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Generalised Airforces on a Cylindrical Shell Oscillating Harmonically in a Uniform Flow

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Summary.

The flow of air past an infinite shell is considered. In its undisturbed state the shell is a circular cylinder. A portion of the shell between two normal cross-sections is flexible, the rest of the shell being rigid and fixed. The flexible portion is closed by planes normal to the axis of the cylinder.

In the absence of any oscillations of the flexible portion there is a uniform flow outside the cylinder, which has the direction of the axis of the cylinder. When the flexible portion oscillates, the uniform flow is perturbed and there is also an acoustic field generated in the space within the flexible portion with the result that there are oscillating perturbation pressures on both sides of the shell surface. A method of determining these perturbation pressures is given when the flexible portion oscillates harmonically with given frequency in a given mode. Expressions for these perturbation pressures are then used to obtain formulae for the associated generalised airforces.

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1. *Introduction.*

When structures are immersed in fluid flow they may vibrate as a result of some excitation such as random pressure fluctuations in a boundary layer, or there may be self-maintained vibration. The vibration in turn induces additional fluctuating pressures which have to be taken into account in any analysis of the nature of the vibration. It has been common practice to obtain the pressure on an oscillating cylindrical shell in supersonic flow by means of piston theory, or modified piston theory, for panel flutter studies. At low supersonic speed the accuracy of piston theory is not good and a more accurate theory should be used. For studies of turbulent boundary-layer excitation, where aerodynamic damping is of particular importance, accurate aerodynamic theory is required.

Randall[1] has considered the supersonic flow past quasi-cylindrical bodies of almost circular cross-section, and the method he used may be extended to obtain the pressure on a cylinder oscillating harmonically in a given uniform flow. Strack and Holt[2] suggest this extension and in fact carry out part of the analysis. The present Report considers an oscillating flexible portion between two cross-sections of a circular cylinder and gives a complete linearised analysis of the determination of generalised airforces acting on the outside surface of this portion.

There is also a discussion of the generalised airforces acting on the inside wall of the flexible portion of the cylinder when this is closed by planes normal to the axis of the cylinder.

2. *Derivation of Generalised Airforces Acting on the Outside Surface of the Cylinder.*

Consider an infinite cylinder as shown in Fig. 1 with its axis along the axis of z. Outside the cylinder there is a fluid flow which is a combination of a uniform flow of Mach number M in the positive direction of z and a superposed perturbation caused by the vibration of a portion of length l of the cylinder surface, intercepted by two planes normal to its axis. Apart from this portion of length l, the cylinder is assumed rigid and fixed.

The origin of co-ordinates is taken at an end of the portion of length l so that the whole of this portion has positive z co-ordinate. Cylindrical polar co-ordinates r, θ, z are used to define the position of a point in space.

The normal displacement at time t of a point (a, θ, z) on the surface of the cylinder in a mode j of vibration will be denoted by $w_j(\theta, z/l, t)$. If we consider harmonic oscillations of circular frequency ω, then we shall write

$$w_j(\theta, \frac{z}{l}, t) = a\, \hat{w}_j(\theta, \frac{z}{l}) \exp(i\omega t) \tag{1}$$

where 'a' is the radius of the outside surface of the cylinder. As is usual, only the real part or the imaginary part of a complex quantity has physical significance.

Because of the assumption that only a portion of length l vibrates, while the rest of the cylinder remains fixed, we must have

$$\hat{w}_j\left(\theta, \frac{z}{l}\right) = 0 \begin{cases} z \geqslant l \\ z \leqslant 0 \end{cases}. \tag{2}$$

Let

$$\tilde{w}_j\left(n, \frac{z}{l}\right) = \int_0^{2\pi} \hat{w}_j\left(\theta, \frac{z}{l}\right) \times \exp(-in\theta)\, d\theta \tag{3}$$

and

$$\bar{w}_j(n,k) = \frac{1}{l}\int_0^l \tilde{w}_j\left(n,\frac{z}{l}\right) \times \exp\left(-\frac{ikz}{l}\right) dz\,. \tag{4}$$

Then

$$\hat{w}_j\left(\theta,\frac{z}{l}\right) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \tilde{w}_j\left(n,\frac{z}{l}\right) \times \exp(in\theta) \tag{5}$$

and also

$$\hat{w}_j\left(\theta,\frac{z}{l}\right) = \frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{w}_j(n,k) \times \exp\left(in\theta+\frac{ikz}{l}\right) dk\,. \tag{6}$$

The perturbation velocity potential ϕ_j at a point (r, θ, z) of the perturbed flow outside the cylinder when it is oscillating according to equation (1) can be written in the form

$$\phi_j = Va\,\hat{\phi}_j\left(\frac{r}{a},\theta,\frac{z}{l},\nu,M,\frac{l}{a}\right)\exp(i\omega t) \tag{7}$$

and the corresponding perturbation pressure p_j in the form

$$p_j = \rho V^2\,\hat{p}_j\left(\frac{r}{a},\theta,\frac{z}{l},\nu,M,\frac{l}{a}\right)\exp(i\omega t) \tag{8}$$

where

$$\nu = \frac{\omega a}{V} \tag{9}$$

is a frequency parameter, V is the main stream speed and $M = V/a_0$ is the Mach number, where a_0 is the speed of sound.

The perturbation velocity potential ϕ_j satisfies the boundary condition

$$\begin{aligned}\left[\frac{\partial\phi_j}{\partial r}\right]_{r=a} &= \left(V\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\right)w_j\left(\theta,\frac{z}{l},t\right)\\ &= a\left(V\frac{\partial}{\partial z}+i\omega\right)\hat{w}_j\left(\theta,\frac{z}{l}\right)\exp(i\omega t)\end{aligned} \tag{10}$$

on the outside surface of the cylinder.

If we substitute for $\hat{w}_j(\theta, z/l)$ from equation (6) into equation (10) we get

$$\left[\frac{\partial \phi_j}{\partial r}\right]_{r=a} = \frac{iV}{4\pi^2}\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{ka}{l}+\nu\right)\bar{w}_j(n,k)\exp\left(in\theta+\frac{ikz}{l}+i\omega t\right)dk \tag{11}$$

Let us consider the velocity potential ϕ of the perturbed flow outside the infinite cylinder, which satisfies the boundary condition

$$\left(\frac{\partial \phi}{\partial r}\right)_{r=a} = -V\exp\left(in\theta+i\kappa\frac{z}{a}+i\omega t\right) \tag{12}$$

on the surface of the cylinder. The coefficient of z in the exponential term on the right of formula (12) is κ/a, rather than k/l which was used in formula (11), for this is the more natural coefficient to take in considering a boundary condition of the form (12) over an infinite cylinder. The minus sign in formula (12) is used so that certain functions derived in Appendix B should collapse to ones obtained by Randall[1] when the frequency of oscillation tends to zero.

The velocity potential ϕ will be of the form

$$\phi = Va\,\bar{\Phi}_n\left(\kappa,\frac{r}{a},\nu,M\right)\exp\left(in\theta+i\kappa\frac{z}{a}+i\omega t\right). \tag{13}$$

Expressions for the function $\bar{\Phi}_n(\kappa, \frac{r}{a}, \nu, M)$ are obtained in Appendix A for positive ν, all integral values of n and all κ. If ν is negative, then we use the relation

$$\bar{\Phi}_n\left(\kappa,\frac{r}{a},\nu,M\right) = \bar{\Phi}_n^*\left(\kappa,\frac{r}{m},-\nu,M\right) \tag{14}$$

where the asterisk denotes the complex conjugate of a function. The relation (14) is easily obtained from (12) and (13) on taking real or imaginary parts, changing the sign of ω, and again forming a complex function from its real or imaginary part.

The perturbation pressure p corresponding to the perturbation velocity potential ϕ of equation (13) will be of the form

$$p = \rho V^2\,\bar{p}_n\left(\kappa,\frac{r}{a},\nu,M\right)\times\exp\left(in\theta+i\kappa\frac{z}{a}+i\omega t\right) \tag{15}$$

where, according to the linearised Bernoulli equation

$$p = \rho\left(V\frac{\partial}{\partial z}+\frac{\partial}{\partial t}\right)\phi \tag{16}$$

we have

$$\bar{p}_n\left(\kappa,\frac{r}{a},\nu,M\right) = i(\kappa+\nu)\,\bar{\Phi}_n\left(\kappa,\frac{r}{a},\nu,M\right). \tag{17}$$

By comparing the boundary conditions (11) and (12) and applying the principle of superposition to the solutions we get

$$p_j = -\frac{i}{4\pi^2}\rho V^2 \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{ka}{l}+v\right) \times \bar{w}_j(n,k) \times \bar{p}_n\left(\frac{ka}{l},\frac{r}{a},v,M\right) \times \exp\left(in\theta+\frac{ikz}{l}+i\omega t\right) \times dk$$

$$= \frac{1}{4\pi^2}\rho V^2 \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{ka}{l}+v\right)^2 \times \bar{w}_j(n,k) \times \bar{\Phi}_n\left(\frac{ka}{l},\frac{r}{a},v,M\right) \times \exp\left(in\theta+\frac{ikz}{l}+i\omega t\right) \times dk. \quad (18)$$

Let us write

$$\tilde{\Phi}_n\left(u,\frac{r}{a},v,M\right) = \frac{\beta}{2\pi}\int_{-\infty}^{\infty} \bar{\Phi}_n\left(\kappa,\frac{r}{a},v,M\right) \times \exp(i\kappa\beta\mu)\,d\kappa \quad (19)$$

where

$$\beta = \sqrt{|M^2-1|}. \quad (20)$$

We may then use the convolution theorem to replace (18) by

$$p_j = -\frac{1}{2\pi\beta}\rho V^2 \sum_{n=-\infty}^{\infty} \frac{1}{a}\int_{-\infty}^{\infty} \left[\left(a\frac{\partial}{\partial z_0}+iv\right)^2 \times \tilde{w}_j\left(n,\frac{z_0}{l}\right)\right] \tilde{\Phi}_n\left(\frac{z-z_0}{\beta a},\frac{r}{a},v,M\right) dz_0 \times$$

$$\times \exp(in\theta+i\omega t) \quad (21)$$

The function $\tilde{w}_j(n, z_0/l)$ is continuous but its first derivative $\partial/\partial z_0\, \tilde{w}_j(n, z_0/l)$ may have discontinuities at $z_0 = 0$ and $z_0 = l$. These will give rise to Dirac delta functions in the expression $(a\,\partial/\partial z_0 + iv)^2$ $\tilde{w}_j(n, z_0/l)$, which must be taken into account in evaluating formula (21).

The generalised airforce coefficient $Q_{jj'}$ given by

$$Q_{jj'}\exp(i\omega t) = \frac{1}{\rho V^2 l}\int_0^{2\pi}\int_0^{l} \Big[p_j\Big]_{r=a} \hat{w}_{j'}\left(\theta,\frac{z}{l}\right) d\theta\, dz$$

$$= \frac{1}{\rho V^2 l}\int_0^{2\pi}\int_{-\infty}^{\infty} \Big[p_j\Big]_{r=a} \hat{w}_{j'}\left(\theta,\frac{z}{l}\right) d\theta\, dz \quad (22)$$

is of interest in studying the dynamics of the cylinder.

If we substitute from (21) into (22) and use the expansion (5) for $\hat{w}_{j'}(\theta, z/l)$ we get

$$Q_{jj'} = -\frac{1}{4\pi^2 \beta a l} \times \sum_{n=-\infty}^{\infty} \sum_{n'=-\infty}^{\infty} \int_0^{2\pi} \exp(in\theta) \times \exp(in'\theta)\, d\theta \int_{-\infty}^{\infty} \tilde{w}_{j'}\left(n', \frac{z}{l}\right) dz \times$$

$$\times \int_{-\infty}^{\infty} \left[\left(a\frac{\partial}{\partial z_0} + iv \right)^2 \tilde{w}_j\left(n, \frac{z_0}{l}\right) \right] \times \tilde{\phi}_n\left(\frac{z-z_0}{\beta a}, 1, v, M \right) dz$$

$$= -\frac{1}{2\pi \beta a l} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\left(a\frac{\partial}{\partial z_0} + iv \right)^2 \tilde{w}_j\left(n, \frac{z_0}{l}\right) \right] \tilde{w}_{j'}\left(-n, \frac{z}{l}\right) \times$$

$$\times \tilde{\phi}_n\left(\frac{z-z_0}{\beta a}, 1, v, M \right) dz_0\, dz$$

$$= -\frac{1}{2\pi\beta}\left(\frac{l}{a}\right) \times \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\phi}_n\left(\frac{ul}{\beta a}, 1, v, M\right) du \int_{-\infty}^{\infty} \left[\left(\frac{a}{l}\frac{\partial}{\partial u} + iv \right)^2 \tilde{w}_j(n, v) \right] \times$$

$$\times \tilde{w}_{j'}(-n, u+v)\, dv$$

$$= -\frac{1}{2\pi\beta}\left(\frac{l}{a}\right) \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\phi}_n\left(\frac{ul}{\beta a}, 1, v, M\right) du \int_{-\infty}^{\infty} \left[\left(\frac{a}{l}\frac{\partial}{\partial v} + iv \right) \tilde{w}_j(n, v) \right] \times$$

$$\times \left[\left(-\frac{a}{l}\frac{\partial}{\partial v} + iv \right) \tilde{w}_{j'}(-n, u+v) \right] dv$$

$$= -\frac{1}{2\pi\beta}\left(\frac{l}{a}\right) \sum_{n=-\infty}^{\infty} \left[\int_{-1}^{0} \tilde{\phi}_n\left(\frac{ul}{\beta a}, 1, v, M\right) du \times \int_{-u}^{1} \left[\left(\frac{a}{l}\frac{\partial}{\partial v} + iv \right) \tilde{w}_j(n, v) \right] \right.$$

$$\left[\left(-\frac{a}{l}\frac{\partial}{\partial v} + iv \right) \tilde{w}_{j'}(-n, u+v) \right] dv + \int_0^1 \tilde{\phi}_n\left(\frac{ul}{\beta a}, 1, v, M\right) du$$

$$\left. \times \int_0^{1-u} \left[\left(\frac{a}{l}\frac{\partial}{\partial v} + iv \right) \tilde{w}_j(n, v) \right] \left[\left(-\frac{a}{l}\frac{\partial}{\partial v} + iv \right) \tilde{w}_{j'}(-n, u+v) \right] dv \right] \tag{23}$$

and in this final form no Dirac delta functions occur.

In order to evaluate the generalised airforce coefficient $Q_{jj'}$ using formula (23) we need the values of $\tilde{\phi}_n(u, 1, v, M)$ in the range $-\frac{l}{\beta a} < u < \frac{l}{\beta a}$. The formula (19) as it stands is not suitable for the evaluation of $\tilde{\phi}_n(u, 1, v, M)$ for a range of values of u since the infinite integral is only slowly convergent. In Appendix B formula (19) is replaced by another which is of more practical value.

