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A Form of the Supersonic Flow Equations for an Ideal Gas

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Summary

Non-linear partial differential equations that govern the steady supersonic three-dimensional flow of an inviscid ideal gas with constant specific heats in an arbitrary curvilinear co-ordinate space are derived. A special form of these equations is constructed for the case when two families of stream surfaces and one family of parallel planes are used as co-ordinates. Characteristic equations equivalent to this special form are obtained. The characteristic equations are suitable for numerical integration of the equations of motion to obtain solutions of mixed initial and boundary value problems. The characteristic equations are a generalized form of similar equations used by Walkden and Caine[4] as the basis of a numerical method of calculating supersonic two-dimensional and axi-symmetric flow fields.

* Replaces A.R.C. 34 160.

LIST OF CONTENTS

1. Introduction

The prediction of pressure distributions generated when aerodynamic body shapes move with steady supersonic speed is an important practical problem which, for complex shapes at least, has not yet been solved satisfactorily.

The non-linear partial differential equations which describe the motion of an inviscid ideal gas with constant specific heats have to be integrated numerically, subject to the condition of zero flow through the body surface. This integration process is often complicated by the existence of numerous shock waves.

When the flow is supersonic everywhere, the equations of steady motion are hyperbolic and the mathematical problem which has to be solved is a mixed initial and boundary-value problem. If a finite difference method is used, the ease and accuracy with which boundary conditions can be applied, and the effectiveness with which shock waves can be treated, depends upon the way in which the equations of motion are formulated prior to the construction of finite difference equations.[1,2]

A wide variety of steady supersonic two-dimensional and axi-symmetric flows containing shock waves have been calculated using a shock-capturing numerical method based on a particular form of the equations governing two-dimensional and axi-symmetric flow.[3,4] The success with which these mixed initial and boundary-value problems have been treated indicates that a similar technique ought to be tried for three-dimensional problems. An appropriate form of the equations which govern the steady supersonic flow of an inviscid ideal gas in three dimensions is derived here.

In Section 2 the equations of motion which express conservation of mass, momentum and energy are formulated so that two families of stream-surfaces and one family of parallel planes are co-ordinates. Then, in Section 3, by treating as parameters all partial derivatives with respect to a variable which defines one family of stream surfaces, it is shown that characteristic curves and corresponding characteristic relations which have the remaining two co-ordinates as independent variables can be constructed. The equations obtained by following the procedure outlined here represent a simple generalisation of similar equations[5] which describe steady supersonic two-dimensional or axi-symmetric flow.

In Appendix B it is verified that the equations of motion given in Section 3 for three-dimensional flow do reduce to the equations of Ref. 5 when the flow is two-dimensional or axi-symmetric.

A brief description of the way in which the equations derived here can be used to construct a method for solving mixed initial and boundary-value problems numerically is given in Section 4.

2. Conservation Law Equations

In this section, starting from an invariant vector form of the equations governing the steady motion of an inviscid ideal gas with constant specific heats, a representation of the equations of motion in a general system of non-orthogonal co-ordinates is constructed. Then the special form of the equations, in which two families of stream surfaces and one family of parallel planes are co-ordinates, is obtained.

If the symbols p, ρ, $\mathbf{v}$ and H are chosen to represent pressure, density, the velocity vector and the total enthalpy respectively then the equations of motion which represent conservation of mass, momentum and energy can be written in the following invariant form:

mass:

$$\operatorname{div}(\rho\mathbf{v}) = 0 \tag{1}$$

momentum:

$$\rho\,\operatorname{grad}(\mathbf{v}\,.\,\mathbf{v}) - 2\rho\mathbf{v} \times \operatorname{curl}\mathbf{v} = -2\,\operatorname{grad} p \tag{2}$$

energy:

$$\rho(\gamma-1)\mathbf{v}\,.\,\mathbf{v} + 2\gamma p = 2\rho H(\gamma-1) \tag{3}$$

where γ is the ratio of specific heats. In most cases of interest in supersonic flow, the total enthalpy H is constant.

At this point, a digression is necessary. Before writing down the form taken by equations (1) to (3) in an arbitrary curvilinear co-ordinate system, some useful relationships concerning co-ordinate systems based on non-orthogonal families of surfaces will be listed. Many results which are only mentioned briefly here are explained fully elsewhere, e.g. by Stratton.[6]

Let

$$\xi^{(i)} = \xi^{(i)}(x^{(1)}, x^{(2)}, x^{(3)}) \qquad i = 1, 2, 3 \tag{4}$$

be three arbitrary functional relationships which describe distinct non-degenerate co-ordinate surfaces $\xi^{(i)} =$ = constant in terms of co-ordinates $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ of either an orthogonal Cartesian system or a cylindrical polar system; and let $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ be unit vectors in the directions of the axes of the orthogonal system in which the co-ordinates are represented by $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$.

Now six vectors associated with the surfaces $\xi^{(i)}$ = constant $i = 1, 2, 3$, can be defined. They are

$$\mathbf{a}_{(i)} = \left(\frac{\partial x^{(1)}}{\partial \xi^{(i)}}\right)\mathbf{i}_1 + \left(\frac{\partial x^{(2)}}{\partial \xi^{(i)}}\right)\mathbf{i}_2 + (x^{(2)})^r\left(\frac{\partial x^{(3)}}{\partial \xi^{(i)}}\right)\mathbf{i}_3 \qquad (i = 1, 2, 3) \tag{5}$$

and

$$\mathbf{a}^{(i)} = \left(\frac{\partial \xi^{(i)}}{\partial x^{(1)}}\right)\mathbf{i}_1 + \left(\frac{\partial \xi^{(i)}}{\partial x^{(2)}}\right)\mathbf{i}_2 + \frac{1}{(x^{(2)})^r}\left(\frac{\partial \xi^{(i)}}{\partial x^{(3)}}\right)\mathbf{i}_3 \qquad (i = 1, 2, 3) \tag{6}$$

where $r = 0$ if $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$ are Cartesian co-ordinates and $r = 1$ if $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$ are cylindrical polar co-ordinates.

The vectors $\mathbf{a}^{(1)}$, $\mathbf{a}^{(2)}$ and $\mathbf{a}^{(3)}$ are normal to the surfaces $\xi^{(1)}$ = constant, $\xi^{(2)}$ = constant and $\xi^{(3)}$ = constant respectively. The vectors $\mathbf{a}_{(1)}$, $\mathbf{a}_{(2)}$ and $\mathbf{a}_{(3)}$ are directed along the intersections of pairs of surfaces e.g. $\mathbf{a}_{(1)}$ is directed along the intersection of $\xi^{(2)}$ = constant and $\xi^{(3)}$ = constant.

It can be shown that

$$\mathbf{a}^{(1)} = (\mathbf{a}_{(2)} \times \mathbf{a}_{(3)})/J, \tag{7}$$

$$\mathbf{a}^{(2)} = (\mathbf{a}_{(3)} \times \mathbf{a}_{(1)})/J \tag{8}$$

and

$$\mathbf{a}^{(3)} = (\mathbf{a}_{(1)} \times \mathbf{a}_{(2)})/J \tag{9}$$

where

$$J = \mathbf{a}_{(1)} \cdot (\mathbf{a}_{(2)} \times \mathbf{a}_{(2)}), \tag{10}$$

so that

$$\mathbf{a}_{(i)} \cdot \mathbf{a}^{(j)} = \delta_i^j \tag{11}$$

where

$$\delta_i^j = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j. \end{cases} \tag{12}$$

Any vector $\mathbf{w}$ can be represented as a linear combination of either the vectors $\mathbf{a}_{(i)}$ $(i = 1, 2, 3)$ or the vectors $\mathbf{a}^{(i)}$ $(i = 1, 2, 3)$. That is the vector $\mathbf{w}$ can be expressed in the form

$$\mathbf{w} = w^{(1)}\mathbf{a}_{(1)} + w^{(2)}\mathbf{a}_{(2)} + w^{(3)}\mathbf{a}_{(3)} \tag{13}$$

or in the form

$$\mathbf{w} = w_{(1)}\mathbf{a}^{(1)} + w_{(2)}\mathbf{a}^{(2)} + w_{(3)}\mathbf{a}^{(3)}. \tag{14}$$

The quantities $w^{(1)}$, $w^{(2)}$ and $w^{(3)}$ are called the contravariant components of $\mathbf{w}$ and $w_{(1)}$, $w_{(2)}$ and $w_{(3)}$ are called the co-variant components of $\mathbf{w}$.

