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A Review and Assessment of Various Formulae for Turbulent Skin Friction in Compressible Flow

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ADDENDA

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1 Addendum to Introduction, p.4

It should be mentioned that many of the theories developed for turbulent skin friction in compressible flow stem from the pioneer work of Frankl and Voishel[14].

2 Extension of range of validity of conclusion 6, p.31

Additional experimental evidence[15-18] now makes it possible to extend the comparisons shown in Figs.3 and 7. This is done in Fig.10, which shows that both Rubesin's interpolation formula (equation 33) and also equation 32b (resulting from the assumption of constancy of velocity profile) give variations of skin friction in reasonable agreement with experiment at least up to $M = 4.5$. All the experimental results were obtained under zero heat transfer conditions, but if conclusion 3 is valid this would mean that the formulae would also be adequate under heat transfer conditions for values of T_w/T_1 up to 5. Experimental checks of the heat transfer case are obviously highly desirable.

Some of the experimental results in Fig.10 are for local skin friction, whereas the curves are based on formulae for mean skin friction. The two are not strictly comparable if the Mach number variation is dependent on the Reynolds number of the test, as is indicated by theory. For example at $T_w/T_1 = 4$ and $Re = 10$ million, the assumption of constancy of velocity profile would give a value of local skin friction ratio C_f/C_{f_1} approximately 5 per cent below the corresponding ratio for mean skin friction as given in Fig.10. However a difference of this order was not considered significant.

Likewise, small variations could also arise from alterations in the index chosen for the viscosity temperature relationship (which was taken as 0.8 in Fig.10)

REFERENCES

No.	Author	Title, etc.
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15	P.F. Brinich and N.S. Diaconis	Boundary layer development and skin friction at Mach number 3.05 NACA Tech. Note 2742 1952
16	D.R. Chapman and R.H. Kester	Measurements of turbulent skin friction on cylinders in axial flow at subsonic and supersonic velocities J.Ae.Sci. 20, 441-448 July, 1953
17	D. Coles and F.E. Goddard	Direct measurement of skin friction on a smooth flat plate at supersonic speeds Paper presented at the eighth International Congress on Theoretical and Applied Mechanics at Istanbul August, 1952
18	J.A. Cole and W.F. Cope	Boundary layer measurements at a Mach number of $2\frac{1}{2}$ in the presence of pressure gradients ARC.12618 May, 1949

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A review and assessment of various formulae for turbulent skin friction in compressible flow

by

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SUMMARY

Despite a lack of experimental evidence, numerous formulae have been developed for the variation of turbulent skin friction on a flat plate in compressible flow, with and without heat transfer.

The present note makes an extended comparison of available formulae and examines the assumptions made in their development, checking against experimental evidence where possible.

It shows that the shearing stress assumption

$$\tau_o = \rho k^2 \left(\frac{du}{dy}\right)^4 \bigg/ \left(\frac{d^2u}{dy^2}\right)$$

gives the values of turbulent skin friction in best agreement with experimental results in the region $1.6 < M < 2.8$ under zero heat transfer conditions. However, the formulae given in Refs.9 and 10 (based on the assumption of constancy of velocity profile) give equally good agreement with experimental results in this range, have the merit of simplicity and under heat transfer conditions the formula of Ref.10 gives a reasonable approximation to results obtained from the above shearing stress assumption.

Therefore it is suggested that the latter formulae should be sufficiently accurate in application probably up to $M = 4$ until further experimental evidence may make refinement possible.

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ADDENDA TO ILLUSTRATIONS

1 Introduction

By contrast with the laminar boundary layer, evaluation of the turbulent boundary layer in compressible flow is hampered by a lack of knowledge of the mechanism of turbulence under such conditions. Even measurements of the mean flow conditions in compressible turbulent boundary layers are few in number, particularly under heat transfer conditions.

However a number of authors have developed formulae for the skin friction in the compressible turbulent boundary layer on a flat plate with zero pressure gradient. There have been two starting points for these developments. The first is to generalise the expression for shearing stress accepted in incompressible flow and to make plausible assumptions concerning the boundary conditions. This was done by Rubesin[2], Ferrari[3], Li[4], Van Driest[5], Clemmow[6], Wilson[11] and by Smith and Harrop[12]. The second is to assume that the log-law velocity distribution found in incompressible flow will apply also in compressible flow if density and viscosity are evaluated at wall temperature. This was done by Cope[9] and Monaghan[10].

As a result, numerous formulae are available, some more complex than others, and comparison is made difficult by differences in notation and in methods of presentation. A comparison of some of the formulae is made by Rubesin, Maydew and Varga in Ref.3. The object of the present note is to extend this comparison and also to examine more fully the back-ground to the assumptions made in the various investigations.

To that end, section 2 gives a survey of the position in incompressible flow and sections 3 and 4 examine the generalisations which are made in going to compressible flow. Section 5 then gives a graphical comparison of the various formulae available under zero heat transfer conditions and section 6 considers the case when heat is being transferred. It will be seen that of the two "starting points" mentioned above, the first is not so far in advance of the second as it might appear, mainly because of the restrictive nature of the boundary conditions assumed.

An attempt has been made to keep the mathematics to a minimum and experimental evidence is drawn upon where available.

2 Velocities and skin friction. Incompressible flow

In the mathematical treatment of turbulent incompressible flow (see Ch.V of Ref.1) it is usually assumed that the motion can be separated into

(a) a mean flow whose components are u, v, w*

and (b) a superposed turbulent flow whose components are u', v', w', the mean values of which are zero.

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* u, v and w are the velocity components parallel to the x, y and z axes respectively. The notation here is changed from that of Ref.1, where U, V and W refer to the mean flow and u, v and w refer to the fluctuating flow.

Thus at any time the velocities at a point in the fluid are $u + u'$, $v + v'$, $w + w'$ and by substitution of these quantities in the equations of motion and of continuity, and by taking mean values, it is found that the resulting equations for the mean flow can be expressed in "laminar" form if stress components such as

$$-\rho\overline{u'v'}$$

are added to the stresses associated with viscosity. These virtual stresses are known as Reynolds stresses. (The bar over the top of a fluctuating quantity denotes a mean value with respect to time at a fixed point. Thus by definition

$$\overline{u'} = \overline{v'} = \overline{w'} = 0$$

but, in general, correlations such as

$$\overline{u'v'}$$

will not be zero).

After making the usual boundary layer approximations it can then be shown that the shearing stress (τ) in a two dimensional incompressible turbulent boundary layer is given by

$$\tau = -\rho\overline{u'v'} + \mu\frac{du}{dy} \tag{1}$$

where ρ is density

and μ is viscosity.

Except in limited regions very close to the wall, the Reynolds stress will outweigh the viscous stress, in which case equation (1) becomes

$$\tau = -\rho\overline{u'v'} \tag{1a}$$

The quantity $\rho\overline{u'v'}$ can be interpreted as representing the rate of transport by turbulence of x-momentum across unit area of a plane normal to the y-axis. For its determination, Prandtl put forward the hypothesis that

$$\overline{u'v'} = -\ell^2\left|\frac{du}{dy}\right|\cdot\left(\frac{du}{dy}\right) \tag{2}$$

if the mean velocity u is a function of y only. Equation (2) serves to define a "mixture length" ℓ and is based on the assumption that

$$\overline{u'^2} \simeq \overline{v'^2}$$

and that u' and v' are closely correlated.

Over a small region of the boundary layer close to the wall we may assume that

$$\tau = \text{const}$$
$$= \tau_o$$

where τ_o is the local shearing stress at the wall.

In the same region there are two hypothesis available for predicting ℓ. The first is that of Prandtl, who took

$$\ell = k_1 y \tag{3a}$$

and the second is that of Von Karman, who took

$$\ell = k_2 \frac{du}{dy} \Big/ \frac{d^2u}{dy^2} \tag{3b}$$

where k_1 and k_2 are constants.

Velocity distributions near the wall can then be determined as follows.

2.1 Velocity distributions

2.11 Assuming the Prandtl mixing length $\ell = k_1 y$

Taking $\tau = \tau_o$, combination of equations (1a), (2) and (3a) gives a first order differential equation which integrates to give the velocity distribution

$$\varphi = \frac{1}{k_1} \ln \eta + \text{const} \tag{4}$$

where $\varphi = \frac{u}{u_\tau}$

$\eta = \frac{y u_\tau}{\nu}$

ν is kinematic viscosity $= \frac{\mu}{\rho}$

and $u_\tau = \sqrt{\left(\frac{\tau_o}{\rho}\right)}$ is the so called "friction velocity".

Because of the assumptions made in its derivation we should only expect this distribution to be valid close to the wall, but in fact it is found that experimentally observed velocity distributions in pipes are well fitted in their entirety (Ref.1, Ch.VIII, pp.336) by taking $k_1 = 0.4$ and the constant to be 5.5, i.e.

$$\begin{aligned} \varphi &= 2.5 \ln \eta + 5.5 \\ &= 5.75 \log_{10} \eta + 5.5 \end{aligned} \tag{5}$$

as plotted in Fig.1

The same experimental results also support the idea that very close to the wall there is a "laminar sub-layer" where viscous stresses predominate. In this region (from equation (1))

$$\tau_o = \mu \frac{du}{dy}$$

and hence the velocity profile is given by

$$\varphi = \eta \qquad (6)$$

as plotted in Fig.1.

Between the laminar sub-layer (equation (6)) and the turbulent core (equation (5)), there is a transition region where the viscous and Reynolds stresses are of comparable magnitude. It happens that equation (4) gives a reasonable fit to the experimental results in this region if we choose $k_1 = 0.2$ and the constant to be -3.05, i.e.

$$\varphi = 5 \ln \eta - 3.05 \qquad (5a)$$

This curve is shown in Fig.1. It meets the turbulent core (equation (5)) at $\eta = 30$ and has a smooth join with the laminar sub-layer (equation (6)) at $\eta = 5$.

Between them, equations (5), (5a) and (6) specify the velocity distribution across the whole of the turbulent boundary layer. This analysis including the transition region is due to Von Karman. Earlier analyses made by Taylor and by Prandtl postulated that there was a sharp boundary between the laminar sub-layer (equation (6)) and the turbulent core (equation (5)). Reference to Fig.1 shows that this would occur at

$$\varphi = \eta = 11.6$$

and this value is usually denoted by "s". This is of interest since it forms a boundary condition much used by authors when deriving formulae for the turbulent boundary layer in compressible flow. It can easily be shown in this case that combining equation (4) with the boundary condition

$$\varphi = \eta = s \qquad (7)$$

at the edge of the laminar sub-layer gives

$$\varphi = \frac{1}{k_1} \ln \eta + \frac{1}{k_1} \ln \frac{e^{k_1 s}}{s} \qquad (4a)$$

as the equation for the velocity distribution in the turbulent core.

2.12 Assuming the Von Karman mixing length $\ell = k_2 \frac{du}{dy} \Big/ \frac{d^2u}{dy^2}$

In this case, combination of equations (1), (2) and (3b) gives a second order differential equation which, by successive integrations gives

$$\frac{d\varphi}{d\eta} = C_1 \, e^{-k_2 \varphi} \qquad (8)$$

and

$$\varphi = \frac{1}{k_2} \ln k_2 C_1 + \frac{1}{k_2} \ln (\eta + C_2) \qquad (9)$$

Thus in distinction to the result obtained from the Prandtl mixing length, there are now two constants (in addition to k_2) available for determining the velocity distribution.

Following the Taylor-Prandtl analysis, one constant is determined by the boundary condition

$$\varphi = \eta = s \qquad (7)$$

At the junction with the laminar sub-layer, and the second constant can be determined by assigning a slope to the profile at the same point, i.e.

$$\frac{d\varphi}{d\eta} = f \qquad (7a)$$

when $\varphi = \eta = s$

Equation (9) then becomes

$$\varphi = \frac{1}{k_2} \ln k_2 f \, e^{ks} + \frac{1}{k_2} \ln \left(\eta + \frac{1}{k_2 f} - s\right) \qquad (9a)$$

The form of the result usually quoted (equation (5)) requires

$$s = \frac{1}{k_2 f}$$

and $k_2 = k_1$

(Thus if $k_1 = 0.4$ and $s = 11.6$ we have $f = 0.216$).

The subscripts to k will therefore be dropped in the subsequent work.

Equation (9a) also permits another solution which is of interest in that it agrees fairly well with experimental results in the transition region as well as in the turbulent core. This is obtained by taking

$$f = 1$$

(i.e. continuity of slope at the junction with the laminar sub-layer) and by choosing s so that the resulting profile agrees with equation (5)

$$\varphi = 5.5 + 2.5 \ln\eta \qquad (5)$$

when η is large. With $k = 0.4$ this means that

$$s = 7.8$$

and equation (9a) becomes

$$\varphi = 5.5 + 2.5 \ln(\eta - 5.3) \qquad (9b)$$

The resulting curve is shown in Fig.1.

The form of equation (9b) is referred to in Ref.2 as the basis of the "Buffer Layer Analysis" and it should be noted that it cannot be obtained from the Prandtl mixing length. (Section 2.11).

2.2 Skin friction

This is obtained by substituting the above velocity profiles in the momentum equation

$$\tfrac{1}{2} c_f = \frac{\tau_o}{\rho u_1^2} = \frac{d}{dx} \int_0^\delta \frac{u}{u_1}\left(1 - \frac{u}{u_1}\right) dy \qquad (10)$$

where c_f is the local skin friction coefficient

and u_1 is the free stream velocity

(This is the form of the momentum equation which is applicable when there are no pressure gradients in the stream direction).

