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Optimum Reinforcement around a Circular Hole in a Flat Sheet under Uniaxial Tension

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Summary

This report determines optimum distributions of compact edge reinforcement around a circular hole in a flat sheet subjected to uniaxial tension. The optimisation is effected by minimising the stress concentration factor, based on the Mises–Hencky criterion, for given total weights of reinforcement.

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* Replaces RAE Technical Report 72013—A.R.C. 33 736.

1. Introduction

The presence of a hole in an otherwise uniformly stressed plane sheet always results in a localised perturbation of the stress field and a consequent weakening of the sheet due to stress concentrations. The stress concentrations can be reduced if the sheet is suitably reinforced but the only way to eliminate them is to adopt a *neutral hole*¹ in which, however, the shape of the hole, as well as the distribution of edge-reinforcement, is determined by the basic stress field. In many instances a circular shape for the hole is specified and the problem then is to determine what distribution of reinforcing material gives the least stress concentration, and if this optimum distribution is too heavy, what then are the optimum distributions for reinforcements of given smaller total weights. This problem was considered by an inverse method of solution by the first author² for infinite sheets subjected to principal stresses in the ratios 1:1 (radial tension), 2:1 (as in a pressurised cylinder), 1:0 (uniaxial tension) and 1:−1 (shear). However, for the particular case of uniaxial tension there are analytical and computational difficulties which stem from the fact that the hoop stress in the reinforcing ring vanishes at certain points on the circumference; if the reinforcing ring is flexible, the radial stress vanishes at the same points and this has the effect of reducing the number of disposable parameters in the analysis. The modern computer now makes it relatively easy to include additional terms to avoid this difficulty. (Note that the vanishing of these stresses when the applied loading is shear is not an analytical handicap because it is a consequence of symmetry.)

2. Assumptions and Method of Solution

It is assumed that the reinforcement is of the same elastic material as the sheet and that it is sufficiently compact for it to be regarded as a line member in the plane of the sheet with finite tensile stiffness but negligible flexural stiffness. This latter feature has, indeed, been shown to be a necessary requirement for minimum weight for the constant annular reinforcement³, for the neutral hole¹ and for a class of optimum reinforced circular holes in sheets of variable thickness⁴. The stress concentration factor S is based on the Mises–Hencky criterion, so that if the applied uniaxial stress is unity

$$S = \max \{ \sqrt{\sigma_\theta^2 + \sigma_r^2 - \sigma_\theta \sigma_r + 3\tau_{r\theta}^2} \}, \quad (1)$$

where σ_θ , σ_r and $\tau_{r\theta}$ are the direct and shear stresses in the sheet.

The boundary conditions at the junction of the sheet and reinforcement (at $r = R$) are conveniently expressed in terms of the stress function Φ and associated stresses as follows:

$$[\Phi]_R = 0, \quad (2)$$

and

$$\frac{A}{Rt} = \left[\frac{\sigma_r}{\sigma_\theta - \nu\sigma_r} \right]_R, \quad (3)$$

where A is the section area of the reinforcement, t is the sheet thickness and ν is Poisson's ratio. The total weight of the reinforcement is expressed non-dimensionally as a fraction of the weight of sheet removed; thus

$$W = \frac{1}{\pi Rt} \int_0^{2\pi} A \, d\theta. \quad (4)$$

The method of solution is an inverse one. A form for the stress function Φ is assumed which satisfies equation (2) and the conditions remote from the hole, and which contains a number of disposable parameters associated with stress perturbations which decay as r^{-2} , r^{-4} , r^{-6} etc. A given value of W is now

chosen which has the effect of reducing the number of disposable parameters. Finally the remaining parameters are chosen to minimise S .

