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The Equations of Motion of a Viscous Compressible Gas Referred to an Arbitrarily Moving Co-ordinate System

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Summary.

The equations of motion of a viscous, compressible, heat conducting fluid are formulated in general co-ordinates ($\xi^0, \xi^1, \xi^2, \xi^3$) where ξ^0 represents time and ξ^1, ξ^2, ξ^3 are arbitrary functions of time and the space co-ordinates in some fixed system. A tensor form for the equations of motion in a fixed cartesian system is derived and the methods of tensor analysis are used to obtain the appropriate representation of the equations of motion in a general co-ordinate system. Two examples are considered to show how the analysis presented may be used to obtain equations of motion appropriate to each case.

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1. *Introduction.*

In this Report the equations of motion of a viscous, compressible, heat conducting fluid are formulated in general co-ordinates ($\xi^0, \xi^1, \xi^2, \xi^3$), where, if x^1, x^2, x^3 are cartesian co-ordinates and x^0 represents time then,

$$\begin{aligned} \xi^0 &= x^0 \\ \xi^\alpha &= \xi^\alpha(x^0, x^1, x^2, x^3) \end{aligned} \tag{1.1}$$

where the functions $\xi^\alpha(x^0, x^1, x^2, x^3)$, for $\alpha = 1, 2, 3$, are arbitrary functions of x^0, x^1, x^2 and x^3.

McVittie[1] has considered the problem of formulating the equations of motion of a fluid in general co-ordinates like those defined by equation (1.1). Since he was interested in meteorological phenomena he included heat conduction effects but neglected viscosity. More recent contributions to the problem of obtaining suitable tensor representations of expressions containing time and space derivatives are papers by Prager[2], Sedov[3], and Koppe and Zimmerman[4]. Here we shall use the method developed by McVittie[1].

Our interest in this problem is in the possibility of using a general formulation of the equations of motion to improve methods of obtaining numerical solutions of certain problems. The notion of working with a general representation of the equations of motion has been found useful in obtaining numerical solutions of a number of steady flow problems[5,6,7,8] and it is to be expected that similar applications of the general form of the unsteady flow equations will be useful.

The problem of deriving equations of motion with ξ^0, ξ^1, ξ^2 and ξ^3 as independent variables is not trivial because the relations

$$\xi^\alpha = \xi^\alpha(x^0, x^1, x^2, x^3) \qquad \alpha = 1, 2, 3$$

will depend upon time (x^0) in general. Then the functions $\xi^\alpha = \xi^\alpha(x^0, x^1, x^2, x^3)$ represent surfaces moving in space. Fluid particles may overtake or be overtaken by these surfaces. In these circumstances, it is clear that the problem of expressing the laws of conservation of mass momentum and energy in terms of the co-ordinates ξ^1, ξ^2, ξ^3 and time (ξ^0) cannot be solved simply by following the procedure used when ξ^1, ξ^2, ξ^3 = constant are surfaces fixed in space—that is, the equations cannot be formulated easily by considering forces acting on the fluid in an elementary region of space and equating these forces to the rate of change of momentum of the fluid in the elementary region.

The representation of systems of partial differential equations in different co-ordinate spaces is the concern of tensor analysis. If a tensor representation of the equations of motion of a viscous, heat conducting, compressible fluid can be found then the problem is solved. A relatively simple tensor representation of the equations of motion is available. This representation is obtained by adding terms of $0\left(\frac{1}{c^2}\right)$ to the equations of motion in a fixed cartesian space. The momentum equations and the continuity equation are combined in a single tensor equation. The energy equation yields a single scalar equation. Elementary tensor analysis is used to obtain the representation of these equations in ξ-space and the limiting form of the equations in ξ-space as $c \to \infty$ represent the equations of motion of a viscous, compressible, heat conducting gas in general co-ordinates. These processes are described in detail in sections 3, 4 and 5. In sections 6 and 7 two examples are given to show how the analysis presented in this Report can be used to derive appropriate equations of motion in special cases. Section 6 contains a formulation of the Lagrangian equations of motion of a viscous fluid and a simple numerical scheme for integrating these equations is proposed. In 7 the equations of motion of an inviscid fluid referred to a rotating co-ordinate system are derived as a special case of the general theory.

2. *Metric and Associated Tensors.*

We shall define three co-ordinate systems: an x-system, an η-system and a ξ-system. We shall regard x^1, x^2, x^3 as space co-ordinates in a cartesian system and η^1, η^2, η^3 as space co-ordinates in some orthogonal system which is fixed relative to x-space. For time we shall write x^0, η^0 or ξ^0 depending on the space we are working in. We shall require that all transformations leave invariant the space time metric

$$-(ds)^2 = (dx^1)^2+(dx^2)^2+(dx^3)^2-c^2(dx^0)^2. \tag{2.1}$$

We shall consider transformations of the type

$$\left.\begin{aligned} \xi^0 &= x^0 \\ \xi^\alpha &= \xi^\alpha(x^0, x^1, x^2, x^3) \end{aligned}\right\} \tag{2.2}$$

and we shall suppose that

and
$$\left.\begin{aligned} \eta^0 &= x^0 \\ \eta^\alpha &= \eta^\alpha(x^1, x^2, x^3) \end{aligned}\right\} \tag{2.3}$$

are given functions such that the surfaces η^1 = constant, η^2 = constant, η^3 = constant are fixed, mutually orthogonal, families of surfaces in x-space†. It follows that ξ and η co-ordinates are related by equations of the form

$$\left.\begin{aligned} \xi^0 &= \eta^0 \\ \xi^\alpha &= \xi^\alpha(\eta^0, \eta^1, \eta^2, \eta^3)\,. \end{aligned}\right\} \tag{2.4}$$

Equations (2.4) are introduced because in some applications of the theory it is useful to consider transformations between η- and ξ-space rather than transformations between x- and ξ-space. Now if

$$t^i_j = \frac{\partial x^i}{\partial \xi^j}$$

and

$$t^{*i}_j = \frac{\partial \xi^i}{\partial x^j}$$

a transformation matrix $\mathscr{F}$ and its inverse $\mathscr{F}^{-1}$ can be defined:—

$$\mathscr{F} = (t^i_j) = \begin{pmatrix} 1 & t^1_0 & t^2_0 & t^3_0 \\ 0 & t^1_1 & t^2_1 & t^3_1 \\ 0 & t^1_2 & t^2_2 & t^3_2 \\ 0 & t^1_3 & t^2_3 & t^3_3 \end{pmatrix} \tag{2.5}$$

†Greek indices take values 1, 2, 3.
Roman indices take values 0, 1, 2, 3.

$$\mathscr{F}^{-1} = (t_j^{*i}) = \begin{pmatrix} 1 & t_0^{*1} & t_0^{*2} & t_0^{*3} \\ 0 & t_1^{*1} & t_1^{*2} & t_1^{*3} \\ 0 & t_2^{*1} & t_2^{*2} & t_2^{*3} \\ 0 & t_3^{*1} & t_3^{*2} & t_3^{*3} \end{pmatrix} \tag{2.6}$$

In general the transformation elements will be finite except, perhaps, at isolated points or along certain curves.

