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by

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SYSTEMS OF COORDINATES SUITABLE FOR THE NUMERICAL CALCULATION OF THREE-DIMENSIONAL FLOW FIELDS

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SUMMARY

During the last few years conformal mapping has successfully been applied to obtain systems of coordinates suitable for the numerical calculation of the inviscid compressible flow past a prescribed profile in two dimensions. Here tensor analysis is used to show that in three-dimensional flow problems certain integrability conditions can be used to calculate three families of coordinate surfaces, one of which contains the surface of a given body shape. They are efficient for numerical work, i.e. they define a 'smooth' system of grid points which are closely spaced near the body and are sparse at large distance from the body. Two cases are considered in some detail, one suitable for wings of finite aspect ratio and one suitable for swept wings of infinite aspect ratio. The latter case is also of mathematical interest, since the theory of complex functions can be used to calculate the coordinate surfaces for a given body shape.

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CONTENTS

1 INTRODUCTION

In recent years it has become apparent that in aircraft design a great deal more information is required on the effects of compressibility on the air-flow past wings and wing body combinations than can be obtained from linearised theory. Transonic small perturbation theory has helped to fill in the gap of theoretical knowledge, in particular since Murman and Cole[1], Krupp and Murman[2], Steger and Lomax[3] have written computer programs for two-dimensional flow. These have recently been extended to cover three-dimensional flows past wings at near sonic speeds (Lomax[4], Bailey and Steger[5]).

On the other hand it has always been highly desirable to obtain solutions of the complete equations of motion for compressible inviscid flow. The method by Sells[6] for two-dimensional subsonic flow and the method by Garabedian[7], which covers also enclosed supersonic systems in a two-dimensional flow are both based on the solution of difference equations in a suitable grid which is obtained by conformal mapping of the exterior of the profile contour on the exterior (or interior) of a circle. The lines of constant radius and constant peripheral angle in the circle plane form a suitable system of coordinates in which a convenient grid can be defined in which mesh size is totally related to the required accuracy of information and also infinity can be treated within a finite working space. For elliptic problems methods based on variational techniques (see review by H. Rasmussen[8]) have been used and the use of finite element techniques - which has proved successful in structural problems - has been advocated in fluid dynamic problems (Argyris[9]). Recently a fairly comprehensive review of this field by Yoshihara[10] has appeared.

In order to obtain results of reasonable accuracy it appears to be advantageous to use a grid in the difference sceheme in which the body surface is a coordinate surface. This has been done by Sells[6] and Garabedian[7] in their difference equations and also by Rasmussen and Heys[11] in their treatment of the variational approach by difference methods. So the application of conformal mapping to obtain suitable grids for two-dimensional problems had several advantages, namely (1) the treatment of infinity was simplified, (2) the treatment of the boundary condition in the body surface was simplified and improved in accuracy and (3) the mesh size in the field was easily adjusted to requirements of accuracy and definition.

This Report is concerned with the question whether some, if not most, of these points can also be achieved in three-dimensional flow problems. Tensor analysis is a useful tool in trying to answer this question, since it provides a fairly simple description of all possible systems of coordinates which can be used in a three-dimensional Cartesian space[12,13]. In section 2 a general system of coordinates is discussed in terms of its associated base vectors and normal vectors and a number of useful relations (for co-variant and contra-variant components of a vector) is given. In section 3 the conservation equations for mass, momentum and energy for inviscid flow[14] are written down in general vector notation.

In order to simplify the description of the directions of the normal base vectors and normal vectors in relation to the Cartesian coordinates, three Euler angles are introduced in section 4.2. They are normally used to describe the motion of a rigid body in three-dimensional space. Here they help to simplify the geometric relations.

In order to define a general system of coordinates certain integrability conditions have to be satisfied. They are discussed in section 4.1. It appears that in general six of the nine integrability conditions in terms of the three Euler angles, the lengths of the three base vectors and the three angles between the three base vectors are used to obtain the components of the fundamental metric tensor. The three degrees of freedom which exist in this system can in many cases be used to obtain a completely orthogonal system. The remaining three integrability conditions are used to establish a coordinate system on the body surface. This system of coordinates would be appropriate for the calculation of a compressible flow past a finite wing or a wing-body combination (Case I, section 2). A system consisting of ellipsoids and two families of hyperboloids (Lamb[15]) is a well-known example.

Another interesting case (Case II, section 5) arises when one tries to establish a system of coordinates, in which one of the Cartesian coordinate planes is retained and only two new families of coordinate surfaces are defined. This system is useful for the investigation of the flow past a semi-infinite body, as was done by Walkden[16], for the calculation of supersonic flows. Another interesting example is a wing of infinite aspect ratio (section 5) on which geometric properties (e.g. sweep) vary along its span. Here one has to satisfy six integrability conditions of which two are to be satisfied on the body (section 5.1). The two remaining degrees of freedom can be used to define conformal mapping within the planes which are retained as coordinate surfaces, with the mapping dependent on this first variable as a parameter (section 5.2).

Here the calculation of the coordinates for a given infinite body shape can be reduced to the computation of two analytic functions of a complex variable.

In both these cases the coordinates can be calculated first and the conservation equations can be solved as a second step. If one chooses stream-surfaces as coordinate surfaces (e.g. Walkden[16]) the integrability conditions must be solved together with the conservation equations.

A number of appendices give some more details of the analysis in the main part of the Report. They are meant to help the reader who has less experience in tensor analysis.

2 GENERAL SYSTEMS OF COORDINATES

Consider a transformation of the form

$$x^i = x^i(\xi^1,\xi^2,\xi^3) \quad ; \qquad i = 1,2,3 \tag{2-1}$$

and its inverse

$$\xi^i = \xi^i(x^1,x^2,x^3) \quad ; \qquad i = 1,2,3 \quad , \tag{2-2}$$

where x^i, $i = 1,2,3$, refer to orthogonal Cartesian coordinates, and ξ^i, $i = 1,2,3$ denote general coordinates[12,13].

The Jacobians of the transformation are denoted respectively by

$$J \equiv \left|\frac{\partial x^i}{\partial \xi^j}\right| \qquad \text{and} \qquad J^* \equiv \left|\frac{\partial \xi^i}{\partial x^j}\right|$$

where $JJ^* = 1$ and $J,J^* \not\equiv 0$ except, perhaps, at some isolated singular points.

At each point P we construct a system of 'base' vectors $\underline{a}_i$, $i = 1,2,3$ (Fig.1) which are tangential to the curves of intersection of the surfaces ξ^{i-1} = const. and ξ^{i+1} = const., and a reciprocal system of 'normal' vectors $\underline{a}^i$, $i = 1,2,3$, normal to the surfaces ξ^i = const. The normal vectors are defined by the relations

$$J\varepsilon_{ijk}\underline{a}^i = \underline{a}_j \times \underline{a}_k \quad . \tag{2-3}$$

We note in equation (2-3) that a summation is carried out over the repeated index i. This convention of tensor analysis will be adopted throughout the presentation that follows unless otherwise stated.

In (2-3) we define

$$\varepsilon_{ijk} = +1 \quad \text{if } ijk \text{ are an even permutation of } 1,2,3$$
$$= -1 \quad \text{if } ijk \text{ are an odd permutation of } 1,2,3$$
$$= 0 \quad \text{in all other cases.}$$

The differential forms of (2-1) and (2-2) will be given by

$$dx^i = \frac{\partial x^i}{\partial \xi^j} d\xi^j \equiv t^i_{\cdot j} d\xi^j \quad , \qquad i = 1,2,3 \tag{2-4}$$

and

$$d\xi^i = \frac{\partial \xi^i}{\partial x^j} dx^j \equiv t^{*i}_{\cdot j} dx^j \quad , \qquad i = 1,2,3 \tag{2-5}$$

respectively. It is shown in Appendix A (see equations (A-2), (A-9)-(A-11)) that the relations between the vectors $\underline{a}_i$, $\underline{a}^j$ and the Cartesian unit base vectors $\underline{c}_i$, $\underline{c}^j$ are of the form

$$\underline{a}_i = t^{\ell}_{\cdot i}\underline{c}_{\ell} \quad , \tag{2-6}$$

and

$$\underline{a}^j = t^{*j}_{n}\underline{c}^n \quad , \tag{2-7}$$

respectively. The fundamental metric is given by

$$ds^2 = g_{ij} d\xi^i d\xi^j \quad , \tag{2-8}$$

where g_{ij} is the symmetric co-variant metric tensor of order two defined by the equation

$$g_{ij} = \underline{a}_i \cdot \underline{a}_j \quad . \tag{2-9}$$

The symmetric contra-variant metric tensor g^{ij} is given by

$$g^{ij} = \underline{a}^i \cdot \underline{a}^j \quad , \tag{2-10}$$

and in Appendix A (equations (A-14)-(A-20)) the following relations are established:

$$g^{\ell i} g_{in} = \delta^{\ell}_{n} \tag{2-11}$$

$$J^2 = |g_{ij}| \tag{2-12}$$

$$J*^2 = |g^{k\ell}| = J^{-2} \tag{2-13}$$

$$\underline{a}_k = g_{k\ell}\underline{a}^{\ell} \tag{2-14}$$

$$\underline{a}^{\ell} = g^{\ell i}\underline{a}_i \quad . \tag{2-15}$$

Furthermore, any vector $\underline{v}$ can be written in the form

$$\underline{v} = v_{\ell}\underline{a}^{\ell} = v^{\ell}\underline{a}_{\ell} \quad , \tag{2-16}$$

where v_{ℓ} and v^{ℓ} are the co-variant and contra-variant components of $\underline{v}$ and from Appendix A (equations (A-25)-(A-28)) we have the relations

$$v_{\ell} = t^{n}_{\ell} u_n \quad , \tag{2-17}$$

$$v^{\ell} = t^{*\ell}_{i} u^i \quad , \tag{2-18}$$

where u_i and u^i are the components of $\underline{v}$ in the Cartesian reference frame. Also, using equations (A-30)-(A-33), of Appendix A we will have

$$v_i = g_{i\ell} v^{\ell} \tag{2-19}$$

$$v^i = g^{is} v_s \tag{2-20}$$

$$v_i = \underline{v} \cdot \underline{a}_i \tag{2-21}$$

$$v^i = \underline{v} \cdot \underline{a}^i \quad . \tag{2-22}$$

If A_i and A^i denote the magnitudes of the vectors $\underline{a}_i$ and $\underline{a}^i$ then from equations (A-35) and (A-38) we obtain the relations

$$t_i^{\,j} = A_i \cos\theta_i^{\,j} \quad , \tag{2-23}$$

and

$$t_j^{*i} = A^i \cos\theta_j^{*i} \quad , \tag{2-24}$$

where $\theta_i^{\,j}$ denotes the angle between the vectors $\underline{a}_i$ and $\underline{c}_j$ and θ_j^{*i} denotes the angle between $\underline{a}^i$ and $\underline{c}^j$. Furthermore, if θ_{ij} denotes the angle between $\underline{a}_i$ and $\underline{a}_j$ and $\theta^{k\ell}$ denotes the angle between $\underline{a}^k$ and $\underline{a}^\ell$ then from equations (A-39)-(A-67) of Appendix A we obtain the following relations

$$\cos\theta_{ij} = \cos\theta_i^{\,k} \cos\theta_j^{\,\ell} \delta_{k\ell} \quad , \tag{2-25}$$

$$\cos\theta^{k\ell} = \delta^{rs} \cos\theta_r^{*k} \cos\theta_s^{*\ell} \quad , \tag{2-26}$$

$$g_{ij} = A_i A_j \cos\theta_{ij} \quad , \tag{2-27}$$

$$g^{k\ell} = A^k A^\ell \cos\theta^{k\ell} \quad , \tag{2-28}$$

$$\begin{aligned} J_0^2 &\equiv (A_1A_2A_3)^{-2} J^2 \\ &= 1 + 2\cos\theta_{12}\cos\theta_{23}\cos\theta_{31} - \cos^2\theta_{12} - \cos^2\theta_{23} - \cos^2\theta_{31} \quad , \end{aligned} \tag{2-29}$$

$$J_0 d_N^{*N} = \sin\theta_{N+1,N+2} > 0 \quad , \tag{2-30}$$

where

$$d_N^{*N} \equiv A_N A^N \quad , \tag{2-31}$$

$$J_0 = \sin\theta_{N,N+1} \sin\theta_{N+1,N+2} \sin\theta^{N+2,N+3} \quad , \tag{2-32}$$

$$\sin\theta_{N,N+1} \sin\theta_{N+1,N+2} \cos\theta^{N+2,N} = \cos\theta_{N,N+1}\cos\theta_{N+1,N+2} - \cos\theta_{N+2,N} \quad , \tag{2-33}$$

$$\sin\theta^{N,N+1} \sin\theta^{N+1,N+2} \cos\theta_{N+2,N} = \cos\theta^{N,N+1}\cos\theta^{N+1,N+2} - \cos\theta^{N+2,N} \quad . \tag{2-34}$$

We note that the use of a capitalized index N in equations (2-30)-(2-34) means no summation over N.

3 EQUATIONS OF MOTION

3.1 General case

The equations of continuity, momentum and energy for a steadily moving compressible inviscid fluid under the influence of no external forces are[14]

$$\frac{\partial}{\partial \xi^k} (J\rho v^k) = 0 \ , \qquad (3\text{-}1)$$

$$\frac{\partial}{\partial \xi^i} (\tfrac{1}{2}q^2) - (\underline{v} \times \underline{\Omega})_i = -\frac{1}{\rho}\frac{\partial p}{\partial \xi^i} \ , \qquad i = 1,2,3 \qquad (3\text{-}2)$$

$$q^2 = v_k v^k \qquad (3\text{-}3)$$

$$v^k \frac{\partial s}{\partial \xi^k} = 0 \ , \qquad (3\text{-}4)$$

where v^i are the contra-variant components of the fluid velocity vector $\underline{v}$, ρ the fluid density, p the fluid pressure, $\underline{\Omega}$ the vorticity vector, q the fluid speed and s the entropy. In addition to equations (3-1)-(3-4) we also have the general gas law

$$dh = \frac{dp}{\rho} + T ds \ , \qquad (3\text{-}5)$$

where h is the enthalpy and T the temperature.

An alternative form of (3-2) is

$$v^k \frac{\partial}{\partial \xi^k} v_i - \frac{1}{2}\left(v^k \frac{\partial}{\partial \xi^i} v_k - v_k \frac{\partial}{\partial \xi^i} v^k\right) = -\frac{1}{\rho}\frac{\partial p}{\partial \xi^i} \qquad (3\text{-}2a)$$

which is useful sometimes.

For irrotational flow we must have $\underline{\Omega} = 0$ so that using (3-5) in (3-2) we obtain the momentum equations (3-2) in the form

$$\frac{\partial}{\partial \xi^i} (h + \tfrac{1}{2}q^2) = T \frac{\partial s}{\partial \xi^i} \ , \qquad i = 1,2,3 \ . \qquad (3\text{-}6)$$

If equation (3-6) is multiplied scalarly by v^i we obtain

$$v^i \frac{\partial}{\partial \xi^i} (h + \tfrac{1}{2}q^2) = Tv^i \frac{\partial s}{\partial \xi^i} . \tag{3-7}$$

Since external forces and viscous effects are neglected the left hand side of (3-6) vanishes since the total energy is constant upstream and remains constant along a streamline. We therefore obtain Bernoulli's equation in the form

$$h + \tfrac{1}{2}q^2 = c , \tag{3-8}$$

where c is a constant except across shock waves. Therefore s remains constant along streamlines. In the case of homentropic flow s will have the same value everywhere. Multiplying (3-2) by v^i we note

$$v^i . (\underline{v} \times \underline{\Omega})_i = 0 , \tag{3-9}$$

since

$$J\Omega^k = \varepsilon^{k\ell m} \frac{\partial v_m}{\partial \xi^\ell} . \tag{3-10}$$

In the case of a perfect gas the equation of state will be of the form

$$p = R\rho T ,$$

where R is the gas constant and the speed of sound, a, will be given by

$$a^2 = \frac{\gamma p}{\rho} = \left(\frac{dp}{d\rho}\right)_{s=\text{const.}} \tag{3-11}$$

where γ is the ratio of specific heats. Also the enthalpy can be written in the form

$$h = \frac{a^2}{\gamma - 1} , \tag{3-12}$$

and the general gas law (3-5) can be written

$$dh = a^2\left(\frac{d\rho}{\rho} + \frac{ds}{\gamma R}\right) . \tag{3-13}$$

Using (3-4), (3-12) and (3-13) in equations (3-1) and (3-8) respectively we obtain the continuity equation and Bernoulli's equation in the form

$$\frac{1}{J}\frac{\partial}{\partial\xi^i}(Jv^i) - \frac{v^i q}{a^2}\frac{\partial q}{\partial\xi^i} = 0 \quad , \tag{3-14}$$

and

$$\frac{a^2}{\gamma - 1} + \tfrac{1}{2}q^2 = \text{const.} \quad . \tag{3-15}$$

The speed of sound is determined from equation (3-15) in terms of q^2 so that, in general, the field equations to be solved are the continuity equation (3-14) and the condition for irrotationality

$$\Omega^k = 0 \quad , \qquad k = 1,2,3 \quad . \tag{3-16}$$

We also note that only two of the equations (3-16) are independent since the identity $(\partial/\partial\xi^k)(J\Omega^k) = 0$ holds everywhere throughout the field. The equations of motion (3-14) and (3-16) must be solved with the appropriate boundary conditions which hold on the body surfaces and at infinity.

