equation:

$$2\Delta v_{zl}(x, y, 0) = \frac{\partial z_t}{\partial x}[v_{xt}^{(1)}(x, y, z_t + z_s) - v_{xt}^{(1)}(x, y, z_t - z_s) + v_{xl}^{(1)}(x, y, z_t + z_s) + v_{xl}^{(1)}(x, y, z_t - z_s)] +$$

$$+ \frac{\partial z_s}{\partial x}[2 + v_{xt}^{(1)}(x, y, z_t + z_s) + v_{xt}^{(1)}(x, y, z_t - z_s) + v_{xl}^{(1)}(x, y, z_t + z_s) - v_{xl}^{(1)}(x, y, z_t - z_s)] +$$

$$+ \frac{\partial z_t}{\partial y}[v_{yt}^{(1)}(x, y, z_t + z_s) - v_{yt}^{(1)}(x, y, z_t - z_s) + v_{yl}^{(1)}(x, y, z_t + z_s) + v_{yl}^{(1)}(x, y, z_t - z_s)] +$$

$$+ \frac{\partial z_s}{\partial y}[v_{yt}^{(1)}(x, y, z_t + z_s) + v_{yt}^{(1)}(x, y, z_t - z_s) + v_{yl}^{(1)}(x, y, z_t + z_s) - v_{yl}^{(1)}(x, y, z_t - z_s)] -$$

$$- 2\alpha^* - 2\alpha_T(y) - [v_{zt}^{(1)}(x, y, z_t + z_s) - v_{zt}^{(1)}(x, y, z_t - z_s)] -$$

$$- [v_{zl}^{(1)}(x, y, z_t + z_s) + v_{zl}^{(1)}(x, y, z_t - z_s)]. \quad (35)$$

The relation between $\Delta v_{zl}(x, y, 0)$ and $\Delta l(x, y)$, similar to equation (27), reads:

$$\frac{1}{8\pi}\iint\limits_S \frac{\Delta l(x', y')}{(y - y')^2}\left[1 + \frac{x - x'}{\sqrt{(x - x')^2 + (y - y')^2}}\right] dx'\, dy' = \Delta v_{zl}(x, y, 0). \quad (36)$$

The values of $\Delta q(x, y)$, $\Delta v_{zl}(x, y, 0)$ given by equations (34) and (35) have strong singularities at the leading edge, similar to those for two-dimensional elliptic aerofoils, which are discussed in Appendix B, *see* equations (B-17) and (B-22). A modification of the values for Δq and Δv_{zl} is therefore required near the leading edge. The steep variation of Δq, Δv_{zl} occurs over a very short distance behind the leading edge, which means that the values given by equations (34), (35) have to be modified only over a short distance. One possibility for achieving such a modification is discussed in Appendix C. In view of the inaccuracies in the flow field near the leading edge implied in a second-order theory, one also may consider taking the values of Δq, Δv_{zl} computed at points away from the leading edge and extrapolating these towards the leading edge in such a manner that $\Delta q(x, y)$ has a square-root singularity similar to $q^{(1)}(x, y)$ and that $\Delta v_{zl}(x, y)$ is finite at the leading edge. Such a crude modification seems reasonable when we deal with a three-dimensional wing, where we want to economize on the computing effort and compute velocity components at only a modest number of chordwise stations.

We may note here that, when we determine Δq and Δv_{zl} from equations (34) and (35), $l^{(1)}(x, y)$ need not be an exact solution of equation (27) but may differ by second-order terms.

When $q^{(2)}(x, y)$ and $l^{(2)}(x, y)$ (*see* equations (28) and (29)) have been determined, one can calculate the total velocity $V(x, y, z_w)$ at the surface of the wing and from this the pressure coefficient $C_p = 1 - V^2$. The velocity components of the undisturbed flow along the chord and normal to it have the values $\cos(\alpha^* + \alpha_T)$ and $\sin(\alpha^* + \alpha_T)$. The total velocity $V(x, y, z_w)$ can therefore be determined from the equation:

$$V^2(x, y, z_w) = [\cos(\alpha^* + \alpha_T) + v_x^{(1)}(x, y, z_w) + \Delta v_x(x, y, z_w)]^2 + [v_y^{(1)}(x, y, z_w) + \Delta v_y(x, y, z_w)]^2 +$$

$$+ [\sin(\alpha^* + \alpha_T) + v_z^{(1)}(x, y, z_w) + \Delta v_z(x, y, z_w)]^2. \quad (37)$$

We have mentioned that we seek to determine the velocity $V(x, y, z_w)$ to $O(\varepsilon^2)$; this means that we need the values of Δv_x, Δv_y, Δv_z only at $z = 0$, at least away from the leading edge. $\Delta v_y(x, y, z_w)$ contributes only a term of third order to $V(x, y, z_w)$, so that we need not compute Δv_y. The term $\Delta v_z(x, y, \pm 0)$ is known from equation (31). We have thus to compute only $\Delta v_x(x, y, 0)$ and can calculate $V(x, y, z_w)$ from

$$V^2(x, y, z_w) = [\cos(\alpha^* + \alpha_T) + v_x^{(1)}(x, y, z_w) + \Delta v_x(x, y, 0)]^2 + [v_y^{(1)}(x, y, z_w)]^2 + \\ + \left\{\frac{\partial z_w}{\partial x}[1 + v_x^{(1)}(x, y, z_w)] + \frac{\partial z_w}{\partial y} v_y^{(1)}(x, y, z_w)\right\}^2. \tag{38}$$

However, this expression may lead to unrealistic values near the leading edge, for example because $\Delta v_{xl}(x, y, 0)$ tends to infinity at the leading edge. To obtain finite values, one may suggest computing not only $v_x^{(1)}, v_y^{(1)}, v_z^{(1)}$ but also Δv_x and Δv_z near the leading edge at the surface and at shifted x-ordinates, as discussed in Appendices B and C.

When one has computed Δv_x, Δv_y, Δv_z at the surface and thus knows $v_x^{(2)}(x, y, z_w)$, $v_y^{(2)}(x, y, z_w)$, $v_z^{(2)}(x, y, z_w)$ then one would insert these into equation (17) to learn how accurate the solution is. If the error

$$\frac{\partial z_w}{\partial x}[1 + v_x^{(2)}(x, y, z_w)] + \frac{\partial z_w}{\partial y} v_y^{(2)}(x, y, z_w) - \alpha^* - \alpha_T - v_z^{(2)}(x, y, z_w) = \Delta(x, y) \tag{39}$$

is too large, then one can consider $\Delta(x, y)$ as a $\Delta^{(2)} v_z(x, y, \pm 0)$ and derive a modification to the source distribution and the load distribution and from these a $\Delta^{(2)} v_x$ and an improved value for $V(x, y, z_w)$. For cambered wings, one must however expect that in certain cases planar singularity distributions cannot represent the shape of the wing near the leading edge to great accuracy. When one decides whether the error $\Delta(x, y)$ is too large, then one has to remember also that for a twisted wing the boundary condition given by equation (17) is only an approximation.

We may note that the question, whether the extension of the Lighthill technique of strained coordinates into three dimensions is permissible, does not arise when we check whether a solution, derived by means of shifted x-ordinates, satisfies the boundary condition to a required accuracy. We use the technique only to reduce the number of steps in the iteration procedure. A check on the error in the boundary condition can only be made when the velocity components on the surface are obtained by computation but not when they are derived by means of Taylor series expansions.

We have mentioned that for the two-dimensional aerofoil the computing effort can be reduced by expressing the velocity components at $z \neq 0$ in terms of the velocity components at $z = 0$. Before we discuss the use of Taylor series expansions for the three-dimensional wing, we examine in the next section how to overcome the present difficulty that we do not yet have a fast method for determining a solution of equation (27) for non-regular planforms.

2.4 Solution for Uncambered Wings derived by using Results from the R.A.E. Standard Method

In the previous section, it is assumed that we have a method for determining a solution $l^{(1)}(x, y)$ which is accurate to first order. The determination of a first-order solution $l^{(1)}(x, y)$ may require a fairly large amount of computation, even though $l^{(1)}(x, y)$ need not be an 'exact' solution of equation (27) ('exact' in the sense that the numerical errors are no greater than some prescribed tolerance).

In this section, therefore we consider an iterative procedure, which aims from the start at a solution of the problem for the wing of finite thickness, where the first approximation can be determined without much effort. Such a first approximation for the wing of finite thickness may be derived by the R.A.E. standard method.[3]

To illustrate the procedure, we study the special case of an uncambered, untwisted wing of finite thickness at an angle of incidence α. For this wing the boundary condition reads:

$$\pm\frac{\partial z_t(x, y)}{\partial x}[\cos\alpha + v_{xt}(x, y, z_t) \pm v_{xl}(x, y, z_t)] \pm \frac{\partial z_t(x, y)}{\partial y}[v_{yt}(x, y, z_t) \pm v_{yl}(x, y, z_t)] \\ = \sin\alpha \pm v_{zt}(x, y, z_t) + v_{zl}(x, y, z_t). \tag{40}$$

By adding the equations for the upper and lower surface, we obtain for v_{zl} the equation

$$v_{zl}(x, y, z_t) = -\sin\alpha + \frac{\partial z_t}{\partial x} v_{xl}(x, y, z_t) + \frac{\partial z_t}{\partial y} v_{yl}(x, y, z_t). \tag{41}$$

By subtracting the equations, we obtain for v_{zt}

$$v_{zt}(x, y, z_t) = \frac{\partial z_t}{\partial x}[\cos \alpha + v_{xt}(x, y, z_t)] + \frac{\partial z_t}{\partial y} v_{yt}(x, y, z_t). \tag{42}$$

We have selected the uncambered wing because $l(x, y)$ is then independent of $q(x, y)$.

We shall consider here how to derive a load distribution which satisfies equation (41) to second-order accuracy. For the determination of the second-order source distribution one can use the procedure of Section 2.3.

We intend to solve equation (41) by an iterative procedure:

$$l^{(n+1)}(x, y) = l^{(n)}(x, y) + \Delta l^{(n)}(x, y), \tag{43}$$

where $l^{(1)}(x, y)$ is to be derived by the standard method.[3] (The load distribution of the standard method would be modified close to the leading edge to produce a square-root singularity since this is assumed in the computer program of Ref. 9.) $\Delta l^{(n)}(x, y)$ is an approximation to the load distribution which corresponds to a distribution of upwash over the wing planform, $\Delta^{(n)} v_{zl}(x, y, z_t)$, given by

$$\Delta^{(n)} v_{zl}(x, y, z_t) = -\sin \alpha - v_{zl}(x, y, z_t; l^{(n)}) + \frac{\partial z_t}{\partial x} v_{xl}(x, y, z_t; l^{(n)}) + \frac{\partial z_t}{\partial y} v_{yl}(x, y, z_t; l^{(n)}), \tag{44}$$

where $v_{zl}(x, y, z_t; l^{(n)})$, v_{xl}, v_{yl} are produced by the known load distribution $l^{(n)}(x, y)$.

Within the iterative procedure, we are free to choose, whether the right-hand side of equation (44) is interpreted as an upwash at $z = 0$ or at $z = z_t$. To determine from $\Delta^{(n)} v_{zl}$ an approximate $\Delta l^{(n)}(x, y)$, one may try to use again the standard method. Having determined from $\Delta^{(1)} v_{zl}$ an approximate $\Delta l^{(1)}$ and hence $l^{(2)}$, one would determine $\Delta^{(2)} v_{zl}$ and continue until $|\Delta^{(n)} v_{zl}(x, y, z_t)|$ is smaller than the required accuracy.

In certain regions of the wing, it is possible that $\Delta^{(n)} v_{zl}(x, y, z_t)$ varies fairly rapidly both in the x- and y-direction; this may imply that the standard method is not suitable for deriving a good approximation to $\Delta l^{(n)}(x, y)$. If this is the case, then one may consider deriving $l^{(1)}(x, y)$ and $\Delta l^{(n)}(x, y)$ by a different method as for example a vortex lattice method. It is possible that $\Delta^{(n)} v_{zl}$ varies still fairly rapidly for example near the centre section of a swept wing, which means that a fairly large number of control points and therefore a large amount of computation could be required unless a somewhat reduced accuracy is accepted near the centre section.

It is not necessary to recompute v_{xl} and v_{yl} in equation (44) for every change $\Delta l^{(n)}$, but sufficient to use v_{xl}, v_{yl} induced by $l^{(1)}(x, y)$, unless it was found that $l^{(1)}$ is a poor first approximation. Within second-order accuracy, we may, away from the leading edge, use $v_{xl}(x, y, 0; l^{(1)})$ and $v_{yl}(x, y, 0; l^{(1)})$ which can be derived from $l^{(1)}(x, y)$ with far less computing effort than is required for computing $v_{xl}(x, y, z_t)$ and $v_{yl}(x, y, z_t)$. With $l(x, y)$, $v_{xl}(x, y, 0)$ is directly known, because

$$v_{xl}(x, y, 0) = \tfrac{1}{4} l(x, y). \tag{45}$$

Since

$$\frac{\partial v_y(x, y, 0)}{\partial x} = \frac{\partial v_x(x, y, 0)}{\partial y}$$

and away from the centre section

$$v_{yl}(x_L(y), y, 0) = -\tan \varphi_L(y) v_{xl}(x_L(y), y, 0), \tag{46}$$

we can derive $v_{yl}(x, y, 0)$ from the equation

$$v_{yl}(x, y, 0) = -\frac{\tan \varphi_L(y)}{4} l(x_L(y), y) + \frac{1}{4} \int_{x_L(y)}^{x} \frac{\partial}{\partial y} l(x', y)\, dx'. \tag{47}$$

Note however that, if (as is frequently the case) $l(x, y) \to \infty$ as $x \to x_L(y)$, it is necessary to use the alternative form of equation (47),

$$v_{yl}(x, y, 0) = \frac{1}{4}\frac{\partial}{\partial y}\int_{x_L(y)}^{x} l(x', y)\, dx'. \tag{47a}$$

Near the leading edge, we would however compute v_{xl}, v_{yl}, v_{zl} at the surface and at shifted x-ordinates.

We have given in Appendix B analytic expressions for the velocity components induced in two-dimensional flow by the flat-plate load distribution, $l(x) = 4\alpha\sqrt{(c - x)/x}$, at any x and z. From equations (B-6), (B-7) we have computed values of $\Delta^{(1)}v_{zl}(x, z_t)$, as defined by equation (44), for a 10 per cent thick R.A.E. 101 and a 10 per cent thick elliptic section; the results are plotted in Fig. 2. We note that $|\Delta^{(1)}v_{zl}(x, z_t)|/\alpha$ is for most of the chord of the magnitude 0·1 to 0·2. From these fairly large values one may conclude for the three-dimensional wing that it may not be advisable to spend a large amount of computing effort to derive a fairly accurate solution of the thin-wing equation, equation (27).

It may also be advisable to start the iteration with an approximate load distribution for the thick wing. For two-dimensional flow past a wing with finite thickness, the $\Delta^{(1)}v_{zl}(x, z_t)$ term is taken into account in a second-order theory by approximating the load distribution by

$$l(x) = 4\alpha\sqrt{\frac{c - x}{x}}[1 + S^{(3)}(x)], \tag{48}$$

with

$$S^{(3)}(x) = \frac{1}{\pi}\int_0^c \left[\frac{dz_t(x')}{dx'} - \frac{z_t(x')}{2x'(c - x')}\right]\frac{dx'}{x - x'}. \tag{49}$$

For the 'sheared wing' region of a finite wing the factor $[1 + S^{(3)}(x, y)/\cos\varphi]$ is included in the standard method; near centre and tip the factor $[1 + S^{(3)}(x, y)/\cos\varphi^*(y)]$ is included to take, at least approximately, account of this interaction between thickness and lift. ($\varphi^*(y) = (1 - |K_2(y)|)\varphi(y)$ is an interpolation function between the value $\varphi^*(y = 0) = 0$ and the local sweep $\varphi(y)$ of the mid-chord line; for the definition of the function $K_2(y)$ we refer the reader to Ref. 3.) The introduction of the factor $[1 + S^{(3)}/\cos\varphi^*]$ is of course not restricted to the load distribution of the standard method, it can be applied to any other solution $l^{(1)}(x, y)$ of the thin-wing problem, equation (27).

To start the iteration with an approximate load distribution for the thick wing instead of an approximate load distribution for the thin wing need of course not improve the convergence of the procedure, but it may reduce the number of steps in the iterative procedure and, as a by-product, one would learn something about the accuracy of the R.A.E. standard method.

Consider the possibility that the standard method produces a load distribution $l^{(1)}(x, y)$ which is wrong to a fairly large degree, so that the values of $|\Delta^{(1)}v_{zl}(x, y, z_t)|$ from equation (44) were on average much larger than the values of $|\Delta v_{zl}(x, y, 0)|$ from equation (35), when $l^{(1)}(x, y)$ is an exact solution of equation (27). This need not imply that we must abandon the suggested procedure. The crucial point would be to find out whether the iterative procedure, where also $\Delta l^{(n)}(x, y)$ is determined from $\Delta^{(n)}v_{zl}(x, y, z_t)$ by the standard method, does converge, or whether another method for deriving $\Delta l^{(n)}$ from $\Delta^{(n)}v_{zl}$ can be devised, which does provide a convergent procedure. We have already stated that it is not necessary to use the standard method to determine a first approximation $l^{(1)}$. If another relatively fast method is available for deriving an approximate solution of equation (27), then this also can be used for determining $l^{(1)}$ and $\Delta l^{(n)}$.

One may be inclined to judge the accuracy of a load distribution $l^{(n)}(x, y)$ by examining the maximum value of $|\Delta^{(n)}v_{zl}(x, y, z_t)|$. We must however keep in mind that we can tolerate a considerably reduced accuracy close to the leading edge, without noticeably reducing the accuracy of the pressure distribution further downstream.

No practical examples are considered in this Report. However, since the work reported here was finished, Sells[17] has implemented the suggested procedure and has determined the pressure distribution on uncambered untwisted wings at an angle of incidence; he has found that 'the iteration cycle converges to acceptable accuracy after 2 steps'.