The requirement that $-(ds)^2$ be invariant leads us to define the following metric tensor which has the form

$$(a_{ik}) = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

in x-space, the form

$$(\gamma_{ik}) = \begin{pmatrix} \gamma_{00} & 0 & 0 & 0 \\ 0 & \gamma_{11} & 0 & 0 \\ 0 & 0 & \gamma_{22} & 0 \\ 0 & 0 & 0 & \gamma_{33} \end{pmatrix} \tag{2.8}$$

in η-space, and the form

$$(g_{ik}) = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} \tag{2.9}$$

in ξ-space.

Now $\gamma_{00} = -c^2$ and

$$\gamma_{\beta\beta} = \left(\frac{\partial x^1}{\partial \eta^\beta}\right)^2 + \left(\frac{\partial x^2}{\partial \eta^\beta}\right)^2 + \left(\frac{\partial x^3}{\partial \eta^\beta}\right)^2 \tag{2.10}$$

$$g_{00} = -c^2\left(1 - \frac{(t_0^1)^2 + (t_0^2)^2 + (t_0^3)^2}{c^2}\right) \tag{2.11}$$

$$g_{\alpha 0} = g_{0\alpha} = t_0^1\, t_\alpha^1 + t_0^2\, t_\alpha^2 + t_0^3\, t_\alpha^3 \tag{2.12}$$

$$g_{\alpha\beta} = g_{\beta\alpha} = t_\alpha^1\, t_\beta^1 + t_\alpha^2\, t_\beta^2 + t_\alpha^3\, t_\beta^3\,. \tag{2.13}$$

Alternatively equations (2.11) to (2.13) may be expressed in the form:

$$g_{00} = -c^2\left(1 - \frac{\gamma_{11}(\tau_0^1)^2 + \gamma_{22}(\tau_0^2)^2 + \gamma_{33}(\tau_0^3)^2}{c^2}\right) \tag{2.14}$$

$$g_{\alpha 0} = g_{0\alpha} = \gamma_{11}\, \tau_0^1\, \tau_\alpha^1 + \gamma_{22}\, \tau_0^2\, \tau_\alpha^2 + \gamma_{33}\, \tau_0^3\, \tau_\alpha^3 \tag{2.15}$$

$$g_{\alpha\beta} = g_{\beta\alpha} = \gamma_{11}\, \tau_\alpha^1\, \tau_\beta^1 + \gamma_{22}\, \tau_\alpha^2\, \tau_\beta^2 + \gamma_{33}\, \tau_\alpha^3\, \tau_\beta^3 \tag{2.16}$$

where

$$\tau_j^i = \frac{\partial \eta^i}{\partial \xi^j}\,.$$

Associated with the tensors a_{ik}, γ_{ik}, and g_{ik} are the tensors a^{ik}, γ^{ik}, and g^{ik} defined so that $a_{ij}\, a^{jk}$, $\gamma_{ij}\, \gamma^{jk}$, and $g_{ij}\, g^{jk}$ are all equal to δ_i^k.

It follows that

$$(a^{ik}) = \begin{pmatrix} -\frac{1}{c^2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.17}$$

$$(\gamma^{ik}) = \begin{pmatrix} -\frac{1}{c^2} & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma_{11}} & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma_{22}} & 0 \\ 0 & 0 & 0 & \frac{1}{\gamma_{33}} \end{pmatrix} \tag{2.18}$$

and

$$(g^{ik}) = \begin{pmatrix} -\frac{1}{c^2} & g^{01} & g^{02} & g^{03} \\ g^{10} & g^{11} & g^{12} & g^{13} \\ g^{20} & g^{21} & g^{22} & g^{23} \\ g^{30} & g^{31} & g^{32} & g^{33} \end{pmatrix} \tag{2.19}$$

where

$$\left.\begin{aligned} g^{\alpha 0} &= g^{0\alpha} = -\frac{t_0^{*\alpha}}{c^2} \\ g^{\alpha\beta} &= g^{\beta\alpha} = t_1^{*\alpha}\, t_1^{*\beta} + t_2^{*\alpha}\, t_2^{*\beta} + t_3^{*\alpha}\, t_3^{*\beta}\,. \end{aligned}\right\} \tag{2.20}$$

Alternatively equations (2.20) can be expressed in the form

$$\left.\begin{aligned} g^{\alpha 0} &= g^{0\alpha} = -\frac{\tau_0^{*\alpha}}{c^2} \\ g^{\alpha\beta} &= g^{\beta\alpha} = \frac{1}{\gamma_{11}}(\tau_1^{*\alpha}\, t_1^{*\beta}) + \frac{1}{\gamma_{22}}(\tau_2^{*\alpha}\, \tau_2^{*\beta}) + \frac{1}{\gamma_{33}}(\tau_3^{*\alpha}\, \tau_3^{*\beta}) \end{aligned}\right\} \tag{2.21}$$

where $\tau_j^{*i} = \dfrac{\partial \xi^i}{\partial \eta^j}$.

The determinant of the transformation matrix $\mathscr{F}$ defined in equation (2.5) is

$$J = \frac{\partial(x^1, x^2, x^3)}{\partial(\xi^1, \xi^2, \xi^3)} \tag{2.22}$$

and this determinant can be expressed in the form

$$J = \sqrt{\gamma_{11}\, \gamma_{22}\, \gamma_{33}}\; \frac{\partial(\eta^1, \eta^2, \eta^3)}{\partial(\xi^1, \xi^2, \xi^3)}. \tag{2.23}$$

In the following Section we shall introduce a contravariant first order tensor (v^i) such that $v^0 = 1$. Neglecting terms of $0\left(\dfrac{1}{c^2}\right)$ compared with those of 0(1) it follows that the associated co-variant tensor is $(-c^2, v_1, v_2, v_3)$ where

$$v_\alpha = g_{\alpha m}\, v^m\,. \tag{2.24}$$

The co-variant derivative of the vector (v_j) is

$$\begin{aligned} v_{j,m} &= \frac{\partial v_j}{\partial \xi^m} - \Gamma_{jm}^k\, v_k \\ &= \frac{\partial v_j}{\partial \xi^m} - \frac{1}{2} g^{kl} \left(\frac{\partial g_{lm}}{\partial \xi^j} + \frac{\partial g_{lj}}{\partial \xi^m} - \frac{\partial g_{jm}}{\partial \xi^l} \right) v_k \\ &= \frac{\partial v_j}{\partial \xi^m} - \frac{1}{2} v^l \left(\frac{\partial g_{lm}}{\partial \xi^j} + \frac{\partial g_{lj}}{\partial \xi^m} - \frac{\partial g_{jm}}{\partial \xi^l} \right) \end{aligned} \tag{2.25}$$

The velocity magnitude is (u_i = cartesian velocity components)

$$q^2 = u_i\, u^i + c^2\,. \tag{2.26}$$

In ξ-space this equation becomes

$$q^2 = v_i v^i + c^2$$
$$= (t_0^1)^2 + (t_0^2)^2 + (t_0^3)^2 + 2g_{0\alpha} v^\alpha + g_{\alpha\beta} v^\alpha v^\beta . \quad (2.27)$$