3.2 Potential flow

We consider the inviscid flow past a wing or a wing-body combination. We introduce a velocity potential Φ by

$$v_i = \frac{\partial\Phi}{\partial\xi^i} \quad , \qquad i = 1,2,3 \tag{3-17}$$

so that equations (3-16) are satisfied. The system of coordinates should be chosen so that the surface $\xi^N = \text{const.}$ corresponds to the body surface. Using equations (A-31) in the continuity equation (3-14) we obtain

$$\frac{1}{J}\frac{\partial}{\partial\xi^i}\left(Jg^{ij}v_j\right) - \frac{g^{ij}v_j q}{a^2}\frac{\partial q}{\partial\xi^i} = 0 \quad , \tag{3-18}$$

which, together with (3-17) and the relation

$$q^2 = g^{ij}v_i v_j \tag{3-19}$$

provides us with an equation for the velocity potential Φ. In the special case when the reference system is orthogonal (see section 4) we obtain $g^{ij} = g_{ij} = 0,\ i \neq j$ and equation (3-18) will have the form

$$\frac{1}{A_1A_2A_3} \frac{\partial}{\partial \xi^i} \left(A_1A_2A_3 (A^i)^2 \frac{\partial \Phi}{\partial \xi^i} \right) - \frac{(A^i)^2}{2a^2} \frac{\partial (q^2)}{\partial \xi^i} \frac{\partial \Phi}{\partial \xi^i} = 0 \ . \tag{3-20}$$

The boundary condition that holds on ξ^N = const. is

$$v^N = 0 \tag{3-21}$$

since there is no flow through the surface of the body. In addition to (3-21) we must impose a suitable condition on the transformation at large distances from the body which leaves the flow field undisturbed. In the case of a lifting wing further boundary conditions must be applied along the vortex sheet behind the wing, which state that there are no forces between the two faces of the sheet.

As will be seen in sections 4 and 5 a system of differential equations, the 'integrability conditions', can be used to determine the coordinates and the quantities A_i and $\cos \theta_{ik}$ for a prescribed body geometry. After this first stage the flow field can be calculated using the equations in this section 3.

3.3 Streamline coordinates

An interesting application of ideas similar to these developed in this Report has been made by F. Walkden, who calculated the three-dimensional supersonic flow past a given body shape by marching in the stream direction. He retained one coordinate $x^1 = \xi^1$ (section 5) and used stream surfaces as coordinate surfaces, so that in his case the integrability conditions and the equations of motion had to be solved simultaneously. We let $\underline{v}$ coincide with $\underline{a}_1$ so that

$$v^2 = v^3 = 0 \tag{3-22}$$

$$v_1 = g_{11}v^1 = (A_1)^2 v^1 \tag{3-23}$$

$$v_2 = g_{21}v^1 = A_2A_1 \cos \theta_{12} v^1 \tag{3-24}$$

$$v_3 = g_{31}v^1 = A_3A_1 \cos \theta_{31} v^1 \tag{3-25}$$

and

$$q^2 = v_1 v^1 = \left(A_1 v^1\right)^2 \quad . \tag{3-26}$$

The conservation equation (3-14) takes the form

$$q \frac{\partial}{\partial \xi^1} \log \left(J_0 A_2 A_3\right) + \left(1 - \frac{q^2}{a^2}\right) \frac{\partial q}{\partial \xi^1} = 0 \tag{3-27}$$

and equations (3-16) are:

$$\frac{\partial}{\partial \xi^3} \left(A_1 q\right) - \frac{\partial}{\partial \xi^1} \left(A_3 \cos \theta_{31} q\right) = 0 \tag{3-28}$$

$$\frac{\partial}{\partial \xi^2} \left(A_1 q\right) - \frac{\partial}{\partial \xi^1} \left(A_2 \cos \theta_{12} q\right) = 0 \quad . \tag{3-29}$$

4 FINITE BODIES (Case I)

4.1 The integrability conditions

The necessary and sufficient conditions for the existence of an integral of equations (2-4) are given by the following nine equations

$$\frac{\partial}{\partial \xi^i} \left(t^k_j\right) = \frac{\partial}{\partial \xi^j} \left(t^k_i\right) \,, \qquad (i \neq j,\ k = 1,2,3) \,. \tag{4-1}$$

In general these equations must hold everywhere in order that transformations of the form (2-1) and (2-2) can be found. The nine equations (4-1) can be written in the form

$$\Omega^{i,k} = 0 \,, \qquad i,k = 1,2,3 \tag{4-2}$$

where

$$\Omega^{i,k} = \varepsilon^{i\ell m} \frac{\partial}{\partial \xi^\ell} \left(t^k_m\right) \quad . \tag{4-3}$$

It will prove useful in what follows to introduce $\bar{\Omega}^{i,k}$ where

$$\bar{\Omega}^{i,k} = \cos \theta^{*k}_{\ell} \Omega^{i,\ell} \quad . \tag{4-4}$$

Next we prove that a necessary and sufficient condition for the integrability of equations (2-4) is that, for any given i, the following six equations hold in the field

$$\bar{\Omega}^{i,k} = 0 \quad , \qquad k = 1,2,3 \quad , \tag{4-5}$$

$$\bar{\Omega}^{i+1,k} = 0 \quad , \qquad k = 1,2,3 \quad , \tag{4-6}$$

and the following three equations hold on some surface ξ^{i+2} = const.

$$\bar{\Omega}^{i+2,k} = 0 \quad , \qquad k = 1,2,3 \quad . \tag{4-7}$$

First we assume that for any given i we have (4-5) and (4-6) satisfied in the field and (4-7) satisfied on the surface ξ^{i+2} = const. Then from (A-48) we have $|\cos \theta_k^{*\ell}| = J_0^* \neq 0$ and it follows from equation (4-4) that

$$\Omega^{i,k} = 0 \quad , \qquad k = 1,2,3 \quad , \tag{4-8}$$

$$\Omega^{i+1,k} = 0 \quad , \qquad k = 1,2,3 \quad , \tag{4-9}$$

at every field point and

$$\Omega^{i+2,k} = 0 \quad , \qquad k = 1,2,3 \tag{4-10}$$

on ξ^{i+2} = const. We also have the identities

$$\frac{\partial}{\partial \xi^\ell} (\Omega^{\ell,k}) = 0 \quad , \qquad k = 1,2,3 \tag{4-11}$$

which hold at every field point. Therefore, using (4-8) and (4-9) in the field we can integrate (4-11) with respect to ξ^{i+2} to obtain, with (4-10),

$$\Omega^{i+2,k} = 0 \quad , \qquad k = 1,2,3 \quad , \tag{4-12}$$

at every point and the nine equations (4-2) are satisfied. The converse follows from equations (4-4) and our assertion is proved.

Our choice of the field equations (4-5)-(4-7) in the subsequent analysis will be

$$\bar{\Omega}^{2,k} = 0 \; , \qquad k = 1,2,3 \; , \tag{4-13}$$

$$\bar{\Omega}^{3,k} = 0 \; , \qquad k = 1,2,3 \; , \tag{4-14}$$

together with

$$\bar{\Omega}^{1,k} = 0 \; , \qquad k = 1,2,3 \tag{4-15}$$

on the surface ξ^1 = const. Using the relations (A-39) we find that the field equations (4-13) and (4-14) will be of the form

$$\frac{\delta_1^k}{A_1 A^k} \frac{\partial A_1}{\partial \xi^3} + A_1 \cos \theta_\alpha^{*k} \frac{\partial}{\partial \xi^3} \left(\cos \theta_1^\alpha \right) = \frac{\delta_3^k}{A_3 A^k} \frac{\partial A_3}{\partial \xi^1} + A_3 \cos \theta_\alpha^{*k} \frac{\partial}{\partial \xi^1} \left(\cos \theta_3^\alpha \right) \; ,$$

$$k = 1,2,3 \; , \tag{4-16}$$

$$\frac{\delta_1^k}{A_1 A^k} \frac{\partial A_1}{\partial \xi^2} + A_1 \cos \theta_\alpha^{*k} \frac{\partial}{\partial \xi^2} \left(\cos \theta_1^\alpha \right) = \frac{\delta_2^k}{A_2 A^k} \frac{\partial A_2}{\partial \xi^1} + A_2 \cos \theta_\alpha^{*k} \frac{\partial}{\partial \xi^1} \left(\cos \theta_2^\alpha \right) \; ,$$

$$k = 1,2,3 \; , \tag{4-17}$$

and the equations (4-15) which hold on ξ^1 = const. will have the form

$$\frac{\delta_2^k}{A_2 A^k} \frac{\partial A_2}{\partial \xi^3} + A_2 \cos \theta_\alpha^{*k} \frac{\partial}{\partial \xi^3} \left(\cos \theta_2^\alpha \right) = \frac{\delta_3^k}{A_3 A^k} \frac{\partial A_3}{\partial \xi^2} + A_3 \cos \theta_\alpha^{*k} \frac{\partial}{\partial \xi^2} \left(\cos \theta_3^\alpha \right) \; ,$$

$$k = 1,2,3 \; . \tag{4-18}$$

In section 4.2 the above form of the integrability conditions will be used for the first special reference system in which $\theta_{23} \not\equiv \pi/2$, $\theta_{12} \equiv \theta_{31} \equiv \pi/2$.

4.2 Nearly orthogonal system (Euler angles)

We choose $\theta_{12} \equiv \theta_{31} \equiv \pi/2$ and obtain $\underline{a}_1/A_1 = \underline{a}^1/A^1$ and $\underline{a}_1 \cdot \underline{a}_3 = \underline{a}_1 \cdot \underline{a}_2 = 0$, $\underline{a}_2 \cdot \underline{a}_3 = A_2 A_3 \cos \theta_{23} \not\equiv 0$ so that the fundamental base vectors $\underline{a}_2$ and $\underline{a}_3$ are both orthogonal to the base vector $\underline{a}_1$. The angle θ_{23} between the base vectors $\underline{a}_2$ and $\underline{a}_3$ is arbitrary. In the case $\theta_{23} = \pi/2$ the coordinate system will be orthogonal (see Fig.3a). In general, the orientation of the base and normal vectors must be expressed in terms of nine direction cosines. However, by introducing the three Euler angles θ, ϕ and ψ (see Appendix B) and the angles θ_{ij} the task of defining the directions of the base and normal vectors can be considerably simplified. In Appendix B the direction cosines of the vectors $\underline{a}_i$, $\underline{a}^i$ are derived in terms of θ, ϕ, ψ and θ_{23} and are tabulated in Tables 1 and 2. Furthermore, in Appendix D the integrability conditions (4-16)-(4-18) are shown to reduce to the following nine equations for A_1, A_2, A_3, θ, ϕ, ψ and θ_{23}:

$$\frac{\partial A_1}{\partial \xi^3} = A_3 \left\{ \sin(\psi - \theta_{23}) \frac{\partial \theta}{\partial \xi^1} - \sin\theta \cos(\psi - \theta_{23}) \frac{\partial \phi}{\partial \xi^1} \right\} \tag{4-19}$$

$$A_1 \left\{ \sin(\psi - \theta_{23}) \sin\theta \frac{\partial \phi}{\partial \xi^3} + \cos(\psi - \theta_{23}) \frac{\partial \theta}{\partial \xi^3} \right\} = - A_3 \left\{ \frac{\partial}{\partial \xi^1} (\psi - \theta_{23}) + \cos\theta \frac{\partial \phi}{\partial \xi^1} \right\} \tag{4-20}$$

$$A_1 \left\{ \sin\psi \sin\theta \frac{\partial \phi}{\partial \xi^3} + \cos\psi \frac{\partial \theta}{\partial \xi^3} \right\} = \sin\theta_{23} \frac{\partial A_3}{\partial \xi^1} - \cos\theta_{23} A_3 \left\{ \frac{\partial}{\partial \xi^1} (\psi - \theta_{23}) + \cos\theta \frac{\partial \phi}{\partial \xi^1} \right\} \tag{4-21}$$

$$\frac{\partial A_1}{\partial \xi^2} = A_2 \left\{ \sin\psi \frac{\partial \theta}{\partial \xi^1} - \sin\theta \cos\psi \frac{\partial \phi}{\partial \xi^1} \right\} \tag{4-22}$$

$$A_1 \left\{ \sin\theta \cos\psi \frac{\partial \phi}{\partial \xi^2} - \sin\psi \frac{\partial \theta}{\partial \xi^2} \right\} = \frac{\partial A_2}{\partial \xi^1} \tag{4-23}$$

$$A_1 \left\{ \cos\psi \frac{\partial \theta}{\partial \xi^2} + \sin\psi \sin\theta \frac{\partial \phi}{\partial \xi^2} \right\} = - A_2 \left\{ \cos\theta \frac{\partial \phi}{\partial \xi^1} + \frac{\partial \psi}{\partial \xi^1} \right\} \tag{4-24}$$

$$A_2 \left\{ \sin\theta \cos\psi \frac{\partial\phi}{\partial\xi^3} - \sin\psi \frac{\partial\theta}{\partial\xi^3} \right\}$$
$$= A_3 \left\{ \sin\theta \cos(\psi - \theta_{23}) \frac{\partial\phi}{\partial\xi^2} - \sin(\psi - \theta_{23}) \frac{\partial\theta}{\partial\xi^2} \right\} \qquad (4\text{-}25)$$

$$\sin\theta_{23} \frac{\partial A_2}{\partial\xi^3} + \cos\theta_{23} A_2 \left\{ \cos\theta \frac{\partial\phi}{\partial\xi^3} + \frac{\partial\psi}{\partial\xi^3} \right\}$$
$$= A_3 \left\{ \frac{\partial(\psi - \theta_{23})}{\partial\xi^2} + \cos\theta \frac{\partial\phi}{\partial\xi^2} \right\} \qquad (4\text{-}26)$$

$$A_2 \left\{ \cos\theta \frac{\partial\phi}{\partial\xi^3} + \frac{\partial\psi}{\partial\xi^3} \right\} = -\sin\theta_{23} \frac{\partial A_3}{\partial\xi^2} + \cos\theta_{23} A_3 \left\{ \cos\theta \frac{\partial\phi}{\partial\xi^2} + \frac{\partial(\psi - \theta_{23})}{\partial\xi^2} \right\} .$$
$$\ldots \quad (4\text{-}27)$$

Equations (4-19)-(4-24) are the integrability conditions governing the reference system at every point not on the surface ξ^1 = const. $\equiv C$. The remaining three equations (4-25)-(4-27) will be satisfied at every point on $\xi^1 = C$. We note that the six integrability conditions are relations between the three A_k, the three Euler angles ϕ, θ, ψ and the angle θ_{23}. So we may choose $\theta_{23} = \pi/2$, which leads to an orthogonal system of coordinates. This statement is consistent with the fact that we have used the three degrees of freedom which exist in a general three-dimensional system of coordinates, by putting $\theta_{12} = \theta_{23} = \theta_{31} = \pi/2$.

The special orthogonal system of coordinates which is obtained through the use of ellipsoidal coordinates is discussed in Appendix E.

Finally we note that the integrability conditions associated with axially symmetric geometries can easily be obtained in a number of different ways (see Appendix F).

$$\text{(a)} \quad \theta = \frac{\pi}{2} , \quad \xi^1 = \phi , \quad A_1 = r$$
$$x^2 = r\cos\xi^1 , \quad x^3 = r\sin\xi^1 \qquad (4\text{-}28)$$

$$\text{(b)} \quad \psi = \theta_{23} = \frac{\pi}{2} , \quad \xi^3 = \phi , \quad A_3 = r$$
$$x^2 = r\sin\xi^3 , \quad x^3 = r\cos\xi^3 \qquad (4\text{-}29)$$

$$\text{(c)}\quad \phi = \frac{\pi}{2}\ , \quad \xi^1 = \theta\ , \quad A_1 = r$$
$$x^1 = r \sin \xi^1\ , \quad x^2 = -r \cos \xi^1 \tag{4-30}$$

$$\text{(d)}\quad \psi = 0\ , \quad \theta_{23} = \frac{\pi}{2}\ , \quad \xi^2 = \phi\ , \quad A_2 = r$$
$$x^2 = r \sin \xi^2\ , \quad x^3 = r \cos \xi^2\ . \tag{4-31}$$

4.3 Possible method of solution

As was stated in section 3.2 the conservation equations for the compressible flow past a given wing can be reduced to one differential equation for the velocity potential Φ with the remaining quantities following from the energy equation (Bernoulli) and the equation of state. For example

$$\frac{a^2}{\gamma - 1} + \tfrac{1}{2}q^2 = \text{const.}\ . \tag{4-32}$$

If, for certain body geometries the system can be made orthogonal then this equation takes the form (3-20) with appropriate boundary conditions on the body and at infinity.

The three parameters A_i and the three Euler angles ϕ, θ, ψ have to be calculated from the six equations (4-19)-(4-24) in the field (with $\theta_{23} = \pi/2$) and the three equations (4-25)-(4-27) on $\xi^1 = C$, where the direction cosines $\cos\theta_1^k$, which according to Table 1 are functions of the Euler angles θ and ϕ, are defined by the geometry of the body shape. We may assume that θ and ϕ are prescribed on $\xi^1 = C$ as functions of two parameters σ^2 and σ^3, say.

An iteration scheme on the following lines could be used in order to calculate the coordinate system on the body $\xi^1 = C$. A suitable guess

$$\xi^2 = \xi^2(\sigma^2,\sigma^3) \qquad \text{and} \qquad \xi^3 = \xi^3(\sigma^2,\sigma^3) \tag{4-33}$$

enables us to determine θ and ϕ as functions of ξ^2 and ξ^3. Then (4-26) and (4-27) can be used to find A_2 and ψ and (4-25) to find A_2/A_3. Since

$$dx^k = A_\ell \cos\theta_\ell^k d\xi^\ell \qquad (d\xi^1 = 0) \tag{4-34}$$

x^k and σ^2 and σ^3 can be related to ξ^2 and ξ^3 and the iteration cycle is closed.