We write

$$\tilde{\phi}_n\left(u, \frac{r}{a}, v, M\right) = \exp\left(\frac{iM^2 v}{\beta} \times u\right) \times \tilde{\psi}_n\left(u, \frac{r}{a}, v, M\right) \tag{24}$$

for $M < 1$ and

$$\tilde{\phi}_n\left(u, \frac{r}{a}, v, M\right) = \exp\left(\frac{-iM^2 v}{\beta} u\right) \times \tilde{\psi}_n\left(u, \frac{r}{a}, v, M\right) \tag{25}$$

for $M > 1$. Formulae from which $\tilde{\psi}_n(u, r/a, v, M)$ can be easily evaluated numerically are given in formula (132) for $M < 1$ and in formulae (194) and (207) for $M > 1$. The $\tilde{\psi}_n(u, r/a, v, M)$ are real for $M > 1$. If $r = a$ these formulae simplify to those given by (133), (195) and (207).

The generalised airforce coefficient $Q_{jj'}$ may then be determined from (23) by evaluating the integrals with respect to u numerically.

3. *Derivation of Generalised Airforces Acting on the Inside Surfaces of the Cylinder.*

We now consider the inside of a section of length l of the cylinder. The ends of the section are assumed to be closed with rigid flat surfaces. The normal displacement of a point (θ, z) on the surface of the cylinder in a mode j of vibration is again given by $w_j(\theta, z/l, t)$ and in a harmonic oscillation formula (1) applies.

The radius of the inside surface will, of course, be smaller than that of the outside surface, but we shall assume the thickness of the cylinder to be so small that, for our purposes, the inside and outside radii can be taken to be the same. It would be an easy matter to take different radii if this were thought necessary.

By equation (3)

$$\tilde{w}_j\left(n, \frac{z}{l}\right) = \int_0^{2\pi} \hat{w}_j\left(\theta, \frac{z}{l}\right) \times \exp(-in\theta)\, d\theta \tag{26}$$

and let

$$\overline{W}_j(n, k) = \frac{1}{l}\int_0^{l} \tilde{w}_j\left(n, \frac{z}{l}\right) \times \cos\left(\frac{\pi k z}{l}\right) dz\,. \tag{27}$$

Then by equation (5)

$$\hat{w}_j\left(\theta, \frac{z}{l}\right) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \tilde{w}_j\left(n, \frac{z}{l}\right) \times \exp(in\theta) \tag{28}$$

and also

$$\hat{w}_j\left(\theta, \frac{z}{l}\right) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \varepsilon_k \overline{W}_j(n, k) \exp(in\theta) \times \cos\left(\frac{\pi k z}{l}\right) \tag{29}$$

where

$$\varepsilon_k = \left.\begin{matrix} 1 \\ 2 \end{matrix}\right\} \begin{matrix} k = 0 \\ k > 0 \end{matrix}. \tag{30}$$

The perturbation velocity potential Φ_j at a point (r, θ, z) of the perturbed flow inside the cylinder when it is oscillating according to equation (1) can be written in the form

$$\Phi_j = Va\,\hat{\Phi}_j\left(\frac{r}{a}, \theta, \frac{z}{l}, \nu_0, \frac{l}{a}\right)\exp(i\omega t) \tag{31}$$

and the perturbation pressure P_j at the point (r, θ, z) corresponding to the oscillation can be written in the form

$$P_j = \rho\, a_0^2\, \hat{P}_j\left(\frac{r}{a}, \theta, \frac{z}{l}, \nu_0, \frac{l}{a}\right)\exp(i\omega t) \tag{32}$$

where

$$\nu_0 = \frac{\omega a}{a_0} \tag{33}$$

is a frequency parameter and a_0 is the speed of sound.

The perturbation velocity potential Φ_j satisfies the boundary conditions

$$\left[\frac{\partial \Phi_j}{\partial r}\right]_{r=a} = \frac{\partial}{\partial t} w_j\left(\theta, \frac{z}{l}, t\right)$$

$$= i\nu_0\, a_0\, \hat{w}_j\left(\theta, \frac{z}{l}\right)\exp(i\omega t) \tag{34}$$

on the inside surface of the cylinder and

$$\left[\frac{\partial \Phi_j}{\partial z}\right]_{z=0} = \left[\frac{\partial \Phi_j}{\partial z}\right]_{z=l} = 0 \tag{35}$$

at the ends of the section of cylinder.

If we substitute for $\hat{w}_j(\theta, z/l)$ from equation (29) into equation (34) we get

$$\left[\frac{\partial \Phi_j}{\partial r}\right]_{r=a} = \frac{1}{2\pi} i\nu_0\, a_0 \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \varepsilon_k \overline{W}_j(n, k) \times \exp(in\theta)\cos\left(\frac{\pi k z}{l}\right) \tag{36}$$

Let us consider the velocity potential Φ of the perturbed flow inside the cylinder, which satisfies the boundary conditions

$$\left[\frac{\partial \Phi}{\partial r}\right]_{r=a} = -a_0 \exp(in\theta)\cos\left(\frac{\pi\kappa z}{a}\right)\exp(i\omega t) \tag{37}$$

$$\left[\frac{\partial \Phi}{\partial z}\right]_{z=0} = \left[\frac{\partial \Phi}{\partial z}\right]_{z=l} = 0 \tag{38}$$

on the inside surface and the ends of the section of cylinder. The velocity potential will be of the form

$$\Phi = aa_0\,\overline{\Phi}_n\left(\kappa, \frac{r}{a}, \nu_0\right) \times \exp(in\theta) \times \cos\left(\frac{\pi\kappa z}{a}\right) \times \exp(i\omega t). \tag{39}$$

An expression for $\overline{\Phi}_n(\kappa, r/a, \nu_0)$ is obtained in Appendix C and is given in formula (231).

The function $\overline{\Phi}_n(\kappa, r/a, \nu_0)$ is a real function and the relations

$$\overline{\Phi}_n\left(\kappa, \frac{r}{a}, \nu_0\right) = \overline{\Phi}_n\left(\kappa, \frac{r}{a}, -\nu_0\right) = \overline{\Phi}_n\left(-\kappa, \frac{r}{a}, \nu_0\right) = \overline{\Phi}_{-n}\left(\kappa, \frac{r}{a}, \nu_0\right) \tag{40}$$

are true.

The perturbation pressure P corresponding to the perturbation velocity potential Φ of equation (39) will be of the form

$$P = \rho\, a_0^2\, \overline{P}_n\left(\kappa, \frac{r}{a}, \nu_0\right) \times \exp(in\theta) \times \cos\left(\frac{\pi\kappa z}{a}\right) \times \exp(i\omega t) \tag{41}$$

where, according to the linearised Bernoulli equation

$$P = \rho\frac{\partial \Phi}{\partial t} \tag{42}$$

we have

$$\overline{P}_n\left(\kappa, \frac{r}{a}, \nu_0\right) = i\nu_0\,\overline{\Phi}_n\left(\kappa, \frac{r}{a}, \nu_0\right). \tag{43}$$

By comparing the boundary conditions (36) and (35) with (37) and (38) and applying the principle of superposition to the solutions, we get

$$P_j = -\frac{1}{2\pi}\rho\, a_0^2\, i\nu_0 \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \varepsilon_k \times \overline{W}_j(n,k) \times \overline{P}_n\left(\frac{ka}{l}, \frac{r}{a}, \nu_0\right) \times \exp(in\theta) \times \cos\left(\frac{\pi kz}{l}\right) \times \exp(i\omega t)$$

$$= \frac{1}{2\pi}\rho\, a_0^2\, \nu_0^2 \sum_{n=-\infty}^{\infty}\sum_{k=0}^{\infty} \varepsilon_k \times \overline{W}_j(n,k) \times \overline{\Phi}_n\left(\frac{ka}{l}, \frac{r}{a}, \nu_0\right) \times \exp(in\theta) \times \cos\left(\frac{\pi kz}{l}\right) \times \exp(i\omega t). \tag{44}$$

We are interested in the airforce coefficient $\tilde{Q}_{jj'}$ given by

$$\tilde{Q}_{jj'} \exp(i\omega t) = \frac{1}{\rho a_0^2 l} \int_0^{2\pi} \int_0^l [P_j]_{r=a} \times \hat{w}_{j'} \left(\theta, \frac{z}{l}\right) d\theta \, dz. \tag{45}$$

If we substitute for $[P_j]_{r=a}$ from (44) into (45) and use the expansion (29) for $\hat{w}_{j'}(\theta, z/l)$ we get

$$\tilde{Q}_{jj'} = \frac{1}{2\pi} v_0^2 \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \overline{W}_j(n, k) \times \overline{W}_{j'}(-n, k) \times \overline{\Phi}_n\left(\frac{ka}{l}, 1, v_0\right) \tag{46}$$

which is the final expression.

It is possible to express $\tilde{Q}_{jj'}$ in terms of the $\tilde{w}_j(n, z/l)$ instead of the $\overline{W}_j(n, k)$ but the result is still a doubly infinite series.

4. *Discussion.*

The generalised forces on a flexible portion of an infinite cylinder oscillating in a uniform flow are expressed in terms of the surface velocity potentials outside and inside the flexible portion. Explicit expressions for these velocity potentials are derived. Some care is required in the numerical evaluation of these expressions since they involve integrals with integrands that are singular at the limits of integration. Apart from this the numerical evaluation is straightforward.

No numerical results for $\tilde{\psi}_n(u, r/a, v, M)$ are given for they would serve little purpose unless very extensive tables of them were given. A programme for a high speed digital computer has been written to determine numerical values of $\tilde{\psi}_n(u, 1, v, M)$ for any given values of u, v and $M > 1$ and for low values of n.

LIST OF SYMBOLS

a	Radius of cylinder
a_0	Speed of sound
l	Length of flexible portion of cylinder
$M = V/a_0$	Mach number of the flow outside the cylinder
p	Perturbation pressure corresponding to the perturbation velocity potential ϕ
p_j	Perturbation pressure corresponding to the perturbation velocity potential ϕ_j
$\hat{p}_j$	Defined in equation (8)
$\bar{p}_n$	Defined in equation (15)
P_j	Perturbation pressure corresponding to the perturbation velocity potential Φ_j
$\hat{P}_j$	Defined in equation (32)
$\bar{P}_n$	Defined in equation (41)
$Q_{jj'}$	Generalised airforce coefficient defined in equation (22)
$\tilde{Q}_{jj'}$	Generalised airforce coefficient defined in equation (45)
r, θ, z	Cylindrical polar co-ordinates
t	Time
V	Speed of undisturbed flow outside the cylinder
$w_j(\theta, z/l, t)$	Normal displacement in mode j of vibration
$\hat{w}_j(\theta, z/l)$	Defined in equation (1)
$\tilde{w}_j(n, z/l)$	Defined in equation (3)
$\bar{w}_j(n, k)$	Defined in equation (4)
$\overline{W}_j(n, k)$	Defined in equation (27)
X_q^n	Function defined in equation (189)
Y_q^n	Function defined in equation (189)
z	Co-ordinate in direction of cylinder axis
β	Defined in equation (20)

ε_k	Defined in equation (30)
$\nu = \dfrac{\omega a}{V}$	A frequency parameter
$\nu_0 = \dfrac{\omega a}{a_0}$	A frequency parameter
ρ	Density of fluid
ϕ	Velocity potential of the perturbed flow outside the cylinder which satisfies boundary condition (12)
ϕ_j	Perturbation velocity potential outside the cylinder when it is oscillating in the j'th mode
$\hat{\phi}_j$	Defined in equation (7)
$\bar{\phi}_n$	Defined in equation (13)
$\tilde{\phi}_n$	Defined in equation (19)
Φ	Velocity potential inside the cylinder which satisfies the boundary conditions (37) and (38)
Φ_j	Velocity potential inside cylinder when it is oscillating in the j'th mode
$\hat{\Phi}_j$	Defined in equation (31)
$\bar{\Phi}_n$	Defined in equation (39)
$\tilde{\psi}_n$	Defined in equation (24) for $M < 1$ and in equation (25) for $M > 1$
ω	Circular frequency

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APPENDIX A

Derivation of Expressions for $\bar{\Phi}_n\left(\kappa, \frac{r}{a}, \nu, M\right)$

We consider an infinite cylinder with its axis along the axis of z. Outside the cylinder there is a fluid flow which is a combination of a uniform flow of Mach number M in the positive direction of z and a superposed perturbation caused by the vibration of the cylinder surface. The velocity potential ϕ of the perturbed flow is assumed to satisfy the boundary condition

$$\left(\frac{\partial \phi}{\partial r}\right)_{r=a} = -V \exp\left(in\theta + i\kappa\frac{z}{a} + i\omega t\right) \tag{47}$$

on the surface of the cylinder. In equation (47), V is the velocity of the uniform flow relative to the axes of co-ordinates. The circular frequency ω is assumed to be a non-negative number in the development below. Results for ω negative are easily obtained from those for ω positive by use of formulae (13) and (14).

The velocity potential ϕ of the perturbed flow satisfies the linear partial differential equation

$$\nabla^2 \phi = M^2\left(\frac{\partial}{\partial z} + \frac{1}{V}\frac{\partial}{\partial t}\right)^2 \phi \tag{48}$$

and behaves like an outgoing wave at infinity.

The function

$$\phi = g\left(\frac{r}{a}\right) \times \exp\left(in\theta + i\kappa\frac{z}{a} + i\omega t\right) \tag{49}$$

is a solution of the partial differential equation (48) provided that the function $g(s)$ satisfies the ordinary differential equation

$$\frac{d^2g(s)}{ds^2}+\frac{1}{s}\frac{dg(s)}{ds}+\left\{\left[M^2\left(1+\frac{\nu}{\kappa}\right)^2-1\right]\kappa^2-\frac{n^2}{s^2}\right\}g(s)=0 \tag{50}$$

where

$$\nu=\frac{\omega a}{V} \tag{51}$$

The differential equation (50) is a Bessel equation. The form in which it is best to write the solution of this equation for our purposes depends on whether $[M^2(1+\nu/\kappa)^2-1]$ is greater than or less than zero.

The boundary condition (47) is appropriate to the case of waves travelling with speed $-\nu/\kappa\times V$ in the direction of positive z on the surface of the cylinder. The speed of the mainstream relative to these waves is supersonic if $[M^2(1+\nu/\kappa)^2-1]$ is greater than zero whereas it is subsonic if $[M^2(1+\nu/\kappa)^2-1]$ is less than zero.