Put $\dfrac{u_1}{u_\tau} = v \left\{ = \sqrt{\left(\dfrac{2}{c_f}\right)} \right\}$

and $z = \dfrac{u}{u_1}$

(Thus $\varphi = \dfrac{u}{u_\tau} = vz$)

then equation (10) can be written

$$\frac{u_1}{\nu} = v^2 \frac{d}{dx}\left\{ v^2 \int_0^1 z\,(1 - z)\, \frac{d\eta}{d\varphi}\, dz \right\} \qquad (10a)$$

and it can easily be shown that use of either the Prandtl or the Karman mixing length gives

$$\frac{d\eta}{d\varphi} = \frac{1}{f} e^{k(\varphi - s)}$$

$$= \frac{1}{f} e^{kv(z - \frac{s}{v})} \qquad (8a)$$

Substituting equation (8a) in equation (10a) and integrating neglecting the very small influence of the laminar sub-layer where $\frac{d\eta}{d\varphi} = 1$) we obtain the final result, valid if v is large,

$$\frac{1}{c_f^{\frac{1}{2}}} = \frac{1}{\sqrt{2}k} \ln \frac{k^2 f e^{ks}}{2} + \frac{1}{\sqrt{2}k} \ln Re_x \, c_f \qquad (11)$$

where $Re_x = \frac{u_1 x}{\nu}$.

Numerical values are as follows, for $k = 0.4$,

(a) from the Taylor-Prandtl analysis which assumes a sharp junction between the laminar sub-layer and turbulent core, giving $s = 11.6$ and $f = 0.216$,

$$\frac{1}{c_f^{\frac{1}{2}}} = 1.04 + 4.07 \log_{10} Re_x \, c_f \qquad (11a)$$

and (b) from the buffer layer analysis, giving $s = 7.8$, and $f = 1$,

$$\frac{1}{c_f^{\frac{1}{2}}} = 1.05 + 4.07 \log_{10} Re_x \, c_f \qquad (11b)$$

and this equation is close to that which would be obtained by modifying the Taylor-Prandtl analysis to include Karman's transition region.

However, Kempf's experimental results for a flat plate support the formula

$$\frac{1}{c_f^{\frac{1}{2}}} = 1.7 + 4.15 \log_{10} Re_x \, c_f$$

(usually quoted as the Karman-Kempf formula, see Ref.1, Ch.VIII, pp.364).

Thus a slight forcing of constants is necessary if equation (11), derived from pipe-flow velocity distributions, is to be applied to give the skin friction on a flat plate.

Turning now to mean skin friction coefficients (C_F) defined by

$$C_F = \frac{\int_0^{\ell} \tau_0 \, dx}{\frac{1}{2} \rho u_1^2 \ell}$$

where $\int_0^{\ell} \tau_0 \, dx$ is the total friction over a length ℓ of the plate, then for boundary layers which are turbulent from the leading edge, experimental results are fitted by Schoenherr's formula

$$\frac{1}{C_F^{\frac{1}{2}}} = 4.13 \log_{10} Re \, C_F \qquad (12)$$

where $$Re = \frac{u_1 \ell}{\nu}$$

Comparison with equation (11) shows that only the constant k appears in this formula. (The value 4.13 corresponds to $k = 0.392$).

Equation (12) is not in a convenient form for making quick estimates of skin friction for a given Reynolds number, so in its place the Prandtl-Schlichting interpolation formula (with constants given by Wieghardt),

$$C_F = 0.46 \, (\log_{10} Re)^{-2.6} \qquad (13)$$

is found useful. A further formula in more general use is

$$C_F = 0.074 \, Re^{-1/5} \qquad (14)$$

given by Blasius.

A comparison of these three formulae (equations (12) to (14)) is made in Fig.2 over the range

$$10^6 < Re < 10^8$$

The Schoenherr (equation (12)) and Prandtl-Schlichting (equation (13)) formulae are in close agreement over the whole of the range whereas the Blasius formula (equation (14)) agrees with them only in the region of $Re = 10^7$. This demonstrates a limitation of the Blasius power-law representation.

3 General remarks on the shearing stress relation in compressible turbulent boundary layers

The picture of the incompressible turbulent boundary layer presented in section 2 was artificial in that apparent stresses were introduced in order to retain a standard form for the equations of the mean flow and a number of assumptions were then made which enabled a simple solution to be obtained. The justification for this procedure rests entirely on the fact that it produced certain results in good agreement with experiment, which implies that any extension of it must also be backed by experiment.

This applies in particular to the turbulent boundary layer in compressible flow. The standard analytical approach has been to extend the incompressible flow picture to include density fluctuations, etc., but when this is done there is still considerable difficulty in formulating an "equivalent laminar motion" for the mean flow, particularly if the continuity equation is to retain its standard form.

Van Driest[5], Clemmow[6] and Young[7] (in chronological order) are among those who have extended the turbulent boundary layer equations to include compressible flow and there are slight variations in the derivation of their final expressions for the shearing stress in the boundary layer on a flat plate. However it is fortunate that the simplest extension of the incompressible flow picture gives the skin friction formulae which are in the best agreement with the limited experimental data at present available (up to $M = 3$, see below), so that the difficulties in derivation have not yet become of major practical importance.

In incompressible flow we had

$$\tau = -\rho \overline{u'v'} \qquad (1a)$$

which arose from generalising the term ρuv in the equations of motion to include fluctuating as well as mean velocities.

In compressible flow, density fluctuations must be included, and we run into difficulties when attempting to generalise terms such as ρuv. Thus if we generalise the whole quantity we have

$$\rho uv + (\rho uv)' = (\rho + \rho')(u + u')(v + v')$$

and by expanding the right hand side of this equation and taking means, we obtain

$$\overline{(\rho uv)'} = \rho \overline{u'v'} + u \overline{\rho'v'} + v \overline{\rho'u'} + \overline{\rho'u'v'}$$

On the other hand, since we are considering a rate of transport of x-momentum (ρu) by a velocity (v) in the y-direction, it might seem more natural to generalise the quantity $(\rho u).v$, which can be shown to give

$$\overline{(\rho u)'v'} = \rho \overline{u'v'} + u \overline{\rho'v'} + \overline{\rho'u'v'}$$

i.e. there is one term less than in the expansion of $\overline{(\rho uv)'}$.

However, if we are considering conditions within a boundary layer then $v \overline{\rho'u'}$ and $\overline{\rho'u'v'}$ are almost certain to be small by comparison with the remaining terms, and the expressions become identical. There is still difficulty in equating either expression with the shearing stress, but if we make this assumption then

$$\tau = -\rho\overline{u'v'} - u\,\overline{\rho'v'} \tag{15}$$

when viscous stresses are neglected.

As in section 2 we shall assume that $\tau = \tau_o$ and that

$$\overline{u'v'} = -\ell^2\left(\frac{du}{dy}\right)^2$$

We now make the additional assumption that

$$\overline{\rho'v'} = -\alpha\,\ell^2\,\frac{du}{dy}\cdot\frac{d\rho}{dy}$$

(which is one of the simplest assumptions that can be made concerning the behaviour of $\overline{\rho'v'}$).

Substituting in equation (15) we obtain

$$\tau_o = \left(\rho\frac{du}{dy} + \alpha u\frac{d\rho}{dy}\right)\ell^2\,\frac{du}{dy} \tag{17}$$

which is in the form used by Li and Nagamatsu[4] and (for $\alpha = 1$) by Clemmow[6].

Fig.3, taken from Li and Nagamatsu's article[4] shows the effect of the factor "α" on the variation of skin friction with Mach number under zero heat transfer conditions. These variations were derived from equation (17) and the momentum equation by a method similar to that described in section 4 below, and were based on the Prandtl mixing length

$$\ell = ky \tag{3a}$$

Also shown in Fig.3 is a representative set of values from the available experimental evidence[2,8,10,11] which indicate that for $M < 3$ (and probably up to $M \doteq 4$) experimental conditions are best represented by taking $\alpha = 0$, i.e. $\overline{\rho'v'}$ negligible by comparison with $\overline{u'v'}$, in which case equation (15) becomes

$$\tau = -\rho\overline{u'v'} \tag{1a}$$

as in incompressible flow.

Li and Nagamatsu suggest that the factor α may vary with Mach number, being zero at low speeds but tending to unity at extremely high speeds. At this stage mention must be made of Ferrari's work[3], which is based on equation (15) but uses less straightforward expressions for $\overline{u'v'}$ and $\overline{\rho'v'}$, the latter containing a suggested arbitrary variation with Mach number. (An appreciation of Ferrari is given by Clemmow[6] who shows that his results are in qualitative agreement with those from equation (17)). However, the worth of such a variation cannot be decided until suitable experimental evidence becomes available at high Mach numbers and to check this and other points, Li and Nagamatsu say that an "intensive experimental programme" will be conducted in the higher supersonic range, using the U.S. Army Ordnance - GALCIT Hypersonic Wind Tunnel.

Meanwhile, the following sections of this note contain a more detailed examination of the effects of various assumptions concerning mixing length, boundary conditions, etc., when applied in conjunction with equations (1a) or (15), but it should be remembered that a thorough examination of the derivation of these equations will become necessary if they are to be applied at Mach numbers greater than about 4.

4 Temperatures, velocities and skin friction. Compressible flow

If, as in incompressible flow, we take

$$\tau = \tau_o = \text{const.}$$

and assume

$$\overline{u'v'} = -\ell^2 \left(\frac{du}{dy}\right)^2 \qquad (2)$$

while

$$\overline{\rho' v'} \ll \overline{u' v'}$$

as was indicated in section 3, then equation (15) gives

$$\tau_o = \rho \ell^2 \left(\frac{du}{dy}\right)^2 \qquad (1b)$$

Two estimates of the mixing length (ℓ) are available from incompressible flow, namely

Prandtl, $$\ell = ky \qquad (3a)$$

and Karman, $$\ell = k\left(\frac{du}{dy}\right) \bigg/ \left(\frac{d^2u}{dy^2}\right) \qquad (3b)$$

but, in distinction to the incompressible flow case, the density (ρ) will now be variable. However, since the static pressure (p) is constant across the boundary layer, the equation of state

$$p = \rho g RT$$

gives

$$\rho T = \text{const.}$$

so that the density distribution can be obtained from the temperature distribution, which will now be considered.

4.1 Temperature distribution across the turbulent boundary layer in compressible flow

An expression for the temperature distribution which has some theoretical and experimental backing is

$$\frac{T}{T_w} = 1 + Bz - A^2 z^2 \qquad (18)$$

where $$z = \frac{u}{u_1}$$

and T_w is the wall (surface) temperature.

Some known values of A and B are as follows.

4.11 Assuming Reynolds analogy, $\sigma = 1$ and $\frac{dp}{dx} = 0$

Reynolds analogy assumes that in turbulent flow momentum and heat will be transferred in the same way. If in addition there is no pressure gradient in the stream direction and if the Prandtl number $\sigma \left(= \frac{C_p \mu}{k} \right)$ is unity, (the latter assumption is necessary because of the presence of a laminar sub-layer and transition region). Then

$$\left. \begin{aligned} B &= \frac{T_{H1}}{T_w} - 1 \\ \text{and} \qquad A^2 &= \frac{T_{H1}/T_1 - 1}{T_w/T_1} \end{aligned} \right\} \qquad (19a)$$

where T_{H1} is the free stream total temperature and is related to the free stream static temperature (T_1) by

$$\frac{T_{H1}}{T_1} = 1 + \frac{\gamma - 1}{2} M_1^2 \qquad (20)$$

where M_1 is the free stream Mach number

and $\gamma = \frac{C_p}{C_v}$ (= 1.4 for air).

Under zero heat transfer conditions, we have

$$T_{H1} = T_w$$

and thus obtain the familiar formula for A,

$$A^2 = \frac{M_1^2}{M_1^2 + 5} \quad \text{if } \gamma = 1.4$$

4.12 From experimental measurements, $\sigma = 0.72$ and $\frac{dp}{dx} = 0$

Experimental measurements[8] of the temperature distribution in a compressible turbulent boundary layer on a flat plate, using air as the working fluid ($\sigma = 0.72$), suggest the values

$$\left.\begin{aligned} B &= \frac{T_{wo}}{T_w} - 1 \\ A^2 &= \frac{T_{wo}/T_1 - 1}{T_w/T_1} \end{aligned}\right\} \qquad (19b)$$

where T_{wo} is the wall temperature for zero heat transfer and is related to T_1 by

$$\frac{T_{wo}}{T_1} = 1 + 0.9\,\frac{\gamma-1}{2}\,M_1^2 \qquad (21)$$

(Thus T_{H1} of section 4.11 is replaced by T_{wo} in the formulae for A and B).

Equation (21) only applies if air is the working fluid, but it is in good agreement with Squire's suggested formula

$$\frac{T_{wo}}{T_1} = 1 + \sigma^{1/3}\,\frac{\gamma-1}{2}\,M_1^2 \qquad (21a)$$

$$= 1 = 0.896\,\frac{\gamma-1}{2}\,M_1^2 \quad \text{if } \sigma = 0.72$$

Thus for fluids other than air it is suggested that equation (21a) should be used for obtaining T_{wo}/T_1 in equation (19b).

4.2 Velocity distribution

Using equation (18) and the fact that

$$\rho\,T = \text{const.}$$

equation (1b) (as given at the beginning of section 4) can now be written

$$\frac{\tau_o}{\rho_w} = \frac{\rho}{\rho_w}\,\ell^2\left(\frac{du}{dy}\right)^2$$

$$= \frac{\ell^2\left(\frac{du}{dy}\right)^2}{\frac{T}{T_w}}$$

$$= \frac{\ell^2\left(\frac{du}{dy}\right)^2}{1 + B\left(\frac{u}{u_1}\right) - A^2\left(\frac{u}{u_1}\right)^2} \qquad (22)$$

In the subsequent work we shall follow the procedure of section 2 and put

$$u_\tau = \sqrt{\left(\frac{\tau_o}{\rho_w}\right)}$$

$$\varphi = \frac{u}{u_\tau}$$

$$v = \frac{u_1}{u_\tau} = \sqrt{\frac{2}{c_{f_w}}}$$

where $$c_{f_w} = \frac{\tau_o}{\frac{1}{2}\ \rho_w u_1^{\,2}}$$

$$z = \frac{u}{u_1} \qquad (\text{thus } \varphi = vz)$$

and $$\eta = \frac{y\ u_\tau}{\nu_w}$$

Thus the symbols will have the same meaning as in section 2 provided the temperature dependent quantities (ρ, μ) are evaluated at wall temperature (T_w).