In the field one would probably use a second iteration scheme, whereby e.g. (4-20) is used to obtain ψ, (4-21) is used to obtain A_3, and (4-24) to obtain A_1/A_2. The remaining equations (4-22) and (4-23) can be used either to obtain A_2 and θ, with ϕ following from (4-19), or to obtain A_2 and ϕ with θ following from (4-19). This is only one of several possible computation schemes. The best one can only be found by practical experience. One problem here is the formulation of the appropriate boundary conditions for these partial differential equations. For shapes with certain geometrical symmetries, conditions not only on $\underline{a}_1$, but also on $\underline{a}_2$ and/or $\underline{a}_3$ may be known. In addition we have to insure that the three A_i and the Euler angles ϕ, θ, ψ are periodic both in ξ^2 and ξ^3.

In view of all these difficulties it appears to be wise to gain experience in the handling of such problems by first considering a wing of finite aspect ratio where the system of coordinates can be obtained as a perturbation to the ellipsoidal coordinates mentioned before (see Appendix E).

5 INFINITE BODIES (Case II)

5.1 Integrability conditions

We represent the infinite body by the equation ξ^2 = const. and we choose a reference system in which the base vectors $\underline{a}_2$ and $\underline{a}_3$ are restricted in such a way that they remain normal to $\underline{c}_1$. The base vector $\underline{a}_1$ is completely unrestricted. The angles θ_{ij} are arbitrary (see Fig.2) and it is shown in Appendix B that in general only one Euler angle $\alpha(\equiv \phi + \psi)$ is required in the description of the coordinate system. In subsection B.3 of Appendix B the direction cosines of the fundamental base and normal vectors are derived in terms of θ_{ij} and α. These are given in Tables 1 and 2 and in Appendix D the integrability conditions are derived in the form

$$\frac{\partial}{\partial \xi^3}\left(A_1 \sin \theta_{31} \sin \theta^{12}\right) = 0 \quad , \tag{5-1}$$

$$\frac{\partial A_3}{\partial \xi^1} = \frac{\partial}{\partial \xi^3}(A_1 \cos \theta_{31}) + A_1 \sin \theta_{31} \cos \theta^{12} \frac{\partial}{\partial \xi^3}(\alpha - \theta_{23}) \quad , \tag{5-2}$$

$$A_3 \frac{\partial(\alpha - \theta_{23})}{\partial \xi^1} = -\frac{\partial}{\partial \xi^3}\left(A_1 \cos \theta^{12} \sin \theta_{31}\right) + A_1 \cos \theta_{31} \frac{\partial(\alpha - \theta_{23})}{\partial \xi^3} \quad , \tag{5-3}$$

$$\frac{\partial}{\partial \xi^2}\left(A_1 \sin \theta_{31} \sin \theta^{12}\right) = 0 \quad , \tag{5-4}$$

$$\frac{\partial A_2}{\partial \xi^1} = \frac{\partial}{\partial \xi^2}(A_1 \cos \theta_{12}) - A_1 \sin \theta_{12} \cos \theta^{31} \frac{\partial \alpha}{\partial \xi^2} , \tag{5-5}$$

$$A_2 \frac{\partial \alpha}{\partial \xi^1} = \frac{\partial}{\partial \xi^2}\left(A_1 \cos \theta^{31} \sin \theta_{12}\right) + A_1 \cos \theta_{12} \frac{\partial \alpha}{\partial \xi^2} , \tag{5-6}$$

$$\frac{\partial A_2}{\partial \xi^3} = \frac{\partial}{\partial \xi^2}(A_3 \cos \theta_{23}) + A_3 \sin \theta_{23} \frac{\partial \alpha}{\partial \xi^2} , \tag{5-7}$$

$$A_2 \frac{\partial \alpha}{\partial \xi^3} = -\frac{\partial}{\partial \xi^2}(A_3 \sin \theta_{23}) + A_3 \cos \theta_{23} \frac{\partial \alpha}{\partial \xi^2} . \tag{5-8}$$

We note that equations (5-1)-(5-8) hold everywhere and, from the derivation in Appendix D, it is clear that equations (4-1) and thus equations (5-1)-(5-8) are necessary and sufficient conditions for the integrability of equations (2-4). The number of conditions is reduced to eight since the equation $\Omega^{1,1} = 0$ is satisfied identically in this case. A further reduction in the number of independent equations is possible since $A^1 = A^1(\xi^1)$ or $A^1 = 1$. This follows by using equations (A-53) and (A-62) in equations (5-1) and (5-4) above to obtain $(\partial/\partial\xi^2)(A^1) = (\partial/\partial\xi^3)(A^1) = 0$. Equations (5-2), (5-3), (5-5)-(5-8) are now the necessary and sufficient conditions for the integrability of equations (2-4).

It is possible to prove an analogous theorem to that proved in section 4 for the six equations (5-2), (5-3) and (5-5)-(5-8). The theorem can be stated in the form. The necessary and sufficient conditions that equations (2-4) possess an integral are that equations (5-5)-(5-8) hold at every point on and external to ξ^2 = const. and that equations (5-2)-(5-3) hold on ξ^2 = const.

The necessity of the above conditions follows immediately from the fact that equations (5-2)-(5-3) and (5-5)-(5-8) must hold everywhere. To prove sufficiency we must replace the system (5-2)-(5-3) and (5-5)-(5-8) with an equivalent system of equations. This equivalent system of equations is obtained from (4-1) by retaining the direction cosines $\cos \theta_1^\ell$ of $\underline{a}_1$. From equations (D-12), (D-13), (D-17), (D-18), (D-21) and (D-22), remembering equations (B-35)-(B-45), we find

$$\frac{\partial}{\partial \xi^1}\left\{A_3 \sin(\alpha - \theta_{23})\right\} = -\frac{\partial}{\partial \xi^3}\left\{A_1 \cos \theta_1^3\right\} , \tag{5-9}$$

$$\frac{\partial}{\partial \xi^1}\left\{A_3 \cos(\alpha - \theta_{23})\right\} = \frac{\partial}{\partial \xi^3}\left\{A_1 \cos \theta_1^2\right\} , \tag{5-10}$$

$$\frac{\partial}{\partial \xi^1}\left\{A_2 \sin \alpha\right\} = -\frac{\partial}{\partial \xi^2}\left\{A_1 \cos \theta_1^3\right\} , \tag{5-11}$$

$$\frac{\partial}{\partial \xi^1}\left\{A_2 \cos \alpha\right\} = \frac{\partial}{\partial \xi^2}\left\{A_1 \cos \theta_1^2\right\} , \tag{5-12}$$

$$\frac{\partial}{\partial \xi^2}\left\{A_3 \cos(\alpha - \theta_{23})\right\} = \frac{\partial}{\partial \xi^3}(A_2 \cos \alpha) , \tag{5-13}$$

$$\frac{\partial}{\partial \xi^2}\left\{A_3 \sin(\alpha - \theta_{23})\right\} = \frac{\partial}{\partial \xi^3}(A_2 \sin \alpha) . \tag{5-14}$$

Differentiating (5-14) with respect to ξ^1 and using (5-11) we obtain, after interchanging the orders of differentiation

$$\frac{\partial}{\partial \xi^2}\left\{\frac{\partial}{\partial \xi^1}\left\{A_3 \sin(\alpha - \theta_{23})\right\} + \frac{\partial}{\partial \xi^3}\left\{A_1 \cos \theta_1^3\right\}\right\} = 0 . \tag{5-15}$$

Therefore if (5-9) holds on ξ^2 = const. then by integrating (5-15) we find that (5-9) holds at every point in the field external to ξ^2 = const. Again, if we differentiate (5-13) with respect to ξ^1 and use (5-12) we obtain

$$\frac{\partial}{\partial \xi^2}\left\{\frac{\partial}{\partial \xi^1}\left\{A_3 \cos(\alpha - \theta_{23})\right\} - \frac{\partial}{\partial \xi^3}\left\{A_1 \cos \theta_1^2\right\}\right\} = 0 , \tag{5-16}$$

so that if (5-10) holds on ξ^2 = const. then (5-10) holds at every point in the field external to ξ^2 = const. This completes the proof.

In a similar manner we can prove the following alternative forms of the theorem. The necessary and sufficient conditions for the integrability of equations (2-4) are that equations (5-2)-(5-3) and (5-7)-(5-8) hold everywhere at points on and external to some surface ξ^3 = const. and that equations (5-5)-(5-6) hold at every point on ξ^3 = const. *or* that equations (5-2)-(5-3) and (5-5)-(5-6) hold everywhere at points on and external to ξ^1 = const. and that equations (5-7)-(5-8) hold at every point on ξ^1 = const. In what follows in sections 5.2 and 5.3 the first form of the theorem will be used.

5.2 Conformal mapping in cross-sectional planes

In the special case where $\theta_{23} = \pi/2$, $A_2 = A_3 = A$ (Ref.17) we can prove the following additional theorem concerning the integrability conditions. The necessary and sufficient conditions that equations (2-4) possess an integral are that the following four equations hold everywhere on and outside the surface ξ^2 = const.

$$\frac{\partial}{\partial \xi^2}\left(A_1 \cos \theta_1^2\right) = \frac{\partial}{\partial \xi^3}\left(A_1 \cos \theta_1^3\right) , \tag{5-17}$$

$$\frac{\partial}{\partial \xi^3}\left(A_1 \cos \theta_1^2\right) = -\frac{\partial}{\partial \xi^2}\left(A_1 \cos \theta_1^3\right) , \tag{5-18}$$

$$\frac{\partial}{\partial \xi^2}(A \cos \alpha) = -\frac{\partial}{\partial \xi^3}(A \sin \alpha) , \tag{5-19}$$

$$\frac{\partial}{\partial \xi^2}(A \sin \alpha) = \frac{\partial}{\partial \xi^3}(A \cos \alpha) , \tag{5-20}$$

and that the following two equations hold on the surface ξ^2 = const., and on the circle $(\xi^2)^2 + (\xi^3)^2 = L^2 : \xi^1$ = const., $L \to \infty$

$$\frac{\partial}{\partial \xi^3}\left(A_1 \cos \theta_1^2\right) = \frac{\partial}{\partial \xi^1}(A \sin \alpha) , \tag{5-21}$$

$$\frac{\partial}{\partial \xi^3}\left(A_1 \cos \theta_1^3\right) = \frac{\partial}{\partial \xi^1}(A \cos \alpha) . \tag{5-22}$$

First we assume that (5-9)-(5-14) hold everywhere on and external to the surface ξ^2 = const. By using (5-9) and (5-12) together with (5-10) and (5-11) we obtain (5-17) and (5-18). The fact that (5-19)-(5-20) hold everywhere on and external to ξ^2 = const. and that (5-21)-(5-22) hold on ξ^2 = const. and on $(\xi^2)^2 + (\xi^3)^2 = L^2 : \xi^1$ = const., $L \to \infty$ follows immediately from equations (5-13)-(5-14) and (5-9)-(5-10). The necessity of the conditions stated in the theorem is therefore established.

To prove sufficiency we note that since equations (5-17)-(5-20) hold everywhere on and outside the surface $\xi^2 = \text{const.}$ then $f_1 \equiv A_1\left(\cos\theta_1^2 + i\cos\theta_1^3\right)$ and $f_2 \equiv Ae^{-i\alpha}$ are analytic functions of $\xi^2 + i\xi^3 \equiv \zeta$ at every point on and external to $\xi^2 = \text{const.}$ for any given value of ξ^1. The function f_2 must depend on ξ^1 in such a way that $\partial f_2/\partial\xi^1$ is analytic. Then $F \equiv (\partial f_1/\partial\xi^3) - i(\partial f_2/\partial\xi^1)$ is analytic at every point on and external to $\xi^2 = \text{const.}$ F vanishes on $\xi^2 = \text{const.}$ and on the circle $(\xi^2)^2 + (\xi^3)^2 = L^2 : \xi^1 = \text{const.}$, $L \to \infty$ by virtue of equations (5-21)-(5-22). By Cauchy's integral formula we obtain $F = 0$ everywhere on and outside $\xi^2 = \text{const.}$ so that equations (5-21)-(5-22) hold everywhere. From these two equations together with equations (5-17)-(5-20) we obtain equations (5-9)-(5-14) and the theorem is proved.

On the surface $\xi^2 = \text{const.}$ the unit vector $\underline{a}^2/A^2$ is specified so that using (B-50)-(B-52) we obtain, on $\xi^2 = \text{const.}$ the relations

$$\cos\theta_1^{*2} = -\frac{\cos\theta_{12}}{\sin\theta_{31}} \equiv -\mu \quad , \tag{5-23}$$

$$\cos\theta_2^{*2} = \frac{J_0\cos\alpha}{\sin\theta_{31}} \equiv \sqrt{1-\mu^2}\cos\alpha \quad , \tag{5-24}$$

$$\cos\theta_3^{*2} = -\frac{J_0\sin\alpha}{\sin\theta_{31}} \equiv -\sqrt{1-\mu^2}\sin\alpha \quad , \tag{5-25}$$

where, since $A^1 = 1$,

$$A_1 = J_0^{-1} = \left(1 - \cos^2\theta_{12} - \cos^2\theta_{31}\right)^{-\frac{1}{2}} = (1-\mu^2)^{-\frac{1}{2}}\sin\theta_{31} \quad . \tag{5-26}$$

Since α is now specified on $\xi^2 = \text{const.}$ by equations (5-24)-(5-25) and $f_2 = Ae^{-i\alpha}$ is analytic in $\xi^2 + i\xi^3$ then $\ln A - i\alpha$ is analytic in $\xi^2 + i\xi^3$ and $\ln A$ can be found on $\xi^2 = \text{const.}$ (a circle in the $\ln\zeta$-plane) by using the well known cotangent integral[17]. The functions A and α can be found uniquely at every point on and external to $\xi^2 = \text{const.}$ We also have, using (B-43)-(B-45)

$$\frac{f_1}{f_2} = \frac{\cos\theta_{12} + i\cos\theta_{31}}{AJ_0} = \frac{\mu}{A\sqrt{1-\mu^2}} + i\,\frac{\cot\theta_{31}}{A\sqrt{1-\mu^2}} \quad , \tag{5-27}$$

and f_1/f_2 is an analytic function of $\xi^2 + i\xi^3$ with a known real part on ξ^2 = const. Therefore, again using the cotangent integral, we can find the imaginary part of f_1/f_2 on ξ^2 = const. and can determine f_1/f_2 uniquely at every point on and external to ξ^2 = const. The four unknowns of the reference system A, α, θ_{12}, θ_{31} can now be found everywhere. These variables must be such that equations (5-21)-(5-22) are satisfied everywhere on ξ^2 = const. and at infinity. However, since F is an analytic function of $\xi^2 + i\xi^3$ then if $\mathrm{Re}F = 0$ *or* $\mathrm{Im}F = 0$ on ξ^2 = const. equations (5-21) and (5-22) are both satisfied. We must restrict ourselves to cases where $F \to 0$ at infinity.

In this special system of coordinates one can use complex function theory to obtain the metric tensor and all relevant information. An alternative way of looking at this result is as follows.

We define the transformation of coordinates by

$$x^1 = \xi^1 \quad , \qquad x^2 + ix^3 = f(\xi^1,\zeta) = z$$

$$\text{with} \quad \zeta = \xi^2 + i\xi^3 \quad .$$

Then $z = f(\xi^1,\zeta)$ defines a conformal map for constant ξ^1 and

$$f_2 = Ae^{-i\alpha} = \frac{\partial f}{\partial \zeta} \quad .$$

The dependency on ξ^1 must be such that

$$f_1 = \frac{\partial f}{\partial \xi^1} = A_1 \left(\cos\theta_1^2 + i\cos\theta_1^3\right)$$

is an analytic function of ζ. Then

$$F = \frac{\partial f_1}{\partial \xi^3} - i\,\frac{\partial f_2}{\partial \xi^1} = i\left(\frac{\partial f_1}{\partial \zeta} - \frac{\partial f_2}{\partial \xi^1}\right)$$

is an analytic function which is identically zero.

If we want to find the coordinates for a given shape $\xi^2 = 0$, we first apply conformal mapping on a circle in cross-sectional planes $x^1 = \xi^1$ = const. This enables us to obtain the arc length elements ds as a function of ξ^1 and ξ^3. Since the direction cosines $\cos\theta_\ell^{*2}$ are known, we find μ (and α)

on the body and the function f_1/f_2 according to (5-27) can be calculated everywhere outside the body $\xi^2 = 0$.

When the coordinate system is established the nonlinear potential equation (3-18) is then solved subject to the following boundary condition on $\xi^2 = \text{const.}$

$$v^2 = 0 \quad , \tag{5-28}$$

i.e.

$$g^{2i} \frac{\partial \Phi}{\partial \xi^i} = 0 \quad , \tag{5-29}$$

and an appropriate boundary condition at infinity.

6 CONCLUDING REMARKS.

The equations of motion and associated kinematic boundary conditions of an inviscid, compressible non-conducting and steadily moving gas have been stated in a general coordinate system. The necessary and sufficient conditions which must be satisfied in the determination of the general reference system have been derived. In two cases the governing field equations and boundary conditions are formulated explicitly in terms of the reference system coordinates, a system of Euler angles and the fluid flow variables. In the first case a system of semi-orthogonal curvilinear coordinates is employed which can be used for the calculation of the flow field past a wing of finite aspect ratio. The second case applies to the problem of potential flow past an infinite body, e.g. a swept wing of infinite aspect ratio. In this case the methods of complex analysis can be exploited in the determination of the system of coordinates.