When solving the thickness problem by the procedure described in Section 2.3, it may also be advantageous to use as a first approximation, $q^{(1)*}(x, y)$ to $q(x, y)$, instead of the source distribution from first-order theory $q^{(1)}(x, y) = 2\,\partial z_t(x, y)/\partial x$, a source distribution which takes already some account of the second-order terms. For the 'sheared wing' region of an uncambered wing, such a source distribution is given by the function (*see* equations (A-6), (A-10) of Appendix A):

$$q^{(1)*}(x, y) = 2\frac{\partial z_t}{\partial x} + \frac{2}{\cos \varphi(x, y)} \frac{\partial}{\partial x}[z_t S^{(1)}(x, y)], \tag{50}$$

where

$$S^{(1)}(x, y) = \frac{1}{\pi}\int_{x_L(y)}^{x_T(y)} \frac{\partial z_t(x', y)}{\partial x'} \frac{dx'}{x - x'} \tag{51}$$

and $\varphi(x, y)$ is the local angle of sweep. We mention in Appendix E how equation (59) can be used, in conjunction with the approximate values for $v_{xt}^{(1)}$ and $v_{yt}^{(1)}$ given by the standard method, to derive a distribution $q^{(1)*}(x, y)$ for any spanwise station. If $q^{(1)*}(x, y)$ were reasonably accurate, then the values of $\Delta q(x, y)$ derived from equation (34) with $q^{(1)*}(x, y)$ would be smaller than those derived with $q^{(1)}(x, y) = 2\,\partial z_t/\partial x$. As a consequence, it is to be expected that the remaining error in the boundary condition, $\Delta(x, y)$ of equation (39), would also be reduced. With a relatively small amount of additional computation, one would obtain a more accurate pressure distribution.

2.5 Simplification by means of Taylor Series Expansions

We have already mentioned that for two-dimensional aerofoils the amount of computation can be reduced if the velocity components induced at $z \neq 0$ are expressed in terms of the velocity components induced at $z = 0$; this is done by expanding the velocity components in powers of ε.

Such expansions are possible also on three-dimensional wings, except at the centre section of swept wings and near wing tips. The velocity components induced at $z = z_w$ are approximated by the first two terms of their Taylor series expansions with respect to the plane $z = 0$:

$$v_x(x, y, z_w) \approx v_x(x, y, 0) + z_w\left(\frac{\partial v_x(x, y, z)}{\partial z}\right)_{z=0}, \tag{52}$$

$$v_y(x, y, z_w) \approx v_y(x, y, 0) + z_w\left(\frac{\partial v_y(x, y, z)}{\partial z}\right)_{z=0}, \tag{53}$$

$$v_z(x, y, z_w) \approx v_z(x, y, 0) + z_w\left(\frac{\partial v_z(x, y, z)}{\partial z}\right)_{z=0}. \tag{54}$$

By making use of the equations of irrotationality and continuity, the derivatives can be expressed in terms of the velocity components in $z = 0$:

$$\left(\frac{\partial v_x(x, y, z)}{\partial z}\right)_{z=0} = \frac{\partial v_z(x, y, 0)}{\partial x}, \tag{55}$$

$$\left(\frac{\partial v_y(x, y, z)}{\partial z}\right)_{z=0} = \frac{\partial v_z(x, y, 0)}{\partial y}, \tag{56}$$

$$\left(\frac{\partial v_z(x, y, z)}{\partial z}\right)_{z=0} = -\frac{\partial v_x(x, y, 0)}{\partial x} - \frac{\partial v_y(x, y, 0)}{\partial y}. \tag{57}$$

We have mentioned in Section 2.3 that we cannot make use of equations (54), (57) near the centre section of swept wings with for example chordwise thickness distributions of the same type along the span; this is due to the fact that, when $(\partial z_t(x, y)/\partial y)_{y=0} \neq 0$, the spanwise derivative $\partial v_{yt}^{(1)}(x, y, 0)/\partial y$ is logarithmically infinite at $y = 0$. A similar behaviour can occur at the tip for wings with finite tip chord. We consider those regions of the wing where $\partial v_y^{(1)}(x, y, 0)/\partial y$ is not large, so that $v_z^{(1)}(x, y, z)$ can be approximated to second-order accuracy by equations (54), (57).

By introducing equations (52) to (57) into equation (34) and ignoring all terms of an order higher than ε^2, we obtain for $\Delta q(x, y)$ the second-order approximation

$$\begin{aligned}\frac{\Delta q(x, y)}{2} &= \frac{\partial z_t}{\partial x}[1 + v_{xt}^{(1)}(x, y, 0)] + \frac{\partial z_s}{\partial x} v_{xl}^{(1)}(x, y, 0) + \frac{\partial z_t}{\partial y} v_{yt}^{(1)}(x, y, 0) + \frac{\partial z_s}{\partial y} v_{yl}^{(1)}(x, y, 0) - \\ &\quad - \left\{ v_{zt}^{(1)}(x, y, 0) - z_t\left[\frac{\partial v_{xt}^{(1)}(x, y, 0)}{\partial x} + \frac{\partial v_{yt}^{(1)}(x, y, 0)}{\partial y}\right]\right\} + z_s\left[\frac{\partial v_{xl}^{(1)}(x, y, 0)}{\partial x} + \frac{\partial v_{yl}^{(1)}(x, y, 0)}{\partial y}\right] \\ &= \frac{\partial z_t(x, y)}{\partial x} - v_{zt}^{(1)}(x, y, 0) + \frac{\partial}{\partial x}[z_t(x, y) v_{xt}^{(1)}(x, y, 0) + z_s(x, y) v_{xl}^{(1)}(x, y, 0)] + \\ &\quad + \frac{\partial}{\partial y}[z_t(x, y) v_{yt}^{(1)}(x, y, 0) + z_s(x, y) v_{yl}^{(1)}(x, y, 0)]. \end{aligned} \tag{58}$$

When the source distribution $q^{(1)}(x, y)$ satisfies equation (24), then we obtain for $\Delta q(x, y)$ the relation:

$$\frac{\Delta q(x, y)}{2} = \frac{\partial}{\partial x}[z_t(x, y) v_{xt}^{(1)}(x, y, 0) + z_s(x, y) v_{xl}^{(1)}(x, y, 0)] + \frac{\partial}{\partial y}[z_t(x, y) v_{yt}^{(1)}(x, y, 0) + z_s(x, y) v_{yl}^{(1)}(x, y, 0)]. \tag{59}$$

If $l^{(1)}(x, y)$ satisfies equations (26), (27) to at least order ε^2, then we obtain from equation (35) for $\Delta v_{zl}(x, y, 0)$ the relation

$$\Delta v_{zl}(x, y, 0) = \frac{\partial}{\partial x}[z_s(x, y) v_{xt}^{(1)}(x, y, 0) + z_t(x, y) v_{xl}^{(1)}(x, y, 0)] + \frac{\partial}{\partial y}[z_s(x, y) v_{yt}^{(1)}(x, y, 0) + z_t(x, y) v_{yl}^{(1)}(x, y, 0)]. \tag{60}$$

If $l^{(1)}(x, y)$ does not satisfy equations (26), (27) to at least order ε^2, then we obtain instead of equation (60) the following one

$$\begin{aligned}\Delta v_{zl}(x, y, 0) &= \frac{\partial z_s(x, y)}{\partial x} - \alpha^* - \alpha_T(y) - v_{zl}^{(1)}(x, y, 0) + \frac{\partial}{\partial x}[z_s(x, y) v_{xt}^{(1)}(x, y, 0) + z_t(x, y) v_{xl}^{(1)}(x, y, 0)] + \\ &\quad + \frac{\partial}{\partial y}[z_s(x, y) v_{yt}^{(1)}(x, y, 0) + z_t(x, y) v_{yl}^{(1)}(x, y, 0)]. \end{aligned} \tag{61}$$

We note that the use of equations (58) to (61) implies that the boundary condition given by equation (31) has been approximated by

$$\frac{\partial z_w}{\partial x}[1 + v_x^{(1)}(x, y, \pm 0)] + \frac{\partial z_w}{\partial y} v_y^{(1)}(x, y, \pm 0) = \alpha^* + \alpha_T + v_z^{(1)}(x, y, z_w) + \Delta v_z(x, y, \pm 0). \tag{62}$$

For wings with the same type of camber but different maximum camber, the use of equation (60) would imply that $\Delta v_{zl}(x, y, 0)$ and with it $\Delta l(x, y)$ and $l^{(2)}(x, y)$ change linearly with the maximum camber. This is not so when equation (35) is applied. For the symmetrical wing at an angle of incidence, α^*, both equation (35) and equation (60) lead to a linear variation of $l(x, y)$ with α^*.

Away from the centre section, the velocity components $v_{xt}^{(1)}(x, y, 0)$, $v_{yt}^{(1)}(x, y, 0)$ are finite, except near the trailing edge. For wings with sharp trailing edge and finite trailing-edge angle, $q^{(1)}(x, y)$ is finite at the trailing edge; therefore, $v_{xt}^{(1)}(x, y, 0)$, $v_{yt}^{(1)}(x, y, 0)$ are logarithmically infinite at the trailing edge. We can therefore modify $v_{xt}^{(1)}(x, y, 0)$, $v_{yt}^{(1)}(x, y, 0)$ in equations (59) and (60) over a very small neighbourhood of the trailing edge, without noticeably altering the values of Δv_x, Δv_y away from the trailing edge.

Away from the centre section, the velocity components $v_{xl}^{(1)}(x, y, 0)$, $v_{yl}^{(1)}(x, y, 0)$ behave near the leading edge like $1/\sqrt{\xi}$, with $\xi = (x - x_L(y))/c(y)$. We consider wings for which $z_t(x, y)$ behaves near the leading edge like $a(y)\sqrt{\xi} + \ldots$ and $z_s(x, y)$ like $b(y)\xi + \ldots$. Therefore, $\Delta q(x, y)$ of equation (59) has the same behaviour near the leading edge as $q^{(1)}(x, y)$, and $\Delta v_{zl}(x, y, 0)$ of equation (60) is finite.

Comparing equation (34) with equations (58), (59) and equation (35) with equations (60), (61), we note a large difference in the computing effort required, in particular for a cambered wing. $v_{xl}^{(1)}(x, y, 0)$, $v_{yl}^{(1)}(x, y, 0)$ are related to $l^{(1)}(x, y)$ by equations (45) and (47); the computation of $v_{xl}^{(1)}(x, y, 0)$, $v_y^{(1)}(x, y, 0)$ therefore requires far less effort than the computation of

$$v_{xl}^{(1)}(x, y, z_t + z_s), \qquad v_{xl}^{(1)}(x, y, z_t - z_s),$$

$$v_{yl}^{(1)}(x, y, z_t + z_s), \qquad v_{yl}^{(1)}(x, y, z_t - z_s).$$

The computation of v_{xt}, v_{yt} at $z = 0$ requires a similar effort as the computation for $z = z_w$; but for a cambered wing, equations (34), (35) require values of $v_{xt}^{(1)}(x, y, z_t + z_s)$ and $v_{xt}^{(1)}(x, y, z_t - z_s)$ whilst equations (58) to (61) require the value of only $v_{xt}^{(1)}(x, y, 0)$.

However, at and near the centre section and near the tip, a Taylor series expansion must not be used for $v_{zt}^{(1)}$ and it seems also not advisable to use it for v_{zl} (at least not near the apex). Therefore a hybrid method where equations (34), (35) are used near the centre and the tips and equations (58) to (61) away from centre and tip may be the most economical with respect to computing time. On the other hand, the hybrid method entails a computer program of larger complexity than the method which does not use Taylor series expansions. It seems therefore advisable to examine first for uncambered wings at an angle of incidence whether the iterative procedure suggested in Section 2.4 is convergent or what method can be used to derive a convergent procedure, when Taylor series expansions are not used. We have already noted that the use of equations (34), (35) does not require that $q^{(1)}(x, y)$ and $l^{(1)}(x, y)$ are the exact solutions of equations (24) to (27). If $l^{(1)}(x, y)$ is not correct to first order, then an iterative procedure, as described in Section 2.4, can be used with equations (35) and (61).

When values of the source and load distributions, $q^{(2)}(x, y)$, $l^{(2)}(x, y)$, correct to second order, have been determined, then we can determine the velocity components $v_x^{(2)}(x, y, z_w)$, $v_y^{(2)}(x, y, z_w)$, which $q^{(2)}$ and $l^{(2)}$ induce at the wing surface. By means of equations (52) to (56) we can approximate $v_x^{(2)}(x, y, z_w)$, $v_y^{(2)}(x, y, z_w)$ to second-order accuracy by

$$v_x^{(2)}(x, y, z_w) = v_x^{(2)}(x, y, \pm 0) + z_w \frac{\partial v_z^{(2)}(x, y, 0)}{\partial x}.$$

To obtain second-order accuracy of $v_x^{(2)}(x, y, z_w)$, the derivative $\partial v_z^{(2)}(x, y, 0)/\partial x$ has to be correct to only first order and can therefore be replaced by $\partial v_z^{(1)}(x, y, 0)/\partial x$, where $v_z^{(1)}(x, y, 0)$ is derived from equation (23). Therefore,

$$v_x^{(2)}(x, y, z_w) = v_x^{(2)}(x, y, \pm 0) + z_w \frac{\partial^2 z_w(x, y)}{\partial x^2}. \tag{63}$$

Similarly,

$$v_y^{(2)}(x, y, z_w) = v_y^{(2)}(x, y, \pm 0) + z_w \frac{\partial^2 z_w(x, y)}{\partial x \, \partial y}. \tag{64}$$

We note that the Taylor series expansions for v_x and v_y can be used also at $y = 0$. We note further that equation (64) produces a discontinuity in $v_y^{(2)}(x, y, z_w)$ at $y = 0$.

From the velocity components $v_x^{(2)}(x, y, z_w)$, $v_y^{(2)}(x, y, z_w)$ we can compute the total velocity at the surface of the wing $V^{(2)}(x, y, z_w)$ to second-order accuracy. Since $v_z^{(2)}(x, y, z_w)$ is related to $v_x^{(2)}$ and $v_y^{(2)}$ by the approximate boundary condition, we obtain

$$[V^{(2)}(x, y, z_w)]^2 = [\cos(\alpha^* + \alpha_T) + v_x^{(2)}(x, y, z_w)]^2 + [v_y^{(2)}(x, y, z_w)]^2 + \\ + \left\{\frac{\partial z_w}{\partial x}[1 + v_x^{(2)}(x, y, z_w)] + \frac{\partial z_w}{\partial y} v_y^{(2)}(x, y, z_w)\right\}^2. \tag{65}$$

The values of $v_x^{(2)}(x, y, z_w)$, $v_y^{(2)}(x, y, z_w)$ derived from the Taylor series expansions, equations (63), (64), have strong singularities at the leading edge; they behave like $1/\xi$. When these values of $v_x^{(2)}$, $v_y^{(2)}$ are inserted into equation (65), then we obtain values for $V^{(2)}$ which are unrealistic near the leading edge.

Making Taylor series expansions implies that one assumes that the perturbation velocities and their derivatives are small. We know however that this assumption does not hold near the leading and trailing edges. The solution given by equations (63) to (65) is therefore not valid near the edges; in particular we have to modify the expression given by equation (65) for $V^{(2)}(x, y, z_w)$ such that we obtain everywhere a finite value and retain the second-order accuracy away from the leading edge. Various rules have been derived[5 to 7] for the modification of $V^{(2)}(x, z_w)$ for the two-dimensional wing. They are summarised in Appendix A, equations (A-20) to (A-22). We would like to have a similar rule for the three-dimensional wing.

For this purpose, we derive first a uniformly valid equation for the second-order theory for the infinite sheared wing. We choose equation (A.22) of Appendix A for the two-dimensional unswept aerofoil. The corresponding equation reads

$$V^2(x, z_w) = \cos^2\alpha \sin^2\varphi + \frac{\left\{\cos\alpha\cos\varphi + \dfrac{v_x^{(2)}(x, 0)}{\cos\varphi} + \cos\varphi\dfrac{\partial}{\partial x}\left(\dfrac{z_w}{\cos\varphi}\dfrac{\partial z_w/\cos\varphi}{\partial x}\right)\right\}^2}{1 + \left(\dfrac{\partial z_w/\cos\varphi}{\partial x}\right)^2}. \quad (66)$$

We denote by V the uniformly valid value for the total velocity on the surface. Equation (66) can be written in the form

$$V^2(x, y, z_w) = \frac{\left[\cos\alpha + v_x^{(2)}(x, 0) + \dfrac{\partial}{\partial x}\left(z_w\dfrac{\partial z_w}{\partial x}\right)\right]^2 + \left[v_y^{(2)}(x, 0) + \dfrac{\partial}{\partial y}\left(z_w\dfrac{\partial z_w}{\partial x}\right)\right]^2 + \tan^2\varphi\cos^2\alpha\left(\dfrac{\partial z_w}{\partial x}\right)^2}{1 + \left(\dfrac{\partial z_w}{\partial x}\right)^2 + \left(\dfrac{\partial z_w}{\partial y}\right)^2}. \quad (67)$$

To obtain a formula for the total velocity $V(x, y, z_w)$ at any spanwise station of an arbitrary wing we follow a procedure suggested by Lock[10] and consider the velocity distribution at the surface of an ellipsoid. It is shown in Appendix D that a uniformly valid formula for the velocity $V(x, y, z_w)$ in terms of the velocity components from second-order theory can be obtained from the following equation:

$$\begin{aligned} V^2(x, y, z_w)\left[1 + \left(\frac{\partial z_w}{\partial x}\right)^2 + \left(\frac{\partial z_w}{\partial y}\right)^2\right] &= \left[\cos\alpha + v_x^{(2)}(x, y, 0) + \frac{\partial}{\partial x}\left(z_w\frac{\partial z_w}{\partial x}\right)\right]^2 + \\ &+ \left[v_y^{(2)}(x, y, 0) + \frac{\partial}{\partial y}\left(z_w\frac{\partial z_w}{\partial x}\right)\right]^2 + \\ &+ \left\{[\cos\alpha + v_{xt}^{(2)}(x, y, 0)]\frac{\partial z_w}{\partial y} - v_{yt}^{(2)}(x, y, 0)\frac{\partial z_w}{\partial x}\right\}^2. \end{aligned} \quad (68)$$

For the two-dimensional sheared wing

$$v_{yt}^{(2)}(x, y, 0) = -\tan\varphi v_{xt}^{(2)}(x, y, 0)$$

and

$$\frac{\partial z_w}{\partial y} = -\tan\varphi\frac{\partial z_w}{\partial x},$$

so that the last term in equation (68) is equal to $(\cos\alpha\tan\varphi\,\partial z_w/\partial x)^2$, which means equations (67) and (68) are identical.