As in section 2, we shall now obtain velocity distributions,

(a) assuming the Prandtl mixing length, $\ell = ky$, (in equation (22))

and (b) assuming the Karman mixing length, $\ell = k \frac{du}{dy} \Big/ \frac{d^2u}{dy^2}$ (in equation (22))

and shall consider the effect of the boundary conditions imposed.

4.21 Velocity distribution assuming $\ell = ky$

Substituting $\ell = ky$ in equation (22), and making the various substitutions listed above, we obtain a first order differential equation between φ and η, which, when integrated in conjunction with the boundary condition

$$\varphi = \eta = s \tag{7a}$$

at the junction of the laminar sub-layer, gives the velocity distribution

$$\eta = s\ \exp\left\{\frac{kv}{A}\ (\sin^{-1} w - \sin^{-1} w_s)\right\} \tag{23}$$

where

$$w = \frac{Az - \frac{B}{2A}}{\sqrt{\left(1 + \frac{B^2}{4A^2}\right)}} \tag{24}$$

and w_s is the value of w for $\varphi = s$ (i.e. for $z = \frac{s}{v}$).

Equations (23) and (24) reduce to the incompressible form (corresponding to equation (4a)).

$$\eta = s \exp\{kv\,(z - \tfrac{s}{v})\}$$

$$= s \exp\{k\,(\varphi - s)\}$$

when $B \to 0$ followed by $A \to 0$*.

4.22 Velocity distribution assuming $\ell = k \frac{du}{dy} \Big/ \frac{d^2u}{dy^2}$

In this case, substitution in equation (22) gives a second order differential equation, which, when integrated in conjunction with the boundary conditions

$$\frac{d\varphi}{d\eta} = f \qquad (7b)$$

when $\varphi = \eta = s$ (7a)

gives

$$\frac{d\varphi}{d\eta} = f.\exp\left\{-\frac{kv}{A}\left(\sin^{-1} w - \sin^{-1} w_s\right)\right\} \qquad (25)$$

and

$$\eta = \frac{\lambda^2}{\lambda^2+1}\;\frac{\sqrt{\left(1+\frac{B^2}{4A^2}\right)}}{kf}\left(\sqrt{1-w^2}+\frac{w}{\lambda}\right)\exp\left\{\lambda(\sin^{-1} w - \sin^{-1} w_s)\right\} + \text{const.} \qquad (26)$$

where w is given by equation (24)

and $\lambda = \frac{kv}{A}$.

Thus, unlike the incompressible flow case, different velocity distributions result from use of the Prandtl or the Karman hypotheses for mixing length. (It can be verified that equation (26) also reduces to equation (4a) if $B \to 0$ followed by $A \to 0$).

4.23 Effect of boundary conditions

In either case there is the choice of boundary conditions s and f.

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* The same result does not follow from $A \to 0$ followed by $B \to 0$, because when $A = 0$ the fundamental equation is of different degree and has a different solution.

If we assume (as is usually done) that these will have the numerical values found in incompressible flow from the Taylor-Prandtl analysis, i.e.

$$f = 0.216$$

and $$s = 11.6,$$

then we are in effect specifying that in compressible flow the velocity profile must pass through the point

$$\varphi = \eta = 11.6$$

and also that it must have the same slope at this point as the curve

$$\left.\begin{aligned}\varphi &= 5.5 + 2.5 \ln\eta \\ &= 5.5 + 5.75 \log_{10}\eta\end{aligned}\right\} \qquad (5)$$

found for incompressible flow in section 2.11, (bearing in mind that in compressible flow, density and viscosity in φ and η are to be evaluated at wall temperature).

Now Cope[9] (for zero heat transfer) and the present author[10] (heat transfer cases included) assumed that the velocity profile in the compressible turbulent boundary layer would always be given by equation (5) if density and viscosity were evaluated at wall temperature.

So it is of interest to compare the profiles obtained

(a) from equation (23) or (26) with the boundary conditions

$$f = 0.216, \quad s = 11.6$$

and (b) from equation (5), with density and viscosity evaluated at wall temperature.

This is done in Fig.4 for some representative cases (with and without heat transfer and choosing equation (26) for the comparison) which show the extent of the approximation afforded by equation (5). Actually this approximation may be better than it seems at first sight, since the "more accurate" solutions can only be expected to apply in the inner portion of the boundary layer because of the assumptions made in their derivation, and (as in the incompressible flow case) any agreement with experiment near the outer edge would be fortuitous.

Thus it appears that despite the more fundamental approach, the velocity distributions given by equation (26) (or (23)) will be of little more value than that given by equation (5) until the variations (if any) of the boundary conditions in compressible flow have been determined by experimental measurement. Unfortunately, limitations in tunnel size have meant that the majority of measurements to date have been of thin layers of the order of 0.25 in. thickness and less, so that the inner regions of the boundary layer remain unexplored.

However, two sets of experimental profiles[8] are plotted in Fig.5 for comparison with the corresponding profiles given by equation (26) with the boundary conditions $f = 0.216$ and $s = 11.6$ (full lines). Under zero heat transfer conditions there is perhaps tolerable agreement but when heat is being transferred (Fig.5b) the boundary conditions are obviously inadequate.

A possibility for improvement might be to assume that in applying the boundary conditions

$$f = 0.216$$

$$s = 11.6$$

density and viscosity (in φ and η) should be evaluated at the temperature at the edge of the laminar sub-layer instead of at wall temperature. Then for φ and η in terms of wall temperature, the boundary conditions would be

$$\frac{d\varphi}{d\eta} = 0.216 \left(\frac{T_w}{T_s}\right)^{0.8}$$

and $$\varphi = 11.6 \left(\frac{T_s}{T_w}\right)^{\frac{1}{2}}$$

if $\mu \propto T^{0.8}$

where T_s is the temperature at the edge of the laminar sub-layer*.

Profiles calculated from equation (26) with these revised boundary conditions are shown by broken lines in Fig.5 and exhibit trends in the same direction as the experimental results. However, it would not be profitable to extend this investigation before further experimental results become available. Meanwhile the analysis will continue with the usual assumption that the boundary conditions are evaluated at wall temperature and it will be shown that the modifications suggested above would have very little effect on the final skin friction results, at least within the range of speeds and temperatures of interest at present.

4.3 Skin friction

This is obtained through the momentum equation

$$\frac{\tau_o}{\rho_w u_1^2} = \frac{d}{dx} \int_o^\delta \frac{\rho u}{\rho_w u_1} \left(1 - \frac{u}{u_1}\right) dy$$

which with the substitutions already defined (at head of section 4.2) becomes

$$\frac{u_1}{\nu_w} = v^2 \frac{d}{dx} \left\{ v^2 \int_o^1 \frac{z(1-z)}{1 + Bz - A^2 z^2} \frac{d\eta}{d\varphi} \; dz \right\} \tag{27}$$

by comparison with equation (10a) in incompressible flow.

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* In deriving these expressions it is assumed that the temperature distribution is given by equations (18), (19b) and (21).

Using the Prandtl mixing length ($\ell = ky$), equation (23) gives

$$\frac{d\eta}{d\varphi} = \frac{s}{v}\frac{d}{dz}\left[\exp\left\{\lambda(\sin^{-1} w - \sin^{-1} w_s)\right\}\right] \tag{28}$$

whereas the Von Karman mixing length $\left(\ell = k\,\frac{du}{dy}\Big/\frac{d^2u}{dy^2}\right)$ gives (equation (25))

$$\frac{d\eta}{d\varphi} = \frac{1}{f}\,.\,\exp\left\{\lambda(\sin^{-1} w - \sin^{-1} w_s)\right\} \tag{25}$$

Consequently, as compared with the incompressible flow case of section 2, estimates of skin friction in compressible flow vary according to the hypothesis used for mixing length. The final results, obtained by integration of equation (27) with either equation (28) or equation (25), and assuming that v is large, gives the following formulae for local skin friction

(a) <u>From $\ell = ky$. (Equations (27) and (28)</u>

$$\frac{\Phi}{c_{f_w}^{\frac{1}{2}}} = \frac{1}{\sqrt{2}k}\ln\frac{k^2 fe^{ks}}{2} + \frac{1}{\sqrt{2}k}\ln Re_{xw}\, c_{f_w}\left(\frac{T_1}{T_w}\right)^{\frac{1}{2}} \tag{29a}$$

where subscript "w" denotes that density and viscosity are evaluated at wall temperature and

$$\Phi = \frac{1}{A}\left[\sin^{-1}\frac{A - B/2A}{\sqrt{1 + B^2/4A^2}} - \sin^{-1}\frac{A\, s/v - B/2A}{\sqrt{(1 + B^2/4A^2)}} + A\,\frac{s}{v}\right] \tag{30}$$

and A and B are the coefficients in the temperature-velocity relation as given in section 4.1.

(b) <u>from $\ell = k\,\frac{du}{dy}\Big/\frac{d^2u}{dy^2}$. (Equations (27) and (25)</u>

$$\frac{\Phi}{c_{f_w}^{\frac{1}{2}}} = \frac{1}{\sqrt{2}k}\ln\frac{k^2 fe^{ks}}{2} + \frac{1}{\sqrt{2}k}\ln Re_{xw}\, c_{f_w} \tag{29b}$$

where Φ is given by equation (30), as in case (a).

The only difference between equations (29a) and (29b) lies in the factor $\left(\frac{T_1}{T_w}\right)^{\frac{1}{2}}$ of the former.

By comparison of these two equations with the incompressible flow formula of section 2

$$\frac{1}{c_f^{\frac{1}{2}}} = \frac{1}{\sqrt{2k}} \ln \frac{k^2 \, f e^{ks}}{2} + \frac{1}{\sqrt{2k}} \ln Re_x \, c_f \qquad (11)$$

we see that if k, f and s are unchanged, the effects of Mach number and heat transfer on local skin friction coefficient can be summarised by the relations

$$\left.\begin{aligned} c_{f_i} &= \frac{c_{f_w}}{\Phi^2} \\ \text{when} \quad Re_{xi} &= Re_{xw} \, \Phi^2 \left(\frac{T_1}{T_w}\right)^m \end{aligned}\right\} \qquad (31)$$

where subscript "i" denotes the incompressible values and

$m = \frac{1}{2}$ for Prandtl mixing length $\ell = ky$

$m = 0$ for Karman mixing length $\ell = k \left(\frac{du}{dy}\right) \Big/ \frac{d^2u}{dy^2}$

Furthermore we shall assume that the same relations will apply to the mean skin friction coefficient.

Variations in k, f and s with Mach number and heat transfer could be included in the function Φ if necessary so that equations (31) are of fairly general validity. Also the notion of "equivalent" incompressible skin friction coefficients and Reynolds numbers can be extended to include the results from other shearing stress assumptions as is shown by the following table (see Appendix I for further details).

/Table

Shearing Stress Assumption	Used by	Replace c_{f_i} by	Replace Re_i by
$\tau_o = \rho \ell^2 \left(\frac{du}{dy}\right)^2$			
(a) $\ell = ky$	Smith and Harrop[12] Van Driest[5] Clemmow[6] Li and Nagamatsu[4]	c_{f_w}/Φ^2 Φ from equation(30)	$Re_{xw}\ \Phi^2 \left(\frac{T_1}{T_w}\right)^{\frac{1}{2}}$
(b) $\ell = k \frac{du/dy}{d^2y/dy^2}$	Wilson[11] (zero heat transfer)	c_{f_w}/Φ^2 Φ from equation (30)	$Re_{xw}\ \Phi^2$
$\tau_o = \ell^2 \frac{du}{dy} \frac{d(\rho u)}{dy}$			
(a) $\ell = ky$	Clemmow[6] Li and Nagamatsu[4]	c_{f_w}/F^2 F from Appendix I equations I.9, I.3	$Re_{xw}\ F^2 \sqrt{(1+A^2)}$ A from temperature - velocity relation, section 4.1
(b) $\ell = k \frac{du/dy}{d^2u/dy^2}$	Clemmow[6]	c_{f_w}/F^2 F from Appendix I equations I.9, I.3	$Re_{xw}\ F^2 . \frac{T_w}{T_1} (1 + A^2)$
Constancy of velocity profile	Cope[9] (Zero heat transfer) Monaghan[10]	c_{f_w}	$Re_{xw}\ \frac{T_1}{T_w}$

It should be noted that the variations ascribed to various authors in the above table are those which result from the shearing stress assumptions which they made and do not necessarily correspond exactly to the formulae quoted in their reports. Thus Smith and Harrop[12] take an erroneous value for A and adopt an unusual approximation when evaluating the momentum integral, which yields a different formula. Also, Clemmow's results[6] are in effect all for the Prandtl mixing length $\ell = ky$ and there are errors in his final conversions to free stream conditions.

Finally, $\frac{s}{v}$ is usually taken to be zero in equation (30) for Φ, which makes calculation easier since Φ is then independent of v (i.e. independent of skin friction). The value of this approximation is considered in section 5 below.

5 Comparison of formulae for mean skin friction (Zero heat transfer)

The analysis of section 4 led to the Karman-Kempf type formulae of equations (29a), (b) for local skin friction in compressible flow. The corresponding formula for mean skin friction would be of the same type, with constants as given by Schoenherr (equation (12)). However these formulae are not particularly amenable to quick calculation, so in the comparisons of this section the various relations tabulated at the end of section 4 have been applied to the Prandtl-Schlichting formula for mean skin friction in incompressible flow

$$C_{F_i} = 0.46\,(\log_{10} Re_i)^{-2.6} \tag{13}$$

to obtain the appropriate formulae for compressible flow.