Appendix A

SOME RELATIONSHIPS FROM TENSOR ANALYSIS

The differential length vector $\underline{ds}$ is given by the equations (see Refs. 12 and 13)

$$\underline{ds} = \underline{c}_k dx^k = \underline{a}_\ell d\xi^\ell \quad . \tag{A-1}$$

Using (2-5) together with (A-1) we find

$$\underline{a}_\ell = t^k_\ell \underline{c}_k \quad . \tag{A-2}$$

Also, from the definitions of $t^j_{\cdot i}$ and $\overset{*}{t}{}^i_\ell$ we obtain

$$t^j_{\cdot i} \overset{*}{t}{}^i_\ell = \delta^j_\ell \quad . \tag{A-3}$$

The Jacobian J can be written in the form

$$\varepsilon_{mjk} J = \varepsilon_{rsn} t^r_{\cdot m} t^s_{\cdot j} t^n_{\cdot k} \quad , \tag{A-4}$$

so that from (2-3) and (A-2) we obtain

$$J\varepsilon_{ijk} \underline{a}^i = \varepsilon_{rsn} t^r_{\cdot i} t^s_{\cdot j} t^n_{\cdot k} \underline{a}^i \quad , \tag{A-5}$$

$$= t^r_{\cdot j} t^s_{\cdot k} \underline{c}_r \times \underline{c}_s \quad , \tag{A-6}$$

i.e.

$$\varepsilon_{rsn} t^r_{\cdot i} t^s_{\cdot j} t^n_{\cdot k} \underline{a}^i = t^r_{\cdot j} t^s_{\cdot k} \varepsilon_{rsn} \underline{c}^n \quad . \tag{A-7}$$

Multiplying (A-7) by $\overset{*}{t}{}^j_\ell \overset{*}{t}{}^k_p$ and using (A-3) we find

$$\varepsilon_{\ell pn} \left(t^n_{\cdot i} \underline{a}^i - \underline{c}^n \right) = 0 \tag{A-8}$$

so that

$$\underline{c}^n = t^n_{\cdot i} \underline{a}^i \quad . \tag{A-9}$$

Again, on multiplying (A-9) by t_n^{*j} and using (A-3) we obtain

$$\underline{a}^j = t_n^{*j}\underline{c}^n \quad . \tag{A-10}$$

The reciprocal relation to (A-2) is found in a similar manner and has the form

$$\underline{c}_n = t_n^{*\ell}\underline{a}_\ell \quad . \tag{A-11}$$

We also have the relations

$$\underline{c}_i = \delta_{ik}\underline{c}^k = \underline{c}^i \quad . \tag{A-12}$$

Using equations (A-2), (A-3) and (A-10) we have

$$\underline{a}_\ell \cdot \underline{a}^i = t_\ell^j t_k^{*i}\delta_j^k = \delta_\ell^i \quad . \tag{A-13}$$

The symmetric co-variant metric tensor g_{ij} is defined by

$$g_{ij} = \underline{a}_i \cdot \underline{a}_j \quad , \tag{A-14}$$

and the symmetric contra-variant metric tensor g^{ij} is given by

$$g^{ij} = \underline{a}^i \cdot \underline{a}^j \quad . \tag{A-15}$$

Using (A-2), (A-3) and (A-10) we have

$$\begin{aligned} g^{\ell i}g_{in} &= (\underline{a}^\ell \cdot \underline{a}^i)(\underline{a}_i \cdot \underline{a}_n) \\ &= \left(\delta^{rs}t_r^{*\ell}t_s^{*i}\right)\left(\delta_{pq}t_i^p t_n^q\right) \quad , \end{aligned}$$

i.e.

$$g^{\ell i}g_{in} = \delta_p^s t_s^{*\ell}t_n^p = \delta_n^\ell \quad . \tag{A-16}$$

Furthermore, we have

$$J^2 = \left|t_i^j\right|^2$$

$$= \left|t_i^j t_k^s \delta_{js}\right| ,$$

$$= \left|\underline{a}_i \cdot \underline{a}_k\right| , \qquad \text{using (A-2)}$$

i.e.

$$J^2 = \left|g_{ik}\right| , \tag{A-17}$$

and, since $JJ^* = 1$, we also obtain using (A-16) and (A-17),

$$J^{*2} = \left|g^{ij}\right| . \tag{A-18}$$

From (A-2) we can write

$$\underline{a}_k = t_k^r \underline{c}_r ,$$

$$= \delta_{rs} t_k^r \underline{c}^s ,$$

$$= \delta_{rs} t_k^r t_\ell^s \underline{a}^\ell , \qquad \text{using (A-9)}$$

i.e.

$$\underline{a}_k = g_{k\ell} \underline{a}^\ell . \tag{A-19}$$

Similarly

$$\underline{a}^\ell = g^{\ell i} \underline{a}_i . \tag{A-20}$$

Any vector $\underline{v}$ can be written in one of the following forms

$$\underline{v} = v_\ell \underline{a}^\ell , \tag{A-21}$$

or

$$\underline{v} = v^\ell \underline{a}_\ell , \tag{A-22}$$

where v_ℓ and v^ℓ are the co-variant and contra-variant components of $\underline{v}$. If u_i and u^i, $i = 1,2,3$, are its co-variant and contra-variant components in the Cartesian reference frame we have

$$\underline{v} = u^i \underline{c}_i \quad , \tag{A-23}$$

$$\underline{v} = u_i \underline{c}^i \quad , \tag{A-24}$$

so that, using (A-2), (A-9)-(A-12) and (A-21)-(A-24) we find

$$u_n = \overset{*}{t}{}_n^\ell v_\ell \quad , \tag{A-25}$$

$$v_\ell = t_\ell^n u_n \quad , \tag{A-26}$$

$$u^i = t_\ell^i v^\ell \quad , \tag{A-27}$$

$$v^\ell = \overset{*}{t}{}_i^\ell u^i \quad . \tag{A-28}$$

From (A-19), (A-21) and (A-22) we have

$$\underline{v} = v_\ell \underline{a}^\ell \quad ,$$

$$= v^\ell g_{\ell s} \underline{a}^s \quad ,$$

i.e.

$$v^\ell g_{\ell s} \underline{a}^s = \underline{a}^s v_s \quad , \tag{A-29}$$

so that

$$v_s = g_{\ell s} v^\ell \quad . \tag{A-30}$$

Similarly it can be shown by inversion that

$$v^i = g^{i\ell} v_\ell \quad . \tag{A-31}$$

Using (A-15), (A-21) and (A-31) we find

$$\underline{v} \cdot \underline{a}^i = \underline{a}^i \cdot \left(v_\ell \underline{a}^\ell\right) ,$$

$$= v_\ell \underline{a}^i \cdot \underline{a}^\ell ,$$

$$= g^{i\ell} v_\ell ,$$

i.e.

$$\underline{v} \cdot \underline{a}^i = v^i . \tag{A-32}$$

Also

$$\underline{v} \cdot \underline{a}_i = v_i . \tag{A-33}$$

If the length of $\underline{a}_i$ is denoted by A_i and the angle between the vectors $\underline{a}_i$ and $\underline{c}_j$ is denoted by θ_i^j then

$$\underline{a}_i \cdot \underline{c}_j = A_i \cos \theta_i^j , \tag{A-34}$$

and from (A-2) we have

$$\underline{a}_i \cdot \underline{c}_j = t_i^j , \tag{A-35}$$

so that

$$t_i^j = A_i \cos \theta_i^j . \tag{A-36}$$

Similarly

$$\underline{a}^i \cdot \underline{c}^j = A^i \cos \theta_j^{*i} , \tag{A-37}$$

and

$$t_j^{*i} = A^i \cos \theta_j^{*i} , \tag{A-38}$$

where θ_j^{*i} is the angle between $\underline{a}^i$ and $\underline{c}^j$.

Using (A-36) and (A-38) in (A-13) we find

$$\begin{aligned} \underline{a}_i \cdot \underline{a}^j &= \left(t_{i}^{k}\underline{c}_k\right) \cdot \left(t_n^{*j}\underline{c}^n\right) \\ &= A_i A^j \cos\theta_i^k \cos\theta_n^{*j}\, \underline{c}_k \cdot \underline{c}^n \\ &= \delta_i^j \ , \end{aligned}$$

i.e.

$$A_i A^j \cos\theta_i^k \cos\theta_k^{*j} = \delta_i^j \ . \tag{A-39}$$

If we set $d_i^j = \cos\theta_i^k \cos\theta_k^{*j}$ then

$$d_i^j = 0 \ , \quad i \neq j \ , \tag{A-40}$$

and

$$d_N^{*N} \equiv \left(d_N^N\right)^{-1} = A_N A^N \ , \tag{A-41}$$

where the use of a capitalized index in equation (A-41) and in what follows means no summation over that index.

Furthermore if θ_{ij} denotes the angle between $\underline{a}_i$ and $\underline{a}_j$ and $\theta^{k\ell}$ denotes the angle between $\underline{a}^k$ and $\underline{a}^\ell$ we have

$$A_i A_j \cos\theta_{ij} = \underline{a}_i \cdot \underline{a}_j = A_i A_j \cos\theta_i^k \cos\theta_j^\ell\, \delta_{k\ell} \ , \tag{A-42}$$

using (A-2) and (A-36), so that

$$\cos\theta_{ij} = \cos\theta_i^k \cos\theta_j^\ell\, \delta_{k\ell} \ . \tag{A-43}$$

Similarly

$$\cos\theta^{k\ell} = \delta^{rs} \cos\theta_r^{*k} \cos\theta_s^{*\ell} \ . \tag{A-44}$$

We also have, from (A-14) and (A-15),

$$g_{ij} = A_i A_j \cos \theta_{ij} \quad , \tag{A-45}$$

and

$$g^{k\ell} = A^k A^\ell \cos \theta^{k\ell} \quad . \tag{A-46}$$

Using (A-17) and (A-45) the Jacobian J will be given by

$$\begin{aligned} J^2 &= |g_{ij}| \quad , \\ &= (A_1 A_2 A_3)^2 J_0^2 \quad , \end{aligned} \tag{A-47}$$

where

$$\begin{aligned} J_0^2 &= \left|\cos \theta_i^j\right| \\ &= 1 + 2 \cos \theta_{12} \cos \theta_{23} \cos \theta_{31} - \cos^2 \theta_{12} - \cos^2 \theta_{23} - \cos^2 \theta_{31} \quad . \end{aligned}$$

$$\dots \text{(A-48)}$$

Also, since $JJ^* = 1$, we obtain using (A-18), (A-41), (A-46) and (A-47), the relation

$$d_1^1 d_2^2 d_3^3 J_0^{-1} = \left|\cos \theta_k^{*\ell}\right| \equiv J_0^* \quad . \tag{A-49}$$

The system of equations (A-16) can be solved for g^{ij} and the solution is expressible in the form

$$\varepsilon_{ijk} J^2 g^{in} = \varepsilon^{nrs} g_{jr} g_{ks} \quad . \tag{A-50}$$

Using (A-45), (A-46) and (A-47) we have

$$\varepsilon_{ijk} J_0^2 A_1^2 A_2^2 A_3^2 A^i A^n \cos \theta^{in} = \varepsilon^{nrs} A_r A_s A_j A_k \cos \theta_{rj} \cos \theta_{sk} \quad . \tag{A-51}$$

Setting $i = n = N$ in (A-51) we obtain

$$J_0^2(A_1A_2A_3)^2(A^N)^2 = A_{N+1}^2A_{N+2}^2 \sin^2 \theta_{N+1,N+2} \quad , \tag{A-52}$$

i.e.

$$J_0 d_N^{*N} = \sin \theta_{N+1,N+2} > 0 \quad . \tag{A-53}$$

Again, setting $i = M \neq N$, $n = N$ in (A-51) we find

$$\varepsilon_{Mjk} J_0^2 d_N^{*N} d_M^{*M} \cos \theta^{NM} = \varepsilon^{Nrs} \cos \theta_{rj} \cos \theta_{sk} \quad , \tag{A-54}$$

i.e.

$$\varepsilon_{Mjk} \sin \theta_{N+1,N+2} \sin \theta_{M+1,M+2} \cos \theta^{NM} = \varepsilon^{Nrs} \cos \theta_{rj} \cos \theta_{sk} \quad , \tag{A-55}$$

so that

$$\sin \theta_{23} \sin \theta_{31} \cos \theta^{12} = \cos \theta_{23} \cos \theta_{31} - \cos \theta_{12} \quad , \tag{A-56}$$

$$\sin \theta_{31} \sin \theta_{12} \cos \theta^{23} = \cos \theta_{31} \cos \theta_{12} - \cos \theta_{23} \quad , \tag{A-57}$$

$$\sin \theta_{12} \sin \theta_{23} \cos \theta^{31} = \cos \theta_{12} \cos \theta_{23} - \cos \theta_{31} \quad . \tag{A-58}$$

If the system of equations (A-16) are solved for g_{ij} the solution will be of the form

$$\varepsilon^{ijk} J^{*2} g_{in} = \varepsilon_{nrs} g^{rj} g^{sk} \quad . \tag{A-59}$$

On using (A-45), (A-46) and (A-47) we find

$$\varepsilon^{ijk} A_i A_n \cos \theta_{in} = J_0^2(A_1A_2A_3)^2 \varepsilon_{nrs} A^r A^s A^j A^k \cos \theta^{rj} \cos \theta^{sk} \quad . \tag{A-60}$$

If we set $i = n = N$ in (A-60) we obtain

$$\varepsilon^{Njk} A_N^2 = J_0^2 (A_1 A_2 A_3)^2 \varepsilon_{Nrs} (A^{N+1})^2 (A^{N+2})^2 \sin^2 \theta^{N+2,N+3} . \qquad \text{(A-61)}$$

On using (A-53) equation (A-61) reduces to

$$J_0 = \sin \theta_{N,N+1} \sin \theta_{N+1,N+2} \sin \theta^{N+2,N+3} . \qquad \text{(A-62)}$$

Again, if we set $i = M \neq N$, $n = N$ in (A-60) we obtain

$$\varepsilon^{Mjk} A_M A_N \cos \theta_{NM} = J_0^2 (A_1 A_2 A_3)^2 \varepsilon_{Nrs} A^r A^s A^j A^k \cos \theta^{rj} \cos \theta^{sk} , \qquad \text{(A-63)}$$

i.e. using (A-53) and (A-62),

$$\varepsilon^{Mjk} \sin \theta^{N+1,N+2} \sin \theta^{M+1,M+2} \cos \theta_{NM} = \varepsilon_{Nrs} \cos \theta^{rj} \cos \theta^{sk} , \qquad \text{(A-64)}$$

so that

$$\sin \theta^{23} \sin \theta^{31} \cos \theta_{12} = \cos \theta^{23} \cos \theta^{31} - \cos \theta^{12} , \qquad \text{(A-65)}$$

$$\sin \theta^{31} \sin \theta^{12} \cos \theta_{23} = \cos \theta^{31} \cos \theta^{12} - \cos \theta^{23} , \qquad \text{(A-66)}$$

$$\sin \theta^{12} \sin \theta^{23} \cos \theta_{31} = \cos \theta^{12} \cos \theta^{23} - \cos \theta^{31} . \qquad \text{(A-67)}$$

Appendix B

SPECIAL SYSTEMS OF COORDINATES

B.1 The Euler angles

In order to simplify the relations between the directional cosines $\cos\theta_k^\ell$, we introduce three Euler angles ϕ, θ, ψ in the following form. Since $\underline{a}^1$ is normal to $\underline{a}_2$ and $\underline{a}_3$ we can consider these three vectors as belonging to two orthogonal systems $\left(\underline{a}^1, \underline{a}_2, \underline{b}_3\right)$ and $\left(\underline{a}^1, \underline{b}_2, \underline{a}_3\right)$ respectively (Fig.2). Here

$$\underline{b}_3 = -\frac{g_{23}}{g_{22}}\underline{a}_2 + \underline{a}_3 \ , \quad \underline{b}_3 \cdot \underline{a}^1 = \underline{b}_3 \cdot \underline{a}_2 = 0$$

$$\underline{b}_2 = \underline{a}_2 - \frac{g_{23}}{g_{33}}\underline{a}_3 \ , \quad \underline{b}_2 \cdot \underline{a}^1 = \underline{b}_2 \cdot \underline{a}_3 = 0 \ .$$

Since

$$\underline{b}_2 \cdot \underline{b}_2 = \frac{g_{22}g_{33} - g_{23}^2}{g_{33}} \ , \quad \underline{b}_3 \cdot \underline{b}_3 = \frac{g_{22}g_{33} - g_{23}^2}{g_{22}} \ ,$$

we have

$$\frac{\underline{b}_2 \cdot \underline{b}_3}{(\underline{b}_2 \cdot \underline{b}_2)^{\frac{1}{2}}(\underline{b}_3 \cdot \underline{b}_3)^{\frac{1}{2}}} = -\frac{g_{23}}{\sqrt{g_{22}g_{33}}} \ , \quad \frac{\underline{a}_2 \cdot \underline{a}_3}{(\underline{a}_2 \cdot \underline{a}_2)^{\frac{1}{2}}(\underline{a}_3 \cdot \underline{a}_3)^{\frac{1}{2}}} = \frac{g_{23}}{\sqrt{g_{22}g_{33}}} \ .$$

The orientation of the vectors in the first system in relation to the Cartesian system can be expressed in terms of the Euler angles ϕ, θ, ψ and that of the second system in terms of the Euler angles ϕ, θ, $\psi + \pi/2 - \theta_{23}$. In Appendix C the general rotation matrix for an orthogonal set of vectors is derived in terms of the Euler angles ϕ, θ and β and the associated direction cosines are expressed in terms of ϕ, θ and β. Setting in turn $\beta = \psi$ and $\beta = \psi + \pi/2 - \theta_{23}$ the nine direction cosines for the vectors $\underline{a}^1/A^1$, $\underline{a}_2/A_2$ and $\underline{a}_3/A_3$ can be written in the form

$$\frac{\underline{a}^1}{A^1} : \begin{cases} \cos\theta_1^{*1} = \cos\theta \ , & \text{(B-1)} \\ \cos\theta_2^{*1} = \sin\theta\sin\phi \ , & \text{(B-2)} \\ \cos\theta_3^{*1} = \sin\theta\cos\phi \ , & \text{(B-3)} \end{cases}$$

$$\frac{\underline{a}_2}{A_2} : \begin{cases} \cos\theta_2^1 = \sin\theta\sin\psi \ , & \text{(B-4)} \\ \cos\theta_2^2 = \cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta \ , & \text{(B-5)} \\ \cos\theta_2^3 = -(\sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta) \ , & \text{(B-6)} \end{cases}$$

$$\frac{\underline{a}_3}{A_3} : \begin{cases} \cos\theta_3^1 = \sin\theta\sin(\psi - \theta_{23}) \ , & \text{(B-7)} \\ \cos\theta_3^2 = \cos(\psi - \theta_{23})\cos\phi - \sin(\psi - \theta_{23})\sin\phi\cos\theta \ , & \text{(B-8)} \\ \cos\theta_3^3 = -(\cos(\psi - \theta_{23})\sin\phi + \sin(\psi - \theta_{23})\cos\phi\cos\theta) \ . & \text{(B-9)} \end{cases}$$

In the subsequent analysis the unit vectors $\underline{a}^1/A^1$, $\underline{a}_2/A_2$ and $\underline{a}_3/A_3$ will be considered as a fundamental triad of base vectors with direction cosines given by (B-1)-(B-9).

