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By

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SUMMARY

Exact solutions are derived for angle of sideslip and fin-and-rudder loads for an aircraft performing two yawing manoeuvres induced by the rudder. Angles of sideslip and fin-and-rudder loads are then calculated for three selected aircraft and compared with results obtained by a simplified method in which rolling motion is neglected. Further calculations are made using a modified method in which the coefficients of the response formulae of the simplified method have been adjusted to take some account of rolling.

The analysis shows that errors of 20% may be incurred if rolling is neglected in the estimation of fin-and-rudder loads for aircraft with swept and delta wings. The errors increase with altitude. The modified method greatly reduces these errors, and may therefore be used where the response of the aircraft is appreciably affected by rolling.

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1 Introduction

In the simplified method previously recommended for the determination of the angles of sideslip and fin-and-rudder loads during the asymmetric manoeuvres specified for design[1], rolling of the aircraft is neglected. This assumption appears to be acceptable for aircraft with unswept wings but its applicability to aircraft with highly swept or delta wings is less certain.

However, in this method it is indicated how the coefficients of the response formulae may be modified to take account of rolling, but up to the present, there is no evidence to show that such a modification gives a more accurate solution. When the method is modified in this way it is referred to throughout as the "modified method" in order to distinguish it from the original method, which is called the "simplified" method.

In the present note, exact solutions for angles of sideslip and fin-and-rudder loads for aircraft in the specified yawing manoeuvres are derived, and compared numerically with those given by the simplified and modified methods. Three aircraft with straight, highly swept, and delta planforms, respectively, are used as examples.

Details of the exact and simplified solutions, together with a discussion of the significance of the modified method for taking account of rolling without recourse to the exact treatment, are given in the Appendices.

2 Scope of Investigation

Details of the exact, simplified and modified methods for estimating angles of sideslip and fin-and-rudder loads for a given aircraft are presented in Appendix I, paragraphs 2.1, 2.2 and 2.3 respectively. Three kinds of aircraft, each typical of existing trends in design are considered in this report, viz.-

Aircraft	Wing Planform
A	straight - slight taper
B	delta
C	highly swept - slight taper

The aerodynamic characteristics of these aircraft, which are all assumed to fly at high altitudes and medium values of C_L , are given in Table I.

The two manoeuvres considered are:-

(1) Instantaneous rudder movement to angle ζ_o .

(2) Sinusoidal rudder movement at the natural frequency of the aircraft $\zeta = \zeta_e \sin J\tau$.

A number of response curves for β and P are included, Figs.1 - 5, but the main results are tabulated in Tables II - IV as "local maxima", that is, values of β and P at $J\tau = \pi$ for the first manoeuvre, and $J\tau = 2\pi$, 3π for the second manoeuvre. Czaykowski[2] shows that, in the simplified method, very close approximations to the true maxima are found if β and P are calculated at these times although strictly speaking, the value of β at $J\tau = \pi$ for the first manoeuvre is the only true maximum.

3 Discussion

3.1 Estimation of Angles of Sideslip

The exact response curves for aircraft "A" and "B" performing the two manoeuvres, and the corresponding curves obtained from the simplified and modified methods, are shown in Figs.1 and 3.

(a) First manoeuvre

Results given by the exact and simplified methods compare favourably at the beginning of the manoeuvre, but differences become noticeable as the manoeuvre develops. Consideration of local maxima (see Table IV) shows that the use of the simplified method for calculations on aircraft "B" and "C" may lead to errors of about 20%. These errors may be reduced to about 5% if the modified method is used. As expected, the use of the simplified method for calculations on aircraft "A" is permissible. However, if the modified method is employed, the accuracy of the results may be improved still further.

It is shown in Appendix II that two of the parameters in the response formulae of the exact method, R' and r_s, which do not occur in the simplified method, have little effect on the numerical values of the coefficients of these formulae. Thus the exact formulae may be considered to be functions of R and J and errors due to the use of the simplified method for certain aircraft instead of the exact method are due primarily to the limitations of the method as a means of estimating R and J. The significance of the derivation of the modified method, in which the exact values of R and J are used in the formulae of the simplified method, is therefore clear. It follows that, in this manoeuvre, the angle of sideslip is a function of $\frac{1}{R^2 + J^2}$ for aircraft with either straight or swept wings.

From a survey of previous work[3], it is concluded that the errors in estimating R and J by the simplified method can be expected to be greatest when $(-\ell_v)$ is large and n_v is small. Reductions in air density will increase these errors. Consequently it may be advisable to use a more rigorous method of estimating R and J in cases where the aircraft has high $(-\ell_v)$ and low n_v, (for example, aircraft with swept wings).

(b) Second manoeuvre

Here again the differences between the simplified and exact solutions become apparent only as the manoeuvre develops. The errors in the maxima, although smaller than for the first manoeuvre, may be as high as 10%. However, these may be reduced to 1% if the modified method is used. Further, rolling has little effect on the times of occurrence of the local maxima (approximately at $J\tau = 2\pi$, 3π etc.). It follows that the basic structures of the coefficients in the response formulae of the exact and simplified methods are very similar, (see paragraph (a) above), and for this manoeuvre, angles of sideslip are directly dependent on $\frac{1}{RJ}$. As already mentioned, the errors in estimation of R and J by the simplified method depend on the magnitudes of $(-\ell_v)$ and n_v. Thus the conclusions drawn in paragraph (a) apply equally well to both manoeuvres.

(c) General

Comparison of the successive maxima (see Table III) shows that the error introduced by neglect of rolling is not always conservative. We have seen that the magnitude of the error is dependent on the relative magnitudes of the exact and simplified values of R and J. The results of previous investigations[3] suggest that the sign of the error is greatly influenced by the signs and magnitudes of n_p and i_E; if they are both negative the error is, in general, conservative.

In Appendix III it is shown that R and J may be obtained accurately without recourse to the stability quartic. However, whether this method, or solution of the quartic by "trial and error", is used, the product of inertia term i_E should be included. If it is not, the modified method may not give expected improvement over the simplified method. Calculations have been made with the present examples to illustrate this point; see Tables III and IV.

3.2 Fin-and-Rudder Loads

Specimen time histories of the fin-and-rudder loads for aircraft "A" and "B" during the two manoeuvres are shown in Figs. 2, 4 and 5. The loads are closely linked with the corresponding angles of sideslip (see Appendix I, paragraph 3) and many of the remarks made in the preceding paragraphs are therefore relevant here. The results are tabulated in Table IV(a) and IV(b).

Consideration of the first manoeuvre, shows that the errors in estimation of the fin-and-rudder loads, introduced through neglect of rolling, are as high as 25%. Thus the error is greater than that arising in the calculation of the corresponding angles of sideslip. This is due to the form of the equation for P. The errors may be reduced to 5% if the modified method is used to determine the angles of sideslip.

In the fish-tail manoeuvre, where the equation for P is slightly different, the errors in estimation of P arising from the neglect of rolling are comparable with those occurring in the corresponding angles of sideslip. The figures in Table IV(b) indicate that, provided the angles of sideslip are calculated by the modified method, the associated fin-and-rudder loads for this manoeuvre will also be accurate.

4 Conclusions

(1) Neglect of rolling motion in manoeuvres induced by the rudder may introduce appreciable errors in the estimation of the angles of sideslip and the associated loads of certain aircraft.

(2) These errors are due primarily to the errors in estimating the damping and frequency parameters of the lateral oscillations, R and J. In this respect, the simplified method is only acceptable when used on aircraft with straight wings.

(3) The errors in estimation of R and J can be expected to be greatest when the aircraft has high $(-\ell_v)$ and low n_v. The errors will increase with altitude.

