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# High Subsonic Flow Past a Steady Two-Dimensional Aerofoil

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LONDON: HER MAJESTY'S STATIONERY OFFICE

1974

PRICE 65p NET

C.P. No.1280*
January 1973

# HIGH SUBSONIC FLOW PAST A STEADY TWO-DIMENSIONAL AEROFOIL

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## SUMMARY

The integral equation method for the prediction of the pressure distribution around aerofoils in transonic flows is examined with a view to developing a uniform approach to the solution of more general problems in transonic flow, steady or unsteady. In this paper the detailed investigation is restricted to the flow around two-dimensional aerofoils in steady high subsonic flow. Both lifting and non-lifting aerofoils are considered.

A simple first approximation to the flow, expressed by

$$\bar{u} - \frac{\bar{u}^2}{4} = \bar{u}_L ,$$

where $\bar{u}$ is the perturbation velocity and $\bar{u}_L$ is a modified linearised value, gives a fair degree of accuracy except in the immediate neighbourhood of the leading edge. An iterated second approximation gives good results for a NACA 0012 aerofoil.

*Replaces A.R.C.34 260

# NOTATION

| | |
|---|---|
| $c_p(x,z)$ | Pressure coefficient |
| $E_c(x)$ | Function given by Eq.(30) |
| $E_T(x)$ | Function given by Eq.(27) |
| $E_{T1}(x)$ | Function given by Eq.(31) |
| $I_c(\bar{x},\bar{u}(\bar{x},\pm 0))$ | Integral defined by Eq.(29) |
| $I_\tau(\bar{x},\bar{u}(\bar{x},\pm 0))$ | Integral defined by Eq.(26) |
| $I(\bar{x},\pm 0)$ | Integral defined by Eq.(34) |
| $k \quad =$ | $(\gamma+1)\ M_\infty^2$ |
| $M(x,z)$ | The local Mach number |
| $M_\infty$ | The freestream Mach number |
| $q(x,z)$ | The total resultant velocity of the fluid |
| $\bar{q}(\bar{x},\bar{z}) \quad =$ | $\frac{k}{\beta^2}\ q(x,z)$ |
| S | The domain in which the Green's theorem is valid |
| $u(x,z)$ | Perturbation velocity in the x direction |
| $\bar{u}(\bar{x},\bar{z}) \quad =$ | $\frac{k}{\beta^2}\ u(x,z)$ |
| $u_i(x,\pm 0)$ | Incompressible limit of u $(x,\pm 0)$ |
| $u_\ell(x,\pm 0)$ | The standard solution for $u(x,\pm 0)$ obtained from linearised theory |
| $\bar{u}_\ell(\bar{x},\pm 0) \quad =$ | $\frac{k}{\beta^2}u\ (x,\pm 0)$ |
| $u_{\ell a}(x,\pm 0)$ | Perturbation velocity defined by Eq.(46) |
| $\bar{u}_L(\bar{x},\pm 0)$ | The modified linearised solution for $\bar{u}(\bar{x},\pm 0)$ given by Eq.(43) |
| $U_\infty$ | The freestream velocity |
| $w(x,z)$ | Perturbation velocity in the z direction |
| $\bar{w}(\bar{x},\bar{z}) \quad =$ | $\frac{k}{\beta^3}\ w(x,z)$ |
| x, z | Cartesian co-ordinate system |
| $\bar{x} \quad =$ | x |
| $\bar{z} \quad =$ | $\beta z$ |

