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Loading Conditions following an Automatic Pilot Failure (Elevator Channel)

By

D. R. Puttock, D.C.Ae.

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SUMMARY

A proposal is made for a standard procedure for calculating the critical loading conditions ensuing from an automatic pilot failure in the elevator channel.

General expressions are derived through response theory for the increments in normal acceleration at the C.G. of the aircraft, normal acceleration at the tail and aerodynamic load on the tailplane which result from the sequence of elevator movements assumed to follow a failure. Analysis of these general expressions leads to formulae suitable for assessing the numerical values of the critical loads on the wing and tailplane.

The influence of the sequence of elevator movements on the loading conditions is discussed with reference to an example.

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1 Introduction

It is recognised that complete reliability of an automatic pilot cannot be expected and that the likelihood of a failure must be accepted[2]. Accordingly the relevant design requirement[1] states that aircraft must have a specified ultimate factor under loading conditions ensuing from a sudden movement of each main control surface - when the failure occurs - and recovery action after a suitable time interval. With a failure in the elevator channel, the critical loading conditions are usually associated with the maximum normal acceleration at the C.G. of the aircraft and with the maximum aerodynamic load on its tailplane.

Since the requirement is couched in general terms, any approach which satisfies all the specified conditions may be used to assess the critical loads and accelerations. It is desirable, however, to establish a standard approach, and to this end, Reddaway[3] interpreted the sudden elevator movement of the requirement as instantaneous movement, and, disregarding the recovery action, derived formulae suitable for the calculation of the resulting critical loads. But, while the choice of instantaneous "runaway" movement was a conservative one with respect to the initial stages of the manoeuvre, the omission of the recovery action might in many cases lead to a serious underestimation of the critical loads occurring during the complete sequence of runaway and recovery.

In the present paper, the sudden movement is interpreted as a gradual movement as rapid as the control servomotor will allow, and recovery action is taken into consideration. The investigation is based on response theory.

The response of the aircraft to the assumed elevator movements, and the formulae necessary for the calculation of the critical loading conditions are derived in Appendix I. This Appendix also contains some discussion of the effects of an automatic pilot failure on possible elevator movements. In a further Appendix the formulae for the critical loading conditions are presented in a form suitable for their direct use by the computor. In this presentation extensive use is made of charts.

An example is given to illustrate the type of response following an automatic pilot failure.

2 Details of the investigation

In a rational approach to the problem of determining the loading conditions following an automatic pilot failure, the investigation may be split into three closely linked parts.

(i) The establishment of a general elevator time history to describe the sequence of movements that occur after a failure.

(ii) The derivation of the response of the aircraft to the established elevator movement.

(iii) The analysis of the response to determine in each case the elevator movements which produce the critical loads, and to obtain workable formulae and charts suitable for routine estimation of these loads.

These three parts are described and discussed in the following sub-paragraphs.

2.1 The elevator movement

Consider first the sequence of events in the case of a typical failure of the elevator channel. At the instant of failure the elevator begins to move under the influence of its servomotor. Its angular rate of movement is initially small because of the friction and inertia effects, but soon builds up to a value approaching the maximum angular rate of the servomotor. This movement gives rise to aerodynamic forces on the elevator, and after a while these forces may become large enough to influence the output of the servomotor, and reduce its angular rate. Finally, a state may be reached in which the external aerodynamic forces are of sufficient magnitude to stall the servomotor. Alternatively, there may be stops in the automatic pilot-elevator circuit to limit the total travel of the elevator under automatic control, in which case, the elevator movement may be arrested before the external forces stall the servomotor. Once the elevator is brought to rest, it remains approximately steady until the recovery is taken. If the recovery is under the direct control of the pilot, it will consist in rapid angular movement of the elevator back to its position before failure or perhaps beyond.

Within this picture the elevator movement and the aircraft response are continuously interdependent; this renders the solution of the overall problem a formidable task, and the final complex formulae would hardly be of any practical use. The following assumptions simplify the mathematical treatment without seriously affecting the accuracy of the final results; they are slightly conservative. It is assumed that the time-history of the elevator movement may be considered as consisting of three stages:

(i) "runaway", during which the elevator moves at a constant rate corresponding to the maximum rate of the servomotor.

(ii) "check", when the elevator is arrested at its maximum deflection, either by a stop or by the fact that aerodynamic forces on the elevator have stalled the servomotor, until

(iii) "recovery", during which the elevator is moved back at a constant rate.

With these assumptions the determination of the elevator time-history becomes a rather straight-forward matter. The following quantities are required: the "runaway" rate, maximum deflection during the "check" period, maximum rate and travel during "recovery", and the time of the beginning of "recovery". A simplified method for calculating the elevator deflection at which the servomotor stalls is given in Appendix I para A.6.6. No specific rate and amount of recovery movement are suggested; they depend on the pilot and the characteristics of the aircraft, and discussion as to their numerical values is outside the scope of the paper. In practice the relevant recovery rate would be that likely to be used by the pilot in such an emergency; the amount of movement is more difficult to assess but some guides to its value are given in Appendix I para A.6.9. The time of the beginning of recovery is discussed in para 2.3 and in Appendix I para A.6.7.

The assumed sequence of movements is illustrated diagrammatically in Fig.1.

2.2 Response of the aircraft

A movement of the elevator produces continuous changes in the angle of incidence of the aircraft and causes the aircraft to fly along a curved path and rotate about its lateral axis. The aircraft is therefore subjected to normal and pitching accelerations which vary as the manoeuvre develops, and its tailplane is subjected to variations in aerodynamic loading, due partly to the movement of the elevator and partly to the movement or response of the aircraft itself.

Thus to assess the effects of an automatic pilot failure, it is necessary to derive the response of the aircraft to the sequence of elevator movements which follow the failure. The required solutions of the equations of motion for the assumed type of disturbance have been obtained in general terms by means of Laplace transformations. The mathematical analysis pertinent to this part of the investigation is given in Appendix I paras A.1 to A.5, where expressions are presented for the incremental values (from steady state) of the normal acceleration at the C.G. of the aircraft, the acceleration at the tail and the aerodynamic load on the tailplane. The type of response in acceleration etc. is illustrated in Fig.2. The data for this example are given in Table I.

2.3 The critical loads

The various response formulae of Appendix I paras A.3, A.4 and A.5 may be used to determine the complete time-histories of the loads and accelerations produced by the general elevator time-history assumed in para 2.1. However, from the airworthiness aspect, it is the various local maxima of these quantities, and in particular their absolute values, which are of major interest. It is therefore desirable to analyse the response formulae and obtain general expressions for the magnitudes of the local maxima. The conclusions deduced from such an analysis are presented in Appendix I para A.6.

They indicate that the loads and accelerations produced by movement of the elevator depend directly on its rate and magnitude, and since the movements defining the runaway and check are the greatest possible in the circumstances, the local maxima that occur in these stages, are also the absolute maxima. The recovery stage must be treated somewhat differently for, in addition to the rate and amount of elevator movement, its time of commencement must also be specified. Once the recovery movement has been chosen, its timing must be selected, from airworthiness considerations to give the most critical loading conditions.

The recovery action always reduces the normal acceleration, and it is therefore only necessary to delay the recovery until the acceleration in the runaway and check stages has reached a mathematical maximum to ensure that the absolute maximum acceleration is obtained (cf Fig.2).

On the other hand, the recovery action increases the acceleration at the tail, and the aerodynamic load on the tail. The extent to which they increase depends on the rate and amount of movement and also on its timing. The method of determining the time of recovery for the greatest tailplane load in the recovery is given in Appendix I para A.6.7. A similar approach may be used to determine the time of recovery for the greatest acceleration at the tail; this case is not considered however, since in practice, only the acceleration at the tail associated with absolute maximum tailplane load is normally required for stressing purposes. Fig.3 illustrates a case in which, by proper timing, the greatest tailplane load in the recovery stage is obtained. It should be noted that the greatest possible normal acceleration at the C.G. is not realised; the recovery is made before the acceleration has reached a peak in the check stage.

In general, the maximum tailplane load in the runaway and that in the recovery are of opposite sign, and provided the parameters defining the recovery movement are chosen realistically, these two loads represent the design loads for the automatic pilot failure case. Both may be significant, since they represent rather different centre of pressure conditions, and therefore affect different aspects of the strength of the tailplane.

The formulae for the magnitudes of the various critical maxima are rather complex, and some effort has been made to reproduce them in a workable graphical form. These charts are discussed at the appropriate points in Appendix I para A.6. Finally, in Appendix II, details of the calculations necessary to determine the critical loadings are presented in a form such that the calculations can be carried out directly by a computor.

3 Concluding remarks

This Note presents a rational method for assessing the loads on an aircraft following automatic pilot failure. It is suggested that it might form the basis of a standard procedure for airworthiness calculations.

NOTATION

A, B, C, C_1	see equations (30), (31) (32) and (33)
$a = \partial C_L / \partial\alpha$	
$a_1 = \partial C_L' / \partial\alpha'$	
$a_2 = \partial C_L' / \partial\eta$	(including the effects of tabs if used)
$\bar{B}$, $\bar{C}$	see equation (53)
$b_1 = \partial C_h / \partial\alpha'$	
$b_2 = \partial C_h / \partial\eta$	(including the effects of tabs if used)
C_L	lift coefficient of aircraft
C_L'	lift coefficient of aircraft tailplane
C_h	hinge moment coefficient of the elevator
C_{hs}	hinge moment coefficient of the elevator corresponding to the stalling torque of the servomotor
C_m	pitching moment coefficient of aircraft
c	standard mean chord of wing
$D = \frac{1}{2}\rho V^2 S \frac{a}{W}$	
$\left(\frac{d\eta}{dt}\right)_s$	"runaway" rate
$\left(\frac{d\eta}{dt}\right)_r = -f\left(\frac{d\eta}{dt}\right)_s$	"recovery" rate
f	ratio of recovery rate to runaway rate (considered positive)
G, H, K, L	auxiliary functions equations (9), (10) and (11)
g	gravity constant
$\bar{H}_s$, $\bar{L}_s$	see equation (39)
I	an imaginary form of J (see equation (4))

J	non-dimensional frequency of the longitudinal short period oscillations
$K_a = \dfrac{1}{\left(\frac{R}{J}\right)^2 + 1}$	
k_B	radius of gyration of aircraft about its lateral axis
ℓ	distance from aircraft C.G. to mean quarter chord point of tailplane
M, N	coefficients of transcendental equation in para A.6.31
$m_q = \dfrac{c}{2\ell} \cdot \dfrac{\partial C_m}{\partial \frac{q\ell}{V}}$	damping derivative in pitch
n	coefficient of normal acceleration at the C.G. of aircraft
n'	coefficient of critical normal acceleration at the C.G. of aircraft
$\bar{n}$	coefficient of normal acceleration at the tailplane due to angular acceleration in pitch
$n_t = n + \bar{n}$	coefficient of total normal acceleration at the tailplane
n_t'	coefficient of total normal acceleration at the tailplane associated with P_3'
$P = P_w + P_\eta$	net aerodynamic load on the tailplane
$P_1', P_2', P_1'', P_3, P_3'$	various maxima of the tailplane load, see para A.6.3
Q_1, T_1	coefficients of equation (40)
q	angular velocity of the aircraft in pitch
$\hat{q} = \hat{t}\, q$	non-dimensional angular velocity in pitch
$R = \frac{1}{2}(\nu + \chi + \frac{a}{2})$	non-dimensional damping factor of the pitching oscillations of the aircraft
S	wing area
S'	tailplane area
t	time in seconds
$\hat{t} = \dfrac{\mu\ell}{V}$	unit of aerodynamic time (seconds)
V	true airspeed of aircraft

W	weight of aircraft
w	velocity component in a vertical plane perpendicular to the initial flight path (positive down)
$\hat{w} = \frac{w}{V}$	incremental incidence at the wing
$\alpha (\simeq \frac{w}{V})$	angle of incidence at the wing
α'_{eff}	effective angle of incidence at tail
γ	an integer, see equation (43)
$\delta = \frac{Wc}{2g\rho S k_B^2} \cdot \frac{S' \ell}{Sc} \cdot a_2$	elevator effectiveness
ε	angle of downwash at the tail
η	elevator angle
$\bar{\eta}$	elevator angle from steady state to limit stops
η_r	maximum angular movement of the elevator in the recovery
$\eta_{r_{crit}}$	see equation (61)
η_s	angular movement of the elevator in the runaway (see para A.6.6)
$\mu = \frac{W}{g\rho S \ell}$	relative density of the aircraft
ν	rotary damping coefficient
ρ	air density
$\tau = \frac{t}{\hat{t}}$	non-dimensional aerodynamic time
χ	downwash damping derivative
$\omega = \frac{-Wc}{2g\rho S k_B^2} \cdot \frac{\partial C_m}{\partial \alpha}$	static stability coefficient

NOTATION (Contd)

Suffices

r	associated with the recovery
s	associated with point at which the runaway is checked
w	due to effective angle of incidence at the tail
η	due to the elevator angle

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APPENDIX I

The mathematical analysis and detailed discussion

A.1 Equations of motion

The non-dimensional linearised differential equations of longitudinal motion of an aircraft may be written (cf Ref.4).

Vertical Forces:
$$\frac{d\hat{w}}{d\tau} + \frac{a}{2}.\hat{w} - \hat{q} = 0 \qquad (1a)$$

Moments:
$$\chi \frac{d\hat{w}}{d\tau} + \omega\hat{w} + \frac{d\hat{q}}{d\tau} + \nu\hat{q} = -\delta.\eta \qquad (1b)$$

Eliminating $\hat{q}$ from equations (1a) and (1b)

$$\frac{d^2\hat{w}}{d\tau^2} + \left(\chi + \nu + \frac{a}{2}\right)\frac{d\hat{w}}{d\tau} + \left(\omega + \frac{a}{2}\nu\right)\hat{w} = -\delta.\eta \qquad (2)$$

$$\frac{d^2\hat{w}}{d\tau^2} + 2R\frac{d\hat{w}}{d\tau} + (R^2 + J^2)\hat{w} = -\delta.\eta \qquad (3)$$

where $R = \frac{1}{2}\left(\chi + \nu + \frac{a}{2}\right)$ is the non-dimensional damping factor of the longitudinal short period oscillations of the aircraft

(4)

and $J = \sqrt{\left(\omega + \frac{a}{2}\nu\right) - R^2}$ is the non-dimensional frequency factor of the longitudinal short period oscillations of the aircraft.
(See para A.6.5 for cases in which J is imaginary).

A.2 Basic solutions

The immediate problem is to solve equation (3) for the sequence of elevator movements defined in para 2.1, and illustrated diagrammatically in Fig.1. Thence such response quantities as the normal acceleration, n, and the aerodynamic load on the tailplane, P, may be determined (cf Ref.4). The elevator movement is composed of three separate stages and the solution to each stage is obtained separately.

Stage I

Here the elevator movement is:

$$\eta = \left(\frac{d\eta}{dt}\right)_s t$$

$$= \left(\frac{d\eta}{d\tau}\right)_s \tau = \left(\frac{\eta_s}{J\tau_s}\right) J\tau \tag{5}$$

where $\tau = \frac{t}{\hat{t}}$ and $\hat{t} = \frac{W}{\rho g S V}$

Note: For the present analysis, and for the presentation of results and graphs, it is more convenient to use the non-dimensional representation of time $J\tau_s$ etc. instead of t_s etc. and this form is used throughout.