It is proposed to use equation (68) also for general wing shapes to derive a uniformly valid formula for the velocity at the wing surface in terms of the second-order velocity components $v_x^{(2)}(x, y, 0)$, $v_y^{(2)}(x, y, 0)$. We stress here that in the treatment described it is essential that the velocity components $v_x^{(2)}(x, y, z_w)$, $v_y^{(2)}(x, y, z_w)$ at the surface must be determined from the components in the plane $z = 0$ by Taylor series expansions, equations (63), (64) (and not by direct computation on $z = z_w$); and the strength of the singularity distributions must be derived either by determining the velocity component $v_z^{(1)}(x, y, z_w)$ also by a Taylor series expansion (equations (58) to (61)) or by modifying the values of $\Delta q(x, y)$ and $\Delta v_{zl}(x, y, 0)$ given by equations (34), (35) in such a way that they behave like $q^{(1)}(x, y)$ and $v_{zl}^{(1)}(x, y, 0)$ near the leading edge.

Further, it is proposed to use equation (68) for wings with general cambered sections, in which case the wing slopes $\partial z_w/\partial x$ and $\partial z_w/\partial y$ should take their separate local values, $\partial(\pm z_t + z_s)/\partial x$, $\partial(\pm z_t + z_s)/\partial y$ on the upper and lower surface respectively. It is not of course possible to justify this procedure with the same degree of rigour as with an uncambered wing; and the same applies whenever the shape of the leading edge differs significantly from elliptical; however it is certainly important that the same definitions of $\partial z_w/\partial x$, $\partial z_w/\partial y$ should be used consistently in both the right- and left-hand sides of equation (68), otherwise the second-order accuracy away from the leading edge will not be maintained.

Lock[10] makes further use of the exact theory for the flow about an ellipsoid and proposes modified expressions for the components of surface velocity. Uniformly valid values of the various velocity components will be required for calculating the effect of the boundary layer.

The computing effort may be reduced by substituting in equation (68) for $v_y^{(2)}(x, y, 0)$ the first-order term $v_y^{(1)}(x, y, 0)$, so that values of $\Delta v_y(x, y, \pm 0)$ induced by $\Delta q(x, y)$ and $\Delta l(x, y)$ need not be computed; this would change the value of $V(x, y, z_w)$ given by equation (68) only by third-order terms.

The term $(\partial z_w/\partial x)^2 + (\partial z_w/\partial y)^2$ on the left-hand side of equation (68) is of second order and can thus be modified away from the leading edge, if the right-hand side is also consistently modified. At the leading edge $\partial z_w/\partial y = -\tan \varphi(y)\, \partial z_w/\partial x$, where $\varphi(y)$ is the local sweep of the leading edge. Without loss of accuracy, we may therefore modify equation (68). Except for wings with large leading-edge sweep or highly tapered wings, we propose to use instead of equation (68) the formula

$$V^2(x, y, z_w) = \frac{\left(\left[\cos(\alpha^* + \alpha_T) + v_x^{(2)}(x, y, 0) + \dfrac{\partial}{\partial x}\left(z_w \dfrac{\partial z_w}{\partial x}\right)\right]^2 + [v_y^{(1)}(x, y, 0)]^2 + \{[1 + v_{xt}^{(1)}(x, y, 0)]\tan\varphi + v_{yt}^{(1)}(x, y, 0)\}^2\left(\dfrac{\partial z_w}{\partial x}\right)^2\right)}{1 + \left(\dfrac{\partial z_w/\partial x}{\cos\varphi}\right)^2}. \tag{69}$$

Equation (69) is similar to the one used in the R.A.E. standard method. The major difference is that instead of the term $v_x^{(2)}(x, y, 0) + \partial/\partial x[z_w(\partial z_w/\partial x)]$ we use in the standard method an approximate value for the first-order term $v_x^{(1)}(x, y, 0)$. (The formula of the standard method[3] contains one second-order term concerning the interference between the wing thickness and the lift due to angle of incidence. Since this term is as yet unknown for the centre and tip region of a swept wing, only an assumed term consistent with the term away from centre and tip has been introduced. It is possible to introduce all second-order terms for the sheared wing in a similar consistent manner. One such possible modification of the basic formula is given in Appendix E.) In the standard method only an approximate value for the first-order term $v_y^{(1)}(x, y, 0)$ is used. Some further differences close to the leading edge are discussed in Appendix F.

For the neighbourhood of the apex, the use of Taylor series expansions for $v_x^{(1)}(x, y, z_w)$, $v_y^{(1)}(x, y, z_w)$ does not produce any reduction in computing effort, since we have to compute $v_x^{(1)}(x, y, z_w)$, $v_y^{(1)}(x, y, z_w)$ at the surface when we derive Δq and Δv_{zl} from equations (34), (35). In view of the unknown accuracy of equation (69) near the apex, it may be advantageous to compute the velocity components induced by Δq and Δl also at the surface and at shifted x-values. As mentioned in Section 2.3, we can then determine the error in the boundary condition, $\Delta(x, y)$ of equation (39), and, if required, improve the singularity distributions and the pressure distribution.

The assumption of small perturbation velocities does not hold close to the trailing edge either. As mentioned above, $v_{xt}^{(2)}(x, y, 0)$ and $v_{yt}^{(2)}(x, y, 0)$ are logarithmically infinite at the trailing edge. The velocity V computed by equation (69) is therefore not correct near the trailing edge. This is unimportant in practice since viscosity alters the inviscid pressure distribution near the trailing edge decisively, even for large Reynolds number.

The suggested procedure also does not apply to the tip edge, because for example, for a wing with a tip of finite chord and a square cut edge, the source distribution must not extend right to the tip edge. We have not yet examined how to derive a representative singularity distribution because viscous effects are large near the tip of a lifting wing, and the pressure distribution near the tip of a non-lifting wing is of less importance than the pressure distribution over the inboard part of the wing.

It has been stated in Section 2.2 that, for wing shapes where the values of $\psi(x, y)$ near the centre section $y = 0$ are of first-order magnitude, we ought to depart from using a single planar singularity distribution and derive more accurate values for the perturbation velocity at a point x, y by placing the singularities in the two half-planes $z = |y| \tan \psi(x, |y|)$.

We could again derive the strength of the singularity distributions in two steps, equations (28), (29), where in the first-order solution the non-zero values of ψ are ignored, which means we can use the source and load distributions of equations (25) to (27). These singularity distributions $q^{(1)}(x, y)$, $l^{(1)}(x, y)$ are then placed in the two planes $z = |y| \tan \psi(x, |y|)$ to compute the velocity components $v_x^{(1)}$, $v_y^{(1)}$, $v_z^{(1)}$ at the surface of the wing, namely at $z = |y| \tan \psi \pm z_t + z_s$.

The approximate boundary condition is still given by equation (31), if z_w denotes $|y| \tan \psi \pm z_t + z_s$. The velocity components Δv_{zt} and Δv_{zl}, induced by Δq and Δl, are of second order and may therefore be taken in the plane $z = 0$, which means they may be cancelled by a source distribution Δq and a load distribution Δl in $z = 0$. Values for $\Delta q(x, y) = 2\Delta v_{zt}(x, y, 0)$ and for $\Delta v_{zl}(x, y, 0)$ can again be obtained by subtracting and adding the two equations which describe the boundary condition on the upper and lower surfaces of the wing. These equations read

$$\left[\pm\frac{\partial z_t}{\partial x} + \frac{\partial z_s}{\partial x}\right][1 + v_{xt}^{(1)}(x, y, |y| \tan \psi \pm z_t + z_s) + v_{xl}^{(1)}(x, y, |y| \tan \psi \pm z_t + z_s)] +$$
$$+ \left[\pm\frac{\partial z_t}{\partial y} + \frac{\partial z_s}{\partial y}\right][v_{yt}^{(1)}(x, y, |y| \tan \psi \pm z_t + z_s) + v_{yl}^{(1)}(x, y, |y| \tan \psi \pm z_t + z_s)]$$
$$= \alpha^* + \alpha_T + v_{zt}^{(1)}(x, y, |y| \tan \psi \pm z_t + z_s) + v_{zl}^{(1)}(x, y, |y| \tan \psi \pm z_t + z_s) \pm \Delta v_{zt}(x, y, 0) + \Delta v_{zl}(x, y, 0). \tag{70}$$

The equations for Δq and Δv_{zl} differ somewhat from equations (34), (35), because the properties of symmetry and antisymmetry which apply to the velocity components for $\psi = 0$ are lost for $\psi \neq 0$.

The values of the second-order term Δv_x in equation (38) can be derived, to second-order accuracy, from the source and load distributions Δq, Δl placed either in $z = 0$ or in $z = |y| \tan \psi$.

For a cambered wing with $\psi(x, y = 0)$ of $O(\varepsilon)$, we would thus modify the hybrid method suggested above, in that we derive first-order singularity distributions $q^{(1)}$, $l^{(1)}$ in the same way as for $\psi(x, y = 0) = 0$, but compute the values of $v_x^{(1)}$ and $v_z^{(1)}$ in equations (70) and (38) from the singularity distributions $q^{(1)}$, $l^{(1)}$ placed in $z = |y| \tan \psi$. Since $v_y^{(1)}$ is required only to first-order accuracy, it can be computed from the singularity distributions in $z = 0$ or in $z = |y| \tan \psi$. For the outer part of the wing, the procedure is the same as before.

3. The Design of Wings with Given Properties

3.1 Wings with Given Load Distribution

We consider two types of design problems. For both, the planform and the thickness distribution are given. In the first case, we consider the task of deriving the shape of a wing for which the load distribution $l(x, y)$ is specified. Again, we mean by $l(x, y)$ the strength of the 'lifting singularities' and not the true load distribution $-\Delta C_p(x, y)$. In the second case, the pressure distribution on the upper surface is given.

We intend to determine the mean camber of the wing, i.e. the spanwise distribution of the twist $\alpha_T(y)$ (α^* being included in α_T) and the camber shape $z_s(x, y)$ such that the boundary condition at the surface of the wing is satisfied to second-order accuracy. We solve the problem in two steps and determine $z_s(x, y)$ as the sum of the camber shape from a first-order theory $z_s^{(1)}(x, y)$ and a second-order correction term $\Delta z(x, y)$:

$$z_s(x, y) = z_s^{(1)}(x, y) + \Delta z_s(x, y) \tag{71}$$

and similarly

$$\alpha_T(y) = \alpha_T^{(1)}(y) + \Delta\alpha_T(y). \tag{72}$$

A first-order form of the boundary condition reads:

$$\frac{\partial z_s^{(1)}(x, y)}{\partial x} - \alpha_T^{(1)}(y) = v_{zl}(x, y, z_t). \tag{73}$$

We choose this form of the boundary condition (instead of $v_{zl}(x, y, z = 0)$ on the right-hand side of equation (73)) because some load distributions, as used in the design of swept wings with straight isobars, produce at the centre section, in the plane $z = 0$, infinite values for the downwash. $v_{zl}(x, y, z_t)$ is the downwash induced by the given load distribution $l(x, y)$. When the load distribution $l(x, y)$ behaves near the leading edge like $c/\sqrt{x - x_L(y)}$, then the downwash $v_{zl}(x, y, z_t(x, y))$ tends to infinity when x tends to $x_L(y)$. To derive a finite value for $v_{zl}(x_L(y), y)$ we may compute v_{zl} at shifted x-values or extrapolate the values computed away from the leading edge.

Since z_s is zero at the leading and trailing edges, we obtain the twist $\alpha_T^{(1)}$ from

$$\alpha_T^{(1)}(y)c(y) = -\int_{x_L(y)}^{x_T(y)} v_{zl}(x, y, z_t)\,dx \tag{74}$$

and the camber $z_s^{(1)}$ from

$$z_s^{(1)}(x, y) = \int_{x_L(y)}^{x} v_{zl}(x', y, z_t)\, dx' + \alpha_T^{(1)}(y)[x - x_L(y)]. \tag{75}$$

In many cases occurring in practice, $(d\alpha_T^{(1)}/dy)_{y=0} = 0(\varepsilon)$, which means

$$\psi^{(1)}(x, y = 0) = 0(\varepsilon).$$

To determine the second-order terms $\Delta z_s(x, y)$ and $\Delta\alpha_T(y)$, we consider therefore the boundary condition as given by equation (70) and approximate it by

$$\left[\pm\frac{\partial z_t}{\partial x} + \frac{\partial z_s^{(1)}}{\partial x}\right][1 + v_{xt}^{(1)}(x, y, |y| \tan\psi^{(1)} \pm z_t + z_s^{(1)}) + v_{xl}(x, y, |y| \tan\psi^{(1)} \pm z_t + z_s^{(1)})] + \frac{\partial \Delta z_s}{\partial x} - \Delta\alpha_T(y) +$$

$$+ \left[\pm\frac{\partial z_t}{\partial y} + \frac{\partial z_s^{(1)}}{\partial y}\right][v_{yt}^{(1)}(x, y, |y| \tan\psi^{(1)} \pm z_t + z_s^{(1)}) + v_{yl}(x, y, |y| \tan\psi^{(1)} \pm z_t + z_s^{(1)})]$$

$$= \alpha_T^{(1)}(y) + v_{zt}^{(1)}(x, y, |y| \tan\psi^{(1)} \pm z_t + z_s^{(1)}) + v_{zl}(x, y, |y| \tan\psi^{(1)} \pm z_t + z_s^{(1)}) \pm \frac{\Delta q(x, y)}{2}. \tag{76}$$

The velocity components v_{xl}, v_{yl}, v_{zl}, are induced by the given load distribution $l(x, y)$ and the velocity components $v_{xt}^{(1)}$, $v_{yt}^{(1)}$, $v_{zt}^{(1)}$ are induced by the source distribution $q^{(1)}(x, y)$ given by equation (25) (or by $q^{(1)*}(x, y)$, *see* equation (50) and Appendix E), with both singularity distributions placed in $z = |y| \tan\psi^{(1)}(x, |y|)$ where

$$\tan\psi^{(1)}(x, y) = \frac{\partial}{\partial y}\{z_T(y) + \tan\alpha_T^{(1)}(y)[x_T(y) - x]\}. \tag{77}$$

For spanwise stations away from the centre section the velocity components can be evaluated from the singularities placed in $z = 0$. The evaluation of the term $\Delta q(x, y)$ is not required for the derivation of the mean surface. We may note that equation (76) differs from equation (70) only by third-order terms.

Adding the two equations (76) which correspond to the upper and the lower surface of the wing respectively, we obtain with equation (73):

$$2\left[\frac{\partial \Delta z_s}{\partial x} - \Delta\alpha_T(y)\right] = v_{zt}^{(1)}(x, y, |y| \tan\psi^{(1)} + z_t + z_s^{(1)}) + v_{zt}^{(1)}(x, y, |y| \tan\psi^{(1)} - z_t + z_s^{(1)}) +$$

$$+ v_{zl}(x, y, |y| \tan\psi^{(1)} + z_t + z_s^{(1)}) + v_{zl}(x, y, |y| \tan\psi^{(1)} - z_t + z_s^{(1)}) - 2v_{zl}(x, y, z_t) -$$

$$- \frac{\partial z_t}{\partial x}[v_{xt}^{(1)}(x, y, |y| \tan\psi^{(1)} + z_t + z_s^{(1)}) - v_{xt}^{(1)}(x, y, |y| \tan\psi^{(1)} - z_t + z_s^{(1)}) +$$

$$+ v_{xl}(x, y, |y| \tan\psi^{(1)} + z_t + z_s^{(1)}) - v_{xl}(x, y, |y| \tan\psi^{(1)} - z_t + z_s^{(1)})] -$$

$$- \frac{\partial z_s^{(1)}}{\partial x}[v_{xt}^{(1)}(x, y, |y| \tan\psi^{(1)} + z_t + z_s^{(1)}) + v_{xt}^{(1)}(x, y, |y| \tan\psi^{(1)} - z_t + z_s^{(1)}) +$$

$$+ v_{xl}(x, y, |y| \tan\psi^{(1)} + z_t + z_s^{(1)}) + v_{xl}(x, y, |y| \tan\psi^{(1)} - z_t + z_s^{(1)})] -$$

$$- \frac{\partial z_t}{\partial y}[v_{yt}^{(1)}(x, y, |y| \tan\psi^{(1)} + z_t + z_s^{(1)}) - v_{yt}^{(1)}(x, y, |y| \tan\psi^{(1)} - z_t + z_s^{(1)}) +$$

$$+ v_{yl}(x, y, |y| \tan\psi^{(1)} + z_t + z_s^{(1)}) - v_{yl}(x, y, |y| \tan\psi^{(1)} - z_t + z_s^{(1)})] -$$

$$- \frac{\partial z_s^{(1)}}{\partial y}[v_{yt}^{(1)}(x, y, |y| \tan\psi^{(1)} + z_t + z_s^{(1)}) + v_{yt}^{(1)}(x, y, |y| \tan\psi^{(1)} - z_t + z_s^{(1)}) +$$

$$+ v_{yl}(x, y, |y| \tan\psi^{(1)} + z_t + z_s^{(1)}) + v_{yl}(x, y, |y| \tan\psi^{(1)} - z_t + z_s^{(1)})]. \tag{78}$$

When we discussed the rather similar expression on the right-hand side of equation (35), we mentioned that some of the terms have a strong singularity near the leading edge. Before we can integrate the expression for $\partial \Delta z_s/\partial x - \Delta\alpha_T$, similar to equations (74), (75), we have to modify the values near the leading edge. This may be done as before by computing the velocity components at shifted x-values. In view of the uncertainty about the correct amount of shift, one may consider taking the values computed away from the leading edge and extrapolating them to a finite value at the leading edge. This would mainly affect the pressure distribution close to the leading edge, where a second-order theory cannot make statements about the accuracy of the pressure distribution on a cambered wing.