B.2 Case I: $\theta_{12} \equiv \theta_{31} \equiv \pi/2, \ \theta_{23} \not\equiv \pi/2$

In this case we have $\underline{a}_1/A_1 = \underline{a}^1/A^1$ and so the direction cosines of the base vector $\underline{a}_1/A_1$ will be given by equations (B-1)-(B-3). The direction cosines of the base vectors $\underline{a}_2/A_2$ and $\underline{a}_3/A_3$ will be given by equations (B-4)-(B-9) (Fig.3, Tables 1 and 2).

Since $\theta_{12} = \theta_{31} = \pi/2$ we have $g_{12} = g_{31} = 0$ and from equations (2-33) and (2-34) we obtain

$$\cos\theta^{12} = \cos\theta^{31} = 0 \ , \qquad \text{(B-10)}$$

$$\cos\theta^{23} = -\cos\theta_{23} \ , \qquad \text{(B-11)}$$

so that

$$g^{12} = g^{31} = 0 \quad , \tag{B-12}$$

$$g^{23} = - A^2 A^3 \cos \theta_{23} \quad , \tag{B-13}$$

$$\theta^{12} = \theta^{31} = \frac{\pi}{2} \quad , \tag{B-14}$$

and

$$\theta^{23} = \pi - \theta_{23} \quad . \tag{B-15}$$

We also obtain from equation (2-32)

$$J_0 = \sin \theta_{23} \quad . \tag{B-16}$$

From equations (2-15) we obtain

$$\underline{a}^2 = g^{22} \underline{a}_2 + g^{23} \underline{a}_3 \quad , \tag{B-17}$$

$$\underline{a}^3 = g^{32} \underline{a}_2 + g^{33} \underline{a}_3 \quad , \tag{B-18}$$

and, using (2-28) and (B-13), we can write

$$\frac{\underline{a}^2}{A^2} = \operatorname{cosec} \theta_{23} \frac{\underline{a}_2}{A_2} - \cot \theta_{23} \frac{\underline{a}_3}{A_3} \quad , \tag{B-19}$$

and

$$\frac{\underline{a}^3}{A^3} = - \cot \theta_{23} \frac{\underline{a}_2}{A_2} + \operatorname{cosec} \theta_{23} \frac{\underline{a}_3}{A_3} \quad , \tag{B-20}$$

i.e.

$$\cos \theta_k^{*2} = \operatorname{cosec} \theta_{23} \cos \theta_2^k - \cot \theta_{23} \cos \theta_3^k \quad , \qquad k = 1,2,3 \quad , \tag{B-21}$$

and

$$\cos \theta_k^{*3} = - \cot \theta_{23} \cos \theta_2^k + \operatorname{cosec} \theta_{23} \cos \theta_3^k \quad , \qquad k = 1,2,3 \quad . \tag{B-22}$$

The direction cosines of the base vectors $\underline{a}^2/A^2$ and $\underline{a}^3/A^3$ are then given by

$$\frac{\underline{a}^2}{A^2}:\begin{cases}\cos\theta_1^{*2} = \sin\theta\,\cos(\psi-\theta_{23}) \,, & \text{(B-23)}\\ \cos\theta_2^{*2} = -(\cos\phi\,\sin(\psi-\theta_{23}) + \sin\phi\,\cos\theta\,\cos(\psi-\theta_{23})) \,, & \text{(B-24)}\\ \cos\theta_3^{*2} = \sin\phi\,\sin(\psi-\theta_{23}) - \cos\phi\,\cos\theta\,\cos(\psi-\theta_{23}) \,, & \text{(B-25)}\end{cases}$$

$$\frac{\underline{a}^3}{A^3}:\begin{cases}\cos\theta_1^{*3} = -\sin\theta\,\cos\psi \,, & \text{(B-26)}\\ \cos\theta_2^{*3} = \sin\psi\,\cos\phi + \sin\phi\,\cos\theta\,\cos\psi \,, & \text{(B-27)}\\ \cos\theta_3^{*3} = -\sin\phi\,\sin\psi + \cos\phi\,\cos\theta\,\cos\psi \,. & \text{(B-28)}\end{cases}$$

When $\theta_{23} \equiv \pi/2$ the reference system will be completely orthogonal in which case we will have

$$\theta_{ij} \equiv \theta^{ij} \equiv \frac{\pi}{2} \,, \qquad i \neq j \,, \tag{B-29}$$

$$g_{ij} \equiv g^{ij} \equiv 0 \,, \qquad i \neq j \,, \tag{B-30}$$

and

$$\theta_j^{*i} \equiv \theta_i^j \,, \qquad i,j = 1,2,3 \,. \tag{B-31}$$

B.3 Case II: $\underline{a}^1/A^1 \equiv \underline{c}^1$; $\theta_{ij} \not\equiv \pi/2$, $i \neq j$

Since $\underline{a}^1/A^1 \equiv \underline{c}^1$ we must have $\cos\theta_2^{*1} = \cos\theta_3^{*1} = 0$ and therefore $\theta = 0$ from equations (B-2) and (B-3). The direction cosines of the base vectors $\underline{a}^1/A^1$, $\underline{a}_2/A_2$ and $\underline{a}_3/A_3$ will now have the form (Fig.4, Tables 1 and 2)

$$\frac{\underline{a}^1}{A^1} : \begin{cases} \cos\,\theta_1^{*1} = 1, & \text{(B-32)} \\ \cos\,\theta_2^{*1} = 0\ , & \text{(B-33)} \\ \cos\,\theta_3^{*1} = 0\ , & \text{(B-34)} \end{cases}$$

$$\frac{\underline{a}_2}{A_2} : \begin{cases} \cos\,\theta_2^{1} = 0\ , & \text{(B-35)} \\ \cos\,\theta_2^{2} = \cos\,\alpha\ , & \text{(B-36)} \\ \cos\,\theta_2^{3} = -\,\sin\,\alpha\ , & \text{(B-37)} \end{cases}$$

$$\frac{\underline{a}_3}{A_3} : \begin{cases} \cos\,\theta_3^{1} = 0\ , & \text{(B-38)} \\ \cos\,\theta_3^{2} = \cos(\alpha - \theta_{23})\ , & \text{(B-39)} \\ \cos\,\theta_3^{3} = -\,\sin\,(\alpha - \theta_{23})\ , & \text{(B-40)} \end{cases}$$

where $\alpha = \phi + \psi$. In this case we will only have one Euler angle α since ϕ and ψ appear in the form $\phi + \psi$.

To find the direction cosines of the base vector $\underline{a}_1/A_1$ we write, using equations (A-19), (A-41) and (A-46)

$$\frac{\underline{a}_1}{A_1} = d_1^1\,\frac{\underline{a}^1}{A^1} - d_1^1 d_2^{*2}\,\cos\,\theta^{12}\,\frac{\underline{a}_2}{A_2} - d_1^1 d_3^{*3}\,\cos\,\theta^{31}\,\frac{\underline{a}_3}{A_3}\ , \qquad \text{(B-41)}$$

i.e.

$$\cos\,\theta_1^k = d_1^1\,\cos\,\theta_k^{*1} - d_1^1 d_2^{*2}\,\cos\,\theta^{12}\,\cos\,\theta_2^k - d_1^1 d_3^{*3}\,\cos\,\theta^{31}\,\cos\,\theta_3^k\ , \qquad \text{(B-42)}$$

$$k = 1,2,3\ .$$

Using equations (B-32)-(B-40), (A-53) and (A-56)-(A-58) we find the direction cosines of the base vector $\underline{a}_1/A_1$ in the form

$$\frac{\underline{a}_1}{A_1} : \begin{cases} \cos\theta_1^1 = J_0/\sin\theta_{23} = \sin\theta^{31}\sin\theta_{12} = \sin\theta^{12}\sin\theta_{31} \quad , & \text{(B-43)} \\ \cos\theta_1^2 = \Big(\cos\theta_{31}\sin\alpha - \cos\theta_{12}\sin(\alpha-\theta_{23})\Big)\Big/\sin\theta_{23} \quad , & \\ \quad = -\sin\alpha\cos\theta^{31}\sin\theta_{12} + \cos\alpha\cos\theta_{12} \quad , & \\ \quad = \sin(\alpha-\theta_{23})\cos\theta^{12}\sin\theta_{31} + \cos(\alpha-\theta_{23})\cos\theta_{31} \quad , & \text{(B-44)} \\ \cos\theta_1^3 = \Big(\cos\theta_{31}\cos\alpha - \cos\theta_{12}\cos(\alpha-\theta_{23})\Big)\Big/\sin\theta_{23} \quad , & \\ \quad = -\cos\alpha\cos\theta^{31}\sin\theta_{12} - \sin\alpha\cos\theta_{12} \quad , & \\ \quad = \cos(\alpha-\theta_{23})\cos\theta^{12}\sin\theta_{31} - \sin(\alpha-\theta_{23})\cos\theta_{31} \quad . & \text{(B-45)} \end{cases}$$

In a similar manner we can write for the base vectors $\underline{a}^2/A^2$ and $\underline{a}^3/A^3$

$$\frac{\underline{a}^2}{A^2} = d_1^{*1}\cos\theta^{12}\frac{\underline{a}_1}{A_1} + d_2^{*2}\frac{\underline{a}_2}{A_2} + d_3^{*3}\cos\theta^{23}\frac{\underline{a}_3}{A_3} \quad , \tag{B-46}$$

and

$$\frac{\underline{a}^3}{A^3} = d_1^{*1}\cos\theta^{31}\frac{\underline{a}_1}{A_1} + d_2^{*2}\cos\theta^{23}\frac{\underline{a}_2}{A_2} + d_3^{*3}\frac{\underline{a}_3}{A_3} \quad . \tag{B-47}$$

Using (B-41) in (B-46) and (B-47) we obtain

$$\cos\theta_k^{*2} = \cos\theta^{12}\cos\theta_k^{*1} + \sin^2\theta^{12}d_2^{*2}\cos\theta_2^k$$
$$+ (\cos\theta^{23} - \cos\theta^{31}\cos\theta^{12})d_3^{*3}\cos\theta_3^k \quad , \qquad k = 1,2,3 \tag{B-48}$$

and

$$\cos\theta_k^{*3} = \cos\theta^{31}\cos\theta_k^{*1} + (\cos\theta^{23} - \cos\theta^{31}\cos\theta^{12})d_2^{*2}\cos\theta_2^k$$
$$+ \sin^2\theta^{31}d_3^{*3}\cos\theta_3^k \quad , \qquad k = 1,2,3 \quad .$$

... (B-49)

The direction cosines of the base vectors $\underline{a}^2/A^2$ and $\underline{a}^3/A^3$ are therefore given by

$$\frac{\underline{a}^2}{A^2}: \begin{cases} \cos\theta_1^{*2} = \cos\theta^{12} = \dfrac{\cos\theta_{23}\cos\theta_{31} - \cos\theta_{12}}{\sin\theta_{23}\sin\theta_{31}}, & \text{(B-50)} \\[2ex] \cos\theta_2^{*2} = -\dfrac{\sin(\alpha-\theta_{23})J_0}{\sin\theta_{23}\sin\theta_{31}} = -\sin(\alpha-\theta_{23})\sin\theta^{12}, & \text{(B-51)} \\[2ex] \cos\theta_3^{*2} = -\dfrac{\cos(\alpha-\theta_{23})J_0}{\sin\theta_{23}\sin\theta_{31}} = -\cos(\alpha-\theta_{23})\sin\theta^{12}, & \text{(B-52)} \end{cases}$$

$$\frac{\underline{a}^3}{A^3}: \begin{cases} \cos\theta_1^{*3} = \cos\theta^{31} = \dfrac{\cos\theta_{12}\cos\theta_{23} - \cos\theta_{31}}{\sin\theta_{12}\sin\theta_{23}}, & \text{(B-53)} \\[2ex] \cos\theta_2^{*3} = \dfrac{\sin\alpha J_0}{\sin\theta_{12}\sin\theta_{23}} = \sin\alpha\sin\theta^{31}, & \text{(B-54)} \\[2ex] \cos\theta_3^{*3} = \dfrac{\cos\alpha J_0}{\sin\theta_{12}\sin\theta_{23}} = \cos\alpha\sin\theta^{31}, & \text{(B-55)} \end{cases}$$

where J_0 is given by (A-48).

In this case we will have

$$\theta_2^{*1} \equiv \theta_3^{*1} \equiv \frac{\pi}{2}, \tag{B-56}$$

and, from (B-50) and (B-53),

$$\theta_1^{*2} \equiv \theta^{12} \tag{B-57}$$

$$\theta_1^{*3} \equiv \theta^{31}. \tag{B-58}$$

In conclusion we note that the direction cosines of the unit base normal vectors which have been derived in this section are tabulated in Tables 1 and 2.

Appendix C

GENERAL RIGID BODY ROTATION MATRIX

At any point P we consider four orthogonal Cartesian frames of reference in which the coordinates of any other point Q will be represented by $\underline{t} = (t^1, t^2, t^3)$, $\underline{u} = (u^1, u^2, u^3)$, $\underline{v} = (v^1, v^2, v^3)$ and $\underline{w} = (w^1, w^2, w^3)$. The frame (P, u^1, u^2, u^3) is obtained by a rigid body rotation ϕ about the t^1 axis and the resulting coordinate transformation will be given by

$$\underline{u} = T_1\underline{t} \, , \tag{C-1}$$

where

$$T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix} . \tag{C-2}$$

Similarly the frames (P, v^1, v^2, v^3) and (P, w^1, w^2, w^3) will be obtained by successive rotations θ and β about the u^2 and v^1 axes respectively. The corresponding coordinate transformations will be given by

$$\underline{v} = T_2\underline{u} \, , \tag{C-3}$$

and

$$\underline{w} = T_3\underline{v} \, , \tag{C-4}$$

where

$$T_2 = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} , \tag{C-5}$$

and

$$T_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} . \tag{C-6}$$

The resultant transformation will be given by

$$\underline{w} = T\underline{t} \quad , \tag{C-7}$$

where

$$T = T_3 T_2 T_1 \quad . \tag{C-8}$$

But

$$w^i = t^k \cos \alpha_k^i \quad , \tag{C-9}$$

where α_k^i is the angle between the ith axis in the (P, w^1, w^2, w^3) frame and the kth axis in the (P, t^1, t^2, t^3) frame. Using (C-7) and (C-9) together with (C-8) we find

$$\cos \alpha_1^1 = \cos \theta \quad , \tag{C-10}$$

$$\cos \alpha_2^1 = \sin \theta \sin \phi \quad , \tag{C-11}$$

$$\cos \alpha_3^1 = \sin \theta \cos \phi \quad , \tag{C-12}$$

$$\cos \alpha_1^2 = \sin \theta \sin \beta \quad , \tag{C-13}$$

$$\cos \alpha_2^2 = \cos \phi \cos \beta - \sin \phi \sin \beta \cos \theta \quad , \tag{C-14}$$

$$\cos \alpha_3^2 = - (\sin \phi \cos \beta + \cos \phi \sin \beta \cos \theta) \quad , \tag{C-15}$$

$$\cos \alpha_1^3 = - \sin \theta \cos \beta \quad , \tag{C-16}$$

$$\cos \alpha_2^3 = \sin \beta \cos \phi + \cos \beta \sin \phi \cos \theta \quad , \tag{C-17}$$

$$\cos \alpha_3^3 = - \sin \beta \sin \phi + \cos \beta \cos \phi \cos \theta \quad . \tag{C-18}$$

Appendix D

DERIVATION OF THE INTEGRABILITY CONDITIONS

D.1 Case I: $\theta_{12} \equiv \theta_{31} \equiv \pi/2,\ \theta_{23} \not\equiv \pi/2$

In equations (4-16) we take $k = 1$ so that

$$\frac{1}{A_1A^1}\frac{\partial A_1}{\partial \xi^3} + A_1 \cos\theta_\alpha^{*1} \frac{\partial}{\partial \xi^3}\left(\cos\theta_1^\alpha\right) = A_3 \cos\theta_\alpha^{*1} \frac{\partial}{\partial \xi^1}\left(\cos\theta_3^\alpha\right) \quad . \tag{D-1}$$

Since $A_1A^1 = 1$, $\cos\theta_\alpha^{*1} = \cos\theta_1^\alpha$, $\alpha = 1,2,3$ and $\cos\theta_\alpha^{*1}\cos\theta_1^\alpha = 1$ then equation (D-1) reduces to

$$\frac{\partial A_1}{\partial \xi^3} = A_3 \cos\theta_\alpha^{*1} \frac{\partial}{\partial \xi^1}\left(\cos\theta_3^\alpha\right) \quad . \tag{D-2}$$

Also, since $\cos\theta_\alpha^{*1}\cos\theta_3^\alpha = 0$ the right hand side of (D-2) can be written as $-A_3\cos\theta_3^\alpha(\partial/\partial\xi^1)\left(\cos\theta_\alpha^{*1}\right)$.