3. *Equations of Motion of a Viscous Compressible Fluid in x-Space.*

We shall suppose that the fluid is an ideal gas so that the equations of motion can be represented in the form:

$$\frac{\partial \rho}{\partial x^0} + \frac{\partial}{\partial x^\beta}(\rho u^\beta) = 0 \quad (3.1)$$

$$\frac{\partial}{\partial x^0}(\rho u_\alpha) + \frac{\partial}{\partial x^\beta}(\rho u_\alpha u^\beta + (p + \frac{2}{3}\mu \Delta)\delta_\alpha^\beta - \mu \delta^{\beta\gamma} e_{\gamma\alpha}) = 0 \quad (3.2)$$

$$\rho C_p \frac{dT}{dx^0} = \Phi + \frac{dp}{dx^0} + \frac{\partial}{\partial x^\alpha}\left(k \frac{\partial T}{\partial x^\alpha}\right) \quad (3.3)$$

where

$$p = \rho R T,$$

$$u^\beta = u_\beta,$$

$$\delta_\alpha^\beta = \begin{cases} 1 & \beta = \alpha \\ 0 & \beta \neq \alpha \end{cases}$$

$$\delta^{\alpha\beta} = \begin{cases} 1 & \beta = \alpha \\ 0 & \beta \neq \alpha \end{cases}$$

$$e_{\alpha\gamma} = \frac{\partial u_\alpha}{\partial x^\gamma} + \frac{\partial u_\gamma}{\partial x^\alpha} \quad (3.4)$$

$$\Delta = \frac{\partial u^\beta}{\partial x^\beta} \quad (3.5)$$

and

$$\Phi = \frac{1}{2}\mu \delta^{\alpha\varepsilon} \delta^{\beta\gamma} e_{\alpha\gamma} e_{\varepsilon\beta} - \frac{2}{3}\mu \Delta^2 . \quad (3.6)$$

If we define a contravariant vector

$$(u^i) = (1, u^1, u^2, u^3)$$

and neglect quantities of $0(1/c^2)$ compared with quantities of 0(1) then equations (3.1) and (3.2) can be represented in the form

$$\frac{\partial T_i^k}{\partial x^k} = 0 \tag{3.7}$$

where

$$T_i^k = \rho\, u_i\, u^k + (p + \frac{2}{3}\mu\,\Delta)\,\delta_i^k - \mu\, a^{mk}\left(\frac{\partial u_i}{\partial x^m} + \frac{\partial u_m}{\partial x^i}\right) \tag{3.8}$$

because

$$\begin{aligned} T_0^0 &= \rho\, u_0\, u^0 + (p + \frac{2}{3}\mu\,\Delta) - \mu\, a^{m0}\left(\frac{\partial u_0}{\partial x^m} + \frac{\partial u_m}{\partial x^0}\right) \\ &= -c^2\left(\rho + 0(\frac{1}{c^2})\right) \end{aligned} \tag{3.9}$$

$$\begin{aligned} T_0^\beta &= \rho\, u_0\, u^\beta - \mu\, a^{m\beta}\frac{\partial u_m}{\partial x^0} \\ &= -c^2\left(\rho\, u^\beta + 0(\frac{1}{c^2})\right) \end{aligned} \tag{3.10}$$

$$\begin{aligned} T_\alpha^0 &= \rho\, u_\alpha\, u^0 + \mu\, a^{m0}\left(\frac{\partial u_\alpha}{\partial x^m} + \frac{\partial u_m}{\partial x^\alpha}\right) \\ &= \rho\, u_\alpha + 0\left(\frac{1}{c^2}\right) \end{aligned} \tag{3.11}$$

$$\begin{aligned} T_\alpha^\beta &= \rho\, u_\alpha\, u^\beta + (p + \frac{2}{3}\mu\,\Delta)\,\delta_\alpha^\beta - \mu\, a^{m\beta}\left(\frac{\partial u_\alpha}{\partial x^m} + \frac{\partial u_m}{\partial x^\alpha}\right) \\ &= \rho\, u_\alpha\, u^\beta + (p + \frac{2}{3}\mu\,\Delta)\,\delta_\alpha^\beta - \mu\, \delta^{\varepsilon\beta}\left(\frac{\partial u_\alpha}{\partial x^\varepsilon} + \frac{\partial u_\varepsilon}{\partial x^\alpha}\right). \end{aligned} \tag{3.12}$$

On substituting in equation (3.7) the expression for T_i^k defined in equations (3.9) to (3.12) we recover equations (3.1) and (3.2) to within terms of $0(1/c^2)$ which are to be neglected. The methods of tensor analysis can be used to generalize the tensor equations (3.7) to arbitrary systems of co-ordinates. This operation is carried out in Section 4. The treatment of the energy equation (3.3) is similar and is considered in detail in Section 5.

4. *Equations of Motion Referred to Arbitrary Co-ordinate Systems.*

In a general system equation (3.7) becomes

$$T_{i,k}^k = 0 \tag{4.1}$$

and equation (3.8) becomes

$$T_i^k = \rho v_i v^k + (p + \frac{2}{3}\mu\,\Delta)\,\delta_i^k - \mu\, g^{mk}\,(v_{i,m} + v_{m,i}) \tag{4.2}$$

where

$$\Delta = \frac{1}{J}\frac{\partial}{\partial \xi^l}(J v^l). \tag{4.3}$$

It is convenient to introduce here

$$T^{jk} = g^{ij}\, T_i^k .$$

From equation (4.2)

$$T^{jk} = \rho v^j v^k + (p + \frac{2}{3}\mu\,\Delta)\, g^{jk} - \mu g^{mk}\, g^{ij}\,(v_{i,m} + v_{m,i}). \tag{4.4}$$

Now $v^k = t_m^{*k}\, u^m$ and $v_k = g_{km}\, v^m$, therefore, if (u^i) is defined as in Section 3 then $v^0 = 1$ and $v_0 = -c^2(1 + 0(1/c^2))$. It should be noted that $\frac{\partial v_0}{\partial \xi^i}$ and $\frac{\partial g_{00}}{\partial \xi^i}$ are both 0(1). The quantity g_{00} is defined in equation (2.11).

The generalized form of the energy equation, considered in detail in Section 5, is

$$\rho\, C_p \frac{dT}{d\xi^0} = \Phi + \frac{dp}{d\xi^0} + \frac{1}{J}\frac{\partial}{\partial \xi^\alpha}\left(k\, J\, g^{\alpha\beta}\frac{\partial T}{\partial \xi^\beta}\right). \tag{4.5}$$

where

$$\frac{dT}{d\xi^0} = \frac{\partial T}{\partial \xi^0} + v^\beta \frac{\partial T}{\partial \xi^\beta} \tag{4.6}$$

$$\Phi = \frac{1}{2}\mu\, g^{\alpha\varepsilon}\, g^{\beta\gamma}\, e_{\alpha\gamma}\, e_{\varepsilon\beta} - \frac{2}{3}\mu\,\Delta^2 \tag{4.7}$$

$$e_{\alpha\gamma} = v_{\alpha,\gamma} + v_{\gamma,\alpha} \tag{4.8}$$

and

$$\Delta = \frac{1}{J}\left(\frac{\partial J}{\partial \xi^0} + \frac{\partial}{\partial \xi^\beta}(J\, v^\beta)\right). \tag{4.9}$$

If quantities of $0(1/c^2)$ in equations (4.1) and (4.5) are neglected we obtain the equations that govern the motion of a viscous compressible gas in ξ-space, where x and η co-ordinates are defined relative to ξ co-ordinates by equations of the form (2.2) and (2.3) respectively.