3. Analysis

The most general form of the stress function Φ , which satisfies equation (2) and the requirements of symmetry, together with the conditions of unit uniaxial stress in the direction $\theta = \frac{1}{2}\pi$ remote from the hole, is given by

$$\Phi = \frac{1}{4} \left\{ 2\alpha R^2 \ln(r/R) + (r^2 - R^2) \left(1 + \cos 2\theta + \frac{\beta R^2}{r^2} \cos 2\theta + \frac{\gamma R^4}{r^4} \cos 4\theta + \frac{\delta R^6}{r^6} \cos 6\theta + \dots \right) \right\} \quad (5)$$

where the parameters $\alpha, \beta, \gamma, \delta \dots$ give rise to progressively more localised perturbations of the general stress field. In what follows, a truncated form of the stress function is chosen which contains only these specified terms.

The stresses are derived from the stress function by the relations

$$\left. \begin{aligned} \sigma_\theta &= \frac{\partial^2 \Phi}{\partial r^2}, \\ \sigma_r &= \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \\ \tau_{r\theta} &= -\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial \Phi}{\partial \theta} \right). \end{aligned} \right\} \quad (6)$$

and

Thus, introducing $\rho = r/R$ we obtain

$$\left. \begin{aligned} \sigma_\theta &= \frac{1}{2}(1 + \cos 2\theta) - \frac{\alpha}{2\rho^2} - \frac{3\beta \cos 2\theta}{2\rho^4} + \gamma \cos 4\theta \left(\frac{3}{2\rho^4} - \frac{5}{\rho^6} \right) + \delta \cos 6\theta \left(\frac{5}{\rho^6} - \frac{21}{2\rho^8} \right), \\ \sigma_r &= \frac{1}{2}(1 - \cos 2\theta) + \frac{\alpha}{2\rho^2} + \frac{(1 - \beta) \cos 2\theta}{\rho^2} + \frac{3\beta \cos 2\theta}{2\rho^4} - \gamma \cos 4\theta \left(\frac{9}{2\rho^4} - \frac{5}{\rho^6} \right) \\ &\quad - \delta \cos 6\theta \left(\frac{10}{\rho^6} - \frac{21}{2\rho^8} \right) \end{aligned} \right\} \quad (7)$$

and

$$\tau_{r\theta} = \frac{1}{2} \sin 2\theta + \frac{(1 - \beta) \sin 2\theta}{2\rho^2} + \frac{3\beta \sin 2\theta}{2\rho^4} - \gamma \sin 4\theta \left(\frac{3}{\rho^4} - \frac{5}{\rho^6} \right) - \frac{3}{2} \delta \sin 6\theta \left(\frac{5}{\rho^6} - \frac{7}{\rho^8} \right).$$

In evaluating A/Rt , it is convenient to express $[\sigma_\theta, \sigma_r]_R$ in powers of $\cos 2\theta (= \mathcal{C})$:

$$\left. \begin{aligned} [\sigma_\theta]_R &= \frac{1}{2}(1 - \alpha + 7\gamma) + \frac{1}{2}(1 - 3\beta + 33\delta)\mathcal{C} - 7\gamma\mathcal{C}^2 - 22\delta\mathcal{C}^3 \\ [\sigma_r]_R &= \frac{1}{2}(1 + \alpha - \gamma) + \frac{1}{2}(1 + \beta - 3\delta)\mathcal{C} + \gamma\mathcal{C}^2 + 2\delta\mathcal{C}^3. \end{aligned} \right\} \quad (8)$$

and

In any optimum variation of A/Rt the hoop stress in the reinforcement vanishes at certain points and hence, because reinforcements with infinite section area are not possible, the stresses $[\sigma_r]_R$ and $[\sigma_\theta]_R$ vanish simultaneously at such points specified by $\mathcal{C} = \varphi_1$, say, and we can write

$$\left. \begin{aligned} [\sigma_\theta]_R &= \frac{1}{2}(\varphi_1 - \mathcal{C})(\varphi_2 + \varphi_3\mathcal{C} + 44\delta\mathcal{C}^2) \\ \text{and} \\ [\sigma_r]_R &= \frac{1}{2}(\varphi_1 - \mathcal{C})(\varphi_4 + \varphi_5\mathcal{C} - 4\delta\mathcal{C}^2), \text{ say.} \end{aligned} \right\} \quad (9)$$

