R. & M. No. 1991
(5286)
A.R.C. Technical Report

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MINISTRY OF AIRCRAFT PRODUCTION

AERONAUTICAL RESEARCH COMMITTEE
REPORTS AND MEMORANDA

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By
D. M. A. LEGGETT, Ph.D.

LONDON: HIS MAJESTY'S STATIONERY OFFICE
Price 1*s*. 3*d*. net

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COMMUNICATED BY THE DIRECTOR OF SCIENTIFIC RESEARCH,
MINISTRY OF AIRCRAFT PRODUCTION

Reports and Memoranda No. 1991
June 1941*

Summary.—Reasons for Investigation.—For an efficient design of spar with thin sheet web it is important to know the load which will just cause the web to buckle. As stiffeners divide the web into panels, it is required to find the buckling stress of rectangular panels bounded on two sides by spar flanges and on the other two sides by stiffeners. Boundary conditions which represent closely this type of edge fixing are clamping (along the flanges) and simple support (along the stiffeners), and the object of this report is to find the critical shear stress for a square panel held in this way.

Conclusions and Further Development.—It is found that the value of the critical shear stress is almost midway between its values when all four edges are clamped and all four edges are simply supported.

The method of solution developed in this report is of very general application, and can be used to investigate the stability of rectangular panels when the loading is any combination of shear and compression or tension, and the edges are clamped or simply supported, and not necessarily all clamped or all simply supported. By an easy extension the method of solution can also be used to find the periods of transverse vibration of rectangular panels for the same types of loading and edge fixing.

§1. *Introduction.*—For an efficient design of spar with thin sheet web, the behaviour of the web must be known in detail. Since its main purpose is to carry shear, a thorough knowledge of its behaviour under this type of loading is therefore necessary. This requires the solution of two problems. The first is to determine the smallest value of the shear for which the panels, into which the web is divided by stiffeners, just become unstable; the second is the behaviour of these panels when buckling has taken place and the applied shear is increased still further[1]. In this report attention is confined to the former of these two problems.

The buckling of flat rectangular panels when the edges are either all clamped[2] or all simply supported[3] has already been fully investigated. The case where two edges are clamped and two are simply supported has not. Since in an ordinary spar the booms are torsionally stiff and restrained by ribs, whereas the stiffeners are torsionally weak, it follows that it is the last mentioned edge conditions which are in best agreement with practice.

§2. *Statement of Problem and Results obtained.*—In this report attention is confined to the initial buckling of a flat square rectangular panel under shear, when one pair of opposite edges

* R.A.E. Report No. A.D. 3176, received 30th August, 1941.

is clamped and the other pair is simply supported. This result and the critical values of the shear for other types of edge condition are given in the adjoining table.

Edge conditions	C.C. & C.C.[2]	C.C. & S.S.	S.S. & S.S.[3]
Critical shear for square plate ..	$15 \cdot 4\ D\pi^2/a^2$	$12 \cdot 6\ D\pi^2/a^2$	$9 \cdot 34\ D\pi^2/a^2$

Here C.C. means that two opposite edges are clamped, S.S. that two opposite edges are simply supported ; D is the flexural rigidity, and a is the length of a side.

It will be noted that the new result of $12 \cdot 6\ D\pi^2/a^2$ is almost midway between the other two, being actually 2 per cent. above their average value.

The method of solution developed in this report and described in detail in the appendix, is an extension of that used by Timoshenko[3] when all the edges are simply supported, and is of very general application. Provided that the rectangular panel does not differ too widely from a square—in which case the computation involved becomes unduly heavy—the method can be used to investigate the stability of rectangular panels when the loading is any combination of shear and compression or tension, and the edges are clamped or simply supported, and not necessarily all clamped or all simply supported. By a simple extension the method of solution can also be used to find the periods of transverse vibration of rectangular panels for the same types of loading and edge fixing.

