R. & M. No. 3563

MINISTRY OF TECHNOLOGY

AERONAUTICAL RESEARCH COUNCIL

The Speed Response of an Aircraft Constrained to Fly along a Straight Path in the Presence of Turbulence at Low Altitude

By J. G. Jones

LONDON: HER MAJESTY'S STATIONERY OFFICE
1969
PRICE 13*s.* 0*d.* NET

The Speed Response of an Aircraft Constrained to Fly along a Straight Path in the Presence of Turbulence at Low Altitude

By J. G. Jones

Reports and Memoranda No. 3563*
September, 1967

Summary.

Effects of speed stability and turbulence upon airspeed response of an aircraft flying under landing approach conditions are investigated by considering response to turbulence when the aircraft is constrained to fly along a straight flight path by means of the elevator. The speed-stable and speed-unstable cases are treated in a uniform manner by considering the way in which airspeed 'diffuses' from an initial known value, results being expressed in terms of the time variation of the probability distribution of airspeed error.

CONTENTS

*Replaces R.A.E. Tech. Report 67 242—A.R.C. 29 912.

1. *Introduction.*

The control of an aircraft during a landing approach is a multivariable task in which speed and height are controlled by co-ordinated use of the throttle and elevator. The stability of the resulting closed loop system has been discussed, for example, by Cromwell and Ashkenas[1], who consider various methods of loop closure. In the present Report we are not concerned directly with the control problem but rather with the assessment of the relative effects of turbulence and aircraft speed stability upon the variability of airspeed. In order to illustrate these effects in a simple manner we have used the first order equation of constrained flight. The concept of stability of partially controlled flight or stability with constraint was introduced by Neumark[2]. In considering the problem of longitudinal stability below minimum-drag speed he considered the dynamics of an aircraft in which a control, for example the elevator, is used to suppress one component of the disturbance. In the case where the aircraft is controlled by the elevator in such a way as to maintain constant height, or more generally a straight glide path, he showed that the equations of longitudinal motion were reduced to a single equation of first order, the time constant of which describes 'speed stability'. He argued that the theory could be used to give approximate solutions of problems in which the pilot moves his control so as to keep deviation from the glide path always as small as possible.

Subsequently, the concept of speed stability, based upon the above theory, has been shown to be physically significant both in flight[3] and in a simulator[4]. Further, the concept has been applied[5] as a theoretical method of estimating minimum comfortable approach airspeeds, and has been adequately correlated with some aspects of pilots' assessments of handling qualities in approaches.

The existing theory[2] considers the response of an aircraft in still air, and in the case of a given initial speed error, allows the time to half amplitude (in the speed stable case) or time to double amplitude (in the speed unstable case) to be calculated.

In the present Report the theory of constrained flight is extended to include the case of aircraft speed response in the presence of turbulence. As in the previous work, the aircraft is assumed to be constrained to fly along a straight path by means of the elevator, and the motion is supposed to take place at constant thrust. Now, however, the aircraft is acted upon by both horizontal and vertical components of turbulence and we consider statistically the way in which speed diverges from an assumed initial trim speed.

In practice, of course, the elevator and throttle are used in a co-ordinated manner to control both speed and height. Indeed, in the case of flight below minimum drag speed under tight constraints such as occur in deck landings it has been found[1] that the throttle is often regarded as the primary flight path controller. On the other hand, for a fairly speed-stable aircraft it appears that it is quite feasible to control altitude using the elevator, control of airspeed requiring only intermittent use of the throttle.

The difficulty of handling an aircraft in a landing approach is compounded of a variety of features, and it would appear to be profitable to consider a variety of approaches to the problem. Multi-loop analysis[1] has been applied to consider the stability of various ways of using the throttle and elevator. This approach has, however, been applied only to the still-air case and has not been directly concerned with sensitivity to turbulence. The longitudinal response of an aircraft may in many cases be considered in terms of two independent modes: the short period and phugoid modes. The short-period mode has little direct effect on speed and height response, but as it is the principal contribution to the difficulty the pilot may have in controlling attitude it indirectly affects the ability of the pilot to control flight path. The phugoid mode, which in the controlled case becomes closely related to the speed stability, defines the basic speed and height response to turbulence. Even if the pilot were able to demand instantaneously any required aircraft attitude (i.e. no short-period difficulties) there is still a basic energy balance to be met: for example, holding flight path (suppressing height disturbances) in the face of an up-gust will increase airspeed. It is this basic energy problem, that would exist even if there were no 'short period' handling difficulties, that concerns us here.

Consequently, the present Report is not intended to present a model of what necessarily happens in controlled flight along a glide path, but to show what *would* happen to the airspeed if the aircraft were constrained to fly along a straight path by means of the elevator. In this way we hope to obtain a measure of the relative effects upon speed variability of aircraft speed stability and turbulence intensity, and it is hoped that the trends indicated will be valid independent of the actual means of control employed. Since we are concerned with trends rather than with quantitative prediction it has been thought adequate to use linearised equations of motion. The main limitation of such a treatment is that speed stability is regarded as a constant throughout the motion. In a more exact theory speed stability depends upon airspeed and this factor could have significant effects on, for example, the time taken for airspeed to diverge by a large amount towards the stall. It is intended to treat this nonlinear problem in a future paper.

2. *Response of the Aircraft to Turbulence.*

From the pilot's point of view the immediate effect of turbulence on longitudinal response is to excite the short-period mode, making attitude control more difficult. However, speed and height response is affected directly to a negligible extent by the short-period mode and can in general be described by simplified equations (the 'phugoid' equations) which neglect effects of pitching inertia.

Both horizontal and vertical components need to be considered if the speed response is to be calculated. The horizontal component of turbulence feeds into the equations of speed response in a direct and obvious way. In particular, because of the inertia of the aircraft, at high frequencies the airspeed fluctuations tend to follow turbulence fluctuations exactly (though the airspeed indicated to the pilot will be filtered by instrument lags). Only after a time governed in the uncontrolled case by the phugoid frequency and in the controlled case by the speed stability time constant do the airspeed variations depend on the properties of the aircraft.

In the case of the vertical component of turbulence the requirement that the flight path be straight will imply that the increase in lift associated with an up-gust is counteracted by a downward elevator deflection and a corresponding nose down attitude change. This results in an acceleration along the flight path which is proportional to the vertical gust velocity.

Of course, in the case of gust components of high frequency (greater, say, than about $\frac{1}{4}$ cycle per sec), the assumption that the pilot can keep the aircraft close to a straight glide path becomes unrealistic. In practice, the pitching inertia of the aircraft, together with lag in pilot control, would effectively provide a low-pass filter in the control equations. However, it can be verified from the results of the present theory and from the phugoid equations that the effects of turbulence on groundspeed and height are only significant at relatively low frequencies so that the present theory does not give physically implausible results.

3. *Equations of Motion.*

The equations of motion for constrained flight at constant height and for flight along a straight (descending) glide path are of identical form. In what follows we will consider specifically the case of flight at constant height. The reason for this choice is that we require the disturbance inputs to the aircraft to be stationary random processes but consider the horizontal and vertical components of turbulence to be height dependent. It is intended that by considering horizontal flight at more than one height a qualitative picture can be deduced of the effects of stability and turbulence on an actual landing approach.

The basic longitudinal equations of motion for flight at constant height, referred to body axes moving in space but fixed in the aircraft and initially coinciding with the undisturbed direction of motion of the aircraft ('wind axes') can be written in the (non-dimensional) standard small perturbation form

$$\frac{d\hat{u}}{d\tau} - x_u(\hat{u}+\hat{u}_g) - x_w(\hat{w}+\hat{w}_g) + k\theta = 0\,, \tag{1}$$

$$-z_u(\hat{u}+\hat{u}_g) + \frac{d\hat{w}}{d\tau} - z_w(\hat{w}+\hat{w}_g) - \hat{q} = 0\,, \tag{2}$$

$$\hat{q} = \frac{d\theta}{d\tau}, \tag{3}$$

$$\theta - \hat{w} = \frac{1}{V}\frac{dH}{dt} = 0, \tag{4}$$

where the usual pitching equation has been replaced by equation (4) which expresses the constraint to constant height.

The above equations can be combined to give

$$\frac{d\hat{u}}{d\tau} - x_u(\hat{u}+\hat{u}_g) - x_w(\hat{w}+\hat{w}_g) + k\hat{w} = 0, \tag{5}$$

$$-z_u(\hat{u}+\hat{u}_g) - z_w(\hat{w}+\hat{w}_g) = 0. \tag{6}$$

Equation (6) expresses the equilibrium of normal-force components, or to first order, equilibrium of vertical-force components, a consequence of the constant height constraint.

Eliminating $\hat{w}$ from equations (5) and (6):

$$\frac{d\hat{u}}{d\tau} - x_u(\hat{u}+\hat{u}_g) + x_w\frac{z_u}{z_w}(\hat{u}+\hat{u}_g) + k\left\{-\hat{w}_g - \frac{z_u}{z_w}(\hat{u}+\hat{u}_g)\right\} = 0. \tag{7}$$

Equation (7) can be written in the form

$$\frac{d\hat{u}}{d\tau} + A\hat{u} = -A\hat{u}_g - B\hat{w}_g \tag{8}$$

where

$$A = -\left\{x_u + \frac{z_u}{z_w}(k - x_w)\right\}, \tag{9}$$

and

$$B = -k. \tag{10}$$

k is defined here by the equation

$$k = \frac{W}{\rho V^2 S}. \tag{11}$$

In the case of a conventional aircraft at moderate incidence (as opposed to partially or entirely jet-borne flight) a consequence of equation (11) is that

$$k = \tfrac{1}{2} C_L.$$

Now, $\hat{u}$ is effectively the ground speed perturbation of the aircraft. If we write

$$\hat{u}_a = \hat{u} + \hat{u}_g, \tag{12}$$

$\hat{u}_a$ is the airspeed perturbation of the aircraft, and equations (8) and (12) lead to the basic equation of airspeed response:

$$\boxed{\frac{d\hat{u}_a}{d\tau}+A\hat{u}_a = \frac{d}{d\tau}\hat{u}_g - B\hat{w}_g\,.} \tag{13}$$

4. *Discussion of Speed-Response Equations.*

Referring to equation (13) we can now see the significant parameters affecting speed response.

$$A = -\left\{x_u+\frac{z_u}{z_w}(k-x_w)\right\} \tag{9}$$

$$= C_D-\frac{C_L}{a}(1+C_D/a)^{-1}\left(2k+a\frac{dC_D}{dC_L}-C_L\right) \tag{14}$$

is the non-dimensional time constant of the subsidence or divergence in still air $\Big($where we have used[2] $x_u = -C_D,\ \ x_w = \frac{1}{2}\left(C_L-a\frac{dC_D}{dC_L}\right),\ \ z_u = -C_L,\ \ z_w = -\frac{1}{2}(a+C_D)\Big)$. In the case of a conventional aircraft at normal attitudes we can put

$$k \doteqdot \tfrac{1}{2}C_L\,.$$

In addition we generally have

$$C_D/a << 1\,,$$

and under these circumstances equation (14) reduces to the usual form[2]

$$A \doteqdot C_D-C_L\frac{dC_D}{dC_L}. \tag{15}$$

The stability criterion, or condition for subsidence is

$$A>0\,,$$

and the speed at which $A = 0$ is the minimum-drag speed.

Equation (8) illustrates the effects of horizontal and vertical gusts on ground-speed response. The ratio of the ground-speed changes due to horizontal and vertical gusts is A/B. In the case of conventional flight, when $k = \frac{1}{2}C_L$,

$$\frac{A}{B} = -2\left(\frac{C_D}{C_L}-\frac{dC_D}{dC_L}\right). \tag{16}$$

In the case of step gusts the asymptotic changes in ground speed in the case of a speed-stable aircraft are respectively and

$$\left.\begin{aligned}\hat{u} &= -\hat{u}_g \\ \hat{u} &= -\frac{B}{A}\hat{w}_g\end{aligned}\right\} \tag{17}$$

Equation (13) illustrates the analogous effects on airspeed response. It is convenient to examine high and low-frequency effects separately. At high frequencies the horizontal component of turbulence has a predominant effect and the fluctuations of airspeed are identical with those of the horizontal turbulence. This result is due to aircraft inertia and is independent of the height constraint: exactly the same result is obtained if the (elevator fixed) phugoid equations are used.