| | | |
|---|---|---|
| $z_{su}(x)$ | | The ordinate of the upper surface of the aerofoil |
| $z_{sl}(x)$ | | The ordinate of the lower surface of the aerofoil |
| $z_c(x)$ | | Camber distribution of the aerofoil |
| $z_\tau(x)$ | | Thickness distribution of the aerofoil |
| $\bar{z}_{su}(\bar{x})$ | = | $\beta z_{su}(x)$ |
| $\bar{z}_{sl}(\bar{x})$ | | $\beta z_{sl}(x)$ |
| $\bar{Z}_\tau(\bar{x})$ | = | $\frac{k}{\beta^3}\ z_\tau(x)$ |
| $\bar{Z}_c(\bar{x})$ | = | $\frac{k}{\beta^3}\ z_c(x)$ |
| $\bar{Z}_+(\bar{x})$, $\bar{Z}_-(\bar{x})$ | | Aerofoil profile distributions defined by Eq.(12) |
| $\alpha$ | | Angle of incidence |
| $\bar{A}$ | = | $\frac{k}{\beta^3}\ \alpha$ |
| $\beta$ | = | $(1-M_\infty^2)^{\frac{1}{2}}$ |
| $\gamma$ | | Ratio of specific heats taken to be 1.4 |
| $\bar{\xi}$, $\bar{\zeta}$ | | Cartesian co-ordinate system corresponding to $(\bar{x},\bar{z})$ |
| $\Lambda(\bar{\xi})$ | | Function defined by Eq.(52) |
| $\phi(x,z)$ | | Perturbation velocity potential defined by Eq.(1) |
| $\bar{\phi}(\bar{x},\bar{z})$ | = | $\frac{k}{\beta^2}\ \phi(x,z)$ |
| $\psi(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})$ | = | $\ln\left[(\bar{x}-\bar{\xi})^2+(\bar{z}-\bar{\zeta})^2\right]^{\frac{1}{2}}$ |

# 1. INTRODUCTION

The inviscid flow around a given wing can be determined to an acceptable accuracy in many practical cases by using a linearised form of the fundamental equations. Linearisation separates the general problem of the uniform flow past an arbitrary wing which is either steady or oscillating in simple harmonic motion into two independent problems, the solutions of which can be superimposed. One is the steady symmetric problem which represents the displacement field due to wing thickness. The other is the antisymmetric problem which represents the lifting field due to camber and incidence, either steady or oscillatory. Considerable expertise is available for the solution of many practical problems for which linearisation of the fundamental equations is valid. If linearisation is not valid then the separation of lifting and non-lifting effects is not possible and alternative means of solution must be used.

One general area for which the application of linearised theory is inadequate is in the transonic regime which covers three distinct sub-regimes; high subsonic shock free flow, mixed flow with supersonic regions embedded in an overall subsonic flow and the low supersonic freestream regime.

One simplification that is valid at transonic speeds is the assumption of isentropic and hence irrotational flow even if shock waves are present since the entropy changes are small compared to changes in the other variables of state if the local Mach number remains close to unity. Thus the flow may be adequately described by the complete, non-linear potential equation. Some further simplification of the potential equation is usually possible but any approximation must retain at least some of the non-linear terms in order to incorporate the essential features of transonic flows.

One of the earliest attempts to find the pressure distribution around wings in transonic flow was the integral equation method of Spreiter and Alksne[1] which followed work by Ostwatitsch[2] and Gullstrand[3]; the application of the method was restricted to the flow around two-dimensional sharp nosed symmetric aerofoils at zero incidence, starting with the transonic form of the two-dimensional potential differential equation in which all but one of the non-linear terms are neglected. The application of Green's theorem inverts the differential equation into integral form and gives the perturbation velocity potential at a general point in the flow field in terms of a line integral of a linear function of the velocity potential around the aerofoil surface and a double (surface) integral of a non-linear function of the velocity potential over the entire flow field. In linear theory this double integral is neglected as a second order term; in transonic flow it must be retained. The double integral is evaluated using suitable approximations for the non-linear term. The results of Spreiter and Alksne[1] are fairly good for the subcritical flow around circular arc aerofoils at zero incidence.

In recent years the advent of high speed computers has greatly facilitated the numerical solution of many problems arising in transonic aerodynamics. Sells[4] has developed a numerical programme for the calculation of the pressure distribution on an arbitrary two-dimensional aerofoil at subcritical speeds. By means of conformal transformations the exterior flow field about the aerofoil is mapped on to the interior of a circle. The governing equations for isentropic irrotational flow are reduced to two coupled non-linear equations for the stream function and the density, and an iterative solution is obtained by using finite difference techniques. Results from the application of Sells'[4] method are very useful as a standard for comparison of results derived by other methods. Conformal transformations cannot be used for three-dimensional or time dependent flows.