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2.	S. Iguchi	Buckling of Rectangular Plates Clamped on all Four Sides by Shear Stresses (*Phys. Maths. Soc. Jap.* Proc. 20, p. 814, Oct., 1938).
3.	S. Timoshenko	Theory of Elastic Stability (McGraw-Hill, 1936, p. 357).

APPENDIX

General Method and Details of Solution

Consider a flat panel acted on by a given system of external forces, and suppose that the panel undergoes a small transverse distortion consistent with the given boundary conditions, and such that there is no stretching of the middle surface. The increase in strain energy of the panel is solely due to bending, and if the panel is just on the point of buckling in the assumed form of distortion, this is equal to the work done by the forces acting in the middle surface.

In the problem considered here the external force system is a constant shearing force $N_{\xi\eta}$ (see Fig. 1),

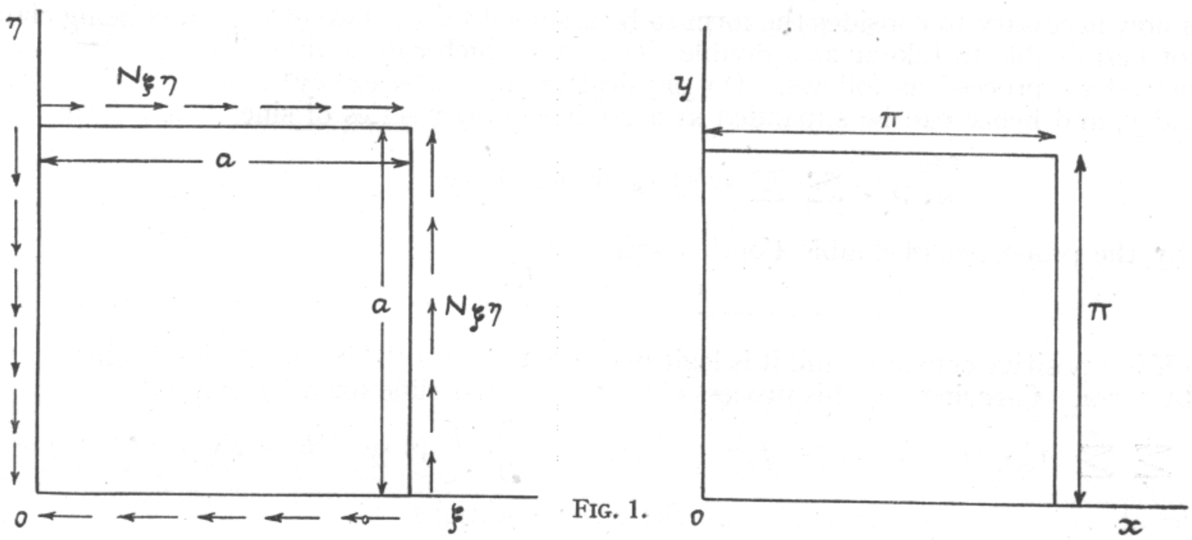

FIG. 1.

and hence, denoting by ω the normal displacement of any point in the middle surface, the fundamental equation from which to obtain $N_{\xi\eta}$ is

$$\frac{D}{2}\int_0^a\int_0^a\left\{\left(\frac{\partial^2 w}{\partial\xi^2}+\frac{\partial^2 w}{\partial\eta^2}\right)^2-2(1-\sigma)\left[\frac{\partial^2 w}{\partial\xi^2}\frac{\partial^2 w}{\partial\eta^2}-\left(\frac{\partial^2 w}{\partial\xi\delta\eta}\right)^2\right]\right\}d\xi\,d\eta$$

$$=-N_{\xi\eta}\int_0^a\int_0^a\frac{\partial w}{\partial\xi}\frac{\partial w}{\partial\eta}\,d\xi\,d\eta$$

which reduces immediately to

$$\frac{D}{2}\int_0^a\int_0^a\left(\frac{\partial^2 w}{\partial\xi^2}+\frac{\partial^2 w}{\partial\eta^2}\right)^2 d\xi\,d\eta=-N_{\xi\eta}\int_0^a\int_0^a\frac{\partial w}{\partial\xi}\frac{\partial w}{\partial\eta}\,d\xi\,d\eta, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