The co-variant and contravariant components of a vector $\mathbf{w}$ are related, e.g.

$$w_{(i)} = \sum_{j=1}^{3} (\mathbf{a}_{(i)} \cdot \mathbf{a}_{(j)}) w^{(j)}. \tag{15}$$

It is usual to write

$$\mathbf{a}_{(i)} \cdot \mathbf{a}_{(j)} = g_{ij}, \tag{16}$$

so that equation (15) takes the form

$$w_{(i)} = \sum_{j=1}^{3} g_{ij} w^{(j)}. \tag{17}$$

It is worth noting too that

$$\mathbf{w} \cdot \mathbf{w} = w_{(1)} w^{(1)} + w_{(2)} w^{(2)} + w_{(3)} w^{(3)} \tag{18}$$

and the divergence, curl and gradient operators are (*see* Stratton)

$$\operatorname{div}(\mathbf{w}) = \frac{1}{J}\{\partial(Jw^{(1)})/\partial\xi^{(1)} + \partial(Jw^{(2)})/\partial\xi^{(2)} + \partial(Jw^{(3)})/\partial\xi^{(3)}\} \tag{19}$$

$$\operatorname{curl} \mathbf{w} = \frac{1}{J}\{(\partial w_3/\partial\xi^{(2)} - \partial w_{(2)}/\partial\xi^{(3)})\mathbf{a}_{(1)} + (\partial w_{(1)}/\partial\xi^{(3)} - \partial w_{(3)}/\partial\xi^{(1)})\mathbf{a}_{(2)} + (\partial w_{(2)}/\partial\xi^{(1)} - \partial w_{(1)}/\partial\xi^{(2)})\mathbf{a}_{(3)}\} \tag{20}$$

and

$$\operatorname{grad} \phi = \partial\phi/\partial\xi^{(1)}\mathbf{a}^{(1)} + \partial\phi/\partial\xi^{(2)}\mathbf{a}^{(2)} + \partial\phi/\partial\xi^{(3)}\mathbf{a}^{(3)}. \tag{21}$$

Now, clearly, by substituting in equations (1) to (3) the expressions for the divergence, curl and gradient operators given in equations (19) to (21), and making use of the relationships given in equations (7) to (9), a representation of equations (1) to (3) in an arbitrary co-ordinate system can be constructed.

The conservation of mass equation takes the form

$$\partial(J\rho v^{(1)})/\partial\xi^{(1)} + \partial(J\rho v^{(2)})/\partial\xi^{(2)} + \partial(J\rho v^{(3)})/\partial\xi^{(3)} = 0. \tag{22}$$

The following three scalar equations are obtained from the vector momentum equation (2):

$$\rho\partial(q)^2/\partial\xi^{(1)} - 2J\rho(v^{(2)}w^{(3)} - v^{(3)}w^{(2)}) = -2\partial p/\partial\xi^{(1)}, \tag{23}$$

$$\rho\partial(q)^2/\partial\xi^{(2)} - 2J\rho(v^{(3)}w^{(1)} - v^{(1)}w^{(3)}) = -2\partial p/\partial\xi^{(2)} \tag{24}$$

and

$$\rho\partial(q)^2/\partial\xi^{(3)} - 2J\rho(v^{(1)}w^{(2)} - v^{(2)}w^{(1)}) = -2\partial p/\partial\xi^{(3)}, \tag{25}$$

where

$$(q)^2 = v_{(1)}v^{(1)} + v_{(2)}v^{(2)} + v_{(3)}v^{(3)} \tag{26}$$

is the square of the velocity magnitude, and

$$w^{(1)} = \frac{1}{J}(\partial v_{(3)}/\partial\xi^{(2)} - \partial v_{(2)}/\partial\xi^{(3)}), \tag{27}$$

$$w^{(2)} = \frac{1}{J}(\partial v_{(1)}/\partial \xi^{(3)} - \partial v_{(3)}/\partial \xi^{(1)}) \tag{28}$$

and

$$w^{(3)} = \frac{1}{J}(\partial v_{(2)}/\partial \xi^{(1)} - \partial v_{(1)}/\partial \xi^{(2)}) \tag{29}$$

are the contravariant components of the vorticity vector curl $\mathbf{v}$.

Equation (3) takes the form

$$\rho(\gamma - 1)(q)^2 + 2\gamma p = 2\rho H(\gamma - 1). \tag{30}$$

Simplified forms of equations (22) to (25) and (30) are required in the special case when the surfaces on which $\xi^{(1)}$ = constant are planes that coincide with the planes $x^{(1)}$ = constant, and the surface $\xi^{(2)}$ = constant and $\xi^{(3)}$ = constant are stream-surfaces.

If $x^{(1)} = \xi^{(1)}$ then of course $\partial x^{(1)}/\partial \xi^{(1)} = 1$ whilst $\partial x^{(1)}/\partial \xi^{(2)} = \partial x^{(1)}/\partial \xi^{(3)} = 0$.

If surfaces $\xi^{(2)}$ = constant are stream-surfaces then the component of velocity normal to them will be zero. The vector $\mathbf{a}^{(2)}$ is normal to $\xi^{(2)}$ = constant and, in terms of its contravariant components, the velocity vector $\mathbf{v} = v^{(1)}\mathbf{a}_{(1)} + v^{(2)}\mathbf{a}_{(2)} + v^{(3)}\mathbf{a}_{(3)}$. It follows that the surfaces $\xi^{(2)}$ = constant will be stream surfaces if, and only if,

$$\mathbf{v} \, . \, \mathbf{a}^{(2)} = 0,$$

i.e.

$$\text{if} \quad v^{(2)} = 0.$$

Similarly surfaces $\xi^{(3)}$ = constant will be stream surfaces if and only if

$$v^{(3)} = 0.$$

Substituting $x^{(1)} = \xi^{(1)}$ and $v^{(2)} = v^{(3)} = 0$ in equations (22) to (25) and in equation (30) yields the required equations of motion. They are

$$\partial(J\rho v^{(1)})/\partial \xi^{(1)} = 0, \tag{31}$$

$$\rho q \partial q/\partial \xi^{(1)} = -\partial p/\partial \xi^{(1)}, \tag{32}$$

$$\rho q \partial q/\partial \xi^{(2)} + \rho v^{(1)}(\partial v_{(2)}/\partial \xi^{(1)} - \partial v_{(1)}/\partial \xi^{(2)}) = -\partial p/\partial \xi^{(2)}, \tag{33}$$

$$\rho q \partial q/\partial \xi^{(3)} - \rho v^{(1)}(\partial v_{(1)}/\partial \xi^{(3)} - \partial v_{(3)}/\partial \xi^{(1)}) = -\partial p/\partial \xi^{(3)} \tag{34}$$

and

$$\rho(\gamma - 1)(q)^2 + 2\gamma p = 2\rho H(\gamma - 1), \tag{35}$$

where

$$(q)^2 = v_{(1)}v^{(1)}, \tag{36}$$

$$v_{(1)} = g_{11}v^{(1)}, \tag{37}$$

$$v_{(2)} = g_{12}v^{(1)} \tag{38}$$

and

$$v_{(3)} = g_{13}v^{(1)}. \tag{39}$$

If transformation elements $t^{(i)}_{(j)}$, $(i, j = 1, 2, 3)$ are defined, so that