A direct numerical solution for steady transonic flow problems has been developed by Murman and Cole[5] who use relaxation methods to determine the flow about a circular arc aerofoil at zero incidence at transonic speeds; the transonic potential equation is used in this method. Subsequent extensions to two-dimensional round-nosed lifting aerofoils have been made by Murman and Krupp[6] and Steger and Lomax[7], who use the exact set of equations for isentropic flow. An important feature of these methods is that a separate difference scheme is used for the subsonic and supersonic regions of the flow, a centred difference scheme being used for the subsonic region and a one-sided difference scheme being used for the supersonic region. Shock waves appear naturally in the solution without any a priori assumptions. The method has recently been extended[8] to include the flow around finite wings. No attempt has yet been made to include time-dependent effects.

Because of the need to develop theoretical methods to predict transonic flow characteristics for a range of wing problems in two and three dimensions and in steady and unsteady conditions, the integral equation method is returned to with the aim of presenting a unified approach which gives approximate results of acceptable accuracy.

In this paper only the shock free two-dimensional flows around lifting and non-lifting aerofoils are considered. Applying Green's theorem to the transonic potential equation and then applying the boundary conditions, two simultaneous integral equations are derived in which the usual symmetric (thickness) and antisymmetric (camber, incidence) effects are coupled in double (surface) integrals involving the second order terms. By a consideration of these second order terms the double integrals are reduced to single integrals over the chord in which coupling between thickness and incidence effects are still retained. Extending the standard linearised techniques, which involves some additional numerical integration procedures, the simultaneous integral equations are solved approximately. An extremely simple formula is obtained for the perturbation surface velocity in terms of the linearised value gives a good first approximation. Results from a second approximation for two test cases show close agreement with results from the more exact high subsonic theory of Sells[4].

## 2. BASIC EQUATIONS

A two-dimensional cartesian co-ordinate system is chosen with the origin at the wing leading edge with the x axis in the free stream direction and z axis normal to free stream, as shown in Fig. 1. The co-ordinates x and z are scaled with respect to the aerofoil chord. The free stream velocity is denoted by $U_\infty$. A non-dimensional perturbation velocity potential, $\phi$, may be defined as

$$\frac{\partial \phi}{\partial x} = u \quad , \qquad \frac{\partial \phi}{\partial z} = w \quad , \qquad (1)$$

where u and w are the non-dimensional perturbation velocities, in the x and z directions respectively, relative to $U_\infty$.

The steady transonic potential equation for inviscid, non-conducting, isentropic, irrotational flow around a two-dimensional aerofoil is

$$(1-M_\infty^2)\ \phi_{xx} + \phi_{zz} = (\gamma+1)M_\infty^2\ \phi_x\ \phi_{xx} \qquad (2)$$

The boundary conditions are:

(i) the resultant flow direction at the aerofoil surface is tangential to the surface.

(ii) the perturbation potential and its derivatives vanish at an infinite distance upstream of the aerofoil.

The Kutta condition, that the pressure is finite and continuous at the trailing edge, is necessary to ensure a unique solution.

The upper and lower surfaces of the aerofoil may be written to a first order

$$z_{su}(x) = -\alpha x + z_c(x) + z_\tau(x)$$
$$z_{sl}(x) = -\alpha x + z_c(x) - z_\tau(x) \qquad (3)$$

where $z = z_{su}(x)$, and $z = z_{sl}(x)$, are the equations of the upper and lower aerofoil surfaces respectively, non-dimensionalised with respect to the chord; $\alpha$ is the aerofoil angle of incidence, $z_c(x)$ is the equation for the camber surface and $z_\tau(x)$ represents the wing thickness distribution. It is assumed that $z_{su}(x)$ and $z_{sl}(x)$ are small compared with unity.

The tangency boundary condition becomes

$$\frac{w\ (x,\ z_{su})}{1 + u\ (x,\ z_{su})} = z'_{su}(x)$$

$$\frac{w\ (x,\ z_{sl})}{1 + u\ (x,\ z_{sl})} = z'_{sl}(x) \qquad (4)$$

where the dash denotes differentiation with respect to x.