Thus, for example, if we assume that

$$\tau_o = \rho\,\ell^2 \left(\frac{du}{dy}\right)^2 \tag{1b}$$

then the table, or equations (31), give

$$\left.\begin{aligned} C_{F_i} &= \frac{C_{F_w}}{\Phi^2} \\ \text{when} \qquad Re_i &= Re_w\,\Phi^2 \left(\frac{T_1}{T_w}\right)^m \end{aligned}\right\} \tag{31}$$

and substituting in equation (13) we obtain

$$\frac{C_{F_w}}{\Phi^2} = 0.46\,\left\{\log_{10} Re_w\,\Phi^2\left(\frac{T_1}{T_w}\right)^m\right\}^{-2.6} \tag{32}$$

as the formula for compressible flow. Likewise the assumption of constancy of velocity profile would also give an equation of the type of equation (32), but with $\Phi = m = 1$.

The validity of replacing the Schoenherr formula by the Prandtl-Schlichting formula was checked by calculating the variations of mean skin friction with Mach number given by both formulae when modified in accordance with the assumption of constancy of velocity profile. The results are given in Fig.6a, which shows that the two formulae give variations within $2\frac{1}{2}$ per cent of each other up to $M = 4$, both for $Re = 10^6$ and for $Re = 10^7$.

On the other hand, Fig.6b shows that a similar generalisation of the **Blasius** power law formula (equation (14)) which gives a single curve for all Reynolds numbers, would not give such good agreement with the generalised Schoenherr formula. As a result, the Prandtl-Schlichting formula was chosen for the main calculations of this section.

Fig.7 then gives a comparison between the variations of mean skin friction obtained for $Re = 10^7$ from the various shearing stress assumptions listed at the end of section 4 and quotes the authors who have used these assumptions. It also includes the variations arising from the assumption of constancy of velocity profile (Cope[9], Monaghan[10]) and from Rubesin's interpolation formula[2].

$$C_F = 0.472\ (\log_{10} Re)^{-2.58} \left(\frac{T_1}{T_w}\right)^{0.467} \qquad (33)$$

Some general points should be noted concerning the structure of this figure, as follows:-

(1) In evaluating Φ or F, $\frac{s}{v}$ was taken to be zero. This corresponds to the procedure used by the authors quoted. (The effect of taking $\frac{s}{v}$ unequal to zero is considered in section 5.1 below).

(2) As already noted at the end of section 4, the variations ascribed to various authors are those which result from the shearing stress assumptions which they made and do not necessarily correspond exactly to the values quoted in their reports.

(3) The viscosity-temperature relation $\mu \propto T$ was used in the calculations and this accounts for the small difference between the comparable curves in Figs.6 and 7 (the curves for constancy of velocity profile) since the relation $\mu \propto T^{0.8}$ was used in Fig.6.

(4) The abscissa is $\frac{T_w}{T_1}\left(= \frac{T_{wo}}{T_1}\right)$ since under zero heat transfer conditions the function Φ or F depends only on $\frac{T_w}{T_1}$. If the abscissa were M_1 then an additional relation linking $\frac{T_w}{T_1}$ and M_1 (equation (20) or (21)) becomes necessary. However, a subsidiary scale of M_1 is given which corresponds to

$$\frac{T_w}{T_1} = 1 + 0.9\,\frac{\gamma-1}{2}\,M_1 \qquad (21)$$

Experimental results obtained by Wilson[11] and Rubesin[2] and included for comparison. The R.A.E. experimental results[8,10] at M_1 = 2.43 and 2.82 are not included since they were obtained at a considerably lower Reynolds number, but they agreed with the variation given at that Reynolds number by equation (32), with $\Phi = 1$, $m = 1$, i.e.

$$C_{F_w} = 0.46 \log_{10} \left(Re_w \frac{T_1}{T_w} \right) \qquad (32b)$$

which is the curve labelled "constancy of velocity profile".

Comparing the theoretical and experimental results, we may say,

(a) as in Fig.3, the shearing stress assumption

$$\tau = -\rho \overline{u'v'} \qquad (1a)$$

combined with

$$\overline{u'v'} = -\ell^2 \left(\frac{du}{dy}\right)^2 \qquad (2)$$

gives results in better agreement with experiment than the assumption

$$\tau = -(\rho \overline{u'v'} + u\, \overline{\rho' v'}) \qquad (15)$$

(b) accepting result (a) then the Karman hypothesis for mixing length

$$\ell = k \frac{du}{dy} \Big/ \frac{d^2u}{dy^2} \qquad (3b)$$

(i.e. Wilson's curve) gives better agreement with experiment than the Prandtl mixing length

$$\ell = ky \qquad (3a)$$

(as used by Smith, Van Driest and Clemmow).

Result (b) is emphasised by the fact that Rubesin's interpolation formula is based on equations (1a), (2) and (3b), above, but was obtained from a set of numerical integrations of the momentum equation instead of by approximate integration as used by Wilson and in section 4. It is therefore of considerable interest that the variation it gives is close to that obtained from the assumption of constancy of velocity profile (equation (32b)), so that the latter can be regarded as a very good approximation to the more fundamental solution.

(It should also be noted that while Rubesin's formula gives the same variation with Mach number for all Reynolds numbers, equation (32b) agrees with Wilson's formula in predicting that the variation with Mach number will also vary with Reynolds number).

5.1 Effect of boundary condition $\frac{s}{v}$

The foregoing results were obtained neglecting terms involving $\frac{s}{v}$ in the formulae for Φ or F, e.g. if, in accordance with the above results, we take

$$\tau = -\rho\overline{u'v'} \tag{1a}$$

$$\overline{u'v'} = -\ell^2\left(\frac{du}{dy}\right)^2 \tag{2}$$

and

$$\ell = k\left(\frac{du}{dy}\right)\Big/\frac{d^2u}{dy^2} \tag{3b}$$

then the results of section 4.3 and equation (32) give

$$\frac{c_{F_w}}{\Phi^2} = 0.46\left[\log_{10} Re_w \quad \Phi^2\right]^{-2.6} \tag{32a}$$

where, under zero heat transfer conditions, equation (30) gives

$$\Phi = \frac{1}{A}\left[\sin^{-1} A - \sin^{-1} A\frac{s}{v} + A\frac{s}{v}\right] \tag{30a}$$

so that neglecting terms involving $\frac{s}{v}$ means assuming

$$\sin^{-1} A\frac{s}{v} \simeq A\frac{s}{v}$$

Now $v\left\{=\sqrt{\left(\frac{2}{c_{f_w}}\right)}\right\}$ is usually of the order of 20, so that with $s = 11.6$ this approximation would only be valid for small values of A, i.e. for small values of $\frac{T_w}{T_1}$ or M_1 since

$$A^2 = \frac{\frac{T_{wo}}{T_1} - 1}{\frac{T_w}{T_1}} \tag{19b}$$

$$\simeq \frac{M_1^2}{M_1^2 + 5} \quad \text{under zero heat transfer conditions.}$$

The extent of the errors involved is shown by Fig.8 which compares the variations obtained from

(1) equations (32a) and (30a), neglecting the terms in $\frac{s}{v}$ (Corresponding to Wilson's curve in Fig.7),

(2) equations (32a) and (30a) taking $s = 11.6$ and appropriate values of v,

and (3) Rubesin's interpolation formula (equation (33)) obtained by numerical integration of the momentum equation.

These show that inclusion of the terms involving $\frac{s}{v}$ gives results much closer to Rubesin's values than did the original analysis for $\frac{s}{v} = 0$. Hence, since the variation obtained from the assumption of constancy of velocity profile has already been shown (Fig.7) to be close to that obtained from Rubesin's formula, it can be said that the variations in velocity profile shown in Fig.4 have little effect on the final estimates of skin friction under zero heat transfer conditions.

What happens when heat is being transferred is considered in the following section.

6 Heat transfer effect on skin friction coefficient

In section 5 it was found that the formula

$$\frac{C_{F_w}}{\Phi^2} = 0.46 \left[\log_{10} Re_w \ \Phi^2 \right]^{-2.6} \qquad (32a)$$

gave the best agreement with experimental results under zero heat transfer conditions, so we shall now apply the same formula to study the additional variations which may arise when heat is being transferred between the plate and the stream. In this case

$$\Phi = \frac{1}{A} \left\{ \sin^{-1} \frac{A - B/2A}{\sqrt{(1 + B^2/4A^2)}} - \sin^{-1} \frac{A\ s/v - B/2A}{\sqrt{(1 + B^2/4A^2)}} + A \frac{s}{v} \right\} \qquad (30)$$

and we shall take values of A and B from Reynolds analogy (section 4.11) i.e.

$$\left. \begin{aligned} A^2 &= \frac{T_{H1}/T_1 - 1}{T_w/T_1} = \frac{(\gamma-1/2) M_1^2}{T_w/T_1} \\ B &= T_{H1}/T_w - 1 \end{aligned} \right\} \qquad (19a)$$

and assume that $\mu \propto T$.

The resulting variations* of mean skin friction with Mach number under zero heat transfer $\left(\frac{T_{H1}}{T_w} = 1\right)$ and two heat transfer $\left(\frac{T_{H1}}{T_w} = 0.5, 2.0\right)$ conditions, for Re = 10^7, are shown by the full line curves in the lower graph of Fig.9. A considerable variation with heat transfer is evident.

At a given Mach number, the skin friction is reduced if heat is flowing from the plate to the air stream $\left(\frac{T_{H1}}{T_w} = 0.5\right)$ and is increased if heat is flowing in the opposite direction.

Now the assumption of constancy of velocity profile,[10] giving

$$\left.\begin{aligned} C_{F_w} &= 0.46 \left[\log_{10} Re_w \frac{T_1}{T_w}\right]^{-2.6} \\ \text{or} \qquad C_F &= 0.46 \frac{T_1}{T_w} \left[\log_{10} Re \left(\frac{T_1}{T_w}\right)^3\right]^{-2.6} \end{aligned}\right\} \qquad (32b)$$

would suggest that for a given Reynolds number, C_F might be a function of $\frac{T_w}{T_1}$ alone. This would mean that instead of C_F being a function of Reynolds number, Mach number and heat transfer rate, it is simply a function of Reynolds number and of the ratio of free stream static to wall temperature. Thus, at a given Reynolds number, the skin friction for M = 2 and zero heat transfer $\left(\text{when } \frac{T_w}{T_1} = 1.8 \text{ if Reynolds analogy is assumed}\right)$ should be identical with that obtained for M = 0 and a heat transfer rate given by $\frac{T_w}{T_1} = 1.8$.

To check this, the skin friction results obtained from equation (32a) above are re-plotted against $\frac{T_w}{T_1}$ in the upper graph of Fig.9. By comparison with the plots against Mach number, this shows a big reduction in the variation with heat transfer. In fact, for the range considered, the skin friction coefficient is always within 10 per cent of its zero heat transfer value.

The above result is based on values of A and B derived from Reynolds analogy, but use of the empirical values of A and B given in section 4.12 should not make any radical alteration to the curves. Another source of error is the fact that the boundary condition "s" was taken to be 11.6 throughout whereas the experimental velocity profiles

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* Values of $v \left\{ = \sqrt{\left(\frac{2}{c_{f_w}}\right)} \right\}$ for substitution in equation (30) were obtained for convenience from the incompressible flow power law formula for local skin friction, modified in accordance with the constancy of velocity profile assumption. Errors thus introduced should be small. The value of "s" was taken to be 11.6.

in Fig.5 suggested that it should vary. To check the order of the error thus introduced, a skin friction coefficient was calculated for $\frac{T_w}{T_1} = 4.0$ and $\frac{T_{H1}}{T_w} = 0.5$, using the revised boundary condition given in section 4.23 (which gave a velocity distribution displaced in similar fashion to the broken lines in Fig.5). The result of the calculation is shown by the cross symbol in Fig.9, whose displacement from the corresponding full line curve is seen to be small.

Thus it may be said that the variation given by the assumption of constancy of velocity profile (equation (32b)) should give results within the order of 10 per cent of the "more fundamental" theoretical formula (equation (32a)) over the range $1 \leqslant \frac{T_w}{T_1} \leqslant 4$ and $0.5 \leqslant \frac{T_{H1}}{T_w} \leqslant 2$. For comparison purposes, the actual variations given by equation (32b) are added as broken lines in the two graphs of Fig.9.

Finally it should be mentioned that the only available experimental results[8] for flat plates at supersonic speeds have shown a variation of skin friction with Mach number but not with heat transfer. These results are for $M = 2.43$ and 2.82 and for $\frac{T_{H1}}{T_w} = 0.74$ and 0.64. On the other hand, skin friction results from flow in pipes at low speeds but at high heat transfer rates have shown a definite variation with heat transfer which has been correlated well[13] by a formula of the type of equation (32b). Further experimental evidence is obviously necessary, but meanwhile it is suggested that the formula (equation (32b)) obtained from the assumption of constancy of velocity profile has the merit of simplicity in application, and therefore that skin friction should be regarded as a function only of Reynolds number and of temperature ratio T_w/T_1. This should be valid up to $\frac{T_w}{T_1} = 4$ (corresponding approximately to $M = 4$ under zero heat transfer conditions). At higher values of $\frac{T_w}{T_1}$ (or of Mach number or of both) density fluctuations may be of importance, as discussed in section 3.

7 Conclusions

1 Of the different shearing stress assumptions made by various authors[2,4,5,6,11,12] the assumption

$$\tau_o = \rho k^2 \left(\frac{du}{dy}\right)^4 \Big/ \left(\frac{d^2u}{dy^2}\right)^2$$

leads to the variation of skin friction with Mach number which is in the best agreement with experimental results in the region $1.6 < M < 2.8$ under zero heat transfer conditions (Fig.7).