$$t^{(i)}_{(j)} = \partial x^{(i)}/\partial \xi^{(j)}, \tag{40}$$

and if

$$t^{(2)}_{(1)} = \alpha_2 \tag{41}$$

and

$$t^{(3)}_{(1)} = \alpha_3/(x^2)^r \tag{42}$$

then, from the definition of g_{ij}, $\mathbf{a}_{(i)}$ and $\mathbf{a}_{(j)}$ it can be seen that

$$g_{11} = 1 + (\alpha_2)^2 + (\alpha_3)^2, \tag{43}$$

$$g_{12} = \alpha_2 t^{(2)}_{(2)} + \alpha_3 (x^{(2)})^r t^{(3)}_{(2)}, \tag{44}$$

$$g_{13} = \alpha_2 t^{(2)}_{(3)} + \alpha_3 (x^{(2)})^r t^{(3)}_{(3)}, \tag{45}$$

$$g_{ij} = t^{(2)}_{(i)} t^{(2)}_{(j)} + (x^{(2)})^{2r} t^{(3)}_{(i)} t^{(3)}_{(j)}, \qquad (i, j = 2, 3) \tag{46}$$

and

$$J = (x^{(2)})^r (t^{(2)}_{(2)} t^{(3)}_{(3)} - t^{(2)}_{(3)} t^{(3)}_{(2)}). \tag{47}$$

In view of equations (36) to (39) and (43) to (47) equations (31) to (35) and (41) to (42) can be regarded as a system of seven equations for the unknowns p, ρ, $v^{(1)}$, α_2, α_3, $x^{(2)}$ and $x^{(3)}$. The equations of this system will now be expressed in a convenient form for use in Section 3.

By differentiating (35) with respect to $\xi^{(1)}$ and combining the resulting equation with (32) it is seen that

$$\partial \rho/\partial \xi^{(1)} = (\rho/\gamma p)\partial p/\partial \xi^{(1)}. \tag{48}$$

Existence of all second derivatives of $x^{(i)}$, $(i = 1, 2, 3)$ involving partial differentiation with respect to the independent variables $\xi^{(1)}$, $\xi^{(2)}$ and $\xi^{(3)}$ in pairs is assumed throughout this report. Relations

$$\partial t^{(i)}_{(j)}/\partial \xi^{(k)} = \partial t^{(i)}_{(k)}/\partial \xi^{(j)} \qquad (i = 1, 2, 3) \qquad (j, k = 1, 2, 3 \quad j \neq k) \tag{49}$$

connecting the transformation elements defined in equation (40) are a consequence of the existence of second derivatives of $x^{(i)}$, $(i = 1, 2, 3)$.

For equation (31), writing $J = (x^{(2)})^r (t^{(2)}_{(2)} t^{(3)}_{(3)} - t^{(2)}_{(3)} t^{(3)}_{(2)})$, carrying out the differentiation with respect to $\xi^{(1)}$, eliminating the $\partial \rho/\partial \xi^{(1)}$ term by using equation (48), and making use of certain of the relationships given by equation (49) shows that equation (31) can be replaced by the equation

$$J\rho \partial v^{(1)}/\partial \xi^{(1)} + \rho v^{(1)}[(x^{(2)})^r t^{(3)}_{(3)} \partial \alpha_2/\partial \xi^{(3)} - t^{(2)}_{(3)} \partial \alpha_3/\partial \xi^{(2)}] + \frac{J\rho v^{(1)}}{\gamma p} \partial p/\partial \xi^{(1)}$$
$$= -\rho v^{(1)}[r t^{(2)}_{(1)} (t^{(2)}_{(2)} \partial x^{(3)}/\partial \xi^{(3)} - t^{(3)}_{(2)} \partial x^{(2)}/\partial \xi^{(3)}) + t^{(2)}_{(2)} \partial \alpha_3/\partial \xi^{(3)} - (x^{(2)})^r t^{(3)}_{(2)} \partial \alpha_2/\partial \xi^{(3)}]. \tag{50}$$

Similarly for equations (32) to (34); writing $q = \sqrt{g_{11}} v^{(1)}$, $v_{(1)} = g_{11} v^{(1)}$, $v_{(2)} = g_{12} v^{(1)}$, $v_3 = g_{13} v^{(1)}$; substituting expressions for g_{11}, g_{12}, g_{13} in terms of transformation elements and the quantities α_2 and α_3; carrying out appropriate differentiation operations; and making use of certain of the relationships given by (49) shows that equations (32) to (34) can be replaced by the equations

$$g_{11} v^{(1)} \partial v^{(1)}/\partial \xi^{(1)} + (v^{(1)})^2 \alpha_2 \partial \alpha_2/\partial \xi^{(1)} + (v^{(1)})^2 \alpha_3 \partial \alpha_3/\partial \xi^{(1)} + (1/\rho)\partial p/\partial \xi^{(1)} = 0, \tag{51}$$

$$g_{12} v^{(1)} \partial v^{(1)}/\partial \xi^{(1)} + (v^{(1)})^2 t^{(2)}_{(2)} \partial \alpha_2/\partial \xi^{(1)} + (v^{(1)})^2 (x^{(2)})^r t^{(3)}_{(2)} \partial \alpha_3/\partial \xi^{(1)} + (1/\rho)\partial p/\partial \xi^{(2)}$$
$$= -r(v^{(1)})^2 \alpha_3 (t^{(2)}_{(1)} t^{(3)}_{(2)} - t^{(2)}_{(2)} t^{(3)}_{(1)}) \tag{52}$$

and

$$g_{13}v^{(1)}\partial v^{(1)}/\partial\xi^{(1)} + (v^{(1)})^2 t^{(2)}_{(3)}\partial\alpha_2/\partial\xi^{(1)} + (v^{(1)})^2(x^{(2)})^r t^{(3)}_{(3)}\partial\alpha_3/\partial\xi^{(1)}$$
$$= -(1/\rho)\partial p/\partial\xi^{(3)} - r(v^{(1)})^2\alpha_3(t^{(2)}_{(1)}t^{(3)}_{(3)} - t^{(3)}_{(1)}t^{(2)}_{(3)}). \quad (53)$$

Equations (50) to (53), (48), (41) and (42) are a system of seven equations for the dependent variables p, ρ, $v^{(1)}$, α_2, α_3, $x^{(2)}$ and $x^{(3)}$. These equations are in a convenient form for use in Section 3.

3. Characteristic Equations

For hyperbolic partial differential equations in two independent variables, characteristic curves exist and along these curves relations connecting rates of change of different dependent variables can be found.

Here, terms containing partial derivatives with respect to $\xi^{(3)}$ in equations [(50) to (53), (48), (41) and (42)] are treated as parameters. Then it is shown that under a condition which can be expected to hold in a variety of practical cases, the equations of motion derived in Section 2 are hyperbolic partial differential equations in the two independent variables $\xi^{(1)}$ and $\xi^{(2)}$. Expressions for both the slopes of characteristic curves and the characteristic relations for the equations of motion given in Section 2 are constructed in this section.

First it is noted that equations (50) to (53) and (48) can be represented in the form

$$A\partial\mathbf{u}/\partial\xi^{(1)} + B\partial\mathbf{u}/\partial\xi^{(2)} = D, \quad (54)$$

where $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$ are four by four matrices and $u = \{u_i\}$ and $D = \{d_i\}$ are column vectors which have four elements.