Now

$$T_0^0 = \rho\, v_0\, v^0 + \left(p + \frac{2}{3}\mu\,\Delta \right) - \mu\, g^{m0}\,(v_{0,m} + v_{m,0})$$

$$= -\rho\, c^2 \left(1 + 0(\frac{1}{c^2}) \right) \tag{4.10}$$

$$T_\alpha^0 = \rho\, v_\alpha\, v^0 - \mu\, g^{0m}\,(v_{\alpha,m} + v_{m,\alpha}) \tag{4.11}$$

$$T_0^\beta = \rho\, v_0\, v^\beta - \mu\, g^{m\beta}\,(v_{0,m} + v_{m,0})$$

$$= -\rho\, v^\beta\, c^2 \left(1 + 0(\frac{1}{c^2}) \right) \tag{4.12}$$

$$T_\alpha^\beta = \rho\, v_\alpha\, v^\beta + \left(p + \frac{2}{3}\mu\,\Delta \right)\delta_\alpha^\beta - \mu\, g^{m\beta}\,(v_{\alpha,m} + v_{m,\alpha})$$

$$= \rho\, v_\alpha\, v^\beta + \left(p + \frac{2}{3}\mu\,\Delta \right)\delta_\alpha^\beta - \mu\, g^{\varepsilon\beta}\,(v_{\alpha,\varepsilon} + v_{\varepsilon,\alpha}) + 0\left(\frac{1}{c^2} \right) \tag{4.13}$$

$$v_{\alpha,\varepsilon} = \frac{\partial v_\alpha}{\partial \xi^\varepsilon} + \frac{1}{2} v^l \left(\frac{\partial g_{l\alpha}}{\partial \xi^\varepsilon} + \frac{\partial g_{l\varepsilon}}{\partial \xi^\alpha} - \frac{\partial g_{\varepsilon\alpha}}{\partial \xi^l} \right).$$

Similarly

$$T^{00} = \rho\, v^0\, v^0 + \left(p + \frac{2}{3}\mu\,\Delta \right) g^{00} - \mu\, g^{n0}\, g^{i0}\,(v_{i,m} + v_{m,i})$$

$$= \rho \left(1 + 0(\frac{1}{c^2}) \right) \tag{4.14}$$

$$T^{0\alpha} = T^{\alpha 0} = \rho\, v^0\, v^\alpha + \left(p + \frac{2}{3}\mu\,\Delta \right) g^{0\alpha} - \mu\, g^{m0}\, g^{i\alpha}\,(v_{i,m} + v_{m,i})$$

$$= \rho\, v^\alpha + 0\left(\frac{1}{c^2} \right) \tag{4.15}$$

$$T^{\alpha\beta} = T^{\beta\alpha} = \rho\, v^\alpha\, v^\beta + \left(p + \frac{2}{3}\mu\,\Delta \right) g^{\alpha\beta} - \mu\, g^{m\alpha}\, g^{i\beta}\,(v_{i,m} + v_{m,i})$$

$$= \rho\, v^\alpha\, v^\beta + \left(p + \frac{2}{3}\mu\,\Delta \right) g^{\alpha\beta} - \mu\, g^{\varepsilon\alpha}\, g^{\gamma\beta}\,(v_{\gamma,\varepsilon} + v_{\varepsilon,\gamma}) + 0\left(\frac{1}{c^2} \right). \tag{4.16}$$

Now equation (4.1) may be expanded to obtain:

$$T^k_{i,k} = \frac{1}{J}\left(\frac{\partial}{\partial\xi^0}(J\,T^0_i)+\frac{\partial}{\partial\xi^\beta}(J\,T^\beta_i)\right)-\Gamma^l{}_{ik}\,T^k_l$$

$$= \frac{1}{J}\left(\frac{\partial}{\partial\xi^0}(J\,T^0_i)+\frac{\partial}{\delta\xi^\beta}(J\,T^\beta_i)\right)-\frac{1}{2}g^{lm}\left(\frac{\partial g_{im}}{\partial\xi^k}+\frac{\partial g_{km}}{\partial\xi^i}-\frac{\partial g_{ik}}{\partial\xi^m}\right)T^k_l$$

$$= \frac{1}{J}\left(\frac{\partial}{\partial\xi^0}(J\,T^0_i)+\frac{\partial}{\partial\xi^\beta}(J\,T^\beta_i)\right)-\frac{1}{2}\left(\frac{\partial g_{im}}{\partial\xi^k}+\frac{\partial g_{km}}{\partial\xi^i}-\frac{\partial g_{ik}}{\partial\xi^m}\right)T^{mk}$$

$$= \frac{1}{J}\left(\frac{\partial}{\partial\xi^0}(J\,T^0_i)+\frac{\partial}{\partial\xi^\beta}(J\,T^\beta_i)\right)-\frac{1}{2}\frac{\partial g_{mk}}{\partial\xi^i}T^{mk}. \qquad (4.17)$$

On putting $i = 0$ in equation (4.17), and substituting appropriate expressions for T^0_0, T^β_0, T^{mk}—defined in equations (4.10), (4.12) and (4.14) to (4.16), and if terms of $0(1/c^2)$ are negligible compared with terms of $0(1)$, then we obtain

$$\frac{\partial}{\partial\xi^0}(J\,\rho)+\frac{\partial}{\partial\xi^\beta}(J\,\rho\,v^\beta) = 0, \qquad (4.18)$$

the generalized equation of continuity.