At very low frequencies the vertical gust has a predominant effect, and in the speed stable case

$$\hat{u}_a \sim -\frac{B}{A}\hat{w}_g . \tag{18}$$

As an illustration we consider the case where the aircraft meets a step gust at $t = 0$, such that

$$\hat{u}_g = \hat{w}_g = \hat{u}^* H(\tau), \tag{19}$$

where $\hat{u}^* = u^*/V$ is a constant, and $H(\tau)$ is a unit step function. Then equation (13) has the solution

$$\hat{u}_a = \hat{u}^* H(\tau)\, e^{-A\tau} - \hat{u}^* \frac{B}{A}(1 - e^{-A\tau}), \tag{20}$$

where the first term comes from the horizontal gust and contains a step of the same size as that in the gust, and the second term comes from the vertical gust and in the speed-stable case ($A > 0$) increases smoothly from zero to the asymptotic value given by equation (18).

In many practical cases the speed-stability time constant A is very small and the transient response of the aircraft becomes indistinguishable from that of a neutrally stable aircraft ($A = 0$) over moderate intervals of time. In this case equation (20) takes the form

$$\hat{u}_a = \hat{u}^* H(\tau) - \hat{u}^* B\tau . \tag{21}$$

It is clear that if B is large the effect of vertical gusts on rate of change of airspeed will be large. To clarify this point it is instructive to write equation (21) in terms of real time. Then we obtain

$$u_a = u^* H(t) + u^* \frac{g}{V} t . \tag{22}$$

Thus the effect of the vertical step gust of magnitude u^* is given by

$$\frac{du_a}{dt} = u^* \frac{g}{V} \tag{23}$$

and is large for low approach speeds V. Thus we can expect relatively large effects of vertical gusts on VTOL and STOL type aircraft.

5. *Speed Response to Turbulence.*

5.1. *Background.*

For the rest of this Report we consider the case of an aircraft in the presence of random turbulence and controlled by the elevator so as to fly along a straight path. The motion is assumed to take place at constant thrust. It is assumed that at $t = 0$ the airspeed error is zero. We then consider the statistics of airspeed error as time increases. Results are expressed in terms of the time dependence of the variance of speed error. It is assumed that the horizontal and vertical components of turbulence velocity can be represented as uncorrelated Gaussian processes. The speed error will then also be Gaussian (since we have used linearised equations) and the mean and variance thus completely define the amplitude probability distribution. The turbulence is defined by its variance, as a measure of intensity, and spectral density.

Since, at $t = 0$, the speed error is assumed to be zero, the amplitude probability distribution is a delta function (this is just a convenient way of regarding an exactly defined quantity as part of a random process). As time increases the probability distribution 'diffuses' in a manner defined by the time variation of the variance. In the speed-stable case the variance will tend asymptotically for large time to the value given by the usual stationary theory (the effect of the constraint at $t = 0$ becoming negligible). We will concentrate in the following sections on the fluctuation of airspeed, based on equation (13), as this is the variable most closely related to the pilot control problem. Analogous results for groundspeed can be derived, based on equation (8).

5.2. *Description of Low Altitude Turbulence.*

The turbulence velocity components u_g and w_g are defined by their variance and power spectra. A simple analytical form for the power spectrum which is adequate for our purposes is

$$G(\Omega) = \sigma^2 \frac{2L}{\pi} \frac{1}{1+\Omega^2 L^2}, \tag{24}$$

where σ^2 is the mean square turbulence velocity, Ω is space frequency in rad.ft^{-1}, and L is the scale of turbulence. In terms of non-dimensional frequency $\hat{\omega}$ (in radians per unit τ) equation (24) becomes

$$\Phi(\hat{\omega}) = \hat{\sigma}^2 \frac{2}{\pi \mu_t} \frac{1}{1+\hat{\omega}^2/\mu_t^2}, \tag{25}$$

where $\hat{\sigma}^2$ is the mean-square non-dimensional velocity and

$$\mu_t = \frac{m}{\rho SL}. \tag{26}$$

The corresponding non-dimensional turbulence autocorrelation function is

$$R(\tau) = \int_0^\infty \Phi(\hat{\omega}) \cos \hat{\omega}\tau \, d\hat{\omega}$$

$$= \hat{\sigma}^2 e^{-\mu_t|\tau|}. \tag{27}$$

In order to obtain quantitative estimates of speed response it remains to substitute numerical values for σ and L in equation (24). Existing experimental data on low altitude turbulence, particularly in the case of the vertical component, is rather sparse and further measurements are an important part of future research. The values we assume are taken from an unpublished note by J. K. Zbrozek describing an approximate model of low altitude turbulence which agrees adequately for our purposes with the trends of existing experimental measurements[6,7]. The values of σ for both vertical and horizontal components of turbulence velocity are expressed in terms of a 'reference turbulence intensity' $(\sigma_w)_{\text{ref}}$, which is the value of the root mean square of the vertical component at the altitude of 250 feet (where the vertical component is of greatest intensity). The probability of exceeding a given value of $(\sigma_w)_{\text{ref}}$ over 'rough terrain', which is appropriate to the vicinity of a typical airfield with close proximity of buildings, etc., is illustrated in Fig. 1. The mean square turbulence velocity is closely correlated with wind statistics, an approximate relationship being

$$(\sigma_w)_{\text{ref}} \doteqdot 0{\cdot}2\, U_{10}$$

where U_{10} is the mean wind speed at 10 metres above the ground. In terms of $(\sigma_w)_{\text{ref}}$, values of σ and L for vertical (w_g) and horizontal (u_g) components of turbulence velocity at altitudes of 40 feet and 500 feet are shown in the following table. These two altitudes are used in Section 7 to illustrate typical effects of altitude on airspeed perturbations.

Altitude	L_w	$\sigma_w/(\sigma_w)_{\text{ref}}$	L_u	$\sigma_u/(\sigma_w)_{\text{ref}}$
40′	75′	0·935	620′	1
500′	620′	0·985	950′	0·985

Power spectra corresponding to the values tabulated above are illustrated in Figs. 2 and 3. At 500 feet the vertical and horizontal components are very similar (Fig. 3) but at very low altitude (Fig. 2) the low-frequency components of the vertical velocity are suppressed. In the model employed, the mean-square value of vertical turbulence velocity decreases by less than 10 per cent in going from 500 feet to 40 feet, the loss of energy at low frequencies being compensated by an increase at higher frequencies (Figs. 2 and 3). Existing data is not adequate to give great confidence in this phenomenon, but in any case the results of the present investigation are insensitive to the higher frequencies in the vertical turbulence.

5.3. *Airspeed Response.*

The equation describing airspeed response to turbulence is

$$\frac{d\hat{u}_a}{d\tau} + A\hat{u}_a = \frac{d}{d\tau}\hat{u}_g - B\hat{w}_g. \tag{13}$$

$\hat{u}_g$ and $\hat{w}_g$ are the (non-dimensional) horizontal and vertical components of turbulence velocity, with autocorrelation function given by equation (27). Since $\hat{u}_g$ and $\hat{w}_g$ are assumed to be uncorrelated, the components of response $\hat{u}_a$ to horizontal and vertical gusts can be considered separately. Since $\hat{u}_g$ and $\hat{w}_g$ are assumed to be Gaussian processes it follows that $\hat{u}_a$ is also Gaussian.

We assume that at $\tau = 0$ the airspeed error is zero, i.e. $\hat{u}_a(o) = 0$. As regards the initial values of turbulence velocity there are two possibilities. We could assume that $\hat{u}_g(o) = 0$ and $\hat{w}_g(o) = 0$, implying that the idealized pilot had contrived to trim airspeed at a moment when the wind velocity was passing through its mean value. Alternatively we could assume that $\hat{u}_g(o)$ and $\hat{w}_g(o)$ are random variables having a Gaussian amplitude probability distribution with variance equal to the mean-square turbulence intensity. In physical terms this assumption implies that the pilot trims to the correct airspeed, but without reference to the gust intensity at the moment of trimming. In practice the above alternative assumptions will provide bounds to what actually happens: the pilot will attempt to trim airspeed relative to the mean wind speed but can never succeed in exactly doing this. At low gust levels, when he is able to watch airspeed fluctuations over a fairly long time interval, he will be able to judge when the wind velocity is near its mean value and the first of the two assumptions will be near to physical reality. However, at high gust levels, when airspeed fluctuations tend to be large, the pilot will only be able to judge the mean of the relatively high frequency fluctuations (due to horizontal gusts). In addition, the necessity of keeping airspeed fluctuations within bounds may force him to adjust the airspeed error to a low value at a moment when the turbulence velocity is not small. In these circumstances the second of the above alternative assumptions may be a better description of the situation.

In view of these two possibilities, the course taken has been to consider the relative effects of the two assumptions in detail in Section 6, using as illustration the response to the horizontal component of turbulence of an aircraft with neutral stability. Having illustrated the relative effects of the two possibilities, we have made the second of the two possible assumptions (i.e. random initial gust velocity) in the remainder of the report.

Throughout the theoretical treatment it is assumed that for $t \geqslant 0$ the thrust takes a predetermined trim value appropriate to the approach. In other words, once the pilot has achieved a zero initial airspeed error $\hat{u}_a(o) = 0$, he returns the throttle immediately to a predetermined setting. As a result, in the case of a speed-stable aircraft, any particular time history of airspeed error will eventually fluctuate about a zero mean value.

For the moment, then, we consider the general problem of the solution of equation (13) with initial condition given by $\hat{u}_a(o) = 0$, with $\hat{u}_g(o)$ and $\hat{w}_g(o)$ random, and find the way in which the variance of the (Gaussian) response $\hat{u}_a(\tau)$ increases from zero as time increases. The analysis of the response to the vertical component of turbulence $\hat{w}_g$ is presented in Appendix A (*see* Fig. 4) and the analogous treatment for the horizontal component $\hat{u}_g$ is in Appendix B (*see* Fig. 5). Combining the results gives

$$\hat{\sigma}^2_{\hat{u}_a}(\tau) = \hat{\sigma}^2_{\hat{u}_g}\frac{\mu_t}{A}\left[1-e^{-2A\tau}-\frac{\mu_t}{\mu_t+A}\left\{1-e^{-2A\tau}\frac{A+\mu_t-2A\,e^{-(\mu_t-A)\tau}}{\mu_t-A}\right\}\right]$$

$$+\frac{B^2\,\hat{\sigma}_{\hat{w}_g}^{\,2}}{A(A+\mu_t)}\left[1-e^{-2A\tau}\frac{A+\mu_t-2A\,e^{-(\mu_t-A)\tau}}{\mu_t-A}\right]. \tag{28}$$

In the speed-stable case ($A > 0$) the variance $\hat{\sigma}^2_{\hat{u}_a}$ tends to a finite value for large values of τ:

$$\hat{\sigma}^2_{\hat{u}_a}(\infty) = \hat{\sigma}^2_{\hat{u}_g}\frac{\mu_t}{\mu_t+A}+\frac{B^2\,\hat{\sigma}_{\hat{w}_g}^{\,2}}{A(\mu_t+A)}. \tag{29}$$

This result can be verified by applying the usual methods for stationary random processes to equation (13).

In the case of an aircraft with neutral-speed stability, we obtain, on taking the limit $A \to 0$ in equation (28):

$$\hat{\sigma}_{\hat{u}_a}^{\,2}(\tau) = 2\,\hat{\sigma}_{\hat{u}_g}^{\,2}\left(1-\mathrm{e}^{-\mu_t\tau}\right)+\frac{2B^2\,\hat{\sigma}_{\hat{w}_g}^{\,2}}{\mu_t^2}\left\{\mu_t\tau-\left(1-e^{-\mu_t\tau}\right)\right. . \tag{30}$$

Since airspeed error $\hat{u}_a$ is a Gaussian process under the present set of assumptions, knowledge of the variance $\hat{\sigma}_{\hat{u}_a}$ for any value of τ (equation (28)) enables us to infer the probability that any arbitrary level of speed error has been exceeded. For example, there is a probability of about 68 per cent that $\hat{u}_a$ satisfies

$$|\hat{u}_a| < \hat{\sigma}_{\hat{u}_a}$$

and a probability of about 95 per cent that

$$|\hat{u}_a| < 2\hat{\sigma}_{\hat{u}_a}.$$

This is perhaps the most convenient way to interpret the numerical values of $\hat{\sigma}_{\hat{u}_a}$ or σ_{u_a} presented in the following sections.