since

$$\int_0^a\int_0^a\left\{\frac{\partial^2 w}{\partial\xi^2}\frac{\partial^2 w}{\partial\eta^2}-\left(\frac{\partial^2 w}{\partial\xi\delta\eta}\right)^2\right\}d\xi\,d\eta$$

$$=\tfrac{1}{2}\int_0^a\int_0^a\left\{\frac{\partial}{\partial\xi}\left[\frac{\partial}{\partial\xi}\left(\frac{\partial w}{\partial\eta}\right)^2-\frac{\partial}{\partial\eta}\left(\frac{\partial w}{\partial\xi}\frac{\partial w}{\partial\eta}\right)\right]+\frac{\partial}{\partial\eta}\left[\frac{\partial}{\partial\eta}\left(\frac{\partial w}{\partial\xi}\right)^2-\frac{\partial}{\partial\xi}\left(\frac{\partial w}{\partial\xi}\frac{\partial w}{\partial\eta}\right)\right]\right\}d\xi\,d\eta$$

$$=\tfrac{1}{2}\oint\left\{\frac{\partial}{\partial\xi}\left(\frac{\partial w}{\partial\eta}\right)^2-\frac{\partial}{\partial\eta}\left(\frac{\partial w}{\partial\xi}\frac{\partial w}{\partial\eta}\right)\right\}d\eta-\tfrac{1}{2}\oint\left\{\frac{\partial}{\partial\eta}\left(\frac{\partial w}{\partial\xi}\right)^2-\frac{\partial}{\partial\xi}\left(\frac{\partial w}{\partial\xi}\frac{\partial w}{\partial\eta}\right)\right\}d\xi$$

and the two line integrals, taken round the edge of the panel, vanish identically by virtue of the boundary conditions.

Making the substitutions

$$x = \frac{\pi \xi}{a},$$

$$y = \frac{\pi \eta}{a},$$

$$S = \frac{N_{\xi\eta}\, a^2}{D\pi^2},$$

equation (1) reduces to

$$\int_0^\pi \int_0^\pi \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 dxdy = -\,2S \int_0^\pi \int_0^\pi \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\, dxdy \quad \ldots \quad \ldots \quad (2)$$

It is now necessary to consider the form to be assumed for w. Two of the edges being clamped, it is not permissible to take w as a double sine series which can be differentiated term by term, and instead we proceed as follows. On physical grounds $\partial^8 \omega / \partial x^4\, \partial y^4$ is a continuous function of x and y, and hence can be expanded as a double Fourier series of sines in the form

$$\frac{\partial^8 w}{\partial x^4 \partial y^4} = \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} m^4 r^4 A_{mr} \sin mx \sin ry. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Then by the properties of double Fourier series

$$|\, A_{mr} \,| < \frac{K}{m^5 r^5}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

where K is a positive constant, and it is legitimate to integrate the infinite series on the right of (3) term by term. Carrying out this process eight times, a possible form for w is

$$\sum_1^{\infty} \sum_1^{\infty} A_{mr} \left\{ \sin mx + t_m + p_m x + q_m x^2 + s_m x^3 \right\} \left\{ \sin ry + h_r + e_r y + f_r y^2 + g_r y^3 \right\}, \quad (5)$$

and the above constants, obtained from the edge conditions

$$w = \frac{\partial^2 w}{\partial x^2} + \sigma \frac{\partial^2 w}{\partial y^2} = 0, \quad x = 0, \pi,$$

$$w = \frac{\partial w}{\partial y} = 0, \qquad y = 0, \pi,$$

are given by the relations

$$\begin{aligned} t_m &= 0, & h_r &= 0, \\ p_m &= 0, & e_r &= -r, \\ q_m &= 0, & f_r &= \frac{1}{\pi}\left\{2 + (-)^r\right\} r, \\ s_m &= 0, & g_r &= -\frac{1}{\pi^2}\left\{1 + (-)^r\right\} r. \end{aligned} \quad \ldots \quad \ldots \quad (6)$$