By comparing coefficients of different powers of \mathcal{C} in equations (8) and (9) it is possible to express $\varphi_1 \dots \varphi_5$ and δ in terms of α, β, γ . Thus φ_1 is determined by the equation

$$\left. \begin{aligned} 4\gamma\varphi_1^2 + (6 + 4\beta)\varphi_1 + (6 + 5\alpha - 2\gamma) &= 0, \\ \text{and} \\ \varphi_2 &= (1 - \alpha + 7\gamma)/\varphi_1, \\ \varphi_3 &= 44\delta\varphi_1 + 14\gamma, \\ \varphi_4 &= (1 + \alpha - \gamma)/\varphi_1, \\ \varphi_5 &= -4\delta\varphi_1 - 2\gamma, \end{aligned} \right\} \quad (10)$$

where

$$\delta = \frac{1 + \beta + 2\gamma\varphi_1 + \varphi_4}{3 - 4\varphi_1^2}.$$

Equations (3), (9) and (10) thus enable A/Rt to be expressed in terms of θ and the parameters α, β, γ :

$$\frac{A}{Rt} = \frac{\varphi_4 + \varphi_5\mathcal{C} - 4\delta\mathcal{C}^2}{(\varphi_2 - \nu\varphi_4) + (\varphi_3 - \nu\varphi_5)\mathcal{C} + (44 + 4\nu)\delta\mathcal{C}^2}, \quad (11)$$

and this may be integrated, as shown in the Appendix, to yield an expression for W of the form

$$W = W(\alpha, \beta, \gamma). \quad (12)$$

We now choose a given numerical value for W , represented by \bar{W} , and use the equation

$$W(\alpha, \beta, \gamma) = \bar{W} \quad (13)$$

to determine γ for given numerical values of α, β . Next we determine S_R at $r = R$ from equations (1) and (7) and find those values, α^*, β^* say, which minimise S_R ; the corresponding values γ^*, δ^* are given by equations (13) and (10). It remains to show that A/Rt is everywhere positive and that S_R is, indeed, the minimum value of S throughout the sheet.

3.1. Modifications to the Analysis

The above procedure has been followed for values of $W = 0.2, 0.25, 0.3 \dots (0.1) \dots 1.0, 1.2$ but it breaks down at the (less practical) extremities of this range. When $W = 0.2$ it is found that this procedure leads

to small negative values of A/Rt in the neighbourhood of $\theta = \frac{1}{2}\pi$ and it is necessary to introduce the constraint

$$\frac{A}{Rt} \geq 0. \quad (14)$$

When $W = 1.0, 1.2$ it is found that peak values of S , albeit exceeding S_R by less than 2 per cent, occur away from the reinforcing ring at about $\rho = 1.05, \theta = 35$ degrees for $W = 1.0$, and $\rho = 1.09, \theta = 40$ degrees for $W = 1.2$. For these values of W it is therefore necessary to choose α^*, β^* to minimise S rather than simply S_R .

4. Numerical Results

Table 1 shows values of $\alpha^*, \beta^*, \gamma^*, \delta^*$ for optimum values of W and S with $\nu = \frac{1}{3}$. $\min \{S\}$ has been determined to an accuracy of about 0.2 per cent, but because the variation of S with α, β is small in the neighbourhood of $\min \{S\}$, the quoted values of α^* , etc., could differ from the 'true' values, by a markedly greater amount.

The variation of S with W is shown in Fig. 2 and compared with the previous 3-parameter solution and with the case of constant reinforcement. It is seen that the introduction of an additional parameter leads to solutions in which S is reduced by some 8 to 12 per cent over the range $0.25 < W < 0.65$. The curve of S is nearly flat at $W = 0.6$ and there is little to be gained by choosing greater values of W .