The solution to equation (3) for the elevator movement defined by equation (5) is:

$$\hat{w} = -\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right) K_a^2 \left(\frac{J\tau}{K_a} + 2\,\frac{R}{J}\left(e^{-\frac{R}{J}J\tau}\cos J\tau - 1\right) + \left(\frac{R^2}{J^2} - 1\right) e^{-\frac{R}{J}J\tau}\sin J\tau\right) \tag{6}$$

thence

$$\frac{d\hat{w}}{d\tau} = -\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right) J.K_a \left(1 - e^{-\frac{R}{J}J\tau}\cos J\tau - \frac{R}{J}\, e^{-\frac{R}{J}J\tau}\sin J\tau\right) \tag{7}$$

and

$$\frac{d^2\hat{w}}{d\tau^2} = -\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right) J^2\, e^{-R\tau} \sin J\tau \tag{8}$$

Introducing the special functions of time:

$$G = K_a^2 \left(\frac{J\tau}{K_a} + 2\,\frac{R}{J}\left(e^{-\frac{R}{J}J\tau}\cos J\tau - 1\right) + \left(\frac{R^2}{J^2} - 1\right) e^{-\frac{R}{J}J\tau}\sin J\tau\right) \tag{9}$$

$$K = K_a\left(1 - e^{-\frac{R}{J}J\tau}\cos J\tau - \frac{R}{J}\, e^{-\frac{R}{J}J\tau}\sin J\tau\right) \tag{10}$$

$$\left.\begin{aligned} H &= e^{-R\tau}\cos J\tau \\ L &= e^{-R\tau}\sin J\tau \end{aligned}\right\} \tag{11}$$

equations (6), (7) and (8) become:

$$\hat{w} = -\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)G \tag{12}$$

$$\frac{d\hat{w}}{d\tau} = -\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)JK \tag{13}$$

and

$$\frac{d^2\hat{w}}{d\tau^2} = -\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)J^2L \tag{14}$$

Stage II

Here

$$\eta = \eta_s \tag{15}$$

and to obtain this condition, an additional elevator movement $-\left(\frac{\eta_s}{J\tau_s}\right)J\tau$ may be superimposed on the existing movement commencing at $J\tau_s$, see Fig.1. Then

$$\hat{w} = -\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)(G - G_s) \tag{16}$$

$$\frac{d\hat{w}}{d\tau} = -\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)J\,(K - K_s) \tag{17}$$

and

$$\frac{d^2\hat{w}}{d\tau^2} = -\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)J^2\,(L - L_s) \tag{18}$$

where the suffices to G, K and L denote that $J\tau$ is replaced by $J(\tau - \tau_s)$ in equations (9), (10) and (11) respectively. These new terms appear, of course, only when $J\tau > J\tau_s$.

Stage III

Here

$$\eta = \eta_s - f\left(\frac{d\eta}{dt}\right)_s\left(t - t_r\right)$$

$$= \eta_s - f\left(\frac{\eta_s}{J\tau_s}\right)J\,(\tau - \tau_r) \tag{19}$$

where (-f) is the ratio of recovery rate to runaway rate.

Equation (19) may be satisfied beyond $J\tau_r$ by superposing a further elevator movement $-f\left(\frac{\eta_s}{J\tau_s}\right) J(\tau - \tau_r)$ from this point. Then

$$\hat{w} = -\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)(G - G_s - f\,G_r) \tag{20}$$

$$\frac{d\hat{w}}{d\tau} = -\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right) J\,(K - K_s - f\,K_r) \tag{21}$$

and

$$\frac{d^2\hat{w}}{d\tau^2} = -\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right) J^2\,(L - L_s - f\,L_r) \tag{22}$$

The suffix r denotes that $J\tau$ is replaced by $J(\tau - \tau_r)$ everywhere in the functions so suffixed.

A.3 Normal acceleration at the C.G. (cf Ref.4)

$$n = D\,\hat{w} \tag{23}$$

where

$$D = \tfrac{1}{2}\rho V^2 S.\frac{a}{W}$$

Thus in Stage III

$$n = -D\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)(G - G_s - f\,G_r) \tag{24}$$

The corresponding equations for Stages II and I may be obtained by deleting $f\,G_r$, and $f\,G_r$ and G_s respectively.

A.4 Acceleration at the tail (cf Ref.4)

A.4.1 Due to pitching alone

$$\bar{n} = -D\left(\frac{2}{\mu a}\;\frac{d^2\hat{w}}{d\tau^2} + \frac{1}{\mu}\;\frac{d\hat{w}}{d\tau}\right) \tag{25}$$

Therefore in Stage III

$$\bar{n} = D\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)\left(\frac{2J^2}{\mu\,a}\,(L - L_s - f\,L_r) + \frac{J}{\mu}\,(K - K_s - f\,K_r)\right) \tag{26}$$

Equations for Stages II and I may be obtained as in A.3 above.

A.4.2 Total

$$n_t = n + \overline{n} \tag{27}$$

A.5 Aerodynamic load on the tailplane (cf Ref.4)

$$P = A\left(B\hat{w} + C\frac{d\hat{w}}{d\tau} + a_2\,\eta\right) \tag{28}$$

$$= AB\left(\hat{w} + \frac{C_1}{J}\,\frac{d\hat{w}}{d\tau}\right) + A\,a_2\,\eta \tag{29}$$

where

$$A = \tfrac{1}{2}\rho V^2 S' \tag{30}$$

$$B = \left(1 - \frac{d\varepsilon}{d\alpha} + \frac{a}{2\mu}\right)a_1 \tag{31}$$

$$C = \left(1 + \frac{d\varepsilon}{d\alpha}\right)\frac{a_1}{\mu} \tag{32}$$

$$C_1 = \frac{CJ}{B} \tag{33}$$

For the present problem it is simpler to write

$$P = P_w + P_\eta \tag{34}$$

where

$$P_w = AB\left(\hat{w} + \frac{C_1}{J}\,\frac{d\hat{w}}{d\tau}\right) \tag{35a}$$

= load due to effective angle of incidence at the tail (α')

and

$$P_\eta = A\,a_2\,\eta \tag{35b}$$

= load due to elevator deflection.

Thus, in Stage I

$$P_w = -AB\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)(G + C_1K) \tag{36a}$$

$$P_\eta = A\,a_2\left(\frac{\eta_s}{J\tau_s}\right)J\tau \tag{36b}$$

in Stage II

$$P_w = -AB\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)\left((G - G_s) + C_1 (K - K_s)\right) \tag{37a}$$

$$P_\eta = A a_2 \eta_s \tag{37b}$$

and in Stage III

$$P_w = -AB\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)\left((G - G_s - f G_r) + C_1 (K - K_s - f K_r)\right) \tag{38a}$$

$$P_\eta = -A a_2 f \left(\frac{\eta_s}{J\tau_s}\right) J(\tau - \tau_r) + A a_2 \eta_s \tag{38b}$$

A.6 Detailed discussion

A.6.1 Introduction

To facilitate this discussion, attention is drawn to Fig.2. This figure illustrates the type of response which is produced by the assumed elevator time-history. The "crosses" are associated with points of discontinuity in the elevator movement, and the chain dotted curves immediately following these points indicate what the response would have been had the elevator movement not been altered. The symbols in the figure illustrate the notation.

A.6.2 Normal acceleration at the C.G.

The response in n is shown in Fig.2(b). If the recovery action is delayed sufficiently long, the acceleration builds up to a local maximum in Stage II, denoted n'. The equation which characterises the response in n in Stage II is:

$$n = -D\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)(G - G_s) \tag{24}$$

and the time of occurrence of n' (i.e. when $\frac{dn}{d\tau} = 0$) is given by the first root beyond $J\tau_s$ of the equation

$$J\tau = J\tau' = \tan^{-1}\left\{\frac{\frac{R}{J}\bar{L}_s - (\bar{H}_s - 1)}{\bar{L}_s + \frac{R}{J}(\bar{H}_s - 1)}\right\} \tag{39}$$

where $\bar{H}_s = e^{R\tau_s} \cos J\tau_s$

and $\bar{L}_s = e^{R\tau_s} \sin J\tau_s$

Equation (39) represents a function of $\frac{R}{J}$ and $J\tau_s$ only, which is independent of the magnitude (or rate) of the runaway movement. Thus $J\tau'$ may be expressed graphically in terms of $\frac{R}{J}$ and $J\tau_s$, see Fig.5(a), and this information may be combined with equation (24) to give n' in terms of $\frac{R}{J}$ and $J\tau_s$, see Fig.4(a).

A.6.3 Aerodynamic load on the tailplane

The response in P is shown in Fig.2(d). Three local maxima, P_1', P_2' and P_3 may be expected.

A.6.31 P_1'

The condition for this local maximum (i.e. when $\frac{dP}{d\tau} = 0$ in Stage I) is given by the first positive root of

$$\cos J\tau + Q_1 \sin J\tau = T_1 e^{R\tau} \tag{40}$$

where $$Q_1 = \frac{R}{J} - \frac{C_1}{K_a}$$

$$T_1 = 1 - \frac{a_2 J^2}{B\delta K_a} .$$

This root, $J\tau_1'$, does not depend on the rate or amount of elevator movement. In some cases it will be found that $J\tau_1'$ is greater than $J\tau_s$ and this means that the runaway is checked before the mathematical maximum occurs, and instead, a smaller, non-mathematical, maximum - also designated P_1' - occurs at $J\tau_s$ (this case is not illustrated).

The significant parameters associated with P_1' are $\frac{R}{J}$, C_1, a_2 and δ and consequently P_1' cannot be easily represented graphically. Instead, $J\tau_1'$ must first be determined. In this connection Fig.6 may be used with $M = \frac{1}{T_1}$ and $N = \frac{Q_1}{T_1}$ (use of this graph is explained in Appendix II). It then remains to calculate P at $J\tau = J\tau_1'$. The relevant equation for P is

$$P = P_W + P_\eta$$

where

$$P_W = - AB \frac{\delta}{J^2} \left(\frac{\eta_s}{J\tau_s}\right) (G + C_1 K) \tag{36a}$$

and

$$P_\eta = A a_2 \left(\frac{\eta_s}{J\tau_s}\right) J\tau \tag{36b}$$

Graphs of K and G, Figs.7 and 8 respectively, may be used in the estimation of P_W at $J\tau = J\tau_1'$.

Note: It may be worthwhile to determine beforehand whether P_1' occurs at or before $J\tau_s$. This may be done by finding the sign of $\frac{dP}{d\tau}$ at $J\tau = J\tau_s$ i.e. applying the following inequalities:

$$\left.\begin{array}{lllll} \text{If} & & & & \\ & (K + C_1\, L)_{J\tau=J\tau_s} < \dfrac{a_2 J^2}{B\delta} & \text{then} & J\tau_1' < J\tau_s & \\ \text{and if} & & & & \\ & (K + C_1\, L)_{J\tau=J\tau_s} > \dfrac{a_2 J^2}{B\delta} & \text{then} & J\tau_1' = J\tau_s & \end{array}\right\} \tag{41}$$

A.6.32 P_2'

P_2' is the value of the local maximum that occurs in Stage II if the recovery action is delayed sufficiently long. It always occurs before n' if the Stage is long enough for both to occur. The position of P_2' (i.e. when $\frac{dP}{d\tau} = 0$ in Stage II) is given by the first root beyond $J\tau_s$ of the equation

$$J\tau = J\tau_2' = \tan^{-1}\left\{\frac{Q_1\overline{L}_s - (\overline{H}_s - 1)}{\overline{L}_s + Q_1(\overline{H}_s - 1)}\right\} \tag{42}$$

The significant parameters are $\frac{R}{J}$, C_1 and $J\tau_s$ and, in Fig.5, $J\tau_2'$ is given as a function of $\frac{R}{J}$ and $J\tau_s$ for four values of C_1. A property of equation (42) is that

$$\left(J\tau_2'\right)_{J\tau_s=\gamma\pi} = \left(J\tau_2'\right)_{J\tau_s=\pi} + (\gamma - 1)\,\pi \qquad \gamma = 1,\ 2 \text{ etc.} \tag{43}$$

This relationship may be used to extend Fig.5 beyond $J\tau_s = \frac{3\pi}{2}$. The values of $J\tau_2'$ from equation (42) may be combined with equation (37a) to obtain P_{W_2}', the contribution of P_W to P_2'; the results are given in Fig.4. The corresponding value of $P\eta$ is $A\, a_2\, \eta_s$.

A.6.33 P_3

P_3 is the value of the local maximum that occurs in Stage III. In this region the overall response in tailplane load may be considered as a combination of two responses, one due to the runaway and check, and the other due to the recovery (this response is identical in character to that in Stage I - of equations (36) and (38)). The maximum P_3 may be a mathematical maximum, satisfying the condition $\frac{dP}{d\tau} = 0$, or a non-mathematical maximum if the recovery motion is arrested before the condition $\frac{dP}{d\tau} = 0$ is reached.

It is clear that the value of P_3 depends on the value of $J\tau_r$ and on the parameters which define the recovery action, and since, in practice, the chief interest lies in the maximum value of P_3 (see para 2) general formulae for P_3 and its time of occurrence are not required. In connection with the determination of the maximum value of P_3 (see para A.6.7) however, the quantity P_1'' is needed.

This quantity is the value of the local maximum of the tailplane load due to the recovery action itself, and since it is identical in character to the local maximum in the runaway stage, the procedure of para A.6.31 may be used in its calculation. Thus if the time of occurrence of P_1'' relative to $J\tau_r$ is $J\tau_1''$ then $J\tau_1''$ is equal to $J\tau_1'$ (see para A.6.31 equation (40)) or $-\eta_r \Big/ f\left(\frac{\eta_s}{J\tau_s}\right)$ (see para A.6.9) whichever of the two is less and

$$P_{W_1}'' = AB\frac{\delta}{J^2} f\left(\frac{\eta_s}{J\tau_s}\right)(G + C_1 K)_{J\tau=J\tau_1''} \tag{44}$$

and

$$P_{\eta_1}'' = -A\, a_2\, f\left(\frac{\eta_s}{J\tau_s}\right) J\tau_1'' \tag{45}$$

whence

$$P_1'' = P_{W_1}'' + P_{\eta_1}'' \quad .$$

A.6.4 Acceleration at the tail

The responses in $\bar{n}$ and n_t are illustrated in Fig.2(c). The total acceleration at the tail associated with the first local maximum of the tailplane load is usually very small, but that associated with P_3 is much greater and may provide inertia relief. It may be calculated from

$$n_t = (n + \bar{n})_{J\tau=J\tau_r+J\tau_1''} \tag{46}$$

$$= -D\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)\left(G - G_s - f\, G_r\right)_{J\tau=J\tau_r+J\tau_1''} \quad .$$

$$+ D\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)\left(\frac{2J^2}{\mu a}(L - L_s - f\, L_r) + \frac{J}{\mu}(K - K_s - f\, K_r)\right)_{J\tau=J\tau_r+J\tau_1''} \tag{47}$$

A.6.5 Modifications when J is imaginary

A.6.51 To the basic formulae of paragraphs A.2, 3, 4 and 5

So far it has been assumed that equation (4) yields a real value of J i.e. that $\omega + \frac{1}{2}(a\nu) > R^2$. However, cases may arise in which this inequality is not satisfied. In these cases equation (3) may be written:

$$\frac{d^2\hat{w}}{d\tau^2} + 2R\frac{d\hat{w}}{d\tau} + (R^2 - I^2)\,\hat{w} = -\delta.\eta \qquad (48)$$

The solution to this equation may be obtained from equations (6), (7) and (8) by making a number of modifications, namely

	(a)	Replace J, J^2 etc. i.e. $J\tau$ becomes $I\tau$	by	I, I^2 etc.
but	(b)	Replace $\sin J\tau$	by	$\sinh I\tau$
	(c)	" $\cos J\tau$	by	$\cosh I\tau$
	(d)	" $\left(\frac{R}{J}\right)^2 + 1$	by	$\left(\frac{R}{I}\right)^2 - 1$
	(e)	" $\left(\frac{R}{J}\right)^2 - 1$	by	$\left(\frac{R}{I}\right)^2 + 1$

A.6.52 To the equations of paragraph A.6.1 to A.6.4

When J is imaginary a number of alterations also have to be made to the equations and conditions for maxima.

(i) n' (A.6.2) occurs after an infinite time, and equation (39) and Fig.4(a) no longer apply. Equation (24) still applies however, but can be simplified to

$$n' = -D\frac{\delta}{I^2} \cdot K_a.\eta_s \qquad (49)$$

where

$$K_a = \frac{1}{\left(\frac{R}{I}\right)^2 - 1}$$

(ii) the condition for P_1' (A.6.31, equation (40)) is modified in accordance with the rules of A.6.51. It becomes

$$\cosh I\tau + Q_1 \sinh I\tau = T_1\, e^{R\tau} \qquad (50)$$

where

$$Q_1 = \frac{R}{I} - \frac{C_1}{K_a} \qquad T_1 = 1 - \frac{a_2 I^2}{B\delta K_a}$$

Figs.6, 7 and 8 no longer apply, but equation (36) is only modified. Equation (50) may give a value of $I\tau_1'$ which is greater than $I\tau_s$, if so the implication is the same as in the J case. In this respect the inequalities, equation (41), suitably modified, still apply.