6. *Discussion of Initial Conditions.*

In order to compare the effects of the two alternative possibilities:

(i) $\hat{u}_a(o) = 0$ and $\hat{u}_g(o) = 0$

(ii) $\hat{u}_a(o) = 0$ and $\hat{u}_g(o)$ random

in choice of initial conditions (discussed in Section 5.3), we now evaluate the ensemble mean-square response to horizontal turbulence in each case for an aircraft with neutral-speed stability ($A = 0$). Since the effects of initial conditions only persist for a relatively short time, the results will be equally valid for any aircraft, stable or unstable, whose speed-stability response time is large (i.e. small A).

In this particular case equation (13) takes the form

$$\frac{d\hat{u}_a}{d\tau} = \frac{d\hat{u}_g}{d\tau}. \tag{31}$$

For any particular realisation of the process, equation (31) implies that

$$\hat{u}_a = \hat{u}_g + \text{constant}. \tag{32}$$

In the case of initial conditions (i) above, the constant in equation (32) is equal to zero and in any particular realisation of the process, the time history of $\hat{u}_a(\tau)$ will fluctuate about a zero mean value. The evolution in time of the variance of $\hat{u}_a$ in this case can be shown by the methods of Appendices A and B to be given by

$$\hat{\sigma}^2_{\hat{u}_a} = \hat{\sigma}^2_{\hat{u}_g}\left(1 - e^{-2u_t\tau}\right). \tag{33}$$

In the case of initial conditions (ii) (above), the constant in equation (32) is a random quantity, equal in magnitude and opposite in sign to the random quantity $\hat{u}_g(o)$. Any particular realisation of $\hat{u}_a$ is thus given by

$$\hat{u}_a(\tau) = \hat{u}_g(\tau) - \hat{u}_g(o). \tag{34}$$

Since the time average of any realisation of $\hat{u}_g(\tau)$ is equal to zero, the time history of $\hat{u}_a(\tau)$ will fluctuate about a random mean of $-\hat{u}_g(o)$. The (ensemble) mean-square value of $\hat{u}_a$ is given in this case by the component of equation (30) due to horizontal turbulence:

$$\hat{\sigma}^2_{\hat{u}_a}(\tau) = 2\,\hat{\sigma}^2_{\hat{u}_g}\left(1 - e^{-\mu_t\tau}\right). \tag{35}$$

The factor of 2 in equation (35) as compared with equation (33) is associated with the random fluctuations about a random mean described above. Although the average value of $\hat{u}_g(o)$ is zero, in a few cases the pilot will have trimmed airspeed, $\hat{u}_a(o) = 0$, at a moment when the initial gust velocity $\hat{u}_g(o)$ is near a peak value. Equation (35) takes account of these possibilities, weighted according to their probabilities. A corresponding effect appears in the time constants in equations (33) and (35), which also differ by a factor of 2.

7. *Discussion of Results.*

The method adopted for the presentation of results has been to choose two types of aircraft, which we shall call Aircraft 1 and Aircraft 2, and to consider each when flying at constant heights of 40 feet and 500 feet in the turbulence environment described in Section 5.2. From our point of view, the most important differences between the two aircraft lie in their approach speeds and associated C_L. In order to illustrate speed-stability effects we have varied the speed-stability parameter A over a wide range in the case of each aircraft. The relevant properties of the two aircraft are summarised in the following table.

	Wing loading $\frac{W}{S}$ (lb/ft^2)	Approach speed V (ft/sec)	C_L
Aircraft 1	40	180	1·1
Aircraft 2	30	230	0·5

In each case the speed stability parameter A has been varied over the following range:

$$A = -0{\cdot}1,\ -0{\cdot}06,\ -0{\cdot}01, 0, 0{\cdot}01, 0{\cdot}06.$$

Corresponding times to half amplitude (when $A > 0$) and double amplitude (when $A < 0$) are summarised in the following table.

A	$-0{\cdot}1$	$-0{\cdot}06$	$-0{\cdot}01$	0	0·01	0·06
Aircraft 1	21·4	35·7	214	∞	214	35·7
Aircraft 2	12·9	21·5	129	∞	129	21·5
	Seconds to double amplitude				Seconds to half amplitude	

We first of all consider the case of neutral stability, $A = 0$, and then show how these results are affected by changes in A. All results are based on equation (28) which, starting from an initial airspeed trim condition $\hat{u}_a(o) = 0$, describes the evolution in time of the ensemble mean square airspeed error $\sigma^2_{u_a}(t)$.

Fig. 6 illustrates $\sigma^2_{u_a}$ for Aircraft 1 at altitude 500 feet in the case of neutral speed stability. The effect of the horizontal component of turbulence is shown for the two alternative types of initial conditions discussed in Section 6, *viz.* (i) $\hat{u}_g(o) = 0$ and (ii) $\hat{u}_g(o)$ random. The main feature is that for a short time after the initial trim condition the predominant effect is that of horizontal turbulence. However, after a time interval of about 5 to 10 seconds (depending upon which of the two types of initial condition is more relevant, i.e. depending upon the extent to which the pilot has managed to trim airspeed relative to mean horizontal windspeed) the predominant effect is that of the vertical component of turbulence.

Figs. 7 and 8 illustrate $\sigma^2_{u_a}$ for Aircraft 1 and 2 at altitudes of 40 feet and 500 feet in the case of neutral-speed stability. The altitude effects are due to the differences in the tubulence power spectra illustrated in Figs. 2 and 3. Altitude has little effect on the airspeed fluctuations due to horizontal turbulence but a very marked effect on the fluctuations due to the vertical component. At altitude 40 feet the vertical component is negligible for at least a minute and so will have little effect on the final phases of an actual landing approach. The marked difference between the airspeed fluctuations due to vertical turbulence at 40 feet and 500 feet is particularly striking as the assumed mean-square values of vertical turbulence velocity at the two altitudes differ by less than 10 per cent (table in Section 5.2). The relatively high frequency energy of the vertical turbulence at 40 feet (Fig. 2) is rendered comparatively innocuous by the low-pass filter effect of the airspeed response to vertical turbulence (equation (13)).

The other effect illustrated by Figs. 7 and 8 is the much larger response to the vertical turbulence of Aircraft 1. This is associated with the lower approach speed V and larger associated value of B. To clarify this point we consider the asymptotic form of the response to vertical turbulence (given by equation (30)):

$$\hat{\sigma}^2_{\hat{u}_a}(\tau) \sim 2B^2\, \hat{\sigma}^2_{\hat{w}_g}\, \tau/\mu_t. \tag{36}$$

In terms of real time t, equation (36) gives:

$$\sigma^2_{u_a}(t) \sim \frac{2g^2\, L\, \sigma^2_{w_g}}{V^3}\, t\,. \tag{37}$$

Thus for given altitude and turbulence intensity the rate of increase of $\sigma^2_{u_a}$ is inversely proportional to V^3. This result should be compared with the analogous result, equation (23), for a vertical step gust, emphasising the increase in airspeed perturbations due to vertical gusts at low approach speeds. In contrast, the initial rate of increase of $\sigma^2_{u_a}$ due to horizontal turbulence (*see* Figs. 7 and 8) is given by equation (30) as:

$$\begin{aligned} \sigma^2_{u_a} &\sim 2\, \sigma^2_{u_g}\, \mu_t\, \tau \\ &= 2\, \sigma^2_{u_g} \frac{Vt}{L} \end{aligned} \tag{38}$$

and is directly proportional to V. This relatively high frequency effect is not particularly relevant to the height holding task, however, and at altitudes where the vertical component of turbulence does not have a negligible effect (500 feet in Figs. 7 and 8) it appears that the effect of height holding on speed will make airspeed control a more difficult task at low approach speeds.

Figs. 9 to 12 illustrate the effects of speed-stability parameter A on $\sigma^2_{u_a}(t)$. For small changes in A ($A = -0{\cdot}01$ and $0{\cdot}01$) the response to horizontal turbulence is negligibly different from that in the zero speed-stability case over the first 60 seconds $\left(\text{although eventually } \dfrac{\sigma^2_{u_a}(t)}{(\sigma_w)^2_{\text{ref}}} \text{ will become indefinitely large in all speed-unstable cases and will tend to } \dfrac{\mu_t}{\mu_t + A} \text{ in all speed-stable cases}\right)$. Figs. 9 to 12 confirm that at altitude 40 feet the effects of vertical gusts are negligible compared with those of horizontal gusts, whereas at altitude 500 feet they dominate the airspeed response over intervals of time greater than about 10 seconds for Aircraft 1 and 20 seconds for Aircraft 2.

At low levels of turbulence and moderate values of speed-stability parameter ($0{\cdot}01 > A > -0{\cdot}01$, say), the horizontal component of turbulence gives rise (Figs. 10 and 12) to fluctuation of airspeed that the pilot could ignore. However, due to the vertical component of turbulence the ensemble mean-square airspeed error eventually either becomes indefinitely large (speed-unstable case) or tends to a limit (in the speed-stable case) given by (equation (29))

$$\frac{\sigma^2_{u_a}}{\sigma^2_{w_g}} \sim \frac{B^2}{A(\mu_t + A)}, \tag{39}$$

which is large for small A. Thus even at low levels of turbulence, except for an aircraft with relatively high speed stability, the low frequency components of vertical turbulence, combined with the height constraint, will continually give rise to the need for correcting power adjustments.

As far as the pilot is concerned, it is clear that task difficulty in control of airspeed will depend both on aircraft speed stability and level of turbulence intensity. The level of instability that will cause concern is evidently a function of gust level. A simple way of looking at this is to choose a fixed value of ensemble root-mean-square airspeed error, say $\sigma^*_{u_a}$, which the pilot would regard as requiring corrective action. We can then find (assuming the glide path constraint *via* elevator) the gust intensity $(\sigma_w)_{ref}$ that would cause $\sigma^*_{u_a}$ to be reached in a given time interval. Fig. 13 shows how the gust intensity $(\sigma_w)_{ref}$ is related to speed-stability parameter A for a sequence of such time intervals, *viz.* 5, 10 and 20 seconds. Obviously the gust intensity which will cause a given level of airspeed error to arise in 5 seconds is greater than the gust intensity which will cause the same level of airspeed error to be reached after 20 seconds. Fig. 13 illustrates how the necessary gust intensity $(\sigma_w)_{ref}$ to cause a given rms airspeed error $\sigma^*_{u_a}$ in a fixed time interval increases in a roughly linear fashion as speed-stability parameter A is increased.

8. *Digital Simulation.*

In order to convey a clearer impression of the effects of speed stability upon airspeed response to horizontal and vertical turbulence, some actual time histories of response, subject to the constant height constraint, have been simulated on a digital computer. The results are illustrated in Figs. 14 and 15. The method employed was to solve the basic differential equation (13) numerically, using turbulence inputs which were obtained by operating on a sub-routine for generating Gaussian random numbers with an appropriate shaping filter in digital form.