The elements of $\mathbf{u}$ are

$$u_1 = v^{(1)}, \qquad u_2 = \alpha_2$$
$$u_3 = \alpha_3, \qquad u_4 = p. \quad (55)$$

The elements of D are

$$d_1 = -r\rho v^{(1)}t^{(2)}_{(1)}(t^{(2)}_{(2)}t^{(3)}_{(3)} - t^{(2)}_{(3)}t^{(3)}_{(2)}) + \rho v^{(1)}(x^{(2)})^r t^{(3)}_{(2)}\partial\alpha_2/\partial\xi^{(3)} - \rho v^{(1)}t^{(2)}_{(2)}\partial\alpha_3/\partial\xi^{(3)},$$

$$d_2 = 0,$$

$$d_3 = -r(v^{(1)})^2\alpha_3(t^{(2)}_{(1)}t^{(3)}_{(2)} - t^{(2)}_{(2)}t^{(3)}_{(1)})$$

and

$$d_4 = -r(v^{(1)})^2\alpha_3(t^{(2)}_{(1)}t^{(3)}_{(3)} - t^{(2)}_{(3)}t^{(3)}_{(1)}) - (1/\rho)\partial p/\partial\xi^{(3)}. \quad (56)$$

The elements of A are

$$\left.\begin{aligned} a_{11} &= J\rho, & a_{12} &= 0 \\ a_{13} &= 0, & a_{14} &= J\rho v^{(1)}/\gamma p \end{aligned}\right\} \quad (57)$$

$$\left.\begin{aligned} a_{21} &= g_{11}v^{(1)}, & a_{22} &= (v^{(1)})^2\alpha_2 \\ a_{23} &= (v^{(1)})^2\alpha_3, & a_{24} &= 1/\rho \end{aligned}\right\} \quad (58)$$

$$\left.\begin{aligned} a_{31} &= g_{12}v^{(1)}, & a_{32} &= (v^{(1)})^2 t^{(2)}_{(2)} \\ a_{33} &= (v^{(1)})^2(x^{(2)})^r t^{(3)}_{(2)}, & a_{34} &= 0 \end{aligned}\right\} \quad (59)$$

$$\left.\begin{aligned} a_{41} &= g_{13}v^{(1)}, & a_{42} &= (v^{(1)})^2 t^{(2)}_{(3)} \\ a_{43} &= (v^{(1)})^2(x^{(2)})^r t^{(3)}_{(3)}, & a_{44} &= 0, \end{aligned}\right\} \quad (60)$$

and finally the elements of B are

$$\left.\begin{aligned} b_{11} &= 0, & b_{12} &= \rho v^{(1)}(x^{(2)})^r t^{(3)}_{(3)} \\ b_{13} &= -\rho v^{(1)} t^{(2)}_{(3)}, & b_{14} &= 0 \end{aligned}\right\} \tag{61}$$

$$\left.\begin{aligned} b_{21} &= 0, & b_{22} &= 0 \\ b_{23} &= 0, & b_{33} &= 0 \end{aligned}\right\} \tag{62}$$

$$\left.\begin{aligned} b_{31} &= 0, & b_{32} &= 0 \\ b_{33} &= 0, & b_{34} &= 1/\rho \end{aligned}\right\} \tag{63}$$

$$\left.\begin{aligned} b_{41} &= 0, & b_{42} &= 0 \\ b_{43} &= 0, & b_{44} &= 0. \end{aligned}\right\} \tag{64}$$

It ought to be remembered that $r = 0$ if $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are Cartesian co-ordinates and $r = 1$ if $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are cylindrical polar co-ordinates.

Characteristic curves associated with equation (54) can be found by introducing a transformation

$$\begin{aligned} \eta^{(1)} &= \eta^{(1)}(\xi^{(1)}_1 \xi^{(2)}), \\ \eta^{(2)} &= \xi^{(2)} \end{aligned} \tag{65}$$

and

$$\eta^{(3)} = \xi^{(3)},$$

so that equations (54) may be written in the form

$$\left(\frac{\partial \eta^{(1)}}{\partial \xi^{(1)}} A + \frac{\partial \eta^{(1)}}{\partial \xi^{(2)}} B\right) \frac{\partial \mathbf{u}}{\partial \eta^{(1)}} + B \frac{\partial \mathbf{u}}{\partial \eta^{(2)}} = D. \tag{66}$$

A finite number of characteristic curves can pass through a given point in space. For equations of the form (54), the characteristic curves through an arbitrary point have slopes $\lambda = -(\partial \eta^{(1)}/\partial \xi^{(1)}/\partial \eta^{(1)}/\partial \xi^{(2)})$, which satisfy the quartic polynomial equation represented by the characteristic determinant

$$|B - \lambda A| = 0. \tag{67}$$

Note:—It must be remembered that the characteristic curves lie in the $(\xi^{(1)}, \xi^{(2)})$ plane and $\lambda = d\xi^{(2)}/d\xi^{(1)}$ is the slope of a characteristic curve in this plane.

When the system (54) is hyperbolic, equation (67) has four real roots. In the case being considered in this report, (67) has the form

$$(\lambda)^2\{(J)^2(1 - (M)^2/g_{11})(\lambda)^2 - 2(g_{13}g_{23} - g_{12}g_{33})\lambda + (g_{11}g_{33} - (g_{13})^2)\} = 0, \tag{68}$$

where $M = \{\rho g_{11}(v^{(1)})^2/\gamma p\}^{\frac{1}{2}}$ is the local Mach number.

Clearly $\lambda = 0$ is a real root of (68) which is repeated twice and, since the quadratic expression in (68) has two distinct real roots if $(M)^2 > g_{11}$, it follows that with elements of U, D, A and B as defined in equations (55) to (64), equations (54) are hyperbolic if $(M)^2 > g_{11}$.

The condition $(M)^2 > g_{11}$ will be satisfied easily for supersonic flow past highly streamlined shapes when g_{11} is close to unity. In other cases, the condition will always be satisfied provided the local Mach number of the flow is high enough.

The two characteristic relations corresponding to $\lambda = 0$ are, trivially, the second and fourth of equations (54) i.e. equations corresponding to (51) and (53) which contain no partial derivatives with respect to $\xi^{(2)}$.

Let λ_+ and λ_- be the non-zero roots of the quartic equation (68), so

$$\lambda_\pm = \frac{(g_{13}g_{23} - g_{12}g_{33}) \pm \sqrt{\{[g_{13}g_{23} - g_{12}g_{33}]^2 - (J)^2[1 - (M)^2/g_{11}][g_{11}g_{33} - (g_{13})^2]\}}}{(J)^2(1 - (M)^2/g_{11})} \tag{69}$$

The characteristic relations corresponding to the characteristic curves whose slopes in a $(\xi^{(1)}, \xi^{(2)})$ plane are λ_+ and λ_- respectively are derived by replacing the elements of the first column of the characteristic determinant $|B - \lambda A|$ by elements of the column vector $C = B\partial\mathbf{u}/\partial\eta^{(2)} - D$. The determinant obtained is evaluated putting $\lambda = \lambda_+$ and $\lambda = \lambda_-$ in turn, and the results equated to zero are the desired characteristic relations.