(iii) P_2' (A.6.32) is reached after an infinite time, and equations (42) and (43) and Fig.4 no longer apply. Instead

$$P_2' = -AB\frac{\delta}{I^2}K_a\,\eta_s + A\,a_2\,\eta_s \tag{51}$$

(iv) the procedure for calculating P_3 is unchanged, but the equation for $I\tau_1''$ is modified as in (ii) above.

A.6.6 Estimation of the amount of elevator movement to stall the servomotor

The stalling torque of the motor is usually known, but the external forces on the elevator (hinge moments) depend on the response of the aircraft, which, in turn, depends on the amount of elevator movement. Thus to calculate the exact amount of elevator movement to stall the servomotor, η_s, a process of "trial and error" would have to be adopted. To ease this labour, two simplified methods are suggested for finding a good approximation to η_s.

(i) From asymptotic conditions:

The general expression for the hinge moment of the elevator in a longitudinal manoeuvre is (cf Ref.4)

$$C_h = \overline{B}\,\hat{w} + \overline{C}\,\frac{d\hat{w}}{d\tau} + b_2\,\eta \tag{52}$$

where

$$\left.\begin{aligned} \overline{B} &= B\,\frac{b_1}{a_1} \\ \overline{C} &= C\,\frac{b_1}{a_1} \end{aligned}\right\} \tag{53}$$

and

The asymptotic conditions depend solely on η_s and the way in which this deflection is reached has no effect whatever. Thus equations (16) and (17) may be used to determine the asymptotic values of $\hat{w}$ and $\frac{d\hat{w}}{d\tau}$ (assuming that the recovery action is not taken).

They are

$$\left.\begin{aligned} \hat{w} &= -\frac{\delta}{J^2}K_a\,\eta_s \\ \frac{d\hat{w}}{d\tau} &= 0 \end{aligned}\right\} \tag{54}$$

Thus the asymptotic value of C_h, if the recovery action is not taken, is

$$C_h = \left(b_2 - \overline{B}\frac{\delta}{J^2}K_a\right)\eta_s \equiv C_{h_s} \qquad \text{when the servomotor stalls.}$$

Thus

$$\eta_s = \frac{C_{h_s}}{\left(b_2 - \overline{B}\frac{\delta}{J^2}K_a\right)} \qquad (55)$$

(ii) From conditions arising from instantaneous elevator movement.

With instantaneous movement of the elevator, the response in $\hat{w}$ and $\frac{d\hat{w}}{d\tau}$ is initially zero, and the hinge moment at $J\tau = 0$ is

$$C_h = b_2\,\eta_s \equiv C_{h_s} \qquad \text{when the servomotor stalls}$$

and

$$\eta_s = \frac{C_{h_s}}{b_2} \qquad (56)$$

Method (i) is conservative when $\overline{B}$ is positive, and method (ii) is conservative when $\overline{B}$ is negative. It is suggested that when determining η_s the sign of $\overline{B}$ should first be examined. The conservative value of η_s (larger than that occurring in practice) may then be calculated.

A.6.7 Choice of $J\tau_r$ for the greatest tailplane load in the recovery

$J\tau_r$ is to be chosen to produce the maximum value of P_3 ($\equiv P_3'$). Since the principle of superposition applies, the response in P in Stage III may be considered as a combination of two basic responses each of which reaches a local maximum at a calculable time, and it is only necessary to ensure that both these maxima occur at the same time ($J\tau_3$) to obtain P_3'. The maxima are P_2' and P_1'', and they occur at $J\tau_2'$ and $J\tau_r + J\tau_1''$ respectively (see paras A.6.32 and A.6.33). These two times must be identical, thus

$$J\tau_r = (J\tau_2' - J\tau_1'') \qquad (57)$$

and

$$J\tau_3 = J\tau_2' \qquad (58)$$

also

$$P_3' = P_2' + P_1'' \qquad (59)$$

In the calculations leading to P_3' the times $J\tau_2'$ and $J\tau_1''$ are needed but $J\tau_r$ is not, in fact it is only required to determine the precise time at which the recovery is made. A case in which P_3' is obtained is illustrated in Fig.3.

In the I case, the greatest acceleration is obtained after an infinite time, but for the majority of practical cases a very close approximation to this acceleration is obtained after a few seconds. P_2' is similarly affected, and thus for finite recovery times equation (59) is slightly conservative.

A.6.8 Comparison of Figs.2 and 3

The response curves in Fig.3 relate to an automatic pilot failure and recovery in which the recovery is timed to produce the greatest tailplane load. The runaway and recovery movements are the same as in Fig.2, but the time of recovery is necessarily earlier. The dotted curves relate to an elevator time-history, in which the runaway and recovery movements are instantaneous, and η_r is equal to η_s ; the recovery is again timed for the maximum tailplane load.

The full curves are similar to those in Fig.2. The maximum value of n is lower than in Fig.2, but n_t is almost unchanged. The greatest value of P is, of course, higher but not much so. The slope of the response curve in P in the neighbourhood of P_2' is very low, and thus P_3 is insensitive to changes in $J\tau_r$. However when $\frac{R}{J}$ is lower there may be considerable variations in P_3' with $J\tau_r$.

Comparison of the full and dotted curves in Fig.3 gives some insight into the effects of the rate of elevator movement. Although the initial responses in n are considerably different the maximum values obtained are not materially changed. The initial responses in $\dot{n}$, n_t and P in Stage I are very different but only in the case of P is this of any consequence. If the first maximum of P is to be predicted accurately, it is essential that a rate of runaway is used which is close to the practical one. For the same reason, the rate of recovery must also be chosen carefully. In the present example, the maximum load for the instantaneous movement case is 38% greater than that from the movement defined in para 2.1 even though η_r is assumed to be less.

A 6.9 Choice of η_r

In certain cases the response in P in Stage III rises to a mathematical maximum and then falls. In these cases P_3 does not depend on η_r and movement of the elevator beyond this deflection does not increase P_3 (cf point X in Fig.3). The condition for this state of affairs is

$$J\tau_1'' < -\frac{\eta_r}{f\left(\frac{\eta_s}{J\tau_s}\right)} \tag{60}$$

If this inequality is not satisfied, the implication is that P_3 is a non-mathematical maximum, and, as such, is a function of η_r.

In the first step towards the selection of a numerical value of η_r it is advisable to calculate $\eta_{r_{crit}}$ where

$$\eta_{r_{crit}} = -f\left(\frac{\eta_s}{J\tau_s}\right)J\tau_1'' \tag{61}$$

For the calculation of the maximum acceleration the values of η_r and f are not required.

APPENDIX II

Computational Procedure

1 Introduction

1.1 This Appendix contains details of the procedure for calculating the maximum normal acceleration and maximum aerodynamic tailplane load that ensue from an automatic pilot failure and subsequent recovery.

1.2 The list of numerical data required is given in para 2 and particulars of the preliminary calculations are given in paras 3 and 4. Finally, in para 5, formulae are given for the response in acceleration and tailplane load and for the maximum acceleration and maximum tailplane load; particulars of various charts which may be used to facilitate the calculations are also given.

1.3 The response quantities are expressed in terms of G, K and L, which are functions of $J\tau$. Since $J\tau$ is a measure of time (see para 3), the numerical values of these quantities can be calculated for any time during the manoeuvre, and complete time-histories of the response quantities may be obtained.

1.4 Two formulae are given for the maximum tailplane load, for P_1' and P_3'; they refer to conditions at different stages of the failure and recovery, and represent rather different centre of pressure cases, thereby affecting different aspects of the strength of the tailplane. These formulae contain contributions due to the incidence of the tailplane (suffix w) and the angular displacement of the elevator (suffix η) and should be distributed according to the usual "x" and "η" chordwise distributions respectively. A formula is also given for the normal acceleration at the tail associated with the load P_3'; the acceleration associated with the load P_1' is usually negligible.

1.5 The accelerations and loads given by the formulae of para 5 are incremental values, and total values are obtained by adding to them the steady accelerations and loads which prevail before the failure.

1.6 A numerical example illustrating the procedure is given in para 6. The data for this example are taken from Table I.

2 List of data required

(cf List of Symbols)

a		$(m_q)_{\text{less tail}}$	
a_1		S	$(ft)^2$
a_2		S'	$(ft)^2$
b_1		V	(ft/sec T.A.S.)
b_2		W	lb
c	(ft)	ρ	$(slugs/ft^3)$
$\frac{\partial C_m}{\partial \alpha}$		C_{h_s}	(usually positive if the elevator movement is negative)

(cf List of Symbols) (Contd)

$\frac{d\varepsilon}{d\alpha}$		η_r	(radians)	see footnote
$g = 32.2$ ft/sec^2		$\bar{\eta}$	(radians)	
k_B	(ft)	$\left(\frac{d\eta}{dt}\right)_s$	(radians/sec)	
ℓ	(ft)	$\left(\frac{d\eta}{dt}\right)_r$	(radians/sec)	

3 Basic quantities to be evaluated

3.1

$$\mu = \frac{W}{g\rho S\ell} \qquad \hat{t} = \frac{\mu\ell}{V}$$

$$B = \left(1 - \frac{d\varepsilon}{d\alpha} + \frac{\varepsilon}{2\mu}\right) a_1 \qquad \bar{B} = B\,\frac{b_1}{a_1}$$

$$C = \left(1 + \frac{d\varepsilon}{d\alpha}\right)\frac{a_1}{\mu} \qquad D = \tfrac{1}{2}\rho V^2\,\frac{a}{W/S}$$

$$F = \frac{WS'}{Sa} \qquad \frac{\partial C_m}{\partial\alpha} = \left(\frac{\partial C_m}{\partial\alpha}\right)_{\text{less tail}} - \frac{S'\ell}{Sc}\left(1 - \frac{d\varepsilon}{d\alpha}\right) a_1$$

$$\omega = -\frac{Wc}{2g\rho S{k_B}^2}\cdot\frac{\partial C_m}{\partial\alpha} \qquad \delta = \frac{Wc}{2g\rho S{k_B}^2}\cdot\frac{S'\ell}{Sc}\cdot a_2$$

$$\nu = \nu_{\text{tail}} + \nu_{\text{less tail}} \qquad \nu_{\text{tail}} = \tfrac{1}{2}\,\frac{S'\ell^2}{S{k_B}^2}\cdot a_1$$

$$\nu_{\text{less tail}} = -\frac{\ell^2}{{k_B}^2}.(m_q)_{\text{less tail}} \qquad \chi = \frac{d\varepsilon}{d\alpha}\cdot\nu_{\text{tail}}$$

$$R = \tfrac{1}{2}\left(\nu + \chi + \tfrac{1}{2}a\right) \qquad J = \sqrt{\left(\omega + \tfrac{1}{2}a\nu\right) - R^2}$$

$$\text{or} \quad I = \sqrt{R^2 - \left(\omega + \tfrac{1}{2}a\nu\right)}$$

$$C_1 = \frac{CJ}{B} \qquad K_a = \frac{1}{\left(\frac{R}{J}\right)^2 + 1} \quad \text{(cf Fig.10)}$$

$$Q_1 = \frac{R}{J} - \frac{C_1}{K_a} \qquad T_1 = 1 - \frac{a_2 J^2}{B\delta K_a}$$

Some discussion of the numerical values of these quantities is given in para 2 of the main text. The quantities are considered negative when the elevator is deflected upwards.

3.2 In addition to these quantities the following auxiliary functions will also have to be evaluated (for the range of $J\tau$ corresponding to the duration of the failure and subsequent recovery) if the time-history in acceleration or tailplane load is required.

$$H = e^{-\frac{R}{J}J\tau} \cos J\tau$$

$$L = e^{-\frac{R}{J}J\tau} \sin J\tau \qquad \text{(cf Fig.9)}$$

$$G = K_a^{\,2}\left(\frac{J\tau}{K_a} + 2\frac{R}{J}(H-1) + \left(\left(\frac{R}{J}\right)^2 - 1\right)L\right) \qquad \text{(cf Fig.8)}$$

$$K = K_a\left(1 - H - \frac{R}{J}L\right) \qquad \text{(cf Fig.7)}$$

$$K_a = \frac{1}{\left(\frac{R}{J}\right)^2 + 1} \qquad \text{(cf Fig.10)}$$

where $J\tau = \frac{Jt}{\hat{t}}$, and t is time in seconds

3.3 When the suffices s and r are used with these auxiliary functions it merely means that the arguments of the trigonometrical and exponential terms are changed from $J\tau$ to $J(\tau - \tau_s)$ and $J(\tau - \tau_r)$ respectively in the suffixed functions, but it should be noted that these functions only appear when $J\tau$ is greater than $J\tau_s$ and $J\tau$ is greater than $J\tau_r$ respectively. Thus if $J\tau$ is less than $J\tau_s$ the response equations will not contain any suffixed auxiliary functions, if $J\tau$ is greater than $J\tau_s$ but less than $J\tau_r$ the response equations will contain auxiliary function with the suffix s in addition to the unsuffixed function, and if $J\tau$ is greater than $J\tau_r$ the response equations will contain auxiliary functions with suffix r in addition to the unsuffixed functions and functions with the suffix s.

3.4 The values of the auxiliary functions at specified values of $J\tau$ are also required in the direct calculation of the maximum accelerations and tailplane loads. Here the graphs 4, 5, 6, 7, 8, 9 and 10 may ease the labour involved.

3.5 In the rare cases where J is imaginary ($= iI$ where $i = \sqrt{-1}$) a number of modifications must be made to the basic quantities and auxiliary functions presented above, namely:

(i) replace J, J^2 etc. by I, I^2 etc.
whence $J\tau$ becomes $I\tau$

but (ii) replace $\sin J\tau$ by $\sinh I\tau$

(iii) " $\cos J\tau$ " $\cosh I\tau$

(iv) replace $\left(\frac{R}{J}\right)^2 + 1$ by $\left(\frac{R}{I}\right)^2 - 1$

and (v) " $\left(\frac{R}{J}\right)^2 - 1$ " $\left(\frac{R}{I}\right)^2 + 1$

Thus when J is imaginary, K_a and H become respectively

$$\frac{1}{\left(\frac{R}{I}\right)^2 - 1} \qquad \text{and} \qquad e^{-\frac{R}{I}I\tau} \cosh I\tau$$

These modifications also apply to the formulae in the following paragraph. Para 5 is divided into two sections to simplify the treatment of the J and I cases within the paragraph.

4 Further quantities to be evaluated

(i)
$$f = -\left(\left(\frac{d\eta}{dt}\right)_r \Big/ \left(\frac{d\eta}{dt}\right)_s\right)$$

(ii) If $\bar{B}$ is positive then

$$\eta_s = \bar{\eta}$$

or $\dfrac{C_{h_s}}{\left(b_2 - \bar{B}\frac{\delta}{J^2}K_a\right)}$ radians, whichever is the less.

But if $\bar{B}$ is negative then

$$\eta_s = \bar{\eta}$$

or $\dfrac{C_{h_s}}{b_2}$ radians, whichever is the less.

(iii)
$$J\tau_s = \frac{J\eta_s}{\hat{t}\left(\frac{d\eta}{dt}\right)_s}$$

$$\bar{H}_s = e^{\frac{R}{J}J\tau_s} \cos J\tau_s$$

$$\bar{L}_s = e^{\frac{R}{J}J\tau_s} \sin J\tau_s$$

5 Formulae for maxima and the response formulae

A similar layout is used in each of the following sections, and where applicable, formulae are given in turn for the maximum value of the particular quantity, its time of occurrence (to be obtained from the condition for maximum), and its response throughout the sequence of failure and recovery. Strictly speaking the full response formula applies only when $J\tau$ is greater than $J\tau_r$ and for lower values of $J\tau$, deletions of the suffixed auxiliary functions must be made in accordance with the instructions of para 3.3 above.

The maximum values of the various quantities may be determined in two ways. They may be calculated directly from the appropriate formulae, using the formulae for their times of occurrence, or they may be obtained from the response curves. With regard to the direct calculation of the maxima, a number of graphs Figs.4-10 are introduced to reduce the labour involved.

In para 5.2, containing information for the treatment of the I case, the only formulae given are for the maximum values of the various quantities.

5.1 J case

5.11 Coefficient of normal acceleration at the C.G.

The maximum acceleration is

$$n' = -D\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)(G - G_s)_{J\tau=J\tau'}$$

where $J\tau'$ is the first root beyond $J\tau_s$ of

$$J\tau = \tan^{-1}\left\{\frac{\left(\frac{R}{J}\right)\overline{L}_s - (\overline{H}_s - 1)}{\overline{L}_s + \left(\frac{R}{J}\right)(\overline{H}_s - 1)}\right\}$$

In most cases n' and $J\tau'$ may be obtained from Figs.4(a) and 5(a) respectively.