The simulated results refer to Aircraft 1 (of Section 7) flying at its approach speed of 180 feet/second with horizontal and vertical components of turbulence corresponding to the model described in Section 5.2. The parameters in the turbulence model correspond to an altitude of 500 feet. As illustrated by the table in Section 5.2, the turbulence mean-square levels are identical for the two components at 500 feet, but the scale length is smaller in the case of the vertical component. Fig. 14(i) illustrates a random sample of the horizontal component of turbulence u_g, expressed as a function of real time, t, in seconds. The value of u_g at $t = 0$ has been arbitrarily chosen to be equal to the rms value σ_{u_g}. Figs. 14(ii) to 14(vi) illustrate the airspeed response u_a, corresponding to the given sample of turbulence, for a range of values of speed-stability parameter A. The initial value of the airspeed error u_a is taken to be zero. This is one of the cases, discussed previously, in which we suppose that the aircraft has been initially 'trimmed' in such a way that the airspeed error is zero at a moment when the horizontal gust velocity is non-zero. The resulting fluctuations in airspeed error clearly follow the fluctuations in horizontal turbulence, and in the cases of nearly neutral stability ($A = 0{\cdot}01$ and $A = -0{\cdot}01$) the airspeed traces are almost identical to the horizontal gust trace except for a constant displacement caused by the initial conditions. The difference between speed stability and speed instability is not apparent in the above cases ($A = 0{\cdot}01$ and $A = -0{\cdot}01$) as the length of the sample is short (60 seconds) as compared with the corresponding speed-stability time constants (214 seconds to half and double amplitude respectively). In the very unstable case ($A = -0{\cdot}1$) however, the divergent trend is clearly illustrated.

Fig. 15(i) illustrates a random sample of the vertical component of turbulence, w_g, over the same time interval of 60 seconds. The initial value of w_g is chosen to be equal to its rms value σ_{w_g}. Figs. 15(ii) to 15(vi) illustrate the corresponding airspeed response, derived from equation (13), for the same range of values of the speed-stability parameter A. In this case the high frequency fluctuations of the gust component do not appear in the airspeed response. The trend towards increasing divergence as the speed instability increases is illustrated in Figs. 15(iv) to 15(vi). Even in the most stable case, $A = 0{\cdot}06$, the airspeed changes after 60 seconds are quite large. This result is not unexpected, as can be verified from equation (39) and Fig. 10.

9. *Conclusions.*

In order to obtain a measure of the relative effects upon airspeed variability of aircraft speed stability and turbulence intensity we have considered what would happen to the airspeed if an aircraft were constrained to fly at constant height by means of elevator control. The results are expressed statistically in terms of the time variation of the probability distribution of airspeed error, expressed in terms of the variance, or ensemble mean square airspeed error, assuming that initially the airspeed error has been trimmed by the pilot to a value of zero. The thrust is assumed to remain constant throughout the ensuing transient motion. The main conclusions are as follows:

(i) The nature of the results justifies the plausibility of the assumptions in that at high frequencies, where the constant height constraint by means of elevator control is unrealistic owing to lags due to pitching inertia for example, the height restraint has negligible effect upon the speed response.

(ii) At high frequencies the inertia of the aircraft causes airspeed fluctuations to follow horizontal turbulence fluctuations exactly. This result is independent of the constant height assumption.

(iii) In the case of the vertical component of turbulence the constant height constraint will imply that the increase in lift associated with an upgust is counteracted by a nose-down change of attitude resulting in an acceleration along the flight path proportional to the vertical gust velocity.

(iv) At low frequencies (time intervals of more than 10 to 20 seconds) the largest contribution to perturbation of airspeed comes from the vertical component of turbulence, except at very low altitudes where the low frequency component of the vertical turbulence has less power.

(v) At relatively higher frequencies (time intervals of less than 10 seconds), where the horizontal component of turbulence is of primary importance, the magnitude of airspeed fluctuations, subsequent to the initial state of zero airspeed error, depends upon the ability of the pilot to judge the mean windspeed and hence contrive to trim the aircraft so that airspeed is correct at a moment when the turbulence velocity is small. This effect has been illustrated by considering two types of initial condition: in one the initial airspeed error and horizontal turbulence velocity are both zero, in the other the initial airspeed error is zero but the initial turbulence velocity is random.

(vi) The rate of increase of airspeed error due to the vertical component of turbulence is greatest at low speeds, the rate of increase of ensemble mean-square airspeed error being inversely proportional to the cube of the speed.

(vii) The initial rate of increase of airspeed error due to the horizontal component of turbulence is greatest at high speeds, the rate of increase of ensemble mean-square airspeed error being directly proportional to the speed. However, this relatively high frequency effect may not noticeably affect task difficulty except in the case of aircraft with a high level of speed instability.

(viii) Even for an aircraft with positive speed stability at low levels of turbulence, the low frequency effects of the vertical component of turbulence combined with the constant height constraint can eventually cause large errors to occur in airspeed if corrective power adjustments are not made.

(ix) The magnitude of the airspeed error which develops over a fixed time interval depends both upon the speed stability (or instability) and upon the level of turbulence. For a given variance of speed error after a fixed time interval the required level of turbulence is least for high levels of instability, the speed stability and turbulence rms being related in a roughly linear fashion, (Fig. 13).

Acknowledgement.

The computer programming for the digital simulation was carried out by Mr. N. Willis, during a period as a sandwich student from the University of Bradford.

LIST OF PRINCIPAL SYMBOLS

a		Lift curve slope
A	=	$-\left\{ x_u + \frac{z_u}{z_w}\left(\frac{1}{2}C_L - x_w\right)\right\} \doteqdot C_D - C_L\frac{dC_D}{dC_L}$ speed-stability parameter
B	=	$-k$
C_D		Drag coefficient
C_L		Lift coefficient
g		Gravity constant, ft/sec^2
$G(\Omega)$		Turbulence power spectrum
H		Altitude
$H(\tau)$		Unit step function
k	=	$\frac{mg}{\rho V^2 S}$
L		Scale of turbulence (ft)
m	=	W/g mass of aircraft
$\hat{q}$	=	$d\theta/d\tau$ Rate of pitch, dimensionless
$R(\tau)$		Autocorrelation function
S		Gross wing area
t		Time (secs)
$\hat{t}$	=	$\frac{m}{\rho S V}$ Unit of aerodynamic time
$u, \hat{u}$	=	$\frac{u}{V}$ Longitudinal increment in velocity
$u_a, \hat{u}_a$	=	$\frac{u_a}{V} = \hat{u} + \hat{u}_g$ Airspeed error
$u_g, \hat{u}_g$	=	u_g/V Horizontal gust velocity
V		Velocity of aircraft (approach speed ft/sec)
$w, \hat{w}$	=	w/V Normal increment in velocity
$w_g, \hat{w}_g$	=	w_g/V Vertical gust velocity
W		Weight of aircraft, lb
x_u, x_w		Longitudinal-force derivatives, dimensionless
z_u, z_w		Normal-force derivatives, dimensionless
θ		Increment of angular displacement of aircraft in pitch
μ_t	=	$\frac{m}{\rho S L}$ Relative density, based on scale of turbulence
ρ		Air density, slugs/ft^3

$\sigma^2, \hat{\sigma}^2$	=	σ^2/V^2 Variance, or ensemble mean square, of velocity increment
$\sigma^2_{u_a}(t)$		Variance of airspeed increment
$(\sigma_w)_{ref}$		Reference turbulence intensity (*see* Section 5.2)
τ	=	$t/\hat{t}$ Aerodynamic time (dimensionless)
Φ		Power spectrum
$\hat{\omega}$		Non-dimensionless frequency (rad/unit τ)
Ω		Space frequency (rad/ft)

REFERENCES

No.	*Author(s)*	*Title, etc.*
1	C. H. Cromwell and I. L. Ashkenas	A systems analysis of longitudinal piloted control in carrier approach. Systems Technology, Inc. Report TR–124–1, (1962).
2	S. Neumark	Problems of longitudinal stability below minimum drag speed and theory of stability under constraint. A.R.C. R. & M. 2983 (1953).
3	K. J. Staples	Flight measurements of the influence of speed stability on the landing approach. AGARD Report 420 (1963).
4	D. H. Perry	A piloted flight simulator study of speed instability during the landing approach. A.R.C. C.P. 980 (1966).
5	D. Lean and R. Eaton	The influence of drag characteristics on the choice of landing approach speeds. AGARD Report 122 (1957).
6	J. L. Lumley and H. A. Panofsky	The structure of atmospheric turbulence. Interscience (1964).
7	A. Burns	Power spectra of low level atmospheric turbulence measured from an aircraft. A.R.C. C.P. 733 (1963).
8	A. Papoulis	Probability, random variables and stochastic processes. McGraw-Hill (1965).

APPENDIX A

Mean-Square Response to the Vertical Component of Turbulence.

In the case of the vertical component of turbulence $\hat{w}_g$, equation (13) takes the form

$$\frac{d\hat{u}_a}{d\tau}+A\hat{u}_a = -B\hat{w}_g = \hat{\eta}\,(\text{say})\,. \qquad \text{(A.1)}$$

$\hat{w}_g$ has an autocorrelation function given by equation (27), and so we take $\hat{\eta}$ to have an autocorrelation function

$$R_{\hat{\eta}}(\tau) = \hat{\sigma}_{\hat{\eta}}^2 \, e^{-\mu_t|\tau|} \tag{A.2}$$

of the same form. We can subsequently put $\hat{\eta}$ equal to $-B\hat{w}_g$.

We have assumed that $\hat{u}_a(o) = 0$ and that $\hat{w}_g(o)$ is random. Since the time history of the input $\hat{\eta}$ previous to $\tau = 0$ can have no effect on the response $\hat{u}_a$ after $\tau = 0$ we can replace (Fig. 4) the input by a function $x(\tau)$ given by

$$x(\tau) = \begin{cases} 0, \tau \leqslant 0 \\ \hat{\eta}, \tau > 0 . \end{cases} \tag{A.3}$$

The input $x(\tau)$ is in effect 'switched on' at $\tau = 0$. $x(\tau)$ is a non-stationary process with autocorrelation function $R_x(\tau_1, \tau_2)$ given by

$$R_x(\tau_1, \tau_2) = \begin{cases} \hat{\sigma}_{\hat{\eta}}^2 \, e^{-\mu_t|\tau_1 - \tau_2|}, \tau_1 \geqslant 0 \text{ and } \tau_2 \geqslant 0 \\ 0 \text{, otherwise.} \end{cases} \tag{A.4}$$

Fig. 4 is a block diagram illustrating equation (A.1) subject to the above conditions. The differential equation is represented by a transfer function $\frac{1}{s+A}$, s being the Laplace transform variable. The positioning of the switch (Fig. 4) ensures that $w_g(o)$ is random.

The following method of analysis is indicated by Papoulis[8].

The impulse response corresponding to the transfer function $\frac{1}{s+A}$ is

$$h(\tau) = e^{-A\tau} . \tag{A.5}$$

Then using equation (A.4) we have for the cross-correlation between x and $\hat{u}_a$:

$$R_{x\hat{u}_a}(\tau_1, \tau_2) = \begin{cases} \hat{\sigma}_{\hat{\eta}}^2 \int_0^{\tau_2} e^{-\mu_t(\tau_1 - \tau_2 + \tau)} \, e^{-A\tau} \, d\tau \; ; \tau_1 \geqslant 0, \tau_2 \geqslant 0, \tau_1 > \tau_2 \; ; \\ \hat{\sigma}_{\hat{\eta}}^2 \left\{ \int_0^{\tau_2 - \tau_1} e^{\mu_t(\tau_1 - \tau_2 + \tau)} \, e^{-A\tau} \, d\tau + \int_{\tau_2 - \tau_1}^{\tau_2} e^{-\mu_t(\tau_1 - \tau_2 + \tau)} \, e^{-A\tau} \, d\tau \right\} \; ; \tau_1 \geqslant 0, \tau_2 \geqslant 0, \tau_1 < \tau_2 ; \\ 0 \; ; \text{otherwise.} \end{cases} \tag{A.6}$$

Evaluation of the integrals involved leads to

$$R_{x\hat{u}_a}(\tau_1,\tau_2)=\begin{cases}\dfrac{\hat{\sigma}^2_{\hat{\eta}}}{\mu_t+A}\left\{e^{-\mu_t\tau_1}\left(e^{\mu_t\tau_2}-e^{-A\tau_2}\right)\right\} & ;\tau_1\geqslant 0,\tau_2\geqslant 0,\tau_1>\tau_2\ ;\\ \hat{\sigma}^2_{\hat{\eta}}\left[\dfrac{1}{A-\mu_t}\left\{e^{-\mu_t(\tau_2-\tau_1)}-e^{-A(\tau_2-\tau_1)}\right\}+\right.\\ \left.+\dfrac{e^{-A\tau_2}}{\mu_t+A}\left\{e^{A\tau_1}-e^{-\mu_t\tau_1}\right\}\right]; & \tau_1\geqslant 0,\tau_2\geqslant 0,\tau_1<\tau_2\ ;\\ 0\ ;\text{otherwise.}\end{cases}\tag{A.7}$$

The variance of $\hat{u}_a$ can be expressed in terms of its autocorrelation function, thus:

$$\begin{aligned}\hat{\sigma}^2_{\hat{u}_a}(\tau) &= R_{\hat{u}_a\hat{u}_a}(\tau,\tau)\\ &= \int_0^\tau R_{x\hat{u}_a}(\tau-\tau',\tau)\,h(\tau')\,d\tau'\\ &= \hat{\sigma}^2_{\hat{\eta}}\int_0^\tau\left[\frac{e^{-A\tau'}}{A-\mu_t}\left\{e^{-\mu_t\tau'}-e^{-A\tau'}\right\}+\frac{e^{-A\tau'}e^{-A\tau}}{\mu_t+A}\left\{e^{A(\tau-\tau')}-e^{-\mu_t(\tau-\tau')}\right\}\right]d\tau'\\ &= \frac{\hat{\sigma}^2_{\hat{\eta}}}{A(A+\mu_t)}\left[1-e^{-2A\tau}\frac{A+\mu_t-2A\,e^{-(\mu_t-A)\tau}}{\mu_t-A}\right].\end{aligned}\tag{A.8}$$

Setting $\hat{\eta}$ equal to $-B\hat{w}_g$ we obtain the component of equation (28) due to $\hat{w}_g$.