It is possible that at and near the centre section Δz_s is not small compared to z_t. We consider load distributions in the plane $z = 0$ which, over the inner wing, are constant along the span. For these, the downwash at the centre section at finite z-values can be approximated by

$$v_{zl}(x, 0, z) = f(x) \ln z + g(x) + \ldots \tag{79}$$

where

$$f(x) = \frac{\tan \varphi}{2\pi} l(x). \tag{80}$$

The term

$$v_{zl}(x, 0, z_t + z_s^{(1)}) + v_{zl}(x, 0, -z_t + z_s^{(1)}) - 2v_{zl}(x, 0, z_t)$$

in equation (78) is then approximately equal to

$$f(x) \ln \left[1 - \left(\frac{z_s^{(1)}(x, 0)}{z_t(x, 0)} \right)^2 \right].$$

In those cases where $|z_s^{(1)}(x, 0)|$ is of a magnitude comparable to $z_t(x, 0)$, the term $\Delta z_s(x, 0)$ derived by equation (78) need not be small, which means that $v_{zl}(x, 0, \pm z_t + z_s^{(1)})$ may differ noticeably from $v_{zl}(x, 0, \pm z_t + z_s^{(1)} + \Delta z_s)$ also away from the leading edge. In this case, we may derive a better approximation to z_s at and near $y = 0$ from

$$z_s(x, y) = z_s^{(1)}(x, y) + \Delta^{(1)} z_s(x, y) + \Delta^{(2)} z_s(x, y) \tag{81}$$

where $z_s^{(1)}(x, y)$ is derived from equation (73), $\Delta^{(1)} z_s(x, y)$ from equation (78) and $\Delta^{(2)} z_s(x, y)$ from

$$\begin{aligned} 2\left[\frac{\partial \Delta^{(2)} z_s(x, y)}{\partial x} - \Delta^{(2)}\alpha_T(y) \right] &= v_{zl}(x, y, |y| \tan \psi^{(1)} + z_t + z_s^{(1)} + \Delta^{(1)} z_s) + \\ &+ v_{zl}(x, y, |y| \tan \psi^{(1)} - z_t + z_s^{(1)} + \Delta^{(1)} z_s) - \\ &- v_{zl}(x, y, |y| \tan \psi^{(1)} + z_t + z_s^{(1)}) - v_{zl}(x, y, |y| \tan \psi^{(1)} - z_t + z_s^{(1)}). \end{aligned} \tag{82}$$

When we can neglect the effect of ψ, when the velocity component v_{zl} induced in $z = 0$ is finite and when $v_{zl}^{(1)}(x, y, z)$ and $v_{zl}(x, y, z)$ can be approximated by the first two terms of their Taylor series expansions (which is the case away from the centre section and other discontinuities in the wing shape), then the computing effort can be greatly reduced.

We derive the first-order approximation $z_s^{(1)}(x, y)$ and $\alpha_T^{(1)}(y)$ from the first-order boundary condition

$$\frac{\partial z_s^{(1)}(x, y)}{\partial x} - \alpha_T^{(1)}(y) = v_{zl}(x, y, 0), \tag{83}$$

instead of equation (73), and from equations (74), (75).

$\Delta z_s(x, y)$ and $\Delta\alpha_T(y)$ are derived from equations (78) by ignoring all terms of a higher order than ε^2. The result reads:

$$\begin{aligned} \frac{\partial \Delta z_s(x, y)}{\partial x} - \Delta\alpha_T(y) = &-\frac{\partial}{\partial x}[z_t(x, y) v_{xl}(x, y, 0) + z_s^{(1)}(x, y) v_{xt}^{(1)}(x, y, 0)] - \\ &-\frac{\partial}{\partial y}[z_t(x, y) v_{yl}(x, y, 0) + z_s^{(1)}(x, y) v_{yt}^{(1)}(x, y, 0)]. \end{aligned} \tag{84}$$

For the type of thickness distribution and load distribution considered in this Report, the right-hand side of equation (84) has finite values, so that Δz_s and $\Delta \alpha_T$ can be derived by integration similar to equations (74), (75).

In a practical case one may use equations (71) to (78) for the neighbourhood of the centre section and equations (83), (84) with (71), (72), (74), (75) away from the centre.

In order to obtain the pressure distribution on the warped wing obtained in this way it is of course necessary to carry out a second-order calculation of $v_x(x, y, z_w)$ in the usual way, which may be done by subtracting the two equations (76) so as to find the correction term $\Delta q(x, y)$ for the source distribution. Since $l(x, y)$ is given, the remainder of the calculation is straightforward; details are therefore not given here.

3.2 Wings with Given Pressure Distribution on the Upper Surface

We consider now the problem, which arises in a practical design, where the thickness distribution and the pressure distribution (and hence the velocity distribution) on the upper surface are prescribed and the task is to determine the mean surface.

We solve the problem again in several steps and determine first a mean surface $z_s^{(1)}(x, y)$ which is correct to first order. For this purpose we determine from the required velocity $V_{req}(x, y, z_t + z_s)$ the value of

$$v_{xl}^{(1)}(x, y, 0) = \sqrt{\left\{V_{req}^2(x, y, z_t + z_s)\left[1 + \left(\frac{\partial z_t/\partial x}{\cos\varphi}\right)^2\right] - \tan^2\varphi\left(\frac{\partial z_t}{\partial x}\right)^2\right\}} - 1 - v_{xt}^{(1)}(x, y, 0) \tag{85}$$

and

$$l^{(1)}(x, y) = 4v_{xl}^{(1)}(x, y, 0). \tag{86}$$

When $v_{xt}^{(1)}(x, y, 0)$ has not been determined previously, but the streamwise velocity $V_{xt}(x, y, z_t)$ and the total velocity $V_t(x, y, z_t)$ at the surface of the uncambered non-lifting wing have been computed, then one might derive an approximate value of $v_{xl}^{(1)}(x, y, 0)$ from the approximate relation

$$V_{req}^2(x, y, z_t + z_s) - V_t^2(x, y, z_t) \approx 2V_{xt}(x, y, z_t) \cdot v_{xl}^{(1)}(x, y, 0). \tag{87}$$

It may be advisable to modify the load distribution $l^{(1)}(x, y)$, derived from equations (85) to (87), close to the leading edge, to ensure that $l^{(1)}$ behaves like $[x - x_L(y)]^{-0.5}$ or $[x - x_L(y)]^{0.5}$.

With $l^{(1)}(x, y)$ we compute by equations (73) or (83) and (74), (75) a first approximation to the mean surface, $z_s^{(1)}(x, y)$ and $\alpha_T^{(1)}(y)$.

For the wing shape derived from this approximate mean surface and $z_t(x, y)$, we compute by second-order theory the velocity on the upper surface $V_{app}(x, y, z_t + z_s^{(1)})$. The difference between the required velocity $V_{req}(x, y, z_t + z_s)$ and the computed $V_{app}(x, y, z_t + z_s^{(1)})$ is a second-order term. From this we can derive a second-order correction to the streamwise velocity component $\Delta v_{xl}(x, y, 0)$.

$$\begin{aligned}
&[V_{req}^2(x, y, z_t + z_s) - V_{app}^2(x, y, z_t + z_s^{(1)})]\left[1 + \left(\frac{\partial(z_t + z_s^{(1)})/\partial x}{\cos\varphi}\right)^2\right] \\
&\approx 2\Delta v_{xl}(x, y, 0)\left\{\cos\alpha_T^{(1)} + v_x^{(1)}(x, y, 0) + \frac{\partial}{\partial x}\left[(z_t + z_s^{(1)})\frac{\partial(z_t + z_s^{(1)})}{\partial x}\right]\right\} \\
&\approx 2\Delta v_{xl}(x, y, 0).
\end{aligned} \tag{88}$$

From

$$\Delta^* l(x, y) = 4\Delta v_{xl}(x, y, 0) \tag{89}$$

we determine by equations (73) or (83), (74), (75) a second-order correction $\Delta^* z_s$, $\Delta^*\alpha_T$ of the mean surface $z_s^{(1)}$, $\alpha_T^{(1)}$.

To obtain the mean surface correct to second order, we have also to determine the second-order terms Δz_s and $\Delta\alpha_T$ by equations (78) or (84) which are related to the load distribution $l^{(1)}(x, y)$, equation (86). Thus the mean surface is given by

$$z_s(x, y) = z_s^{(1)}(x, y) + \Delta z_s(x, y) + \Delta^* z_s(x, y) \tag{90}$$

$$\alpha_T(y) = \alpha_T^{(1)}(y) + \Delta\alpha_T(y) + \Delta^*\alpha_T(y). \tag{91}$$

It may be advisable to compute Δz_s and $\Delta\alpha_T$ before $\Delta^* z_s$, $\Delta^*\alpha_T$ are computed and to derive a V_{app} for $z_s^{(1)} = z_s^{(1)} + \Delta z_s$.

4. Résumé

It is shown how the incompressible second-order theory for two-dimensional aerofoils can be extended to deal with finite swept wings. The flow field is represented by singularity distributions on the 'chord surface' which contains the chord at each spanwise station. The induced velocity field is approximated by the velocities induced by planar singularity distributions, which can be calculated on and off the surface by the computer programs of Refs. 8 and 9.

For a twisted wing, the boundary condition of zero normal velocity is approximated by an equation in terms of the velocity components computed from planar singularity distributions. The boundary condition contains the value of the velocity component v_z at the surface of the wing. Away from the centre section, this can be expressed, by means of a Taylor series expansion, in terms of the velocity components in the plane $z = 0$ (as in two dimensions); but this is not so for the centre section. Therefore, a solution of the problem is derived first without using Taylor series expansions.

The strength of the singularity distributions is determined in at least two steps. A formal solution which is correct to first-order is considered first and then a second-order correction is added to it. The choice of a first-order source distribution, $q^{(1)}(x, y)$, is straightforward. The derivation of the first-order load distribution $l^{(1)}(x, y)$, which satisfies the first-order boundary condition exactly, involves the solution of an integral equation in two variables. Since this problem of 'lifting-surface theory' has not yet been satisfactorily solved for general swept wings, it is proposed to use an approximation to the first-order load distribution, and solve directly the second-order problem. This procedure has the possible advantage that near the apex of the wing the downwash is computed only at finite values of z; further the interference effect between thickness and lift can be taken into account approximately from the start. It is suggested that the second-order load distribution should be determined by an iterative procedure. No practical examples are considered in this Report but the decisive question about convergence has been answered positively by Sells.[17,18] The determination of the second-order correction to $q^{(1)}(x, y)$ requires only the computation of the various velocity components induced by $q^{(1)}$ and $l^{(1)}$. To derive a solution which is valid also near the leading edge, where the perturbation velocity is not small, it is suggested that Lighthill's technique of 'strained coordinates' be applied in an approximate form by computing the perturbation velocity near the leading edge at shifted values of x and then to examine whether the solution satisfies the boundary condition to a required accuracy; if this is not the case, then a further modification of the singularities may be made.

It is further shown that by using Taylor series expansions, which is possible away from centre and tip, the computing effort can be considerably reduced, in particular for cambered wings. Therefore the use of a hybrid method is suggested, which uses the general relations near the centre and the simpler relations away from the centre even though this requires a more complicated computer program. From the known singularity distributions the total velocity at the surface of the wing can be computed. However, the formal solution derived by Taylor series expansion breaks down near the leading and trailing edges. The approximate and the exact velocity distributions for yawed ellipsoids have been compared and a formula has been derived for modifying the approximate velocity distribution for a general three-dimensional wing which gives a finite velocity at the leading edge and a velocity which is correct to second order away from the leading edge.

The problem of designing a wing of given thickness distribution which has a prescribed pressure distribution on the upper surface is also treated. As a preliminary step, the problem of designing a wing with given load distribution $l(x, y)$ is solved.

LIST OF SYMBOLS

x^*, y^*, z^*	rectangular coordinate system, where $z^* = 0$ is a median wing plane
x, y, z	rectangular coordinate system, where $z = 0$ is the plane which contains the singularity distributions; the x-axis is along the chord at each spanwise station; in second-order theory the difference between y and y^* is ignored
$z_w^*(x^*, y^*)$	ordinate of the wing surface
$z_c^*(x^*, y^*)$	ordinate of the chord surface
$z_t(x, y)$	thickness distribution
$z_s(x, y)$	camber distribution
$z_w(x, y)$	$= \pm z_t(x, y) + z_s(x, y)$
$c(y)$	wing chord
ξ	$= \dfrac{x - x_L(y)}{c(y)}$
ρ	nose radius
S	plan area
α^*	angle of incidence with respect to the plane $z^* = 0$
$\alpha_T(y)$	twist distribution
φ	angle of sweep
ψ	$= \tan^{-1}\left(\dfrac{\partial z_c^*(x^*, y^*)}{\partial y^*}\right)$
V	total velocity
V_0	free stream velocity, taken as unity
$v_{x^*}, v_{y^*}, v_{z^*}$; v_x, v_y, v_z	components of the perturbation velocity with respect to the various axes
$l(x, y), \Delta l(x, y)$	strength of the distribution of lifting singularities
$q(x, y), \Delta q(x, y)$	strength of source distribution
Suffices:	
L	leading edge
T	trailing edge
l	velocity induced by load distribution
t	velocity induced by source distribution
Superscripts:	(except Section 3)
(1)	related to singularity distribution from first-order theory
(2)	related to singularity distribution from second-order theory (except Section 2.4)

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APPENDIX A

Second-Order Theory for Two-Dimensional Aerofoils

Lighthill,[5] van Dyke[6] and Gretler[7] have treated the two-dimensional aerofoil in incompressible flow by second-order theory and derived expressions for the components of the perturbation velocity in powers of the thickness-to-chord ratio, the camber ratio and the angle of attack.

The resulting flow field can be interpreted as the velocity field induced by a singularity distribution on the x-axis. Lighthill has shown that it is essential that the x-axis is chosen such that it passes through the leading and trailing edges. Taylor series expansions are used to express the velocity components on the surface of the aerofoil $z_w(x) = \pm z_t(x) + z_s(x)$ in terms of the velocity components on $z = 0$.

$$\begin{aligned} v_x(x, z_w) &= v_x(x, \pm 0) + z_w\left(\frac{\partial v_x(x, z)}{\partial z}\right)_{z=0} \\ &= v_x(x, \pm 0) + z_w\frac{\partial v_z(x, \pm 0)}{\partial x}, \end{aligned} \tag{A-1}$$

$$\begin{aligned} v_z(x, z_w) &= v_z(x, \pm 0) + z_w\left(\frac{\partial v_z(x, z)}{\partial z}\right)_{z=0} \\ &= v_z(x, \pm 0) - z_w\frac{\partial v_x(x, \pm 0)}{\partial x}. \end{aligned} \tag{A-2}$$

In deriving these equations, use has been made of the equations of continuity and irrotationality. These expansions break down near stagnation points. Therefore a formal solution is derived first and this is then modified near the stagnation points.

For the formal solution the components of the perturbation velocity are written as the sum of the first-order term and a second-order correction term

$$v_x^{(2)}(x, \pm 0) = v_x^{(1)}(x, \pm 0) + \Delta v_x(x, \pm 0), \tag{A-3}$$

$$v_z^{(2)}(x, \pm 0) = v_z^{(1)}(x, \pm 0) + \Delta v_z(x, \pm 0). \tag{A-4}$$

The first-order terms are derived from the first-order boundary condition:

$$v_z^{(1)}(x, \pm 0) = -\alpha \pm \frac{dz_t(x)}{dx} + \frac{dz_s(x)}{dx}. \tag{A-5}$$

The solution reads:

$$v_x^{(1)}(x, \pm 0) = S^{(1)}(x) \pm S^{(4)}(x) \pm \alpha\sqrt{\frac{1-x}{x}} \tag{A-6}$$

where

$$S^{(1)}(x) = \frac{1}{\pi}\int_0^1 \frac{dz_t}{dx'}\frac{dx'}{x - x'}, \tag{A-7}$$

$$S^{(4)}(x) = \sqrt{\frac{1-x}{x}}\,\frac{1}{\pi}\int_0^1 \frac{dz_s}{dx'}\sqrt{\frac{x'}{1-x'}}\,\frac{dx'}{x - x'}. \tag{A-8}$$

The $\pm$ signs refer to the upper and lower surface of the aerofoil. Inserting equations (A-2), (A-4), (A-5), into the second-order approximation to the boundary condition

$$\left(\pm\frac{dz_t}{dx} + \frac{dz_s}{dx}\right)[1 + v_x^{(1)}(x, \pm 0)] = -\alpha + v_z^{(2)}(x, z_w) \tag{A-9}$$

leads to

$$\Delta v_z(x, \pm 0) = \frac{d}{dx}[v_x^{(1)}(x, \pm 0)z_w]. \tag{A-10}$$

The resulting expression for the second-order approximation to the x-component of the total velocity, V_x, can be written

$$V_x^{(2)}(x, z_w) = \cos\alpha\left[1 + S^{(1)}(x) \pm S^{(4)}(x) + S^{(40)}(x) \pm S^{(41)}(x) + z_w\frac{d^2z_w}{dx^2}\right] +$$
$$+ \sin\alpha\left[S^{(44)}(x) \pm \sqrt{\frac{1-x}{x}}(1 - S^{(3)}(x))\right] \tag{A-11}$$

where

$$S^{(3)}(x) = \frac{1}{\pi}\int_0^1 \frac{d}{dx'}\left[z_t\sqrt{\frac{1-x'}{x'}}\right]\sqrt{\frac{x'}{1-x'}}\,\frac{dx'}{x-x'}, \tag{A-12}$$

$$S^{(40)}(x) = \frac{1}{\pi}\int_0^1 \frac{d}{dx'}[S^{(1)}z_t + S^{(4)}z_s]\frac{dx'}{x-x'}, \tag{A-13}$$

$$S^{(41)}(x) = \sqrt{\frac{1-x}{x}}\,\frac{1}{\pi}\int_0^1 \frac{d}{dx'}[S^{(1)}z_s + S^{(4)}z_t]\sqrt{\frac{x'}{1-x'}}\,\frac{dx'}{x-x'}, \tag{A-14}$$

$$S^{(44)}(x) = \frac{1}{\pi}\int_0^1 \frac{d}{dx'}\left[z_s\sqrt{\frac{1-x'}{x'}}\right]\frac{dx'}{x-x'}. \tag{A-15}$$

The total velocity V at the surface of the aerofoil is

$$V(x, z_w) = V_x(x, z)\sqrt{1 + \left(\frac{dz_w}{dx}\right)^2}. \tag{A-16}$$

A second-order approximation to this is

$$V^{(2)}(x, z_w) = V_x^{(2)}(x, z) + \frac{1}{2}\left(\frac{dz_w}{dx}\right)^2. \tag{A-17}$$

We note that $V^{(2)}$ contains two terms, $z_w(d^2z_w/dx^2)$, $\frac{1}{2}(dz_w/dx)^2$, which are singular near the leading edge, $x = 0$, like $1/x$; thus $V^{(2)}$ has a stronger singularity than $V^{(1)} = 1 + v_x^{(1)}(x, 0)$ which is singular like $1/\sqrt{x}$.