2 The assumption of constancy of velocity profile[9,10] gives skin friction results in equally good agreement with experiment under zero heat transfer conditions (Fig.7).

3 Under heat transfer conditions the same assumption gives a reasonable approximation to results obtained from the shearing stress assumption of

conclusion 1 (Fig.9). This would mean that skin friction may conveniently be regarded as a function only of Reynolds number and of the ratio of free stream static to wall temperature (instead of as a function of Reynolds number, Mach number and heat transfer rate).

4 The explanation of conclusions 2 and 3 is that velocity profiles obtained from the assumption of conclusion 1 (or from any other shearing stress assumption) are usually forced by artificial boundary conditions to agree with the "constant" velocity profile at points near the wall (Fig.4).

5 Extensive further measurements of turbulent boundary layers in compressible flow would be necessary before these restrictions could be overcome. This is particularly true for flows with heat transfer.

6 Meanwhile it is suggested that the formula

$$C_{F_w} = 0.46\left[\log_{10} Re_w \frac{T_1}{T_w}\right]^{-2.6}$$

obtained from the assumption of constancy of velocity profile has the merit of simplicity and should be sufficiently accurate in application up to $\frac{T_w}{T_1} = 4$ (corresponding approximately to $M = 4$ under zero heat transfer conditions).

7 For Mach numbers $\left(\text{and possibly } \frac{T_w}{T_1}\right)$ greater than four it will probably become necessary to make a thorough examination of the derivation of the equations for the turbulent boundary layer in compressible flow since density fluctuations may assume importance.

LIST OF SYMBOLS

x,y	distances parallel and normal to plate
u,v	mean velocity components parallel and normal to plate
ρ	density
T	temperature
μ	viscosity
ν	kinematic viscosity
Subscript "1"	denotes free stream conditions (outside boundary layer)
Subscript "w"	denotes wall temperature conditions (i.e. at surface of plate)
Subscript "s"	denotes temperature conditions at outer edge of laminar sub-layer

u_τ	friction velocity $\left(= \sqrt{\frac{\tau_o}{\rho_w}}\right)$
z	is $\frac{u}{u_1}$
φ	is $\frac{u}{u_\tau}$
η	is $\frac{y u_\tau}{\nu_w}$
k	is constant in expression for mixing length (= 0.4 in incompressible flow)
s	is value of φ (or η) at edge of laminar sub-layer (= 11.6 in incompressible flow)
f	is value of $\frac{d\varphi}{d\eta}$ at edge of laminar sub-layer (= 0.216 in incompressible flow)
c_f	local skin friction coefficient $\left(= \frac{\tau_o}{\frac{1}{2}\rho_1 u_1^2}\right)$
C_F	mean skin friction coefficient $\left(= \frac{F}{\frac{1}{2}\rho_1 u_1^2 x}\right)$
Re	Reynolds number
v	$\sqrt{\left(\frac{2}{c_{f_w}}\right)}$
A^2,B	coefficients in temperature-velocity distribution

$$\frac{T}{T_w} = 1 + B_z - A^2 z^2$$

{In general $B = \frac{T_{wo}}{T_w} - 1$

$$A^2 = \frac{T_{wo}/T_1 - 1}{T_w/T_1}$$

where T_{wo} is wall temperature for zero heat transfer}.

List of Symbols (Contd.)

λ	is $\frac{kv}{A}$
w	is $\dfrac{Az - B/_{2A}}{\sqrt{(1 + B^2/4A^2)}}$
Φ	defined by equation (30), section 4.3
F	defined in Appendix I

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APPENDIX I

Relations between compressible and incompressible skin friction coefficients

Based on the shearing stress relation

$$\tau_o = -\rho\overline{u'v'}$$

and with

$$\overline{u'v'} = -\ell^2\left(\frac{du}{dy}\right)^2$$

the analysis of section 4 gave the relations

$$\left.\begin{aligned} c_{f_i} &= \frac{c_{f_w}}{\Phi^2} \\ \text{when} \qquad Re_{xi} &= Re_{xw}\,\Phi^2\left(\frac{T_1}{T_w}\right)^m \end{aligned}\right\} \qquad (31)$$

between incompressible (subscript "i") and compressible skin friction coefficients and Reynolds numbers, where subscript "w" denotes that density and viscosity are evaluated at wall temperature T_w and T_1 is the static temperature of the stream outside the boundary layer. Φ is a function of Mach number and of temperature, being defined by equation (30) of the main text and the index "m" takes the values

$$m = \tfrac{1}{2} \text{ for Prandtl mixing length } \ell = ky$$

and

$$m = 0 \text{ for Karman mixing length } \ell = k\,\frac{du}{dy}\bigg/\frac{d^2u}{dy^2}$$

This Appendix considers the modifications introduced into equations (31)

(1) by taking the shearing stress relation

$$\tau_o = -\rho\overline{u'v'} - u\,\overline{\rho'v'}$$

or (2) by assuming constancy of velocity profile as in Ref.10.

1 Shearing stress relation $\tau_o = -\rho\overline{u'v'} - u\overline{\rho'v'}$

We shall consider the form obtained by assuming

$$\overline{u'v'} = -\ell^2\left(\frac{du}{dy}\right)^2$$

and

$$\overline{\rho'v'} = -\ell^2\,\frac{du}{dy}\cdot\frac{d\rho}{dy}$$

which gives

$$\tau_o = \left(\rho \frac{du}{dy} + u \frac{d\rho}{dy}\right) \ell^2 \frac{du}{dy}$$

$$= \ell^2 \frac{du}{dy} \frac{d(\rho u)}{dy} \qquad \text{I.1}$$

The effect of the various values of "ℓ" are as follows.

1a Prandtl value $\ell = ky$

In this case by substituting for ℓ in equation I.1, by making the substitutions listed at the beginning of section 4.2, by taking

$$\frac{\rho_w}{\rho} = \frac{T}{T_w} = 1 + Bz - A^2 z^2$$

where

$$z = \frac{u}{u_1}$$

and by integrating, we obtain the velocity distribution

$$\eta = C\, e^{kv\psi} \qquad \text{I.2}$$

where

$$\psi = \int \frac{\sqrt{1 + A^2 z^2}}{1 + Bz - A^2 z^2}\, dz$$

$$= \frac{1}{A}\left\{\frac{a}{\sqrt{a^2+1}} \sinh^{-1} \frac{1 + aAz}{a - Az} - \frac{b}{\sqrt{b^2+1}} \sinh^{-1} \frac{1 - bAz}{b + Az} - \sinh^{-1} Az\right\} \qquad \text{I.3}$$

with

$$a = \sqrt{\left(1 + \frac{B^2}{4A^2}\right)} + \frac{B}{2A}$$

and

$$b = \sqrt{\left(1 + \frac{B^2}{4A^2}\right)} - \frac{B}{2A}$$

The constant C is defined by conditions at the edge of the laminar sub-layer; giving

$$C = s\, e^{-kv\psi_s} \qquad \text{I.4}$$

where ψ_s denotes that $z = \frac{s}{v}$ in equation I.3.

In incompressible flow the constant is

$$C_i = s\,e^{-ks} \quad \left(= \frac{1}{kf}\,e^{ks} \quad \text{if} \quad s = \frac{1}{kf}\right)$$

so that equation I.4 can be written

$$C = C_i\,e^{-kv(\psi_s - \frac{s}{v})} \qquad \text{I.5}$$

The momentum equation

$$\frac{\tau_o}{\rho_w u_1^2} = \frac{d}{dx} \int_o^1 \frac{\rho u}{\rho_w u_1}\left(1 - \frac{u}{u_1}\right) dy$$

becomes

$$\frac{1}{v^2} = \frac{\nu_w}{u_1}\frac{d}{dx}\left[C\,v \int_o^1 \frac{z\,(1-z)}{1 + Bz - A^2 z^2}\frac{d}{dz}\left(e^{kv\psi}\right) dy\right] \qquad \text{I.6}$$

and by integration by parts and assuming v large, we obtain the final form

$$\frac{\psi_1 - \psi_s + \frac{s}{v}}{c_{f_w}^{\frac{1}{2}}} = \frac{1}{\sqrt{2k}} \ln \frac{k^2 f e^{ks}}{2} + \frac{1}{\sqrt{2k}} \ln \left(Re_{xw}\, c_{f_w} \sqrt{1 + A^2}\right) \qquad \text{I.7}$$

where ψ_1 denotes that $z = 1$ in equation I.3.

Therefore by comparison with the incompressible flow equation

$$\frac{1}{c_{f_i}^{\frac{1}{2}}} = \frac{1}{\sqrt{2k}} \ln \frac{k^2 f e^{ks}}{2} + \frac{1}{\sqrt{2k}} \ln \left(Re_{xi}\, c_{f_i}\right)$$

we obtain the relations

$$\left.\begin{aligned} c_{f_i} &= \frac{c_{f_w}}{F^2} \\ \text{when} \qquad Re_{xi} &= Re_{xw}\, F^2 \sqrt{1 + A^2} \end{aligned}\right\} \qquad \text{I.8}$$

where $\qquad F = \psi_1 - \psi_s + \frac{s}{v} \qquad$ I.9

and $\qquad \psi$ is given by equation I.3.

1b $\quad$ Karman value $\ell = k \frac{du}{dy} \Big/ \frac{d^2u}{dy^2}$

Proceeding as before we obtain

$$\frac{d\eta}{d\varphi} = C_2 \, e^{kv\psi} \qquad \text{I.10}$$

where

$$C_2 = \frac{1}{f} \cdot e^{-kv\psi_s}$$

and ψ is defined by equation I.3 as before.

Taking $s = \frac{1}{kf}$ and relating C_2 to C_i as given in section 1a, we obtain

$$C_2 = k \, C_i \, e^{-kv(\psi_s - \frac{s}{v})} \qquad \text{I.11}$$

Substituting from equation I.10 in the momentum equation $\left(\text{remembering that } \frac{d}{d\varphi} = \frac{1}{v}\frac{d}{dz}\right)$ we obtain

$$\frac{1}{v^2} = \frac{\nu_w}{u_1} \frac{d}{dx}\left[C_2 \, v^2 \int_0^1 \frac{z(1-z)}{1 + Bz - A^2z^2} \, e^{kv\psi} \, dz\right] \qquad \text{I.12}$$

and this time the final form is

$$\frac{\psi_1 - \psi_s + \frac{s}{v}}{c_{f_w}^{\frac{1}{2}}} = \frac{1}{\sqrt{2k}} \ln \frac{k^2 f \, e^{ks}}{2} + \frac{1}{\sqrt{2k}} \ln \left\{Re_{xw} \, c_{f_w} \frac{T_w}{T_1}\left(1 + A^2\right)\right\} \qquad \text{I.13}$$

so that the relation between incompressible and compressible flow values is

$$\left.\begin{aligned} c_{f_i} &= \frac{c_{f_w}}{F^2} \\ \text{when} \quad Re_{xi} &= Re_{xw} \, F^2 \frac{T_w}{T_1} (1 + A^2) \end{aligned}\right\} \qquad \text{I.14}$$

where F is given by equation I.9.

2 Assumption of constancy of velocity profile

In this we assume that the velocity profile in compressible flow is given by

$$\varphi = A + B \ln \eta \tag{I.15}$$

where A and B are the experimental constants found valid in incompressible flow, (but remembering that density and viscosity in φ and η are to be evaluated at wall temperature).

Using equation I.15 the momentum equation gives (see Ref.10) a skin friction formula which is related to the incompressible flow formula by

$$\left.\begin{aligned} c_{f_i} &= c_{f_w} \\ \text{when} \quad & \\ Re_{xi} &= Re_{xw} \frac{T_1}{T_w} \end{aligned}\right\} \tag{I.16}$$