Using the equations (B-1)-(B-9) the direction cosines $\cos\theta_3^\alpha$ and $\cos\theta_\alpha^{*1}$, $\alpha = 1, 2, 3,$ can be expressed in terms of θ, ϕ, ψ and θ_{23} and we obtain

$$\begin{aligned}
&- A_3 \cos\theta_3^\alpha \frac{\partial}{\partial\xi^1}\left(\cos\theta_\alpha^{*1}\right) \\
&= - A_3 \Bigg\{\sin\theta\sin(\psi - \theta_{23})\frac{\partial}{\partial\xi^1}(\cos\theta) \\
&\qquad + \Big(\cos(\psi - \theta_{23})\cos\phi - \sin(\psi - \theta_{23})\sin\phi\cos\theta\Big)\frac{\partial}{\partial\xi^1}(\sin\theta\sin\phi) \\
&\qquad - \Big(\cos(\psi - \theta_{23})\sin\phi + \sin(\psi - \theta_{23})\cos\phi\cos\theta\Big)\frac{\partial}{\partial\xi^1}(\sin\theta\cos\phi)\Bigg\} \\
&= - A_3 \Bigg\{\sin(\psi - \theta_{23})\Bigg[\sin\theta\frac{\partial}{\partial\xi^1}(\cos\theta) - \sin\phi\cos\theta\frac{\partial}{\partial\xi^1}(\sin\theta\sin\phi) \\
&\qquad\qquad - \cos\phi\cos\theta\frac{\partial}{\partial\xi^1}(\sin\theta\cos\phi)\Bigg] \\
&\qquad + \cos(\psi - \theta_{23})\Bigg[\cos\phi\frac{\partial}{\partial\xi^1}(\sin\theta\sin\phi) - \sin\phi\frac{\partial}{\partial\xi^1}(\sin\theta\cos\phi)\Bigg]\Bigg\} \\
&= A_3\left\{\sin(\psi - \theta_{23})\frac{\partial\theta}{\partial\xi^1} - \sin\theta\cos(\psi - \theta_{23})\frac{\partial\phi}{\partial\xi^1}\right\} \quad ,
\end{aligned} \tag{D-3}$$

i.e.

$$\frac{\partial A_1}{\partial \xi^3} = A_3 \left\{ \sin(\psi - \theta_{23}) \frac{\partial \theta}{\partial \xi^1} - \sin\theta \cos(\psi - \theta_{23}) \frac{\partial \phi}{\partial \xi^1} \right\} , \qquad (D-4)$$

which is equation (4-19).

In a similar manner if we take $k = 2$ and $k = 3$ in equations (4-16) and use the expressions for the direction cosines in Appendix B and Tables 1 and 2 we will obtain equations (4-20) and (4-21). The remaining equations (4-22)-(4-27) are derived by taking the systems of equations (4-17)-(4-18), setting $k = 1,2$ and 3 in each system and proceeding in the same manner as in the derivation of (4-19)-(4-21). The resulting six equations will be of the form

$$\frac{\partial A_1}{\partial \xi^2} = A_2 \left\{ \sin\psi \frac{\partial \theta}{\partial \xi^1} - \sin\theta \cos\psi \frac{\partial \phi}{\partial \xi^1} \right\} , \qquad (D-5)$$

$$A_1 \left\{ \sin\theta \cos\psi \frac{\partial \phi}{\partial \xi^2} - \sin\psi \frac{\partial \theta}{\partial \xi^2} \right\} = \frac{\partial A_2}{\partial \xi^1} , \qquad (D-6)$$

$$A_1 \left\{ \cos\psi \frac{\partial \theta}{\partial \xi^2} + \sin\psi \sin\theta \frac{\partial \phi}{\partial \xi^2} \right\} = - A_2 \left\{ \cos\theta \frac{\partial \phi}{\partial \xi^1} + \frac{\partial \psi}{\partial \xi^1} \right\} , \qquad (D-7)$$

$$A_2 \left\{ \sin\theta \cos\psi \frac{\partial \phi}{\partial \xi^3} - \sin\psi \frac{\partial \theta}{\partial \xi^3} \right\}$$

$$= A_3 \left\{ \sin\theta \cos(\psi - \theta_{23}) \frac{\partial \phi}{\partial \xi^2} - \sin(\psi - \theta_{23}) \frac{\partial \theta}{\partial \xi^2} \right\} , \qquad (D-8)$$

$$\sin\theta_{23} \frac{\partial A_2}{\partial \xi^3} + \cos\theta_{23} A_2 \left\{ \cos\theta \frac{\partial \phi}{\partial \xi^3} + \frac{\partial \psi}{\partial \xi^3} \right\} = A_3 \left\{ \frac{\partial(\psi - \theta_{23})}{\partial \xi^2} + \cos\theta \frac{\partial \phi}{\partial \xi^2} \right\} ,$$

$$\ldots \quad (D-9)$$

$$A_2 \left\{ \cos\theta \frac{\partial \phi}{\partial \xi^3} + \frac{\partial \psi}{\partial \xi^3} \right\} = - \sin\theta_{23} \frac{\partial A_3}{\partial \xi^2} + \cos\theta_{23} A_3 \left\{ \cos\theta \frac{\partial \phi}{\partial \xi^2} + \frac{\partial(\psi - \theta_{23})}{\partial \xi^2} \right\} .$$

$$\ldots \quad (D-10)$$

D.2 Case II: $\underline{a}^1/A^1 \equiv \underline{c}^1$; $\theta_{ij} \neq \pi/2$, $i \neq j$

To derive equations (5-1)-(5-8) it is convenient to start with equations (4-1). The first three integrability conditions are obtained by setting $i = 3$, $j = 1$ and $k = 1,2,3$, and using the expressions for the direction cosines given in Appendix B by equations (B-35)-(B-40) and (B-43)-(B-45). We obtain, using (A-62),

$$\frac{\partial}{\partial \xi^3}\left(A_1 \sin \theta_{31} \sin \theta^{12}\right) = 0 \ , \tag{D-11}$$

$$\frac{\partial}{\partial \xi^3}\left[A_1\left(\sin(\alpha - \theta_{23}) \cos \theta^{12} \sin \theta_{31} + \cos(\alpha - \theta_{23}) \cos \theta_{31}\right)\right] = \frac{\partial}{\partial \xi^1}\left[A_3 \cos(\alpha - \theta_{23})\right] \ . \tag{D-12}$$

$$\frac{\partial}{\partial \xi^3}\left[A_1\left(\cos(\alpha - \theta_{23}) \cos \theta^{12} \sin \theta_{31} - \sin(\alpha - \theta_{23}) \cos \theta_{31}\right)\right] = \frac{\partial}{\partial \xi^1}\left[- A_3 \sin(\alpha - \theta_{23})\right] \ . \tag{D-13}$$

If we form the sum (D-12) $\times \cos(\alpha - \theta_{23})$ - (D-13) $\times \sin(\alpha - \theta_{23})$ we obtain after some algebra the relation

$$\frac{\partial A_3}{\partial \xi^1} = \frac{\partial}{\partial \xi^3}(A_1 \cos \theta_{31}) + A_1 \sin \theta_{31} \cos \theta^{12} \frac{\partial}{\partial \xi^3}(\alpha - \theta_{23}) \ . \tag{D-14}$$

Again, if we form the sum (D-12) $\times \sin(\alpha - \theta_{23})$ + (D-13) $\times \cos(\alpha - \theta_{23})$ and proceed as above we obtain

$$A_3 \frac{\partial(\alpha - \theta_{23})}{\partial \xi^1} = - \frac{\partial}{\partial \xi^3}\left(A_1 \cos \theta^{12} \sin \theta_{31}\right) + A_1 \cos \theta_{31} \frac{\partial(\alpha - \theta_{23})}{\partial \xi^3} \ . \tag{D-15}$$

Similarly, if we set $i = 2$, $j = 1$ and $k = 1,2,3$ in equations (4-1) we obtain, after using equations (B-35)-(B-40), (B-43)-(B-45) and (A-62)

$$\frac{\partial}{\partial \xi^2}\left(A_1 \sin \theta_{31} \sin \theta^{12}\right) = 0 \ , \tag{D-16}$$

$$\frac{\partial}{\partial\xi^2}\left[A_1\left(-\sin\alpha\cos\theta^{31}\sin\theta_{12}+\cos\alpha\cos\theta_{12}\right)\right] = \frac{\partial}{\partial\xi^1}\left[A_2\cos\alpha\right] , \quad \text{(D-17)}$$

$$\frac{\partial}{\partial\xi^2}\left[A_1\left(-\cos\alpha\cos\theta^{31}\sin\theta_{12}-\sin\alpha\cos\theta_{12}\right)\right] = \frac{\partial}{\partial\xi^1}\left[-A_2\sin\alpha\right] . \quad \text{... (D-18)}$$

Again, if we form the sums (D-17) × cos α − (D-18) × sin α and (D-17) × sin α + (D-18) × cos α we obtain the relations

$$\frac{\partial A_2}{\partial\xi^1} = \frac{\partial}{\partial\xi^2}(A_1\cos\theta_{12}) - A_1\sin\theta_{12}\cos\theta^{31}\frac{\partial\alpha}{\partial\xi^2} , \quad \text{(D-19)}$$

and

$$A_2\frac{\partial\alpha}{\partial\xi^1} = \frac{\partial}{\partial\xi^2}\left(A_1\cos\theta^{31}\sin\theta_{12}\right) + A_1\cos\theta_{12}\frac{\partial\alpha}{\partial\xi^2} . \quad \text{(D-20)}$$

Finally, if we take $i = 3,\ j = 2,\ k = 1,2,3$ in equations (4-1) we obtain an identity when $k = 1$ and the following two equations for $k = 2$ and $k = 3$

$$\frac{\partial}{\partial\xi^2}\left[A_3\cos(\alpha-\theta_{23})\right] = \frac{\partial}{\partial\xi^3}\left[A_2\cos\alpha\right] , \quad \text{(D-21)}$$

and

$$\frac{\partial}{\partial\xi^2}\left[A_3\sin(\alpha-\theta_{23})\right] = \frac{\partial}{\partial\xi^3}\left[A_2\sin\alpha\right] . \quad \text{(D-22)}$$

Forming the sums (D-21) × cos α + (D-22) × sin α and − (D-21) × sin α + (D-22) × cos α and proceeding as before we obtain the following two equations

$$\frac{\partial A_2}{\partial\xi^3} = \frac{\partial}{\partial\xi^2}(A_3\cos\theta_{23}) + A_3\sin\theta_{23}\frac{\partial\alpha}{\partial\xi^2} , \quad \text{(D-23)}$$

and

$$A_2\frac{\partial\alpha}{\partial\xi^3} = -\frac{\partial}{\partial\xi^2}(A_3\sin\theta_{23}) + A_3\cos\theta_{23}\frac{\partial\alpha}{\partial\xi^2} . \quad \text{(D-24)}$$

The eight integrability conditions will be given by equations (D-11), (D-14), (D-15), (D-16), (D-19), (D-20), (D-23) and (D-24).

Appendix E

ELLIPSOIDAL COORDINATES

In this special orthogonal curvilinear coordinate system equations (2-1) will have the form[15,18]

$$x^1 = \left(1 + \xi^{1^2}\right)^{\frac{1}{2}} \mathrm{dn}\xi^2 \overline{\mathrm{sn}}\xi^3 \quad , \tag{E-1}$$

$$x^2 = \left(\sigma + \xi^{1^2}\right)^{\frac{1}{2}} \mathrm{cn}\xi^2 \overline{\mathrm{cn}}\xi^3 \quad , \tag{E-2}$$

$$x^3 = \xi^1 \mathrm{sn}\xi^2 \overline{\mathrm{dn}}\xi^3 \quad , \tag{E-3}$$

$$(0 < \xi^1 < \infty;\ -2K \leqslant \xi^2 \leqslant 2K;\ -2K^1 \leqslant \xi^3 \leqslant 2K^1) \ .$$

Here σ is a positive parameter such that $0 \leqslant \sigma \leqslant 1$ and σ, $(1 - \sigma)$ are the parameters associated with the variables ξ^2 and ξ^3 which arise in the introduction of the Jacobian elliptic functions. These functions possess the following well known properties:

$$\mathrm{sn}\xi^2 = \sin\varphi \quad , \tag{E-4}$$

$$\mathrm{cn}\xi^2 = \cos\varphi \quad , \tag{E-5}$$

$$\mathrm{dn}\xi^2 = (1 - \sigma\sin^2\varphi)^{\frac{1}{2}} \quad , \tag{E-6}$$

where

$$\xi^2 = \int_0^{\varphi} \frac{d\gamma}{(1 - \sigma\sin^2\gamma)^{\frac{1}{2}}} \quad , \tag{E-7}$$

and

$$\overline{\mathrm{sn}}\xi^3 = \sin\bar{\varphi} \quad , \tag{E-8}$$

$$\overline{\mathrm{cn}}\xi^3 = \cos\bar{\varphi} \quad , \tag{E-9}$$

$$\overline{\mathrm{dn}}\xi^3 = (1 - (1 - \sigma)\sin^2\bar{\varphi})^{\frac{1}{2}} \quad , \tag{E-10}$$

where

$$\xi^3 = \int_0^{\bar{\varphi}} \frac{d\gamma}{(1 - [1 - \sigma] \sin^2 \gamma)^{\frac{1}{2}}} \quad . \tag{E-11}$$

From (E-4)-(E-11) it follows that

$$dn^2\xi^2 = 1 - \sigma sn^2\xi^2 \quad , \tag{E-12}$$

$$sn^2\xi^2 = 1 - cn^2\xi^2 \quad , \tag{E-13}$$

$$dn^2\xi^2 = (1 - \sigma) + \sigma cn^2\xi^2 \quad , \tag{E-14}$$

$$\overline{dn}^2\xi^3 = 1 - (1 - \sigma)\overline{sn}^2\xi^3 \quad , \tag{E-15}$$

$$\overline{sn}^2\xi^3 = 1 - \overline{cn}^2\xi^3 \quad , \tag{E-16}$$

and

$$\overline{dn}^2\xi^3 = \sigma + (1 - \sigma)\overline{cn}^2\xi^3 \quad . \tag{E-17}$$

Also the functions $sn\xi^2$, $cn\xi^2$ are periodic in ξ^2 with period $4K$ and $dn\xi^2$ has a period $2K$ where K is given by

$$K = \int_0^{\pi/2} \frac{d\gamma}{(1 - \sigma \sin^2 \gamma)^{\frac{1}{2}}} \quad . \tag{E-18}$$

The derivatives of the Jacobian elliptic functions are given by

$$\frac{d}{d\xi^2} (sn\xi^2) = cn\xi^2 dn\xi^2 \quad , \tag{E-19}$$

$$\frac{d}{d\xi^2} (cn\xi^2) = - sn\xi^2 dn\xi^2 \quad , \tag{E-20}$$

and

$$\frac{d}{d\xi^2} (dn\xi^2) = - \sigma sn\xi^2 cn\xi^2 \quad . \tag{E-21}$$

Furthermore $\mathrm{sn}\xi^2$, $\mathrm{cn}\xi^2$ and $\mathrm{dn}\xi^2$ are analytic for all real values of ξ^2 and $\mathrm{sn}\xi^2$ and $\mathrm{cn}\xi^2$ have simple zeroes at the points $0, \pm 2K, \pm 4K, \ldots$ and $\pm K, \pm 3K, \pm 5K, \ldots$ respectively. The function $\mathrm{dn}\xi^2$ has no real zeroes.

A similar set of properties hold for the functions $\overline{\mathrm{sn}}\xi^3$, $\overline{\mathrm{cn}}\xi^3$ and $\overline{\mathrm{dn}}\xi^3$ with σ replaced by $1 - \sigma$ and K by K^1 where K^1 is given by

$$K^1 = \int_0^{\pi/2} \frac{d\gamma}{(1 - (1 - \sigma) \sin^2 \gamma)^{\frac{1}{2}}} . \tag{E-22}$$

The complete properties of the Jacobian elliptic functions and their associated functions can be found outlined in Refs. 15 and 18.