By means of a Taylor series expansion Eqs. (4) may be expressed as

$$w\ (x,\ +0) = z'_{su}(x) + u(x,\ +0)\ z'_{su}(x) - \frac{\partial w(x,\ +0)}{\partial z}\ z_{su}(x)\ \ldots.$$
$$(5)$$
$$w\ (x,\ -0) = z'_{sl}(x) + u(x,\ -0)\ z'_{sl}(x) - \frac{\partial w(x,\ -0)}{\partial z}\ z_{sl}(x)\ \ldots.$$

A first order approximation to the tangency boundary condition, Eqs. (4), is

$$w\ (x,\ +0) = \left(\frac{\partial\phi}{\partial z}\right)_{z=+0} = z'_{su}(x) = -\alpha + z'_c(x) + z'_\tau(x)$$
$$(6)$$
$$w\ (x,\ -0) = \left(\frac{\partial\phi}{\partial z}\right)_{z=-0} = z'_{sl}(x) = -\alpha + z'_c(x) - z'_\tau(x)$$

The boundary conditions are therefore applied on the plane z = 0 rather than on the aerofoil surface.

The pressure coefficient, $c_p(x,z)$ can be found by using Bernoulli's equation; thus

$$c_p(x,z) = \frac{2}{\gamma M_\infty^2} \{\{1 + \frac{(\gamma-1)M_\infty^2}{2} \left[1 - q^2(x,z)\right] \}^{\frac{\gamma}{\gamma-1}} - 1\} \tag{7}$$

where

$$q(x,z) = \left[(1 + u(x,z))^2 + w^2(x,z)\right]^{\frac{1}{2}}$$

which is the resultant velocity non-dimensionalised with respect to the freestream velocity.

A first order approximation to the pressure coefficient is given by

$$c_p(x,z) = - 2\, u(x,z) \tag{8}$$

Eq. (2) can be transformed, introducing the parameters $\beta^2 = 1-M_\infty^2$ $k = (\gamma+1)\, M_\infty^2$ and the variables

$$\bar{\phi} = \frac{k}{\beta^2}\, \phi \;,\; \bar{u} = \frac{k}{\beta^2}\, u \;,\; \bar{w} = \frac{k}{\beta^3}\, w \;, \tag{9}$$
$$\bar{x} = x,\; \bar{z} = \beta z$$

With this transformation it is noted that

$$\bar{u} = \frac{\partial\bar{\phi}}{\partial\bar{x}} \;,\; \bar{w} = \frac{\partial\bar{\phi}}{\partial\bar{z}}$$

On substitution of the variables defined in Eq. (9), Eq. (2) becomes

$$\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{z}\bar{z}} = \bar{\phi}_{\bar{x}}\, \bar{\phi}_{\bar{x}\bar{x}} \tag{10}$$

while the boundary conditions in Eq. (6) become

$$(\bar{\phi}_{\bar{z}})_{\bar{z}=+0} = \bar{w}(\bar{x}, +0) = \frac{k}{\beta^3}\, (-\alpha+z_c'(x) + z_\tau'(x)) = \bar{Z}_+'(\bar{x})$$

$$(\bar{\phi}_{\bar{z}})_{\bar{z}=-0} = \bar{w}(\bar{x}, -0) = \frac{k}{\beta^3}\, (-\alpha+z_c'(x) - z_\tau'(x)) = \bar{Z}_-'(\bar{x}) \tag{11}$$

The functions $\bar{Z}_+(\bar{x})$, $\bar{Z}_-(\bar{x})$ are introduced for convenience and imply the upper and lower surfaces of a modified profile. Following the previous convention

$$\bar{Z}_+(\bar{x}) = -\bar{A}\bar{x} + \bar{Z}_c(\bar{x}) + \bar{Z}_\tau(\bar{x})$$
$$\bar{Z}_-(\bar{x}) = -\bar{A}\bar{x} + \bar{Z}_c(\bar{x}) - \bar{Z}_\tau(\bar{x}) \quad (12)$$

where, from Eqs. (11)

$$\bar{A} = \frac{k}{\beta^3}\alpha \ , \ \bar{Z}_c(\bar{x}) = \frac{k}{\beta^3} z_c(x) \ , \ \bar{Z}_\tau(\bar{x}) = \frac{k}{\beta^3} z_\tau(x)$$