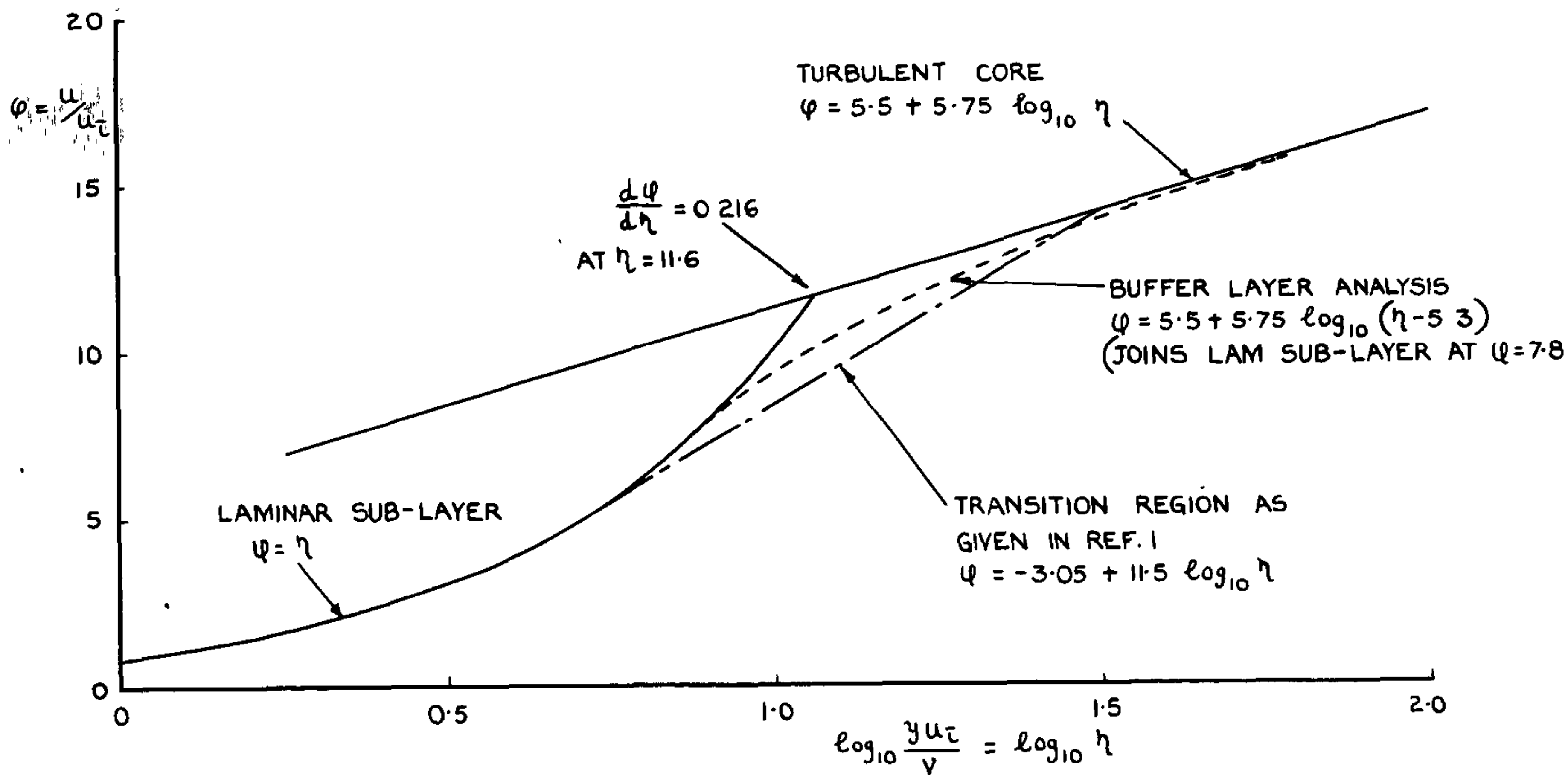


FIG. 1. VELOCITY DISTRIBUTION IN INCOMPRESSIBLE TURBULENT BOUNDARY LAYER.

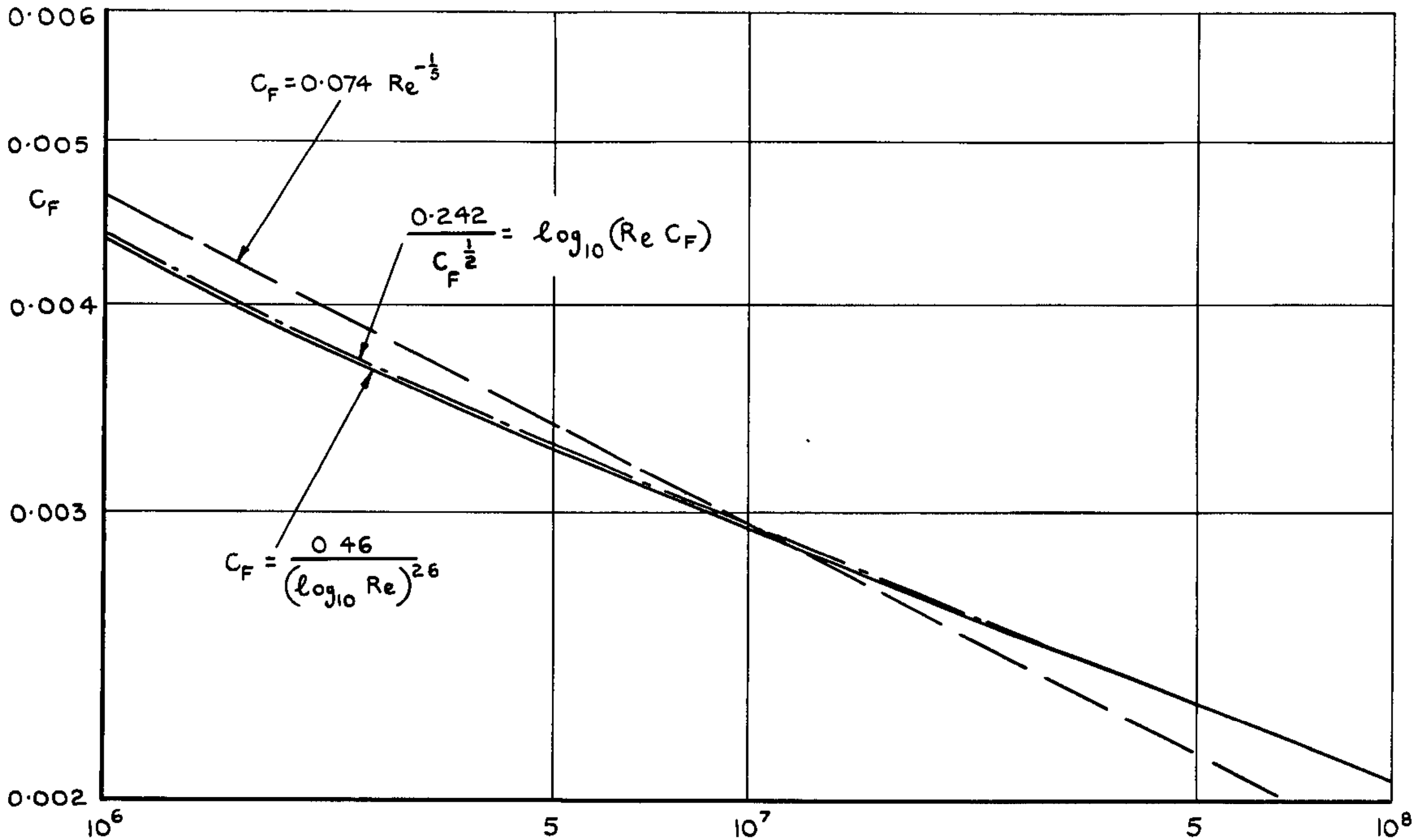


FIG.2. COMPARISON OF VARIOUS FORMULAE FOR MEAN SKIN FRICTION IN INCOMPRESSIBLE FLOW.

EXPERIMENTAL DATA		
SYMBOL	SOURCE	MEAN $Re \times 10^{-6}$
⊙	WILSON[11]	10
×	RUBESIN[2]	5
+	R A E[8,10]	2

CURVES FROM ARTICLE BY LI AND NAGAMATSU[4]
FOR Re = 11 MILLION

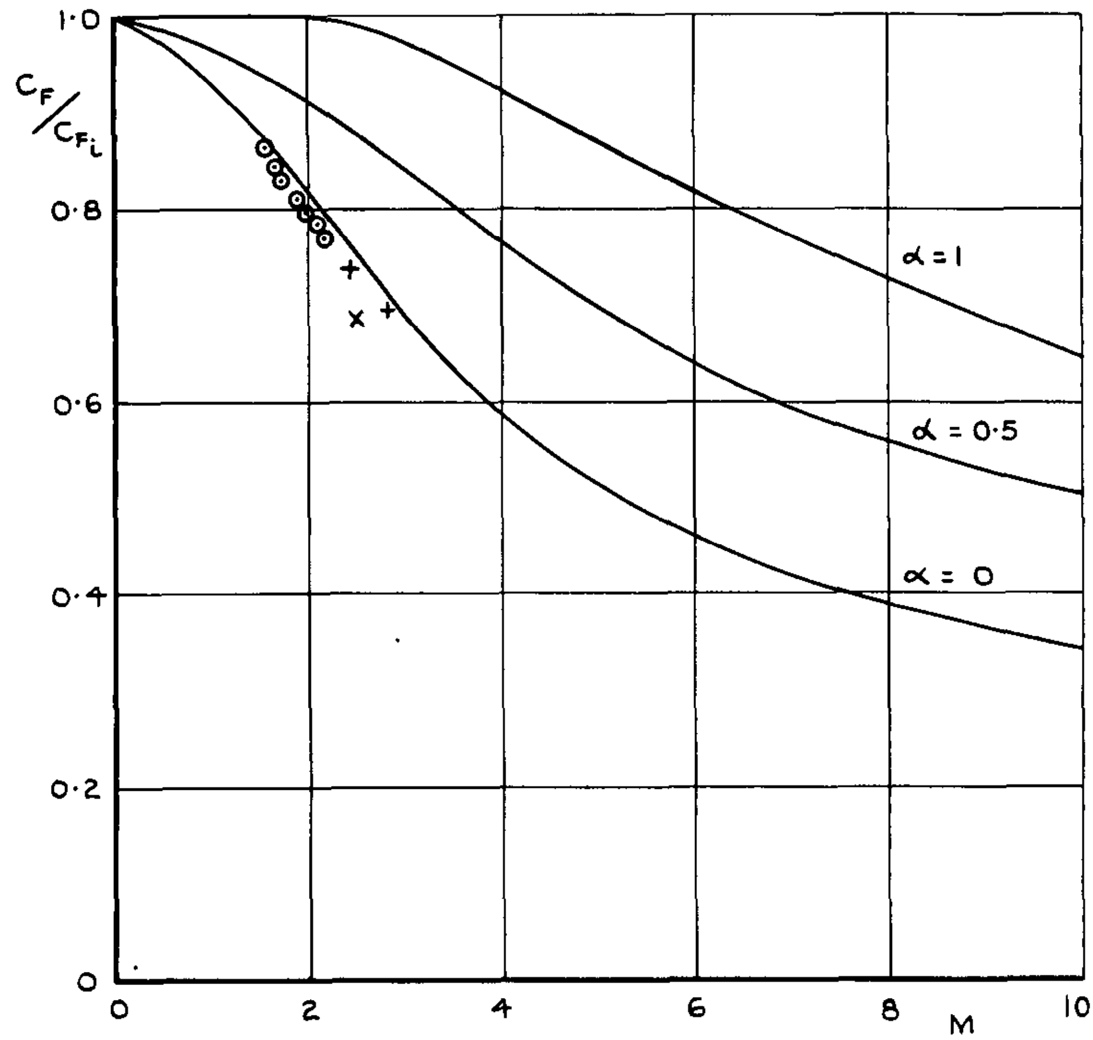
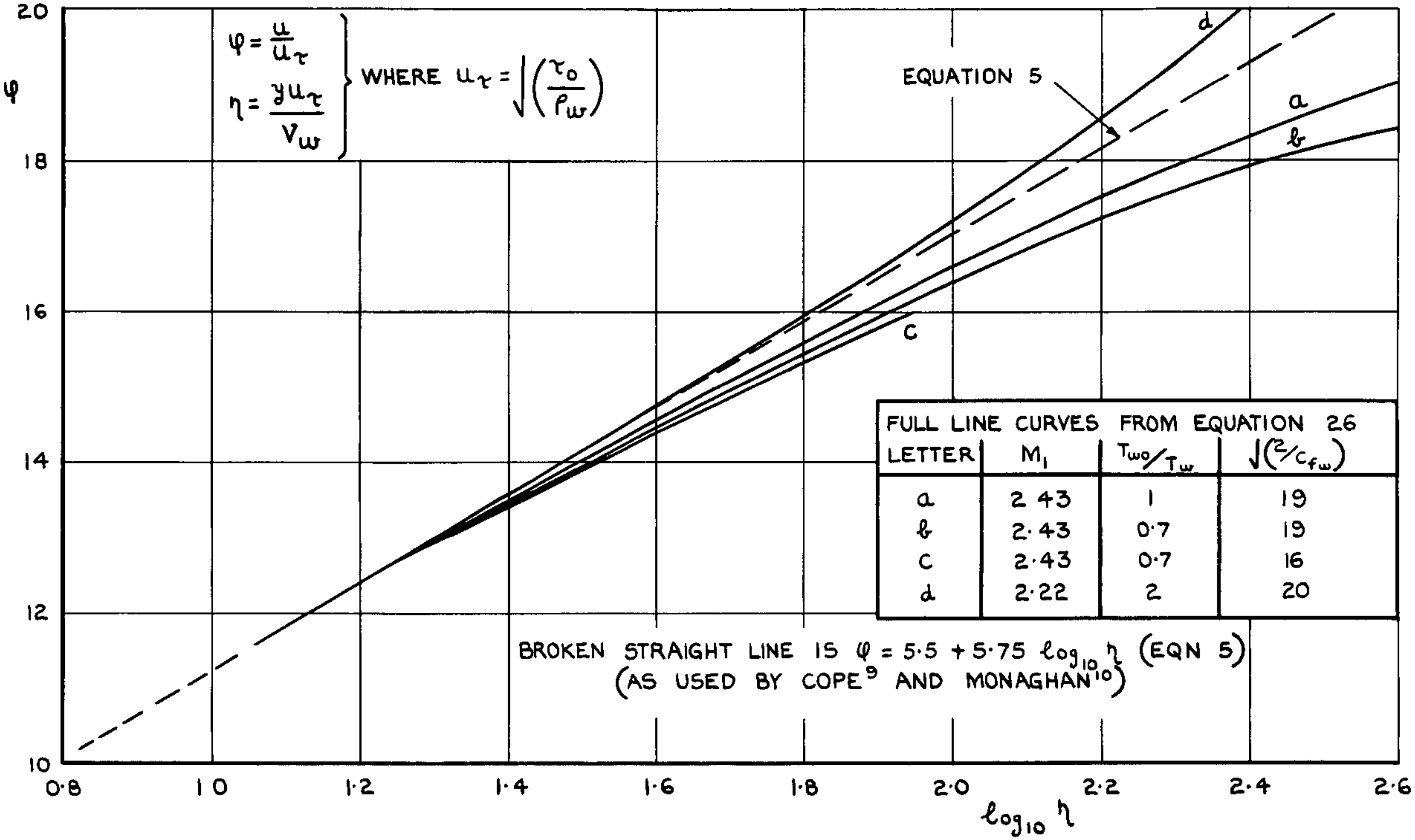


FIG.3. INFLUENCE OF DENSITY FLUCTUATIONS ON VARIATION OF MEAN SKIN FRICTION WITH MACH NUMBER UNDER ZERO HEAT TRANSFER CONDITIONS AND COMPARISON WITH EXPERIMENTAL RESULTS.