The complete time-history is

$$n = -D\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)(G - G_s - f\,G_r)$$

and n' is obtained when $J\tau_r$ is assumed equal to or greater than $J\tau'$.

5.12 Net aerodynamic load on the tailplane

Two maxima should be considered.

(a) P_1' , $\dfrac{\text{which occurs at } J\tau_1'}{\text{(a download if negative)}}$

$$P_1' = P_{w_1}' + P_{\eta_1}'$$

where

$$P_{w_1}' = -\,DFB\,\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)(G + C_1\ K)_{J\tau=J\tau_1'}$$

$$P_{\eta_1}' = DFa_2\left(\frac{\eta_s}{J\tau_s}\right)J\tau_1'$$

and $J\tau_1'$, expressed in radians, is either the first positive root of

$$\cos J\tau + Q_1\ \sin J\tau = T_1\ e^{\frac{R}{J}J\tau}$$

or $J\tau_s$ whichever of the two is less. The last equation may be solved graphically with the aid of Fig.6 (putting $M = \frac{1}{T_1}$ and $N = \frac{Q_1}{T_1}$) as follows: draw a circle through the origin with co-ordinates of its centre $(\frac{1}{2}M, \frac{1}{2}N)$ and then draw a straight line from the origin to the point of intersection of the circle and the appropriate $\frac{R}{J}$ curve, extending it to the peripheral scale; the required value of $J\tau_1'$ (in degrees) may be read directly from this scale. If there is no point of intersection then $J\tau_1' = J\tau_s$; if there are two intersections, the one corresponding to the lowest value of $J\tau$ should be considered as the required root of the equation. Figs.7 and 8 may be used to evaluate G and K at $J\tau = J\tau_1'$.

(b) P_3' , $\dfrac{\text{which occurs at } J\tau_2'}{\text{(an upload if positive)}}$

$$P_3' = P_{w_2}' + P_{w_1}'' + P_{\eta_2}' + P_{\eta_1}''$$

where

$$P_{w_2}' = -\,DFB\,\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)\Big(G - G_s + C_1\ (K - K_s)\Big)_{J\tau=J\tau_2'}$$

$$P_{w_1}'' = DFB\,\frac{\delta}{J^2}\,f\left(\frac{\eta_s}{J\tau_s}\right)\Big(G + C_1\ K\Big)_{J\tau=J\tau_1''}$$

$$P_{\eta_2}' = DFa_2\eta_s$$

$$P_{\eta_1}'' = -\,DFa_2 f\left(\frac{\eta_s}{J\tau_s}\right)J\tau_1''$$

$J\tau_2'$. expressed in radians is the first root beyond $J\tau_s$ of

$$J\tau = \tan^{-1}\left\{\frac{Q_1\bar{L}_s - (\bar{H}_s - 1)}{\bar{L}_s + Q_1(\bar{H}_s - 1)}\right\}$$

and $J\tau_1''$ is equal to $J\tau_1'$ (see (a) above) or $-\eta_r \Big/ f\left(\frac{\eta_s}{J\tau_s}\right)$ whichever of the two is less.

Figs.4, 5, 7 and 8 may be used to evaluate $J\tau_2'$, P_{W_2}' and G and K at $J\tau = J\tau_1''$ respectively.

The complete time-history is

$$P = P_W + P_\eta$$

where

$$P_W = - DFB\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)\left\{(G - G_s - f\,G_r) + Q_1(K - K_s - f\,K_r)\right\}$$

and

$$P_\eta = DFa_2\left(\frac{\eta_s}{J\tau_s}\right)\left\{J\tau - J\,(\tau - \tau_s) - fJ\,(\tau - \tau_r)\right\}$$

and the maxima P_1' and P_3' are obtained when $J\tau_r = J\tau_2' - J\tau_1''$ (see (b) above).

5.13 <u>Coefficient of normal acceleration at the tail</u>

The acceleration associated with P_3' is

$$n_t' = - D\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)(G - G_s - f\,G_r)_{J\tau=J\tau_2'}$$

$$+ D\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)\left(\frac{2J^2}{\mu\,a}(L - L_s - f\,L_r) + \frac{J}{\mu}(K - K_s - f\,K_r)\right)_{J\tau=J\tau_2'}$$

where $J\tau_r = J\tau_2' - J\tau_1''$.

For the value of $J\tau_2'$ and $J\tau_1''$, see para 5.12 (b). Figs.7, 8 and 9 may be used to evaluate G, G_s etc. at $J\tau = J\tau_2'$.

$J\tau_2'$, expressed in radians is the first root beyond $J\tau_s$ of

$$J\tau = \tan^{-1}\left\{\frac{Q_1\bar{L}_s - (\bar{H}_s - 1)}{\bar{L}_s + Q_1(\bar{H}_s - 1)}\right\}$$

and $J\tau_1''$ is equal to $J\tau_1'$ (see (a) above) or $-\eta_r \Big/ f\left(\frac{\eta_s}{J\tau_s}\right)$ whichever of the two is less.

Figs.4, 5, 7 and 8 may be used to evaluate $J\tau_2'$, F_{W_2}' and G and K at $J\tau = J\tau_1''$ respectively.

The complete time-history is

$$P = P_W + P_\eta$$

where

$$P_W = -DFB\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)\left\{(G - G_s - f\,G_r) + C_1(K - K_s - f\,K_r)\right\}$$

and

$$P_\eta = DFa_2\left(\frac{\eta_s}{J\tau_s}\right)\left\{J\tau - J(\tau - \tau_s) - fJ(\tau - \tau_r)\right\}$$

and the maxima P_1' and P_3' are obtained when $J\tau_r = J\tau_2' - J\tau_1''$ (see (b) above).

5.13 Coefficient of normal acceleration at the tail

The acceleration associated with P_3' is

$$n_t' = -D\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)(G - G_s - f\,G_r)_{J\tau = J\tau_2'}$$

$$+ D\frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)\left(\frac{2J^2}{\mu\,a}(L - L_s - f\,L_r) + \frac{J}{\mu}(K - K_s - f\,K_r)\right)_{J\tau = J\tau_2'}$$

where $J\tau_r = J\tau_2' - J\tau_1''$.

For the value of $J\tau_2'$ and $J\tau_1''$, see para 5.12 (b). Figs.7, 8 and 9 may be used to evaluate G, G_s etc. at $J\tau = J\tau_2'$.

The complete time-history is

$$n_t = n + \bar{n}$$

where

$$n = - D \frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)(G - G_s - f\, G_r)$$

and

$$\bar{n} = D \frac{\delta}{J^2}\left(\frac{\eta_s}{J\tau_s}\right)\left(\frac{2J^2}{\mu\, a}(L - L_s - f\, L_r) + \frac{J}{\mu}(K - K_s - f\, K_r)\right)$$

5.2 I Case

5.21 Coefficient of normal acceleration at the C.G.

The maximum is

$$n' = - D \frac{\delta}{I^2} K_a\, \eta_s$$

K_a may be estimated from Fig.10 (n' occurs after an infinite time)

5.22 Aerodynamic load on the tailplane

The maxima are

(i) P_1' (which occurs at $I\tau_1'$)

$$P_1' = P_{w_1}' + P_{\eta_1}'$$

where

$$P_{w_1}' = - DFB \frac{\delta}{I^2}\left(\frac{\eta_s}{I\tau_s}\right)(G + C_1\, K)_{I\tau = I\tau_1'}$$

$$P_{\eta_1}' = DFa_2\left(\frac{\eta_s}{I\tau_s}\right) I\tau_1'$$

and $I\tau_1'$, expressed in radians, is the first positive root of

$$\left(\frac{1 + Q_1}{2}\right)e^{I\tau} + \left(\frac{1 - Q_1}{2}\right)e^{-I\tau} = T_1\, e^{\frac{R}{I}I\tau_s}$$

or $I\tau_s$ whichever of the two is the less.

(ii) P_3' (which occurs after an infinite time)

$$P_3' = P_{w_2}' + P_{w_1}' + P_{\eta_2}' + P_{\eta_1}''$$

where

$$P_{w_2}' = - DFB \frac{\delta}{I^2} K_a \eta_s$$

$$P_{w_1}'' = DFB \frac{\delta}{I^2} f\left(\frac{\eta_s}{I\tau_s}\right) (G + C_1 K)_{I\tau = I\tau_1''}$$

$$P_{\eta_2}' = DFa_2 \eta_s$$

$$P_{\eta_1}'' = - DFa_2 f\left(\frac{\eta_s}{I\tau_s}\right) I\tau_1''$$

and $I\tau_1''$ is equal to $I\tau_1'$ or $- \eta_s \Big/ f\left(\frac{\eta_s}{I\tau_s}\right)$ whichever of the two is the less.

5.23 Coefficient of normal acceleration at the tail associated with P_3'

$$n_t = n' + D \frac{\delta}{I^2} f\left(\frac{\eta_s}{I\tau_s}\right) \left(\frac{2J^2}{\mu a} L + \frac{J}{\mu} K\right)_{I\tau = I\tau_1''}$$

6 Numerical example

In this example use is made of the formulae and charts of para 5. The data is contained in Table I.

6.1 From the data in Table I (cf para 4)

(i) $f = 4$

(ii) $\frac{C_{h_s}}{b_2} = -0.1265$ radians, i.e. less than $\bar{\eta}$ so that

$\eta_s = -0.1265$ radians.

(iii) $J\tau_s = 2.6178$ radians.

6.2 Maximum normal acceleration (cf para 5.11)

From Fig.4(a), with $\frac{R}{J} = 0.815$ and $J\tau_s = 2.6178 = 0.833\pi$,

$$\frac{n'}{D \frac{\delta}{J^2} \eta_s} = -0.625 \quad \text{and} \quad \frac{n'}{D} = -\frac{35.93}{3.816^2} \times (-0.1265) \times 0.625 = \underline{0.195}$$

finally $\quad n' = 14.75 \times 0.195 = 2.88$.

From Fig.5(a)

$J\tau' = 283^\circ = 4.939$ radians so that the maximum occurs after

$\frac{4.939 \times \hat{t}}{J}$ secs, i.e. <u>1.83 secs.</u>

6.3 Maxima of the aerodynamic tailplane load (cf para 5.12)

(i) P_1'

From Table I, $Q_1 = -0.0332$ and $T_1 = 0.2383$, and thus, from Fig.6 the required root of the equation $\cos J\tau + Q_1 \sin J\tau = T_1 e^{\frac{R}{J} J\tau}$ is $J\tau = 56^\circ$, i.e. less than $J\tau_s$ so that $J\tau_1' = 56^\circ = 0.977$ radians.

From Figs.7 and 8 K and G at $J\tau = 56^\circ$ are 0.27 and 0.10 respectively.

Thus

$$\frac{P_{w_1}'}{DF} = -2.39 \times \frac{35.93}{3.816^2} \times \left(-\frac{0.1265}{2.6178}\right)(0.10 + 0.511 \times 0.27)$$

$$= \underline{0.069}$$

$$\frac{P_{\eta_1}'}{DF} = 2.7 \times \left(-\frac{0.1265}{2.6178}\right) \times 0.977$$

$$= -0.128$$

and

$$\frac{P_1'}{DF} = +0.069 - 0.128 = \underline{-0.059}$$

so that $P_1' = 23860 \times 0.059$ lb $= 1410$ lb; it occurs after $\frac{0.977 \hat{t}}{J}$ secs, i.e. <u>0.36</u> secs.

(ii) P_3'

By interpolation of Fig.5(a), with C_1 = 0.511

$$J\tau_2' = 243^\circ = 4.241 \text{ radians}$$

By interpolation of Fig.4(a)

$$\frac{P_{w_2}'}{DFB \frac{\delta}{J^2} \eta_s} = -0.64 \quad \text{and} \quad \frac{P_{w_2}'}{DF} = -2.39 \times \frac{35.93}{3.816^2} \times (-0.1265) \times 0.64 = \underline{0.478}$$

At the same time

$$P_{\eta_2}' = +2.7 \times (-0.1265) = \underline{-0.342}$$

Since

$$\frac{-\eta_r}{f\left(\frac{\eta_s}{J\tau_s}\right)} = 62.1^\circ, \quad J\tau_1'' = J\tau_1' = 56^\circ$$

and

$$\frac{P_{w_1}''}{DF} = -f\frac{P_{w_1}'}{DF} = -4 \times 0.069 = \underline{-0.276}$$

also

$$\frac{P_{\eta_1}''}{DF} = -f\frac{P_{\eta_1}''}{DF} = 4 \times 0.128 = \underline{0.512}$$

Finally $\frac{P_3'}{DF}$ = 0.478 - 0.342 - 0.276 + 0.512 = $\underline{0.372}$ and

P_3' = 23860 × 0.372 lb = 8,900 lb. This maximum occurs after $\frac{4.241\ \hat{t}}{J}$ secs, i.e. $\underline{1.57 \text{ secs}}$.

6.4 <u>Acceleration at the tail associated with P_3' (cf para 5.13)</u>

From para 6.4

$$J\tau_r = J\tau_2' - J\tau_1'' = 187^\circ = 3.26 \text{ radians}$$

From Figs. 7, 8 and 9

K, G and L at $J\tau = 243^{\circ}$ are 0.6, 1.97 and -0.028 respectively.

K_s, G_s and L_s at " (ie $J(\tau - \tau_s) = 93^{\circ}$) " 0.475, 0.348 and 0.263 "

and K_r, G_r " L_r " " (ie $J(\tau - \tau_r) = 56^{\circ}$) " 0.27, 0.10 " 0.37 "

so that

$$\frac{n_t'}{D} = -\frac{35.93}{3.816^2} \times \left(-\frac{0.1265}{2.6178}\right)(1.97 - 0.348 - 4 \times 0.1)$$

$$+ \frac{35.93}{3.816^2} \times \left(-\frac{0.1265}{2.6178}\right)\Big(0.492\,(-0.028 - 0.263 - 4 \times 0.37)$$

$$+ 0.2943\,(0.6 - 0.475 - 4 \times 0.27)\Big)$$

$$= 0.146 + 0.137 = \underline{0.283}$$

and

$$n_t' = 0.283 \times 14.75 = 4.18 .$$

TABLE I

a	= 4.57	η_s	=	-7.25°	= -0.1265 rads
a_1	= 3.0	$\bar{\eta}$	=	-10°	= -0.1745 rads
a_2	= 2.7	η_r	=	12°	= 0.2094 rads
b_1	= -0.1	$\left(\frac{d\eta}{dt}\right)_s$	=	-7.5°/sec	= -0.1308 rads/sec
b_2	= -0.3	$\left(\frac{d\eta}{dt}\right)_r$	=	30°/sec	= 0.5232 rads/sec
B	= 2.39	$J\tau_s$	=	150°	= 2.6178 rads
$\bar{B}$	= -0.080	$J\tau_r$	=	300°	= 5.24 rads (Fig.2)
C_1	= 0.511		=	187°	= 3.26 rads (Fig.3)
R	= 3.11	D	=	14.75	
J	= 3.816	DF	=	23,860 lb	
$\frac{R}{J}$	= 0.815				
$\hat{t}$	= 1.41 secs				
μ	= 13				
δ	= 35.93				
C_{h_s}	= 0.038				