The optimum variations of A/Rt over the quadrant $0 \leq \theta \leq \frac{1}{2}\pi$ are shown in Figs. 3a and b for various values of W . It is seen that for low values of W it is preferable to distribute most of the reinforcement in the range $0 < \theta < \frac{1}{4}\pi$. For somewhat greater values of W the reinforcement reaches a maximum value away from $\theta = 0$.

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APPENDIX

In evaluating W it is convenient to express A/Rt , see equation (11), in partial fractions in the form

$$\frac{A}{Rt} = \frac{N_1}{1 + \omega_1 \mathcal{C}} + \frac{N_2}{1 + \omega_2 \mathcal{C}} - \frac{1}{11 + v},$$

where

$$N_1 = \frac{4\varepsilon + \omega_1 \varphi_5 - \omega_1^2 \varphi_4}{\omega_1(\omega_2 - \omega_1)(\varphi_2 - v\varphi_4)},$$

$$N_2 = \frac{4\varepsilon + \omega_2 \varphi_5 - \omega_2^2 \varphi_4}{\omega_2(\omega_1 - \omega_2)(\varphi_2 - v\varphi_4)}$$

and

$$\omega_{1,2} = a \pm (a^2 - B)^{\frac{1}{2}}$$

where

$$a = \frac{\varphi_3 - v\varphi_5}{2(\varphi_2 - v\varphi_4)},$$

$$B = \frac{4\varepsilon(11 + v)}{\varphi_2 - v\varphi_4}.$$

(15)

Now

$$\int_0^\pi \frac{d\theta}{1 + \omega \cos 2\theta} = \frac{\pi}{\sqrt{1 - \omega^2}}$$

and accordingly

$$W = \frac{2N_1}{\sqrt{1 - \omega_1^2}} + \frac{2N_2}{\sqrt{1 - \omega_2^2}} - \frac{2}{11 + v}. \quad (16)$$

If the roots $\omega_{1,2}$ are complex, the above expression for W retains its validity but it is advisable for computational purposes to re-arrange it in purely real terms by introducing

$$b = \sqrt{B - a^2}$$

and

$$e = \{(1 - a^2 + b^2)^2 + 4a^2b^2\}^{\frac{1}{2}}$$

so that

$$W = \frac{2(bc - as)}{be(11 + v)} + \frac{2\{\varphi_4(as + bc) - \varphi_5s\}}{be(\varphi_2 - v\varphi_4)} - \frac{2}{11 + v},$$

(17)

where

$$c = \cos \frac{1}{2}\Omega, \quad s = \sin \frac{1}{2}\Omega$$

and

$$\Omega = \tan^{-1} \left(\frac{2ab}{1 - a^2 + b^2} \right).$$

TABLE 1
 Values of α^* , β^* , γ^* , δ^* for optimum values of W , S with $\nu = 1/3$

W	S	α^*	β^*	γ^*	δ^*
0.2	1.814	-0.698	-0.5854	+0.09628	-0.01609
0.25	1.717	-0.646	-0.5238	+0.09796	-0.02396
0.3	1.665	-0.605	-0.4754	+0.09113	-0.02880
0.4	1.599	-0.555	-0.4104	+0.06305	-0.02297
0.5	1.552	-0.515	-0.3635	+0.04130	-0.01801
0.6	1.518	-0.485	-0.3310	+0.02435	-0.01312
0.7	1.493	-0.457	-0.3056	+0.01191	-0.01040
0.8	1.474	-0.438	-0.2887	+0.00231	-0.00736
0.9	1.459	-0.420	-0.2751	-0.00502	-0.00550
1.0	1.451	-0.405	-0.2655	-0.01127	-0.00427
1.2	1.440	-0.385	-0.2541	-0.01975	-0.00194