APPENDIX B

Mean-Square Response to the Horizontal Component of Turbulence.

In the case of the horizontal component of turbulence equation (13) reduces to

$$\frac{d\hat{u}_a}{d\tau}+A\hat{u}_a=\frac{d\hat{u}_g}{d\tau}.\tag{B.1}$$

From equation (27) we have for the autocorrelation function of $\hat{u}_g$

$$R_{\hat{u}_g}(\tau)=\hat{\sigma}^2_{\hat{u}_g}e^{-\mu_t|\tau|}.\tag{B.2}$$

It follows that the autocorrelation function of $\dfrac{d\hat{u}_g}{d\tau}$ is

$$R_{\dot{\hat{u}}_g}(\tau)=\hat{\sigma}^2_{\hat{u}_g}(2\mu_t\,\delta(\tau)-\mu_t^2\,e^{-\mu_t|\tau|}),\tag{B.3}$$

where $\delta(\tau)$ is the delta function.

We have assumed that $\hat{u}_a(o) = 0$ and that $\hat{u}_g(o)$ is random. As in Appendix A we replace the input by a function that is 'switched on' at $\tau = 0$, *viz*:

$$y(\tau) = \begin{cases} 0\,, \tau \leqslant 0 \\ \dfrac{d\hat{u}_g}{d\tau}\,, \tau > 0\,. \end{cases} \tag{B.4}$$

$y(\tau)$ is thus a non-stationary process with auto-correlation function $R_y(\tau_1,\tau_2)$ given by

$$R_y(\tau_1,\tau_2) = \begin{cases} \hat{\sigma}^2_{\hat{u}_g}\,(2\mu_t\,\delta(\tau_1-\tau_2)-\mu_t^2\;e^{-\mu_t|\tau_1-\tau_2|})\,, \tau_1 \geqslant 0 \text{ and } \tau_2 \geqslant 0\,, \\ 0\,, \text{otherwise.} \end{cases} \tag{B.5}$$

Fig. 5 is a block diagram illustrating equation (B.1) subject to the above conditions. Then the cross correlation between y and $\hat{u}_a$ is given in the case $\tau_1 \geqslant 0$, $\tau_2 \geqslant 0$, $\tau_1 < \tau_2$ by

$$R_{y\hat{u}_a}(\tau_1,\tau_2) = 2\mu_t\,\hat{\sigma}^2_{\hat{u}_g}\int_0^{\tau_2} \delta\,(\tau_2-\tau_1-\tau)\,e^{-A\tau}\,d\tau -$$

$$-\mu_t^2\,\hat{\sigma}^2_{\hat{u}_g}\left\{\int_0^{\tau_2-\tau_1} e^{\mu_t(\tau_1-\tau_2+\tau)}\,e^{-A\tau}\,d\tau + \right.$$

$$\left. + \int_{\tau_2-\tau_1}^{\tau_2} e^{-\mu_t(\tau_1-\tau_2+\tau)}\,e^{-A\tau}\,d\tau\right\}. \tag{B.6}$$

The variance of $\hat{u}_a$ can now be expressed in terms of its autocorrelation function, thus:

$$\hat{\sigma}^2_{\hat{u}_a}(\tau) = R_{\hat{u}_a\,\hat{u}_a}(\tau,\tau)$$

$$= \int_0^{\tau} R_{y\,\hat{u}_a}(\tau-\tau',\tau)\,h(\tau')\,d\tau'$$

$$= \hat{\sigma}^2_{\hat{u}_g}\left[\frac{\mu_t}{A}(1-e^{-2A\tau}) - \frac{\mu_t^2}{A(A+\mu_t)}\left\{1-e^{-2A\tau}\frac{A+\mu_t-2A\,e^{-(\mu_t-A)\tau}}{\mu_t-A}\right\}\right]. \tag{B.7}$$

Equation (B.7) gives the component of equation (28) due to the horizontal component of turbulence $\hat{u}_g$.

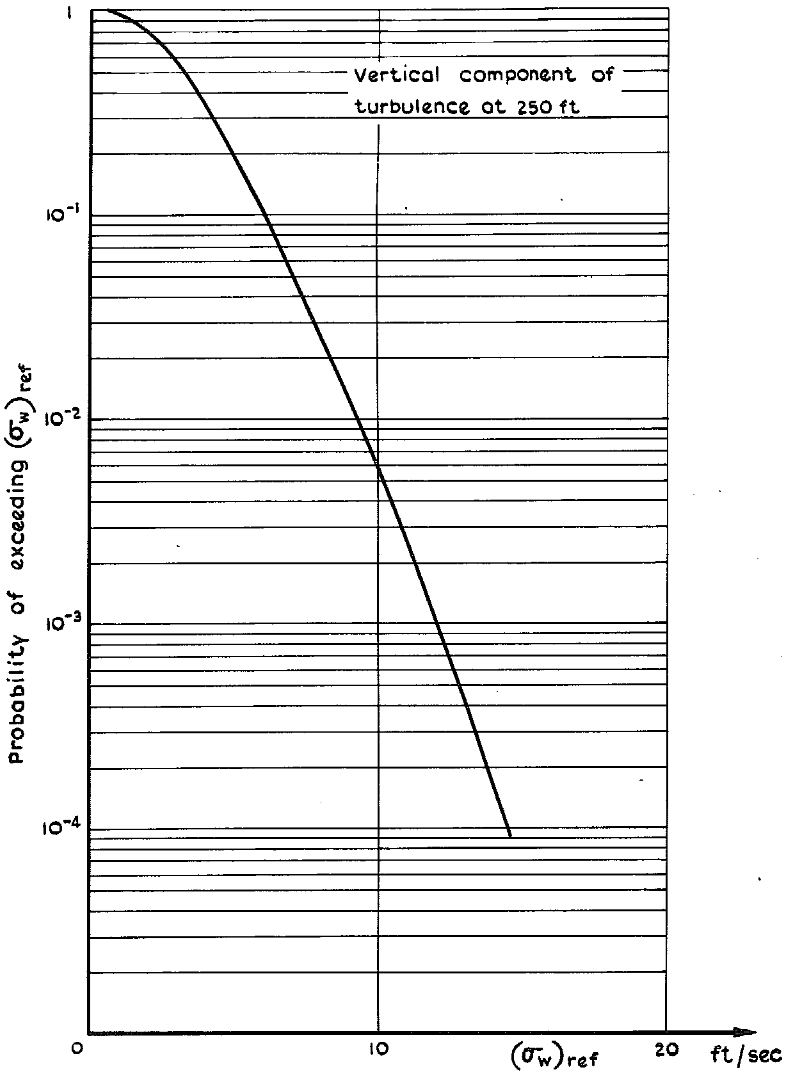

FIG. 1. Probability of exceeding given value of $(\sigma_w)_{ref}$.

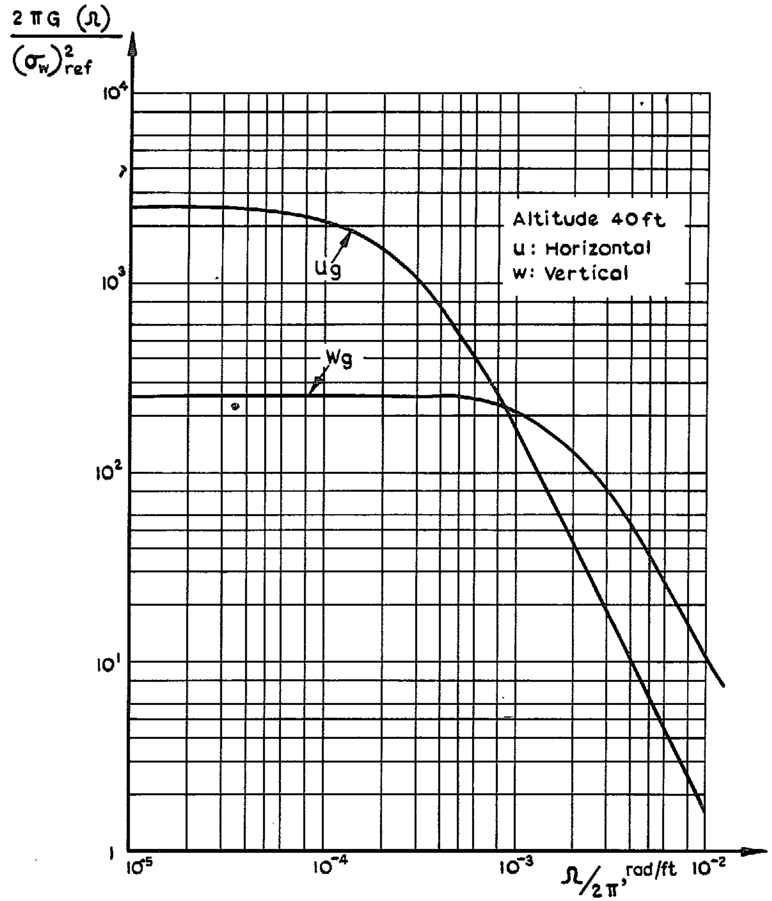

FIG. 2. Spectra of horizontal and vertical components of turbulence at altitude 40ft.

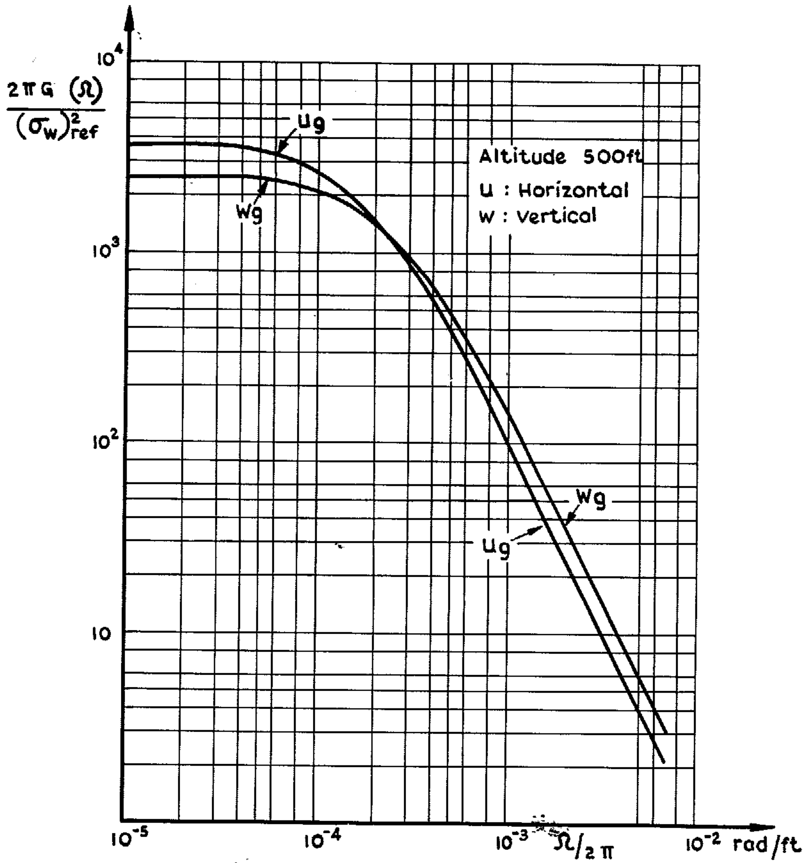

FIG. 3. Spectra of horizontal and vertical components of turbulence at altitude 500ft.