For the equations derived in Section 2, the characteristic relations obtained by following the procedure outlined above have the form

$$
\begin{aligned}
&-\frac{(v^{(1)})^2(x^{(2)})^r t^{(3)}_{(3)}}{J}\left[(J)^2 + \frac{1}{\lambda}(g_{12}g_{33} - g_{13}g_{23})\right]\left[\frac{\partial\alpha_2}{\partial\xi^{(2)}} + \frac{1}{\lambda}\frac{\partial\alpha_2}{\partial\xi^{(1)}}\right] + \\
&+\frac{(v^{(1)})^2 t^{(2)}_{(3)}}{J}\left[(J)^2 + \frac{1}{\lambda}(g_{12}g_{33} - g_{13}g_{23})\right]\left[\frac{\partial\alpha_3}{\partial\xi^{(2)}} + \frac{1}{\lambda}\frac{\partial\alpha_3}{\partial\xi^{(1)}}\right] - \\
&-\frac{1'}{\rho}\left[\frac{(M)^2}{g_{11}}(g_{12}g_{33} - g_{13}g_{23}) + g_{33}/\lambda\right]\left[\frac{\partial p}{\partial\xi^{(2)}} + \frac{1}{\lambda}\frac{\partial p}{\partial\xi^{(1)}}\right] + \\
&+ c_4\left[\frac{J(M)^2}{g_{11}}(\alpha_2(x^{(2)})^r t^{(3)}_{(2)} - \alpha_3 t^{(2)}_{(2)}) + \frac{g_{23}}{\lambda} + \frac{g_{13}}{(\lambda)^2}\right] + \\
&+ \frac{d_1}{\rho J}\left[(J)^2 + \frac{1}{\lambda}(g_{12}g_{33} - g_{13}g_{23})\right] - \\
&- c_3^{(1)}\left[\frac{(M)^2}{g_{11}}(g_{12}g_{33} - g_{13}g_{23}) + \frac{g_{33}}{\lambda}\right] = 0
\end{aligned}
\tag{70}
$$

where

$$c_4 = r(v^{(1)})^2\alpha_3(t^{(2)}_{(1)}t^{(3)}_{(3)} - t^{(2)}_{(3)}t^{(3)}_{(1)}) + \frac{1}{\rho}\frac{\partial p}{\partial\xi^{(3)}} \tag{71}$$

$$c_3^{(1)} = r(v^{(1)})^2\alpha_3(t^{(2)}_{(1)}t^{(3)}_{(2)} - t^{(2)}_{(2)}t^{(3)}_{(2)}) \tag{72}$$

and

$$d_1 = -r\rho v^{(1)}t^{(2)}_{(1)}(t^{(2)}_{(2)}t^{(3)}_{(3)} - t^{(2)}_{(3)}t^{(3)}_{(2)}) + \rho v^{(1)}(x^{(2)})^r t^{(3)}_{(2)}\partial\alpha_2/\partial\xi^{(3)} - \rho v^{(1)}t^{(2)}_{(2)}\partial\alpha_3/\partial\xi^{(3)}. \tag{73}$$

Substituting in equation (70), $\lambda = \lambda_+$ from equation (69), yields the characteristic relation corresponding to the characteristic curve whose slope is $\lambda = \lambda_+$. Similarly, substituting $\lambda = \lambda_-$ in equation (70) yields the characteristic relation corresponding to the characteristic curve whose slope is $\lambda = \lambda_-$.

Although the derivation of equations (69) and (70) is simple in principle, the analysis is fairly complicated. The analysis is given in Appendix A.

4. Discussion

The object of the investigation described in this report was to derive a version of the equations which govern the supersonic inviscid flow of an ideal gas. These equations were required in a form suitable for applications in the construction of numerical solutions to mixed initial and boundary-value problems associated with hyperbolic partial-differential equations.

For the seven dependent variables $v^{(1)}$, α_2, α_3, p, ρ, $x^{(2)}$ and $x^{(3)}$:

(70) with $\lambda = \lambda_+$ (*see* (69))		(74)
(70) with $\lambda = \lambda_-$		(75)
(48)		(76)
(51)		(77)
(53)		(78)

$$\frac{\partial x^{(2)}}{\partial\xi^{(1)}} = \alpha_2 \tag{79}$$

and

$$\frac{\partial x^{(3)}}{\partial \xi^{(1)}} = \alpha_3/(x^{(2)})^r, \tag{80}$$

form a suitable system for solving mixed initial and boundary-value problems numerically.

A detailed account of the construction of a numerical method of solving equations (74) to (80) will not be given now. Here, it is noted simply that,

(1) when derivatives with respect to $\xi^{(3)}$ that appear in equations (74) to (80) are replaced by finite difference approximations, (74) to (80) yield a system of simultaneous equations in two independent variables ($\xi^{(1)}$ and $\xi^{(2)}$), and
(2) the system of equations obtained by replacing $\xi^{(3)}$-derivatives by finite difference approximations is hyperbolic when $(M)^2 > g_{11}$.

It follows then that when $(M)^2 > g_{11}$, the discretisation method used by Walkden and Caine[4] to solve mixed initial and boundary-value problems associated with systems of hyperbolic partial-differential equations with two independent variables can be applied to the equations obtained from (74) to (80) by introducing finite difference expressions for $\xi^{(3)}$-derivatives.

For two-dimensional flow, when $r = 0$ so that $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ are co-ordinates in Cartesian space, or when $r = 1$, so that $x^{(1)}$, $x^{(2)}$ and $x^{(3)}$ are cylindrical polar co-ordinates, the characteristic slopes λ_+ and λ_- given by equation (69) together with the characteristic relations derived from equation (70) can be reduced to the form given by Poole and Walkden.[5] Appendix B contains the analysis which verifies that both the two-dimensional and axi-symmetric flow forms of equations (69) and (70) are equivalent to the forms given in Ref. 5.

APPENDIX A

A.1. Derivation of Equation (69)

In order to obtain equation (69) the determinant $|B - \lambda A|$ has to be evaluated for the case when the elements of the matrices B and A take values given in equations (57) to (64). In this case

$$|B - \lambda A| =$$

$$= \begin{vmatrix} -J\rho\lambda & \rho v^{(1)}(x^{(2)})^r t^{(3)}_{(3)} & -\rho v^{(1)} t^{(2)}_{(3)} & -\lambda J\rho v^{(1)}/\gamma p \\ -g_{11}v^{(1)}\lambda & -\lambda(v^{(1)})^2\alpha_2 & -\lambda(v^{(1)})^2\alpha_3 & -\lambda/\rho \\ -g_{12}v^{(1)}\lambda & -\lambda(v^{(1)})^2 t^{(2)}_{(2)} & -\lambda(v^{(1)})^2(x^2)^r t^{(3)}_{(2)} & 1/\rho \\ -g_{13}v^{(1)}\lambda & -\lambda(v^{(1)})^2 t^{(2)}_{(3)} & -\lambda(v^{(1)})^2(x^{(2)})^r t^{(3)}_{(3)} & 0 \end{vmatrix}$$

$$= \Delta_1 + \Delta_2 + \Delta_3, \tag{A-1}$$

where

$$\Delta_1 = \frac{J\rho(v^{(1)})^6(\lambda)^4}{\gamma p} \begin{vmatrix} g_{11} & \alpha_2 & \alpha_3 \\ g_{12} & t^{(2)}_{(2)} & (x^{(2)})^r t^{(3)}_{(2)} \\ g_{13} & t^{(2)}_{(3)} & (x^{(2)})^r t^{(3)}_{(3)} \end{vmatrix}$$

$$= \frac{J\rho(v^{(1)})^6(\lambda)^4}{\gamma p}\{g_{11}[(x^{(2)})^r t^{(2)}_{(2)} t^{(3)}_{(3)} - (x^{(2)})^r t^{(3)}_{(2)} t^{(2)}_{(3)}] - g_{12}[\alpha_2(x^{(2)})^r t^{(3)}_{(3)} - \alpha_3 t^{(2)}_{(3)}] + g_{13}[\alpha_2(x^{(2)r} t^{(3)}_{(2)} - \alpha_3 t^{(2)}_{(2)}]\}$$

$$= (v^1)^4(\lambda)^4\{[g_{11} - (\alpha_2)^2 - (\alpha_3)^2](J)^2(v^1)^2\rho/\gamma p\}$$