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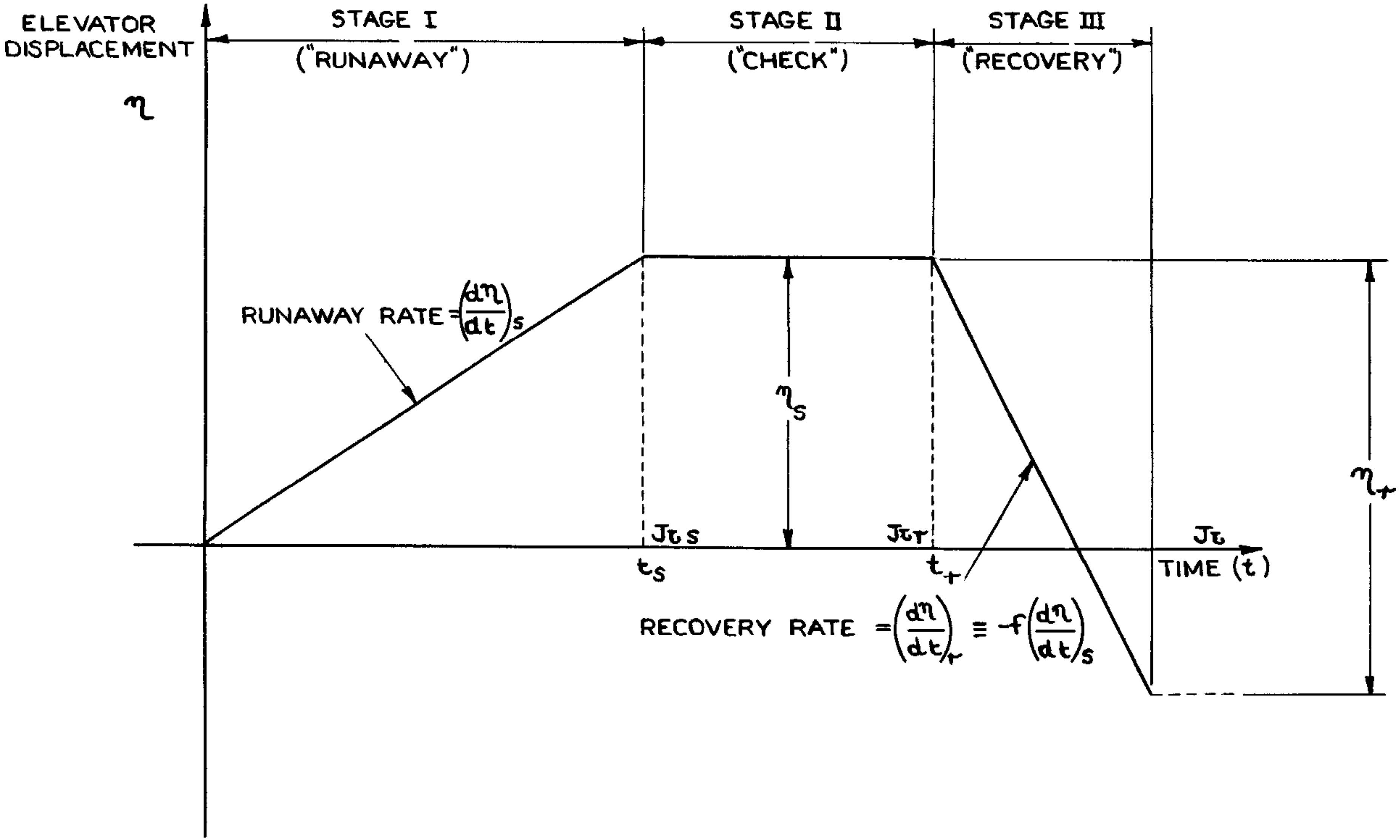


FIG 1. THE ASSUMED ELEVATOR TIME-HISTORY.

FIG.1.

FIG 2 (a - d)

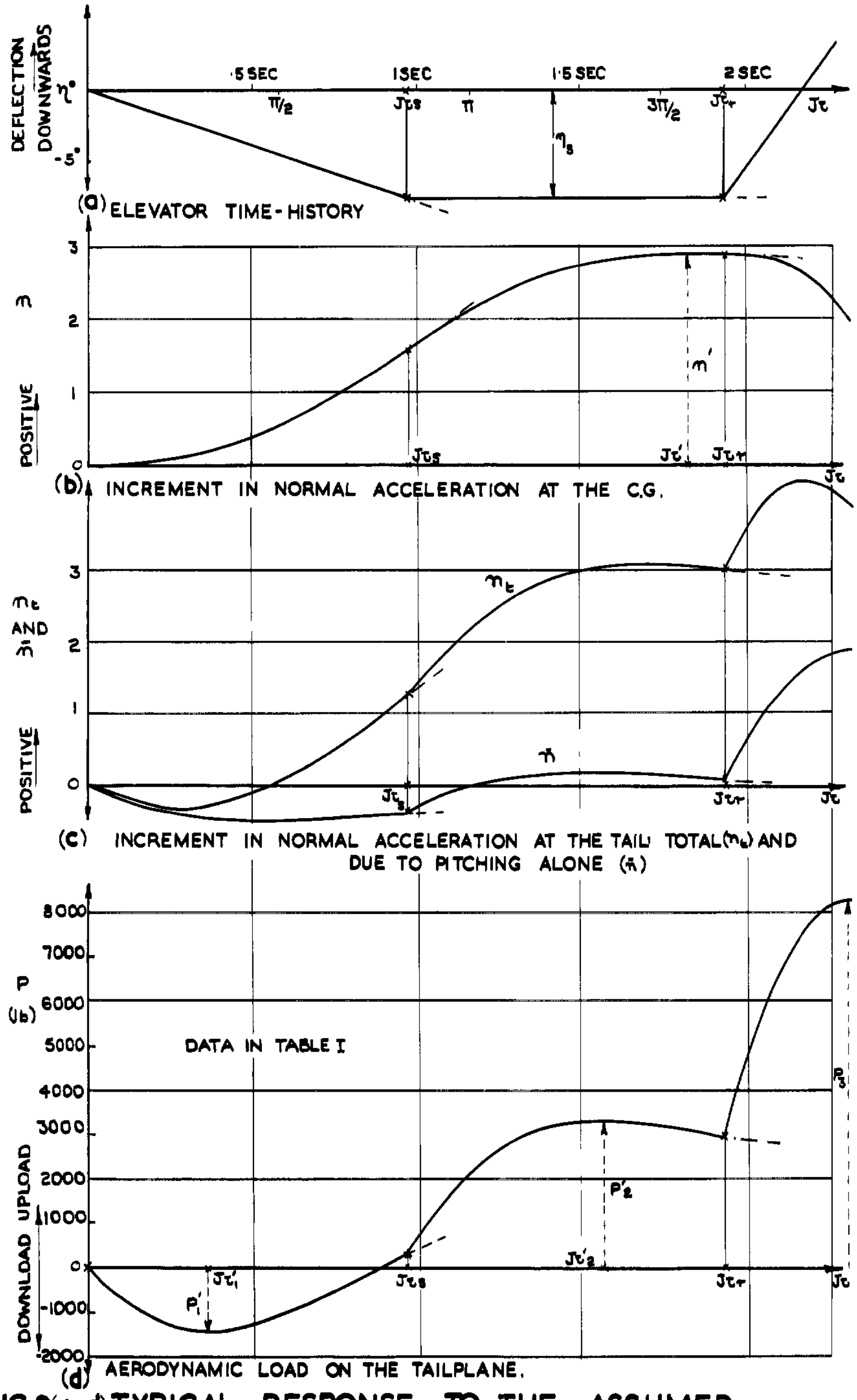


FIG 2(a-d) TYPICAL RESPONSE TO THE ASSUMED ELEVATOR TIME-HISTORY.

FIG 3.(a–c)

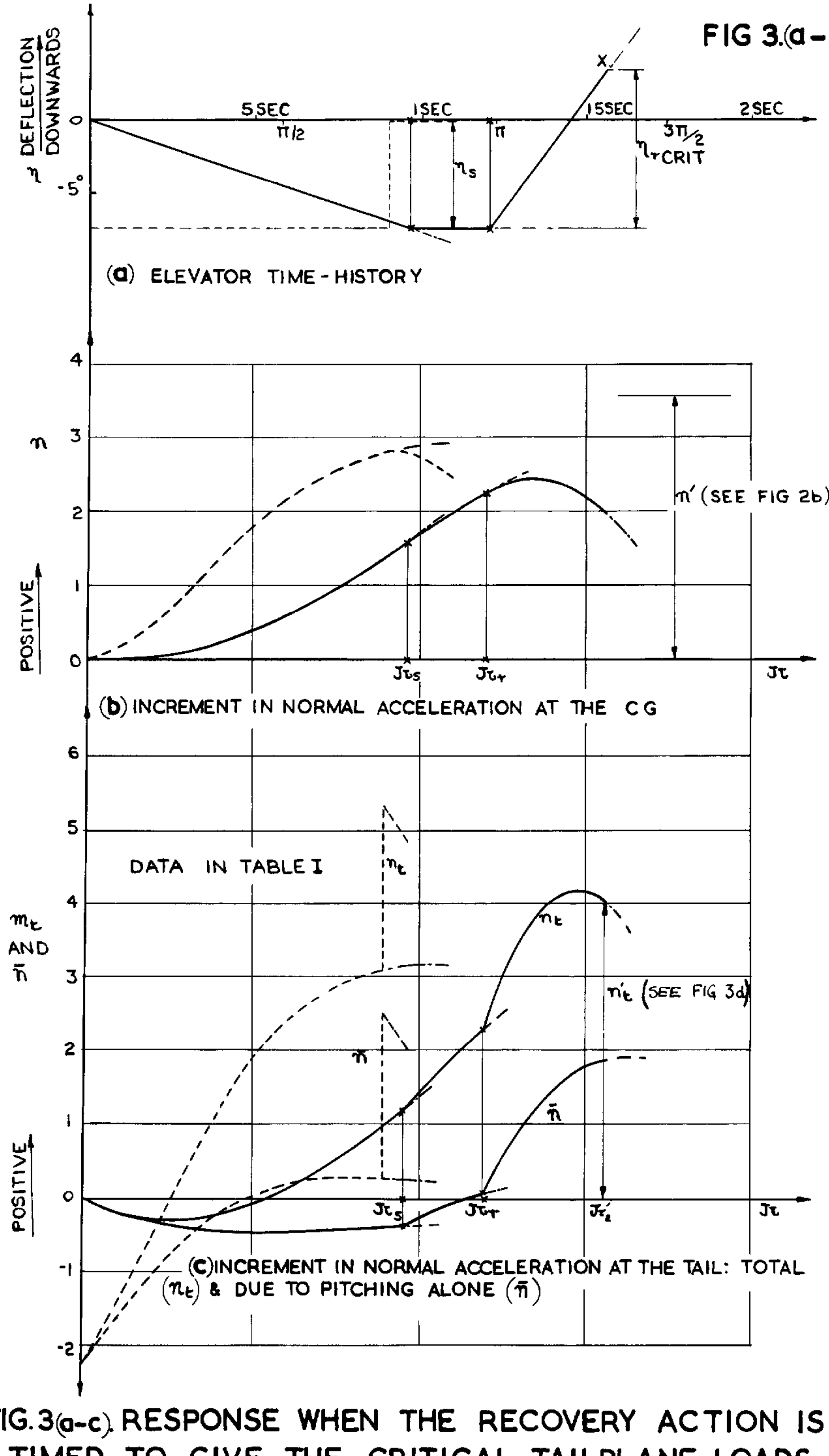


FIG. 3(a–c). RESPONSE WHEN THE RECOVERY ACTION IS TIMED TO GIVE THE CRITICAL TAILPLANE LOADS.

FIG 3 (a & d).

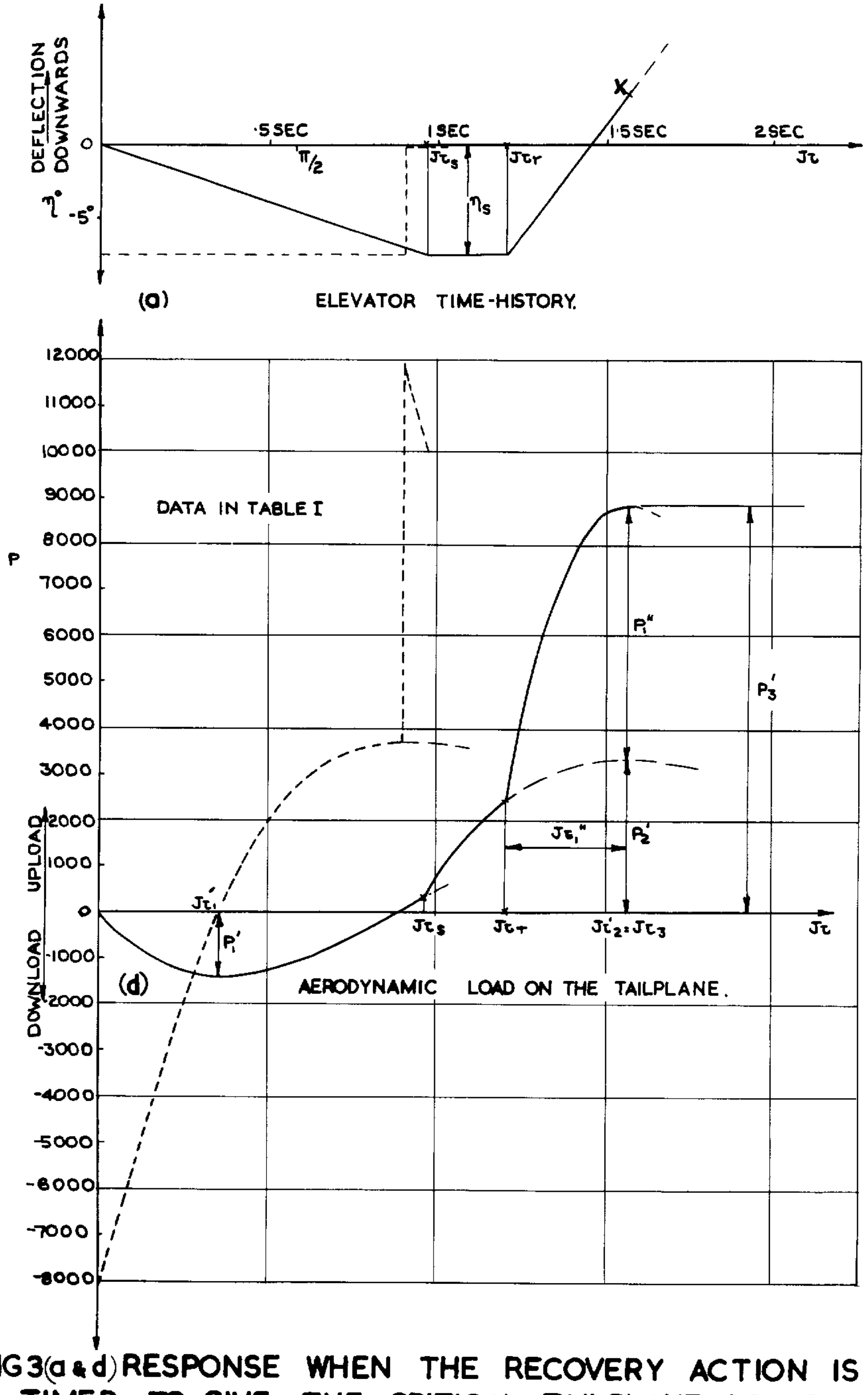


FIG 3 (a & d) RESPONSE WHEN THE RECOVERY ACTION IS TIMED TO GIVE THE CRITICAL TAILPLANE LOADS (CONT)

FIG 4 (a & b)