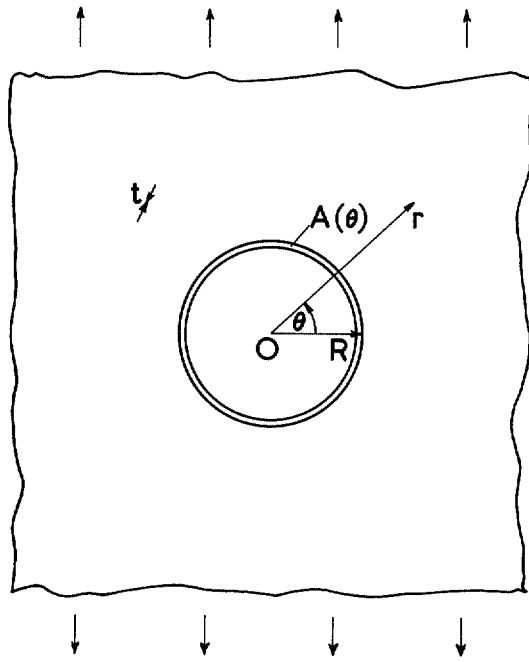


FIG. 1. Diagram showing notation.

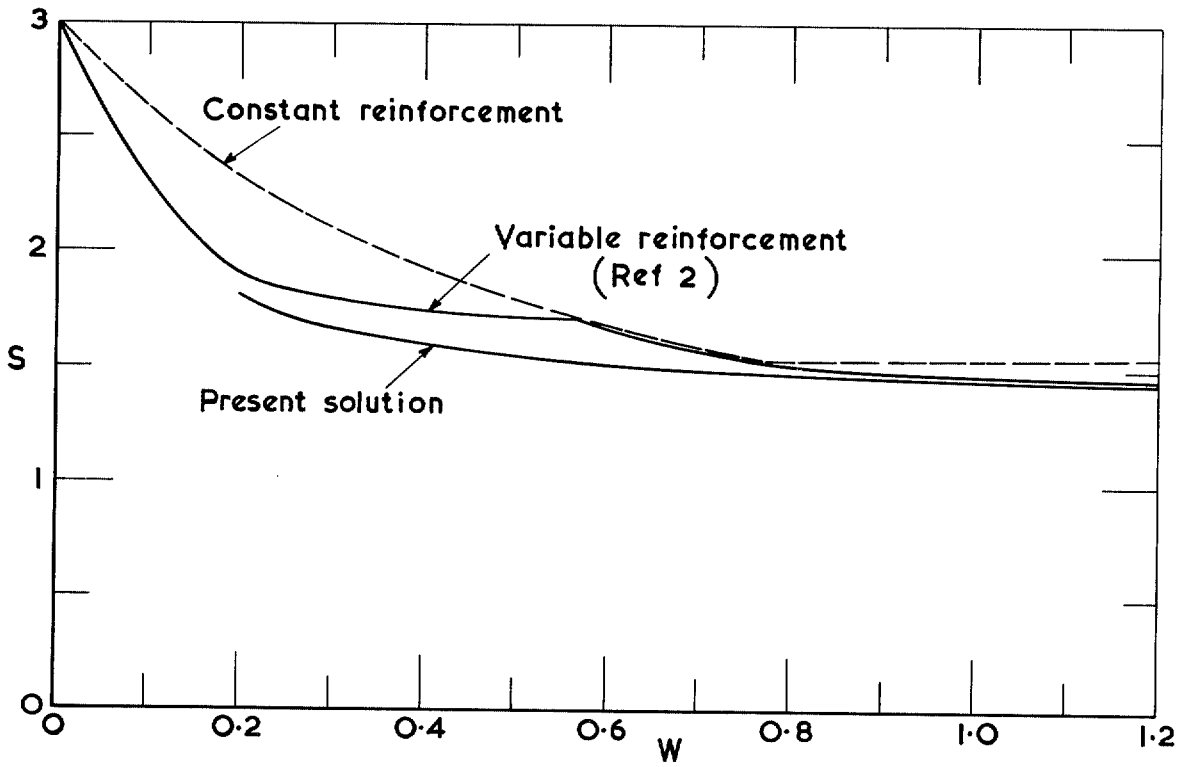


FIG. 2. Variation of stress concentration factor S with weight of reinforcement W .