$$= (v^{(1)})^4(\lambda)^4(M)^2(J)^2/g_{11}, \tag{A-2}$$

$$\Delta_2 = \frac{-(v^{(1)})^4(\lambda)^2}{\rho}\begin{vmatrix} J\rho & \rho(x^{(2)})^r t_{(3)}^{(3)} & -\rho t_{(3)}^{(2)} \\ g_{12} & -t_{(2)}^{(2)}\lambda & -(x^{(2)})^r t_{(2)}^{(3)}\lambda \\ g_{13} & -t_{(3)}^{(2)}\lambda & -(x^{(2)})^r t_{(3)}^{(3)}\lambda \end{vmatrix}$$

$$= -(v^{(1)})^4(\lambda)^4(J)^2 - (v^{(1)})^4(\lambda)^3\{g_{12}[(x^{(2)})^{2r}(t_{(3)}^{(3)})^2 + (t_{(3)}^{(2)})^2] - g_{13}[(x^{(2)})^{2r}t_{(3)}^{(3)}t_{(2)}^{(3)} + t_{(2)}^{(2)}t_{(3)}^{(2)}]\}$$

$$= -(v^{(1)})^4(\lambda)^4(J)^2 - (v^{(1)})^4(\lambda)^3\{g_{12}g_{33} - g_{13}g_{23}\} \tag{A-3}$$

$$\Delta_3 = \frac{\lambda}{\rho}\begin{vmatrix} J\rho & \rho v^{(1)}(x^{(2)})^r t_{(3)}^{(3)} & -\rho v^{(1)} t_{(3)}^{(2)} \\ g_{11}v^{(1)} & -(v^{(1)})^2\alpha_2\lambda & -(v^{(1)})^2\alpha_3\lambda \\ g_{13}v^{(1)} & -(v^{(1)})^2 t_{(3)}^{(2)}\lambda & -(v^{(1)})^2(x^{(2)})^r t_{(3)}^{(3)}\lambda \end{vmatrix}$$

$$= -(v^{(1)})^4 J(\lambda)^3\{\alpha_2(x^{(2)})^r t_{(3)}^{(3)} - \alpha_3 t_{(3)}^{(2)}\} - \frac{(v^{(1)})^4(\lambda)^2}{\rho}\{\rho g_{11}[(x^{(2)})^{2r}(t_{(3)}^{(3)})^2 + (t_{(3)}^{(2)})^2] - \rho g_{13}[\alpha_3(x^{(2)})^r t_{(3)}^{(3)} + \alpha_2 t_{(3)}^{(2)}]\}$$

$$= -(v^{(1)})^4(\lambda)^3\{g_{12}g_{33} - g_{13}g_{23}\} - (v^{(1)})^4(\lambda)^2\{g_{11}g_{33} - (g_{13})^2\}. \tag{A-4}$$

Now

$$|B - \lambda A| = \Delta_1 + \Delta_2 + \Delta_3$$

$$= (v^{(1)})^4(\lambda)^4(J)^2\left[\frac{(M)^2}{g_{11}} - 1\right] - 2(v^{(1)})^4(\lambda)^3(g_{12}g_{33} - g_{13}g_{23}) - (v^{(1)})^4(\lambda)^2(g_{11}g_{33} - (g_{13})^2). \tag{A-5}$$

Since $v^{(1)} \neq 0$, it follows that the characteristic equation $|B - \lambda A|$ can be expressed in the form (69).

A.2. Derivation of Equation (70)

In order to obtain equation (70), the elements of the first column of the matrix $(B - \lambda A)$ will be replaced by the elements of the column vector

$$C = (B\partial\mathbf{u}/\partial\eta^{(2)} - D)$$

and the determinant of the resulting matrix will be evaluated and equated to zero.

First it is noted that,

$$c_1 = \rho v^{(1)}(x^{(2)})^r t_{(3)}^{(3)}\partial\alpha_2/\partial\eta^{(2)} - \rho v^{(1)} t_{(3)}^{(2)}\partial\alpha_3/\partial\eta^{(3)} - d_1, \tag{A-6}$$

$$c_2 = 0, \tag{A-7}$$

$$c_3 = (1/\rho)\partial p/\partial\eta^{(2)} + c_3^{(1)} \tag{A-8}$$

and

$$c_4 = r(v^{(1)})^2\alpha_3(t_{(1)}^{(2)}t_{(3)}^{(3)} - t_{(3)}^{(2)}t_{(1)}^{(3)}) + (1/\rho)\partial p/\partial\xi^{(3)} \tag{A-9}$$

where

$$c_3^{(1)} = r(v^{(1)})^2\alpha_3(t_{(1)}^{(2)}t_{(2)}^{(3)} - t_{(2)}^{(2)}t_{(1)}^{(3)}) \tag{A-10}$$

and

$$d_1 = -r\rho v^{(1)}t_{(1)}^{(2)}(t_{(2)}^{(2)}t_{(3)}^{(3)} - t_{(3)}^{(2)}t_{(2)}^{(3)}) + \rho v^{(1)}(x^{(2)})^r t_{(2)}^{(3)}\partial\alpha_2/\partial\xi^{(3)} - \rho v^{(1)}t_{(2)}^{(2)}\partial\alpha_3/\partial\xi^{(3)}. \tag{A-11}$$

It follows that the required characteristic relations are given by

$$\begin{vmatrix} c_1 & \rho v^{(1)}(x^{(2)})^r t^{(3)}_{(3)}/\lambda & -\rho v^{(1)} t^{(2)}_{(3)}/\lambda & -J\rho v^{(1)}/\gamma p \\ c_2 & -(v^{(1)})^2\alpha_2 & -(v^{(1)})^2\alpha_3 & -1/\rho \\ c_3 & -(v^{(1)})^2 t^{(2)}_{(2)} & -(v^1)^2(x^2)^r t^{(3)}_{(2)} & 1/(\rho\lambda) \\ c_4 & -(v^{(1)})^2 t^{(2)}_{(3)} & -(v^{(1)})^2(x^{(2)})^r t^{(3)}_{(3)} & 0 \end{vmatrix} = 0, \tag{A-12}$$

i.e.

$$c_1\Delta_4 + \left(\frac{1}{\rho}\frac{\partial p}{\partial \eta^{(2)}} + c_3^{(1)}\right)\Delta_5 - c_4\Delta_6 = 0 \tag{A-13}$$

where

$$\Delta_4 = \begin{vmatrix} (v^{(1)})^2\alpha_2 & (v^{(1)})^2\alpha_3 & -1/\rho \\ (v^{(1)})^2 t^{(2)}_{(2)} & (v^{(1)})^2(x^{(2)})^r t^{(3)}_{(2)} & 1/(\rho\lambda) \\ (v^{(1)})^2 t^{(2)}_{(3)} & (v^{(1)})^2(x^{(2)})^r t^{(3)}_{(3)} & 0 \end{vmatrix}$$

$$= -\frac{(v^{(1)})^4}{\rho}\left([(t^{(2)}_{(2)})(x^{(2)})^r t^{(3)}_{(3)} - (x^{(2)})^r t^{(3)}_{(2)} t^{(2)}_{(3)}] + \frac{1}{\lambda}[\alpha_2(x^{(2)})^r t^{(3)}_{(3)} - \alpha_3 t^{(2)}_{(3)}]\right)$$

$$= -\frac{(v^{(1)})^4}{\rho J}\left((J)^2 + \frac{1}{\lambda}(g_{12}g_{33} - g_{13}g_{23})\right), \tag{A-14}$$

$$\Delta_5 = -\begin{vmatrix} -\rho v^{(1)}(x^{(2)})^r t^{(3)}_{(3)}/\lambda & \rho v^{(1)} t^{(2)}_{(3)}/\lambda & J\rho v^{(1)}/\gamma p \\ (v^{(1)})^2\alpha_2 & (v^{(1)})^2\alpha_3 & 1/\rho \\ (v^{(1)})^2 t^{(2)}_{(3)} & (v^{(1)})^2(x^{(2)})^r t^{(3)}_{(3)} & 0 \end{vmatrix}$$