From (5) and (6) the expression for w can be put in the form

$$\sum_{m=1}^{\infty} \sin mx \left\{ \sum_{r=1}^{\infty} A_{mr} \sin ry + E_m y + F_m y^2 + G_m y^3 \right\}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

where

$$E_m = -\sum_{r=1}^{\infty} r A_{mr},$$

$$F_m = \frac{1}{\pi} \sum_{r=1}^{\infty} \left\{2 + (-)^r\right\} r\, A_{mr},$$

$$G_m = -\frac{1}{\pi^2} \sum_{r=1}^{\infty} \left\{1 + (-)^r\right\} r\, A_{mr}.$$

It is now possible to simplify the form for w still further by utilising the fact that

$$w\,(x,y) = w\,(\pi - x, \pi - y).$$

For by substituting in this relation the expression for ω given by (7), it can be shown that $m + r$ must be even.

The next step is to evaluate the two double integrals on the right and left hand sides of equation (2), denoted below by L and M respectively. From (4) it follows that all the series involved are absolutely and uniformly convergent, so that term by term integration or differentiation is legitimate in every case. The evaluation is accordingly quite straightforward, but the work involved is heavy, and only the final expressions are given here. They are

$$L = \underset{(m+n \text{ odd})}{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}} \frac{2nm}{(n^2-m^2)} \Bigg[\underset{(r+s \text{ odd})}{\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}} \frac{2rs}{(r^2-s^2)} + \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{\pi^2}{60} rs\Big\{(-)^r - (-)^s\Big\}$$

$$+ \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{r^2-s^2}{rs}\Big\{1 - (-)^r(-)^s\Big\}$$

$$+ \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{6}{\pi^2}\Big\{\frac{s}{r^3}\Big(1+(-)^s\Big)\Big(1-(-)^r\Big) - \frac{r}{s^3}\Big(1-(-)^s\Big)\Big(1+(-)^r\Big)\Big\}\Bigg] A_{mr}A_{ns},$$

$$M = \left(\frac{\pi}{2}\right)^2 \sum_{m=1}^{\infty}\sum_{r=1}^{\infty} (m^2+r^2)^2 A_{mr}{}^2$$

$$+ \frac{\pi}{2}\sum_{m=1}^{\infty}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\Bigg[\Big\{\frac{rs}{\pi} - \frac{(r^4+s^4)}{\pi r^3 s^3}(m^2+r^2)(m^2+s^2)\Big\}$$

$$\times \Big\{4 + 2(-)^r + 2(-)^s + 4(-)^r(-)^s\Big\}$$

$$+ \frac{m^2\pi rs}{60}\Big\{16 - 4(-)^r - 4(-)^s + 16(-)^r(-)^s\Big\}$$

$$+ \frac{m^4\pi^3 rs}{420}\Big\{4 - 3(-)^r - 3(-)^s + 4(-)^r(-)^s\Big\}\Bigg] A_{mr}A_{ms}.$$

S is now given in terms of the A's by substituting for L and M in (2), and the next step is to find for what type of distortion S is a minimum. Since S is stationary when regarded as a function of the A's, there results the following system of equations from which to determine the ratio of the A's and the corresponding value of the critical shear

$$2S\frac{\partial L}{\partial A_{mr}} + \frac{\partial M}{\partial A_{mr}} = 0.$$

In general the only solution of this infinite set of equations linear in the A's, is that in which all the A's are zero. If, however, the infinite determinant formed by eliminating the A's vanishes, an exception arises and there exists a non zero solution. Since it is only this case which is significant, it follows that, by equating the infinite determinant to zero, there results an equation from which to determine the critical values of S. Being of indefinitely large order, this equation has an infinite number of roots, but in practice it is only the smallest of these roots that is of interest.