The remaining equations are more complicated. For the case $i = \alpha, \alpha = 1, 2, 3$, we obtain from equation (4.17)

$$\frac{1}{J}\left(\frac{\partial}{\partial\xi^0}(J\,T^0_\alpha)+\frac{\partial}{\partial\xi^\beta}(J\,T^\beta_\alpha)\right)-\frac{1}{2}\frac{\partial g_{mk}}{\partial\xi^\alpha}T^{mk} = 0.$$

Taking the terms in this equation singly we see that

$$\frac{1}{J}\left(\frac{\partial}{\partial\xi^0}(J\,T^0_\alpha)+\frac{\partial}{\partial\xi^\beta}(J\,T^\beta_\alpha)\right)$$

$$= \frac{1}{J}\left[\frac{\partial}{\partial\xi^0}(J\,\rho\,v_\alpha)+\frac{\partial}{\partial\xi^\beta}\left(J\left(\rho\,v_\alpha\,v^\beta+(p+\frac{2}{3}\mu\,\Delta)\,\delta^\beta_\alpha-\mu\,g^{\varepsilon\beta}\,(v_{\alpha,\varepsilon}+v_{\varepsilon,\alpha})\right)\right)\right] + 0\left(\frac{1}{c^2}\right)$$

and, similarly,

$$\frac{1}{2}\frac{\partial g_{mk}}{\partial\xi^\alpha}T^{mk} = \frac{1}{2}\frac{\partial g_{00}}{\partial\xi^\alpha}T^{00}+\frac{\partial g_{0\beta}}{\partial\xi^\alpha}T^{0\beta}+\frac{1}{2}\frac{\partial g_{\varepsilon\beta}}{\partial\xi^\alpha}T^{\varepsilon\beta}$$

$$= \frac{1}{2}\rho\frac{\partial g_{00}}{\partial\xi^\alpha}+\rho\,v^\beta\frac{\partial g_{0\beta}}{\partial\xi^\alpha}+\frac{1}{2}\frac{\partial g_{\varepsilon\beta}}{\partial\xi^\alpha}\left(\rho\,v^\varepsilon\,v^\beta+(p+\frac{2}{3}\mu\,\Delta)\,g^{\varepsilon\beta}-\mu\,g^{\omega\varepsilon}\,g^{\gamma\beta}\,(v_{\gamma,\omega}+v_{\omega,\gamma})\right) + 0\left(\frac{1}{c^2}\right).$$

Therefore, neglecting terms of $0(1/c^2)$ the generalized momentum equations are:

$$\frac{1}{J}\left[\frac{\partial}{\partial\xi^0}(J\,\rho\,v_\alpha)+\frac{\partial}{\partial\xi^\beta}\left(J\left(\rho\,v_\alpha\,v^\beta+(p+\frac{2}{3}\mu\,\Delta)\,\delta^\beta_\alpha-\mu\,g^{\varepsilon\beta}\;(v_{\alpha,\varepsilon}+v_{\varepsilon,\alpha})\right)\right)\right]$$

$$-\frac{1}{2}\rho\frac{\partial g_{00}}{\partial\xi^\alpha}-\rho\,v^\beta\frac{\partial g_{0\beta}}{\partial\xi^\alpha}-\frac{1}{2}\frac{\partial g_{\varepsilon\beta}}{\partial\xi^\alpha}\left(\rho\,v^\varepsilon\,v^\beta+(p+\frac{2}{3}\mu\,\Delta)\,g^{\varepsilon\beta}-\mu\,g^{\omega\varepsilon}\,g^{\gamma\beta}\;(v_{\gamma,\omega}+v_{\omega,\gamma})\right)=0 \qquad (4.19)$$

Equations (4.5), (4.18) and the three equations (4.19) together with the equation of state $p = \rho\,R\,T$ represent the equations of motion of an ideal, viscous, compressible gas referred to an arbitrary system of moving co-ordinates.

5. *The Energy Equation.*

In x-space the energy equation can be expressed in the form of equation (3.3) and we shall show that to within terms of $0(1/c^2)$ equation (3.3) is equal to the following equation:

$$\rho\,C_p\,u^i\frac{\partial T}{\partial x^i} = \Phi_1+u^i\frac{\partial p}{\partial x^i}+\frac{\partial}{\partial x^i}\left(a^{ij}\,k\frac{\partial T}{\partial x^j}\right) \qquad (5.1)$$

where

$$\Phi_1 = \frac{1}{2}\mu\,a^{ik}\,a^{mn}\,e_{km}\,e_{in}-\frac{2}{3}\mu\,\Delta^2 \qquad (5.2)$$

$$e_{ik} = \frac{\partial u_i}{\partial x^k}+\frac{\partial u_k}{\partial x^i} \qquad (5.3)$$

and

$$\Delta = \frac{\partial u^i}{\partial x^i}. \qquad (5.4)$$

The four dimensional vector (u^i) is defined so that $u^0 = 1$. The remaining three components are the contravariant velocity components in x-space.

Now

$$u^i\frac{\partial T}{\partial x^i} = u^0\frac{\partial T}{\partial x^0}+u^\alpha\frac{\partial T}{\partial x^\alpha}$$

$$= \frac{dT}{dx^0}$$

and similarly

$$u^i\frac{\partial p}{\partial x^i} = \frac{dp}{dx^0}.$$

Further

$$\frac{\partial}{\partial x^i}\left(a^{ij}\,k\,\frac{\partial T}{\partial x^j}\right) = \frac{\partial}{\partial x^0}\left(a^{0j}\,k\,\frac{\partial T}{\partial x^j}\right) + \frac{\partial}{\partial x^\alpha}\left(a^{\alpha j}\,k\,\frac{\partial T}{\partial x^j}\right)$$

where

$$a^{0j} = 0 \text{ if } j \neq 0$$

$$a^{00} = -\frac{1}{c^2}$$

$$a^{\alpha j} = \begin{cases} 1 & j = \alpha \\ 0 & j \neq \alpha \end{cases}$$

hence

$$\frac{\partial}{\partial x^i}\left(a^{ij}\,k\,\frac{\partial T}{\partial x^j}\right) = \frac{\partial}{\partial x^\alpha}\left(k\,\frac{\partial T}{\partial x^\alpha}\right) + 0\left(\frac{1}{c^2}\right).$$

Now

$$\Phi_1 = \frac{1}{2}\mu\, a^{ik}\, a^{mn}\, e_{km}\, e_{in} - \frac{2}{3}\mu\,\Delta^2$$

where

$$\Delta = \frac{\partial u^i}{\partial x^i}.$$

Because $u^0 = 1$, and $\frac{\partial u^0}{\partial x^0} = 0$, it follows that

$$\Delta = \frac{\partial u^\beta}{\partial x^\beta}$$

whilst

$$\frac{1}{2}\mu\, a^{ik}\, a^{mn}\, e_{km}\, e_{in} = \frac{1}{2}\mu\,\delta^{\varepsilon\alpha}\,\delta^{\beta\gamma}\,e_{\varepsilon\beta}\,e_{\alpha\gamma} + 0\left(\frac{1}{c^2}\right).$$

It follows that

$$\Phi_1 = \Phi + 0\left(\frac{1}{c^2}\right)$$

where

$$\Phi = \frac{1}{2}\mu\,\delta^{\alpha\varepsilon}\,\delta^{\beta\gamma}\,e_{\varepsilon\beta}\,e_{\alpha\gamma}.$$