We have that

$$\left[M^2\left(1+\frac{\nu}{\kappa}\right)^2-1\right]>0 \tag{52}$$

when

$$\kappa>-\frac{M\nu}{M+1}\quad\text{or}\quad\kappa<-\frac{M\nu}{M-1} \tag{53}$$

if $M>1$ and when

$$-\frac{M\nu}{1+M}<\kappa<\frac{M\nu}{1-M} \tag{54}$$

if $M<1$.

In these cases the function ϕ given by (49) will be written in the form

$$\phi=\left[A\,H_n^{(1)}\left\{\frac{r}{a}\sqrt{\left[M^2\left(1+\frac{\nu}{\kappa}\right)^2-1\right]\kappa^2}\right\}+B\,H_n^{(2)}\left\{\frac{r}{a}\sqrt{\left[M^2\left(1+\frac{\nu}{\kappa}\right)^2-1\right]\kappa^2}\right\}\right]\times$$
$$\times\exp\left(in\theta+i\kappa\frac{z}{a}+i\omega t\right) \tag{55}$$

where A and B are constants of integration appearing in the general solution of the differential equation (50). The values of A and B are determined from the boundary conditions.

To consider the radiation condition we consider the solution (55) in a frame of reference moving with the main stream flow. The point with co-ordinates (r,θ,z) with respect to the original frame is at time t the point with co-ordinates (r,θ,Z) with respect to the frame moving with the mainstream flow, where

$$z=Z+Vt. \tag{56}$$

The solution (55) then becomes

$$\phi = \left[A H_n^{(1)} \left\{\frac{r}{a}\sqrt{\left[M^2\left(1+\frac{\nu}{\kappa}\right)^2-1\right]\kappa^2}\right\} + B H_n^{(2)} \left\{\frac{r}{a}\sqrt{\left[M^2\left(1+\frac{\nu}{\kappa}\right)^2-1\right]\kappa^2}\right\}\right] \times$$

$$\times \exp\left\{in\theta + i\kappa\frac{Z}{a} + i\frac{V}{a}(\kappa+\nu)\,t\right\} \tag{57}$$

and this will be the complete velocity potential with respect to the frame moving with the mainstream since the unperturbed velocity potential is now identically zero.

For large values of r/a we can use the asymptotic expansion of Hankel functions to obtain

$$\phi = \sqrt{\frac{2a}{\pi r\sqrt{\left[M^2\left(1+\frac{\nu}{\kappa}\right)^2-1\right]\kappa^2}}} \times \left\{1+0\left(\frac{a}{r}\right)\right\} \exp\left\{in\theta + i\kappa\frac{Z}{a} + i\frac{V}{a}(\kappa+\nu)\,t\right\} \times$$

$$\times \left[A \exp i\left\{\frac{r}{a}\sqrt{\left[M^2\left(1+\frac{\nu}{\kappa}\right)^2-1\right]\kappa^2} - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\right\} + \right.$$

$$\left. + B\exp -i\left\{\frac{r}{a}\sqrt{\left[M^2\left(1+\frac{\nu}{\kappa}\right)^2-1\right]\kappa^2} - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi\right\}\right]. \tag{58}$$

Now when

$$\kappa > -\frac{M\nu}{M+1} \tag{59}$$

we have

$$\kappa+\nu = \frac{\nu}{M+1}$$

$$> 0 \tag{60}$$

so that in this case we can have an outgoing wave at infinity only if $A = 0$.

When

$$\kappa < -\frac{M\nu}{M-1} \tag{61}$$

and

$$M > 1 \tag{62}$$

we have

$$\kappa+\nu < -\frac{\nu}{M-1}$$

$$< 0 \tag{63}$$

so that in this case we can have an outgoing wave at infinity only if $B = 0$.

The remaining constant is determined by substituting the form (55) for ϕ into the boundary condition (47). This then leads to the following expressions for ϕ:

(i) if

$$M > 1 \text{ and } \kappa > -\frac{Mv}{M+1} \tag{64}$$

or

$$M < 1 \text{ and } -\frac{Mv}{1+M} < \kappa < \frac{Mv}{1-M} \tag{65}$$

we have

$$\phi = -\frac{Va}{\sqrt{\left[M^2\left(1+\frac{v}{\kappa}\right)^2-1\right]\kappa^2}} \times \frac{H_n^{(2)}\left\{\frac{r}{a}\sqrt{\left[M^2\left(1+\frac{v}{\kappa}\right)^2-1\right]\kappa^2}\right\}}{H_n^{(2)\prime}\left\{\sqrt{\left[M^2\left(1+\frac{v}{\kappa}\right)^2-1\right]\kappa^2}\right\}} \times$$

$$\times \exp\left(in\theta + i\kappa\frac{z}{a} + i\omega t\right). \tag{66}$$

(ii) if

$$M > 1 \text{ and } \kappa < -\frac{Mv}{M-1} \tag{67}$$

we have

$$\phi = -\frac{Va}{\sqrt{\left[M^2\left(1+\frac{v}{\kappa}\right)^2-1\right]\kappa^2}} \times \frac{H_n^{(1)}\left\{\frac{r}{a}\sqrt{\left[M^2\left(1+\frac{v}{\kappa}\right)^2-1\right]\kappa^2}\right\}}{H_n^{(1)\prime}\left\{\sqrt{\left[M^2\left(1+\frac{v}{\kappa}\right)^2-1\right]\kappa^2}\right\}} \times$$

$$\times \exp \quad in\theta + i\kappa\frac{z}{a} + i\omega t \quad . \tag{68}$$

We have that

$$\left[M^2\left(1+\frac{v}{\kappa}\right)^2-1\right] < 0 \tag{69}$$

when

$$-\frac{Mv}{M-1} < \kappa < -\frac{Mv}{M+1} \tag{70}$$

if $M > 1$ and when

$$\kappa > \frac{M\nu}{1-M} \text{ or } \kappa < -\frac{M\nu}{M+1} \tag{71}$$

if $M < 1$.

In these cases the function ϕ given by (49) is better written in the form

$$\phi = \left[A\,K_n\left\{\frac{r}{a}\sqrt{\left[1-M^2\left(1+\frac{\nu}{\kappa}\right)^2\right]\kappa^2}\right\} + B\,I_n\left\{\frac{r}{a}\sqrt{\left[1-M^2\left(1+\frac{\nu}{\kappa}\right)^2\right]\kappa^2}\right\}\right] \times$$

$$\times \exp\left(in\theta + i\kappa\frac{z}{a} + i\omega t\right) \tag{72}$$

where A and B are constants of integration appearing in the general solution of equation (50). The values of A and B are again determined from the boundary conditions.

We must have $B = 0$ since $I_n\left\{\frac{r}{a}\sqrt{\left[1-M^2\left(1+\frac{\nu}{\kappa}\right)^2\right]\kappa^2}\right\}$ tends to infinity when $\frac{r}{a}$ tends to infinity. The constant A is determined from the boundary condition (47). This then leads to the following expression for ϕ:

(iii) if

$$M > 1 \text{ and } -\frac{M\nu}{M-1} < \kappa < -\frac{M\nu}{M+1} \tag{73}$$

or if

$$M < 1 \text{ and } \kappa > \frac{M\nu}{1-M} \text{ or } \kappa < \frac{M\nu}{1+M} \tag{74}$$

we have

$$\phi = -\frac{Va}{\sqrt{\left[1-M^2\left(1+\frac{\nu}{\kappa}\right)^2\right]\kappa^2}} \times \frac{K_n\left\{\frac{r}{a}\times\sqrt{\left[1-M^2\left(1+\frac{\nu}{\kappa}\right)^2\right]\kappa^2}\right\}}{K_n'\left\{\sqrt{\left[1-M^2\left(1+\frac{\nu}{\kappa}\right)^2\right]\kappa^2}\right\}}. \tag{75}$$

Collecting results, and writing all the Hankel functions in terms of modified Bessel functions of the second kind, we have that the perturbation velocity potential ϕ in the flow of mainstream Mach number M, which satisfies the boundary condition (47) is given by

$$\phi = Va\,\bar{\phi}\left(\kappa, \frac{r}{a}, \nu, M\right) \times \exp\left(in\theta + i\kappa\frac{z}{a} + i\omega t\right) \tag{76}$$

where

(1) If $M > 1, \kappa > -\dfrac{M\nu}{M+1}$

$$\bar{\Phi}\left(\kappa, \frac{r}{a}, \nu, M\right) = -\frac{1}{i\sqrt{(M^2-1)\left\{\left(\kappa+\dfrac{M^2\nu}{M^2-1}\right)^2-\dfrac{M^2\nu^2}{(M^2-1)^2}\right\}}}\times$$

$$\times\frac{K_n\left[\dfrac{r}{a}\sqrt{(M^2-1)\left\{\left(\kappa+\dfrac{M^2\nu}{M^2-1}\right)^2-\dfrac{M^2\nu^2}{(M^2-1)^2}\right\}}\times\exp\left(\dfrac{i\pi}{2}\right)\right\}}{K_n'\left[\sqrt{(M^2-1)\quad\left(\kappa+\dfrac{M^2\nu}{M^2-1}\right)^2-\dfrac{M^2\nu^2}{(M^2-1)^2}\right\}}\times\exp\left(\dfrac{i\pi}{2}\right)\right\}}. \tag{77}$$

(2) If $M > 1, -\dfrac{M\nu}{M-1} < \kappa < -\dfrac{M\nu}{M+1}$

$$\bar{\Phi}\left(\kappa, \frac{r}{a}, \nu, M\right) = -\frac{1}{\sqrt{(M^2-1)\left\{\dfrac{M^2\nu^2}{(M^2-1)^2}-\left(\kappa+\dfrac{M^2\nu}{M^2-1}\right)^2\right\}}}\times$$

$$\times\frac{K_n\left[\dfrac{r}{a}\times\sqrt{(M^2-1)\left\{\dfrac{M^2\nu^2}{(M^2-1)^2}-\left(\kappa+\dfrac{M^2\nu}{M^2-1}\right)^2\right\}}\right]}{K_n'\left[\sqrt{(M^2-1)\left\{\dfrac{M^2\nu^2}{(M^2-1)^2}-\left(\kappa+\dfrac{M^2\nu}{M^2-1}\right)^2\right\}}\right]}. \tag{78}$$

(3) If $M > 1, \kappa < -\dfrac{M\nu}{M-1}$

$$\left(\bar{\Phi}\quad\kappa, \frac{r}{a}, \nu, M\right) = \frac{1}{i\sqrt{(M^2-1)\left\{\left(\kappa+\dfrac{M^2\nu}{M^2-1}\right)^2-\dfrac{M^2\nu^2}{(M^2-1)^2}\right\}}}\times$$

$$\times\frac{K_n\left[\dfrac{r}{a}\times\sqrt{(M^2-1)\left\{\dfrac{M^2\nu^2}{(M^2-1)^2}-\left(\kappa+\dfrac{M^2\nu}{M^2-1}\right)^2\right\}}\exp\left(-\dfrac{i\pi}{2}\right)\right]}{K_n'\left[\sqrt{(M^2-1)\left\{\dfrac{M^2\nu^2}{(M^2-1)^2}-\left(\kappa+\dfrac{M^2\nu}{M^2-1}\right)^2\right\}}\times\exp\left(-\dfrac{i\pi}{2}\right)\right]}. \tag{79}$$

(4) If $M < 1, \kappa > \dfrac{Mv}{1-M}$ or $\kappa < -\dfrac{Mv}{1+M}$

$$\bar{\Phi}\left(\kappa, \frac{r}{a}, v, M\right) = -\frac{1}{\sqrt{(1-M^2)\left\{\left(\kappa - \dfrac{M^2 v}{1-M^2}\right)^2 - \dfrac{M^2 v^2}{(1-M^2)^2}\right\}}} \times$$

$$\times \frac{K_n\left[\dfrac{r}{a} \times \sqrt{(1-M^2)\left\{\left(\kappa - \dfrac{M^2 v}{1-M^2}\right)^2 - \dfrac{M^2 v^2}{(1-M^2)^2}\right\}}\right]}{K_n'\left[\sqrt{(1-M^2)\left\{\left(\kappa - \dfrac{M^2 v}{1-M^2}\right)^2 - \dfrac{M^2 v^2}{(1-M^2)^2}\right\}}\right]}. \tag{80}$$

(5) If $M < 1, -\dfrac{Mv}{1+M} < \kappa < \dfrac{Mv}{1-M}$

$$\bar{\Phi}\left(\kappa, \frac{r}{a}, v, M\right) = -\frac{1}{i\sqrt{(1-M^2)\left\{\dfrac{M^2 v^2}{(1-M^2)^2} - \left(\kappa - \dfrac{M^2 v}{1-M^2}\right)^2\right\}}} \times$$

$$\times \frac{K_n\left[\dfrac{r}{a}\sqrt{(1-M^2)\left\{\dfrac{M^2 v^2}{(1-M^2)^2} - \left(\kappa - \dfrac{M^2 v}{(1-M^2)}\right)^2\right\}} \exp\left(\dfrac{i\pi}{2}\right)\right]}{K_n'\left[\sqrt{(1-M^2)\left\{\dfrac{M^2 v^2}{(1-M^2)^2} - \left(\kappa - \dfrac{M^2 v}{(1-M^2)}\right)^2\right\}} \exp\left(\dfrac{i\pi}{2}\right)\right]}. \tag{81}$$

APPENDIX B

Derivation of Expressions for $\tilde{\Phi}_n\left(u, \frac{r}{a}, v, M\right)$

In this Appendix we obtain a numerical means of evaluating

$$\tilde{\Phi}_n\left(u, \frac{r}{a}, v, M\right) = \frac{\beta}{2\pi}\int_{-\infty}^{\infty} \bar{\Phi}_n\left(\kappa, \frac{r}{a}, v, M\right) \times \exp(i\kappa\beta u)\, d\kappa. \tag{82}$$

Expressions for $\bar{\Phi}_n(\kappa, r/a, v, M)$ have been obtained in Appendix A, but insertion of these into the integrand of the integral on the right of equation (82) and straightforward evaluation of the integral is not practical since the convergence of the integral is very slow. The integral expression on the right of equation (82) will, in what follows, be transformed into other expressions by means of complex contour integration and we do this separately for $M < 1$ and $M > 1$ since the processes are different for these two cases. The resulting expressions are much more amenable to numerical evaluation than is the original expression on the right of equation (82).