To derive from the formal solution of equations (A-11), (A-17) a uniformly valid approximation, Lighthill has applied his technique[13] for rendering approximate solutions uniformly valid and found that for the two-dimensional aerofoil it is sufficient to shift the x-ordinate by half the nose radius of the aerofoil, ρ, if cubes of the perturbation velocities are ignored. Uniformly valid values of the perturbation velocities are thus obtained from

$$v_x(x, z_w) = v_x^{(2)}\left(x - \frac{\rho}{2}, z_w\right), \tag{A-18}$$

$$v_z(x, z_w) = v_z^{(2)}\left(x - \frac{\rho}{2}, z_w\right). \tag{A-19}$$

Lighthill shows further that one can obtain from the formula

$$V = \sqrt{\frac{x}{x+\rho/2}}\left[\cos\alpha + v_x^{(2)}(x, z_w) + \frac{1}{2}\left(\frac{dz_w}{dx}\right)^2 + \frac{\rho}{4x}\right]$$

$$= \sqrt{\frac{x}{x+\rho/2}}\left[\cos\alpha + v_x^{(2)}(x, 0) + z_w\frac{d^2 z_w}{dx^2} + \frac{1}{2}\left(\frac{dz_w}{dx}\right)^2 + \frac{\rho}{4x}\right] \qquad \text{(A-20)}$$

a uniformly valid result which is correct to first order near the leading edge and correct to second order away from the leading edge.

Van Dyke[6] has compared the exact solution for a parabola with the formal result from second-order theory. He has derived the following modification:

$$V = \sqrt{\frac{x \pm b_0\sqrt{2\rho x}}{x + \rho/2 \pm b_0\sqrt{2\rho x}}}\left[\cos\alpha + v_x^{(2)}(x, 0) + z_w\frac{d^2 z_w}{dx^2} + \frac{1}{2}\left(\frac{dz_w}{dx}\right)^2 + \frac{\rho}{4x}\right] \qquad \text{(A-21)}$$

where b_0 is the slope of the camber line at $x = 0$. Equation (A-21) yields a uniformly valid approximation which is correct to second order also near the leading edge if the rate of change of curvature of the section shape is continuous. This condition is satisfied for those sections where z_t behaves like $a_0\sqrt{x} + a_2\sqrt{x^3} + \ldots$ near the leading ledge. For some N.A.C.A. sections z_t behaves like $a_0\sqrt{x} + a_1 x + \ldots$; for these equation (A-21) yields results of only first-order accuracy near the leading edge. Van Dyke suggest therefore using the simpler formula provided by equation (A-20).

Gretler[7] has suggested a somewhat different modification; it can be interpreted as a replacement of $\rho/2x$ in equation (A-20) by $(dz_w/dx)^2$. The leading term in $(dz_w/dx)^2$ near the leading edge is $\rho/2x$, since from

$$z = \pm\sqrt{2\rho x} + (\pm a_1 + b_0)x + \ldots$$

$$\left(\frac{dz}{dx}\right)^2 = \frac{\rho}{2x} + \sqrt{\frac{2\rho}{x}}(a_1 \pm b_0) + \ldots.$$

Gretler's formula reads

$$V = \frac{\cos\alpha + v_x^{(2)}(x, 0) + z_w\dfrac{d^2 z_w}{dx^2} + \left(\dfrac{dz_w}{dx}\right)^2}{\sqrt{1 + (dz_w/dx)^2}}$$

$$= \frac{\cos\alpha + v_x^{(2)}(x, 0) + \dfrac{d}{dx}\left(z_w\dfrac{dz_w}{dx}\right)}{\sqrt{1 + (dz_w/dx)^2}}. \qquad \text{(A-22)}$$

We note that for those cambered sections where equation (A-21) is applicable, values from equation (A-22) differ from those of equation (A-21) for the very small region near the leading edge where x is smaller or of order $2\rho b_0^2$.

Equation (A-22) is similar to the result from first-order theory when this has been made uniformly valid by the 'Riegels factor';

$$V = \frac{1 + v_x^{(1)}(x, 0)}{\sqrt{1 + (dz_w/dx)^2}}. \qquad \text{(A-23)}$$

For the special case of an elliptic aerofoil, equations (A-22) and (A-23) agree with the exact value of V since

$$v_x^{(2)}(x, 0) - v_x^{(1)}(x, 0) = -\frac{d}{dx}\left(z_w\frac{dz_w}{dx}\right) = t^2.$$

Van Dyke has also derived a formula to modify the formal results near the trailing edge of sections with finite trailing-edge angle. We do not intend to consider the details near the trailing edge since the error of the formal solution is less important and the boundary layer modifies the inviscid solution near the rear stagnation point drastically.

We note that equations (A-20) to (A-22) give satisfactory results only when $v_x^{(2)}(x, z)$ is derived from the Taylor series expansion, equation (A-1); the equations must not be used with values of $v_x^{(2)}(x, z)$ computed at $z \neq 0$, from the formal singularity distributions. We note further that the correction made by equation (A-22) to the approximate result

$$V(x, z) = V_x^{(2)}(x, z)\sqrt{1 + \left(\frac{dz}{dx}\right)^2}$$

namely

$$\Delta V = \frac{\cos\alpha + v_x^{(2)}(x, z_w) + \left(\frac{dz_w}{dx}\right)^2}{\sqrt{1 + (dz_w/dx)^2}} - V_x^{(2)}(x, z_w)\sqrt{1 + \left(\frac{dz_w}{dx}\right)^2}$$

$$= \left(\frac{dz_w}{dx}\right)^2 \frac{1 - \cos\alpha - v_x^{(2)}(x, z_w)}{\sqrt{1 + (dz_w/dx)^2}}$$

decreases rapidly when moving away from the leading edge.

In Fig. 14 of Ref. 14 it is shown that, for cambered Karman–Trefftz aerofoils, second-order theory produces fairly accurate results even when the ratio between maximum camber and chord is about 0·08.

APPENDIX B

Comparison between some Approximate Results and the Exact Velocity Distribution on a Two-Dimensional Elliptic Aerofoil

When determining the strength of the singularity distributions $q^{(2)}(x, y)$, $l^{(2)}(x, y)$ one uses usually the boundary condition in the approximate form

$$\frac{\partial z_w}{\partial x}[1 + v_x^{(1)}(x, y, \pm 0)] + \frac{\partial z_w}{\partial y} v_y^{(1)}(x, y, \pm 0) = \alpha + v_z^{(1)}(x, y, z_w) + \Delta v_z(x, y, \pm 0) \tag{B-1}$$

(*see* equations (62) and (A-9)). This equation is derived under the assumption that the first-order singularity distributions $q^{(1)}(x, y)$, $l^{(1)}(x, y)$ induce nearly the same velocity components $v_x^{(1)}$ and $v_y^{(1)}$ at $z = z_w$ as at $z = 0$. This assumption does not however hold near the leading edge. When deriving equations (59) and (60) for $\Delta q(x, y)$ and $\Delta v_{zl}(x, y)$, we approximate $v_z^{(1)}(x, y, z_w)$, by the first two terms of the Taylor series expansion with respect to z, which implies that we assume that the higher-order terms produce differences of small magnitude; this assumption also does not hold near the leading edge.

To examine this matter, we consider the two-dimensional flow past a section of elliptic shape, for which the exact flow field is known.

We have derived analytic expressions for the velocity components which are induced at non-zero values of z by the source distribution $q^{(1)}(x)$ related to the thickness distribution of an ellipse and the load distribution $l(x)$ of a flat plate at an angle of incidence:

$$q(x) = t\frac{1 - 2x}{\sqrt{x(1 - x)}}, \tag{B-2}$$

$$l(x) = 4\alpha\sqrt{\frac{1 - x}{x}} \tag{B-3}$$

where t is the thickness-to-chord ratio of the ellipse and the chord c is chosen as unity.

These singularities induce the velocity components

$$v_{xt}(x, z) = \frac{1}{2\pi}\int_0^1 q(x')\frac{x - x'}{(x - x')^2 + z^2}\,dx',$$

$$v_{zt}(x, z) = \frac{1}{2\pi}\int_0^1 q(x')\frac{z}{(x - x')^2 + z^2}\,dx',$$

$$v_{xl}(x, z) = \frac{1}{4\pi}\int_0^1 l(x')\frac{z}{(x - x')^2 + z^2}\,dx',$$

$$v_{zl}(x, z) = \frac{1}{4\pi}\int_0^1 l(x')\frac{x - x'}{(x - x')^2 + z^2}\,dx'.$$

The integrals can be evaluated by contour integration, similar to the procedure of Section 7.1 in Ref. 9. The following relations are obtained:

$$v_{xt}(x, z) = t + \frac{t}{2\sqrt{2}}\,\frac{\dfrac{x}{\sqrt{x^2 + z^2}} + \dfrac{1 - x}{\sqrt{(1 - x)^2 + z^2}} - 2\sqrt{x^2 + z^2} - 2\sqrt{(1 - x)^2 + z^2}}{\sqrt{\sqrt{x^2 + z^2}\sqrt{(1 - x)^2 + z^2} - x(1 - x) + z^2}}, \tag{B-4}$$

$$v_{zt}(x, z) = \pm\frac{t}{2\sqrt{2}}\sqrt{\sqrt{x^2 + z^2}\sqrt{(1 - x)^2 + z^2} + x(1 - x) - z^2}\left[\frac{1}{\sqrt{x^2 + z^2}} - \frac{1}{\sqrt{(1 - x)^2 + z^2}}\right], \tag{B-5}$$

$$v_{xl}(x, z) = \pm\alpha\sqrt{\frac{\sqrt{x^2 + z^2}\sqrt{(1 - x)^2 + z^2} + x(1 - x) - z^2}{2(x^2 + z^2)}}, \tag{B-6}$$

$$v_{zl}(x, z) = -\alpha + \alpha\sqrt{\frac{\sqrt{x^2 + z^2}\sqrt{(1 - x)^2 + z^2} - x(1 - x) + z^2}{2(x^2 + z^2)}}. \tag{B-7}$$

The related second-order formulae, derived by developing the above relations into series with respect to powers of z, read

$$v_{xt}^{(2*)}(x, z) = t - \frac{zt}{4\sqrt{x(1-x)^3}}, \tag{B-8}$$

$$v_{zt}^{(2*)}(x, z) = \pm \frac{t(1-2x)}{2\sqrt{x(1-x)}}, \tag{B-9}$$

$$v_{xl}^{(2*)}(x, z) = \pm\alpha\sqrt{\frac{1-x}{x}}, \tag{B-10}$$

$$v_{zl}^{(2*)}(x, z) = -\alpha + \frac{z\alpha}{2x\sqrt{x(1-x)}}. \tag{B-11}$$

We have added the asterisks to ensure that the expressions are not confused with the velocity components from second-order theory (the velocity components $v_{xt}^{(2*)}, \ldots$ are induced by $q^{(1)}(x)$, $l^{(1)}(x)$ not by $q^{(2)}(x)$, $l^{(2)}(x)$).

We quote in Tables 1 and 2 values of the difference between the exact velocity components induced at the surface of elliptic aerofoils and their second-order approximations, related to the first-order values. We have chosen elliptic sections with values of the thickness-to-chord ratio of 0·1 and 0·15 and note that the nose shape of a 15 per cent thick ellipse is very similar to that of a 12 per cent thick R.A.E. 101 section. The table shows that small values for the higher-order terms are only obtained at a distance downstream of the leading edge appreciably greater than the nose radius $\rho = 0{\cdot}5t^2$.

According to equation (B-9), the second-order Taylor series expansion of $v_{zt}^{(1)}(x, z_t)$ is for the elliptic section equal to the first-order term $v_{zt}^{(1)}(x, 0)$.

If we use the boundary condition in the form of equation (B-1) and the Taylor series expansion for $v_{zt}^{(1)}(x, z_t)$, we therefore obtain for $\Delta q(x)$ the well-known result

$$\frac{\Delta q(x)}{2} = t\frac{dz_t}{dx} = t\frac{q^{(1)}(x)}{2}. \tag{B-12}$$

It is also known that $\Delta q(x)$ from equation (B-12) leads (with the Riegels factor, equation (A-23)) to the exact pressure distribution on the aerofoil.

We want to learn how important it is whether we use for $v_{zt}^{(1)}(x, z_t)$ the exact values at z_t or the values derived by Taylor series expansion. We consider therefore the source distribution

$$\frac{\Delta q^*(x)}{2} = -[v_{zt}^{(1)}(x, z_t) - v_{zt}^{(1)}(x, 0)]. \tag{B-13}$$

(Values of $-\Delta q^*(x)/q^{(1)}(x) = [v_z(x, z) - v_z^{(2*)}(x, z)]/v_z(x, 0)$ are given in Table 1.) We have computed the chordwise velocity component $\Delta v_{xt}^*(x, 0)$, which would be induced by the source distribution $\Delta q^*(x)$, from

$$\Delta v_{xt}^*(x, 0) = \frac{1}{2\pi}\int_0^1 \Delta q^*(x')\frac{dx'}{x - x'}. \tag{B-14}$$

The result, plotted in Fig. 1, shows that even at mid-chord Δv_{xt}^* is of the same magnitude as $\Delta v_{xt}(x, 0) = t^2$, the velocity induced by the second-order source distribution $\Delta q(x) = 2t\, dz_t/dx$, equation (B-12). That $\Delta q^*(x)$, which away from the leading edge is a term of order t^3, produces such large values of $\Delta v_{xt}^*(x, 0)$ is due to the fact that for $x \to 0$, $\Delta q^*(x)$ tends to $q^{(1)}(x)$.

When we apply the boundary condition in the form of equation (31) instead of equation (B-1), i.e. when we do not approximate $v_{xt}^{(1)}(x, z_t)$ at the surface by the value at $z = 0$, $v_{xt}^{(1)}(x, 0)$, then we obtain for Δq the relation, (*see* equation (34)):

$$\frac{\Delta q(x)}{2} = t\frac{dz_t}{dx} + \frac{dz_t}{dx}[v_{xt}^{(1)}(x, z_t) - v_{xt}^{(1)}(x, 0)] - [v_{zt}^{(1)}(x, z_t) - v_{zt}^{(1)}(x, 0)]. \tag{B-15}$$

We have considered the source distribution

$$\frac{\Delta q^{**}(x)}{2} = \frac{dz_t}{dx}[v_{xt}^{(1)}(x, z_t) - v_{xt}^{(1)}(x, 0)] - [v_{zt}^{(1)}(x, z_t) - v_{zt}^{(1)}(x, 0)]. \quad \text{(B-16)}$$

The two terms on the right-hand side of equation (B-16) are of opposite sign, as a consequence $|\Delta q^{**}(x)|$ is noticeably smaller than $|\Delta q^{*}(x)|$. We have computed $\Delta v_x^{**}(x, 0)$ induced by $\Delta q^{**}(x)$ and have plotted $\Delta v_x^{**}(x, 0)$ also in Fig. 1. We note that the source distribution $\Delta q^{**}(x)$ produces a noticeably smaller error than $\Delta q^{*}(x)$. We have computed Δv_x^{**} for several values of t and quote some results.

t	$\frac{\Delta v_x^{**}(x = 0{\cdot}1, 0)}{t^2}$	$\frac{\Delta v_x^{**}(x = 0{\cdot}5, 0)}{t^2}$
0·05	0·136	0·046
0·1	0·154	0·058
0·15	0·157	0·074
0·2	0·136	0·086

These results show that, when we use the boundary condition in the form of equations (31), (34), the error in the velocity, away from the leading edge, is of the order $0{\cdot}1t^2$.

We draw the tentative conclusion that, for general aerofoil shapes and three-dimensional wings, we are likely to obtain a more accurate source distribution if with computed value of $v_{zt}^{(1)}(x, y, z_w)$ we use for $v_{xt}^{(1)}(x, y, z_w)$ and $v_{yt}^{(1)}(x, y, z_w)$ also values computed at the surface, and that it seems advisable to use the values $v_{xt}^{(1)}(x, y, 0)$, $v_{yt}^{(1)}(x, y, 0)$ in $z = 0$ together with values of $v_z^{(1)}(x, y, z_w)$ approximated by Taylor series expansion. The values of $[v_x(x, z) - v_x^{(2^*)}(x, z)]/v_x(x, 0)$ given in Table 1 show that it is not advisable to use near the leading edge for $v_{xt}^{(1)}(x, y, z)$, $v_{yt}^{(1)}(x, y, z)$ values derived by Taylor series expansions.

For the elliptic section at an angle of incidence, the boundary condition (B-1) together with the Taylor series expansion for $v_{zl}^{(1)}(x, z)$ leads to

$$\Delta v_{zl}(x, 0) = -\alpha t. \quad \text{(B-17)}$$

It is known that this value for Δv_{zl} leads (with the Riegels factor) to the exact velocity distribution.

Equation (27) gives for $\Delta v_{zl}(x, 0)$:

$$\begin{aligned}\Delta v_{zl}(x, 0) &= \frac{dz_t}{dx} v_{xl}^{(1)}(x, z_t) - [v_{zl}^{(1)}(x, z_t) - v_{zl}^{(1)}(x, 0)] \\ &= -\alpha t + \frac{dz_t}{dx}[v_{xl}^{(1)}(x, z_t) - v_{xl}^{(1)}(x, 0)] - \left[v_{zl}^{(1)}(x, z_t) - v_{zl}^{(1)}(x, 0) - \frac{\alpha t}{2x}\right].\end{aligned} \quad \text{(B-18)}$$

The second and third term on the right-hand side of equation (B-18) are of opposite sign (*see* Table 2), so that it is again advisable to combine values of $v_{xl}^{(1)}(x, z)$ with computed values of $v_{zl}^{(1)}(x, z)$ and values of $v_{xl}^{(1)}(x, 0)$ with values of $v_{zl}^{(1)}(x, z)$ approximated by Taylor series expansion.

However, the values of Δv_{zl} given by equations (B-18) are not acceptable close to the leading edge, since Δv_{zl} tends to infinity as $-\alpha t/2x$.

This inadequate behaviour is due to the fact that a singularity distribution on $z = 0$ which is to represent the flow past an ellipse exactly, must not extend into the leading edge nor into the trailing edge. It is known that the conformal transformation

$$\zeta = \zeta_1 + \frac{R^2}{\zeta_1}$$

transforms the ellipse with axes $c = 1$ and t, and centre at the origin of the coordinate system in the ζ-plane, into a circle of radius $r = (1 + t)/4$ and the slit $-2R < \zeta < 2R$ into the circle of radius R, where $R = \sqrt{1 - t^2}/4$. The flow past the circle of radius r can be represented by a source and vortex distribution on the circle of radius R, and therefore the flow past the ellipse by a source and vortex distribution on the slit. The length of the slit is $4R = \sqrt{1 - t^2}$, i.e. for small values of the thickness-to-chord ratio $\approx 1 - t^2/2$.