The generalized form of (5.1) is

$$\rho\, C_p\, v^i \frac{\partial T}{\partial \xi^i} = \Phi_1 + v^i \frac{\partial p}{\partial \xi^i} + \frac{1}{J}\, \frac{\partial}{\partial \xi^i}\left(J g^{ij}\, k \frac{\partial T}{\partial \xi^j}\right) \tag{5.5}$$

where

$$v^i \frac{\partial T}{\partial \xi^i} = \frac{dT}{d\xi^0}$$

$$\Phi_1 = \frac{1}{2}\mu\, g^{ik}\, g^{mn}\, e_{km}\, e_{in} - \frac{2}{3}\mu\, \Delta^2$$

and

$$e_{km} = v_{m,k} + v_{k,m}\,.$$

Neglecting quantities of $0(1/c^2)$ compared with quantities of $0(1)$, equation (5.5) becomes:

$$\rho\, C_p \frac{dT}{d\xi^0} = \Phi + \frac{dp}{d\xi^0} + \frac{1}{J}\, \frac{\partial}{\partial \xi^\alpha}\left(J\, g^{\alpha\beta}\, k \frac{\partial T}{\partial \xi^\beta}\right) \tag{5.6}$$

where

$$\Phi = \frac{1}{2}\mu\, g^{\alpha\varepsilon}\, g^{\beta\gamma}\, e_{\varepsilon\beta}\, e_{\alpha\gamma} - \frac{2}{3}\mu\, \Delta^2$$

$$\Delta = \frac{1}{J}\left(\frac{\partial J}{\partial \xi^0} + \frac{\partial}{\partial \xi^\beta}(J\, v^\beta)\right)$$

and

$$e_{\alpha\gamma} = v_{\alpha,\gamma} + v_{\gamma,\alpha}\,.$$

6. *Lagrangian Form of Equations of Motion.*

In this section the Langrangian equations of motion of a viscous, compressible, heat conducting fluid are derived from the general expression in Section 4, and, for an ideal gas, a simple minded scheme for the numerical solution of these equations is considered.

Let $F(\xi^0, \xi^1, \xi^2, \xi^3)$ be any scalar function of ξ^0, ξ^1, ξ^2 and ξ^3. It follows that

$$\frac{dF}{d\xi^0} = v^i \frac{\partial F}{\partial \xi^i}$$

$$= \frac{\partial F}{\partial \xi^0} + v^\beta \frac{\partial F}{\partial \xi^\beta}\,. \tag{6.1}$$

Now

$$\frac{dF}{d\xi^0} = \frac{\partial F}{\partial \xi^0} + \frac{d\xi^\beta}{d\xi^0}\, \frac{\partial F}{\partial \xi^\beta}\,.$$

Since these relations are true for all scalar functions F it follows that

$$v^{\beta} \equiv \frac{d\xi^{\beta}}{d\xi^{0}}. \tag{6.2}$$

The surface ξ^{β} = constant, $\beta = 1, 2, 3$, is generated by curves which are particle paths if $\frac{d\xi^{\beta}}{d\xi^{0}} = 0$, i.e. if $v^{\beta} = 0$. Thus if $v^1 = v^2 = v^3 = 0$ and if the quantities $x^1 = x^1(\xi^0, \xi^1, \xi^2, \xi^3)$, $x^2 = x^2(\xi^0, \xi^1, \xi^2, \xi^3)$ and $x^3 = x^3(\xi^0, \xi^1, \xi^2, \xi^3)$ are calculated for constant values of ξ^1, ξ^2, ξ^3 then, as ξ^0 varies, the point (x^1, x^2, x^3) traces out a particle path. The quantities (ξ^1, ξ^2, ξ^3) are the Lagrangian co-ordinates of a particle.

On substituting $v^1 = v^2 = v^3 = 0$ in the continuity equation we obtain

$$\frac{\partial(J\rho)}{\partial\xi^0} = 0\,. \tag{6.3}$$

The momentum equations ($\alpha = 1, 2, 3$) are:

$$\frac{1}{J}\left[\frac{\partial}{\partial\xi^0}\,(J\,\rho\,v_\alpha) + \frac{\partial}{\partial\xi^\beta}\left(J\left(\left(p+\frac{2}{3}\mu\,\Delta\right)\delta^\beta_\alpha - \mu\,g^{\varepsilon\beta}\,(v_{\alpha,\varepsilon}+v_{\varepsilon,\alpha})\right)\right)\right]$$

$$= \frac{1}{2}\rho\,\frac{\partial g_{00}}{\partial\xi^\alpha} + \frac{1}{2}\,\frac{\partial g_{\varepsilon\beta}}{\partial\xi^\alpha}\left(\left(p+\frac{2}{3}\mu\,\Delta\right)g^{\varepsilon\beta} - \mu\,g^{\omega\varepsilon}\,g^{\gamma\beta}\,(v_{\gamma,\omega}+v_{\omega,\gamma})\right) \tag{6.4}$$

where, in this case,

$$\Delta = \frac{1}{J}\,\frac{\partial J}{\partial\xi^0} \tag{6.5}$$

$$v_{\alpha,\varepsilon} = \frac{\partial v_\alpha}{\partial\xi^\varepsilon} - \frac{1}{2}\left(\frac{\partial g_{0\varepsilon}}{\partial\xi^\alpha} + \frac{\partial g_{0\alpha}}{\partial\xi^\varepsilon} + \frac{\partial g_{\alpha\varepsilon}}{\partial\xi^0}\right) \tag{6.6}$$

$$v_\alpha = g_{0\alpha} \tag{6.7}$$

The energy equation is

$$\rho\,C_p\,\frac{\partial T}{\partial\xi^0} = \Phi + \frac{\partial p}{\partial\xi^0} + \frac{1}{J}\,\frac{\partial}{\partial\xi^\alpha}\left(k\,J\,g^{\alpha\beta}\,\frac{\partial T}{\partial\xi^\beta}\right) \tag{6.8}$$

where

$$\Phi = \frac{1}{2}\mu\,g^{\alpha\varepsilon}\,g^{\beta\gamma}\,e_{\alpha\gamma}\,e^{\varepsilon\beta} - \frac{2}{3}\mu\,\Delta^2 \tag{6.9}$$

and

$$e_{\alpha\gamma} = v_{\alpha,\gamma} + v_{\gamma,\alpha}\,. \tag{6.10}$$

Equations (6.3), (6.4) and (6.8) are the Lagrangian equations of motion of a viscous, heat conducting, compressible fluid.

If $x^1, x^2, x^3, g_{01}, g_{02}, g_{03}, \rho$ and T are known on a surface ξ^0 = constant then the derivatives of these quantities with respect to ξ^0 can be evaluated on the surface ξ^0 = constant in the following way. Equation (6.14) can be used to eliminate p from equation (6.8):

$$\rho\, C_v \frac{\partial T}{\partial \xi^0} = \Phi + R\, T \frac{\partial \rho}{\partial \xi^0} + \frac{1}{J} \frac{\partial}{\partial \xi^\alpha}\left(k\, J\, g^{\alpha\beta} \frac{\partial T}{\partial \xi^\beta} \right). \tag{6.15}$$