The surfaces $\xi^1, \xi^2, \xi^3 = \text{const.}$ will be, respectively, an ellipsoid, hyperboloid of one sheet and a hyperboloid of two sheets. The equations defining these surfaces are

$$\frac{{x^1}^2}{1 + {\xi^1}^2} + \frac{{x^2}^2}{\sigma + {\xi^1}^2} + \frac{{x^3}^2}{{\xi^1}^2} = 1 , \tag{E-23}$$

$$\frac{{x^1}^2}{\mathrm{dn}^2\xi^2} + \frac{{x^2}^2}{\sigma \mathrm{cn}^2\xi^2} - \frac{{x^3}^2}{\sigma \mathrm{sn}^2\xi^2} = 1 , \tag{E-24}$$

and

$$\frac{{x^1}^2}{(1 - \sigma)\overline{\mathrm{sn}}^2\xi^3} - \frac{{x^2}^2}{(1 - \sigma)\overline{\mathrm{cn}}^2\xi^3} - \frac{{x^3}^2}{\overline{\mathrm{dn}}^2\xi^3} = 1 . \tag{E-25}$$

We also have, from (2-4), (2-23), (B-1)-(B-9), (E-1)-(E-3) and (E-19)-(E-21) the relations

$$t_1^1 = \xi^1\left(1 + {\xi^1}^2\right)^{-\frac{1}{2}} \mathrm{dn}\xi^2 \overline{\mathrm{sn}}\xi^3 = A_1 \cos \theta_1^1 = A_1 \cos \theta , \tag{E-26}$$

$$t_1^2 = \xi^1\left(\sigma + {\xi^1}^2\right)^{-\frac{1}{2}} \mathrm{cn}\xi^2 \overline{\mathrm{cn}}\xi^3 = A_1 \cos \theta_1^2 = A_1 \sin \theta \sin \phi , \tag{E-27}$$

$$t_1^3 = \mathrm{sn}\xi^2 \overline{\mathrm{dn}}\xi^3 = A_1 \cos \theta_1^3 = A_1 \sin \theta \cos \phi , \tag{E-28}$$

$$t_2^1 = -\sigma\left(1 + {\xi^1}^2\right)^{\frac{1}{2}} \overline{\mathrm{sn}}\xi^3 \mathrm{sn}\xi^2 \mathrm{cn}\xi^2 = A_2 \cos \theta_2^1 = A_2 \sin \theta \sin \psi , \tag{E-29}$$

$$t_2^2 = -\left(\sigma + \xi^{1^2}\right)^{\frac{1}{2}} \overline{\mathrm{cn}}\xi^3 \mathrm{sn}\xi^2 \mathrm{dn}\xi^2 = A_2 \cos\theta_2^2$$

$$= A_2 (\cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta) \quad ,$$

$$\ldots \text{(E-30)}$$

$$t_2^3 = \xi^1 \overline{\mathrm{dn}}\xi^3 \mathrm{cn}\xi^2 \mathrm{dn}\xi^2 = A_2 \cos\theta_2^3 = -A_2 (\sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta) \quad ,$$

$$\ldots \text{(E-31)}$$

$$t_3^1 = \left(1 + \xi^{1^2}\right)^{\frac{1}{2}} \mathrm{dn}\xi^2 \overline{\mathrm{cn}}\xi^3 \overline{\mathrm{dn}}\xi^3 = A_3 \cos\theta_3^1 = -A_3 \cos\psi\sin\theta \quad , \qquad \text{(E-32)}$$

$$t_3^2 = -\left(\sigma + \xi^{1^2}\right)^{\frac{1}{2}} \mathrm{cn}\xi^2 \overline{\mathrm{sn}}\xi^3 \overline{\mathrm{dn}}\xi^3 = A_3 \cos\theta_3^2$$

$$= A_3 (\sin\psi\cos\phi + \cos\psi\sin\phi\cos\theta) \quad ,$$

$$\ldots \text{(E-33)}$$

$$t_3^3 = -\xi^1 (1-\sigma) \mathrm{sn}\xi^2 \overline{\mathrm{sn}}\xi^3 \overline{\mathrm{cn}}\xi^3 = -A_3 (\sin\psi\sin\phi - \cos\psi\cos\phi\cos\theta) \quad .$$

$$\ldots \text{(E-34)}$$

From (E-26)-(E-34) we obtain

$$A_1 = B_2 B_3 \left(1 + (\xi^1)^2\right)^{-\frac{1}{2}} \left(\sigma + (\xi^1)^2\right)^{-\frac{1}{2}} \quad , \qquad \text{(E-35)}$$

$$A_2 = B_1 B_3 \quad , \qquad \text{(E-36)}$$

$$A_3 = B_1 B_2 \quad , \qquad \text{(E-37)}$$

with

$$B_1 \equiv \left(\sigma \mathrm{cn}^2\xi^2 + (1-\sigma)\overline{\mathrm{cn}}^2\xi^3\right)^{\frac{1}{2}} \quad , \qquad \text{(E-38)}$$

$$B_2 \equiv \left(\overline{\mathrm{dn}}^2\xi^3 + (\xi^1)^2\right)^{\frac{1}{2}} \quad , \qquad \text{(E-39)}$$

$$B_3 \equiv \left(\sigma \mathrm{sn}^2\xi^2 + (\xi^1)^2\right)^{\frac{1}{2}} \quad , \qquad \text{(E-40)}$$

$$J = A_1A_2A_3 = \frac{(B_1B_2B_3)^2}{\left(1 + (\xi^1)^2\right)^{\frac{1}{2}}\left(\sigma + (\xi^1)^2\right)^{\frac{1}{2}}}$$

$$= \frac{\left(\sigma \mathrm{sn}^2\xi^2 + (\xi^1)^2\right)^{\frac{1}{2}}\left(\overline{\mathrm{dn}}^2\xi^3 + (\xi^1)^2\right)^{\frac{1}{2}}\left(\sigma \mathrm{cn}^2\xi^2 + (1-\sigma)\overline{\mathrm{cn}}^2\xi^3\right)^{\frac{1}{2}}}{\left(1 + (\xi^1)^2\right)^{\frac{1}{2}}\left(\sigma + (\xi^1)^2\right)^{\frac{1}{2}}} . \quad \text{... (E-41)}$$

Also, the Euler angles are given by

$$\cos\theta = \frac{\xi^1\left(\sigma + (\xi^1)^2\right)^{\frac{1}{2}} \mathrm{dn}\xi^2\overline{\mathrm{sn}}\xi^3}{\left(\sigma \mathrm{sn}^2\xi^2 + (\xi^1)^2\right)^{\frac{1}{2}}\left(\overline{\mathrm{dn}}^2\xi^3 + (\xi^1)^2\right)^{\frac{1}{2}}} , \tag{E-42}$$

$$\tan\phi = \frac{\xi^1 \mathrm{cn}\xi^2\overline{\mathrm{cn}}\xi^3}{\left(\sigma + (\xi^1)^2\right)^{\frac{1}{2}} \mathrm{sn}\xi^2\overline{\mathrm{dn}}\xi^3} , \tag{E-43}$$

and

$$\tan\psi = \frac{\sigma \mathrm{sn}\xi^2 \mathrm{cn}\xi^2\overline{\mathrm{sn}}\xi^3\left(\overline{\mathrm{dn}}^2\xi^3 + (\xi^1)^2\right)^{\frac{1}{2}}}{\mathrm{dn}\xi^2\overline{\mathrm{cn}}\xi^3\overline{\mathrm{dn}}\xi^3\left(\sigma \mathrm{sn}^2\xi^2 + (\xi^1)^2\right)^{\frac{1}{2}}} . \tag{E-44}$$

From (E-38)-(E-41) we find that the singular points of the transformation, i.e. the points for which $J = 0$, occur when $B_1 = 0$ or $B_3 = 0$. The equations $B_1 = 0$ and $B_3 = 0$ imply $\xi^2 = \pm K$, $\xi^3 = \pm K^1$, and $\xi^1 = 0$, $\xi^2 = 0$ respectively so that from (E-1)-(E-3) we see that the singular points of the transformation lie on the hyperbola given by the equations

$$\frac{(x^1)^2}{1-\sigma} - \frac{(x^3)^2}{\sigma} = 1 , \qquad x^2 = 0 \tag{E-45}$$

and the ellipse given by the equations

$$(x^1)^2 + \frac{(x^2)^2}{\sigma} = 1 , \qquad x^3 = 0 . \tag{E-46}$$

In Tables 3 and 4 we find the values of $\cos\theta$, $\tan\phi$, $\tan\psi$, B_1, B_2 and B_3 for any given ξ^1 and special values of the arguments ξ^2 and ξ^3. The expressions shown in Tables 3 and 4 will hold for all values of σ except where otherwise stated and the values of the functions at other points can be found from these tables and the properties of the Jacobian elliptic functions given above and in Refs.15 and 18.

In the special cases when $\sigma \to 0$ and $\sigma \to 1$ we obtain, from the above formulae:

Case $\sigma \to 0$

$$\mathrm{sn}\xi^2 \to \sin\xi^2 \ , \tag{E-47}$$

$$\mathrm{cn}\xi^2 \to \cos\xi^2 \ , \tag{E-48}$$

$$\mathrm{dn}\xi^2 \to 1 \ , \tag{E-49}$$

$$\overline{\mathrm{sn}}\xi^3 \to \tanh\xi^3 \ , \tag{E-50}$$

$$\overline{\mathrm{cn}}\xi^3, \overline{\mathrm{dn}}\xi^3 \to \mathrm{sech}\,\xi^3 \ , \tag{E-51}$$

$$K \to \frac{\pi}{2} \ , \quad K^1 \to \infty \ , \tag{E-52}$$

$$B_1 = \mathrm{sech}\,\xi^3 \ , \tag{E-53}$$

$$B_2 = \left((\xi^1)^2 + \mathrm{sech}^2\,\xi^3\right)^{\frac{1}{2}} \ , \tag{E-54}$$

$$B_3 = \xi^1 \ , \tag{E-55}$$

$$\cos\theta = \frac{\xi^1 \tanh\xi^3}{\left((\xi^1)^2 + \mathrm{sech}^2\,\xi^3\right)^{\frac{1}{2}}} \ , \tag{E-56}$$

$$\tan\phi = \cot\xi^2 \ , \tag{E-57}$$

$$\tan\psi = 0 \ . \tag{E-58}$$

The singular points of the transformation are along the axis $x^2 = 0$, $x^3 = 0$.

Case $\sigma \to 1$

$$\mathrm{sn}\xi^2 \to \tanh \xi^2 \quad , \tag{E-59}$$

$$\mathrm{cn}\xi^2, \mathrm{dn}\xi^2 \to \mathrm{sech}\ \xi^2 \quad , \tag{E-60}$$

$$\overline{\mathrm{sn}}\xi^3 \to \sin \xi^3 \quad , \tag{E-61}$$

$$\overline{\mathrm{cn}}\xi^3 \to \cos \xi^3 \quad , \tag{E-62}$$

$$\overline{\mathrm{dn}}\xi^3 \to 1 \quad , \tag{E-63}$$

$$K \to \infty \quad , \quad K^1 \to \frac{\pi}{2} \quad , \tag{E-64}$$

$$B_1 = \mathrm{sech}\ \xi^2 \quad , \tag{E-65}$$

$$B_2 = \left((\xi^1)^2 + 1\right)^{\frac{1}{2}} \tag{E-66}$$

$$B_3 = \left((\xi^1)^2 + \tanh^2 \xi^2\right)^{\frac{1}{2}} \quad , \tag{E-67}$$

$$\cos\theta = \frac{\xi^1\ \mathrm{sech}\ \xi^2 \sin \xi^3}{\left((\xi^1)^2 + \tanh^2 \xi^2\right)^{\frac{1}{2}}} \quad , \tag{E-68}$$

$$\tan\phi = \frac{\xi^1\ \mathrm{cosech}\ \xi^2 \cos \xi^3}{\left(1 + \xi^{1^2}\right)^{\frac{1}{2}}} \quad , \tag{E-69}$$

$$\tan\psi = \frac{\left(1 + \xi^{1^2}\right)^{\frac{1}{2}} \tanh \xi^2}{\left(\tanh^2 \xi^2 + (\xi^1)^2\right)^{\frac{1}{2}}} \tan \xi^3 \quad . \tag{E-70}$$

The singular points of the transformation lie on the plane $x^3 = 0 \left(x^{1^2} + x^{2^2} = 1\right)$ and along the axis $x^1 = x^2 = 0$.

Appendix F

THE INTEGRABILITY CONDITIONS FOR SOME AXIALLY SYMMETRIC GEOMETRIES

F.1 Case $\theta = \pi/2$ $(\theta_{12} = \theta_{31} = \pi/2)$

Here we set

$$x^2 = r \cos \xi^1 \tag{F-1}$$

$$x^3 = r \sin \xi^1 \tag{F-2}$$

$$r = A_1 \tag{F-3}$$

and

$$\phi = -\xi^1 \quad . \tag{F-4}$$

We find, after some algebra, that equations (4-19) and (4-22) will be satisfied and that (4-25) is an identity. Equations (4-20), (4-21), (4-23) and (4-24) state that $\psi - \theta_{23}$, A_3, A_2 and ψ are independent of ξ^1, i.e. independent of ϕ. The remaining integrability conditions will be of the form

$$\sin \theta_{23} \frac{\partial A_2}{\partial \xi^3} + \cos \theta_{23} A_2 \frac{\partial \psi}{\partial \xi^3} = A_3 \frac{\partial}{\partial \xi^2} (\psi - \theta_{23}) \quad , \tag{F-5}$$

$$A_2 \frac{\partial \psi}{\partial \xi^3} = - \sin \theta_{23} \frac{\partial A_3}{\partial \xi^2} + \cos \theta_{23} A_3 \frac{\partial}{\partial \xi^2} (\psi - \theta_{23}) \quad , \tag{F-6}$$

and equations (F-5), (F-6) will reduce to the Riemann-Cauchy conditions in the special case $A_2 = A_3 \equiv A$, $\theta_{23} = \pi/2$ so that $Ae^{-i\psi}$ is an analytic function of $\xi^2 + i\xi^3$ for any value of ξ^1.

F.2 Case $\psi = \theta_{23} = \pi/2$ $(\theta_{12} = \theta_{31} = \pi/2)$

In this case we set

$$x^2 = r \sin \xi^3 \tag{F-7}$$

$$x^3 = r \cos \xi^3 \tag{F-8}$$

$$r = A_3 \tag{F-9}$$

and

$$\phi = \xi^3 \tag{F-10}$$

Equations (4-21) and (4-27) will be satisfied and (4-24) will be identically satisfied. Equations (4-19), (4-20), (4-25) and (4-26) state that A_1, θ and A_2 are independent of ξ^3 and the two remaining equations (4-22) and (4-23) become

$$\frac{\partial A_1}{\partial \xi^2} = A_2 \frac{\partial \theta}{\partial \xi^1} , \tag{F-11}$$

and

$$\frac{\partial A_2}{\partial \xi^1} = - A_1 \frac{\partial \theta}{\partial \xi^2} . \tag{F-12}$$

In the special case when $A_1 = A_2 \equiv A$ equations (F-11) and (F-12) reduce to the Riemann-Cauchy equations and $Ae^{-i\theta}$ will be an analytic function of $\xi^1 + i\xi^2$. The body can be represented again by $\xi^2 = \text{const}$.

Case $\phi = \pi/2$

We set

$$x^1 = r \sin \xi^1 , \tag{F-13}$$

$$x^2 = - r \cos \xi^1 , \tag{F-14}$$

$$r = A_1 , \tag{F-15}$$

and

$$\theta = \xi^1 . \tag{F-16}$$

Equations (4-19) and (4-22) are satisfied and (4-25) becomes an identity. Also equations (4-20), (4-21), (4-23) and (4-24) state that $\psi - \theta_{23}$, A_3, A_2 and ψ are independent of ξ^1. The remaining integrability conditions will be of the same form as (F-5) and (F-6) and reduce to the Riemann-Cauchy conditions when $A_2 = A_3 = A$ and $\theta_{23} = \pi/2$ as in the case $\theta = \pi/2$.

F.4 Case $\psi = 0, \theta_{23} = \pi/2$

We set

$$x^2 = r \sin \xi^2$$

$$x^3 = r \cos \xi^2 \qquad A_2 = r, \ \xi^2 = \phi .$$

The integrability conditions are

$$\frac{\partial A_1}{\partial \xi^3} = - A_3 \frac{\partial \theta}{\partial \xi^1}$$

$$A_1 \frac{\partial \theta}{\partial \xi^3} = \frac{\partial A_3}{\partial \xi^1} ,$$

with A_1, A_3, θ independent of ξ^2.