1. We consider first the case $M < 1$.

If in the integrand on the right of equation (82) we make the change of variable

$$\kappa = \frac{\kappa'}{\beta} + \frac{M^2 v}{\beta} \tag{83}$$

and write

$$\bar{\phi}_n\left(\kappa, \frac{r}{a}, v, M\right) = \bar{\psi}_n\left(\kappa', \frac{r}{a}, v, M\right) \tag{84}$$

then we get

$$\tilde{\phi}_n\left(u, \frac{r}{a}, v, M\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\psi}_n\left(\kappa', \frac{r}{a}, v, M\right) \times \exp\left\{ i\left(\kappa' + \frac{M^2 v}{\beta}\right)u \right\} d\kappa'$$

$$= \exp\left(i\frac{M^2 v}{\beta} \times u\right) \times \tilde{\psi}_n\left(u, \frac{r}{a}, v, M\right) \tag{85}$$

where

$$\tilde{\psi}_n\left(u, \frac{r}{a}, v, M\right) = \frac{1}{2\pi} \times \int_{-\infty}^{\infty} \bar{\psi}_n\left(\kappa', \frac{r}{a}, v, M\right) \times \exp\left(i\kappa' u\right) \times d\kappa' . \tag{86}$$

The function $\bar{\psi}_n(\kappa', r/a, v, M)$, when continued analytically from the real axis into the complex κ'-plane, has branch points at

$$\kappa' = -\frac{Mv}{\beta} \quad \text{and} \quad \kappa' = \frac{Mv}{\beta} \tag{87}$$

and these are symmetrically placed with respect to the origin in the complex κ'-plane.

We observe from formulae (80) and (81) that

$$\bar{\psi}_n\left(\kappa', \frac{r}{a}, v, M\right) = \bar{\psi}_n\left(-\kappa', \frac{r}{a}, v, M\right) \tag{88}$$

and hence from (86) we have

$$\tilde{\psi}_n\left(u, \frac{r}{a}, v, M\right) = \tilde{\psi}_n\left(-u, \frac{r}{a}, v, M\right) . \tag{89}$$

In order to continue $\bar{\psi}_n(\kappa', r/a, v, M)$ analytically from the real axis into the complex κ'-plane we write

$$\kappa' + \frac{Mv}{\beta} = r_1 \times \exp(i\theta_1) \tag{90}$$

$$\kappa' - \frac{Mv}{\beta} = r_2 \times \exp(i\theta_2) \tag{91}$$

take

$$\left[\kappa'^2 - \frac{M^2 v^2}{\beta^2}\right]^{\frac{1}{2}} = \sqrt{r_1 r_2} \times \exp\left\{\frac{i(\theta_1+\theta_2)}{2}\right\} \tag{92}$$

and define

$$F(\kappa') = -\frac{1}{\left[\kappa'^2 - \frac{M^2 v^2}{\beta^2}\right]^{\frac{1}{2}}} \times \frac{K_n\left\{\frac{r}{a} \times \left[\kappa'^2 - \frac{M^2 v^2}{\beta^2}\right]^{\frac{1}{2}}\right\}}{K_n'\left\{\left[\kappa'^2 - \frac{M^2 v^2}{\beta^2}\right]^{\frac{1}{2}}\right\}}. \tag{93}$$

In this Report we use the square root sign $\sqrt{\quad}$ to denote only the positive square root of a positive number. A quantity raised to the power $\frac{1}{2}$ will be the more general complex function which is two-valued and which may be complex.

In order to be able to distinguish between the different branches of $F(\kappa')$ which are defined by equations (90) to (93) we make the further definitions

$$F_1(\kappa') = F(\kappa') \quad \text{for} \quad \begin{matrix} 0 \leqslant \theta_1 \leqslant \pi \\ -\frac{3\pi}{2} \leqslant \theta_2 \leqslant -\pi \end{matrix} \tag{94}$$

$$F_2(\kappa') = F(\kappa') \quad \text{for} \quad \begin{matrix} 0 \leqslant \theta_1 \leqslant \frac{\pi}{2} \\ 0 \leqslant \theta_2 \leqslant \pi \end{matrix}. \tag{95}$$

Now, consider only $u > 0$.

From Cauchy's theorem of residues we get (*see* Fig. 2)

$$\left(\int_{AP_1} + \int_{\Gamma_1^+} + \int_{P_2B} + \int_{BC} + \int_{CA}\right) F_1(\kappa') \exp(i\kappa' u) \times d\kappa' = 2\pi i \sum R_1 \tag{96}$$

$$\left(\int_{DP_3} + \int_{\Gamma_2} + \int_{P_4E} + \int_{EF} + \int_{FD}\right) F_2(\kappa') \times \exp(i\kappa' u) \times d\kappa' = 2\pi i \sum R_2 \tag{97}$$

where ΣR_1 is the sum of the residues of $F_1(\kappa') \times \exp(i\kappa' u)$ at its poles in the upper left quadrant of the complex κ'-plane, and ΣR_2 is the sum of the residues of $F_2(\kappa') \times \exp(i\kappa' u)$ at its poles in the upper right quadrant of the complex κ'-plane.

It is an easy matter to show that

$$\lim_{\varepsilon \to 0} \int_{\Gamma_1^+} F_1(\kappa') \times \exp(i\kappa' u) \times d\kappa' = 0 \tag{98}$$

and

$$\lim_{\varepsilon \to 0} \int_{\Gamma_2^+} F_2(\kappa') \times \exp(i\kappa' u) \times d\kappa' = 0 \tag{99}$$

where ε is the radius of the circular arcs Γ_1^+ and Γ_2^+.

When the radius R of the circular arcs CA and EF (Fig. 2) becomes indefinitely large, then we have on CA

$$F_1(\kappa') \sim \frac{1}{\kappa'} \sqrt{\frac{a}{r}} \exp\left\{ \kappa'\left(\frac{r}{a} - 1\right)\right\} \tag{100}$$

and on EF

$$F_2(\kappa') \sim \frac{1}{\kappa'} \sqrt{\frac{a}{r}} \exp\left\{ -\kappa'\left(\frac{r}{a} - 1\right)\right\} \tag{101}$$

as is easily shown using asymptotic expansions of the modified Bessel functions. Thus, for $u > 0$, we have

$$\lim_{R \to \infty} \int_{CA} F_1(\kappa') \times \exp(i\kappa' u) \times d\kappa' = 0 \tag{102}$$

and

$$\lim_{R \to \infty} \int_{EF} F_2(\kappa') \times \exp(i\kappa' u) \times d\kappa' = 0. \tag{103}$$

Neither CA nor EF in formulae (102) and (103) could be replaced by the whole semi-circle of radius R and centre origin in the upper half-plane since the behaviours of $F_1(\kappa')$ and $F_2(\kappa')$ are still given by formulae (100) and (101) over the whole semi-circle and there would be divergence of the limits.

If we proceed to the limits $\varepsilon = 0$ and $R = \infty$ in formulae (96) and (97) we therefore get for $u > 0$

$$\int_{-\infty}^{-\frac{M\nu}{\beta}} + \int_{-\frac{M\nu}{\beta}}^{0} + \int_{0}^{i\infty} F_1(\kappa') \times \exp(i\kappa' u) \times d\kappa' = 2\pi i \sum R_1 \tag{104}$$

and

$$\int_{0}^{\frac{M\nu}{\beta}} + \int_{\frac{M\nu}{\beta}}^{\infty} + \int_{0}^{i\infty} F_2(\kappa') \times \exp(i\kappa' u) \times d\kappa' = 2\pi i \sum R_2 . \tag{105}$$

The function $F_1(\kappa')$ has been defined so that

$$F_1(\kappa') = \bar{\psi}_n\left(\kappa', \frac{r}{a}, \nu, M\right) \tag{106}$$

on the part of the real axis

$$\kappa' < -\frac{Mv}{\beta} \tag{107}$$

and the function $F_2(\kappa')$ has been defined so that

$$F_2(\kappa') = \bar{\psi}_n\left(\kappa', \frac{r}{a}, v, M\right) \tag{108}$$

on the part of the real axis

$$\kappa' > \frac{Mv}{\beta} \tag{109}$$

in the κ'-plane. It so happens that the formula (108) is true also on the part of the real axis

$$0 < \kappa' < \frac{Mv}{\beta}. \tag{110}$$

From formula (86) we then get

$$\tilde{\psi}_n\left(u, \frac{r}{a}, v, M\right) = \frac{1}{2\pi}\int_{-\infty}^{-\frac{Mv}{\beta}} F_1(\kappa') \times \exp(i\kappa' u) \times d\kappa' +$$

$$+\frac{1}{2\pi}\int_{-\frac{Mv}{\beta}}^{0} \bar{\psi}_n\left(\kappa', \frac{r}{a}, v, M\right) \times \exp(i\kappa' u)\, d\kappa' +$$

$$+\frac{1}{2\pi}\int_{0}^{\infty} F_2(\kappa') \times \exp(i\kappa' u) \times d\kappa' \tag{111}$$

and, using (104) and (105) this reduces for $u > 0$ to

$$\tilde{\psi}_n\left(u, \frac{r}{a}, v, M\right) = \frac{1}{2\pi}\int_{-\frac{Mv}{\beta}}^{0} \left\{\bar{\psi}_n\left(\kappa', \frac{r}{a}, v, M\right) - F_1(\kappa')\right\} \exp(i\kappa' u)\, d\kappa' +$$

$$+\frac{1}{2\pi}\int_{0}^{i\infty} \left\{F_2(\kappa') - F_1(\kappa')\right\} \times \exp(i\kappa' u) \times d\kappa' +$$

$$+ i\left\{\sum R_1 + \sum R_2\right\}. \tag{112}$$

On BC we may write

$$\kappa' + \frac{Mv}{\beta} = \sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \times \exp\left[i\gamma(s)\right] \tag{113}$$

$$\kappa' - \frac{Mv}{\beta} = \sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \times \exp\left[-i(\pi + \gamma(s))\right] \tag{114}$$

where s is real positive and $\gamma(s)$ is the acute angle

$$\gamma(s) = \tan^{-1} \frac{\beta s}{Mv}\ . \tag{115}$$

Hence, using (90) to (93), we get on BC

$$F_1(\kappa') = \frac{1}{i\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2}} \times \frac{K_n\left[\frac{r}{a} \times \sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \times \exp\left(-\frac{i\pi}{2}\right)\right]}{K_n'\left[\sqrt{s^2 + \left(\frac{Mv}{2}\right)^2} \times \exp\left(-\frac{i\pi}{2}\right)\right]} \tag{116}$$

On DF we may write

$$\kappa' + \frac{Mv}{\beta} = \sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \times \exp\left[i\gamma(s)\right] \tag{117}$$

$$\kappa' - \frac{Mv}{\beta} = \sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \times \exp\left[i(\pi - \gamma(s))\right] \tag{118}$$

where s is real positive and $\gamma(s)$ is the acute angle given by formula (115).
Hence, using (90) to (93), we get on DF

$$F_2(\kappa') = -\frac{1}{i\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2}} \times \frac{K_n\left[\frac{r}{a} \times \sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \times \exp\left(\frac{i\pi}{2}\right)\right]}{K_n'\left[\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \times \exp\left(\frac{i\pi}{2}\right)\right]}. \tag{119}$$

Therefore

$$\frac{1}{2\pi}\int_0^{i\infty} \left\{F_2(\kappa') - F_1(\kappa')\right\} \exp(i\kappa' u)\, d\kappa'$$

$$= \frac{1}{2\pi}\int_0^{\infty} \left\{ \frac{K_n\left[\frac{r}{a}\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \exp\left(\frac{i\pi}{2}\right)\right]}{K_n'\left[\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \exp\left(\frac{i\pi}{2}\right)\right]} + \frac{K_n\left[\frac{r}{a}\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \exp\left(-\frac{i\pi}{2}\right)\right]}{K_n'\left[\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \exp\left(-\frac{i\pi}{2}\right)\right]} \right\} \times$$

$$\times \frac{\exp(-us)\,ds}{\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}}$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\left\{J_n\left[\frac{r}{a}\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]Y_n'\left[\frac{r}{a}\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]-Y_n\left[\frac{r}{a}\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]J_n'\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}}{\left\{J_n'\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}^2+\left\{Y_n'\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}^2}\times$$

$$\times \frac{\exp(-us)\,ds}{\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}} \tag{120}$$

On P_2B we may write

$$\kappa'+\frac{Mv}{\beta}=\frac{Mv}{\beta}(1+s)\exp(i0) \tag{121}$$

$$\kappa'-\frac{Mv}{\beta}=\frac{Mv}{\beta}(1-s)\exp(-i\pi) \tag{122}$$

where s is a real number in the range

$$-1 \leqslant s \leqslant 0. \tag{123}$$

Hence, using (90) to (93), we get on P_2B

$$F_1(\kappa')=\frac{1}{i\frac{Mv}{\beta}\times\sqrt{1-s^2}}\times\frac{K_n\left[\frac{r}{a}\times\frac{Mv}{\beta}\times\sqrt{1-s^2}\times\exp\left(-\frac{i\pi}{2}\right)\right]}{K_n'\left[\frac{Mv}{\beta}\times\sqrt{1-s^2}\times\exp\left(-\frac{i\pi}{2}\right)\right]}. \tag{124}$$

Also on P_2B, according to (81) and (84) we have

$$\bar{\psi}_n\left(\kappa',\frac{r}{a},v,M\right)=-\frac{1}{i\frac{Mv}{\beta}\times\sqrt{1-s^2}}\times\frac{K_n\left[\frac{r}{a}\times\frac{Mv}{\beta}\times\sqrt{1-s^2}\times\exp\left(\frac{i\pi}{2}\right)\right]}{K_n'\left[\frac{Mv}{\beta}\times\sqrt{1-s^2}\times\exp\left(\frac{i\pi}{2}\right)\right]} \tag{125}$$

where

$$\kappa'=\frac{Mv}{\beta}s \tag{126}$$

and s is a real number in the range (123).

Therefore

$$\frac{1}{2\pi}\int_{-\frac{Mv}{\beta}}^{0}\left\{\bar{\psi}_n\left(\kappa',\frac{r}{a},v,M\right)-F_1(\kappa')\right\}\exp(i\kappa' u)\,d\kappa'$$

$$= -\frac{1}{2\pi i}\int_{-1}^{0}\left\{\frac{K_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\exp\left(\frac{i\pi}{2}\right)\right]}{K'_n\left[\frac{Mv}{\beta}\sqrt{1-s^2}\exp\left(\frac{i\pi}{2}\right)\right]}+\frac{K_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\exp\left(\frac{-i\pi}{2}\right)\right]}{K'_n\left[\frac{Mv}{\beta}\sqrt{1-s^2}\exp\left(-\frac{i\pi}{2}\right)\right]}\right\}\times$$

$$\times\frac{\exp\left(i\frac{Mv}{\beta}su\right)ds}{\sqrt{1-s^2}}$$

$$= -\frac{1}{\pi}\int_{0}^{1}\frac{J_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\right]Y'_n\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]-Y_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\right]J'_n\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\}}{\left\{J'_n\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}^2+\left\{Y'_n\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}^2}$$

$$\times\frac{\exp\left(-i\frac{Mv}{\beta}su\right)ds}{\sqrt{1-s^2}}. \tag{127}$$

It now remains to determine the residues R_1 and R_2 at the poles of $F_1(\kappa')\exp(i\kappa' u)$ and $F_2(\kappa')\exp(i\kappa' u)$ respectively in the upper left and right quadrants.