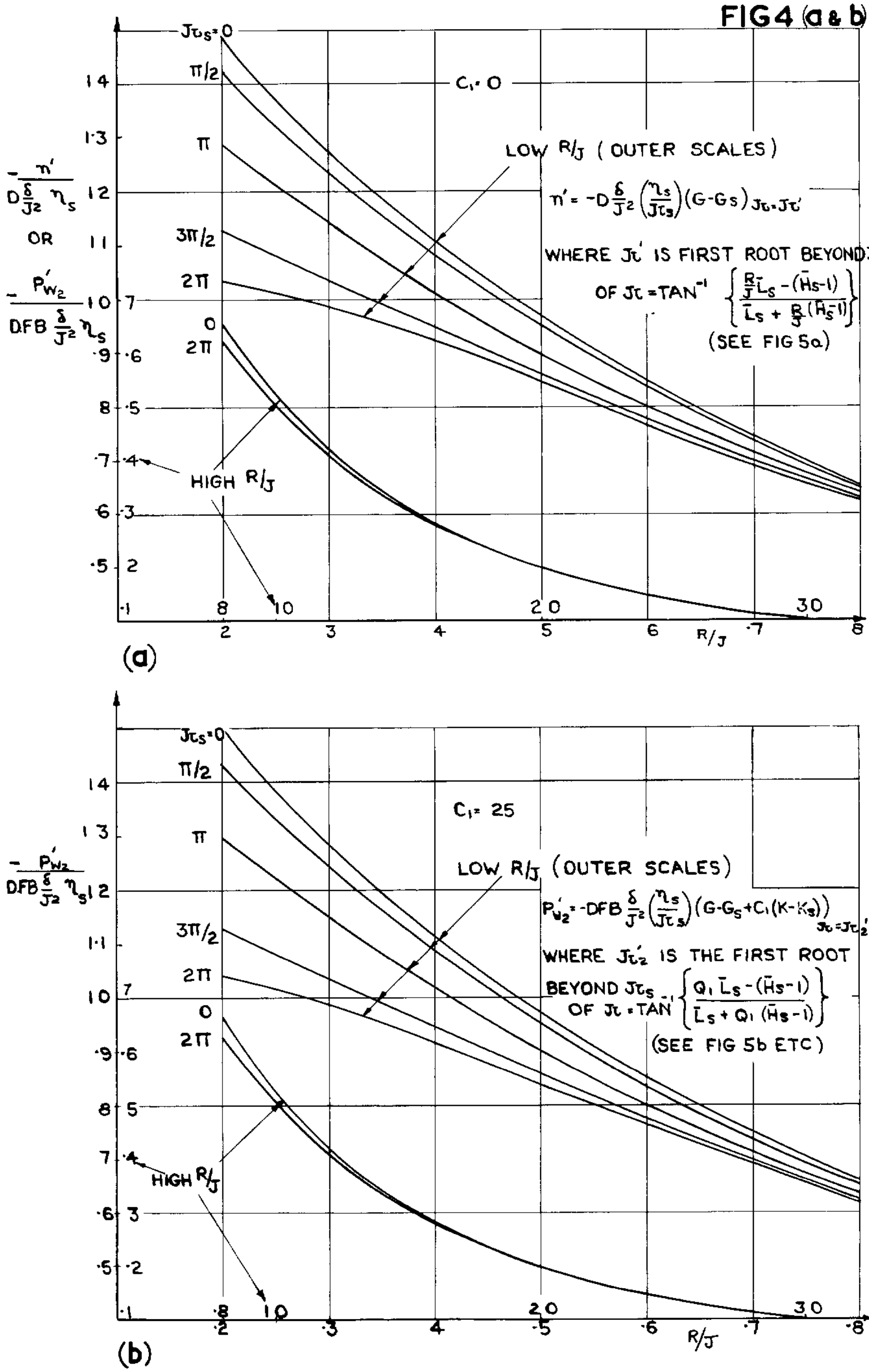
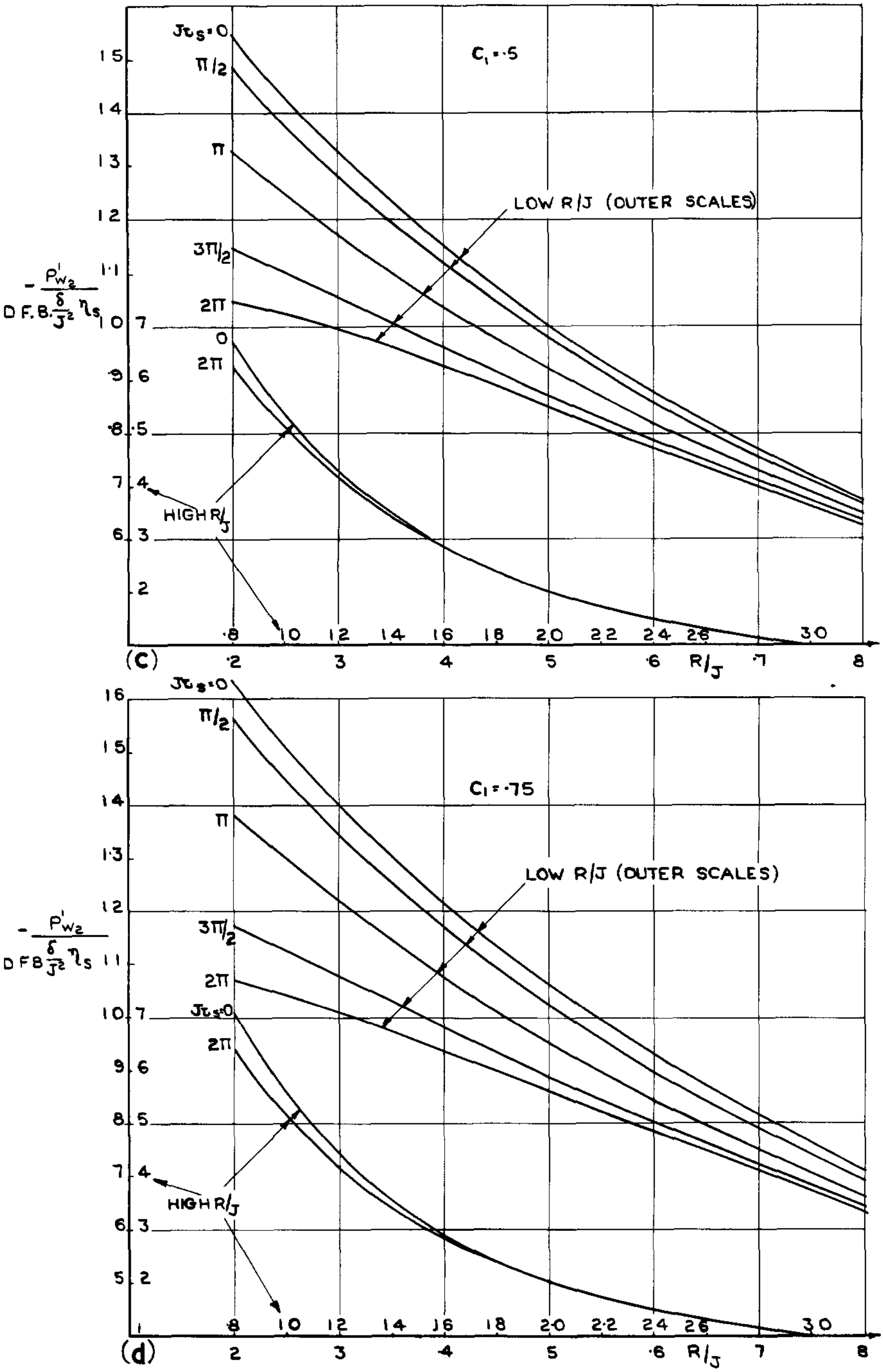


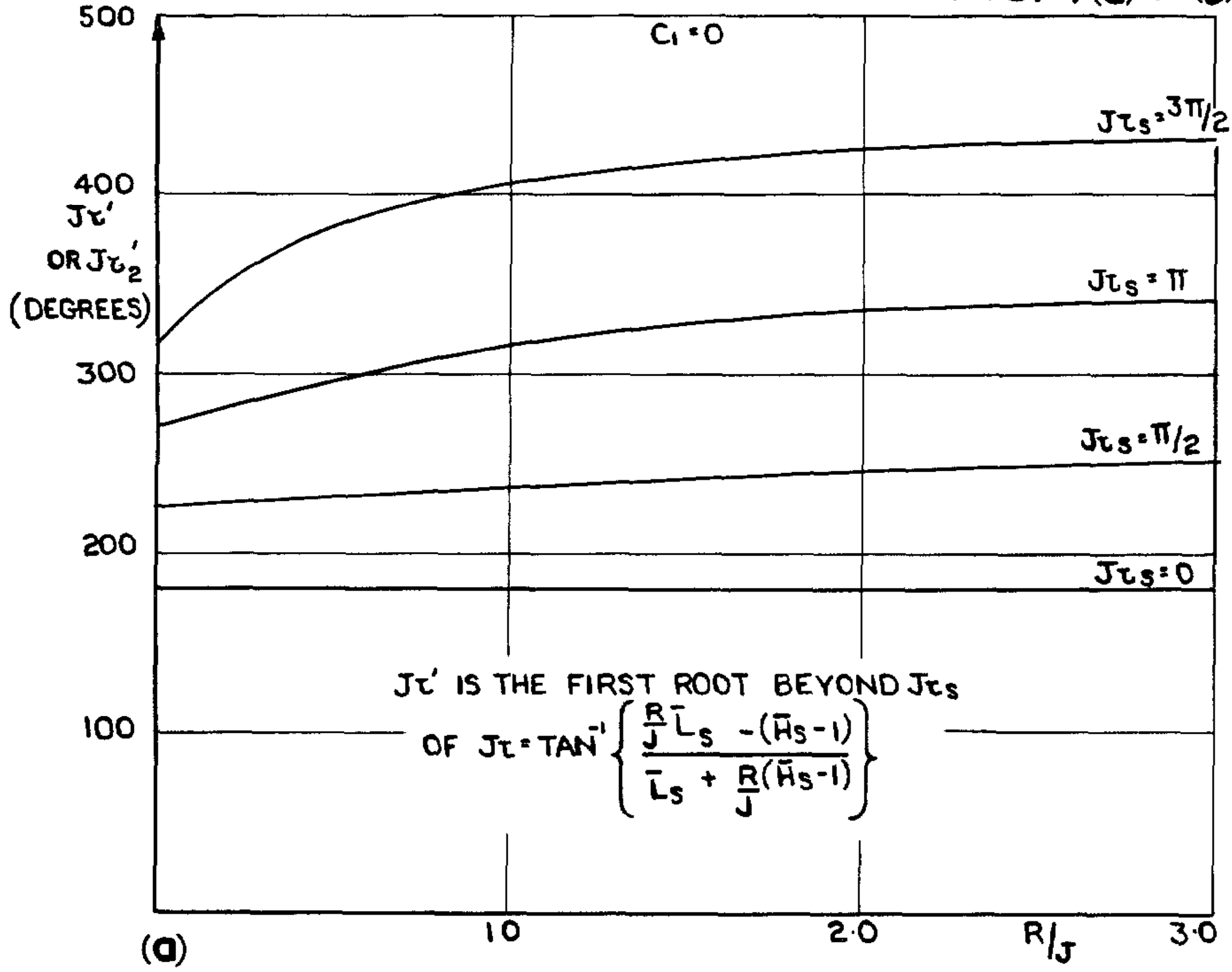
FIG 4 (a & b) CURVES OF THE GREATEST ACCELERATION (n') AND THE P_W COMPONENT OF P_2 (DUE TO THE EFFECTIVE ANGLE OF INCIDENCE AT THE TAIL)

FIG 4 (c) & (d)



Jτs=0
π/2
π
3π/2
2π
0
2π
C1=·5
LOW R/J (OUTER SCALES)
HIGH R/J
(c)
R/J
Jτs=0
π/2
π
3π/2
2π
Jτs=0
2π
C1=·75
LOW R/J (OUTER SCALES)
HIGH R/J
(d)
R/J

FIG.5. (a) & (b)



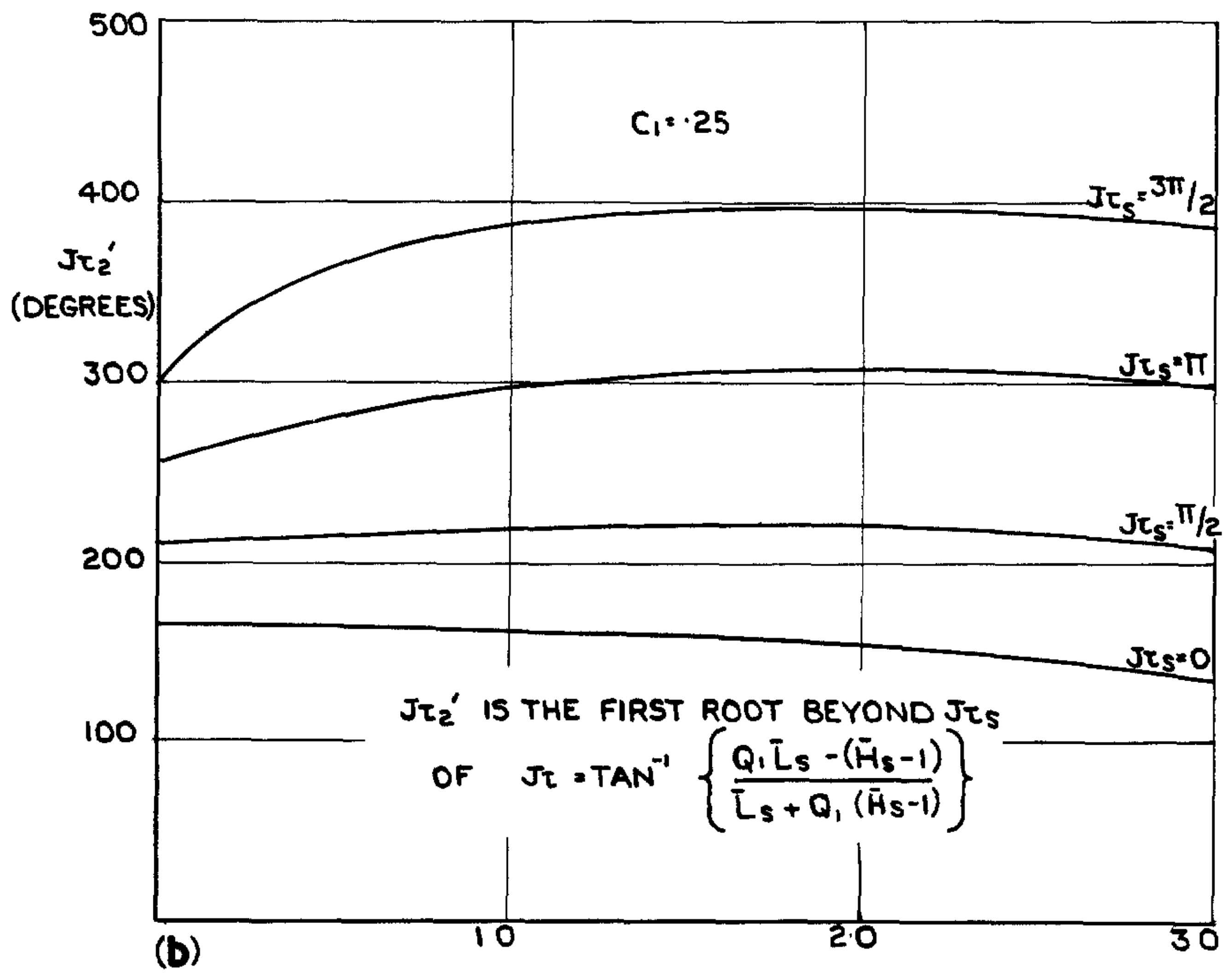
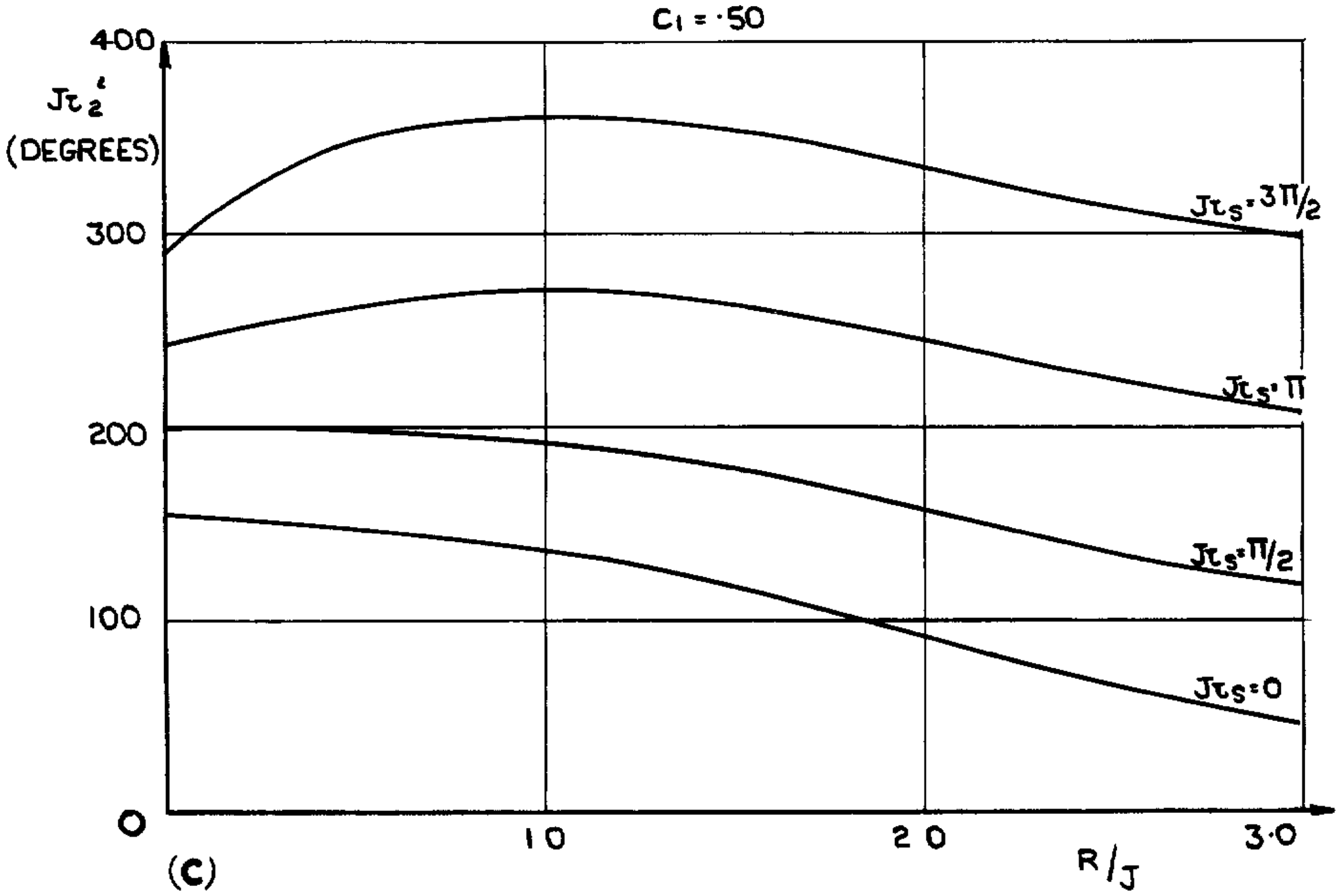
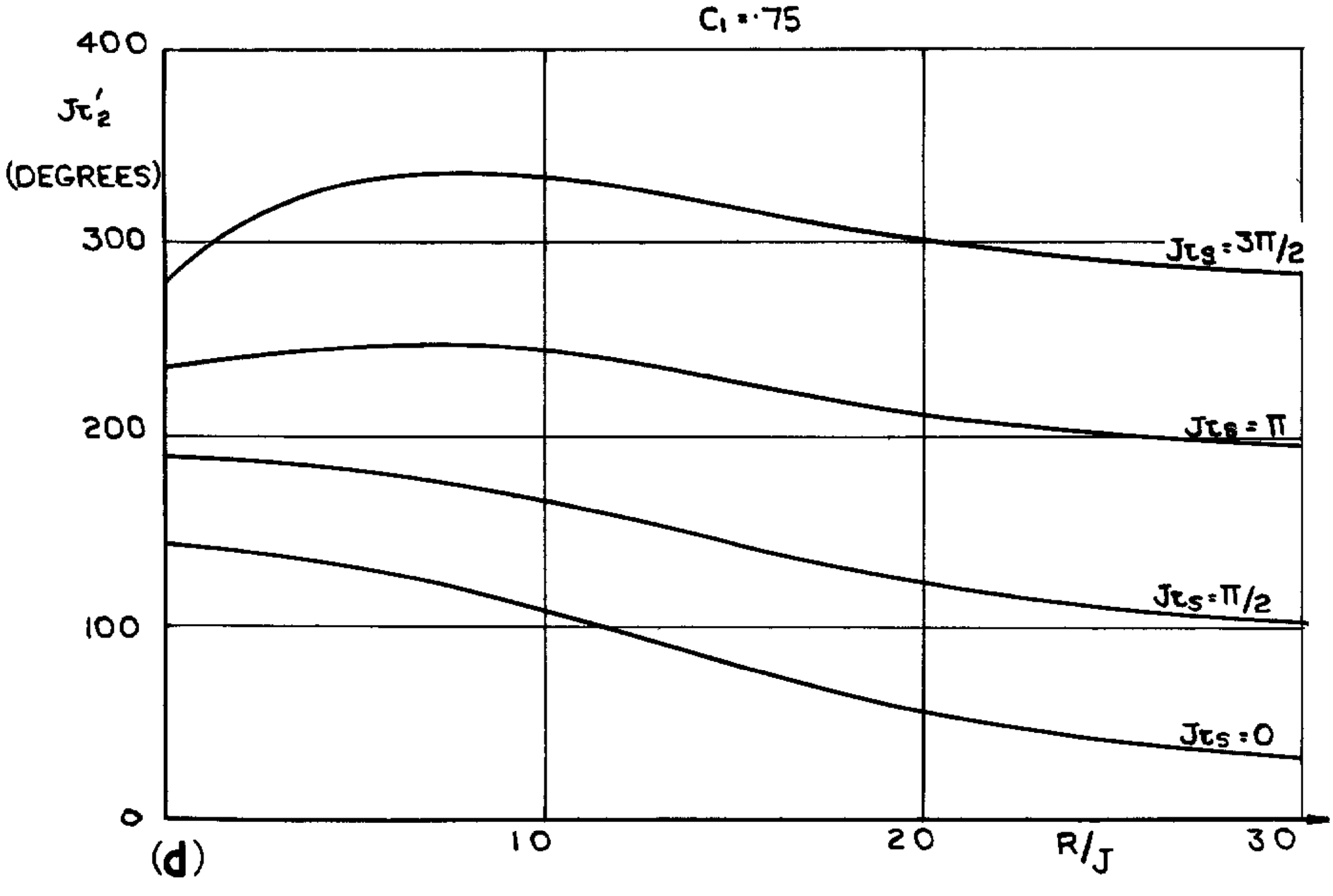