(4) It is confirmed that a simple method for reducing the errors due to neglect of rolling is to use the exact values of R and J in the formulae of the simplified method.

(5) The product of inertia term i_E should be included in any estimate of the exact values of R and J. If this term is neglected the modified method may not prove any more accurate than the simplified method.

(6) The procedure suggested by Neumark (see Appendix III) is perhaps the simplest for finding the exact values of R and J if the main interest is in angles of sideslip and fin and rudder loads in yawing manoeuvres.

NOTATION

A_o, B_o, C_o, D_o, E_o	:	coefficients of equation (8)
A_1, B_1, C_1, D_1, E_1, F_1	:	coefficients of equation (10)
B_2, C_2, D_2, E_2	:	coefficients of quartic equation (2)
F_2, G_2	:	coefficients in equation (2)
H_0, H_1, H_2, H_3, etc.	:	coefficients of equation (20)
I_1, I_2, I_3, I_4	:	factors in quartic influenced by the inertia coupling term i_E
$a_1 = -\frac{\partial C_{Y_f}}{\partial \beta}$		
$a_2 = +\frac{\partial C_{Y_f}}{\partial \zeta}$		
b	:	wing span
C_L	:	lift coefficient (total)
C_{Y_f}	:	side force coefficient
f, h	:	coefficients of equation (6) or equation (13)
g	:	gravity constant
$i = \sqrt{-1}$		
i_A	:	inertia coefficient about x axis

i_C	:	inertia coefficient about z axis
i_E	:	inertia coupling about x - z axis - coefficient of product of inertia
J	:	non-dimensional frequency of lateral oscillations - also circular frequency of disturbance
$k = \frac{C_L}{2}$		
ℓ	.	fin-and-rudder arm
ℓ_R	:	distance of C.P. of fin-and-rudder load due to rudder deflection to C.G. of aircraft
ℓ_p	:	damping derivative in roll
ℓ_r	:	rolling moment derivative due to yaw
ℓ_v	·	dihedral stability derivative
n_p	:	yawing moment derivative due to roll
n_r	.	damping derivative in yaw
n_v	:	static stability derivative in yaw
$\hat{p} = p\,\hat{t}$	:	angular velocity in roll (non-dimensional)
P	:	fin-and-rudder load
R	:	damping factor of lateral oscillations
R'	:	damping factor of rolling subsidence
r_s	:	damping factor of spiral motion
$\hat{r} = r\,\hat{t}$	:	angular velocity in yaw (non-dimensional)
S	.	wing area
S"	:	fin-and-rudder area
t	:	time in seconds
$\hat{t} = \frac{W}{g\rho S V}$	:	unit of aerodynamic time in seconds
V	:	true velocity of C.G. of aircraft

v	:	velocity of sideslip
$\overline{V}_R = \frac{S'' \ell_R}{S b}$	:	fin-and-rudder volume coefficient
W	:	weight of aircraft
$\overline{y}_v = - y_v$	:	lateral force derivative due to sideslip
$\beta = \frac{v}{V}$	:	angle of sideslip
$\delta_n = - \frac{\mu_2 \overline{V}_R a_2}{i_C}$	:	rudder effectiveness
$\varepsilon = R' - \nu_\ell$	:	see equation (20)
ζ	:	rudder angle
λ	:	stability root
$\mu_2 = \frac{2W}{g \rho S b} = \frac{2V\hat{t}}{b}$	:	relative density of aircraft (referred to semi span)
$\mu_3 = \frac{W}{g \rho S \ell} = \frac{b}{2\ell} \cdot \mu_2$	:	relative density of aircraft (referred to length)
$\nu_\ell = - \frac{\ell_p}{i_A}$		
$\nu_{\ell r} = \frac{\ell_r}{i_A}$		
$\nu_n = - \frac{n_r}{i_C}$		
$\nu_{np} = - \frac{n_p}{i_C}$		
ρ	:	air density
τ	:	aerodynamic time (non-dimensional)
ϕ	:	angle of bank
$\omega_\ell = - \frac{\mu_2 \ell_v}{i_A}$		

$$\omega_n = \frac{\mu_2 \, n_v}{i_C}$$

Suffices

o : maximum rudder angle in first manoeuvre and quantities due to that angle

e : maximum rudder angle in second manoeuvre and quantities due to that angle

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APPENDIX I

Mathematical Analysis of the Problem

1 General Equations of Motion

The linearised differential equations of lateral motion of an aircraft may be written (cf. Refs. 5 and 6).

$$\left.\begin{aligned}
&\text{Sideslip:-} && \frac{d\beta}{d\tau} + \bar{y}_v\,\beta + \hat{r} - k\phi = 0 \\
&\text{Roll:-} && \omega_\ell\,\beta + \frac{d\hat{p}}{d\tau} + \nu_\ell\,\hat{p} - \frac{i_E}{i_A}\frac{d\hat{r}}{d\tau} + \nu_{\ell r}\hat{r} = 0 \\
&\text{Yaw:-} && -\omega_n\,\beta - \frac{i_E}{i_A}\frac{d\hat{p}}{d\tau} + \nu_{np}\,\hat{p} + \frac{d\hat{r}}{d\tau} + \nu_n\,\hat{r} + \delta_n\,\zeta = 0 \\
&\text{Kinematic Relationship:-} && -\hat{p} + D\phi = 0
\end{aligned}\right\} \quad (1)$$

The last term in the yawing equation is the disturbing function expressed in terms of the applied rudder angle. Similar terms in the other two equations, representing the effects of rudder movement on the sideforce and rolling moments respectively, have been neglected. The effects of any displacement between the wind axes and principal axes of inertia are included.

2 Solutions

2.1 Exact Solution for Angle of Sideslip

If equations (1) are written as a function of β alone, we have, expressing the result in a form suitable for the application of the Laplace Transform:-

$$\frac{d^4\beta}{d\tau^4} + B_2\frac{d^3\beta}{d\tau^3} + C_2\frac{d^2\beta}{d\tau^2} + D_2\frac{d\beta}{d\tau} + E_2\,\beta = \delta_n\left(\frac{d^2\zeta}{d\tau^2} + F_2\frac{d\zeta}{d\tau} + G_2\,\zeta\right) \quad (2)$$

where

$$\left.\begin{aligned}
B_2 &= \nu_\ell + \nu_n + \bar{y}_v + I_1 \\
C_2 &= \bar{y}_v(\nu_\ell + \nu_n) + (\nu_{\ell r}\,\nu_{np} + \nu_n\,\nu_\ell) + \omega_n + I_2 \\
D_2 &= \bar{y}_v(\nu_{\ell r}\,\nu_{np} + \nu_n\,\nu_\ell) + \omega_\ell(\nu_{np} + k) + \omega_n\,\nu_\ell + I_3 \\
E_2 &= k(\omega_\ell\,\nu_n - \nu_{\ell r}\,\omega_n) \\
F_2 &= \nu_\ell + I_4 \\
G_2 &= -\nu_{\ell r}\,k
\end{aligned}\right\} \quad (3)$$

$$
\left.
\begin{aligned}
I_1 &= \frac{i_E}{i_A} \nu_{np} - \frac{i_E}{i_C} \nu_{\ell r} \\
I_2 &= \bar{y}_v I_1 - \omega_\ell \frac{i_E}{i_C} \\
I_3 &= - \omega_n \frac{i_E}{i_A} k \\
I_4 &= - \frac{i_E}{i_A}
\end{aligned}
\right\} \text{terms containing effects of inertia coupling } i_E \qquad (4)
$$