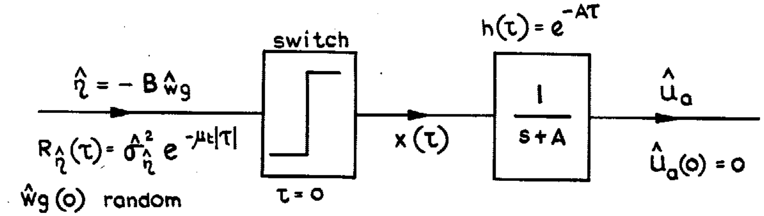

FIG. 4. Block diagram for response to vertical component of turbulence (Appendix A).

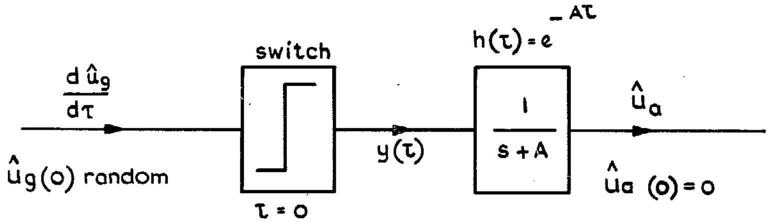

FIG. 5. Block diagram for response to horizontal component of turbulence (Appendix B).

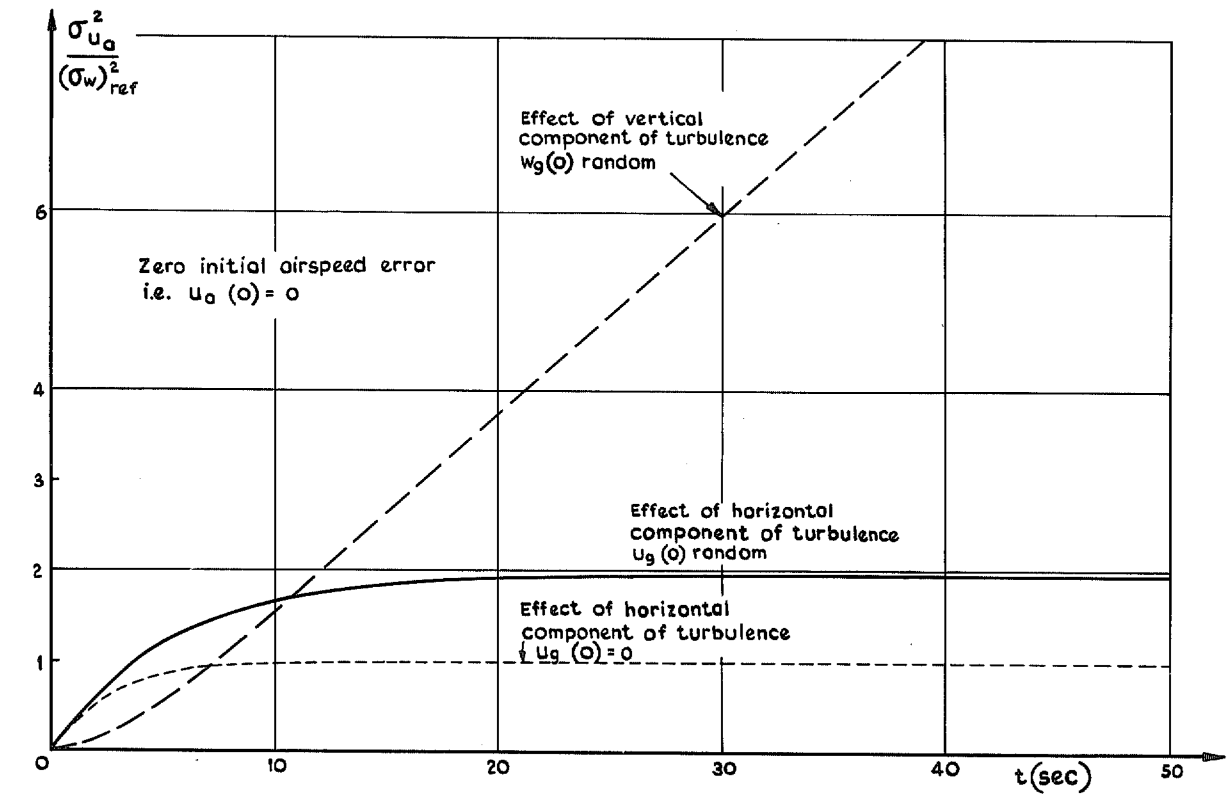

FIG. 6. Ensemble mean-square airspeed error. Aircraft 1, zero-speed stability ($A = 0$), altitude 500ft.

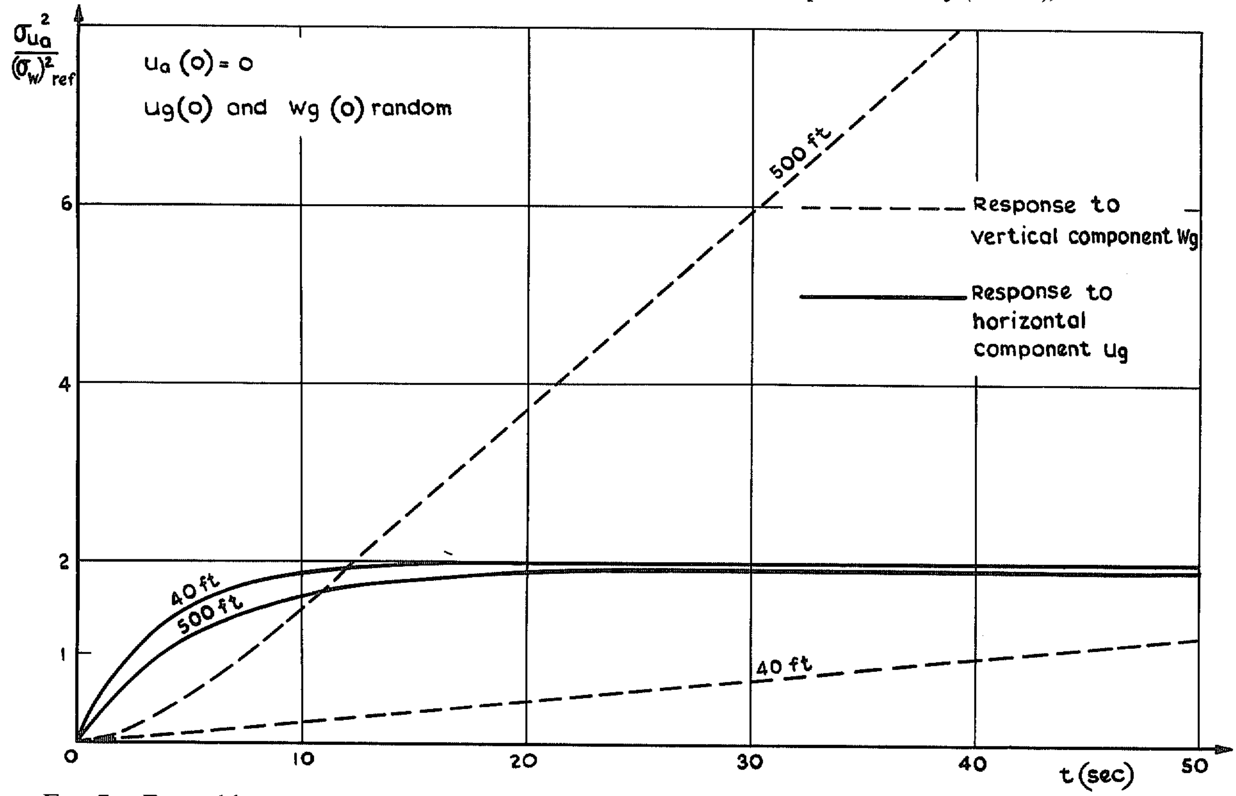

FIG. 7. Ensemble mean-square airspeed error. Aircraft 1, zero-speed stability ($A = 0$), altitudes 40ft. & 500ft.

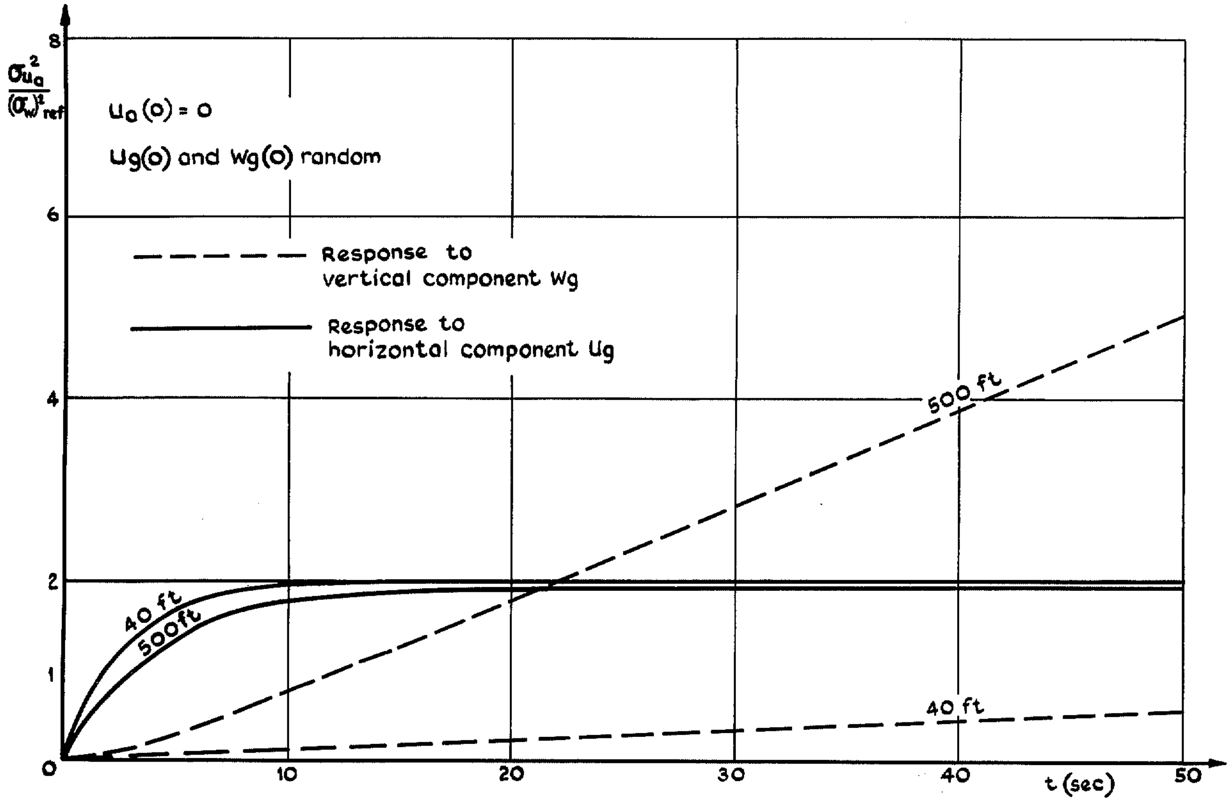

FIG. 8. Ensemble mean-square airspeed error. Aircraft 2, zero-speed stability ($A = 0$), altitudes 40ft. & 500ft.