["α" IS FACTOR OF PROPORTIONALITY IN SHEARING STRESS RELATION $\tau_0 = \left(\rho \frac{du}{dy} + \alpha u \frac{d\rho}{dy}\right) \ell^2 \frac{du}{dy}$]

FIG.4.



FULL LINE CURVES FROM EQUATION 26

LETTER	M_1	T_{wo}/T_w	$\sqrt{(2/c_{fw})}$
a	2·43	1	19
b	2·43	0·7	19
c	2·43	0·7	16
d	2·22	2	20

FIG.4. VELOCITY DISTRIBUTIONS RESULTING FROM EQUATION 26 WITH BOUNDARY CONDITIONS $s = 11{\cdot}6$, $\dot{f} = 0{\cdot}216$.

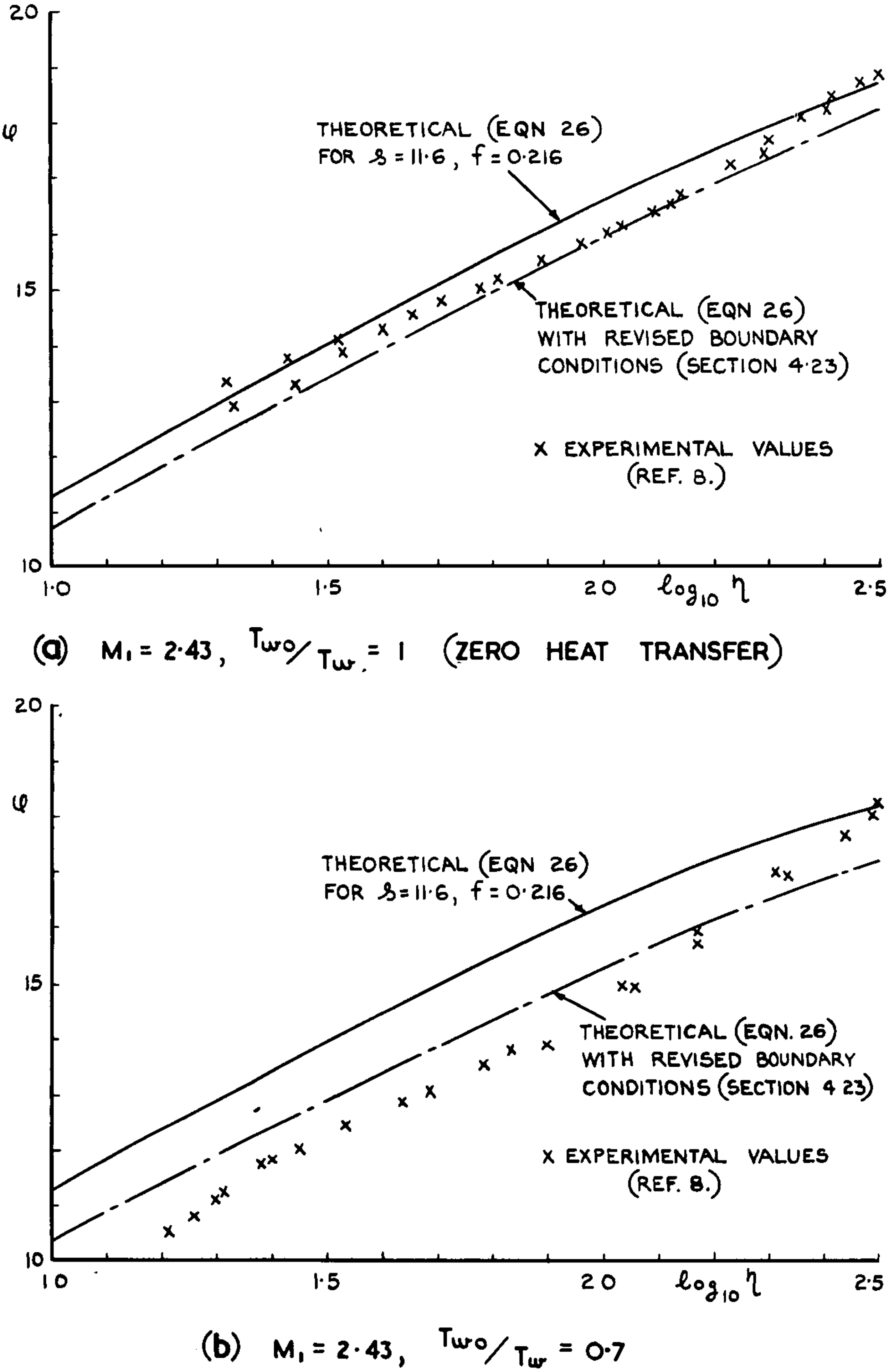
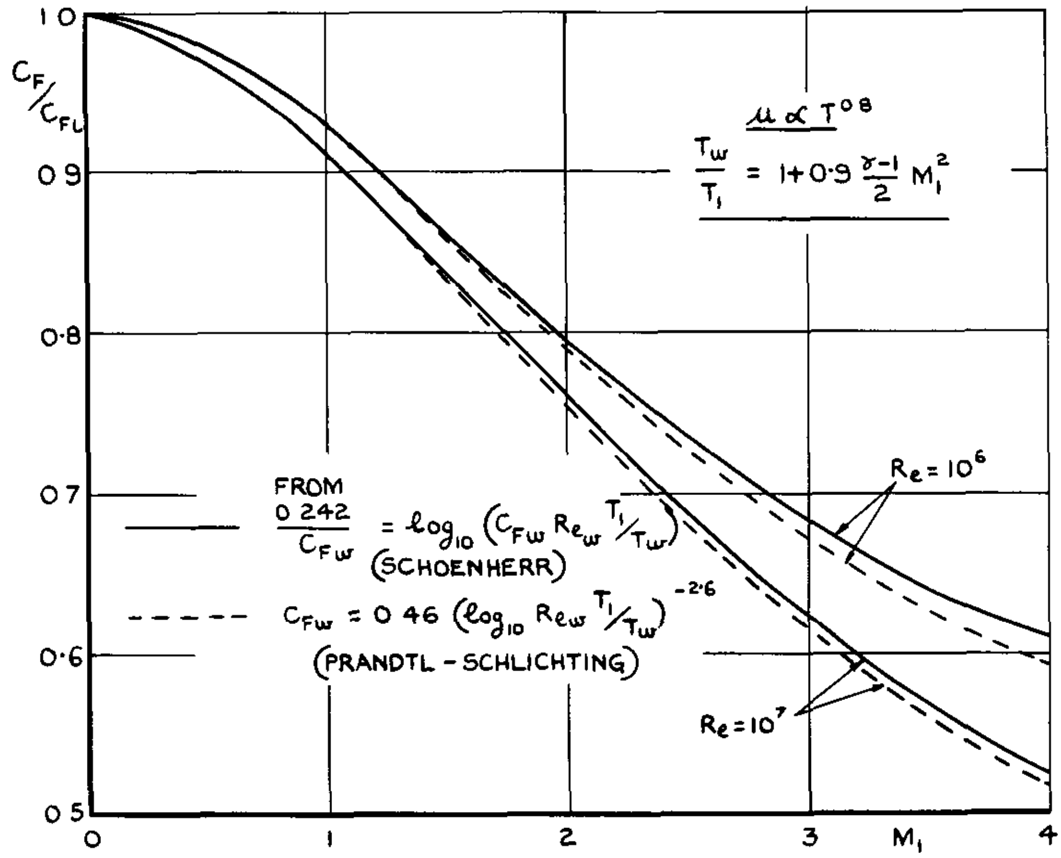
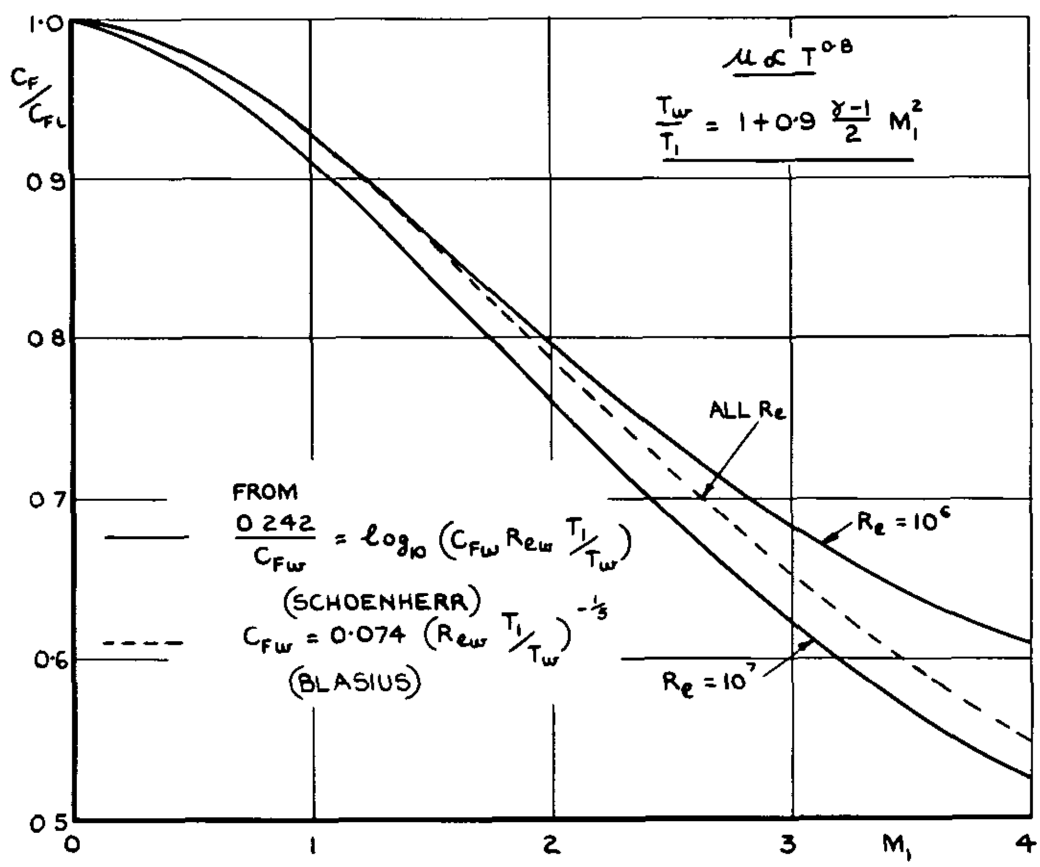


FIG.5 (a&b) COMPARISON OF THEORETICAL (EQN 26) AND EXPERIMENTAL VELOCITY DISTRIBUTIONS, AND EFFECT OF REVISED BOUNDARY CONDITIONS.

FIG.6.(a&b)



(a) SCHOENHERR AND PRANDTL – SCHLICHTING FORMULAE



(b) SCHOENHERR AND BLASIUS FORMULAE.

FIG.6.(a&b) EFFECT OF CHOICE OF INCOMPRESSIBLE FLOW FORMULA ON VARIATION OF MEAN SKIN FRICTION WITH MACH NUMBER UNDER ZERO HEAT TRANSFER CONDITIONS.

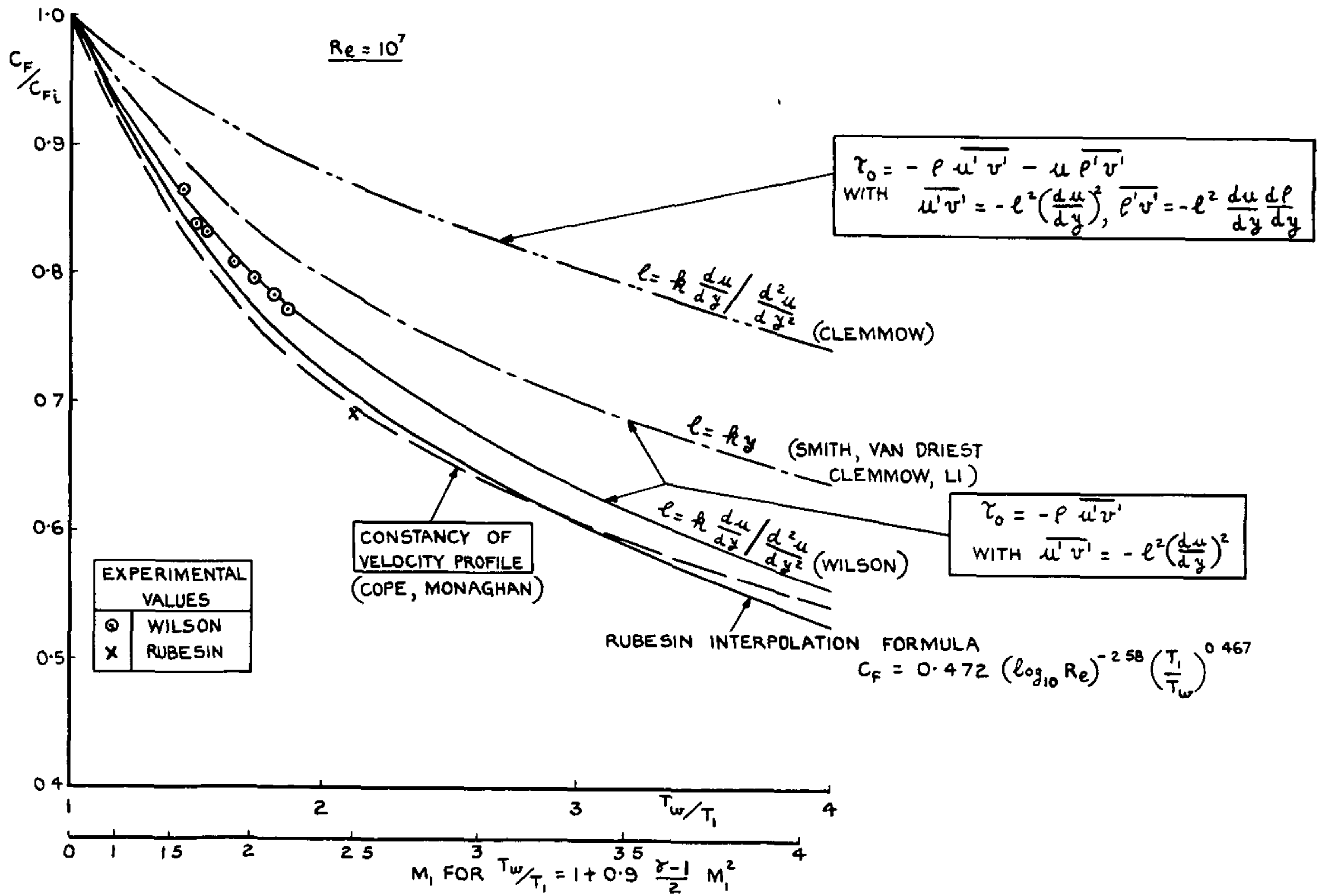


FIG.7. EFFECTS OF DIFFERENT SHEARING STRESS ASSUMPTIONS ON VARIATION OF MEAN SKIN FRICTION WITH MACH NUMBER UNDER ZERO HEAT TRANSFER CONDITIONS.