The corresponding stability quartic is

$$\lambda^4 + B_2 \lambda^3 + C_2 \lambda^2 + D_2 \lambda + E_2 = 0 \qquad (5)$$

which may, in general, be factorised to

$$(\lambda + r_s)(\lambda + R')(\lambda^2 + f \lambda + h) = 0 \qquad (6)$$

The four roots are then

$$
\left.
\begin{aligned}
\lambda_1 &= - r_s && = \text{damping factor of the spiral motion} \\
\lambda_2 &= - R' && = \text{damping factor of the rolling motion} \\
\lambda_{3,4} &= - \tfrac{1}{2} f \pm i \sqrt{h - \frac{f^2}{4}} && = \text{complex roots of lateral oscillation} \\
&= - R \pm i J
\end{aligned}
\right\} (7)
$$

where R = damping factor of the lateral oscillation

J = frequency factor of the lateral oscillation

If we solve equation (2) for the two specified manoeuvres we obtain:-

(i) For the first manoeuvre:-

Using equations (2) and (6)

with $\zeta = \zeta_0$

and initial conditions $p = \phi = r = \beta = 0$

at $\tau = 0$

$$\frac{\beta}{\delta_n \zeta_o} = A_o + B_o e^{-R'\tau} + C_o e^{-r_s \tau} + e^{-R\tau}\left\{D_o \cos J\tau + \frac{E_o - D_o R}{J} \sin J\tau\right\} \tag{8}$$

The coefficients A_o , B_o , C_o , D_o and E_o may be deduced from the following equations:-

$$\left.\begin{aligned}
B_o + C_o + D_o + \frac{G_2}{E_2} &= 0 \\
(r_s+f)\, B_o + (R'+f)\, C_o + (R'+r_s)\, D_o + E_o + \frac{G_2}{E_2} \cdot B_2 &= 0 \\
(r_s f+h)\, B_o + (R'f+h)\, C_o + R' r_s\, D_o + (R'+r_s)\, E_o + \frac{G_2}{E_2} \cdot C_2 &= 1 \\
r_s h\, B_o + R'h\, C_o + R' r_s\, E_o + \frac{G_2}{E_2} \cdot D_2 &= F_2 \\
A_o &= \frac{G_2}{E_2}
\end{aligned}\right\} \tag{9}$$

(ii) For the second manoeuvre, similarly:-

with $\zeta = \zeta \sin J\tau$

and initial conditions $p = \phi = r = \beta = 0$

at $\tau = 0$

$$\frac{1}{J}\left(\frac{\beta}{\delta_n \zeta_e}\right) = A_1 e^{-R'\tau} + B_1 e^{-r_s \tau} + e^{-R\tau}\left\{C_1 \cos J\tau + \frac{D_1 - C_1 R}{J} \sin J\tau\right\}$$

$$+ E_1 \cos J\tau + \frac{F_1}{J} \sin J\tau \tag{10}$$

and the corresponding equations for the coefficients are:-

$$
\left.
\begin{array}{l}
A_1 + B_1 + C_1 + E_1 = 0 \\
(f+r_s) A_1 + (f+R') B_1 + (R'+r_s) C_1 + D_1 + B_2 E_1 + F_1 = 0 \\
(J^2+h+r_s f) A_1 + (J^2+h+R'f) B_1 + (R'r_s+J^2) C_1 + (R'+r_s) D_1 + C_2 E_1 + B_2 F_1 = 0 \\
\{J^2 f+r_s(h+J^2)\} A_1 + \{J^2 f+R'(h+J^2)\} B_1 + J^2(R'+r_s) C_1 + (R'r_s+J^2) D_1 + D_2 E_1 + C_2 F_1 = 1 \\
J^2(h+fr_s) A_1 + J^2(h+fR') B_1 + R'r_s J^2 C_1 + J^2(R'+r_s) D_1 + E_2 E_1 + D_2 F_1 = F_2 \\
r_s hJ^2 A_1 + R'hJ^2 B_1 + R'r_s J^2 D_1 + E_2 F_1 = G_2
\end{array}
\right\} \qquad (11)
$$

In the present investigation equations (9) and (11) are solved numerically for the three aircraft.

Note:- The coefficients of the stability quartic B_2 , C_2 , D_2 and E_2 may also be written

$$B_2 = R' + r_s + f$$

$$C_2 = R' r_s + h + f\,(R' + r_s)$$

$$D_2 = h\,(R' + r_s) + f\,R'\,r_s$$

$$E_2 = R'\,r_s\,h$$

2.2 Simplified Method (cf. Refs.1 and 2)

If rolling motion is neglected completely, equation (2) becomes

$$\frac{d^2\beta}{d\tau^2} + (\bar{y}_v + \nu_n)\frac{d\beta}{d\tau} + (\omega_n + \nu_n\,\bar{y}_v)\,\beta = \delta_n\,\zeta \qquad (12)$$

The corresponding stability quadratic is

$$\left.\begin{aligned} \lambda^2 + f\lambda + h &= 0 \\ f &= \bar{y}_v + \nu_n \\ h &= \omega_n + \nu_n\,\bar{y}_v \end{aligned}\right\} \qquad (13)$$

and the roots are

$$\lambda_1\,,\,\lambda_2 = -\frac{f}{2} \pm i\sqrt{h - \frac{f^2}{4}}$$

$$= -R \pm i\,J \qquad (14)$$

where R and J are again the damping and frequency factors of the lateral oscillations. In view of the assumption of zero rolling motion these roots will not be exact.

The corresponding complete solutions for the two manoeuvres considered are:-

(i) $$\frac{\beta}{\delta_n\,\zeta_0} = \frac{1}{R^2 + J^2}\left[1 - e^{-R\tau}\left(\cos J\tau + \frac{R}{J}\sin J\tau\right)\right] \qquad (15)$$

(ii) $$\frac{1}{J}\left(\frac{\beta}{\delta_n \zeta_e}\right) = \frac{1}{R(4J^2+R^2)}\left[\frac{R}{J}\sin J\tau - 2\cos J\tau + e^{-R\tau}\left(2\cos J\tau + \frac{R}{J}\sin J\tau\right)\right] \quad (16)$$

2.3 Modified Method (Refs.1 and 2)

When the simplified method for the estimation of sideslip angles and fin-and-rudder loads was proposed, it was thought that acceptable solutions would be obtained in most cases. For the outstanding cases, where rolling motion might have a marked effect on the response, it was suggested that a closer approximation would be reached if the exact values of R and J, obtained from a rigorous solution of the quartic, were used in the coefficients of the response formulae equations (15) and (16).

3 Fin-and-Rudder Loads

The general expression for the aerodynamic load on the fin-and-rudder during a manoeuvre may be written

$$P = \tfrac{1}{2}\rho V^2 S'' \left(-a_1 \beta + \frac{\ell}{V} r a_1 + a_2 \zeta\right) \quad (17)$$

$$= A\left(-B\beta - C\frac{d\beta}{d\tau} + a_2 \zeta\right) \quad (18)$$

where

$$A = \tfrac{1}{2}\rho V^2 S''$$

$$B = \left(1 + \frac{\bar{y}_v}{\mu_3}\right) a_1$$

$$C = \frac{1}{\mu_3} a_1$$

APPENDIX II

Comparison of Results of Exact, Simplified and Modified Methods

First Manoeuvre

Consider the formulae of the exact and simplified methods

$$\frac{\beta}{\delta_n \zeta_o} = \left(A_o + C_o e^{-r_s \tau}\right) + e^{-R\tau}\left\{D_o \cos J\tau + \frac{E_o - D_o R}{J} \sin J\tau\right\} + B_o e^{-R'\tau} \quad (8)$$

$$\frac{\beta}{\delta_n \zeta_o} = \frac{1}{R^2 + J^2} + e^{-R\tau}\left\{-\frac{1}{R^2 + J^2} \cos J\tau - \frac{R}{J(R^2 + J^2)} \sin J\tau\right\} \quad (15)$$

The term $e^{-R'\tau}$, in the exact solution, is very small since the damping in roll R' is usually large. Numerical results indicate that the coefficient B_o is also very small. Thus the term $B_o e^{-R'\tau_o}$ may be neglected in the exact solution. The spiral term $C_o e^{-r_s \tau}$ must be retained since $e^{-r_s \tau}$ is approximately unity whilst calculations show that A_o and C_o are comparable but of opposite sign. Thus the general structures of the two formulae are essentially similar.