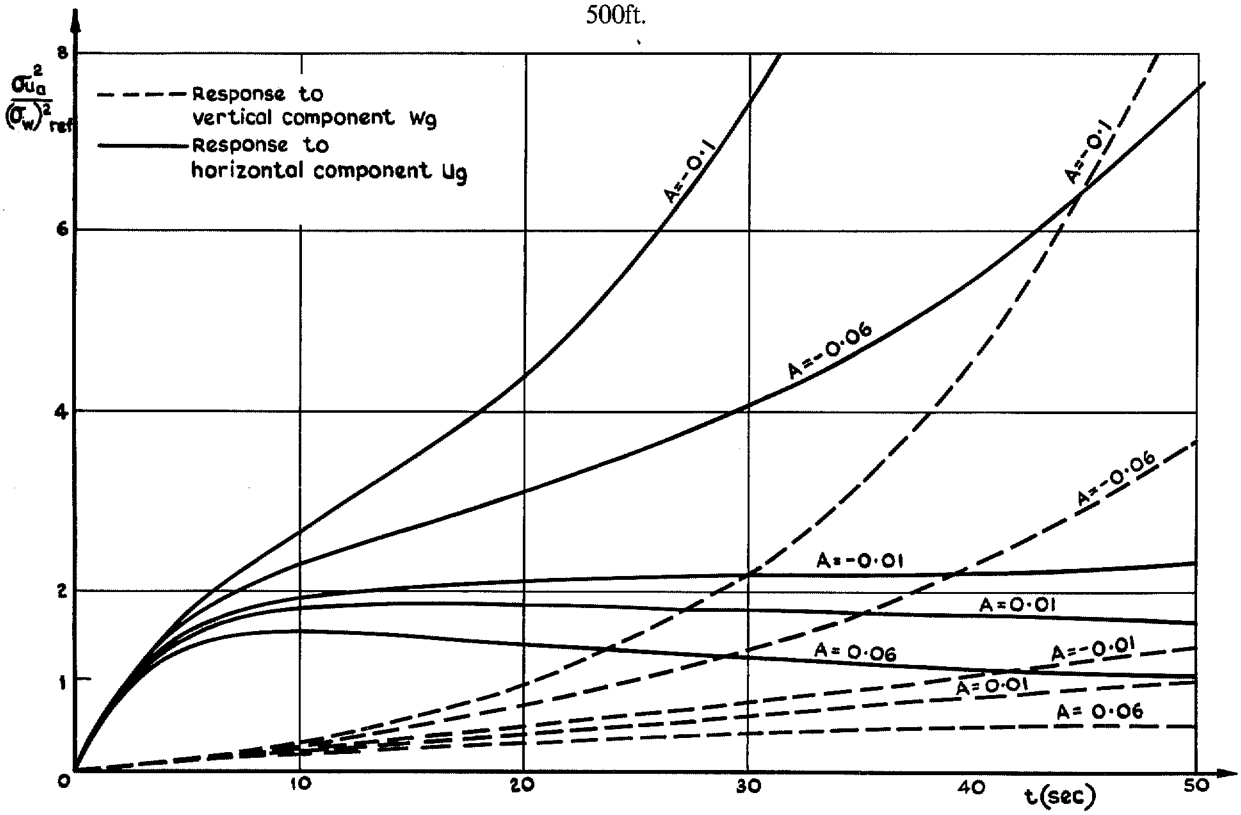

FIG. 9. Ensemble mean-square airspeed error. Aircraft 1, variable-speed stability, altitude 40ft.

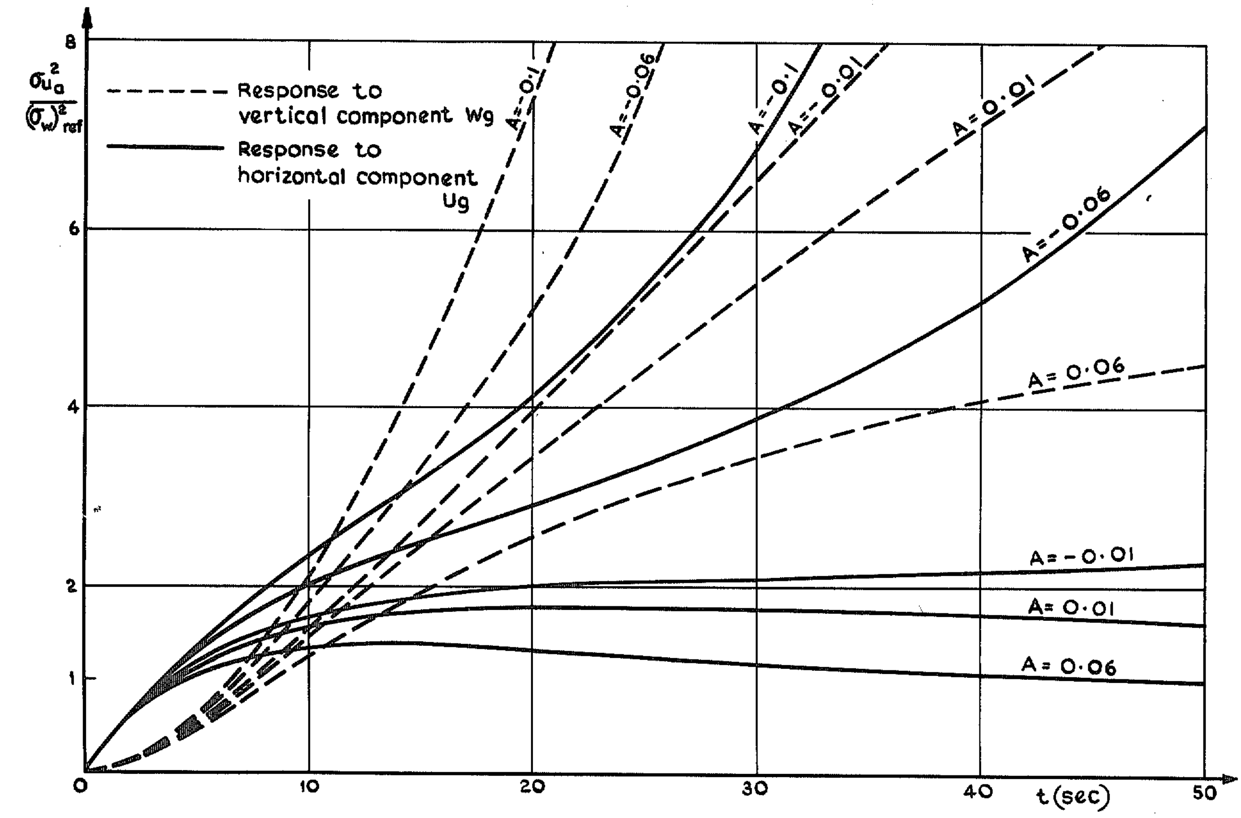

FIG. 10. Ensemble mean-square airspeed error. Aircraft 1, variable-speed stability, altitude 500ft.

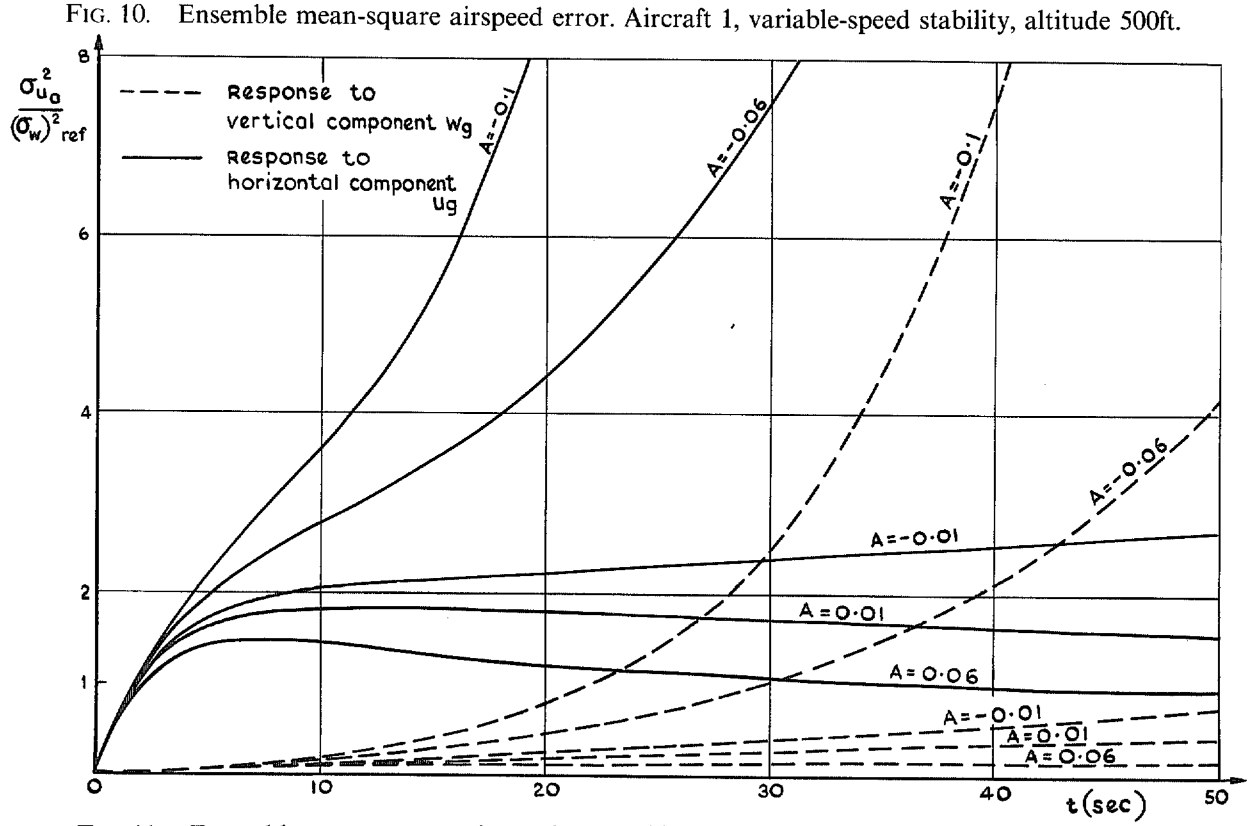

FIG. 11. Ensemble mean-square airspeed error. Aircraft 2, variable-speed stability, altitude 40ft.

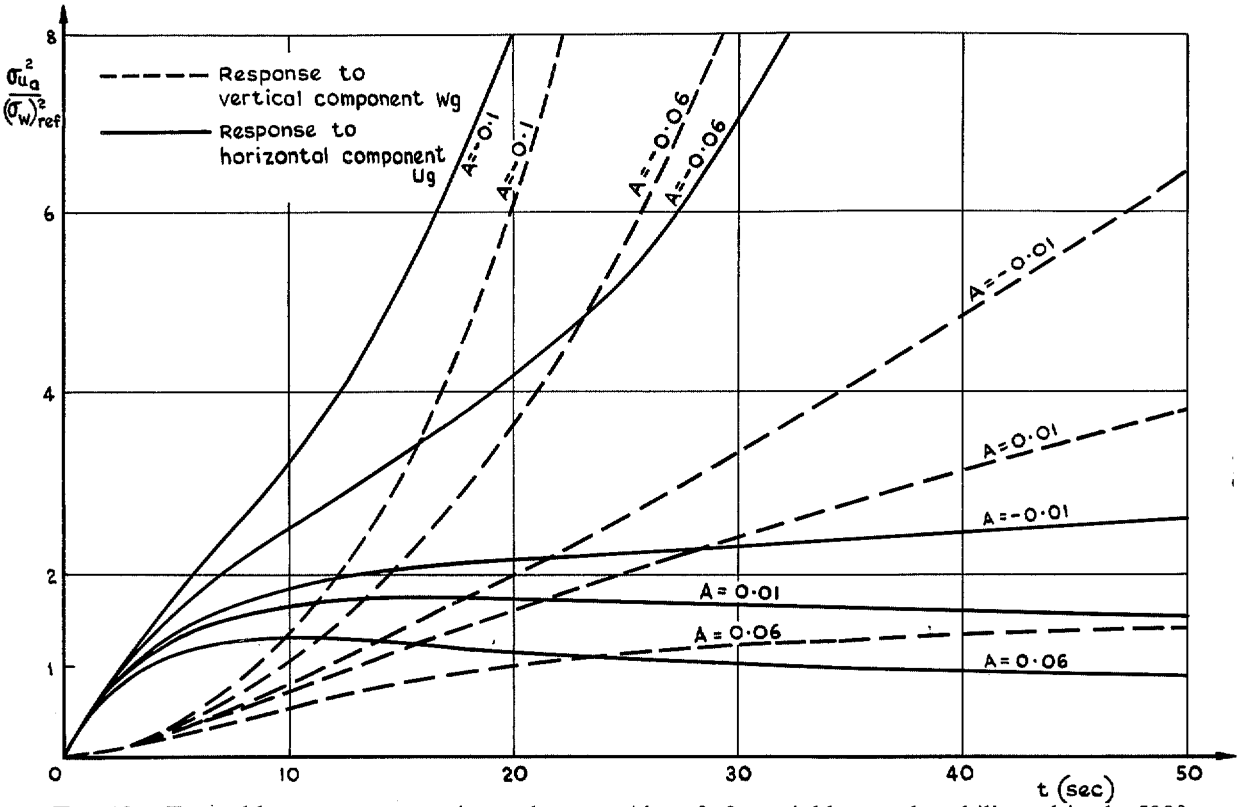

FIG. 12. Ensemble mean-square airspeed error. Aircraft 2, variable-speed stability, altitude 500ft.