The singularity distribution on the chord of the elliptic aerofoil begins thus at a distance of about $t^2/4$ behind the leading edge and ends at a distance of about $t^2/4$ in front of the trailing edge. The nose radius of the elliptic aerofoil is $t^2/2$.

This result is related to one derived by Lighthill.[5] He considers a general aerofoil, expands the velocity components in powers of ε, introduces these expansions into the boundary condition and derives the strength of the singularity distributions. He applies then his technique[13] for rendering approximate solutions uniformly valid and finds that it is sufficient to shift the singularity distribution downstream (parallel to the chord) through a distance equal to half the leading-edge radius of curvature, if cubes of the perturbation velocities are to be ignored. Lighthill derives thus the strength of the singularity distributions by using the Taylor series expansion for $v_z^{(1)}(x, z)$ and the boundary condition (B-1) and uses the shifted x-coordinate to determine uniformly valid values for the pressure distribution by computing the velocity components from the second-order singularity distributions.

$$v_x(x, z_w) = v_x^{(2)}\left(x - \frac{\rho}{2}, z_w\right),$$

$$v_z(x, z_w) = v_z^{(2)}\left(x - \frac{\rho}{2}, z_w\right).$$

For three-dimensional flow we cannot always use a Taylor series expansion for $v_z^{(1)}(x, y, z)$. We have therefore examined for the two-dimensional elliptic aerofoil the behaviour of Δv_z determined from the boundary condition of equation (31) with $v_x^{(1)}(x, z)$ and $v_z^{(1)}(x, z)$ computed from the shifted first-order singularity distributions $q^{(1)}$, $l^{(1)}$, i.e. we have derived $v_x^{(1)}(x, z_w)$ and $v_z^{(1)}(x, z_w)$ from the formulae (B-4) to (B-7) by substituting $[x - t^2/4]/[1 - t^2/2]$ for x and $t\sqrt{x(1 - x)}$ for z. The ratio between the values of Δv_{zt} determined in this way and the values of $t\, dz_t/dx$, departs by only a small amount from 1; the largest difference occurs at $x = 0$, where the difference is $[\sqrt{1 - t^2/4} - 1]/t\sqrt{1 - t^2/4}$, which is approximately $-t/8$. The values for Δv_{zt} are everywhere finite; at $x = 0$, the value is $-\alpha t(3 - t^2)/2\sqrt{1 - t^2/4}$, whilst equation (B-17) gives $-\alpha t$. The difference between Δv_{zl} and $-\alpha t$ is only noticeable close to the leading edge; for $x = t^2$, $\Delta v_{zl} = -0{\cdot}86\alpha t(1 + 0(t^2))$.

We draw from these results the conclusion that, for those cases where the Taylor series expansions for $v_{zt}^{(1)}$ and $v_{zl}^{(1)}$ are not used, we can improve the accuracy of the singularity distributions by using the boundary condition given by equation (31) with the velocity components $v_x^{(1)}$, $v_z^{(1)}$ derived from the shifted first-order singularities.

We note that it follows from equation (B-4) that the source distribution $q^{(1)}(x)$, related to an elliptic aerofoil, induces at the distance $\rho/2 = t^2/4$ upstream of the singularity distribution the velocity

$$v_{xt}^{(1)}\left(x = -\frac{\rho}{2}, 0\right) = -\frac{1 + t^2/2}{\sqrt{1 + t^2/4}} + t$$

which means the stagnation point of the total flow lies at about $x = -\rho/2$. For aerofoils with non-elliptic noses, the distance x_s of the point for which $v_{xt}^{(1)}(-x_s, 0) = -1$ and the leading edge, $x = 0$, of $q^{(1)}(x)$, may differ from $\rho/2$, so that it is likely that a shift of the singularity distributions by x_s instead of by $\rho/2$ may improve the results. For aerofoils where the nose shape differs noticeably from an ellipse, we have still to examine how the results derived in this way compare with exact results and also how the results derived by equations (A-11) and (A-22) compare with exact results.

One might consider modifying the values of Δv_z very close to the leading edge, obtained from equation (31) without a shift of the x-ordinate, by multiplying the values for example by $[1 + (dz_t/dx)^2]^{-1}$; however the resulting values, though finite, vary too rapidly for a numerical computation. For the elliptic aerofoil, the term $\Delta v_{zt}[1 + (dz/dx)^2]^{-1}$, with Δv_{zt} from equation (B-18), behaves near the leading edge like

$$[2\alpha/t][1 - 4x/t^2 + \ldots].$$

It may be useful to summarise the results for two-dimensional aerofoils. We can derive valid approximations to the velocity near the leading edge in the following ways:

(1) We start with the exact solutions, $q^{(1)}(x)$, $l^{(1)}(x)$, of the first-order boundary condition, equation (A-5); compute $v_x^{(1)}(x, 0)$; derive $v_z^{(1)}(x, z_w)$ by means of a Taylor series, equation (A-2); determine the solution $q^{(2)}(x)$, $l^{(2)}(x)$ of the second-order boundary condition, equation (B-1); compute $v_x^{(2)}(x, 0)$; derive a uniformly valid value of V from equation (A-20) or equation (A-22).

(2) We determine the singularity distributions $q^{(2)}(x)$, $l^{(2)}(x)$ as for (1) and compute the velocity at the surface of the aerofoil from the shifted singularities $q^{(2)}(x)$, $l^{(2)}(x)$.

(3) We start with the singularity distributions $q^{(1)}(x)$, $l^{(1)}(x)$ as for (1); compute $v_x^{(1)}(x, z_w)$, $v_z^{(1)}(x, z_w)$ from the shifted singularities; determine $q^{(2)}(x)$, $l^{(2)}(x)$ from the boundary condition, equation (31); compute the velocity at the surface of the aerofoil from the shifted singularities $q^{(2)}(x)$, $l^{(2)}(x)$.

We note that method (1) needs the least amount of computation, but the method does not allow to check the accuracy of the solution which is possible with methods (2) and (3) if one computes not only $v_x^{(2)}(x, z_w)$ but also $v_z^{(2)}(x, z_w)$ from the shifted singularities $q^{(2)}(x)$, $l^{(2)}(x)$. Method (2) requires less computation than method (3), but it uses a Taylor series expansion for $v_z^{(1)}(x, z_w)$, which does not always exist in three dimensions. Further, with method (3) it is allowable that $l^{(1)}(x)$ differs from the exact solution of the first-order boundary condition, equation (23), by second-order terms; this is of no advantage for two-dimensional flow but it can be important for three-dimensional flow.

APPENDIX C

Application of Lighthill's Technique of 'Strained Coordinates' in an Approximate Form

In Appendix B, it is shown that the strong singular behaviour of $\Delta q(x)$ and $\Delta v_{zl}(x, 0)$ given by equations (B-15) and (B-18) can be avoided by applying Lighthill's technique[5,13] of 'strained coordinates'.

For three-dimensional flow, the values of $\Delta q(x, y)$ and $\Delta v_{zl}(x, y, 0)$ given by equations (34), (35) will also have strong singularities at the leading edge. Therefore the question arises whether one can determine an appropriate shift of the first-order singularity distributions and obtain values of $\Delta q(x, y)$ and $\Delta v_{zl}(x, y, 0)$ which behave similar to $q^{(1)}(x, y)$ and $v_{zl}^{(1)}(x, y, 0)$ near the leading edge. Lighthill's technique has not yet been properly extended to a finite swept wing. (It is somewhat uncertain whether the same method is applicable in a case where Taylor series expansions are not possible for all velocity components.) We therefore aim at an approximation.

For the two-dimensional aerofoil, the shift of the singularity distribution produces a noticeable modification of the velocity field only locally close to the leading edge. We may therefore expect that we obtain for the three-dimensional wing a reasonably accurate approximation for most of the wing span by shifting the singularity distributions at each spanwise station rearwards by the amount required for the corresponding two-dimensional sheared wing.

Consider a sheared wing, where the angle of sweep is φ and for which the streamwise section has the nose radius ρ and the chord c. The section normal to the leading edge has the nose radius ρ_N and the chord $\epsilon_N = c \cos \varphi$; since $\rho_N/c_N = (1/\cos^2 \varphi)\rho/c$, it follows that $\rho_N = (1/\cos \varphi)\rho$. The singularity distribution has to be shifted by $\frac{1}{2}\rho_N$ normal to the leading edge, this means by $\frac{1}{2}(1/\cos \varphi)\rho_N = \frac{1}{2}(1/\cos^2 \varphi)\rho$ in the streamwise direction.

We must expect that this amount of shift produces the least satisfactory approximation close to the apex. To obtain some guidance on how to improve the approximation near the apex, we consider an uncambered swept wing at zero angle of incidence for which the streamwise section shape is constant along the span. For this wing the boundary condition reads for $y > 0$:

$$\frac{dz_t}{dx}[1 + v_{xt}(x, y, z_t) - \tan \varphi v_{yt}(x, y, z_t)] = v_{zt}(x, y, z_t).$$

(We note that the term $v_{xt} - \tan \varphi v_{yt}$ is equal to $1/\cos \varphi$ times the perturbation velocity normal to the leading edge.) We have mentioned above that for the elliptic aerofoil the distance $\rho/2$ is approximately equal to the distance between the point, $-x_s$, where $v_{xt}^{(1)}(-x_s, z = 0) = -1$ and the leading edge, $x = 0$, of the distribution $q^{(1)}(x)$. For the swept wing, one would therefore like to know how the distance between the leading edge $x = y \tan\varphi, y, z = 0$ and the points $x_s(y), y, z = 0$ for which

$$v_{xt}^{(1)}(x_s(y), y, 0) - \tan \varphi v_{yt}^{(1)}(x_s(y), y, 0) = -1$$

varies along the span. We have not yet determined values of $x_s(y)$, except for the centre section where the velocity $v_{xt}^{(1)}(x, 0, 0)$ for points in front of the leading edge of the source distribution $q^{(1)}(x, y)$ is equal to $\cos \varphi$ times the velocity $[v_{xt}^{(1)}(x, 0)]_{2D}$ of the two-dimensional unswept source distribution. This implies that the stagnation point at the centre section of the swept wing is closer to the edge of the source distribution than for the two-dimensional unswept section. For a wing with elliptic section shape, the velocity

$$\cos \varphi[v_{xt}^{(1)}(-x_s, 0)]_{2D}$$

is approximately -1 for $x_s = \cos^2 \varphi\, t^2/4 = \cos^2 \varphi\, \rho/2$. (It follows from equation (B-4) that for $x < 0, z = 0$: $v_{xt} = -t(1 + 2|x|)/2\sqrt{|x|}\sqrt{1 + |x| + t}$.) We therefore expect that an appropriate shift of the singularities at the centre section is $\cos^2 \varphi\, \rho(y = 0)/2$, whilst away from the centre section it is $(1/\cos^2 \varphi)\rho(y)/2$. We can expect that the variation from $\rho \cos^2 \varphi$ at $y = 0$ to $\rho/\cos^2 \varphi$ for large y is very rapid near $y = 0$, similar to the rapid change of the measured pressure coefficient at the leading edge.

The varying shift $x_s(y)$ of the source distribution mentioned above would produce a curved leading edge for the source distribution. The computer programs of Refs. 8 and 9 are written for singularity distributions with straight edges. We would therefore obtain a simpler computer program, and probably results of similar accuracy, if we compute from the unshifted singularities the velocity components at the shifted points $x - \frac{1}{2}\rho(y)/[\cos \varphi(y)]^2$.

It is to be expected that, when the velocity components $v_x^{(1)}(x, y, z_w)$, $v_y^{(1)}(x, y, z_w)$, $v_z^{(1)}(x, y, z_w)$ are computed at the shifted points, equations (34), (35) would lead to improved values for $\Delta q(x, y)$ and $\Delta v_{zl}(x, y, 0)$ near the leading edge. Numerical calculations using such a procedure have not yet been done, so that we cannot yet say whether further modifications near the leading edge are necessary. Numerical examples will also show how far behind the leading edge we have to go before it is sufficient to approximate $v_x^{(1)}(x, y, z)$ and $v_y^{(1)}(x, y, z)$ in equations (34), (35) by their Taylor series expansions with respect to z.

APPENDIX D

Velocity Distribution at the Surface of an Ellipsoid

We consider the ellipsoid

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{z_w^2}{c^2} = 1, \tag{D-1}$$

where the rectangular coordinate system ξ, η, z is derived from the system x, y, z by a rotation, i.e.

$$\left.\begin{aligned} \xi &= x \cos \varphi - y \sin \varphi \\ \eta &= x \sin \varphi + y \cos \varphi \end{aligned}\right\}. \tag{D-2}$$

The velocity of the undisturbed flow is parallel to the planes $y = \text{const.}$

Lock[10] has shown (*see* also Refs. 15, 16) that the components of the velocity at the surface of the ellipsoid can be determined from the relations

$$V_x(x, y, z_w)\left[1 + \left(\frac{\partial z_w}{\partial x}\right)^2 + \left(\frac{\partial z_w}{\partial y}\right)^2\right] = (A \cos^2 \varphi + B \sin^2 \varphi) \cos \alpha \left[1 + \left(\frac{\partial z_w}{\partial y}\right)^2\right] + $$

$$+ (A - B) \sin \varphi \cos \varphi \cos \alpha \frac{\partial z_w}{\partial x} \frac{\partial z_w}{\partial y} + C \sin \alpha \frac{\partial z_w}{\partial x}, \tag{D-3}$$

$$V_y(x, y, z_w)\left[1 + \left(\frac{\partial z_w}{\partial x}\right)^2 + \left(\frac{\partial z_w}{\partial y}\right)^2\right] = -(A \cos^2 \varphi + B \sin^2 \varphi) \cos \alpha \frac{\partial z_w}{\partial x} \frac{\partial z_w}{\partial y} -$$

$$- (A - B) \sin \varphi \cos \varphi \cos \alpha \left[1 + \left(\frac{\partial z_w}{\partial x}\right)^2\right] + C \sin \alpha \frac{\partial z_w}{\partial y}, \tag{D-4}$$

$$V_z(x, y, z_w)\left[1 + \left(\frac{\partial z_w}{\partial x}\right)^2 + \left(\frac{\partial z_w}{\partial y}\right)^2\right] = (A \cos^2 \varphi + B \sin^2 \varphi) \cos \alpha \frac{\partial z_w}{\partial x} -$$

$$- (A - B) \sin \varphi \cos \varphi \cos \alpha \frac{\partial z_w}{\partial y} + C \sin \alpha \left[\left(\frac{\partial z_w}{\partial x}\right)^2 + \left(\frac{\partial z_w}{\partial y}\right)^2\right]. \tag{D-5}$$

A, B, C are constants which are related to the lengths of the axes a, b, c of the ellipsoid by the following equations

$$\left.\begin{aligned} A &= \frac{2}{2 - \alpha_0} \\ B &= \frac{2}{2 - \beta_0} \\ C &= \frac{2}{2 - \gamma_0} \end{aligned}\right\} \tag{D-6}$$

and

$$\left.\begin{aligned} \alpha_0 &= abc \int_0^\infty \frac{d\lambda}{\sqrt{(a^2 + \lambda)^3 (b^2 + \lambda)(c^2 + \lambda)}} \\ \beta_0 &= abc \int_0^\infty \frac{d\lambda}{\sqrt{(a^2 + \lambda)(b^2 + \lambda)^3 (c^2 + \lambda)}} \\ \gamma_0 &= abc \int_0^\infty \frac{d\lambda}{\sqrt{(a^2 + \lambda)(b^2 + \lambda)(c^2 + \lambda)^3}} \end{aligned}\right\}. \tag{D-7}$$

We note from equations (D-3), (D-4) that the terms due to incidence $C \sin \alpha \, \partial z_w/\partial x$, $C \sin \alpha \, \partial z_w/\partial y$ in the expressions for V_x, V_y do not vanish at the 'trailing edge' (defined by $z = 0$), i.e. a Kutta–Joukowski condition, as required for a wing with sharp trailing edge, is not satisfied. The given velocity field due to angle of incidence is antisymmetric with respect to the planes $\xi = 0$ and $\eta = 0$, which means the total lift on the ellipsoid is zero.

The equation for the total velocity on the ellipsoid can be written in the form

$$V^2(x, y, z_w)\left[1 + \left(\frac{\partial z_w}{\partial x}\right)^2 + \left(\frac{\partial z_w}{\partial y}\right)^2\right] = \left[(A \cos^2 \varphi + B \sin^2 \varphi) \cos \alpha + C \sin \alpha \frac{\partial z_w}{\partial x}\right]^2 +$$
$$+ \left[-(A - B) \sin \varphi \cos \varphi \cos \alpha + C \sin \alpha \frac{\partial z_w}{\partial y}\right]^2 +$$
$$+ \left[(A \cos^2 \varphi + B \sin^2 \varphi) \cos \alpha \frac{\partial z_w}{\partial y} + \right.$$
$$\left. + (A - B) \sin \varphi \cos \varphi \cos \alpha \frac{\partial z_w}{\partial x}\right]^2. \tag{D-8}$$

We consider now ellipsoids which are similar to wings in that the thickness-to-chord ratio of sections normal to the η-axis, c/a, is small compared to unity. For small values of c/a and $b > a$, the constants A and B have values close to 1·0, i.e. the differences $A - 1{\cdot}0$, $B - 1{\cdot}0$ are of order c/a (*see* e.g. Figs. 2 and 3 of Ref. 16). We want to rewrite the expression for the velocity $V(x, y, z_w)$, given by equation (D-8), in terms of the velocity components $V_x^{(2)}(x, y, z_w)$, $V_y^{(2)}(x, y, z_w)$ derived from a second-order theory. We learn from equations (D-3), (D-4) that approximations to V_x and V_y which are correct to second order are given by the relations

$$V_x^{(2)}(x, y, z_w) = (A \cos^2 \varphi + B \sin^2 \varphi) \cos \alpha + C \sin \alpha \frac{\partial z_w}{\partial x} - \left(\frac{\partial z_w}{\partial x}\right)^2, \tag{D-9}$$

$$V_y^{(2)}(x, y, z_w) = -(A - B) \sin \varphi \cos \varphi \cos \alpha + C \sin \alpha \frac{\partial z_w}{\partial y} - \frac{\partial z_w}{\partial x} \frac{\partial z_w}{\partial y}. \tag{D-10}$$

With these relations, we can write the equation (D-8) for the total velocity $V(x, y, z_w)$ in the form

$$V^2(x, y, z)\left[1 + \left(\frac{\partial z_w}{\partial x}\right)^2 + \left(\frac{\partial z_w}{\partial y}\right)^2\right]$$
$$= \left[V_x^{(2)}(x, y, z_w) + \left(\frac{\partial z_w}{\partial x}\right)^2\right]^2 + \left[V_y^{(2)}(x, y, z_w) + \frac{\partial z_w}{\partial x} \frac{\partial z_w}{\partial y}\right]^2 +$$
$$+ \left\{\left[V_x^{(2)}(x, y, z_w) + \left(\frac{\partial z_w}{\partial x}\right)^2\right]\frac{\partial z_w}{\partial y} - \left[V_y^{(2)}(x, y, z_w) + \frac{\partial z_w}{\partial x} \frac{\partial z_w}{\partial y}\right]\frac{\partial z_w}{\partial x}\right\}^2. \tag{D-11}$$

Following Lock's suggestion, we make use of this relation between the total velocity $V(x, y, z_w)$ and the velocity components $V_x^{(2)}(x, y, z_w)$, $V_y^{(2)}(x, y, z_w)$ by second-order theory also for general wing shapes. Therefore, we consider some details of equation (D-11).