If we now examine the numerical values of the respective coefficients, using the exact values of R and J (modified approach) in the simplified formula, (equation (15)) we have the following results.

Aircraft	"A"	"B"	"C"
A_o	-0.38853	-0.23888	-0.09194
C_o	0.44478	0.32525	0.138819
B_o	0.00015	0.00063	0.0001
$\frac{1}{R^2 + J^2}$	0.0565	0.0876	0.04694
$\left(A_o + C_o e^{-r_s \tau}\right)_{J\tau = \pi}$	0.05625	0.08637	0.04688
$-\frac{1}{R^2 + J^2}$	-0.0565	-0.0876	-0.04694
D_o	-0.0564	-0.08701	-0.04698
$-\frac{R}{J(R^2 + J^2)}$	-0.0053	-0.00348	-0.00264
$\frac{E_o - D_o R}{J}$	-0.00501	-0.00192	-0.00135

The respective coefficients are almost identical in all cases. Thus it may be inferred that the corresponding coefficients of the two formulae are similar functions of R and J and that the slight variations between the numerical values of the coefficients are due to the parameters R' and r_s. The errors arising from the use of the simplified method for certain aircraft instead of the exact method are therefore due primarily to the limitations of the method as a means of estimating R and J.

Second Manoeuvre

The two response formulae for β may be written

$$\frac{\beta}{\delta_n \zeta_e} = J\left(A_1 e^{-R'\tau} + B_1 e^{-r_s \tau}\right) + e^{-R\tau}\left(JC_1 \cos J\tau + [D_1 - C_1 R] \sin J\tau\right) + JE_1 \cos J\tau + F_1 \sin J\tau \qquad (10)$$

$$\frac{\beta}{\delta_n \zeta_e} = e^{-R\tau}\left(2 H_e \cos J\tau + \frac{R}{J} H_e \sin J\tau\right) - 2H_e \cos J\tau + \frac{R}{J} H_e \sin J\tau \qquad (16)$$

where

$$H_e = \frac{J}{R(4J^2 + R^2)}$$

Numerical results show that $J\left(A_1 e^{-R'\tau} + B_1 e^{-r_s \tau}\right)$ may be disregarded (see table below) and then the general forms of the two equations are identical. If the numerical values of the coefficients are calculated, again using the exact values of R and J, modified method, we have:-

/Table

Aircraft	"A"	"B"	"C"
$J\left(A_1 e^{-R'\tau} + B_1 e^{-r_s\tau}\right)$	-0.00013	-0.00085	-0.0011
$2\ H_e$	0.30109	1.10462	0.41895
$J\ C_1$	0.30023	1.09811	0.41889
$(D_1 - C_1\ R)$	0.01251	0.00145	0.0003
$\frac{R}{J}\ H_e$	0.01423	0.02192	0.01176
$-\ 2\ H_e$	-0.30109	-1.10462	-0.41895
$J\ E_1$	-0.30001	-1.0969	-0.41766
$\frac{R}{J}$	0.01423	0.0219	0.01176
F_1	0.0158	0.0418	0.02319

The numerical values of the respective coefficients are almost identical. It follows that for this manoeuvre also the errors arising from the use of the simplified method are due primarily to its limitations as a method of estimating R and J.

APPENDIX III

Factorization of the Quartic Equation (5)

In the main text it is shown that acceptable solutions to the response of an aircraft in yaw can be obtained if the exact values of R and J are known. In the simplified theory, the expressions for R and J are explicit, but the inclusion of rolling motion destroys this mathematical simplicity. Hence an accurate estimation of angle of sideslip and fin-and-rudder loads is reduced to the problem of deriving the exact values of R and J, i.e. the factorization of the quartic, and the use of the simplified formulae.

(1) If the numerical values of the lateral stability coefficients B_2, C_2, D_2 and E_2 are known, the quartic may be solved by "trial and error" using the characteristics of the lateral quartic for selecting the first approximations to λ_1 and λ_2, i.e.

$$\lambda_1 \simeq -\frac{E_2}{D_2} \qquad \lambda_2 \simeq -B_2 \tag{19}$$

(2) If the main interest is in the response, rather than the stability of the aircraft, however, it may be more convenient to use the method suggested by Neumark[5]. This method can be readily tabulated for computational purposes. The relevant formulae are presented here, with certain additions covering the effects of inertia coupling terms (neglected in the original report).

$$\varepsilon = \frac{H_o}{H_1} + \left(\frac{H_o}{H_1}\right)^2 \frac{H_2}{H_1} - \left(\frac{H_o}{H_1}\right)^3 \left\{\frac{H_3}{H_1} - 2\frac{H_2}{H_1}\right\} + \text{etc.}$$

where

$$\left.\begin{aligned}
H_o &= \omega_\ell(\nu_{np} + k) - (\nu_\ell - \bar{y}_v)\,\nu_{\ell r}\,\nu_{np} - \frac{E_2}{\nu_\ell} + [I_3 - \nu_\ell(I_2 - \nu_\ell I_1)] \\
H_1 &= \omega_n + (\nu_\ell - \nu_n)(\nu_\ell - \bar{y}_v) + \frac{\omega_\ell(\nu_{np}+k) + \bar{y}_v \nu_{\ell r}\nu_{np}}{\nu_\ell} - \frac{2E_2}{\nu_\ell^2} + \left[\frac{I_3}{\nu_\ell} - \nu_\ell I_1\right] \\
H_2 &= -\nu_\ell + \frac{D_2}{\nu_\ell^2} - \frac{3E_2}{\nu_\ell^3} \qquad H_3 = \frac{D_2}{\nu_\ell^3} - \frac{4E_2}{\nu_\ell^4} \quad \text{etc.}
\end{aligned}\right\} \tag{20}$$

Then

$$R' = \nu_{\ell} + \varepsilon \tag{21}$$

$$r_s = \frac{E_2}{D_2 - E_2/R'} \tag{22}$$

$$f = (\nu_n + \bar{y}_v) + I_1 - (r_s + \varepsilon) \tag{23}$$

$$h = (\omega_n + \bar{y}_v \nu_n) + \nu_{\ell r}\, \nu_{np} + (R' - \nu_n - \bar{y}_v) - R'\, I_1 + I_2 - r_s\, f \tag{24}$$

These equations include the product of inertia term i_E , Thorpe[4] has recently shown that serious errors may arise in the estimation of R if this term is omitted. It has been convenient to check this point in the present investigation and the results are included in Tables II - IV. The calculations show that both R and J are affected by such an omission (Table II), and, in view of the remarks in Appendix II, the omission of i_E must introduce appreciable errors into any lateral response calculations made on aircraft flying at moderate and high values of C_L .

The results in Tables III and IV suggest that if R and J are calculated on the assumption that i_E is negligible and are then substituted into the simplified expressions for β etc., the errors incurred may be comparable with those associated with the results obtained when rolling is ignored.

Since the coupling term has such a powerful influence on the estimated response of an aircraft, the relative inclinations of the principal and body axes should be determined before response calculations are attempted.