Since

$$1 - M^2(x,z) = 1 - M_\infty^2 - ku(x,z) = (1-M_\infty^2)(1 - \bar{u}(\bar{x},\bar{z})),$$

where M(x,z) is the local Mach number, then

$$\frac{1 - M^2(\bar{x},\bar{z})}{1 - M_\infty^2} = 1 - \bar{u}(\bar{x},\bar{z})$$

so the sign of $(1-\bar{u})$ indicates whether the local flow is subsonic or supersonic; Eq. (10) is elliptic for $\bar{u} < 1$ and hyperbolic for $\bar{u} > 1$.

Eq. (10) can be inverted to an integral form by using Green's theorem.

Thus, as shown in the Appendix,

$$\bar{\phi}(\bar{x},\bar{z}) = \frac{1}{2\pi}\int_0^1 \left[\psi(\bar{x},\bar{\xi};\bar{z},0)\Delta\bar{\phi}_{\bar{\zeta}}(\bar{\xi}) - \psi_{\bar{\zeta}}(\bar{x},\bar{\xi};\bar{z},0)\Delta\bar{\phi}(\bar{\xi})\right] d\bar{\xi}$$
$$- \frac{1}{4\pi}\iint_S \psi_{\bar{\xi}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\,\bar{u}^2(\bar{\xi},\bar{\zeta})\,dS \quad (14)$$

where

$$\psi(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = \ln\left[(\bar{x}-\bar{\xi})^2 + (\bar{z}-\bar{\zeta})^2\right]^{\frac{1}{2}} \quad (15)$$

and

$$\Delta\bar{\phi}(\bar{\xi}) = \bar{\phi}(\bar{\xi}, +0) - \bar{\phi}(\bar{\xi}, -0)$$

$$\Delta\bar{\phi}_{\bar{\zeta}}(\bar{\xi}) = \bar{\phi}_{\bar{\zeta}}(\bar{\xi}, +0) - \bar{\phi}_{\bar{\zeta}}(\bar{\xi}, -0)$$

and S is the domain of integration shown in Fig. 2.

The surface integral in Eq. (14) is defined for $\bar{z} > 0$ as

$$\iint_S F dS = \lim_{\varepsilon \to 0} \left\{ \int_{-\infty}^{\bar{x}-\varepsilon} \left( \int_0^{\infty} F d\bar{\zeta} \right) d\bar{\xi} + \int_{\bar{x}+\varepsilon}^{\infty} \left( \int_0^{\infty} F d\bar{\zeta} \right) d\bar{\xi} + \int_{-\infty}^{+\infty} \left( \int_{-\infty}^{0} F d\bar{\zeta} \right) d\bar{\xi} \right.$$

$$\left. + \int_{\bar{x}-\varepsilon}^{\bar{x}+\varepsilon} \left[ \int_0^{\bar{z}- \varepsilon^2-(\bar{x}-\bar{\xi})^2 \frac{1}{2}} F d\bar{\zeta} + \int_{\bar{z}+ \varepsilon^2-(\bar{x}-\xi)^2 \frac{1}{2}}^{\infty} F d\bar{\zeta} \right] d\bar{\xi} \right\} \tag{16a}$$

while for $\bar{z} < 0$

$$\iint_S F dS = \lim_{\varepsilon \to 0} \left\{ \int_{-\infty}^{+\infty} \left( \int_0^{\infty} F d\bar{\zeta} \right) d\bar{\xi} + \int_{-\infty}^{\bar{x}-\varepsilon} \left( \int_{-\infty}^{0} F d\bar{\zeta} \right) d\bar{\xi} + \int_{x+\varepsilon}^{\infty} \int_{-\infty}^{0} F d\bar{\zeta}) d\bar{\xi} \right.$$

$$\left. + \int_{\bar{x}-\varepsilon}^{\bar{x}+\varepsilon} \left[ \int_{-\infty}^{\bar{z}- \left[\varepsilon^2-(\bar{x}-\bar{\xi})^2\right]^{\frac{1}{2}}} F d\bar{\zeta} + \int_{\bar{z}+\left[\varepsilon^2-(\bar{x}-\bar{\xi})^2\right]^{\frac{1}{2}}}^{0} F d\bar{\zeta} \right] d\bar{\xi} \right\} \tag{16b}$$