According to formulae (90) to (93) we have

$$F(\kappa') = -\frac{1}{\sqrt{r_1 r_2}\exp\left\{\frac{i(\theta_1+\theta_2)}{1}\right\}}\times\frac{K_n\left[\frac{r}{a}\sqrt{r_1 r_2}\exp\left\{\frac{i(\theta_1+\theta_2)}{1}\right\}\right]}{K'_n\left[\sqrt{r_1 r_2}\exp\left\{\frac{i(\theta_1+\theta_2)}{2}\right\}\right]}. \tag{128}$$

Therefore, according to formulae (94) and (128) the poles of $F_1(\kappa')\exp(i\kappa' u)$ in the upper left quadrant must occur at the values of r_1, r_2, θ_1 and θ_2 with

$$-\frac{\pi}{2}\leqslant\frac{\theta_1+\theta_2}{2}\leqslant 0 \tag{129}$$

for which $K'_n\left[\sqrt{r_1 r_2}\exp\left\{\frac{i(\theta_1+\theta_2)}{2}\right\}\right]$ is zero.

Also, according to formulae (95) and (128) the poles of $F_2(\kappa')\exp(i\kappa' u)$ in the upper right quadrant must occur at the values of r_1, r_2, θ_1 and θ_2 with

$$0\leqslant\frac{\theta_1+\theta_2}{2}\leqslant\frac{\pi}{2} \tag{130}$$

for which $K_n'\left[\sqrt{r_1 r_2}\exp\left\{\dfrac{i(\theta_1+\theta_2)}{2}\right\}\right]$ is zero.

But (*see* Randall[1]) $K_n'\left[\sqrt{r_1 r_2}\exp\left\{\dfrac{i(\theta_1+\theta_2)}{2}\right\}\right]$ has no zeroes when $\left(\dfrac{\theta_1+\theta_2}{2}\right)$ lies in either of the ranges (129) or (130).

Hence

$$\sum R_1+\sum R_2 = 0. \tag{131}$$

Substituting the results (120), (127) and (131) into formula (112) we then get for $u > 0$

$$\begin{aligned}
&\tilde{\psi}_n\left(u,\frac{r}{a},v,M\right)\\
&=\frac{1}{\pi}\int_0^\infty \frac{\left\{J_n\left[\frac{r}{a}\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]Y_n'\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]-Y_n\left[\frac{r}{a}\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]J_n'\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}}{\left\{J_n'\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}^2+\left\{Y_n'\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}^2}\times\\
&\quad\times\frac{\exp(-us)\,ds}{\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}}\\
&\quad-\frac{i}{\pi}\int_0^1 \frac{J_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\right]Y_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]-Y_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\right]J_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]}{\left\{J_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}^2+\left\{Y_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}^2}\times\\
&\quad\times\frac{\exp\left(-i\frac{Mv}{\beta}us\right)ds}{\sqrt{1-s^2}}.
\end{aligned}\tag{132}$$

If $r = a$, then expression (132) simplifies to

$$\begin{aligned}
\tilde{\psi}_n(u,1,v,M) = \frac{2}{\pi^2}\int_0^\infty &\frac{1}{\left\{J_n'\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}^2+\left\{Y_n'\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}^2}\times\\
&\times\frac{\exp(-us)\,ds}{s^2+\left(\frac{Mv}{\beta}\right)^2}\\
&-\frac{2i}{\pi}\int_0^1\frac{1}{\left\{J_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}^2+\left\{Y_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}^2}\times
\end{aligned}$$

$$\times \frac{\exp\left(-i\frac{Mv}{\beta}us\right)ds}{1-s^2}. \tag{133}$$

An expression for $\tilde{\psi}_n(u, r/a, v, M)$ for $u < 0$ can be obtained in an analogous manner, but it is quite unnecessary to follow the procedure through since the expression is easily obtained from (132) by use of formula (89).

2. Secondly we consider the case $M > 1$.

If in the integrand of the integral on the right of equation (82) we make the change of variable

$$\kappa = \frac{\kappa'}{\beta} - \frac{M^2 v}{\beta^2} \tag{134}$$

and write

$$\bar{\Phi}_n\left(\kappa, \frac{r}{a}, v, M\right) = \bar{\Psi}_n\left(\kappa', \frac{r}{a}, v, M\right) \tag{135}$$

then we get

$$\tilde{\phi}_n\left(u, \frac{r}{a}, v, M\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\Psi}_n\left(\kappa', \frac{r}{a}, v, M\right) \exp\left\{i\left(\kappa' - \frac{M^2 v}{\beta}\right)u\right\} d\kappa'$$

$$= \exp\left(-i\frac{M^2 v}{\beta}u\right)\tilde{\psi}_n\left(u, \frac{r}{a}, v, M\right) \tag{136}$$

where

$$\tilde{\psi}_n\left(u, \frac{r}{a}, v, M\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\Psi}_n\left(\kappa', \frac{r}{a}, v, M\right) \exp(i\kappa' u)\, d\kappa'. \tag{137}$$

The function $\bar{\Psi}_n(\kappa', r/a, v, M)$, when continued analytically from the real axis into the complex κ'-plane, has branch points at

$$\kappa' = -\frac{Mv}{\beta} \text{ and } \kappa' = \frac{Mv}{\beta} \tag{138}$$

and these are symmetrically placed with respect to the origin in the complex κ'-plane.

In order to continue $\bar{\Psi}_n(\kappa', r/a, v, M)$ analytically from the real axis into the complex κ'-plane we write

$$\kappa' + \frac{Mv}{\beta} = r_1 \exp(i\theta_1) \tag{139}$$

$$\kappa' - \frac{Mv}{\beta} = r_2 \exp(i\theta_2) \tag{140}$$

take

$$\left[\frac{M^2 v^2}{\beta^2} - \kappa'^2\right]^{\frac{1}{2}} = \sqrt{r_1 r_2} \exp\left\{\frac{i(\pi+\theta_1+\theta_2)}{2}\right\} \tag{141}$$

and define

$$F(\kappa') = -\frac{1}{\left[\frac{M^2 v^2}{\beta^2} - \kappa'^2\right]^{\frac{1}{2}}} \times \frac{K_n\left\{\frac{r}{a}\left[\frac{M^2 v^2}{\beta^2} - \kappa'^2\right]^{\frac{1}{2}}\right\}}{K_n'\left\{\left[\frac{M^2 v^2}{\beta^2} - \kappa'^2\right]^{\frac{1}{2}}\right\}}. \tag{142}$$

In order to be able to distinguish between the different branches of $F(\kappa')$ which are defined by equations (139) to (142) we make the further definitions

$$F_1(\kappa') = F(\kappa') \quad \begin{array}{l} -2\pi \leqslant \theta_1 \leqslant -\pi \\ -\frac{3\pi}{2} \leqslant \theta_2 \leqslant -\pi \end{array} \tag{143}$$

$$F_2(\kappa') = F(\kappa') \quad \begin{array}{l} 0 \leqslant \theta_1 \leqslant \frac{\pi}{2} \\ 0 \leqslant \theta_2 \leqslant \pi \end{array} \tag{144}$$

$$F_3(\kappa') = F(\kappa') \quad \begin{array}{l} -\pi \leqslant \theta_1 \leqslant 0 \\ -\pi \leqslant \theta_2 \leqslant 0 \end{array}, \tag{145}$$

(i) Consider contours in the upper half κ'-plane as in Fig. 3. From Cauchy's theorem of residues we get (*see* Fig. 3)

$$\left\{\int_{AP_1} + \int_{\Gamma_1^\dagger} + \int_{P_2B} + \int_{BC} + \int_{CA}\right\} F_1(\kappa') \exp(i\kappa' u)\, d\kappa' = 2\pi i \sum R_1 \tag{146}$$

$$\left\{\int_{DP_3} + \int_{\Gamma_2} + \int_{P_4E} + \int_{EF} + \int_{FD}\right\} F_2(\kappa') \exp(i\kappa' u)\, d\kappa' = 2\pi i \sum R_2 \tag{147}$$

where $\sum R_1$ is the sum of the residues of $F_1(\kappa') \exp(i\kappa' u)$ at its poles in the upper left quadrant of the complex κ'-plane, and $\sum R_2$ is the sum of the residues of $F_2(\kappa') \exp(i\kappa' u)$ at its poles in the upper right quadrant of the complex κ'-plane.

It is an easy matter to show that

$$\lim_{\varepsilon \to 0} \int_{\Gamma_1^\dagger} F_1(\kappa') \exp(i\kappa' u)\, d\kappa' = 0 \tag{148}$$

and

$$\lim_{\varepsilon \to 0} \int_{\Gamma_2^\ddagger} F_2(\kappa') \exp(i\kappa' u)\, d\kappa' = 0 \tag{149}$$

where ε is the radius of the circular arcs Γ_1^+ and Γ_2^+.

When the radius R of the circular arcs CA and EF (Fig. 2) becomes indefinitely large, then we have on CA

$$F_1(\kappa') \sim -\frac{1}{i\kappa'} \exp \left\{ -i\kappa'\left(\frac{r}{a}-1\right) \right\} \tag{150}$$

and on EF

$$F_2(\kappa') \sim -\frac{1}{i\kappa'} \exp \left\{ -i\kappa'\left(\frac{r}{a}-1\right) \right\} \tag{151}$$

as is easily shown using asymptotic expansions of the modified Bessel functions.

Thus, for

$$u > \frac{r}{a} - 1 \tag{152}$$

we have

$$\lim_{R \to \infty} \int_{\mathrm{CA}} F_1(\kappa') \exp(i\kappa' u)\, d\kappa' = 0 \tag{153}$$

and

$$\lim_{R \to \infty} \int_{\mathrm{EF}} F_2(\kappa') \exp(i\kappa' u)\, d\kappa' = 0\,. \tag{154}$$

If we proceed to the limits $\varepsilon = 0$ and $R = \infty$ in formulae (146) and (147) we therefore get for $u > \frac{r}{a} - 1$

$$\int_{-\infty}^{-\frac{M}{\beta\nu}} + \int_{-\frac{M\nu}{\beta}}^{0} + \int_{0}^{i\infty} F_1(\kappa') \exp(i\kappa' u)\, d\kappa' = 2\pi i \sum R_1 \tag{155}$$

$$\int_{0}^{\frac{M\nu}{\beta}} + \int_{\frac{M\nu}{\beta}}^{0} - \int_{0}^{i\infty} F_2(\kappa') \exp(i\kappa' u)\, d\kappa' = 2\pi i \sum R_2\,. \tag{156}$$

The function $F_1(\kappa')$ has been defined so that

$$F_1(\kappa') = \bar{\psi}_n\left(\kappa', \frac{r}{a}, \nu, M\right) \tag{157}$$

on the part of the real axis

$$\kappa' < -\frac{M\nu}{\beta} \tag{158}$$

and the function $F_2(\kappa')$ has been defined so that

$$F_2(\kappa') = \bar{\psi}_n\left(\kappa', \frac{r}{a}, \nu, M\right) \tag{159}$$

on the part of the real axis

$$\kappa' > \frac{M\nu}{\beta}. \tag{160}$$

From formula (137) we then get

$$\tilde{\psi}_n\left(u, \frac{r}{a}, \nu, M\right) = \frac{1}{2\pi}\int_{-\infty}^{-\frac{M\nu}{\beta}} F_1(\kappa') \exp(i\kappa' u)\, d\kappa' + \frac{1}{2\pi}\int_{-\frac{M\nu}{\beta}}^{\frac{M\nu}{\beta}} \bar{\psi}_n\left(\kappa', \frac{r}{a}, \nu, M\right) \exp(i\kappa' u)\, d\kappa' +$$

$$+\frac{1}{2\pi}\int_{\frac{M\nu}{\beta}}^{\infty} F_2(\kappa') \exp(i\kappa' u)\, d\kappa' \tag{161}$$

and, using (155) and (156) this reduces for $u > \frac{r}{a} - 1$ to

$$\tilde{\psi}_n\left(u, \frac{r}{a}, \nu, M\right) = \frac{1}{2\pi}\int_{-\frac{M\nu}{\beta}}^{0} \left\{\bar{\psi}_n\left(\kappa', \frac{r}{a}, \nu, M\right) - F_1(\kappa')\right\} \exp(i\kappa' u)\, d\kappa' +$$

$$+\frac{1}{2\pi}\int_{0}^{\frac{M\nu}{\beta}} \left\{\bar{\psi}_n\left(\kappa', \frac{r}{a}, \nu, M\right) - F_2(\kappa')\right\} \exp(i\kappa' u)\, d\kappa' +$$

$$+\frac{1}{2\pi}\int_{0}^{i\infty} \left\{F_2(\kappa') - F_1(\kappa')\right\} \exp(i\kappa' u)\, d\kappa' + i\left\{\sum R_1 + \sum R_2\right\}. \tag{162}$$

On BC we may write

$$\kappa' + \frac{M\nu}{\beta} = \sqrt{s^2 + \left(\frac{M\nu}{\beta}\right)^2} \exp\{-i(2\pi - \gamma(s))\} \tag{163}$$

$$\kappa' - \frac{Mv}{\beta} = \sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \exp\{-i(\pi + \gamma(s))\} \tag{164}$$

where s is real positive and $\gamma(s)$ is the acute angle

$$\gamma(s) = \tan^{-1}\left(\frac{\beta s}{Mv}\right). \tag{165}$$

Hence, using (139) to (142), we get on BC

$$F_1(\kappa') = \frac{1}{\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2}} \times \frac{K_n\left[\frac{r}{a}\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \exp(-i\pi)\right]}{K_n'\left[\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \exp(-i\pi)\right]}. \tag{166}$$

On DF we may write

$$\kappa + \frac{Mv}{\beta} = \sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \exp\{i\gamma(s)\} \tag{167}$$

$$\kappa - \frac{Mv}{\beta} = \sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \exp[i(\pi - \gamma(s))] \tag{168}$$

where s is real positive and $\gamma(s)$ is the acute angle given by formula (165).
Hence, using (139) to (142), we get on DF

$$F_2(\kappa') = \frac{1}{\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2}} \times \frac{K_n\left[\frac{r}{a}\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \exp(i\pi)\right]}{K_n'\left[\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \exp(i\pi)\right]} \tag{169}$$