TABLE I

Relevant Aerodynamic Data

	Aircraft "A"	Aircraft "B"	Aircraft "C"
C_L	0.147	0.2	0.3
μ_2	36.8	50.02	91.4
$\hat{t}$	1.34	1.601	2.654
k	0.0735	0.1	0.15
ℓ_v	-0.04	-0.062	-0.081
ℓ_p	-0.34	-0.21	-0.258
ℓ_r	0.04	0.055	0.139
n_v	0.07	0.055	0.086
n_p	0.05	-0.011	0
n_r	-0.08	-0.07	-0.157
$\bar{y}_v$	0.23	0.177	0.168
a_1	2.5	2.35	2.78
a_2	1.8	0.85	0.316
i_A	0.07	0.063	0.055
i_C	0.14	0.278	0.290
i_E	0.005	-0.0056	-0.0058
ω_n	18.4	9.895	17.965
ω_ℓ	20.98	49.137	134.85
ν_ℓ	4.85	3.355	4.699
$\nu_{\ell r}$	0.57	0.879	2.532
ν_{np}	-0.29	0.04	0
ν_n	0.57	0.252	0.541
δ_n	22.53	5.995	8.794
B_2	5.6098	3.7974	5.4587
C_2	21.5024	12.3986	24.0544
D_2	85.2019	40.3188	105.3632
E_2	0.1079	0.3679	4.1286

TABLE II

Roots of the Lateral Dynamic Stability Equation

Aircraft	Method of Solution	ε	r_s	R'	f	h	R	J	Remarks
"A"	Exact	-0.0331	0.00127	4.8169	0.79158	17.6823	0.39579	4.1864	Quartic factorized by "trial and error" method
	Exact	-0.03308	0.00127	4.81692	0.79156	17.6824	0.39578	4.1864	Quartic factorized by Neumark's method (two approximations)
	Simplified	-	-	-	-	-	0.400	4.2928	Simplified values of R and J
	Modified	-	-	-	-	-	0.39579	4.1864	Exact values of R and J
	Exact	0.1018	0.00127	4.9518	0.69699	18.7873	0.34849	4.3204	Omission of coupling terms from quartic
"B"	Exact	0.1654	0.00915	3.520	0.26825	11.4197	0.13412	3.3766	Quartic factorized by "trial and error" method
	Exact	0.16597	0.00915	3.5206	0.26768	11.4216	0.13384	3.3769	Quartic factorized by Neumark's method (two approximations)
	Simplified	-	-	-	-	-	0.2144	3.1455	Simplified values of R and J
	Modified	-	-	-	-	-	0.13412	3.3766	Exact values of R and J
	Exact	0.30708	0.00917	3.6617	0.11255	10.9667	0.0563	3.3111	Omission of coupling terms from quartic
"C"	Exact	0.2022	0.03954	4.9016	0.51756	21.3033	0.25878	4.6083	Quartic factorized by "trial and error" method
	Exact	0.2024	0.03950	4.9018	0.51738	21.3042	0.25869	4.6084	Quartic factorized by Neumark's method (two approximations)
	Simplified	--	-	-	-	-	0.3547	4.2344	Simplified values of R and J
	Modified	-	-	-	-	-	0.25878	4.6083	Exact values of R and J
	Exact	0.47471	0.03959	5.1741	0.19510	20.1674	0.0976	4.4898	Omission of coupling terms from quartic

TABLE III

Maximum Sideslip Angles (Local)

Aircraft	Method of Solution	R	J	$\frac{R}{J}$	Manoeuvre (1): $\left(\frac{\beta}{\zeta_o}\right)_{J\tau=\pi}$	Manoeuvre (1): % Error	Manoeuvre (2) - (Fish-Tail): $\left(\frac{\beta}{\zeta_e}\right)_{J\tau=2\pi}$	Manoeuvre (2) - (Fish-Tail): % Error	Manoeuvre (2) - (Fish-Tail): $\left(\frac{\beta}{\zeta_e}\right)_{J\tau=3\pi}$	Manoeuvre (2) - (Fish-Tail): % Error	Remarks
"A"	Exact	0.39579	4.1864	0.0945	2.2024	-	-3.0290	-	3.9820	-	Equations (8) and (10) respectively
	Simplified	0.40	4.2928	0.0932	2.1169	-3.90	-2.9023	-4.17	3.8264	-3.91	Equations (15) and (16) respectively
	Modified	0.39579	4.1864	0.0945	2.2213	+0.90	-3.0398	-0.37	4.0016	0.50	Exact values of R and J in Equations (15) and (16)
	Modified	0.34849	4.3204	0.0807	2.1303	-3.27	-2.9703	-1.92	3.9778	-0.10	R and J modified by omission of i_E ; Table II
"B"	Exact	0.13412	3.3766	0.0397	0.9616	-	-1.4525	-	2.0430	-	Equations (8) and (10) respectively
	Simplified	0.2144	3.1455	0.0682	1.090	13.35	-1.5466	6.63	2.1042	2.99	Equations (15) and (16) respectively
	Modified	0.13412	3.3766	0.0397	0.9884	2.79	-1.4627	0.84	2.0662	1.13	Equations (15) and (16) Exact values of R and J
	Modified	0.0563	3.3111	0.0170	1.0648	10.7	-1.6291	12.31	2.3808	16.53	R and J modified by omission of i_E ; Table II
"C"	Exact	0.2588	4.6083	0.0562	0.7261	-	-1.0942	-	1.4931	-	Equations (8) and (10) respectively
	Simplified	0.3547	4.2344	0.0838	0.8614	18.63	-1.1959	9.3	1.5955	6.86	Equations (15) and (16) respectively
	Modified	0.2588	4.6083	0.0562	0.7589	4.5	-1.0953	0.10	1.5141	1.4	Equations (15) and (16) Exact values of R and J
	Modified	0.0976	4.4898	0.0217	0.8438	16.15	-1.2808	17.05	1.8590	24.51	R and J modified by omission of i_E ; Table II

TABLE IV(a)

Local Maximum Fin and Rudder Loads - "Instantaneous Rudder Deflection"

Aircraft	Method of Solution	$\left(\frac{\beta}{\zeta_o}\right)_{J\tau=\pi}$	$\frac{d}{d\tau}\left(\frac{\beta}{\zeta_o}\right)_{J\tau=\pi}$	B	C	$-B\frac{\beta}{\zeta_o}$	$-C\frac{d}{d\tau}\left(\frac{\beta}{\zeta_o}\right)$	a_2	$\frac{P}{A\zeta_o}$	% Error	Remarks
"A"	Exact	2.2024	-0.0357	2.5167	0.0727	-5.5418	0.0026	1.8	-3.7392	-	
	Simplified	2.1170	0	2.5167	0.0727	-5.3279	0	1.8	-3.5279	-5.65	
	Modified	2.2213	0	2.5167	0.0727	-5.5860	0	1.8	-3.7860	1.25	Exact values of R and J
	Modified	2.1303	0	2.5167	0.0727	-5.3613	0	1.8	-3.5613	-4.76	R and J modified by omission of i_E
"B"	Exact	0.9616	-0.0456	2.3554	0.0304	-2.2650	0.0014	0.85	-1.4136	-	
	Simplified	1.090	0	2.3554	0.0304	-2.5674	0	0.85	-1.7174	21.49	
	Modified	0.9884	0	2.3554	0.0304	-2.3281	0	0.85	-1.4781	4.56	Exact values of R and J
	Modified	1.0648	0	2.3554	0.0304	-2.5080	0	0.85	-1.6580	17.3	R and J modified by omission of i_E
"C"	Exact	0.7261	-0.091	2.7851	0.03	-2.022	0.0027	0.316	-1.7036	-	
	Simplified	0.8614	0	2.7851	0.03	-2.3991	0	0.316	-2.0831	22 28	
	Modified	0.7589	0	2.7851	0.03	-2.1136	0	0.316	-1.7976	5.52	Exact values of R and J
	Modified	0.8434	0	2.7851	0.03	-2.3488	0	0.316	-2.0328	19.33	R and J modified by omission of i_E