FIG. 5. (a) & (b) CURVES OF $J\tau'$ & $J\tau_2$

FIG. 5.(c) & (d)



C1 = ·50
400
300
200
100
0
Jτ2'
(DEGREES)
Jτs = 3π/2
Jτs = π
Jτs = π/2
Jτs = 0
1·0
2·0
3·0
R/J
(c)



C1 = ·75
400
300
200
100
0
Jτ2'
(DEGREES)
Jτs = 3π/2
Jτs = π
Jτs = π/2
Jτs = 0
1·0
2·0
3·0
R/J
(d)

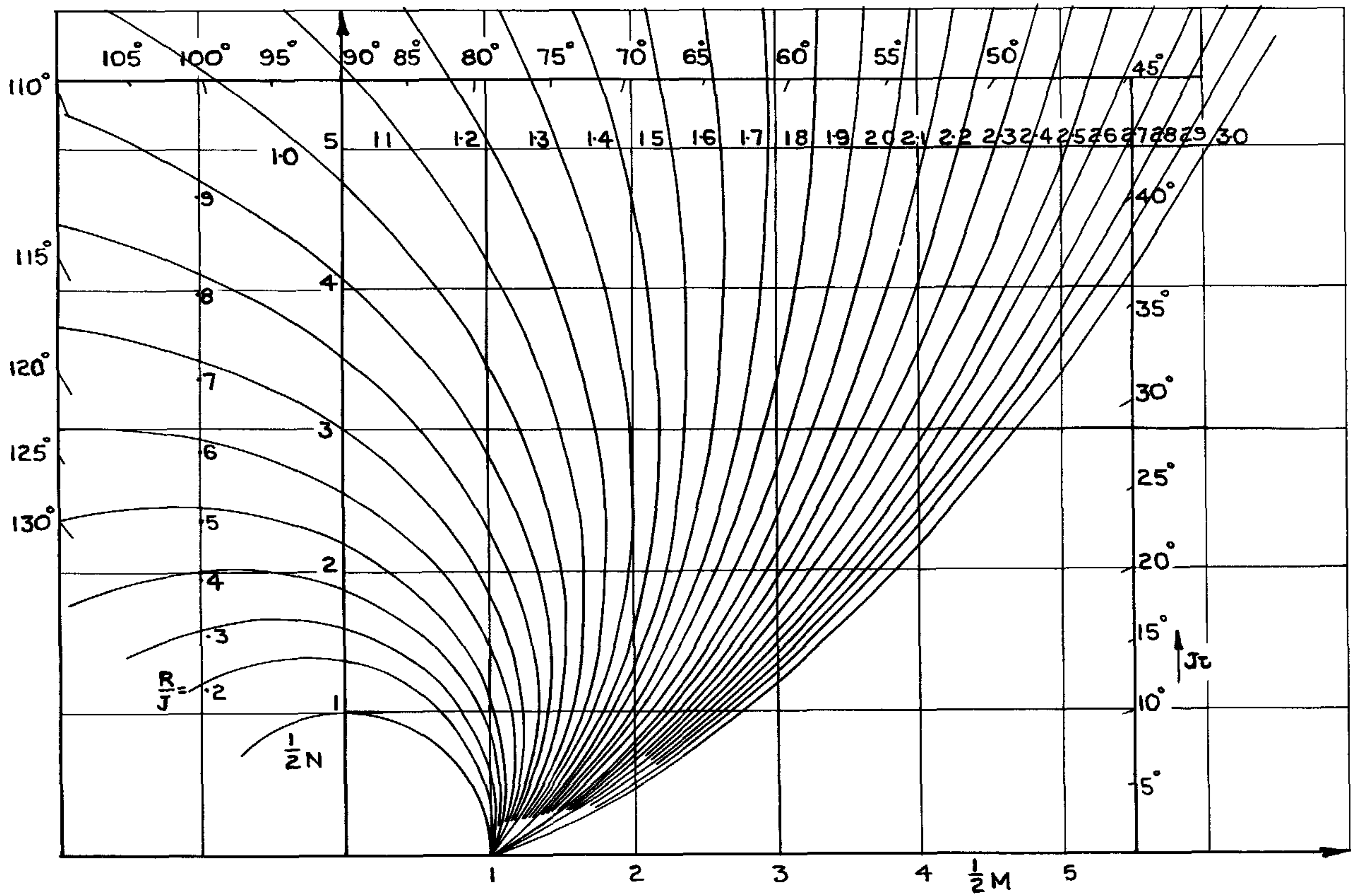


FIG. 6. CURVES FOR THE SOLUTION OF $M \cos J\tau + N \sin J\tau = e^{R\tau}$

FIG. 6.

FIG. 7.

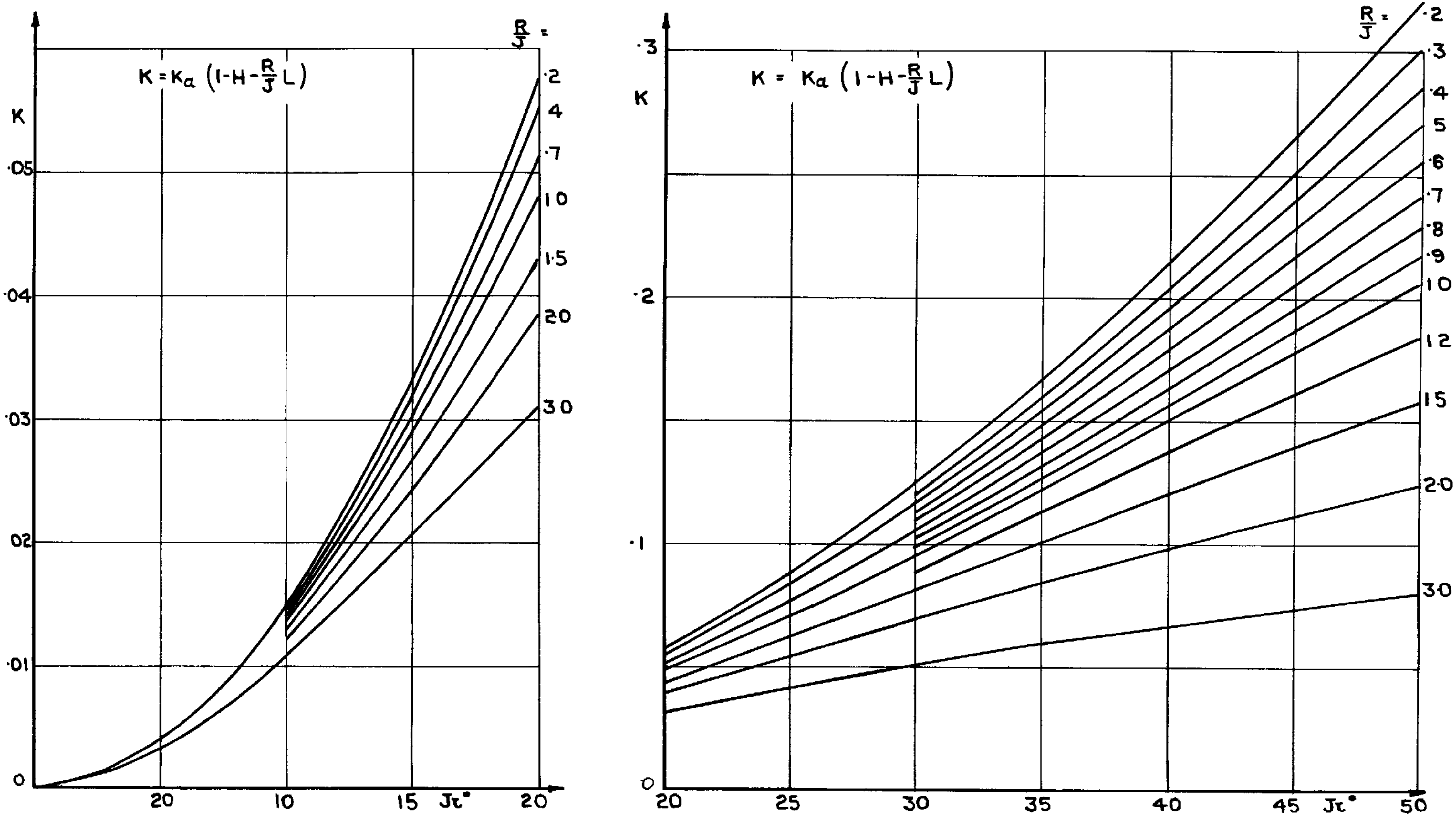
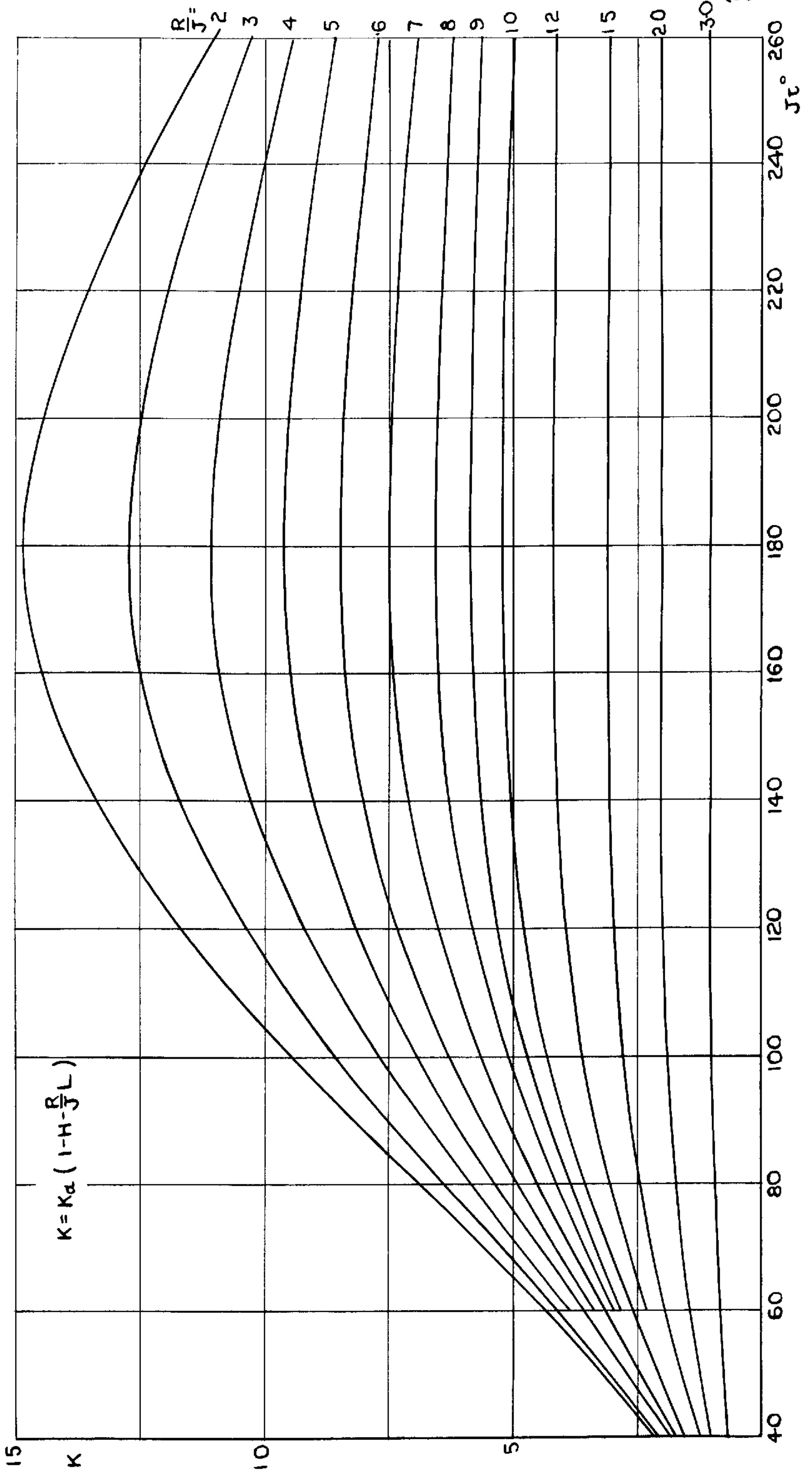


FIG. 7. AUXILIARY FUNCTION K.