The determinantal equation is of the form

$$\begin{array}{l} \sin x \sin y \\ \sin 2x \sin 2y \\ \sin 3x \sin y \\ \sin x \sin 3y \\ \sin 3x \sin 3y \\ \sin 4x \sin 2y \\ \sin 2x \sin 4y \\ \sin 5x \sin y \\ \sin x \sin 5y \end{array} \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\ a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99} \end{vmatrix} = 0, \quad \ldots \quad \ldots \quad (8)$$

where a_{ij} is the coefficient of $A_{mr}\ A_{ns}$ in $2SL + M$ after m, n, r, s have been given their appropriate values. Since L and M are homogeneous quadratic expressions in the A's

$$a_{ij} = a_{ji}.$$

Since the solution as expressed by the equation (8) is merely a formal one, it is now desirable to consider the convergence of the determinant comprising its left hand side. For some infinite determinants it is often possible to construct a purely analytical proof of convergence. This, however, does not appear practicable here, and instead we proceed otherwise. Denote by S_n the smallest root of the equation for S when the infinite determinant (8) is replaced by the finite determinant containing its first n^2 elements. Then since the type of distortion corresponding to S_{n+1} is a more general one than that corresponding to S_n, actually including it as a special case, it follows that $S_{n+1} \leqslant S_n$. Since in addition S_n is essentially positive, it follows that $S_n(n = 1, 2, - - \infty)$ is a decreasing sequence which is bounded below and accordingly tends to a limit ; and it is this limit which is the critical value of S required. In order to show the nature of the determinants involved, the one corresponding to n equal to 9 is given below.

$$\begin{vmatrix} 0{\cdot}7829 & -0{\cdot}3410\,S & 0 & -10{\cdot}67 & 0 & -0{\cdot}1364\,S & 1{\cdot}007\,S & 0 & -16{\cdot}25 \\ -0{\cdot}3410\,S & 36{\cdot}34 & 0{\cdot}6138\,S & 6{\cdot}061\,S & -10{\cdot}91\,S & 0 & -126{\cdot}1 & 0{\cdot}2436\,S & 5{\cdot}070\,S \\ 0 & 0{\cdot}6138\,S & 9{\cdot}849 & 0 & -134{\cdot}7 & -0{\cdot}8768\,S & -1{\cdot}812\,S & 0 & 0 \\ -10{\cdot}67 & 6{\cdot}061\,S & 0 & 264{\cdot}5 & 0 & 2{\cdot}424\,S & -21{\cdot}82\,S & 0 & 34{\cdot}21 \\ 0 & -10{\cdot}91\,S & -134{\cdot}7 & 0 & 2123 & 15{\cdot}58\,S & 39{\cdot}28\,S & 0 & 0 \\ -0{\cdot}1364\,S & 0 & -0{\cdot}8768\,S & 2{\cdot}424\,S & 15{\cdot}58\,S & 174{\cdot}8 & 0 & 1{\cdot}137\,S & 2{\cdot}028\,S \\ 1{\cdot}007\,S & -126{\cdot}1 & -1{\cdot}812\,S & -21{\cdot}82\,S & 39{\cdot}28\,S & 0 & 968{\cdot}7 & -0{\cdot}7192\,S & -3{\cdot}087\,S \\ 0 & 0{\cdot}2436\,S & 0 & 0 & 0 & 1{\cdot}137\,S & -0{\cdot}7192\,S & 62{\cdot}88 & 0 \\ -16{\cdot}25 & 5{\cdot}070\,S & 0 & 34{\cdot}21 & 0 & 2{\cdot}028\,S & -3{\cdot}087\,S & 0 & 1732 \end{vmatrix}$$

The values of S_n, from which the limiting value is obtained, are given in the adjoining table.

n	4	5	7	9	∞
S_n	13·3(7)	12·6(8)	12·6(7)	12·6	12·6

Owing to the very large amount of multiplication which has to be performed in the evaluation of the determinants, a cumulative error occurs, and no weight is to be attached to the fourth figure.

(72176) Wt. 9/7116 3/46 Hw. G.377/1