TABLE IV(b)

Local Maximum Fin and Rudder Loads - "Fish-Tail Manoeuvre"

Aircraft	Method of Solution	$J\tau = 2\pi$: $-B\frac{\beta}{\zeta_e}$	$J\tau = 2\pi$: $-C\frac{d}{d\tau}\left(\frac{\beta}{\zeta_e}\right)$	$J\tau = 2\pi$: $\frac{P}{A\,\zeta_e}$	$J\tau = 2\pi$: % Error	$J\tau = 3\pi$: $-B\frac{\beta}{\zeta_e}$	$J\tau = 3\pi$: $-C\frac{d}{d\tau}\left(\frac{\beta}{\zeta_e}\right)$	$J\tau = 3\pi$: $\frac{P}{A\,\zeta_e}$	$J\tau = 3\pi$: % Error	Remarks
"A"	Exact	7.6218	-0.0481	7.5737	-	-10.02	0.0636	-9.9579	-	
	Simplified	7.3042	-0.0422	7.2620	-4.1	-9.630	0.0556	-9.5743	-3.85	
	Modified	7.6503	-0.0437	7.6066	0.43	-10.07	0.0576	-10.013	0.55	Exact values of R and J
	Modified	7.4753	-0.0376	7.4377	-1.8	-10.011	0.0503	-9.9607	0.03	R and J modified by omission of i_E
"B"	Exact	3.4165	-0.0055	3.4110	-	-4.8123	0.0079	-4.804	-	
	Simplified	3.6429	-0.005	3.6380	6.7	-4.9562	0.0069	-4.9494	3.02	
	Modified	3.4452	-0.003	3.4422	0.9	-4.8666	0.0042	-4.8624	1.01	Exact values of R and J
	Modified	3.8372	-0.0014	3.8358	12.5	-5.6078	0.0020	-5.6058	16.7	R and J modified by omission of i_E
"C"	Exact	3.0474	-0.008	3.0392	-	-4.1584	0.0115	-4.1469	-	
	Simplified	3.3308	-0.0064	3.3244	9.38	-4.4436	0.0086	-4.4350	6.95	
	Modified	3.0505	-0.0030	3.0475	0.30	-4.2168	0.0059	-4.2109	1.54	Exact values of R and J
	Modified	3.5670	-0.0019	3.5651	17.3	-5.1774	0.0027	-5.1746	24.80	R and J modified by omission of i_E

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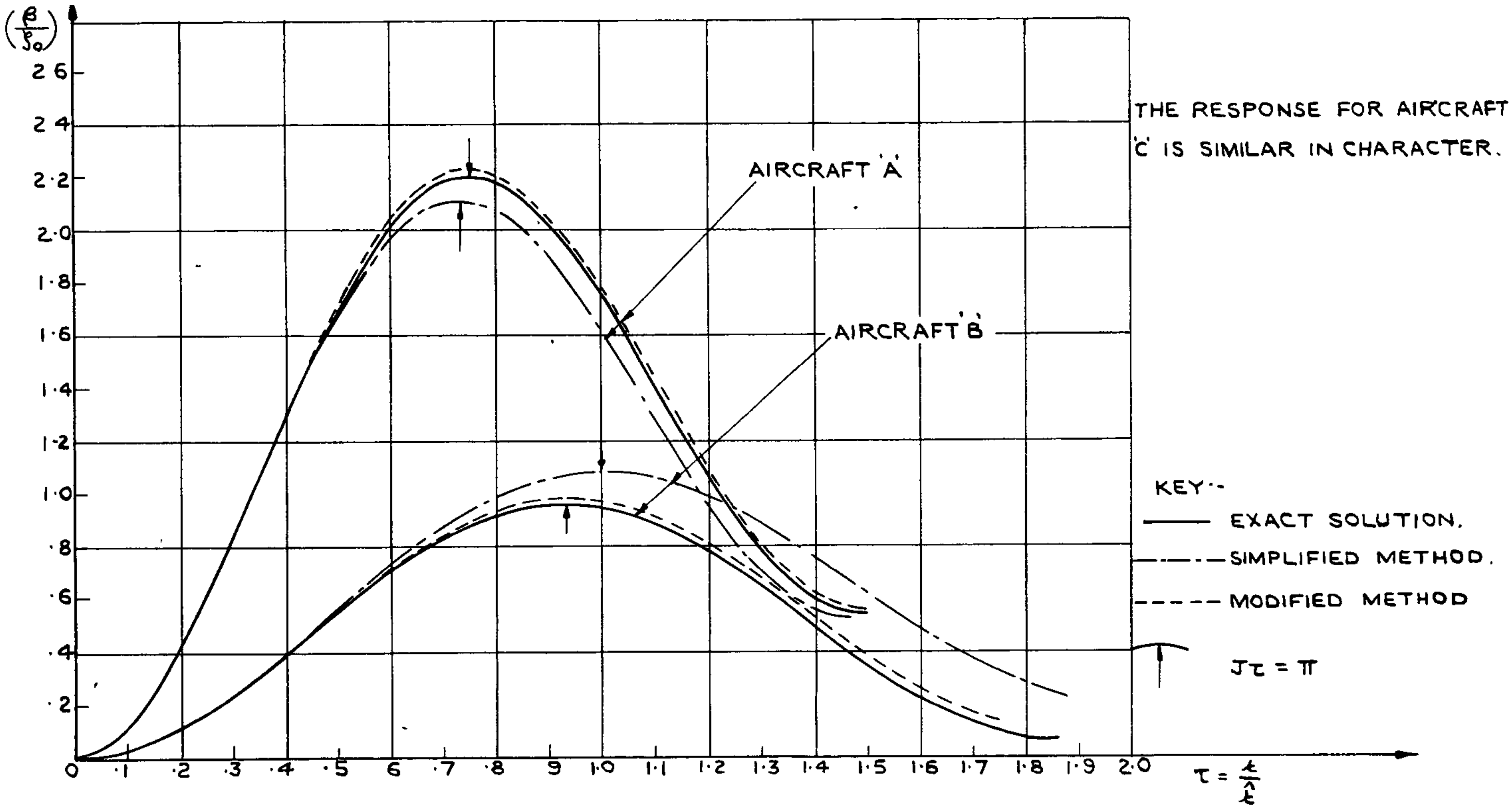

FIG 1. RESPONSE TO AN INSTANTANEOUS RUDDER DEFLECTION ζ_0

FIG 1.