FIG 7
(CONTINUED)



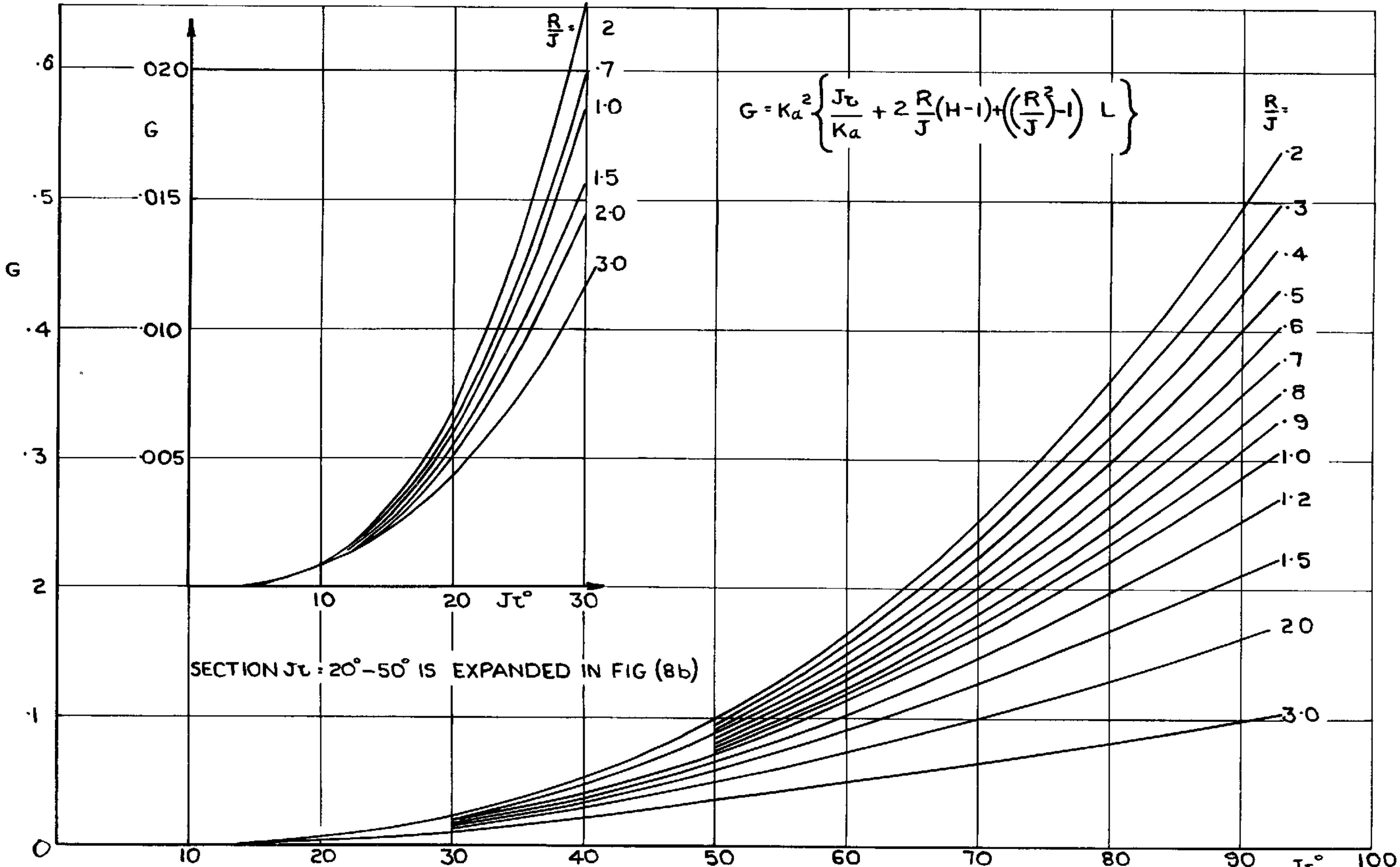


FIG.8. AUXILIARY FUNCTION G

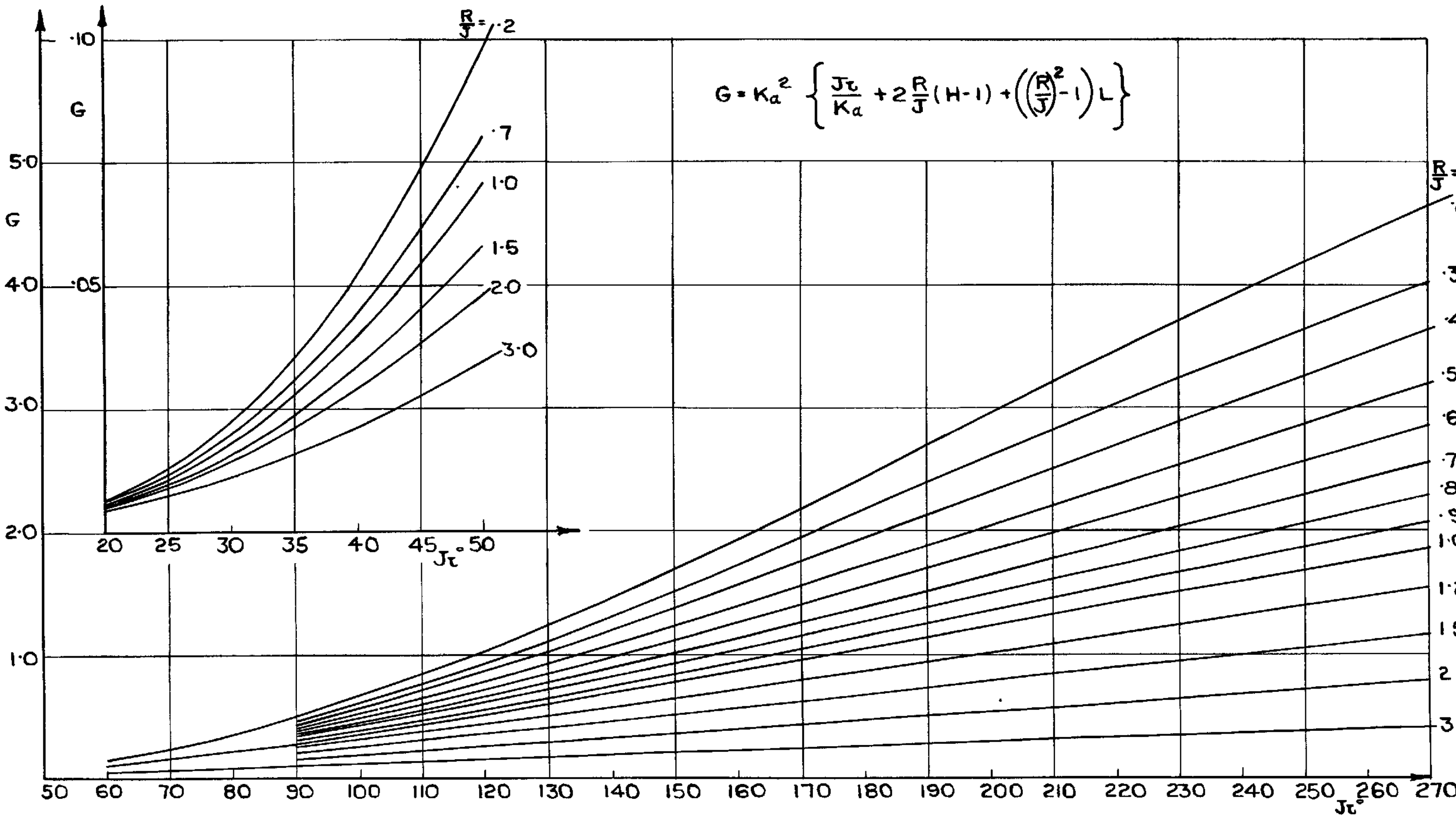


FIG. 8. (CONTINUED).

FIG. 9.

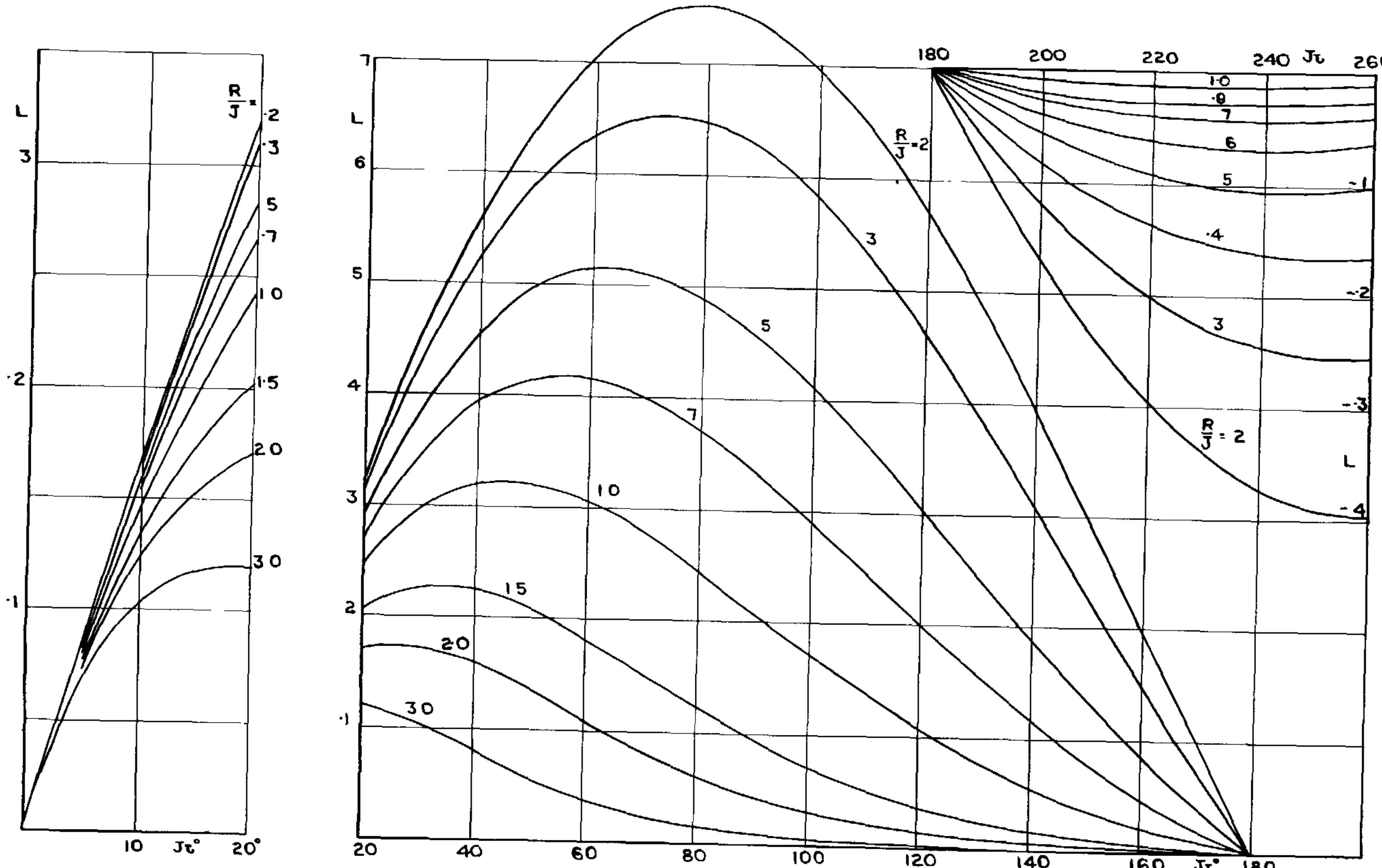


FIG. 9. AUXILIARY FUNCTION $L = e^{-\frac{R}{J}J\tau} \sin J\tau$

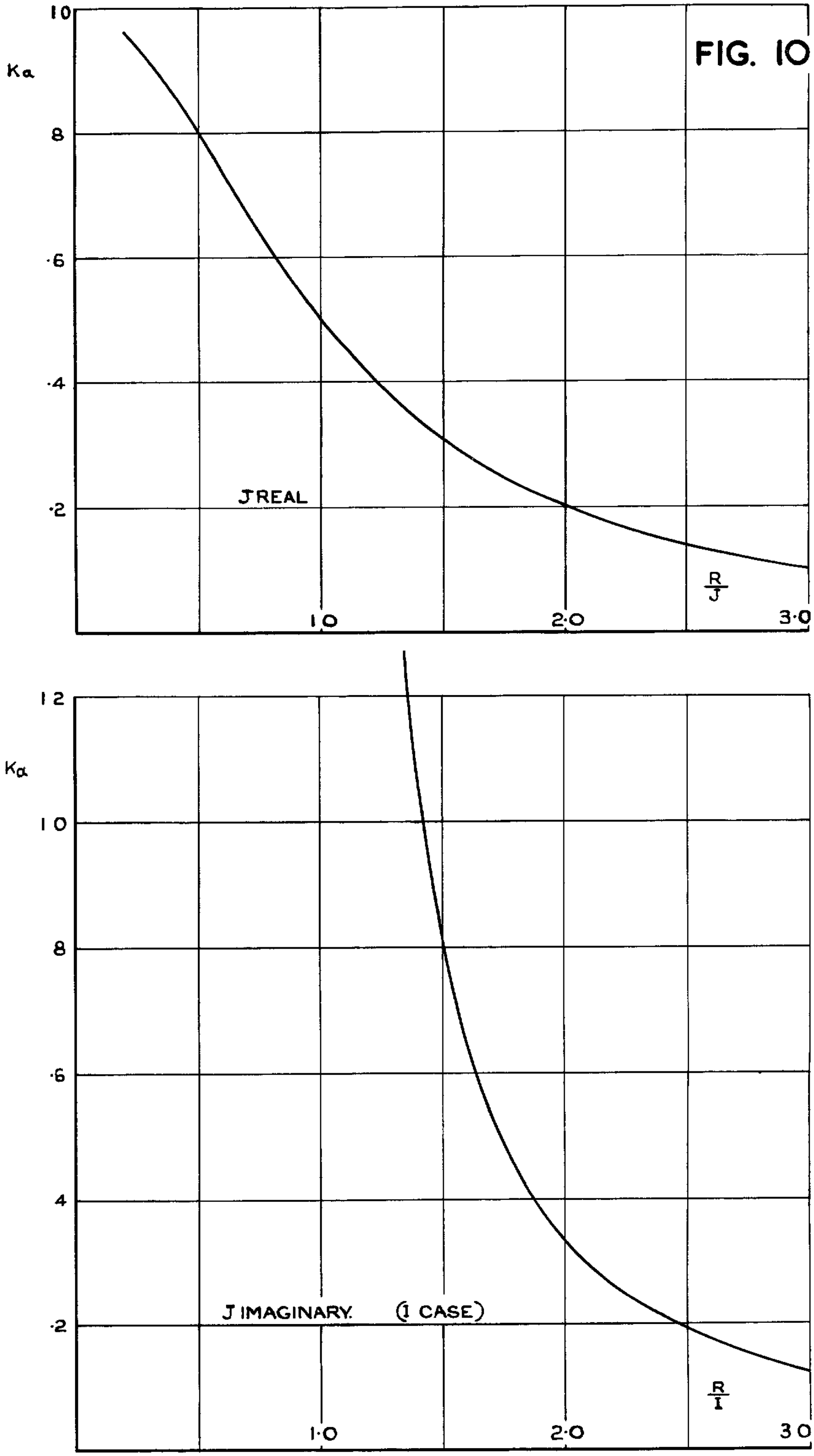


FIG. 10. CURVES OF K_a.

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