Eq. (14) is very similar to the equation obtained by Spreiter and Alksne[1] in their integral equation method. In their work the acceleration potential, $\bar{u}\ (\bar{x},\bar{z})$, is used instead of the velocity potential, $\bar{\phi}(\bar{x},\bar{z})$, and the surface integral is defined as

$$\iint_S F dS = \lim_{\varepsilon \to 0} \left\{ \int_0^{\infty} \left( \int_{-\infty}^{\bar{x}-\varepsilon} F d\bar{\xi} \right) d\bar{\zeta} + \int_0^{\infty} \left( \int_{\bar{x}+\varepsilon}^{\infty} F d\bar{\xi} \right) d\bar{\zeta} + \int_{-\infty}^{0} \left( \int_{-\infty}^{+\infty} F d\bar{\xi} \right) d\bar{\zeta} \right\} \tag{17a}$$

for $\bar{z} > 0$, and

$$\iint_S F dS = \lim_{\varepsilon \to 0} \left\{ \int_0^{\infty} \left( \int_{-\infty}^{+\infty} F d\bar{\xi} \right) d\bar{\zeta} + \int_{-\infty}^{0} \left( \int_{-\infty}^{\bar{x}-\varepsilon} F d\bar{\xi} \right) d\bar{\zeta} + \int_{-\infty}^{0} \left( \int_{\bar{x}+\varepsilon}^{\infty} F d\bar{\xi} \right) d\bar{\zeta} \right\} \tag{17b}$$

for $\bar{z} < 0$.

The difference between the definitions of Eq. (16) and Eq. (17) is in the manner of exclusion of the singularity at $(\bar{x}, \bar{z})$ from the domain S; Eq. (16) excludes $(\bar{x}, \bar{z})$ by a circle of small radius whereas Eq. (17) excludes it by an infinite strip of small width. As seen later this difference leads to some significant results.

The usual linearisation approximation implies that the non-linear double integral in Eq. (14) is neglected. On differentiating this linear form first with respect to $\bar{x}$ and then with respect to $\bar{z}$, and applying the tangency flow boundary condition, the two usual independent equations are formulated for the symmetric and antisymmetric components. A similar procedure is now followed for the non-linear form in Eq. (14).

On differentiation with respect to $\bar{x}$, Eq. (14) gives

$$\bar{\phi}_{\bar{x}}(\bar{x},\bar{z}) = \frac{1}{2\pi}\int_0^1 \{ \frac{(\bar{x}-\bar{\xi})\Delta\ \bar{\phi}_{\bar{\zeta}}(\bar{\xi})}{(\bar{x}-\bar{\xi})^2+\bar{z}^2} - \frac{2\ \bar{z}(\bar{x}-\bar{\xi})}{\left[(\bar{x}-\bar{\xi})^2+\bar{z}^2\right]^2}\ \Delta\bar{\phi}(\bar{\xi})\}d\bar{\xi}$$

$$+ \frac{\bar{u}^2(\bar{x},\bar{z})}{4} - \frac{1}{4\pi}\iint_S \psi_{\bar{\xi}\bar{x}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\bar{u}(\bar{\xi},\bar{\zeta})dS. \tag{18}$$

The term $\frac{\bar{u}^2(\bar{x},\bar{z})}{4}$ arises from the differentiation of the limits around the singular point $(\bar{x},\bar{z})$ in the double integral.