Since $x^1, x^2, x^3, g_{01}, g_{02}, g_{03}, \rho$ and T are known on ξ^0 = constant functions of these quantities and their derivatives with respect to ξ^1, ξ^2 and ξ^3 can be evaluated at points on ξ^0 = constant. Hence the transformation elements t^α_β ($\alpha, \beta = 1, 2, 3$) may be evaluated on ξ^0 = constant. Then the equations

$$t^1_0\, t^1_1 + t^2_0\, t^2_1 + t^3_0\, t^3_1 = g_{01} \tag{6.16}$$

$$t^1_0\, t^1_2 + t^2_0\, t^2_2 + t^3_0\, t^3_2 = g_{02} \tag{6.17}$$

$$t^1_0\, t^1_3 + t^2_0\, t^2_3 + t^3_0\, t^3_3 = g_{03} \tag{6.18}$$

can be solved simultaneously to obtain

$$t^1_0 = \frac{\partial x^1}{\partial \xi^0} \tag{6.19}$$

$$t^2_0 = \frac{\partial x^2}{\partial \xi^0} \tag{6.20}$$

and

$$t^3_0 = \frac{\partial x^3}{\partial \xi^0}. \tag{6.21}$$

Equation (6.3) states that $J\rho$ is independent of ξ^0 so, in view of equation (6.7), equations (6.4) yield values for

$$\frac{\partial g_{01}}{\partial \xi^0}, \quad \frac{\partial g_{02}}{\partial \xi^0}, \quad \text{and} \quad \frac{\partial g_{03}}{\partial \xi^0}.$$

All the coefficients and the remaining partial derivatives in equation (6.4) can be evaluated at points on ξ^0 = constant. Pressure is given in terms of ρ and T by equation (6.14). Now

$$J^2 = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix} \tag{6.22}$$

and differentiating this expression with respect to ξ^0 leads to an equation which can be solved to yield $\frac{\partial J}{\partial \xi^0}$ at points on ξ^0 = constant. This value for $\frac{\partial J}{\partial \xi^0}$ can be substituted in equation (6.3) to obtain $\frac{\partial \rho}{\partial \xi^0}$. The quantity $\frac{\partial T}{\partial \xi^0}$ follows from equation (6.15).

This procedure could form the basis of a simple numerical scheme for solving an initial value problem in which the quantities $x^1, x^2, x^3, g_{01}, g_{02}, g_{03}, \rho$ and T are given on a surface ξ^0 = constant, and the equations of motion are to be integrated to obtain the values of these quantities for all values of ξ^0 greater than this constant. Usually, however, the values of the cartesian velocity components t_0^1, t_0^2, t_0^3 would be given on the initial surface rather than g_{01}, g_{02} and g_{03}—the co-variant velocity components. The basic procedure described for finding partial derivatives of $\frac{\partial x^\alpha}{\partial \xi^0}, \frac{\partial g_{0\alpha}}{\partial \xi^0}$, for $\alpha = 1, 2, 3, \frac{\partial \rho}{\partial \xi^0}$ and $\frac{\partial T}{\partial \xi^0}$ can still be used even in this case because the quantities $g_{0\alpha}$, $\alpha = 1, 2, 3$, are easily obtained in terms of t_0^1, t_0^2, t_0^3 and the partial derivatives of x^1, x^2 and x^3 with respect to ξ^1, ξ^2 and ξ^3 from equations (6.16) to (6.18).

It should be noted that complications arise except in cases such that neighbouring particles at time $t = 0$ remain neighbouring particles for all time—in such cases a uniform mesh of points set up at time $t = 0$ is not subsequently severely distorted. If an initial mesh is distorted by large amounts the finite difference equations constructed from the equations described here, with such an initial mesh as basis, are not valid approximations to the Lagrangian equations of motion for all time—they are valid only whilst the distortion of the initial mesh is small. In such cases it would be necessary either to create new Lagrangian meshes at suitable intervals of time or to rearrange the calculation so that in order to predict the motion of a given particle only conditions at neighbouring particles are used.

7. *Equations of Motion Referred to Rotating Co-ordinates.*

We shall suppose that x^1, x^2, x^3 are cartesian co-ordinates and that ξ^1, ξ^2, ξ^3 are cartesian co-ordinates in a system rotating with angular velocity Ω about the x^3-axis of the x-system. It follows that

$$x^0 = \xi^0$$

$$x^1 = \xi^1 \cos \Omega\, \xi^0 - \xi^2 \sin \Omega\, \xi^0$$

$$x^2 = \xi^1 \sin \Omega\, \xi^0 + \xi^2 \cos \Omega\, \xi^0$$

$$x^3 = \xi^3 .$$

Hence

$$t_0^0 = 1$$
$$t_0^1 = -\Omega\, \xi^1 \sin \Omega\, \xi^0 - \Omega\, \xi^2 \cos \Omega\, \xi^0$$
$$t_0^2 = \Omega\, \xi^1 \cos \Omega\, \xi^0 - \Omega\, \xi^2 \sin \Omega\, \xi^0$$
$$t_0^3 = 0$$
$$t_1^0 = 0$$
$$t_1^1 = \cos \Omega\, \xi^0$$
$$t_2^1 = -\sin \Omega\, \xi^0$$
$$t_3^1 = 0$$
$$t_2^0 = 0$$
$$t_1^2 = \sin \Omega\, \xi^0$$
$$t_2^2 = \cos \Omega\, \xi^0$$
$$t_3^2 = 0$$
$$t_3^0 = t_1^3 = t_2^3 = 0$$

and

$$t_3^3 = 1.$$

If follows that

$$J = \begin{vmatrix} 1 & t_0^1 & t_0^2 & 0 \\ 0 & t_1^1 & t_1^2 & 0 \\ 0 & t_2^1 & t_2^2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= 1$$

and

$$g_{00} = -c^2\left(1-\frac{\Omega^2((\xi^1)^2+(\xi^2)^2)}{c^2}\right)$$

$$g_{01} = -\Omega\,\xi^2$$

$$g_{02} = \Omega\,\xi^1$$

$$g_{03} = 0$$

$$g_{11} = g_{22} = g_{33} = 1$$

$$g_{12} = g_{13} = g_{23} = 0\,.$$

Since ξ^1, ξ^2 and ξ^3 are co-ordinates in a cartesian system it follows that $v^\beta = \dfrac{d\xi^\beta}{d\xi^0}$ is the velocity along the ξ^β axis. In the following analysis we shall suppose that $\mu = 0$. The continuity and momentum equations are:

$$\frac{1}{J}\left(\frac{\partial(J\rho)}{\partial\xi^0}+\frac{\partial}{\partial\xi^\beta}(J\rho\,v^\beta)\right) = 0 \tag{7.1}$$

$$\begin{aligned} \rho\frac{\partial v_1}{\partial\xi^0}+\rho\,v^\beta\frac{\partial v_1}{\partial\xi^\beta} &= -\frac{\partial p}{\partial\xi^1}+\frac{1}{2}\rho\cdot 2\Omega^2\,\xi^1+\rho\,v^1\frac{\partial g_{01}}{\partial\xi^1}+\rho\,v^2\frac{\partial g_{02}}{\partial\xi^1} \\ &= -\frac{\partial p}{\partial\xi^1}+\rho\,\Omega^2\,\xi^1+\rho\,v^2\,\Omega \end{aligned} \tag{7.2}$$

$$\rho\frac{\partial v_2}{\partial\xi^0}+\rho\,v^\beta\frac{\partial v_2}{\partial\xi^\beta} = -\frac{\partial p}{\partial\xi^2}+\frac{1}{2}\rho\cdot 2\Omega^2\,\xi^2+\rho\,v^1\frac{\partial g_{01}}{\partial\xi^2}+\rho\,v^2\frac{\partial g_{02}}{\partial\xi^2} \tag{7.3}$$

$$\rho\frac{\partial v_3}{\partial\xi^0}+\rho\,v^\beta\frac{\partial v_3}{\partial\xi^\beta} = -\frac{\partial p}{\partial\xi^3}\,. \tag{7.4}$$