Therefore

$$\frac{1}{2\pi}\int_0^{i\infty} \left\{F_2(\kappa') - F_1(\kappa')\right\} \exp(i\kappa' u)\, d\kappa'$$

$$= \frac{i}{2\pi}\int_0^{\infty} \left\{ \frac{K_n\left[\frac{r}{a}\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \exp(i\pi)\right]}{K_n'\left[\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \exp(i\pi)\right]} - \frac{K_n\left[\frac{r}{a}\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \exp(-i\pi)\right]}{K_n'\left[\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2} \exp(-i\pi)\right]} \right\} \frac{\exp(-us)\, ds}{\sqrt{s^2 + \left(\frac{Mv}{\beta}\right)^2}}$$

$$= (-1)^n \int_0^\infty \frac{\left\{K_n\left[\frac{r}{a}\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right] I_n'\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right] - I_n\left[\frac{r}{a}\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right] K_n'\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}}{\left\{K_n'\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}^2 + \pi^2\left\{I_n'\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}^2} \times$$

$$\times \frac{\exp(-us)\,ds}{\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}}. \tag{170}$$

On P_2B we may write

$$\kappa' + \frac{Mv}{\beta} = \frac{Mv}{\beta}(1+s)\exp(-2i\pi) \tag{171}$$

$$\kappa' - \frac{Mv}{\beta} = \frac{Mv}{\beta}(1-s)\exp(-i\pi) \tag{172}$$

where s is a real number in the range

$$-1 \leqslant s \leqslant 0. \tag{173}$$

Hence, using (139) to (143), we get on P_2B

$$F_1(\kappa') = \frac{1}{\frac{Mv}{\beta}\sqrt{1-s^2}} \; \frac{K_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\exp(-i\pi)\right]}{K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\exp(-i\pi)\right]}. \tag{174}$$

On DP_3 we may write

$$\kappa' + \frac{Mv}{\beta} = \frac{Mv}{\beta}(1+s)\exp(i0) \tag{175}$$

$$\kappa' - \frac{Mv}{\beta} = \frac{Mv}{\beta}(1-s)\exp(i\pi) \tag{176}$$

where s is a real number in the range

$$0 \leqslant s \leqslant 1. \tag{177}$$

Hence, using (139) to (142), we get on DP_3

$$F_2(\kappa') = \frac{1}{\frac{Mv}{\beta}\sqrt{1-s^2}} \; \frac{K_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\exp(i\pi)\right]}{K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\exp(i\pi)\right]}. \tag{178}$$

Also on P_2B and DP_3, according to (78) and (135) we have

$$\bar{\Psi}_n \;\; \kappa', \frac{r}{a}, v, M \;\; = -\frac{1}{\frac{Mv}{\beta}\sqrt{1-s^2}} \; \frac{K_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\right]}{K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]} \tag{179}$$

where

$$\kappa' = \frac{Mv}{\beta}s \tag{180}$$

and s is a real number in the range (173) over P_2B and in the range (177) over DP_3.
Therefore

$$\frac{1}{2\pi}\int_{-\frac{Mv}{\beta}}^{0} \left\{\bar{\Psi}_n\left(\kappa', \frac{r}{a}, v, M\right) - F_1(\kappa')\right\} \exp(i\kappa' u)\, d\kappa'$$

$$= -\frac{1}{2\pi}\int_{-1}^{0} \left\{ \frac{K_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\right]}{K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]} + \frac{K_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\exp(-i\pi)\right]}{K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\exp(-i\pi)\right]} \right\} \frac{\exp\left(i\frac{Mv}{\beta}s\,u\right)ds}{\sqrt{1-s^2}}$$

$$= -\frac{i}{2}\int_{0}^{1} \frac{K_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\right]\times I_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right] - I_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\right]\times K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]}{K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\left\{(-1)^n K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right] + \pi i\, I_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}} \times$$

$$\times \frac{\exp\left(-i\frac{Mv}{\beta}s\,u\right)ds}{\sqrt{1-s^2}} \tag{181}$$

and

$$\frac{1}{2\pi}\int_{0}^{\frac{Mv}{\beta}} \left\{\bar{\Psi}_n\left(\kappa', \frac{r}{a}, v, M\right) - F_2(\kappa')\right\} \exp(i\kappa' u)\, d\kappa'$$

$$= -\frac{1}{2\pi}\int_{0}^{1} \left\{ \frac{K_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\right]}{K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]} + \frac{K_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\exp(i\pi)\right]}{K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\exp(i\pi)\right]} \right\} \frac{\exp\left(i\frac{Mv}{\beta}s\,u\right)ds}{\sqrt{1-s^2}}$$

$$= \frac{i}{2}\int_0^1 \frac{\left\{K_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\right]\times I_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]-I_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\right]\times K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}}{K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\left\{(-1)^n K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]-\pi i I_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}}\times$$

$$\times\frac{\exp\left(i\frac{Mv}{\beta}s\,u\right)ds}{\sqrt{1-s^2}}. \tag{182}$$

If we add the results (181) and (182) we get

$$\frac{1}{2\pi}\int_{-\frac{Mv}{\beta}}^{0}\left\{\bar{\psi}_n\left(\kappa',\frac{r}{a},v,M\right)-F_1(\kappa')\right\}\exp(i\kappa'u)\,d\kappa'+$$

$$+\frac{1}{2\pi}\int_0^{\frac{Mv}{\beta}}\left\{\bar{\psi}_n\left(\kappa',\frac{r}{a},v,M\right)-F_2(\kappa')\right\}\exp(i\kappa'u)\,d\kappa'$$

$$= -\int_0^1 \frac{\left\{K_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\right]I_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]-I_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\right]K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}}{\left\{K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}^2+\pi^2\left\{I_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}^2}\times$$

$$\times\left\{(-1)^n\sin\left(\frac{Mv}{\beta}s\,u\right)+\pi\frac{I_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]}{K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]}\cos\left(\frac{Mv}{\beta}s\,u\right)\right\}\frac{ds}{\sqrt{1-s^2}}. \tag{183}$$

It now remains to determine the residues R_1 and R_2 at the poles of $F_1(\kappa')\exp(i\kappa'u)$ and $F_2(\kappa')\exp(i\kappa'u)$ respectively in the upper left and right quadrants.

According to formulae (139) to (142) we have

$$F(\kappa') = \frac{1}{\sqrt{r_1 r_2}\exp\left\{\frac{i(\pi+\theta_1+\theta_2)}{2}\right\}}\;\frac{K_n\left[\frac{r}{a}\sqrt{r_1 r_2}\exp\left\{\frac{i(\pi+\theta_1+\theta_2)}{2}\right\}\right]}{K_n'\left[\sqrt{r_1 r_2}\exp\left\{\frac{i(\pi+\theta_1+\theta_2)}{2}\right\}\right]}. \tag{184}$$

Therefore, according to formulae (143) and (184) the poles of $F_1(\kappa')\exp(i\kappa'u)$ in the upper left quadrant must occur at the values of r_1, r_2, θ_1 and θ_2 with

$$-\pi \leqslant \frac{(\pi+\theta_1+\theta_2)}{2} \leqslant -\frac{\pi}{2} \tag{185}$$

for which $K_n'\left[\sqrt{r_1 r_2}\exp\left\{\frac{i(\pi+\theta_1+\theta_2)}{2}\right\}\right]$ is zero.

Also, according to formulae (144) and (184) the poles of $F_2(\kappa')\exp(i\kappa' u)$ in the upper right quadrant must occur at the values of r_1, r_2, θ_1 and θ_2 with

$$\frac{\pi}{2} \leqslant \frac{(\pi+\theta_1+\theta_2)}{2} \leqslant \pi \tag{186}$$

for which $K_n'\left[\sqrt{r_1 r_2}\exp\left\{\frac{i(\pi+\theta_1+\theta_2)}{2}\right\}\right]$ is zero.

Now (*see* Randall[1]) $K_n'\left[\sqrt{r_1 r_2}\exp\left\{\frac{i(\pi+\theta_1+\theta_2)}{2}\right\}\right]$ has $\left[\frac{|n|+1}{2}\right]$ simple zeroes in each of the ranges (185) and (186), where the square brackets [ ] are to be interpreted as denoting the integral part of the number $\frac{|n|+1}{2}$ appearing within them.

Let the zeros of $K_n'\left[\sqrt{r_1 r_2}\exp\left\{\frac{i(\pi+\theta_1+\theta_2)}{2}\right\}\right]$ in the range (185) be at

$$\left.\begin{aligned}\sqrt{r_1 r_2} &= \chi_q^n \\ \frac{\pi+\theta_1+\theta_2}{2} &= -\pi+\phi_q^n\end{aligned}\right\} q = 1, 2, \ldots, \left[\frac{|n|+1}{2}\right] \tag{187}$$

where the ϕ_q are positive acute angles. Then the zeros of $K_n'\left[\sqrt{r_1 r_2}\exp\left\{\frac{i(\pi+\theta_1+\theta_2)}{2}\right\}\right]$ in the range (186) are at

$$\left.\begin{aligned}\sqrt{r_1 r_2} &= \chi_q^n \\ \frac{\pi+\theta_1+\theta_2}{2} &= \pi-\phi_q^n\end{aligned}\right\} q = 1, 2, \ldots, \left[\frac{|n|+1}{2}\right] \tag{188}$$

Note that there are no zeros at all if $n = 0$, and the zeros for positive n are the same as those for negative n since $K_n(z) = K_{-n}(z)$. The values of the moduli χ_q^n and of the phases ϕ_q^n can be determined as functions of n and q only. In fact $z = -\chi_q^n \cos\phi_q^n \pm i\,\chi_q^n \sin\phi_q^n$ are zeros of $K_n'(z)$.

We can now determine $\kappa_q'^n$, the value of κ' corresponding to a zero (187), by use of equations (139) and (140), and write

$$\kappa_q^n = i\left[X_q^n\left(\frac{Mv}{\beta}\right)+i\,Y_q^n\left(\frac{Mv}{\beta}\right)\right] \tag{189}$$

where $X_q^n\left(\frac{Mv}{\beta}\right)$ and $Y_q^n\left(\frac{Mv}{\beta}\right)$ are positive functions for $\frac{Mv}{\beta}$ positive. Then corresponding to a zero (188) the value $\bar{\kappa}_q^n$ of κ' will be

$$\bar{\kappa}_q^n = i\left[X_q^n\left(\frac{Mv}{\beta}\right)-i\,Y_q^n\left(\frac{Mv}{\beta}\right)\right] \tag{190}$$

We note that when $v = 0$ we get

$$X_q^n(0) + i\, Y_q^n(0) = \chi_q^n \exp(i\, \phi_q^n). \tag{191}$$

The residue of $F_1(\kappa') \exp(i\kappa' u)$ at the pole q in the upper left quadrant is then found to be

$$R_1^{q,n} = \frac{1}{i} \frac{1}{\left\{ X_q^n\left(\frac{Mv}{\beta}\right) + i\, Y_q^n\left(\frac{Mv}{\beta}\right) \right\}} \frac{K_n\left[\frac{r}{a} X_q^n \exp\left\{-i\left(\pi - \phi_q^n\right)\right\}\right]}{K_n''\left[\chi_q^n \exp\left\{-i\left(\pi - \phi_q^n\right)\right\}\right]} \times$$

$$\times \exp\left[-u\left\{X_q^n\left(\frac{Mv}{\beta}\right) + i\, Y_q^n\left(\frac{Mv}{\beta}\right)\right\}\right]$$

$$= \frac{1}{i} \frac{\left\{X_q^n(0) + i\, Y_q^n(0)\right\}^2}{\left\{X_q^n\left(\frac{Mv}{\beta}\right) + i\, Y_q^n\left(\frac{Mv}{\beta}\right)\right\}} \frac{1}{\left[n^2 + \left\{X_q^n(0) + i\, Y_q^n(0)\right\}^2\right]} \times$$

$$\times \frac{K_n\left[\frac{r}{a} \chi_q^n \exp\left\{-i\left(\pi - \phi_q^n\right)\right\}\right]}{K_n\left[\chi_q^n \exp\left\{-i\left(\pi - \phi_q^n\right)\right\}\right]}$$

$$\times \exp\left[-u\left\{X_q^n\left(\frac{Mv}{\beta}\right) + i\, Y_q^n\left(\frac{Mv}{\beta}\right)\right\}\right] \tag{192}$$

The residue of $F_2(\kappa') \exp(i\kappa' u)$ at the pole q in the upper right quadrant is found to be

$$R_2^{q,n} = \frac{1}{i} \frac{1}{\left\{X_q^n\left(\frac{Mv}{\beta}\right) - i\, Y_q^n\left(\frac{Mv}{\beta}\right)\right\}} \frac{K_n\left[\frac{r}{a} \chi_q^n \exp\left\{i\left(\pi - \phi_q^n\right)\right\}\right]}{K_n''\left[\chi_q^n \exp\{i(\pi - \phi_q^n)\}\right]} \times$$

$$\times \exp\left[-u\left\{X_q^n\left(\frac{Mv}{\beta}\right) - i\, Y_q^n\left(\frac{Mv}{\beta}\right)\right\}\right]$$

$$= \frac{1}{i} \frac{\{X_q^n(0) - i\, Y_q^n(0)\}^2}{\left\{X_q^n\left(\frac{Mv}{\beta}\right) - i\, Y_q^n\left(\frac{Mv}{\beta}\right)\right\}} \frac{1}{[n^2 + \{X_q^n(0) - i\, Y_q^n(0)\}^2]} \times$$

$$\times \frac{K_n\left[\frac{r}{a} \chi_q^n \exp\left\{i\left(\pi - \phi_q^n\right)\right\}\right]}{K_n[\chi_q^n \exp\{i(\pi - \phi_q^n)\}]} \times \exp\left[-u\left\{X_q^n\left(\frac{Mv}{\beta}\right) - i\, Y_q^n\left(\frac{Mv}{\beta}\right)\right\}\right] \tag{193}$$

Substituting the results (170), (183), (192) and (193) into formula (162) we get for $u > \frac{r}{a} - 1$

$$\tilde{\psi}_n\left(u, \frac{r}{a}, v, M\right)$$

$$= (-1)^n \int_0^\infty \frac{\left\{K_n\left[\frac{r}{a}\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right] I'_n\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right] - I_n\left[\frac{r}{a}\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right] K'_n\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}}{\left\{K'_n\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}^2 + \pi^2\left\{I'_n\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}^2}$$

$$\times \frac{\exp(-us)\, ds}{\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}}$$

$$-\int_0^1 \frac{K_n\left\{\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\right] I'_n\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right] - I_n\left[\frac{r}{a}\frac{Mv}{\beta}\sqrt{1-s^2}\right] K'_n\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}}{\left\{K'_n\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}^2 + \pi^2\left\{I'_n\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}^2} \times$$

$$\times \left\{(-1)^n \sin\left(\frac{Mv}{\beta}us\right) + \pi \frac{I'_n\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]}{K'_n\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]} \cos\left(\frac{Mv}{\beta}us\right)\right\} \frac{ds}{\sqrt{1-s^2}}$$

$$+ \sum_{q=1}^{\left[\frac{|n|+1}{2}\right]} \frac{\{X_q^n(0) + i\, Y_q^n(0)\}^2}{\left\{X_q^n\left(\frac{Mv}{\beta}\right) + i\, Y_q^n\left(\frac{Mv}{\beta}\right)\right\}} \frac{1}{[n^2 + \{X_q^n(0) + i\, Y_q^n(0)\}^2]} \times$$

$$\times \frac{K^n\left[\frac{r}{a}\chi_q^n \exp\left\{-i\left(\pi - \phi_q^n\right)\right\}\right]}{K_n[\chi_q^n \exp\{-i(\pi - \phi_q^n)\}]} \exp\left[-u\left\{X_q^n\left(\frac{Mv}{\beta}\right) + i\, Y_q^n\left(\frac{Mv}{\beta}\right)\right\}\right] +$$

$$+ \sum_{q=1}^{\left[\frac{|n|+1}{2}\right]} \frac{\{X_q^n(0) - i\, Y_q^n(0)\}^2}{\left\{X_q^n\left(\frac{Mv}{\beta}\right) - i\, Y_q^n\left(\frac{Mv}{\beta}\right)\right\}} \frac{1}{[n^2 + \{X_q^n(0) - i\, Y_q^n(0)\}^2]} \times$$

$$\times \frac{K_n\left[\frac{r}{a}\chi_q^n \exp\left\{i\left(\pi - \phi_q^n\right)\right\}\right]}{K_n[\chi_q^n \exp\{i(\pi - \phi_q^n)\}]} \exp\left[-u\left\{X_q^n\left(\frac{Mv}{\beta}\right) - i\, Y_q^n\left(\frac{Mv}{\beta}\right)\right\}\right] \qquad (194)$$

Since $K_n(\chi e^{-i\phi})$ is the complex conjugate of $K_n(\chi e^{i\phi})$, the expression (194) for $\tilde{\psi}_n\left(u, \frac{r}{a}, v, M\right)$ is wholly real.