Table 1

UNIT BASE VECTORS $A_k^{-1} \underline{a}_k = \underline{c}_\ell \cos \theta_k^\ell$

$$\left(J_0^2 = 1 + 2 \cos \theta_{12} \cos \theta_{23} \cos \theta_{31} - \cos^2 \theta_{12} - \cos^2 \theta_{23} - \cos^2 \theta_{31}\right)$$

	$\cos \theta_k^\ell$	$\ell = 1$	$\ell = 2$	$\ell = 3$
Case I	$k = 1$	$\cos \theta$	$\sin \theta \sin \phi$	$\sin \theta \cos \phi$
	$k = 2$	$\sin \theta \sin \psi$	$\cos \phi \cos \psi - \sin \phi \cos \theta \sin \psi$	$-\sin \phi \cos \psi - \cos \phi \cos \theta \sin \psi$
	$k = 3$	$\sin \theta \sin (\psi - \theta_{23})$	$\cos \phi \cos(\psi - \theta_{23}) - \sin \phi \cos \theta \sin (\psi - \theta_{23})$	$-\sin \phi \cos(\psi - \theta_{23}) - \cos \phi \cos \theta \sin (\psi - \theta_{23})$
Case II $(\alpha = \phi + \psi)$	$k = 1$	$\dfrac{J_0}{\sin \theta_{23}}$	$\dfrac{\cos \theta_{31} \sin \alpha - \cos \theta_{12} \sin (\alpha - \theta_{23})}{\sin \theta_{23}}$	$\dfrac{\cos \theta_{31} \cos \alpha - \cos \theta_{12} \cos(\alpha - \theta_{23})}{\sin \theta_{23}}$
	$k = 2$	0	$\cos \alpha$	$-\sin \alpha$
	$k = 3$	0	$\cos(\alpha - \theta_{23})$	$-\sin (\alpha - \theta_{23})$

Table 2

UNIT NORMAL VECTORS $(A^k)^{-1} \underline{a}^k = \cos \theta_\ell^{*k} \underline{c}^\ell$

	$\cos \theta_\ell^{*k}$	$\ell = 1$	$\ell = 2$	$\ell = 3$
Case I	$k = 1$	$\cos \theta$	$\sin \theta \sin \phi$	$\sin \theta \cos \phi$
	$k = 2$	$\sin \theta \cos(\psi - \theta_{23})$	$- \cos \phi \sin (\psi - \theta_{23}) - \sin \phi \cos \theta \cos(\psi - \theta_{23})$	$\sin \phi \sin (\psi - \theta_{23}) - \cos \phi \cos \theta \cos(\psi - \theta_{23})$
	$k = 3$	$- \sin \theta \cos \psi$	$\cos \phi \sin \psi + \sin \phi \cos \theta \cos \psi$	$- \sin \phi \sin \psi + \cos \phi \cos \theta \cos \psi$
Case II ($\alpha = \phi + \psi$)	$k = 1$	1	0	0
	$k = 2$	$\dfrac{\cos \theta_{23} \cos \theta_{31} - \cos \theta_{12}}{\sin \theta_{23} \sin \theta_{31}}$	$- \dfrac{J_0 \sin (\alpha - \theta_{23})}{\sin \theta_{23} \sin \theta_{31}}$	$- \dfrac{J_0 \cos(\alpha - \theta_{23})}{\sin \theta_{23} \sin \theta_{31}}$
	$k = 3$	$\dfrac{\cos \theta_{12} \cos \theta_{23} - \cos \theta_{31}}{\sin \theta_{12} \sin \theta_{23}}$	$\dfrac{J_0 \sin \alpha}{\sin \theta_{12} \sin \theta_{23}}$	$\dfrac{J_0 \cos \alpha}{\sin \theta_{12} \sin \theta_{23}}$

Table 3

VALUES OF $\cos\theta$, $\tan\phi$, $\tan\psi$ AT $\xi^2 = 0, k$, $\xi^3 = 0, k^1$

	$\xi^2 = 0$	$\xi^2 = k$	$\xi^3 = 0$	$\xi^3 = k^1$
$\cos\theta$	$\dfrac{\left(\sigma + {\xi^1}^2\right)^{\frac{1}{2}} \overline{\mathrm{sn}}\xi^3}{\left(\overline{\mathrm{dn}}^2\xi^3 + {\xi^1}^2\right)^{\frac{1}{2}}}$	$\dfrac{\xi^1 (1-\sigma)^{\frac{1}{2}} \overline{\mathrm{sn}}\xi^3}{\left(\overline{\mathrm{dn}}^2\xi^3 + {\xi^1}^2\right)^{\frac{1}{2}}}$	0	$\dfrac{\xi^1 \mathrm{dn}\xi^2}{\left(\sigma \mathrm{sn}^2\xi^2 + {\xi^1}^2\right)^{\frac{1}{2}}}$
$\tan\phi$	∞	0	$\dfrac{\xi^1 \mathrm{cn}\xi^2}{\left(\sigma + {\xi^1}^2\right)^{\frac{1}{2}} \mathrm{sn}\xi^2}$	0
$\tan\psi$	0	0 $\left(\begin{matrix}\xi^3 \neq \pm k^1, \\ \sigma \neq 1\end{matrix}\right)$,	0	∞ $(\xi^2 \neq \pm k)$

Table 4

VALUES OF B_1, B_2, B_3 AT $\xi^2 = 0,k$; $\xi^3 = 0,k^1$

	$\xi^2 = 0$	$\xi^2 = k$	$\xi^3 = 0$	$\xi^3 = k^1$
B_1	$(1 - (1-\sigma)\overline{\mathrm{sn}}^2\xi^3)^{\frac{1}{2}}$	$(1-\sigma)^{\frac{1}{2}}\overline{\mathrm{cn}}\xi^3 > 0$	$(1 - \sigma\,\mathrm{sn}^2\xi^2)^{\frac{1}{2}}$	$\sigma^{\frac{1}{2}}\mathrm{cn}\xi^2 > 0$
B_2	$\left(\overline{\mathrm{dn}}^2\xi^3 + \xi^{1^2}\right)^{\frac{1}{2}}$	$\left(\overline{\mathrm{dn}}^2\xi^3 + \xi^{1^2}\right)^{\frac{1}{2}}$	$\left(1 + \xi^{1^2}\right)^{\frac{1}{2}}$	$\left(\sigma + \xi^{1^2}\right)^{\frac{1}{2}}$
B_3	ξ^1	$\left(\sigma + \xi^{1^2}\right)^{\frac{1}{2}}$	$\left(\sigma\,\mathrm{sn}^2\xi^2 + \xi^{1^2}\right)^{\frac{1}{2}}$	$\left(\sigma\,\mathrm{sn}^2\xi^2 + \xi^{1^2}\right)^{\frac{1}{2}}$

SYMBOLS

a	speed of sound
$\underline{a}_i$	base vector
$\underline{a}^i$	normal vector
A	$= A_2 = A_3$ (subsection 5.2)
A_i	magnitude of base vector $\underline{a}_i$
A^i	magnitude of normal vector $\underline{a}^i$
$\underline{b}_2$, $\underline{b}_3$	vectors in orthogonal systems (Fig. 1)
B_i	magnitude related to A_i (Appendix E)
c	constant in equation (3-7)
$\underline{c}_i$	Cartesian unit base vector
$\underline{c}^i$	Cartesian unit normal vector
cn	Jacobian elliptic function with parameter σ and argument ξ^2 (Appendix E)
$\overline{\text{cn}}$	Jacobian elliptic function with parameter $1 - \sigma$ and argument ξ^3 (Appendix E)
$\underline{ds}$	differential length vector, equation (A-1)
ds	magnitude of differential length vector, equation (2-8)
dn	Jacobian elliptic function with parameter σ and argument ξ^2 (Appendix E)
$\overline{\text{dn}}$	Jacobian elliptic function with parameter $1 - \sigma$ and argument ξ^3 (Appendix E)
d_N^{*N}	$A_N A^N$
d_N^N	$\left(d_N^{*N}\right)^{-1}$
F	$\dfrac{\partial f_1}{\partial \xi^3} - i \dfrac{\partial f_2}{\partial \xi^1}$
f_1	$A_1 \left(\cos \theta_1^2 + i \cos \theta_1^3\right)$
f_2	$Ae^{-i\alpha}$
g_{ij}	co-variant metric tensor of order 2
g^{ij}	contra-variant metric tensor of order 2
h	specific enthalpy
i	$\sqrt{-1}$

SYMBOLS (continued)

J, J^*	Jacobians of general transformations
J_0	$(A_1A_2A_3)^{-1}J$
J_0^*	$d_1^1d_2^2d_3^3J_0^{-1}$
K, K^1	quarter periods of elliptic functions with parameters σ and $1 - \sigma$
ℓ	radius of large circle (subsection 5.2)
p	fluid pressure
P	point in Euclidean three space
q	fluid speed
r	polar coordinate (Appendix F)
R	gas constant
s	specific entropy
sn	Jacobian elliptic function with parameter σ and argument ξ^2 (Appendix E)
$\overline{\mathrm{sn}}$	Jacobian elliptic function with parameter $1 - \sigma$ and argument ξ^3 (Appendix E)
T	temperature
T_1, T_2, T_3	rigid body rotation matrices (Appendix C)
$\underline{t}$	vector in Cartesian reference frame
t^i	i^{th} contra-variant component of vector $\underline{t}$
t_i^j	$\dfrac{\partial x^j}{\partial \xi^i}$
t_i^{*j}	$\dfrac{\partial \xi^j}{\partial x^i}$
$\underline{u}, \underline{v}, \underline{w}$	vectors in Cartesian reference system
u_i	i^{th} co-variant component of vector $\underline{u}$
u^i	i^{th} contra-variant component of vector $\underline{u}$
v_i	co-variant component of vector $\underline{v}$
v^i	contra-variant component of vector $\underline{v}$
x^i	Cartesian coordinate
α	$\phi + \psi$
α_i^j	angle between i^{th} axis in (P, w^1, w^2, w^3) Cartesian frame and k^{th} axis in (P, t^1, t^2, t^3) Cartesian frame

SYMBOLS (continued)

β	Euler angle
γ	ratio of specific heats
δ_{ij}	co-variant Kronecker delta symbol $= 0,\ i \neq j$ $= 1,\ i = j$
δ^{ij}	contra-variant Kronecker delta symbol $= 0,\ i \neq j$ $= 1,\ i = j$
δ_i^j	mixed Kronecker delta symbol $= 0,\ i \neq j$ $= 1,\ i = j$
ξ^i	i^{th} general coordinate
ε_{ijk}, ε^{ijk}	$= 1$ for i, j, k an even permutation of 1, 2, 3 $= -1$ for i, j, k an odd permutation of 1, 2, 3 $= 0$ if any two of i, j, k are the same
θ	Euler angle
θ_{ij}	angle between the vectors $\underline{a}_i$ and $\underline{a}_j$
θ^{ij}	angle between the vectors $\underline{a}^i$ and $\underline{a}^j$
θ_i^j	angle between the vectors $\underline{a}_i$ and $\underline{c}_j$
θ_i^{*j}	angle between the vectors $\underline{a}^j$ and $\underline{c}_i$
μ	$\dfrac{\cos\theta_{12}}{\sin\theta_{31}}$
ρ	fluid density
σ	parameter of Jacobian elliptic function
φ, $\bar{\varphi}$	amplitude of Jacobian elliptic functions with parameters σ and $1 - \sigma$
ϕ	Euler angle
Φ	velocity potential function
ψ	Euler angle
$\underline{\Omega}$	vorticity vector
$\Omega^{i,k}$	$\varepsilon^{i\ell m} \dfrac{\partial}{\partial \xi^\ell}\left(t_m^k\right)$
$\bar{\Omega}^{i,k}$	$\cos\theta_\ell^{*k}\Omega^{i,\ell}$

Subscripts

i, j, k	indices running from 1 to 3
N	index used in definition of d_N^N (no summation over N)

SYMBOLS (concluded)

Superscripts

i, j, k	indices running from 1 to 3
N	index used in definition of d_N^N (no summation over N)

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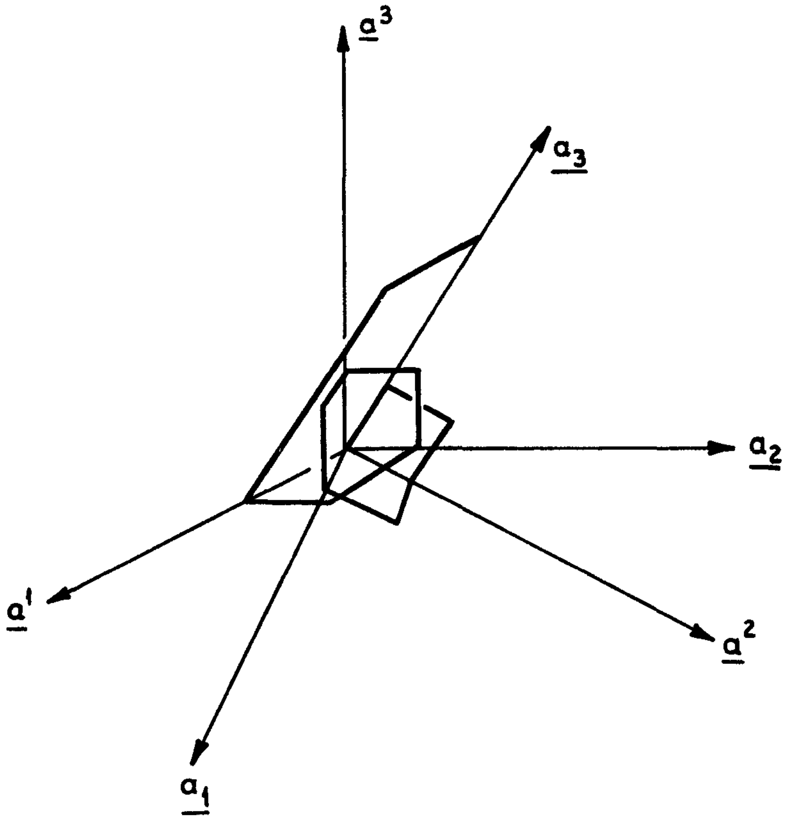

Fig.1 Base and normal vectors
(right angles indicated by parallelograms)

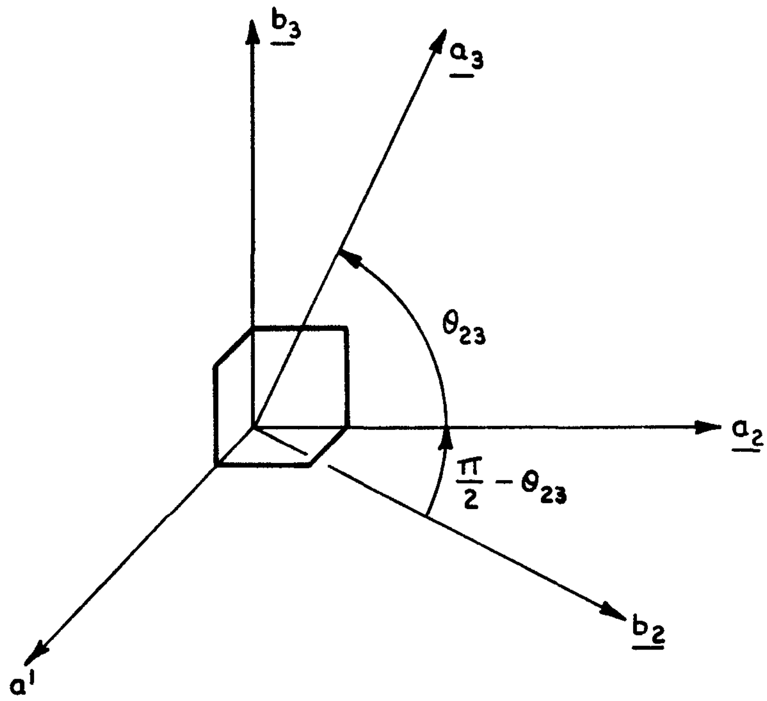

Fig.2 Two othogonal systems

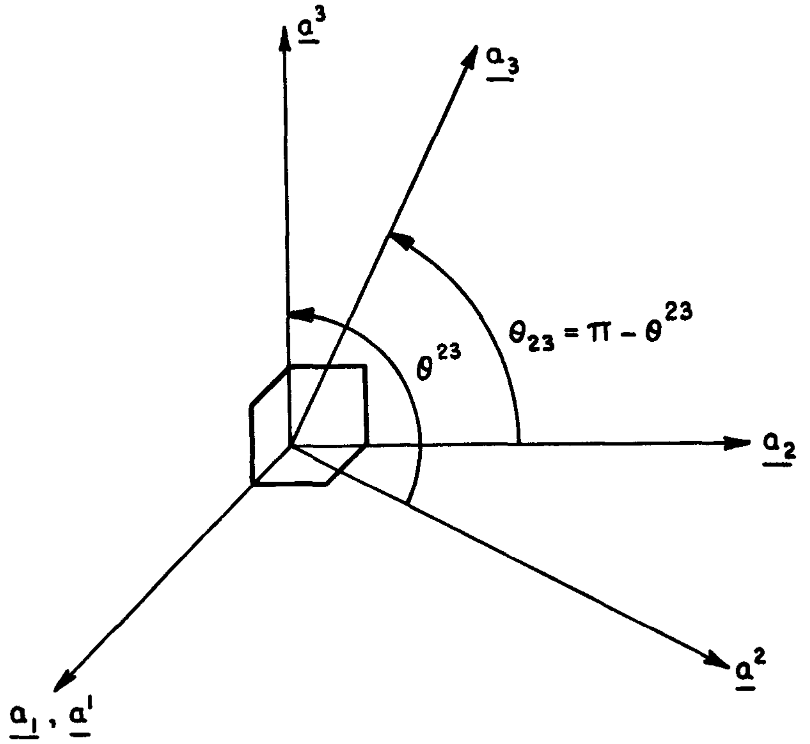

Fig.3 Base and normal vectors (case I)

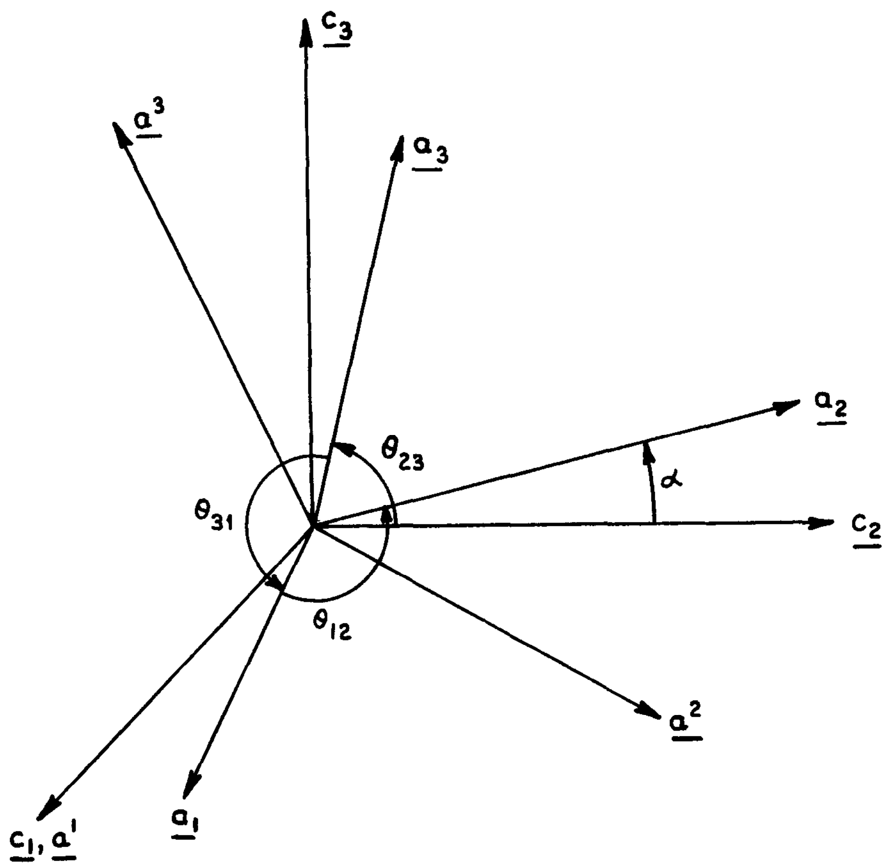

Fig.4 Base and normal vectors (case II)

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