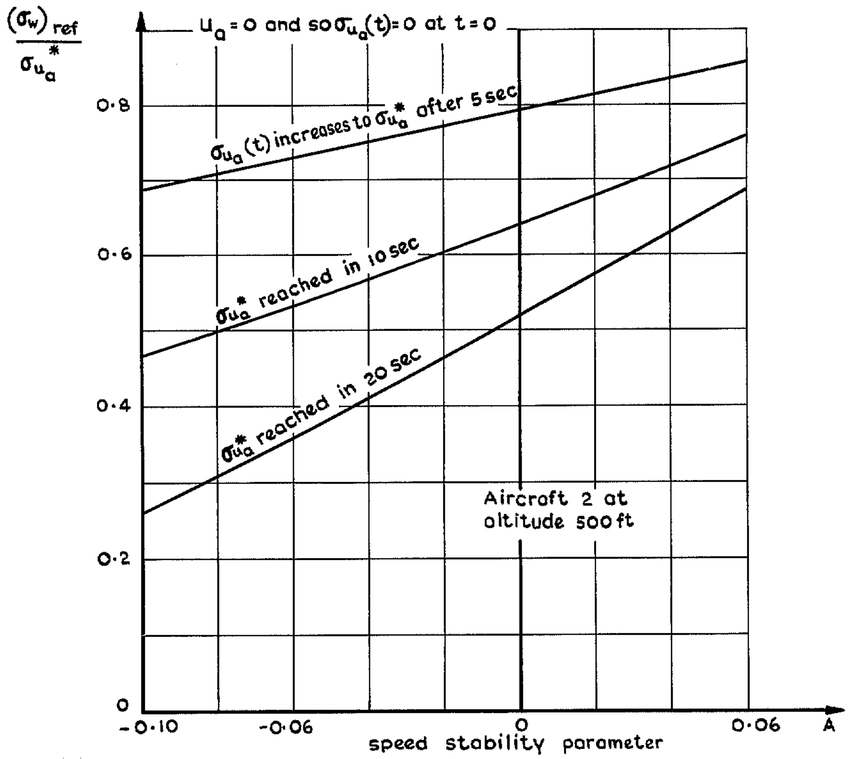

FIG. 13. Turbulence intensity $(\sigma_w)_{ref}$ to cause airspeed error rms $\sigma_{u_a}{}^*$.

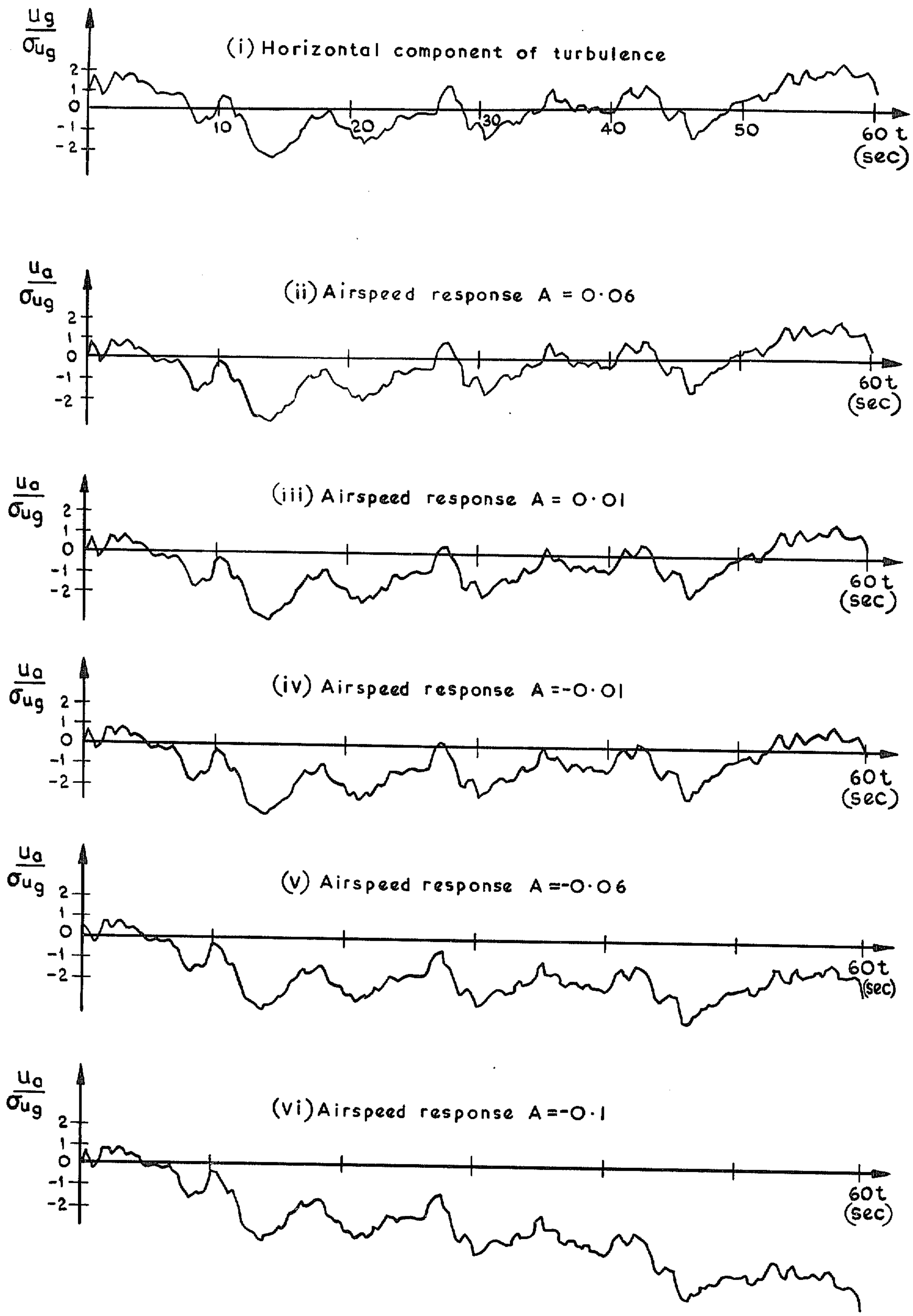

FIG. 14. Digital simulation, aircraft 1. Airspeed response to horizontal turbulence.

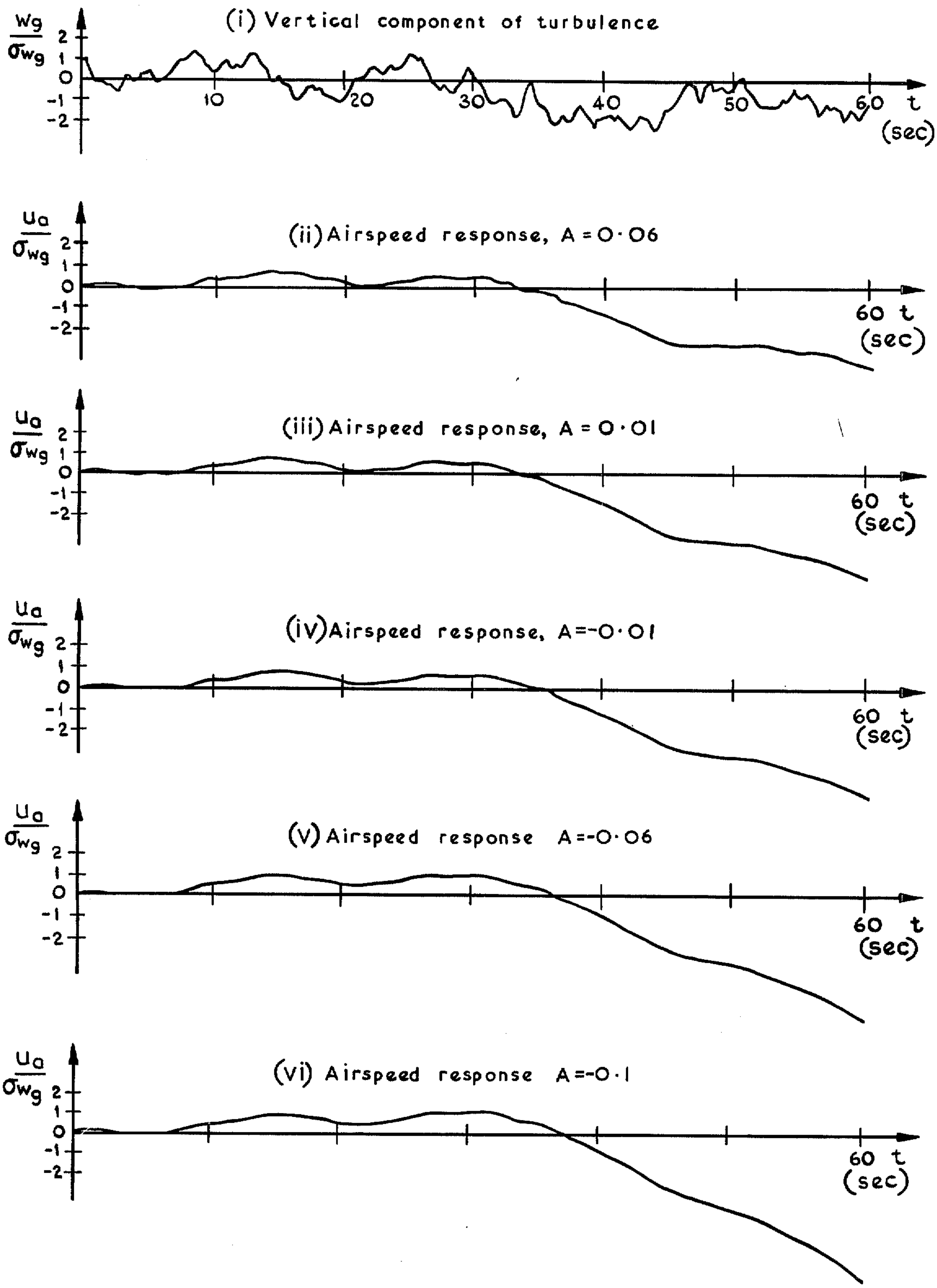

FIG. 15. Digital simulation, aircraft 1. Airspeed response to vertical turbulence.

Printed in Wales for Her Majesty's Stationery Office by Allens Printers (Wales) Limited

5646 K5

R. & M. No. 3563

Published by
HER MAJESTY'S STATIONERY OFFICE

To be purchased from
49 High Holborn, London W.C.1
13A Castle Street, Edinburgh 2
109 St. Mary Street, Cardiff CF1 1JW
Brazennose Street, Manchester M60 8AS
50 Fairfax Street, Bristol BS1 3DE
258 Broad Street, Birmingham 1
7 Linenhall Street, Belfast BT2 8AY
or through any bookseller

R. & M. No. 3563

S.O. Code No. 23-3563