If $r = a$, then expression (194) simplifies to

$$\tilde{\psi}_n(u, 1, v, M)$$

$$= (-1)^n \int_0^\infty \frac{1}{\left\{K_n'\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}^2 + \pi^2\left\{I_n'\left[\sqrt{s^2+\left(\frac{Mv}{\beta}\right)^2}\right]\right\}^2} \times$$

$$\times \frac{\exp(-us)}{s^2+\left(\frac{Mv}{\beta}\right)^2} ds$$

$$- \frac{\beta}{Mv}\int_0^1 \frac{1}{\left\{K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}^2 + \pi^2\left\{I_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]\right\}^2} \times$$

$$\times \left\{(-1)^n \sin\left(\frac{Mv}{\beta}us\right) + \pi \frac{I_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]}{K_n'\left[\frac{Mv}{\beta}\sqrt{1-s^2}\right]} \cos\left(\frac{Mv}{\beta}us\right)\right\} \frac{ds}{1-s^2} +$$

$$+2\sum_{q=1}^{\left[\frac{|n|+1}{2}\right]} \left\{ \left(X_q^n\left(\frac{Mv}{\beta}\right)[(\{X_q^n(0)\}^2 - \{Y_q^n(0)\}^2)(n^2 + \{X_q^n(0)\}^2 - \{Y_q^n(0)\}^2) + \right.\right.$$

$$\left. +4\{X_q^n(0)\}^2\{Y_q^n(0)\}^2] + 2n^2\, Y_q^n\left(\frac{Mv}{\beta}\right) X_q^n(0)\, Y_q^n(0) \right) \cos\left[u\, Y_q^n\left(\frac{Mv}{\beta}\right)\right] +$$

$$+ \left(2n^2 X_q^n\left(\frac{Mv}{\beta}\right) X_q^n(0)\, Y_q^n(0) - Y_q^n\left(\frac{Mv}{\beta}\right)[(\{X_q^n(0)\}^2 - \{Y_q^n(0)\}^2) \times \right.$$

$$\left. \times (n^2 + \{X_q^n(0)\}^2 - \{Y_q^n(0)\}^2) + 4\{X_q^n(0)\}^2\{Y_q^n(0)\}^2] \right) \sin\left[u\, Y_q^n\left(\frac{Mv}{\beta}\right)\right] \times$$

$$\times \frac{\exp\left[-u\, X_q^n\left(\frac{Mv}{\beta}\right)\right]}{\left(\left\{X_q^n\left(\frac{Mv}{\beta}\right)\right\}^2 + \left\{Y_q^n\left(\frac{Mv}{\beta}\right)\right\}^2\right)\{(n^2 + \{X_q^n(0)\}^2 - \{Y_q^n(0)\}^2)^2 + 4\{X_q^n(0)\}^2\{Y_q^n(0)\}^2}.$$

(195)

It may be noted that

$$\lim_{v \to 0} \tilde{\psi}_n\left(u, \frac{r}{a}, v, M\right) = V_n\left(u, \frac{r}{a}\right) \tag{196}$$

where the $V_n\left(u, \frac{r}{a}\right)$ are the functions obtained by Randall[1].

(ii) Consider a contour in the lower half-plane as in Fig. 3.
From Cauchy's theorem of residues we get (*see* Fig. 3)

$$\int_{GQ_1} + \int_{\Gamma_1^-} + \int_{Q_2Q_3} + \int_{\Gamma_1^-} + \int_{Q_4I} + \int_{IHG} F_3(\kappa') \exp(i\kappa' u)\, d\kappa' = -2\pi i \sum R_3 \tag{197}$$

where $\sum R_3$ is the sum of the residues of $F_3(\kappa') \exp(i\kappa' u)$ at its poles in the lower half-plane.
It is an easy matter to show that

$$\lim_{\varepsilon \to 0} \int_{\Gamma_1^-} + \int_{\Gamma_2^-} F_3(\kappa') \exp(i\kappa' u)\, d\kappa' = 0 \tag{198}$$

where ε is the radius of the circular arcs Γ_1^- and Γ_2^-.
When the radius R of the circular arc IHG becomes indefinitely large, then we have on IHG

$$F_3(\kappa') \sim \frac{1}{i\kappa'} \exp \left\{-i\kappa'\left(\frac{r}{a} - 1\right)\right\} \tag{199}$$

as is easily shown using asymptotic expansions of the modified Bessel functions.
Thus, for

$$u < \frac{r}{a} - 1 \tag{200}$$

we have

$$\lim_{R \to \infty} \int_{IHG} F_3(\kappa') \exp(i\kappa' u)\, d\kappa' = 0. \tag{201}$$

If we proceed to the limits $\varepsilon = 0$ and $R = \infty$ in formula (197) we therefore get for $u < \frac{r}{a} - 1$

$$\int_{-\infty}^{\infty} F_3(\kappa') \exp(i\kappa' u)\, d\kappa' = -2\pi i \sum R_3. \tag{202}$$

The function $F_3(\kappa')$ has been defined so that

$$F_3(\kappa') = \bar{\psi}_n\left(\kappa', \frac{r}{a}, \nu, M\right) \tag{203}$$

on the whole of the real axis. From formulae (137) and (202) we therefore get for $u < \frac{r}{a} - 1$

$$\tilde{\psi}_n\left(u, \frac{r}{a}, \nu, M\right) = -2\pi i \sum R_3 . \tag{204}$$

It now remains to determine the residues R_3 at the poles of $F_3(\kappa')\exp(i\kappa' u)$ in the lower half-plane.

According to formulae (145) and (184) the poles of $F_3(\kappa')\exp(i\kappa' u)$ in the lower half-plane must occur at the values of r_1, r_2, θ_1 and θ_2 with

$$-\frac{\pi}{2} \leqslant \frac{(\pi+\theta_1+\theta_2)}{2} \leqslant \frac{\pi}{2} \tag{205}$$

for which $K_n'\left[\sqrt{r_1 r_2}\exp\left\{\frac{i(\pi+\theta_1+\theta_2)}{2}\right\}\right]$ is zero.

But (*see* Randall[1]) $K_n'\left[\sqrt{r_1 r_2}\exp\left\{\frac{i(\pi+\theta_1+\theta_2)}{2}\right\}\right]$ has no zeros in the range (205).

Hence

$$\sum R_3 = 0 \tag{206}$$

and consequently for $u < \frac{r}{a} - 1$

$$\tilde{\psi}_n\left(u, \frac{r}{a}, \nu, M\right) = 0 \tag{207}$$

which is to be expected since disturbances are confined to $z > \beta\left(\frac{r}{a} - 1\right)$ in supersonic flow.

We can obtain a power series solution for $\tilde{\psi}_n\left(u, \frac{r}{a}, \nu, M\right)$ which is valid for small u by means of the following procedure.

If we take the Fourier inverse of formula (137) we get

$$\bar{\psi}_n\left(\kappa', \frac{r}{a}, \nu, M\right) = \int_{-\infty}^{\infty} \tilde{\psi}_n\left(u, \frac{r}{a}, \nu, M\right) \exp(-i\kappa' u)\, du . \tag{208}$$

The formula (208) is in the first place valid for real κ', but because of formula (207) the integral on the right hand side remains convergent when the quantity κ' becomes a complex number with negative imaginary part. By analytic continuation we can then say that formula (208) is valid in the lower half-plane.

The analytic continuation of $\bar{\psi}_n\left(\kappa', \frac{r}{a}, \nu, M\right)$ into the lower half-plane is given by

$$\bar{\psi}_n \left(\kappa', \frac{r}{a}, \nu, M\right) = F_3(\kappa') \tag{209}$$

where $F_3(\kappa')$ is given by formulae (139) to (142) and (145).

On JH (*see* Fig. 2) we may write

$$\kappa' + \frac{M\nu}{\beta} = \sqrt{s^2 + \left(\frac{M\nu}{\beta}\right)^2} \exp\left\{-i\gamma(s)\right\} \tag{210}$$

$$\kappa' - \frac{M\nu}{\beta} = \sqrt{s^2 + \left(\frac{M\nu}{\beta}\right)^2} \exp\left\{-i(\pi - \gamma(s))\right\} \tag{211}$$

where s is real positive and $\gamma(s)$ is the acute angle given be formula (165).
Hence, using (141), we get on JH

$$\left[\frac{M^2\nu^2}{\beta^2} - \kappa'^2\right]^{\frac{1}{2}} = \sqrt{s^2 + \left(\frac{M\nu}{\beta}\right)^2} \tag{212}$$

and, from (142) and (145)

$$F_3(\kappa') = -\frac{1}{\sqrt{s^2 + \left(\frac{M\nu}{\beta}\right)^2}} \frac{K_n\left\{\frac{r}{a}\sqrt{s^2 + \left(\frac{M\nu}{\beta}\right)^2}\right\}}{K_n'\left\{\sqrt{s^2 + \left(\frac{M\nu}{\beta}\right)^2}\right\}}. \tag{213}$$

If s is very large compared with unity and with $\frac{M\nu}{\beta}$ we can expand (213) in the asymptotic expansion

$$\begin{aligned}F_3(\kappa') \sim \exp\left\{-\left(\frac{r}{a}-1\right)s\right\} \Bigg[\frac{1}{s} &+ \left\{\frac{(4n^2-1)}{8}\frac{a}{r} - \frac{(4n^2+3)}{8} - \frac{1}{2}\left(\frac{r}{a}-1\right)\left(\frac{M\nu}{\beta}\right)^2\right\}\frac{1}{s^2} + \\ &+ \left\{\frac{(16n^4-40n^2+9)}{12}\frac{a^2}{r^2} - \frac{(16n^4+8n^2-3)}{64}\frac{a}{r} + \frac{(16n^4-8n^2+33)}{128} + \right. \\ &+ \left(\frac{(4n^2-1)}{16}\frac{a}{r} - \frac{(4n^2+5)}{8} + \frac{(4n^2+3)}{16}\frac{r}{a}\right)\left(\frac{M\nu}{\beta}\right)^2 + \\ &\left. + \frac{1}{8}\left(\frac{r}{a}-1\right)^2\left(\frac{M\nu}{\beta}\right)^4\right\}\frac{1}{s^3} + \\ &+ \left(\frac{(64n^6-560n^4+1036n^2-225)}{3072}\frac{a^3}{r^3} - \frac{(64n^6-112n^4-84n^2+27)}{1024}\frac{a^2}{r^2}\right. \\ &+ \frac{(64n^6-48n^4+140n^2-33)}{1024}\frac{a}{r} - \frac{(64n^6-368n^4-1364n^2+747)}{3072} +\end{aligned}$$

$$+\left(\frac{(16n^4-40n^2+9)}{256}\frac{a^2}{r^2}-\frac{(48n^4+104n^2-29)}{256}\frac{a}{r}+\right.$$

$$\left.+\frac{(48n^4+136n^2+123)}{256}-\frac{(16n^4-8n^4+33)}{256}\frac{r}{a}\right)\left(\frac{Mv}{\beta}\right)^2+$$

$$+\left(\frac{(4n^2-1)}{64}\frac{a}{r}-\frac{(12n^2+25)}{64}+\frac{(12n^2+25)}{64}\frac{r}{a}+\right.$$

$$\left.-\frac{(4n^2+3)}{64}\frac{r^2}{a^2}\right)\left(\frac{Mv}{\beta}\right)^4-\frac{1}{48}\left(\frac{r}{a}-1\right)^3\left(\frac{Mv}{\beta}\right)^6\Bigg\}\left(\frac{1}{s^4}\right)+ \qquad (214)$$

Now, from (208) we get on using (207) that on JH (Fig. 2)

$$F_3(\kappa')=\tilde{\psi}_{k_2}\left(\kappa',\frac{r}{a},v,M\right)=\int_{\frac{r}{a}-1}^{\infty}\tilde{\psi}_n\left(u,\frac{r}{a},v,M\right)\exp(-su)\,du$$

$$=\exp\left\{-\left(\frac{r}{a}-1\right)s\right\}\int_0^{\infty}\tilde{\psi}_n\left(u+\frac{r}{a}-1,\frac{r}{a},v,M\right)\exp(-su)\,du. \qquad (215)$$

If, for $u>0$, $\tilde{\psi}_n\left(u+\frac{r}{a}-1,\frac{r}{a},v,M\right)$ has the power series expansion

$$\tilde{\psi}_n\left(u+\frac{r}{a}-1,\frac{r}{a},v,M\right)=\sum_{n=0}^{\infty}a_n u^n \qquad (216)$$

then, according to Watson's lemma, we have the asymptotic expansion

$$\int_0^{\infty}\tilde{\psi}_n\left(u+\frac{r}{a}-1,\frac{r}{a},v,M\right)\exp(-su)\,du\sim\sum_{n=0}^{\infty}\frac{n!a_n}{s^{n+1}}. \qquad (217)$$