On differentiation with respect to $\bar{z}$, Eq. (14) gives

$$\bar{\phi}_{\bar{z}}(\bar{x},\bar{z}) = \frac{1}{2\pi}\int_0^1 \{ \frac{\bar{z}\ \Delta\bar{\phi}_{\bar{\zeta}}(\bar{\xi})}{\left[(\bar{x}-\bar{\xi})^2+\bar{z}^2\right]} - \frac{\left[\bar{z}^2-(\bar{x}-\bar{\xi})^2\right]}{\left[(\bar{x}-\bar{\xi})^2+\bar{z}^2\right]^2}\ \Delta\bar{\phi}(\bar{\xi})\}d\bar{\xi}$$

$$- \frac{1}{4\pi}\iint_S \psi_{\bar{\xi}\bar{z}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\bar{u}^2(\bar{\xi},\bar{\zeta})dS \tag{19}$$

Both the limit as $\bar{z}\rightarrow +0$ and the limit as $\bar{z}\rightarrow -0$ of Eq. (18) are now taken, and on addition of these limiting forms

$$\bar{u}(\bar{x},+0) + \bar{u}(\bar{x}-0) - \frac{\left[\bar{u}^2(\bar{x},+0) + \bar{u}^2(\bar{x},-0)\right]}{4}$$

$$= \frac{1}{\pi}\int_0^1 \frac{\Delta\bar{w}(\bar{\xi})}{(\bar{x}-\bar{\xi})} - \lim_{\bar{z}\to+0} \frac{1}{4\pi}\iint_S \psi_{\bar{\xi}\bar{x}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\left[\bar{u}^2(\bar{\xi},\bar{\zeta})+\bar{u}^2(\bar{\xi},-\bar{\zeta})\right]dS \tag{20}$$

As $\bar{z}\rightarrow \pm 0$ the limiting procedure with the appropriate formula Eq. (16a) or Eq. (16b) must be used.

The double integral in Eq. (20) arises because on changing the sign of $\bar{z}$ and $\bar{\zeta}$

$$\lim_{\bar{z}\to -0} \iint_S \psi_{\bar{\xi}\bar{x}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\bar{u}^2(\bar{\xi},\bar{\zeta})dS = \lim_{\bar{z}\to +0} \iint_S \psi_{\bar{\xi}\bar{x}}(\bar{x},\bar{\xi};-\bar{z},-\bar{\zeta})\bar{u}^2(\bar{\xi},-\bar{\zeta})dS$$

$$= \lim_{z\to +0} \iint_S \psi_{\bar{\xi}\bar{x}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\ \bar{u}^2(\bar{\xi},-\bar{\zeta})dS$$

because $\psi_{\bar{\xi}\bar{x}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})$ is a function of $(\bar{z}-\bar{\zeta})^2$.

A similar operation can be performed on Eq. (19) which, after integrating the single integral by parts, gives

$$\bar{w}(\bar{x},+0) + \bar{w}(\bar{x},-0) = -\frac{1}{\pi}\int_0^1 \frac{[\bar{u}(\bar{\xi},+0) - \bar{u}(\bar{\xi},-0)]}{(\bar{x}-\bar{\xi})}\, d\bar{\xi}$$

$$- \lim_{z\to +0} \frac{1}{4\pi} \iint \psi_{\bar{\xi}\bar{z}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\left[\bar{u}^2(\bar{\xi},\bar{\zeta})-\bar{u}^2(\bar{\xi},-\bar{\zeta})\right] dS \tag{21}$$

Eqs. (20, 21) express two fundamental exact relationships for two-dimensional high subsonic flows. If the non-linear terms are negligible these equations reduce to the standard linearised integral equations, replacing $\bar{w}(\bar{x},+0)$ and $\bar{w}(\bar{x},-0)$ by the usual boundary conditions from Eqs. (11, 12).

The non-linear terms in Eqs. (20, 21) can be regarded as correction terms to the standard linearised equations which are valid for

$$\bar{u}(\bar{\xi},\bar{\zeta}) << 1$$

At high subsonic Mach numbers $\bar{u}(\bar{\xi},\bar{\zeta})$ is not small compared to unity and it is known that the pressure distribution derived from the linearised theory is of the order of 25% in error. If the correction terms can be found to the same order of accuracy as the linearised solution then the ultimate solution should be accurate to within at least 5% which is a significant improvement on the linearised theory; optimistically one may hope for even better accuracy.