$$= -J\rho v^{(1)}/\gamma p\{(v^{(1)})^4[\alpha_2(x^{(2)})^r t^{(3)}_{(3)} - \alpha_3 t^{(2)}_{(3)}]\} + 1/\rho\{-\rho(v^{(1)})^3[(x^{(2)})^{2r}(t^{(3)}_{(3)})^2/\lambda + (t^{(2)}_{(3)})^2/\lambda)\}$$

$$= -(v^{(1)})^3\left(\frac{(M)^2}{g_{11}}(g_{12}g_{33} - g_{13}g_{23}) + g_{33}/\lambda\right), \tag{A-15}$$

$$\Delta_6 = -\begin{vmatrix} -\rho v^{(1)}(x^{(2)})^r t^{(3)}_{(3)}/\lambda & \rho v^{(1)} t^{(2)}_{(3)}/\lambda & J\rho v^{(1)}/\gamma p \\ (v^{(1)})^2\alpha_2 & (v^{(1)})^2\alpha_3 & 1/\rho \\ (v^{(1)})^2 t^{(2)}_{(2)} & (v^{(1)})^2(x^{(2)})^r t^{(3)}_{(2)} & -1/(\rho\lambda) \end{vmatrix}$$

$$= -J\rho v^{(1)}/\gamma p\{(v^{(1)})^4[\alpha_2(x^{(2)})^r t^{(3)}_{(2)} - \alpha_3 t^{(2)}_{(2)}]\} -$$

$$-\frac{(v^{(1)})^3}{\lambda}\{(x^{(2)})^{2r} t^{(3)}_{(3)} t^{(3)}_{(2)} + t^{(2)}_{(3)} t^{(2)}_{(2)}\} - \frac{(v^{(1)})^3}{\lambda^2}\{\alpha_3(x^{(2)})^r t^{(3)}_{(3)} + \alpha_2 t^{(2)}_{(3)}\}$$

$$= -(v^{(1)})^3\left(\frac{J(M)^2}{g_{11}}[\alpha_2(x^{(2)})^r t^{(3)}_{(2)} - \alpha_3 t^{(2)}_{(2)}] + \frac{g_{23}}{\lambda} + \frac{g_{13}}{(\lambda)^2}\right)\cdot \quad (\lambda)^2 \tag{A-16}$$

Substituting in equation (A-13) the expressions given in this appendix for c_1, $c_3^{(1)}$, c_4, Δ_4, Δ_5 and Δ_6 yields the equation,

$$-\frac{(v^{(1)})^5(x^{(2)})^r t^{(3)}_{(3)}}{J}\left\{(J)^2 + \frac{1}{\lambda}(g_{12}g_{33} - g_{13}g_{23})\right\}\frac{\partial\alpha_2}{\partial\eta^{(2)}} + \frac{(v^{(1)})^5 t^{(2)}_{(3)}}{J}\left\{(J)^2 + \frac{1}{\lambda}(g_{12}g_{33} - g_{13}g_{23})\right\}\frac{\partial\alpha_3}{\partial\eta^{(2)}} -$$

$$-\frac{(v^{(1)})^3}{\rho}\left\{\frac{(M)^2}{g_{11}}(g_{12}g_{33} - g_{13}g_{23}) + \frac{g_{33}}{\lambda}\right\}\frac{\partial p}{\partial\eta^{(2)}} + \frac{d_1(v^{(1)})^4}{\rho J}\left\{(J)^2 + \frac{1}{\lambda}(g_{12}g_{33} - g_{13}g_{23})\right\} -$$

$$- c_3^{(1)}(v^{(1)})^3\left\{\frac{(M)^2}{g_{11}}(g_{12}g_{33} - g_{13}g_{23}) + \frac{g_{33}}{\lambda}\right\} + c_4(v^{(1)})^3\left\{\frac{J(M)^2}{g_{11}}[\alpha_2(x^{(2)})^r t_{(2)}^{(3)} - \alpha_3 t_{(2)}^{(2)}] + \frac{g_{23}}{\lambda} + \frac{g_{13}}{(\lambda)^2}\right\} = 0. \tag{A-17}$$

Equation (70) follows if equation (A-17) is divided by $(v^{(1)})^3$, and the operator $\partial/\partial\eta^{(2)}$ is replaced wherever it appears in (A-17) by the operator $\partial/\partial\xi^{(2)} + (1/\lambda)\partial/\partial\xi^{(1)}$.

APPENDIX B

B.1. Characteristic Slopes for Two-dimensional and Axi-symmetric Flow

When the flow is axi-symmetric or two-dimensional, the surfaces $\xi^{(3)}$ = constant can be chosen to be planes such that $x^{(3)} = \xi^{(3)}$ and then $x^{(2)} = x^{(2)}(\xi^{(1)}, \xi^{(2)})$.

It follows that

$$J = (x^{(2)})^r(t_{(2)}^{(2)}t_{(3)}^{(3)} - t_{(3)}^{(2)}t_{(2)}^{(3)}) = (x^{(2)})^r t_{(2)}^{(2)},$$

$$g_{13} = 0,$$

$$g_{33} = (x^{(2)})^{2r},$$

$$g_{23} = 0,$$

$$g_{11} = 1 + (\alpha_2)^2$$

and

$$g_{12} = t_{(1)}^{(2)}t_{(2)}^{(2)}.$$

The streamline slope is

$$t_{(1)}^{(2)} = \alpha_2$$

$$= \tan\theta$$

in this appendix.

Now the quadratic equation

$$(\lambda)^2(J)^2(1 - (M)^2/g_{11}) - 2\lambda(g_{13}g_{23} - g_{12}g_{33}) + (g_{11}g_{33} - (g_{13})^2) = 0, \tag{B-1}$$

for the characteristic slopes λ_+ and λ_-, reduces to

$$(x^{(2)})^{2r}(t_{(2)}^{(2)})^2(1 - (M)^2/(1 + \tan^2\theta))(\lambda)^2 + 2(x^{(2)})^{2r}\tan\theta t_{(2)}^{(2)}\lambda + (x^{(2)})^{2r}(1 + \tan^2\theta) = 0. \tag{B-2}$$

The roots of this equation are λ_+ and λ_- where

$$\begin{aligned}\frac{1}{\lambda_\pm} &= -\frac{t_{(2)}^{(2)}\tan\theta \mp \sqrt{\{(t_{(2)}^{(2)})^2\tan^2\theta - (t_{(2)}^{(2)})^2\sec^2\theta(1 - (M)^2/\sec^2\theta)\}}}{\sec^2\theta}\\ &= -t_{(2)}^{(2)}(\sin\theta\cos\theta \pm \cos^2\theta\sqrt{\{\tan^2\theta - (\sec^2\theta - \operatorname{cosec}^2\mu)\}})\\ &= -t_{(2)}^{(2)}\cos\theta/\sin\mu(\sin\theta\sin\mu \pm \cos\theta\cos\mu).\end{aligned} \tag{B-3}$$

In equation (B-3), $\mu = \sin^{-1}(1/M)$.

From equation (B-3) it is clear that:

$$\lambda_{\pm} = \frac{-\sin\mu}{t^{(2)}_{(2)}\cos\theta(\sin\theta\sin\mu \pm \cos\theta\cos\mu)}$$

$$= \mp\frac{\sec^2\theta\sin\mu}{t^{(2)}_{(2)}\cos\mu(1 \pm \tan\theta\tan\mu)}$$

$$= \mp\frac{g_{11}\tan\mu}{t^{(2)}_{(2)}(1 \pm t^2_1\tan\mu)}. \tag{B-4}$$

The expressions for λ_- and λ_+ obtained from (B-4) are identical to those given in Ref. 5 for the slopes of the left and right hand characteristics respectively.