Now

$$v_1 = g_{01}+g_{11}\,v^1$$

$$v_2 = g_{02}+g_{22}\,v^2$$

and

$$v_3 = v^3\,.$$

Therefore equations (7.1) to (7.4) become

$$\frac{\partial \rho}{\partial \xi^0}+\frac{\partial}{\partial \xi^\beta}(\rho\, v^\beta) = 0 \tag{7.5}$$

$$\rho\,\frac{\partial v^1}{\partial \xi^0}+\rho\, v^\beta \frac{\partial v^1}{\partial \xi^\beta} = -\frac{\partial p}{\partial \xi^1}+\rho\,\Omega^2\,\xi^1+2\rho\, v^2\,\Omega \tag{7.6}$$

$$\rho\,\frac{\partial v^2}{\partial \xi^0}+\rho\, v^\beta \frac{\partial v^2}{\partial \xi^\beta} = -\frac{\partial p}{\partial \xi^2}+\rho\,\Omega^2\,\xi^2-2\rho\, v^1\,\Omega \tag{7.7}$$

$$\rho\,\frac{\partial v^3}{\partial \xi^0}+\rho\, v^\beta \frac{\partial v^3}{\partial \xi^\beta} = -\frac{\partial p}{\partial \xi^3} \tag{7.8}$$

Equations (7.6) to (7.8) should be compared with the following equations for motion in a rotating co-ordinate frame:

$$\rho\,\frac{d\boldsymbol{v}}{d\xi^0} = -2\rho(\boldsymbol{\omega}\times\boldsymbol{v})-\rho(\boldsymbol{\omega}\times\boldsymbol{\omega}\times\boldsymbol{r})-\nabla p \tag{7.9}$$

where $\boldsymbol{v} = (v^1, v^2, v^3)$

$\boldsymbol{\omega} = (0, 0, \Omega)$

and $\boldsymbol{r} = (\xi^1, \xi^2, \xi^3)$.

When $\boldsymbol{v}$, $\boldsymbol{\omega}$ and $\boldsymbol{r}$ take the above form equations (7.9) and (7.6) to (7.8) are identical.

8. *Concluding Remarks.*

The equations of motion of a viscous, compressible, heat conducting gas in a general co-ordinate system have been derived. An obvious possible application of the theory is to unsteady flow problems in which one or more boundary conditions have to be satisfied on an arbitrary surface. The analysis presented here can be used to ensure that this surface is a co-ordinate surface. This remark applies even if the shape of the surface is unknown in the first place (as in the case of a free vortex sheet) and depends upon the solution of the governing partial differential equations.

i, j, k, l, m	Indices which take values 0, 1, 2, 3
$\alpha, \beta, \gamma, \varepsilon, \omega$	Indices which take values 1, 2, 3
(x^i)	Point in x-space, x^0 represents time
(η^i)	Point in η-space, η^0 represents time
(ξ^i)	Point in ξ-space, ξ^0 represents time
ds	Measures separation of events in space time
c	Speed of light
$t^i_j = \dfrac{\partial x^i}{\partial \xi^j}$	
$t^{*i}_j = \dfrac{\partial \xi^i}{\partial x^j}$	
$\mathscr{F} = (t^i_j)$	A 4 × 4 matrix
$\mathscr{F}^{-1} = (t^{*j}_i)$	Inverse matrix to $\mathscr{F}$
(a_{ik})	Metric tensor in x-space
(γ_{ik})	Metric tensor in η-space
(g_{ik})	Metric tensor in ξ-space
τ^i_j	$= \dfrac{\partial \eta^i}{\partial \xi^j}$
τ^{*i}_j	$= \dfrac{\partial \xi^i}{\partial \eta^j}$
$(a^{ik}), (\gamma^{ik}), (g^{ik})$	Tensors associated with the tensors (a_{ik}), (γ_{ik}) and (g_{ik}) respectively
J	$= \lvert\mathscr{F}\rvert = \dfrac{\partial(x^1, x^2, x^3)}{\partial(\xi^1, \xi^2, \xi^3)}$
(v^i)	Contravariant vector in ξ-space
(v_i)	Co-variant vector in ξ-space associated with (v^i) $v_i = g_{ij} v^j$
Γ^k_{jm}	Christoffel symbol of the 2nd kind
(u^i)	Contravariant vector in x-space
(u_i)	Co-variant vector in x-space, $u_\alpha = u^\alpha$
q	Velocity magnitude
ρ	Density
μ	Coefficient of viscosity
Δ	Dilation
k	Thermal conductivity
$e_{\gamma\alpha}$	Rate of strain tensor

$\delta^{\alpha\beta}, \delta^{\beta}_{\alpha}$	Unity when $\beta = \alpha$, zero when $\beta \neq \alpha$
T	Temperature
R	Universal gas constant
C_v	Specific heat at constant volume
C_p	Specific heat at constant pressure
Φ	Dissipation function
p	Pressure
T^{ij}, T^{j}_{i}	Stress tensors
Φ_1	$= \Phi+0\left(\frac{1}{c^2}\right)$
$F(\xi^0, \xi^1, \xi^2, \xi^3)$	A scalar function of ξ^0, ξ^1, ξ^2 and ξ^3
Ω	Angular velocity

REFERENCES

No.	*Author(s)*	*Title, etc.*
1	G. C. McVittie	A systematic treatment of moving axes in hydrodynamics. *Proc. Roy. Soc.* A *196*, p. 285, 1949.
2	W. Prager	An elementary discussion of definitions of stress rate. *Quart. Appl. Math. 18*, p. 403, 1960.
3	L. I. Sedov	Different definitions of the rate of change of a tensor. *Prikl. Mat. Mekh 24*, p. 579, 1960.
4	E. Koppe and G. Zimmerman	On the tensorial description of time derivatives (in German). *Acta Mechanica,* Vol. 1, No. 1, p. 16, 1965.
5	F. Walkden and J. M. Howie	A new method for calculating the supersonic flow past a body. R.A.E. Tech. Note Maths. 89. A.R.C. 24 350, 1962.
6	D. Catherall and K. W. Mangler	The integration of the two-dimensional laminar boundary-layer equations past the point of vanishing skin friction. *J. Fluid Mechanics,* Vol. 26, Pt. 1, p. 163, September 1966.
7	F. Walkden and B. A. L. Hart	Supersonic flow fields produced by conical bodies of arbitrary cross section. R.A.E. Tech. Report No. 66 129. A.R.C. 28 325, 1966.
8	D. Catherall and K. W. Mangler	An indirect method for the solution of the Navier-Stokes equations for laminar incompressible flow at large Reynolds numbers. R.A.E. Report Aero 2683. A.R.C. 25 599, 1963.

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