FIG.7.

FIG 8

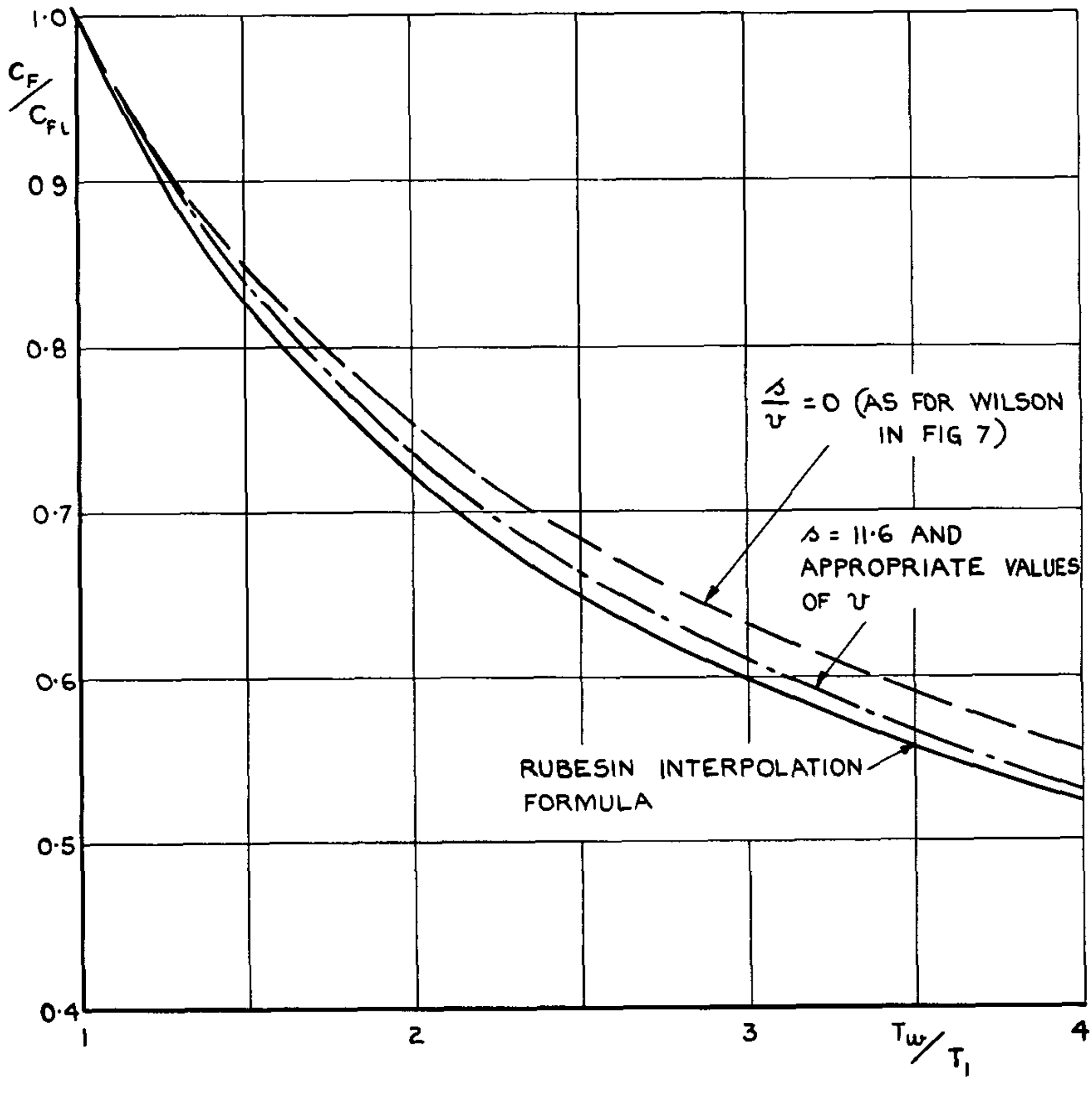


FIG 8 EFFECT OF INCLUSION OF BOUNDARY CONDITION "s" IN MEAN SKIN FRICTION FORMULA

(ACCEPTING $\tau_0 = \rho k^2 \left(\frac{du}{dy}\right)^4 \Big/ \left(\frac{d^2u}{dy^2}\right)^2$ FOR SHEARING STRESS)

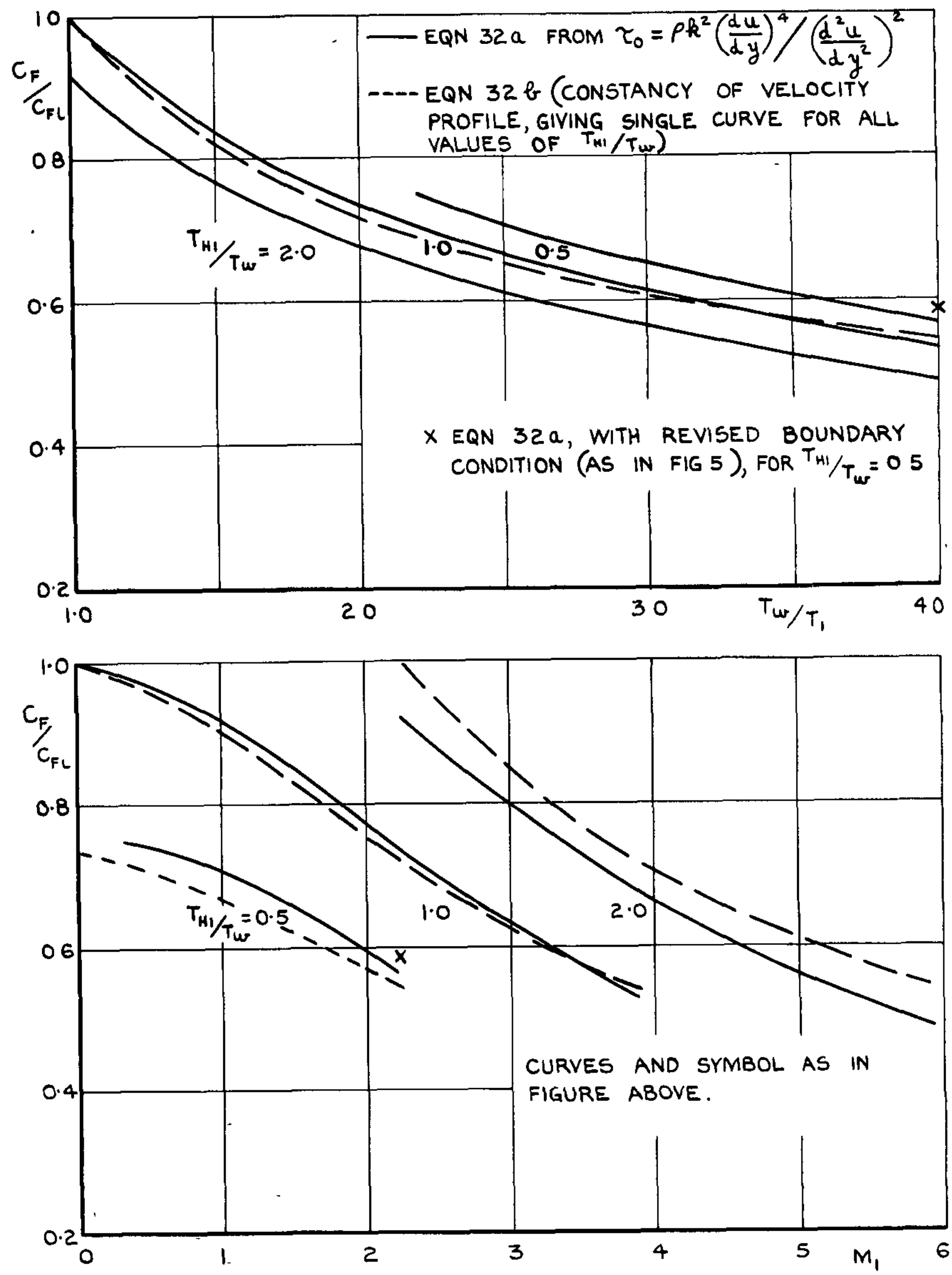


FIG 9 VARIATION OF MEAN SKIN FRICTION WITH MACH NUMBER AND HEAT TRANSFER

FIG. 10

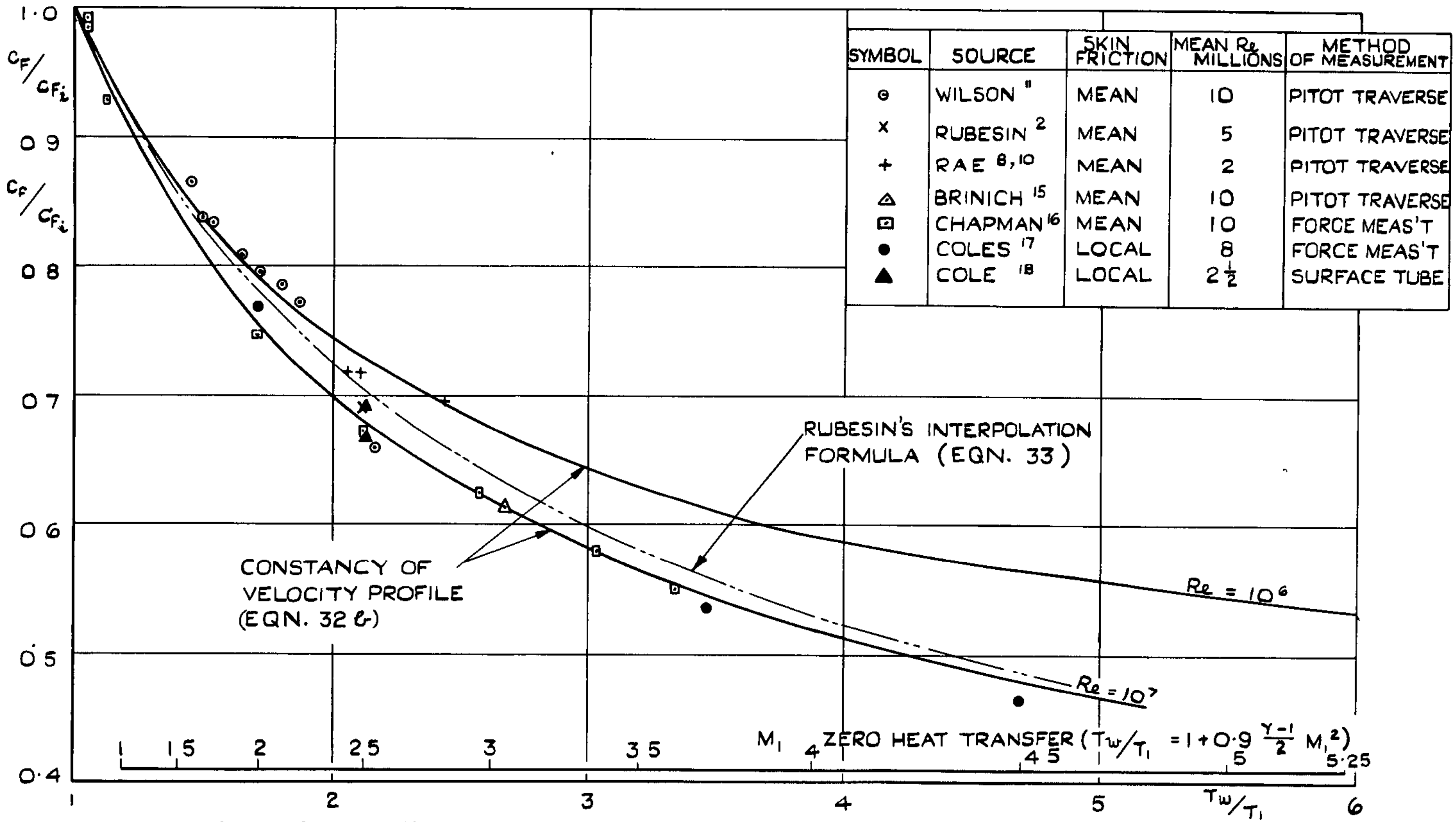


FIG. 10. VARIATION OF SKIN FRICTION COEFFICIENT WITH MACH № UNDER ZERO HEAT TRANSFER CONDITIONS. (EXTENSION OF FIG. 7.)

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