B.2. Characteristic Relations for Two-dimensional or Axi-symmetric Flow

If $(q)^2 = g_{11}(v^{(1)})^2$ then when the flow is two-dimensional or axi-symmetric equation (70) reduces to

$$-\frac{(q)^2}{t^{(2)}_{(2)}g_{11}}\left[(x^{(2)})^{2r}(t^{(2)}_{(2)})^2 + \frac{1}{\lambda_\pm}(x^{(2)})^{2r}t^{(2)}_{(1)}t^{(2)}_{(2)}\right]\left[\frac{\partial\alpha_2}{\partial\xi^{(2)}} + \frac{1}{\lambda_\pm}\frac{\partial\alpha_2}{\partial\xi^{(1)}}\right] -$$

$$-\frac{1}{\rho}\left[\frac{(M)^2(x^{(2)})^{2r}t^{(2)}_{(1)}t^{(2)}_{(2)}}{g_{11}} + \frac{(x^{(2)})^{2r}}{\lambda_\pm}\right]\left[\frac{\partial p}{\partial\xi^{(2)}} + \frac{1}{\lambda_\pm}\frac{\partial\alpha_2}{\partial\xi^{(1)}}\right] -$$

$$-\frac{r(q)^2t^{(2)}_{(1)}t^{(2)}_{(2)}}{g_{11}(x^{(2)})^r t^{(2)}_{(2)}}[(x^{(2)})^{2r}(t^{(2)}_{(2)})^2 + t^{(2)}_{(1)}t^{(2)}_{(2)}(x^{(2)})^{2r}/\lambda] = 0$$

i.e.,

$$-\frac{q^2}{g_{11}}[t^{(2)}_{(2)} + t^{(2)}_{(1)}/\lambda_\pm]\left[\frac{\partial\alpha_2}{\partial\xi^{(2)}} + \frac{1}{\lambda_\pm}\frac{\partial\alpha_2}{\partial\xi^{(1)}}\right] - \frac{1}{g_{11}\rho}\left[(M)^2t^{(2)}_{(1)}t^{(2)}_{(2)} + \frac{g_{11}}{\lambda_\pm}\right]\left[\frac{\partial p}{\partial\xi^{(2)}} + \frac{1}{\lambda_\pm}\frac{\partial p}{\partial\xi^{(1)}}\right]$$

$$= \frac{r(q)^2t^{(2)}_{(1)}t^{(2)}_{(2)}}{g_{11}(x^{(2)})^r}[t^{(2)}_{(2)} + t^{(2)}_{(1)}/\lambda_\pm]. \tag{B-5}$$

This equation can be expressed in the form

$$q^2[t^{(2)}_{(2)}\lambda_\pm + t^{(2)}_{(1)}]\left[\frac{\partial\alpha_2}{\partial\xi^{(2)}}\lambda_\pm + \frac{\partial\alpha_2}{\partial\xi^{(1)}}\right] + \frac{1}{\rho}[(M)^2t^{(2)}_{(1)}t^{(2)}_{(2)}\lambda_\pm + g_{11}]\left[\frac{\partial p}{\partial\xi^{(2)}}\lambda_\pm + \frac{\partial p}{\partial\xi^{(1)}}\right]$$

$$= -\frac{r(q)^2t^{(2)}_{(1)}t^{(2)}_{(2)}}{(x^{(2)})^r}\lambda_\pm[t^{(2)}_{(2)}\lambda_\pm + t^{(2)}_{(1)}]. \tag{B-6}$$

Now

$$\frac{g_{11} \mp g_{11}t^{(2)}_{(1)}\tan\mu \pm t^{(2)}_{(1)}\tan\mu(M)^2g_{11}}{t^{(2)}_{(1)} \pm \tan\mu}$$

$$= \frac{g_{11}[1 \mp t^{(2)}_{(1)}\tan\mu \pm t^{(2)}_{(1)}\tan\mu(M)^2]}{[t^{(2)}_{(1)} \pm \tan\mu]}$$

$$= \frac{g_{11}[1 \pm t^{(2)}_{(1)}\cot\mu]}{[t^{(2)}_{(1)} \pm \tan\mu]}$$

$$= \pm g_{11}\cot\mu.$$

Therefore, for two-dimensional flow ($r = 0$) or axi-symmetric flow ($r = 1$), the characteristic relation (70) takes the form

$$\frac{1}{g_{11}}\left[\frac{\partial\alpha_2}{\partial\xi^{(1)}}t^{(2)}_{(2)}(1 \mp t^{(2)}_{(1)}\tan\mu) + \frac{\partial\alpha_2}{\partial\xi^{(2)}}(\pm g_{11}\tan\mu)\right] \pm \frac{\cot\mu}{\rho(q)^2}\left[\frac{\partial p}{\partial\xi^{(1)}}t^{(2)}_{(2)}(1 \mp t^{(2)}_{(1)}\tan\mu) + \frac{\partial p}{\partial\xi^{(2)}}(\pm g_{11}\tan\mu)\right]$$

$$= \mp rt^{(2)}_{(1)}t^{(2)}_{(2)}\tan\mu/(x^{(2)})^r. \tag{B-7}$$

Equation (B-7) is equivalent to

$$\left[\frac{\partial \alpha_2}{\partial \xi^{(1)}} t^{(2)}_{(2)}(1 \mp t^{(2)}_{(1)} \tan \mu) + \frac{\partial \alpha_2}{\partial \xi^{(2)}}(\pm g_{11} \tan \mu)\right][t^{(2)}_{(1)} \pm \tan \mu] +$$

$$+ \frac{1}{\rho(q)^2}\left[\frac{\partial p}{\partial \xi^{(1)}} t^{(2)}_{(2)}(1 \mp t^{(2)}_{(1)} \tan \mu) + \frac{\partial p}{\partial \xi^{(2)}}(\pm g_{11} \tan \mu)\right] \times [g_{11} \mp g_{11} \tan \mu \pm t^{(2)}_{(1)} \tan \mu (M)^2 g_{11}]$$

$$= \mp r t^{(2)}_{(1)} t^{(2)}_{(2)} g_{11} \tan \mu [t^{(2)}_{(1)} \pm \tan \mu]/(x^{(2)})^r. \qquad \text{(B-8)}$$

On dividing this equation by $g_{11}(t^{(2)}_{(1)} \pm \tan \mu)$ it is seen that

$$\frac{1}{g_{11}}\left[\frac{\partial \alpha_2}{\partial \xi^{(1)}} t^{(2)}_{(2)}(1 \mp t^{(2)}_{(1)} \tan \mu) + \frac{\partial \alpha_2}{\partial \xi^{(2)}}(\pm g_{11} \tan \mu)\right] +$$

$$+ \frac{1}{\rho(q)^2 g_{11}}\left[\frac{\partial p}{\partial \xi^{(1)}} t^{(2)}_{(2)}(1 \mp t^{(2)}_{(1)} \tan \mu) + \frac{\partial p}{\partial \xi^{(2)}}(\pm g_{11} \tan \mu)\right] \times \frac{[g_{11} \mp g_{11}({}^{(2)}_{(1)} \tan \mu \pm t^{(2)}_{(1)} \tan \mu (M)^2 g_{11}]}{[t^{(2)}_{(1)} \pm \tan \mu]}$$

$$= \mp r t^{(2)}_{(1)} t^{(2)}_{(2)} \tan \mu/(x^{(2)})^r. \qquad \text{(B-9)}$$

Equation (B-9) can be compared directly with the characteristic relations given by Poole and Walkden.[5]

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