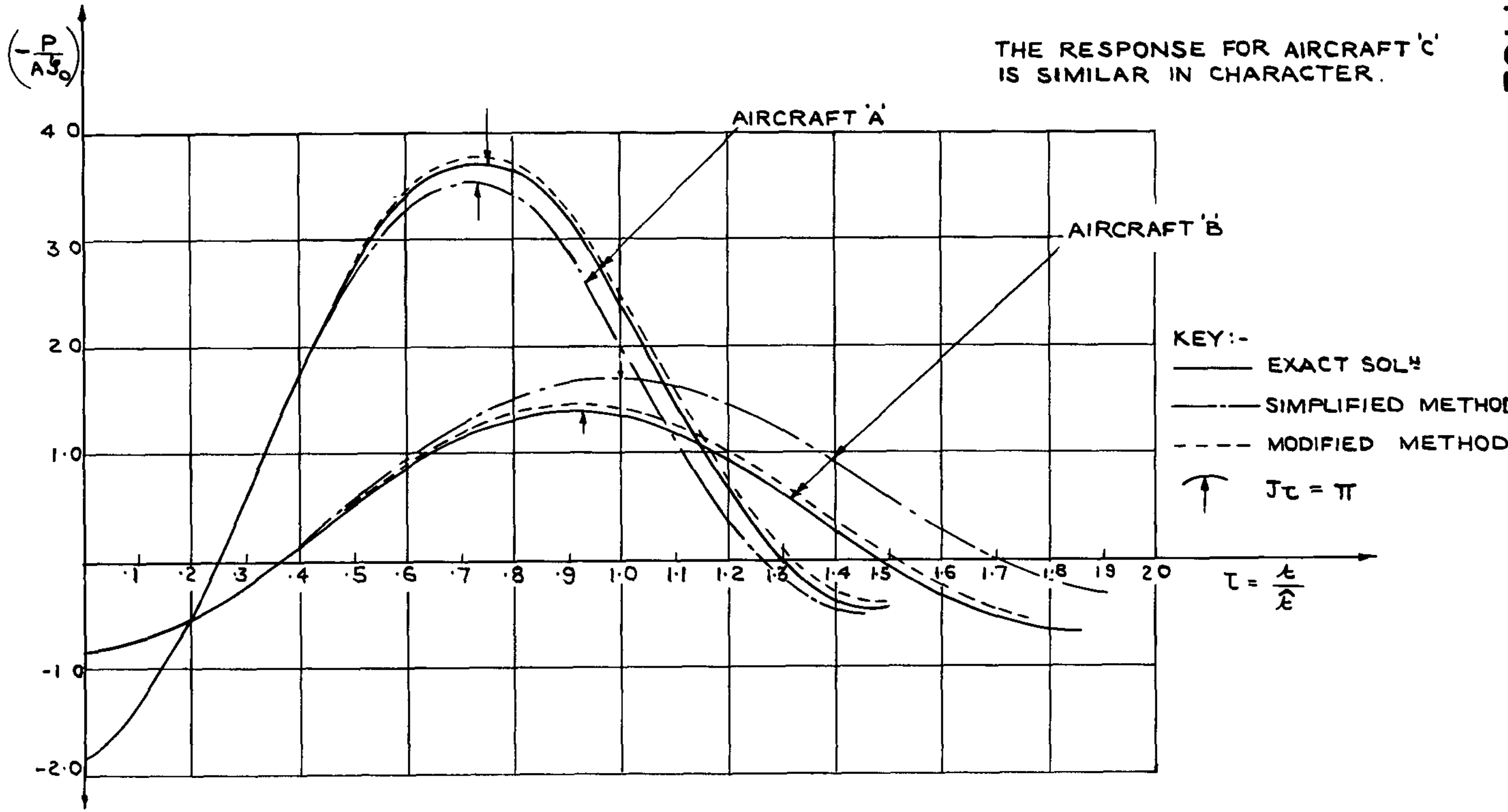

FIG 2. TIME HISTORY OF FIN & RUDDER LOADS AFTER AN INSTANTANEOUS RUDDER DEFLECTION ζ_0

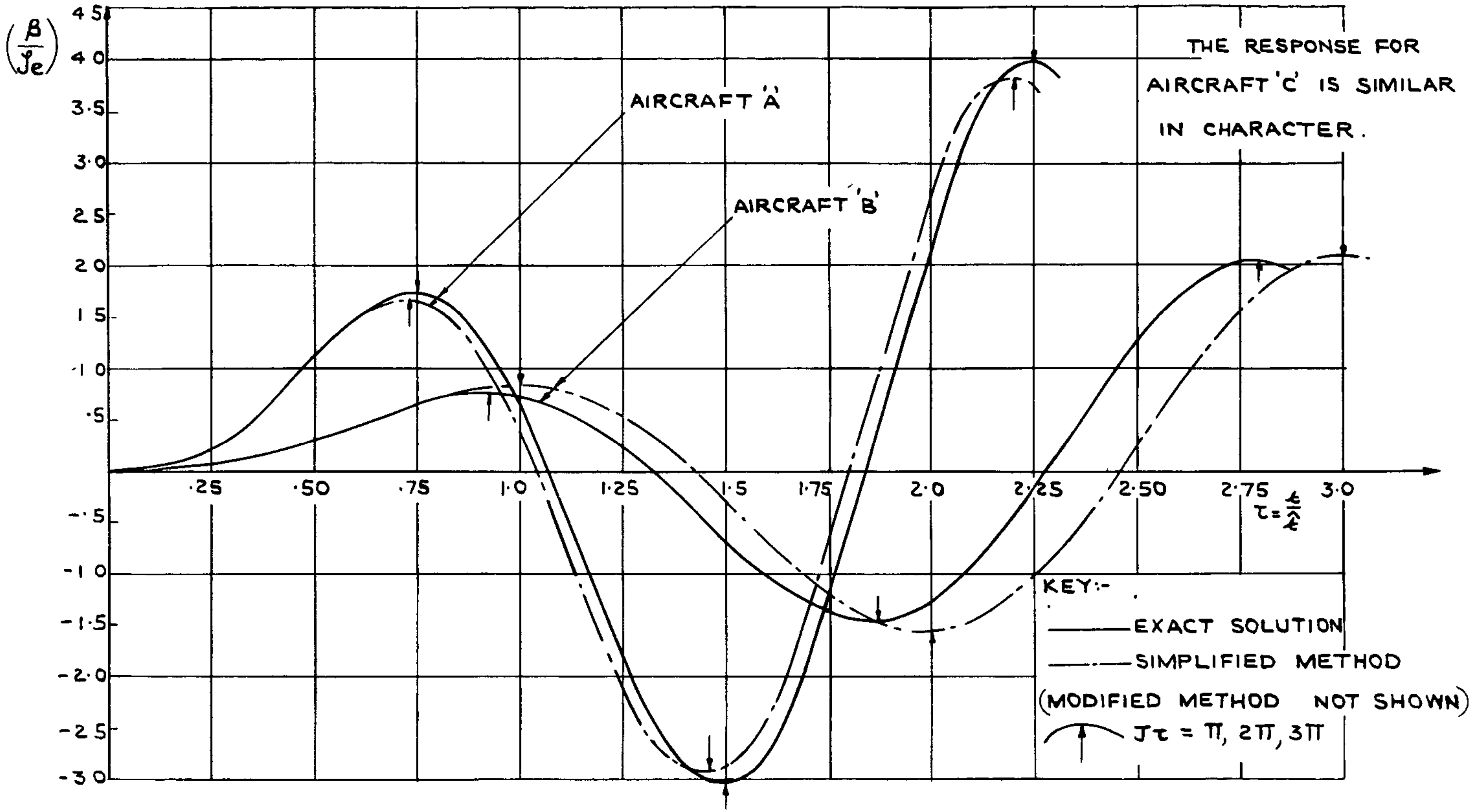

FIG 3. RESPONSE TO A SINUSOIDAL RUDDER DEFLECTION $\zeta = \zeta_e \sin J\tau$

FIG 3.