Velocity components $V_x^{(2)}(x, y, z_w)$, $V_y^{(2)}(x, y, z_w)$ which are correct to second order are not uniquely defined, because they can differ by third- or higher-order terms. If we were to compute for an ellipsoid at zero angle of incidence a source distribution $q^{(2)}(x, y)$ as described in Section 2.3 and from this the velocity components $v_{xt}^{(2)}(x, y, z_w)$, $v_{yt}^{(2)}(x, y, z_w)$, then the terms

$$V_x^{(2)}(x, y, z_w) + \left(\frac{\partial z_w}{\partial x}\right)^2 = 1 + v_{xt}^{(2)}(x, y, z_w) + \left(\frac{\partial z_w}{\partial x}\right)^2$$

and

$$V_y^{(2)}(x, y, z_w) + \frac{\partial z_w}{\partial x} \frac{\partial z_w}{\partial y} = v_{yt}^{(2)}(x, y, z_w) + \frac{\partial z_w}{\partial x} \frac{\partial z_w}{\partial y}$$

would not be constant as are those defined by equations (D-9), (D-10); they would have strong singularities at the leading and trailing edge. However, if we were to compute first $v_{xt}^{(1)}(x, y, 0)$, $v_{yt}^{(1)}(x, y, 0)$ (which are independent of x and y) and use equation (59) to derive $\Delta q(x, y)$, then also $v_{xt}^{(2)}(x, y, 0)$ and $v_{yt}^{(2)}(x, y, 0)$ are independent of x and y. If we use then the Taylor series expansions, equations (63), (64), to derive the velocity components at the surface, we obtain

$$V_x^{(2)}(x, y, z_w) + \left(\frac{\partial z_w}{\partial x}\right)^2 = 1 + v_{xt}^{(2)}(x, y, 0) + \frac{\partial}{\partial x}\left(z_w \frac{\partial z_w}{\partial x}\right), \tag{D-12}$$

$$V_y^{(2)}(x, y, z_w) + \frac{\partial z_w}{\partial x}\frac{\partial z_w}{\partial y} = v_{yt}^{(2)}(x, y, 0) + \frac{\partial}{\partial y}\left(z_w \frac{\partial z_w}{\partial x}\right). \tag{D-13}$$

The terms on the right-hand sides of equations (D-12), (D-13) do not contain singularities but are constant over the planform of the ellipsoid. If the terms given by equations (D-12), (D-13) are inserted into equation (D-11), then we obtain uniformly valid values for the velocity V at the surface of the ellipsoid, which are correct to second order.

If we were to determine a second-order solution for the ellipsoid at an angle of incidence for which the Kutta–Joukowski condition is satisfied at the trailing edge, by using equations (25), (27) and (60), then we can expect that $v_{xl}^{(2)}(x, y, 0)$ would behave near the leading edge like $1/\sqrt{x - x_L(y)}$, i.e. like $\partial z_w/\partial x$ and $v_{yl}^{(2)}(x, y, 0)$ like $\partial z_w/\partial y$. The expressions

$$V_x^{(2)}(x, y, z_w) + \left(\frac{\partial z_w}{\partial x}\right)^2 = \cos\alpha + v_{xt}^{(2)}(x, y, 0) \pm v_{xl}^{(2)}(x, y, 0) + \frac{\partial}{\partial x}\left(z_w \frac{\partial z_w}{\partial x}\right) \tag{D-14}$$

and

$$V_y^{(2)}(x, y, z_w) + \frac{\partial z_w}{\partial x}\frac{\partial z_w}{\partial y} = v_{yt}^{(2)}(x, y, 0) \pm v_{yl}^{(2)}(x, y, 0) + \frac{\partial}{\partial y}\left(z_w \frac{\partial z_w}{\partial x}\right) \tag{D-15}$$

(where $v_{xt}^{(2)}(x, y, 0)$, $v_{yt}^{(2)}(x, y, 0)$, $v_{xl}^{(2)}(x, y, 0)$, $v_{yl}^{(2)}(x, y, 0)$ are determined from equations (25) to (27), (59), (60)) have thus the same behaviour as the terms given by equations (D-9), (D-10). By inserting equations (D-14), (D-15) into equation (D-11), we therefore obtain uniformly valid values of the velocity at the surface of an ellipsoid which are correct to second order.

The third term in equation (D-11)

$$\begin{aligned}
&\left\{\left[V_x^{(2)}(x, y, z_w) + \left(\frac{\partial z_w}{\partial x}\right)^2\right]\frac{\partial z_w}{\partial y} - \left[V_y^{(2)}(x, y, z_w) + \frac{\partial z_w}{\partial x}\frac{\partial z_w}{\partial y}\right]\frac{\partial z_w}{\partial x}\right\}^2 \\
&= \left\{\left[\cos\alpha + v_{xt}^{(2)}(x, y, 0) \pm v_{xl}^{(2)}(x, y, 0) + \frac{\partial}{\partial x}\left(z_w \frac{\partial z_w}{\partial x}\right)\right]\frac{\partial z_w}{\partial y} - \right. \\
&\left. - \left[v_{yt}^{(2)}(x, y, 0) \pm v_{yl}^{(2)}(x, y, 0) + \frac{\partial}{\partial y}\left(z_w \frac{\partial z_w}{\partial x}\right)\right]\frac{\partial z_w}{\partial x}\right\}^2
\end{aligned}$$

can be modified, without losing the second-order accuracy, into

$$\left\{[\cos\alpha + v_{xt}^{(2)}(x, y, 0)]\frac{\partial z_w}{\partial y} - [v_{yt}^{(2)}(x, y, 0)]\frac{\partial z_w}{\partial x}\right\}^2$$

because the term $v_{xl}^{(2)}(x, y, 0)\partial z_w/\partial y - v_{yl}^{(2)}(x, y, 0)\partial z_w/\partial x$ vanishes, at least near the leading edge. Further, it follows from equations (D-1), (D-2) that

$$\frac{\partial}{\partial x}\left(z_w \frac{\partial z_w}{\partial x}\right) = -\frac{c^2}{a^2}\left[\cos^2\varphi + \frac{a^2}{b^2}\sin^2\varphi\right],$$

$$\frac{\partial}{\partial y}\left(z_w \frac{\partial z_w}{\partial x}\right) = \frac{c^2}{a^2}\sin\varphi\cos\varphi\left[1 - \frac{a^2}{b^2}\right],$$

which means that these terms are of second order.

Summarising, we can state that we obtain uniformly valid values for the velocity at the surface of an ellipsoid which are correct to second order from the relation

$$V^2(x, y, z_w)\left[1 + \left(\frac{\partial z_w}{\partial x}\right)^2 + \left(\frac{\partial z_w}{\partial y}\right)^2\right] = \left[\cos\alpha + v_{xt}^{(2)}(x, y, 0) \pm v_{xl}^{(2)}(x, y, 0) + \frac{\partial}{\partial x}\left(z_w \frac{\partial z_w}{\partial x}\right)\right]^2 +$$

$$+ \left[v_{yt}^{(2)}(x, y, 0) \pm v_{yl}^{(2)}(x, y, 0) + \frac{\partial}{\partial y}\left(z_w \frac{\partial z_w}{\partial x}\right)\right]^2 +$$

$$+ \left\{[\cos\alpha + v_{xt}^{(2)}(x, y, 0)]\frac{\partial z_w}{\partial y} - v_{yt}^{(2)}(x, y, 0)\frac{\partial z_w}{\partial x}\right\}^2. \qquad \text{(D-16)}$$

APPENDIX E

A Modification to the 'Basic Formula' in the RAE Standard Method[3] to include all Second-Order Terms for the Sheared Wing

Applying equations (A-11) and (A-22) of Appendix A to the infinite sheared wing, we obtain

$$V^2 = \cos^2\alpha\sin^2\varphi + \frac{\left\{\cos\alpha\cos\varphi\left[1 + \frac{S^{(1)}}{\cos\varphi} \pm \frac{S^{(4)}}{\cos\varphi} + \frac{S^{(40)}}{\cos^2\varphi} \pm \frac{S^{(41)}}{\cos^2\varphi} + \frac{1}{\cos^2\varphi}\frac{\partial}{\partial x}\left(z_w\frac{\partial z_w}{\partial x}\right)\right] + \sin\alpha\left[\frac{S^{(44)}}{\cos\varphi} \pm \sqrt{\frac{1-x}{x}}\left(1 + \frac{S^{(3)}}{\cos\varphi}\right)\right]\right\}^2}{1 + \left(\frac{\partial z_w/\partial x}{\cos\varphi}\right)^2}. \quad \text{(E-1)}$$

This equation can be written in the form of equation (69) with

$$v_x^{(2)}(x, z_w) = \cos\varphi S^{(1)} \pm \cos\varphi S^{(4)} + S^{(40)} \pm S^{(41)} + z_w\frac{\partial^2 z_w}{\partial x^2} + \sin\alpha\left[S^{(44)} \pm \cos\varphi\sqrt{\frac{1-x}{x}} \pm S^{(3)}\sqrt{\frac{1-x}{x}}\right], \quad \text{(E-2)}$$

$$v_y^{(2)}(x, z_w) = -\tan\varphi v_x^{(2)}(x, z_w). \quad \text{(E-3)}$$

We write $v_x^{(2)}$ as the sum of the terms which are included in the formula of the standard method given in Ref. 3 and additional terms:

$$v_x^{(2)}(x, z_w) = \cos\varphi S^{(1)}(x) \pm \cos\varphi S^{(4)}(x) \pm \sin\alpha\cos\varphi\sqrt{\frac{1-x}{x}}\left[1 + \frac{S^{(3)}(x)}{\cos\varphi}\right] + \left[S^{(40)} + z_t\frac{\partial^2 z_t}{\partial x^2} + z_s\frac{\partial^2 z_s}{\partial x^2} + \sin\alpha S^{(44)}\right] \pm \left[S^{(41)} + z_t\frac{\partial^2 z_s}{\partial x^2} + z_s\frac{\partial^2 z_t}{\partial x^2}\right]. \quad \text{(E-4)}$$

The formula for the velocity at any spanwise station as given in Ref. 3 reads:

$$\begin{aligned} V^2\left[1 + \left(\frac{\partial z_w/\partial x}{\cos\varphi^*}\right)^2\right] = &\left[\cos\alpha_e\left\{1 + K_3\cos\varphi_t S^{(1)} - \frac{K_2\cos\varphi_t f(\varphi_t)\,\partial z_t/\partial x}{\sqrt{1 + (\partial z_t/\partial x)^2}}\right\} \pm \left\{\frac{\cos\alpha_e\cos\varphi_m}{\cos(\lambda\varphi_m)}\frac{\gamma}{2} + \right.\right. \\ &\left.\left. + H\sin\alpha_e\cos\varphi_m\left[1 + \frac{S^{(3)}}{\cos\varphi^*}\right]\left(\frac{1-x}{x}\right)^n\right\}\left[1 + \left(\frac{\partial z_w/\partial x}{\cos\varphi^*}\right)^2\right]^{0\cdot5-n}\right]^2 + \\ &+ \left[\cos\alpha_e(1 - |K_2|)\sin\varphi_t S^{(1)} \pm \left\{\cos\alpha_e\frac{\sqrt{\cos^2(\lambda\varphi_m) - \cos^2\varphi_m}}{\cos(\lambda\varphi_m)}\frac{\gamma}{2} + \right.\right. \\ &\left.+ H\sin\alpha_e\sqrt{\cos^2(\lambda\varphi_m) - \cos^2\varphi_m}\left[1 + \frac{S^{(3)}}{\cos\varphi^*}\right]\left(\frac{1-x}{x}\right)^n\right\} \times \\ &\left.\times\left[1 + \left(\frac{\partial z_w/\partial x}{\cos\varphi^*}\right)^2\right]^{0\cdot5-n}\right]^2 + (1 - K_2^2)\sin^2\varphi_t\cos^2\alpha_e\left(\frac{\partial z_w/\partial x}{\cos\varphi^*}\right)^2. \end{aligned} \quad \text{(E-5)}$$

We note the similarity of this formula with equation (69). We suggest modifying only the first term on the right-hand side of equation (E-5), which means modifying only the v_x contribution into

$$v_x^{(2)}(x, y, 0) + \frac{\partial}{\partial x}\left(z_w\frac{\partial z_w}{\partial x}\right).$$

To do this, we add to the symmetrical term (i.e. the approximate $v_{xt}^{(1)}(x, y, 0)$) a term corresponding to the term

$$\Delta v_{xt} = S^{(40)} + \alpha_e S^{(44)} + \frac{\partial}{\partial x}\left[z_t\frac{\partial z_t}{\partial x} + z_s\frac{\partial z_s}{\partial x}\right] \quad \text{(E-6)}$$

of the sheared wing and to the antisymmetric term (i.e. the approximate $v_{xl}^{(1)}(x, y, 0)$) a term corresponding to

$$\Delta v_{xl} = S^{(41)} + \frac{\partial}{\partial x}\left(z_t \frac{\partial z_s}{\partial x} + z_s \frac{\partial z_t}{\partial x}\right). \tag{E-7}$$

For the sheared wing, the term $S^{(40)} + \alpha S^{(44)}$ takes account of the source distribution Δq given by equation (59), which reads in this case

$$\frac{\Delta q(x, y)}{2} = \frac{1}{\cos^2 \varphi} \frac{\partial}{\partial x}[z_t(x, y) v_{xt}^{(1)}(x, y, 0) + z_s(x, y) v_{xl}^{(1)}(x, y, 0)]. \tag{E-8}$$

Equation (59) however is not applicable at the centre section if $v_{yt}^{(1)}(x, y, 0)$ is the value of the spanwise velocity in the plane $z = 0$ derived from the source distribution $q^{(1)}(x, y) = 2\,\partial z_t/\partial x$ because $\partial v_{yt}^{(1)}/\partial y$ is infinite for $y \to 0$. The singularity is only logarithmic; if one uses instead of $v_{yt}^{(1)}(x, y, 0)$ an approximate value v_{yt}^* for which $\partial v_{yt}^*/\partial y$ is finite for $y \to 0$, then one may expect that equation (59) produces a reasonably accurate approximation to Δq for any spanwise station of an uncambered wing. Such an approximate value of v_{yt}^* is given by the standard method, where

$$v_{yt}^* = -(1 - |K_2(y)|) \sin \varphi_t S^{(1)} \tag{E-9}$$

and

$$v_{xt}^* = \cos \varphi_t S^{(1)} - K_2 \cos \varphi_t f(\varphi_t) \frac{\partial z_t}{\partial x}. \tag{E-10}$$

The approximate values of $v_{xl}^{(1)}$, $v_{yl}^{(1)}$, given by the standard method, read

$$v_{xl}^* = \pm \frac{\cos \varphi_m}{\cos (\lambda \varphi_m)} \frac{\gamma}{2} \pm H\alpha_e \cos \varphi_m \left(\frac{1 - x}{x}\right)^n, \tag{E-11}$$

$$v_{yl}^* = -\tan \varphi_v v_{xl}^*, \tag{E-12}$$

where φ_v is the angle of sweep of the bound vortices. φ_v varies between $\varphi_v = 0$ at $y = 0$ and $\varphi_v = \varphi_m$ far away from the centre section. The standard method uses the interpolation function

$$\tan \varphi_v = \frac{\sqrt{\cos^2 (\lambda \varphi_m) - \cos^2 \varphi_m}}{\cos \varphi_m}. \tag{E-13}$$

The spanwise derivative of v_{yl}^* as given by equations (E-12), (E-13) is infinite for $y \to 0$. If we want to use v_{xl}^* and v_{yl}^* in conjunction with equations (59) and (61), we have to modify the function for $\tan \varphi_v$. One possible choice would be

$$\begin{aligned} \tan\varphi_v(y) &= \frac{dx_{ac}(y)}{dy} \\ &= \tan \varphi_m + \frac{\varphi_m}{2\pi} \frac{d\lambda(y)}{dy} \\ &= \tan \varphi_m \frac{\dfrac{2\pi \tan \varphi_m}{\varphi_m} y}{\sqrt{1 + \left(\dfrac{2\pi \tan \varphi_m}{\varphi_m} y\right)^2}}. \end{aligned} \tag{E-14}$$

When $l^{(1)}(x, y)$ is an exact solution of equation (27) and $v_{yl}^{(1)}(x, y, 0)$ is determined from equation (47), then we may expect that the spanwise derivative of $v_{yl}^{(1)}$ is finite for $y \to 0$.