If we substitute for $\int_0^{\infty}\tilde{\psi}_n\left(u+\frac{r}{a}-1,\frac{r}{a},v,M\right)\exp(-su)\,du$ from (217) into (215) and compare the result term by term with (214) we get expressions for the a_n. We then have the power series expansion

$$\tilde{\psi}_n\left(u,\frac{r}{a},v,M\right)=1$$

$$+\left\{\frac{(4n^2-1)}{8}\frac{a}{r}-\frac{(4n^2+3)}{8}-\frac{1}{2}\left(\frac{r}{a}-1\right)\left(\frac{Mv}{\beta}\right)^2\right\}\left(u-\frac{r}{a}+1\right)$$

$$+\frac{1}{2}\left\{\frac{(16n^4-40n^2+9)}{128}\frac{a^2}{r^2}-\frac{(16n^4+8n^2-3)}{64}\frac{a}{r}+\frac{(16n^4-8n^2+33)}{128}\right.$$

$$\left.+\left(\frac{(4n^2-1)}{16}\frac{a}{r}-\frac{(4n^2+5)}{8}+\frac{(4n^2+3)}{16}\frac{r}{a}\right)\left(\frac{Mv}{\beta}\right)^2+\frac{1}{8}\left(\frac{r}{a}-1\right)^2\left(\frac{Mv}{\beta}\right)^4\right\}\left(u-\frac{r}{a}+1\right)^2$$

$$+\frac{1}{6}\left\{\frac{(64n^6-560n^4+1036n^2-225)}{3072}\frac{a^3}{r^3}-\frac{(64n^6-112n^4-84n^2+27)}{1024}\frac{a^2}{r^2}\right.$$

$$+\frac{(64n^6-48n^4+140n^2-33)}{1024}\frac{a}{r}-\frac{(64n^6-368n^4-1364n^2+747)}{3072}$$

$$+\left(\frac{(16n^4-40n^2+9)}{256}\frac{a^2}{r^2}-\frac{(48n^4+104n^2-29)}{256}\frac{a}{r}\right.$$

$$\left.+\frac{(48n^4+136n^2+123)}{256}-\frac{(16n^4-8n^2+33)}{256}\frac{r}{a}\right)\left(\frac{Mv}{\beta}\right)^2+$$

$$+\left(\frac{(4n^2-1)}{64}\frac{a}{r}-\frac{(12n^2+25)}{64}+\frac{(12n^2+25)}{64}\frac{r}{a}-\frac{(4n^2+3)}{64}\frac{r^2}{a^2}\right)\left(\frac{Mv}{\beta}\right)^4-$$

$$\left.-\frac{1}{48}\left(\frac{r}{a}-1\right)^3\left(\frac{Mv}{\beta}\right)^6\right\}\left(u-\frac{r}{a}+1\right)^3 \qquad (218)$$

for $u > \frac{r}{a}-1$.

If $r = a$ the expression (218) reduces to

$$\tilde{\psi}_n(u, 1, v, M) = 1-\frac{1}{2}u-\frac{1}{2}\left[\frac{(4n^2-3)}{8}+\frac{1}{2}\left(\frac{Mv}{\beta}\right)^2\right]u^2+\frac{1}{6}\left[\frac{(8n^2-3)}{8}+\frac{1}{2}\left(\frac{Mv}{\beta}\right)^2\right]u^4+\dots \qquad (219)$$

for $u > 0$.

In particular, we have

$$\tilde{\psi}_1(u, 1, 0, M) = 1-\frac{1}{2}u-\frac{1}{16}u^2+\frac{5}{48}u^3+\dots \qquad (220)$$

The first three terms of formula (220) should agree with the three terms of formula (54) given in Ward[3]. In fact the coefficients of the third term are different, being $-\frac{1}{16}$ in formula (220) of this paper and $-\frac{11}{32}$ in formula (54) of Ward[3].

APPENDIX C

Derivation of the Expression for $\overline{\Phi}_n\left(\kappa, \frac{r}{a}, \nu_0\right)$

The velocity potential Φ inside the cylinder is assumed to satisfy the boundary conditions,

$$\left(\frac{\partial \Phi}{\partial r}\right)_{r=a} = -a_0 \exp(in\theta) \cos\left(\frac{\pi\kappa z}{a}\right) \exp(i\omega t) \tag{221}$$

and

$$\left(\frac{\partial \Phi}{\partial z}\right)_{z=0} = \left(\frac{\partial \Phi}{\partial z}\right)_{z=l} = 0 \tag{222}$$

on the surface and ends of the cylinder. The internal radius of the cylinder has been taken to be the same as the external diameter on the assumption that the thickness of the cylinder wall is very small compared with the radius. It is quite easy to take different values for the internal and external radius if this is desired.

The velocity potential Φ satisfies the linear partial differential equation

$$\nabla^2 \Phi = \frac{1}{a_0^2} \frac{\partial^2 \phi}{\partial t^2}. \tag{223}$$

The function

$$\Phi = g\left(\frac{r}{a}\right) \exp(in\theta) \cos\left(\frac{\pi\kappa z}{a}\right) \exp(i\omega t) \tag{224}$$

is a solution of the partial differential equation (223) provided that the function $g(s)$ satisfies the ordinary differential equation

$$\frac{d^2 g(s)}{ds^2} + \frac{1}{s}\frac{d}{ds} g(s) + \left\{\nu_0^2 - \pi^2\kappa^2 - \frac{n^2}{s^2}\right\} g(s) = 0 \tag{225}$$

where

$$\nu_0 = \frac{\omega a}{a_0} \tag{226}$$

The differential equation (225) is a Bessel equation. The form in which it is best to write the solution of this equation for our purposes depends on whether $(\nu_0^2 - \pi^2\kappa^2)$ is greater than or less than zero.

The boundary condition (221) is appropriate to the case of standing waves on the surface of the cylinder. These standing waves are equivalent to a super-position of waves travelling in the direction of positive and negative z with speed $\frac{\nu_0 a_0}{\pi\kappa}$ and this speed is supersonic if $(\nu_0^2 - \pi^2\kappa^2)$ is greater than zero, whereas it is subsonic if $(\nu_0^2 - \pi^2\kappa^2)$ is less than zero.

If $\nu_0^2 - \pi^2\kappa^2$ is greater than zero we write the solution of (226) in the form

$$g(s) = A\, J_n\left[s\sqrt{\nu_0^2 - \pi^2\kappa^2}\right] + B\, Y_n\left[s\sqrt{\nu_0^2 - \pi^2\kappa^2}\right] \tag{227}$$

and if $v_0^2 - \pi^2 \kappa^2$ is less than zero we write the solution of (226) in the form

$$g(s) = A\, I_n\,[s\sqrt{\pi^2 \kappa^2 - v_0^2}] + B\, K_n\,[s\sqrt{\pi^2 \kappa^2 - v_0^2}] \tag{228}$$

Since the velocity potential must be finite on the axis of the cylinder, we must have $B = 0$ in both (227) and (228).

The function (224) satisfies the boundary conditions (222) automatically, if $\kappa = \frac{ak}{l}$ and k is an integer. The constants A in formulae (227) and (228) are determined by making the function (224) satisfy the boundary conditions (221). We then get

$$\Phi = \begin{cases} -\dfrac{a\, a_0}{\sqrt{v_0^2 - \pi^2 \kappa^2}} \dfrac{J_n\left[\dfrac{r}{a}\sqrt{v_0^2 - \pi^2 \kappa^2}\right]}{J_n'[\sqrt{v_0^2 - \pi^2 \kappa^2}]} \exp(in\theta) \cos\left(\dfrac{\pi \kappa z}{a}\right) \exp(i\omega t) & \text{for } v_0^2 - \pi^2 \kappa^2 > 0 \\[2ex] -\dfrac{a\, a_0}{\sqrt{\pi^2 \kappa^2 - v_0^2}} \dfrac{I_n\left[\dfrac{r}{a}\sqrt{\pi^2 \kappa^2 - v_0^2}\right]}{I_n'[\sqrt{\pi^2 \kappa^2 - v_0^2}]} \exp(in\theta) \cos\left(\dfrac{\pi \kappa z}{a}\right) \exp(i\omega t) & \text{for } v_0^2 - \pi^2 \kappa^2 < 0 \end{cases} \tag{229}$$

If we write

$$\Phi = a\, a_0\, \overline{\Phi}_n \left(\kappa, \frac{r}{a}, v_0\right) \exp(in\theta) \cos\left(\frac{\pi \kappa z}{a}\right) \exp(i\omega t) \tag{230}$$

then

$$\overline{\Phi}_n \left(\kappa, \frac{r}{a}, v_0\right) = \begin{cases} -\dfrac{1}{\sqrt{v_0^2 - \pi^2 \kappa^2}} \dfrac{J_n\left[\dfrac{r}{a}\sqrt{v_0^2 - \pi^2 \kappa^2}\right]}{J_n'[\sqrt{v_0^2 - \pi^2 \kappa^2}]} \\[2ex] -\dfrac{1}{\sqrt{\pi^2 \kappa^2 - v_0^2}} \dfrac{I_n\left[\dfrac{r}{a}\sqrt{\pi^2 \kappa^2 - v_0^2}\right]}{I_n'[\sqrt{\pi^2 \kappa^2 - v_0^2}]} \end{cases} \tag{231}$$

We may note that

$$J_n'[\sqrt{v_0^2 - \pi^2 \kappa^2}] = 0 \tag{232}$$

when

$$\sqrt{v_0^2 - \pi^2 \kappa^2} = j_{n,m}' \tag{233}$$

or

$$\nu_0 = \sqrt{(j'_{n,m})^2 + \pi^2 \kappa^2} \tag{234}$$

where

$$j'_{n,m} \qquad m = 1,2,\ldots \tag{235}$$

are the zeros of $J'_n(z)$.

The velocity potential (230) becomes indefinitely large for the value (234) of ν_0 so that in physical reality no oscillation satisfying (221) and (222) is possible. This corresponds to a resonant condition within the cylinder.

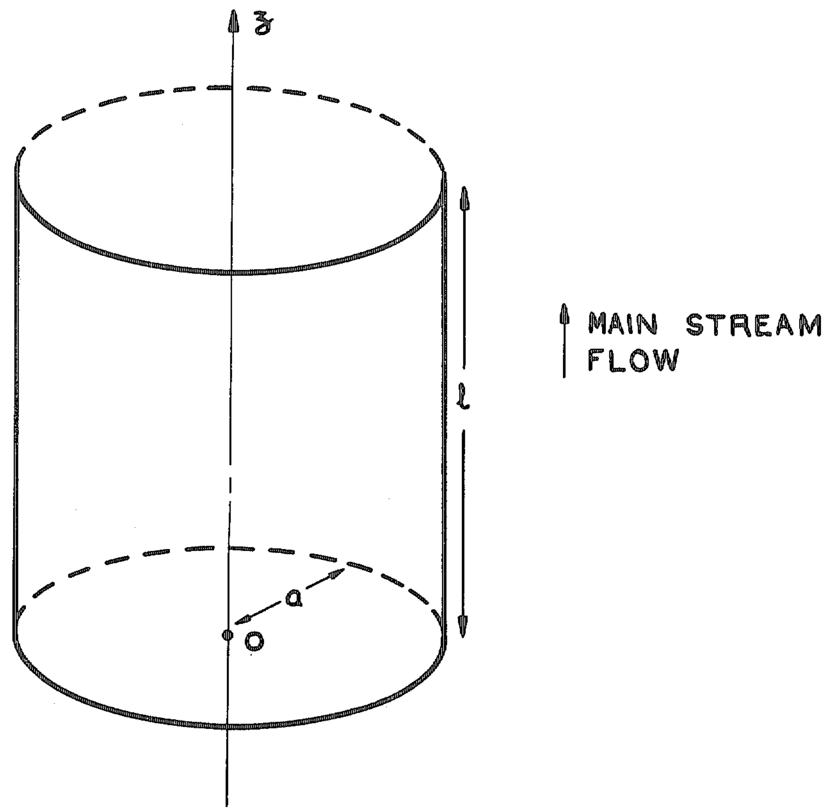

FIG. 1. Diagram of flexible part of the infinite cylinder.

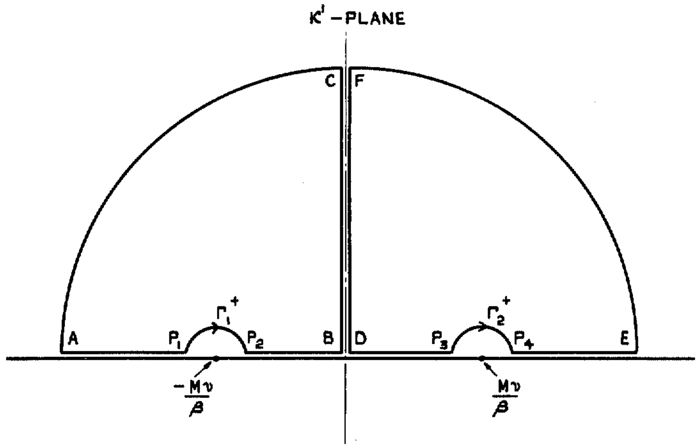

FIG. 2. Contours of integration in K' – plane for subsonic flow.

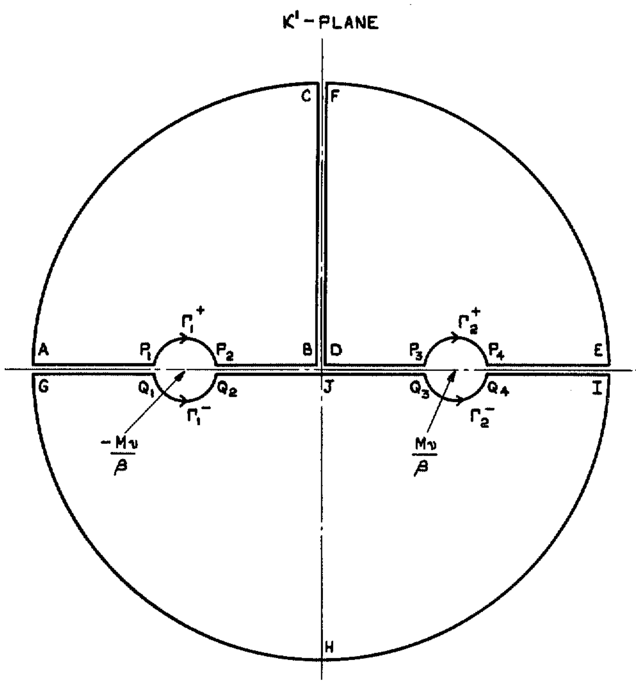

FIG. 3. Contours of integration in K' – plane for supersonic flow.

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