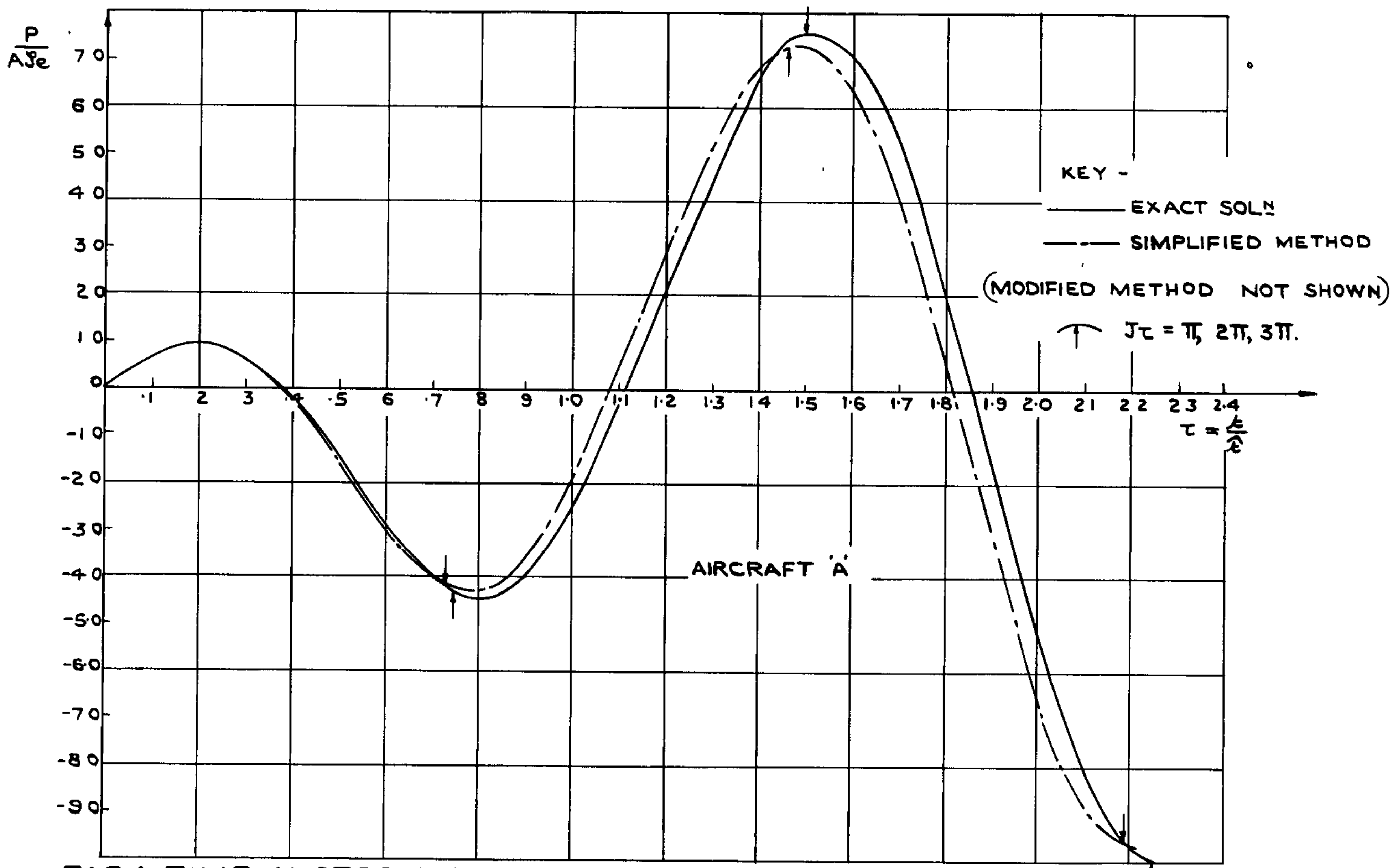

FIG4. TIME HISTORY OF FIN & RUDDER LOADS PRODUCED BY A SINUSOIDAL RUDDER DEFLECTION $\zeta = \zeta_e \sin J\tau$.

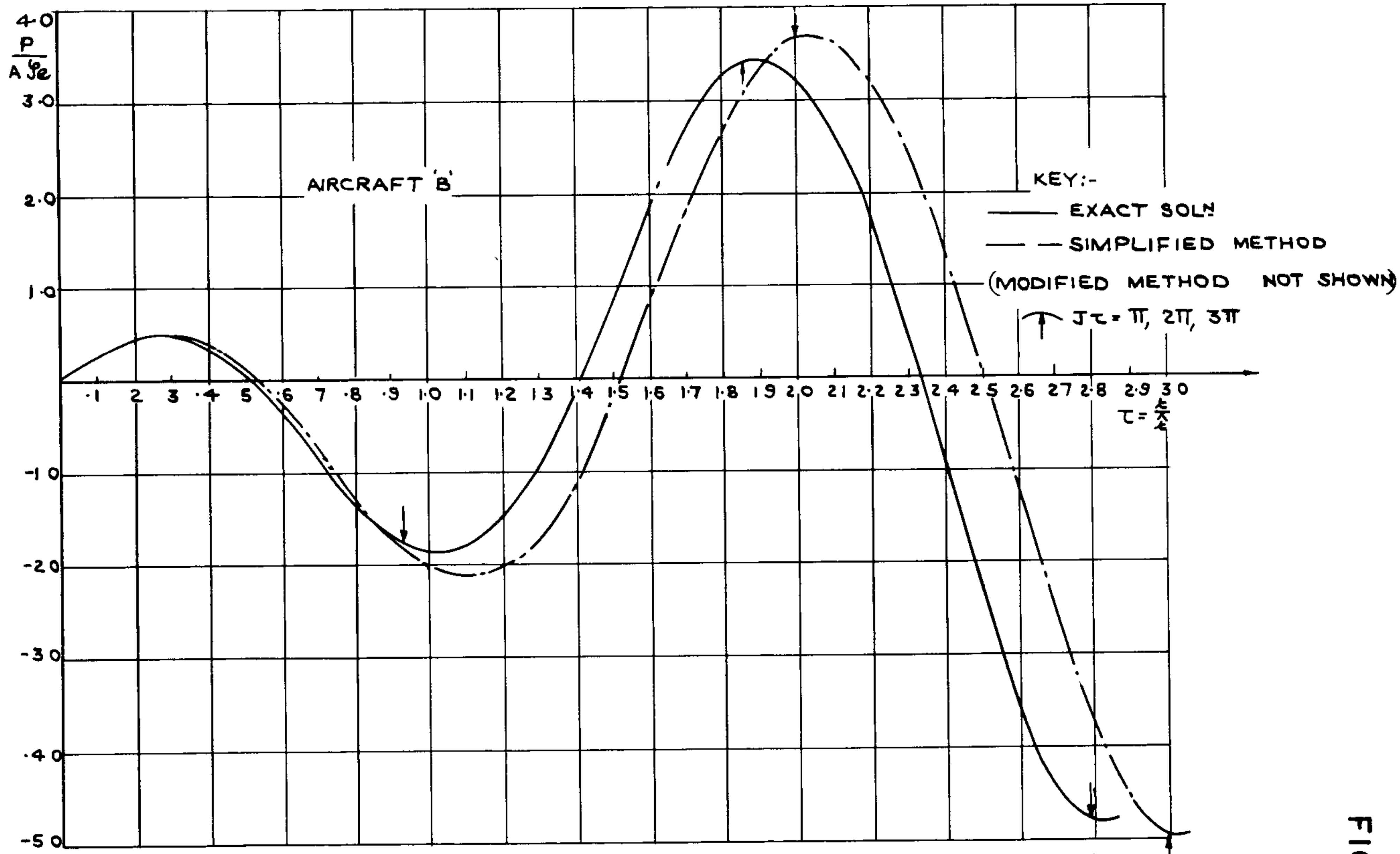

FIG 5. TIME HISTORY OF FIN & RUDDER LOADS PRODUCED BY A SINUSOIDAL RUDDER DEFLECTION $\zeta = \zeta_e \sin J\tau$

FIG 5

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