When we insert the values of v^*_{xt}, v^*_{yt}, v^*_{xl}, v^*_{yl}, given by equations (E-9) to (E-12), (E-14), into equations (59), (60), then we obtain values for an additional source distribution Δq and an additional required upwash Δv_{zl}. One might consider to determine from Δq and Δv_{zl} values of Δv_{xt} and Δv_{xl} in a manner which is similar to that for determining v^*_{xt} and v^*_{xl} from $q^{(1)}(x, y) = 2\,\partial z_t/\partial x$ and $v^{(1)}_{zl} = \partial z_s/\partial x - \alpha$. We suggest deferring the corresponding modification of equation (E-5) until we have done some sample calculations using the methods of Sections 2.3 to 2.5. We therefore include only the second-order terms which are appropriate to a 'sheared wing', this means we propose to use Δv_{xt} from equation (E-6) and Δv_{xl} from equation (E-8), where the terms $S^{(40)}$, $S^{(41)}$, $S^{(44)}$ are derived from the section shape at the spanwise station y under consideration and equations (A-7), (A-8), (A-13) to (A-15). The modified formula which would replace equation (3) of Ref. 3 and equation (E-5) reads then

$$\begin{aligned}
V^2\left[1 + \left(\frac{\partial z_w/\partial x}{\cos\varphi^*}\right)^2\right] = \Bigg[\cos\alpha_e\Bigg\{1 &+ K_3\cos\varphi_t S^{(1)} - \frac{K_2\cos\varphi_t f(\varphi_t)\,\partial z_t/\partial x}{\sqrt{1 + (\partial z_t/\partial x)^2}} + S^{(40)} + \alpha_e S^{(44)} + \\
&+ \frac{\partial}{\partial x}\left[z_t\frac{\partial z_t}{\partial x} + z_s\frac{\partial z_s}{\partial x}\right]\Bigg\} \pm \Bigg\{\frac{\cos\alpha_e\cos\varphi_m}{\cos(\lambda\varphi_m)}\frac{\gamma}{2} + H\sin\alpha_e\cos\varphi_m \times \\
&\times\left[1 + \frac{S^{(3)}}{\cos\varphi^*}\right]\left(\frac{1-x}{x}\right)^n + H\frac{\cos\varphi_m}{\cos\varphi^*}\left(\frac{1-x}{x}\right)^{n-0\cdot5} S^{(41)} + \\
&+ \frac{\partial}{\partial x}\left[z_t\frac{\partial z_s}{\partial x} + z_s\frac{\partial z_t}{\partial x}\right]\Bigg\}\left[1 + \left(\frac{\partial z_w/\partial x}{\cos\varphi^*}\right)^2\right]^{0\cdot5-n}\Bigg]^2 + \\
&+ \Bigg[\cos\alpha_e(1 - |K_2|)\sin\varphi_t S^{(1)} \pm \Bigg\{\cos\alpha_e\frac{\sqrt{\cos^2(\lambda\varphi_m) - \cos^2\varphi_m}}{\cos(\lambda\varphi_m)}\frac{\gamma}{2} + \\
&+ H\sin\alpha_e\sqrt{\cos^2(\lambda\varphi_m) - \cos^2\varphi_m}\left[1 + \frac{S^{(3)}}{\cos\varphi^*}\right]\left(\frac{1-x}{x}\right)^n\Bigg\} \times \\
&\times\left[1 + \left(\frac{\partial z_w/\partial x}{\cos\varphi^*}\right)^2\right]^{0\cdot5-n}\Bigg]^2 + \\
&+ (1 - K_2^2)\sin^2\varphi_t\cos^2\alpha_e\left(\frac{\partial z_w/\partial x}{\cos\varphi^*}\right)^2. \qquad \text{(E-15)}
\end{aligned}$$

One could of course modify the higher order terms in equation (E-15) such that $V(x, y, z_w)$ behaves near the leading edge like the velocity given by equation (69). Such a formula may read

$$\begin{aligned}
V^2(x, y, z_w)\left[1 + \left(\frac{\partial z_w/\partial x}{\cos\varphi^{**}}\right)^2\right] = \Bigg[\cos\alpha_e\Bigg\{1 &+ K_3\cos\varphi_t S^{(1)} - \frac{K_2\cos\varphi_t f(\varphi_t)\,\partial z_t/\partial x}{\sqrt{1 + (\partial z_t/\partial x)^2}} + S^{(40)} + \alpha_e S^{(44)} + \\
&+ \frac{\partial}{\partial x}\left[z_t\frac{\partial z_t}{\partial x} + z_s\frac{\partial z_s}{\partial x}\right]\Bigg\} \pm \Bigg\{\frac{\cos\alpha_e\cos\varphi_m}{\cos(\lambda\varphi_m)}\frac{\gamma}{2} + H\sin\alpha_e\cos\varphi_m \times \\
&\times\left[1 + \frac{S^{(3)}}{\cos\varphi^*}\right]\left(\frac{1-x}{x}\right)^n + H\frac{\cos\varphi_m}{\cos\varphi^*}\left(\frac{1-x}{x}\right)^{n-0\cdot5} S^{(41)} + \\
&+ \frac{\partial}{\partial x}\left[z_t\frac{\partial z_s}{\partial x} + z_s\frac{\partial z_t}{\partial x}\right]\Bigg\}\left[1 + \left(\frac{\partial z_w/\partial x}{\cos\varphi^{**}}\right)^2\right]^{0\cdot5-n}\Bigg]^2 + \\
&+ \Bigg[\cos\alpha_e(1 - |K_2|)\sin\varphi_t S^{(1)} \pm \Bigg\{\cos\alpha_e\frac{\sqrt{\cos^2(\lambda\varphi_m) - \cos^2\varphi_m}}{\cos(\lambda\varphi_m)}\frac{\gamma}{2} + \\
&+ H\sin\alpha_e\sqrt{\cos^2(\lambda\varphi_m) - \cos^2\varphi_m}\left[1 + \frac{S^{(3)}}{\cos\varphi^*}\right]\left(\frac{1-x}{x}\right)^n\Bigg\} \times \\
&\times\left[1 + \left(\frac{\partial z_w/\partial x}{\cos\varphi^{**}}\right)^2\right]^{0\cdot5-n}\Bigg]^2 + \\
&+ \Bigg\{\left[1 + K_3\cos\varphi_t S^{(1)} - \frac{K_2\cos\varphi_t f(\varphi_t)\,\partial z_t/\partial x}{\sqrt{1 + (\partial z_t/\partial x)^2}}\right]\tan\varphi^{**} - \\
&- (1 - |K_2|)\sin\varphi_t S^{(1)}\Bigg\}^2\left(\frac{\partial z_w}{\partial x}\right)^2 \qquad \text{(E-16)}
\end{aligned}$$

with

$$\varphi^{**}(y) = \sqrt{1 - K_2^2(y)}\varphi(y). \tag{E-17}$$

APPENDIX F

Some Comments on Equation (69)

We have mentioned in Section 2.5 that equation (69) is rather similar to the basic formula of the standard method, equation (E-5), but that, close to the leading edge, there are some differences. With the term $(\partial z_w/\partial x)^2/\cos^2\varphi$ in the denominator of equation (69), we use in equation (69) the geometric angle of sweep of the leading edge, whilst in Ref. 3 we use for φ a function, $\varphi^* = [1 - |K_2(y)|]\varphi_t(x)$, which varies between $\varphi = 0$ at the centre section and the sweep $\varphi_t(x)$ of lines of constant percentage chord, $[x - x_L(y)]/c(y) = \text{const.}$, further outboard. We cannot yet say whether the results from equation (69) would be improved or otherwise, if we were to change the value of $\varphi(y)$.

Instead of the term

$$\{[1 + v_{xt}^{(1)}(x, y, 0)] \tan\varphi + v_{yt}^{(1)}(x, y, 0)\}^2 \left(\frac{\partial z_w}{\partial x}\right)^2$$

in equation (69) the standard method contains the term

$$[1 - K_2^2(y)] \frac{\sin^2\varphi_t}{\cos^2\varphi^*} \left(\frac{\partial z_w}{\partial x}\right)^2.$$

For an uncambered wing at zero angle of incidence, equation (69) gives for the velocity at the leading edge away from the centre section:

$$V(x_L, y, 0) = [1 + v_x^{(1)}(x_L, y, 0)] \sin\varphi + v_y^{(1)}(x_L, y, 0) \cos\varphi \tag{F-1}$$

whilst the standard method gives

$$V(x_L, y, 0) = \sqrt{1 - K_2^2(y)} \sin\varphi. \tag{F-2}$$

Equation (F-2) was chosen to represent the rapid spanwise variation near the apex of the pressure measured at the leading edge and to give at the apex of an uncambered wing at zero angle of incidence the correct value $V(x_L, 0, 0) = 0$. Equation (F-2) does not however represent the fact that the spanwise variation of $V(x_L, y, 0)$ will depend on the thickness of the wing. Such a dependence is given by equation (F-1).

Equation (69) does not however give the correct value of $V(x_L(y), y \to 0)$ at the centre section of an uncambered swept wing at zero angle of incidence, namely zero. This is due to the fact that $v_{xt}^{(1)}(x, y = 0, 0)$ contains a term proportional to $q^{(1)}(x, 0)$, i.e. to $2\,\partial z_t(x, 0)/\partial x$. If we modify $\Delta q(x, y)$ from equation (34) so that it behaves similarly to $q^{(1)}(x, y)$ near the leading edge, then $\Delta v_{xt}(x, 0, 0)$ and $v_{xt}^{(2)}(x, 0, 0)$ behave also like $1/\sqrt{\xi}$ near the leading edge. Therefore, $V(x, 0, z_w)$ from equation (69) is infinite at the leading edge. This 'fault' of equation (69) is in practice not a serious one, since it affects only a minute area of the wing; the failure is not surprising because equation (68), derived from the flow past an ellipsoid, is not applicable.

TABLE 1

Difference between the exact values of the velocity components induced at the surface of elliptic aerofoils ($z = t\sqrt{x(1-x)}$) by the source distribution

$$q(x) = t\frac{1-2x}{\sqrt{x(1-x)}}$$

and their second-order approximations.

x	$\frac{v_{xt}(x,z) - v_{xt}^{(2*)}(x,z)}{v_{xt}(x,0)}$		$\frac{v_{zt}(x,z) - v_{zt}^{(2*)}(x,z)}{v_{zt}(x,0)}$	
	$t = 0{\cdot}1$	$t = 0{\cdot}15$	$t = 0{\cdot}1$	$t = 0{\cdot}15$
0	∞	∞	−1	−1
0·01	0·885	1·983	−0·227	−0·350
0·02	0·277	0·705	−0·143	−0·247
0·03	0·134	0·364	−0·105	−0·194
0·04	0·078	0·222	−0·084	−0·160
0·05	0·051	0·149	−0·070	−0·137
0·06	0·036	0·107	−0·060	−0·120
0·07	0·027	0·080	−0·053	−0·107
0·08	0·020	0·062	−0·047	−0·097
0·09	0·016	0·049	−0·043	−0·090
0·10	0·013	0·040	−0·039	−0·082
0·15	0·006	0·018	−0·028	−0·061
0·20	0·003	0·009	−0·023	−0·050
0·25	0·002	0·006	−0·020	−0·043
0·30	0·001	0·004	−0·018	−0·039
0·35	0·001	0·003	−0·016	−0·036
0·40	0·001	0·002	−0·015	−0·034
0·45		0·002	−0·015	−0·033
0·50		0·002		

TABLE 2

Difference between the exact values of the velocity components induced at the surface of elliptic aerofoils ($z = t\sqrt{x(1 - x)}$) by the load distribution

$$l(x) = 4\alpha\sqrt{\frac{1 - x}{x}}$$

and their second-order approximations.

x	$\frac{v_{xl}(x, z) - v_{xl}^{(2*)}(x, z)}{v_{xl}(x, 0)}$		$\frac{v_{zl}(x, z) - v_{zl}^{(2*)}(x, z)}{v_{zl}(x, 0)}$	
	$t = 0{\cdot}1$	$t = 0{\cdot}15$	$t = 0{\cdot}1$	$t = 0{\cdot}15$
0	−1	−1	∞	∞
0·01	−0·223	−0·344	1·765	3·955
0·02	−0·139	−0·240	0·552	1·403
0·03	−0·101	−0·186	0·266	0·723
0·04	−0·079	−0·151	0·156	0·441
0·05	−0·065	−0·128	0·101	0·296
0·06	−0·055	−0·110	0·071	0·211
0·07	−0·048	−0·097	0·052	0·158
0·08	−0·042	−0·087	0·040	0·122
0·09	−0·038	−0·078	0·031	0·097
0·10	−0·034	−0·071	0·025	0·079
0·15	−0·023	−0·049	0·011	0·034
0·20	−0·017	−0·036	0·006	0·018
0·25	−0·013	−0·029	0·003	0·011
0·30	−0·011	−0·023	0·002	0·007
0·35	−0·009	−0·019	0·001	0·005
0·40	−0·007	−0·016	0·001	0·003
0·45	−0·006	−0·013	0·001	0·002
0·50	−0·005	−0·011		0·002

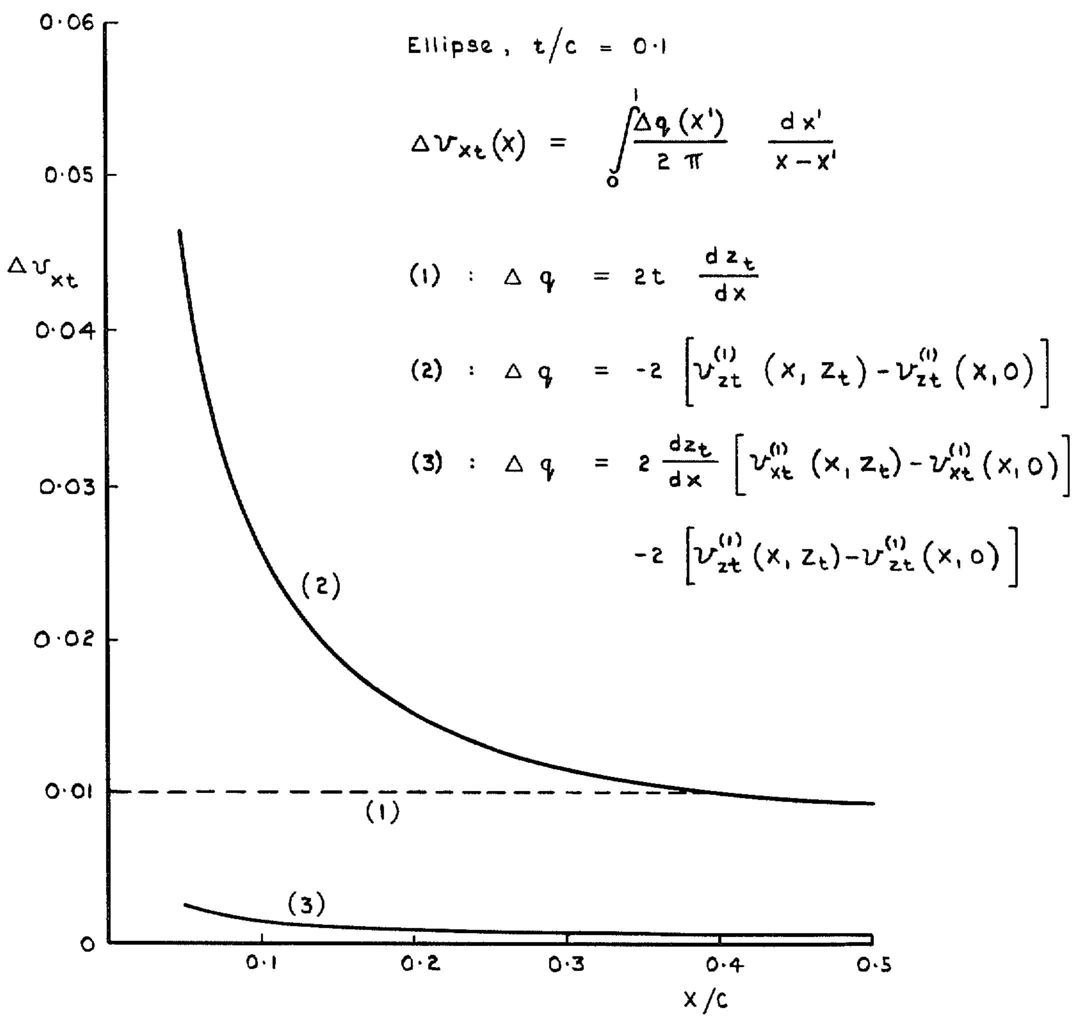

FIG. 1 Various incremental velocities for the ellipse at zero angle of incidence, see equations (B-12) to (B-16).

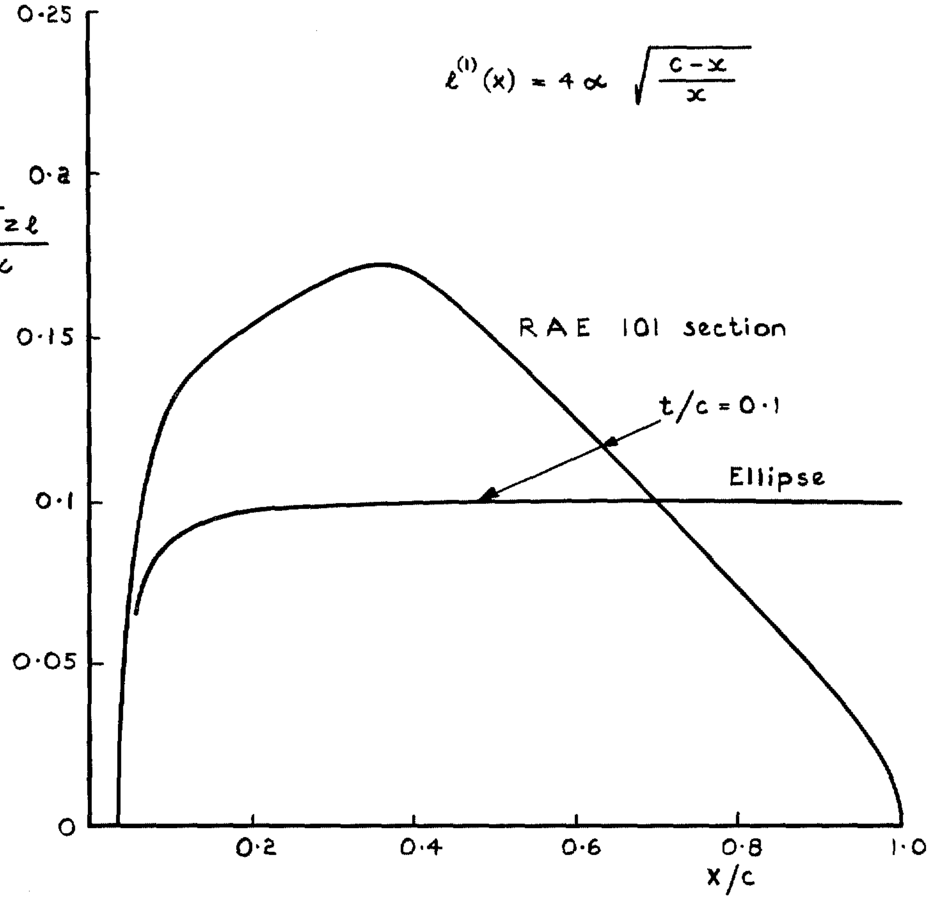

FIG. 2 Downwash to be produced by the additional load distribution $\Delta l(x)$ for two-dimensional aerofoils at an angle of incidence, see equation (44).

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