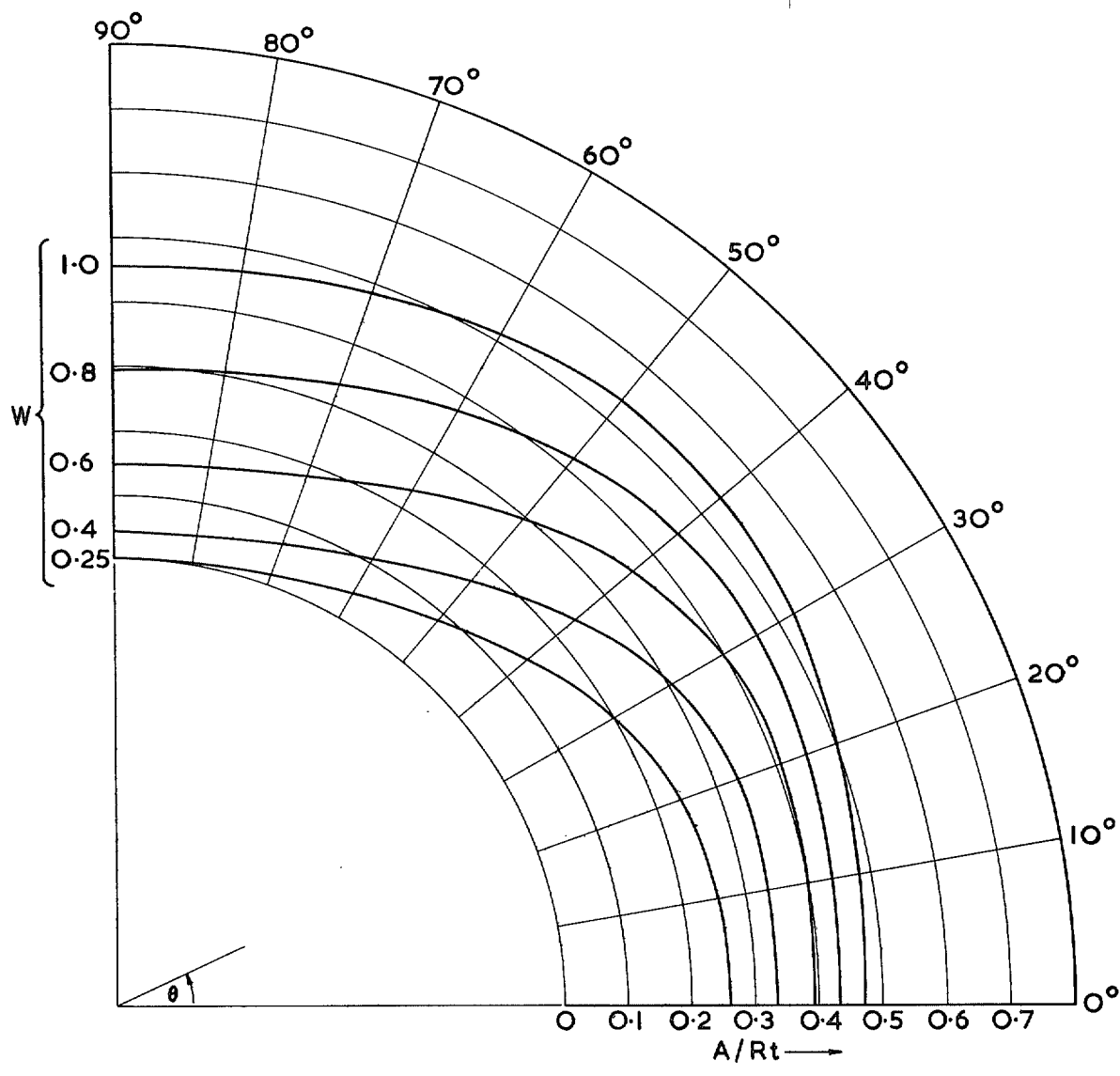


FIG. 3a. Optimum variations of A/Rt .

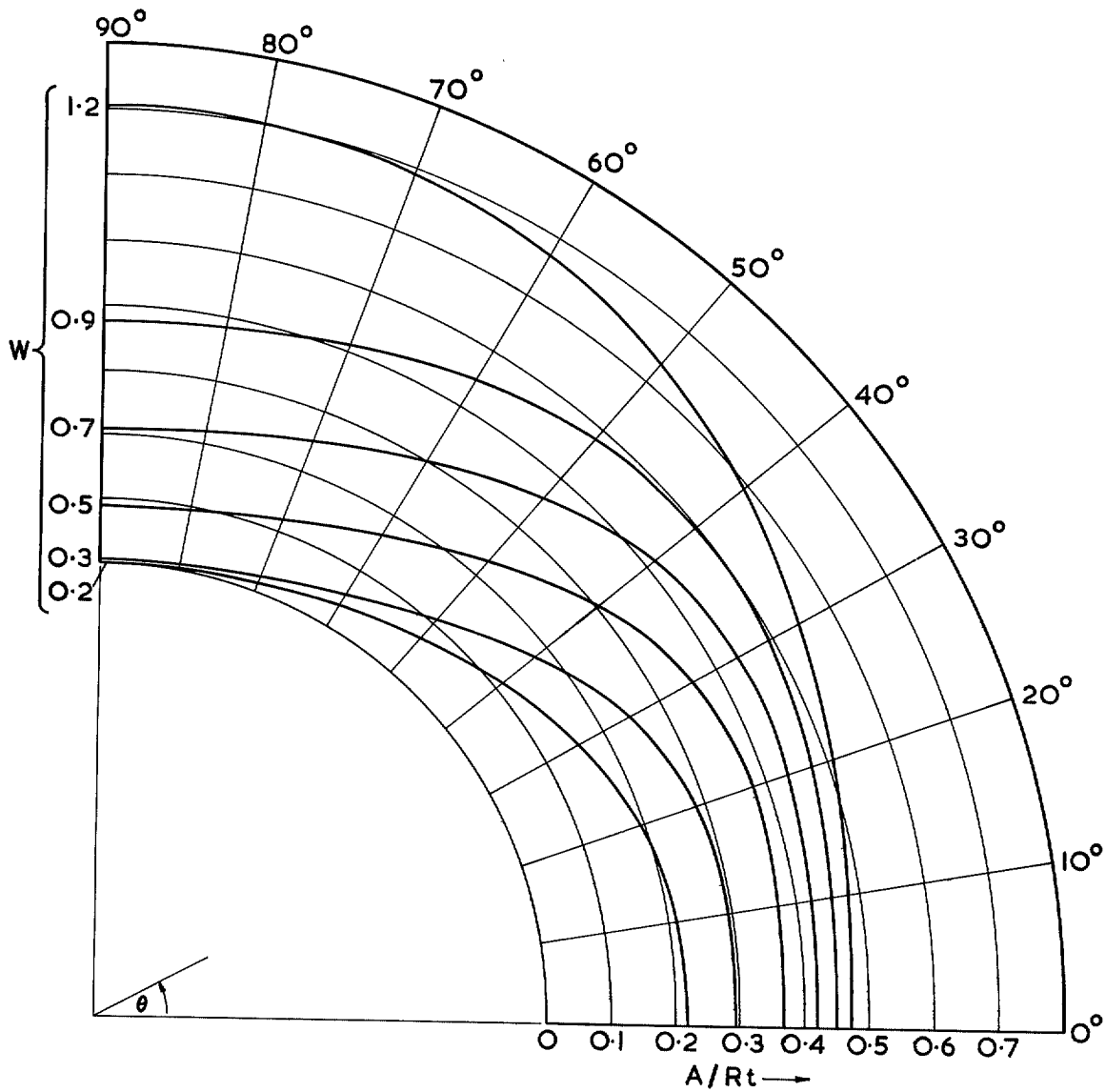


FIG. 3b. Optimum variations of A/Rt .

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