## 2.1 Approximating Functions

In the double integral in Eq. (20) the kernel function

$$\psi_{\bar{\xi}\bar{x}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = \frac{\left[(\bar{x}-\bar{\xi})^2 - (\bar{z}-\bar{\zeta})^2\right]}{\left[(\bar{x}-\bar{\xi})^2 + (\bar{z}-\bar{\zeta})^2\right]^2}$$

tends to zero as $|\bar{\zeta}|^{-2}$ as $|\bar{\zeta}|\to\infty$. This result suggests that if the variation of $\bar{u}(\bar{\xi},\bar{\zeta})$ is approximated in the double integral by a function that is exact for small $\bar{\zeta}$ then the total integral may be reasonably accurate since any error in the approximation away from the surface is offset by the rapidly vanishing kernel function.

The approximating functions are chosen so that the variation of $\bar{u}(\bar{\xi},\bar{\zeta})$ with $\bar{\zeta}$ is expressed by a known function of $\bar{\zeta}$, the value of $\bar{u}(\bar{\xi},\bar{\zeta})$ on the aerofoil surface and the aerofoil geometry. The functions are assumed to give an exact representation of $\bar{u}(\bar{\xi},\bar{\zeta})$ for small $\bar{\zeta}$ if after expansion as a power series in $\bar{\zeta}$ to a specified number of terms this expansion is identical to the McLaurin's series expansion of $\bar{u}(\bar{\xi},\bar{\zeta})$ itself in powers of $\bar{\zeta}$. The choice of approximating function is further restricted by imposing the condition that the qualitative behaviour of $\bar{u}(\bar{\xi},\bar{\zeta})$ as $|\bar{\zeta}|\to\infty$ must be adequately represented. For a non-lifting aerofoil $\bar{u}(\bar{\xi},\bar{\zeta})$ is assumed to approach zero as $|\bar{\zeta}|\to\infty$ like $|\bar{\zeta}|^{-2}$; for a lifting aerofoil $\bar{u}(\bar{\xi},\bar{\zeta})$ is assumed to approach zero as $|\bar{\zeta}|\to\infty$ like $|\bar{\zeta}|^{-1}$.

It is possible to devise quite elaborate approximation functions that give a fairly good representation of the variation of $\bar{u}(\bar{\xi},\bar{\zeta})$ with $\bar{\zeta}$, see for example that of Gullstrand[9], but in the present analysis simple (and less accurate) functions are used since it is shown later that because of the arrangement of the fundamental equations used the final solution is not significantly affected by a moderate decrease in the accuracy of the integrals.

The McLaurin's expansion for $\bar{u}(\bar{\xi},\bar{\zeta})$ for $\bar{\zeta} \geqslant 0$ is

$$\bar{u}(\bar{\xi},\bar{\zeta}) = \bar{u}(\bar{\xi}, +0) + \bar{\zeta}\, \bar{u}_{\bar{\zeta}}(\bar{\xi}, +0) + \ldots.. \tag{22a}$$

Since

$$\bar{u}_{\bar{\zeta}}(\bar{\xi}, +0) = \bar{w}_{\bar{\xi}}(\bar{\xi}, +0)$$

for irrotational flow, then on using the boundary condition Eq. (11), Eq. (22a) can be written for $0 \leqslant \bar{\xi} \leqslant 1$ as

$$\bar{u}(\bar{\xi},\bar{\zeta}) = \bar{u}(\bar{\xi}, +0) + \bar{\zeta}\, \bar{Z}_+''(\bar{\xi}) + \ldots \tag{22b}$$

Since interest is to be concerned with conditions at the wing surface it is argued that it is necessary to represent $\bar{u}(\bar{\xi},\bar{\zeta})$ with more accuracy in the neighbourhood of the wing surface than in front or aft of the wing where in any case $\bar{u}(\bar{\xi},\bar{\zeta})$ will be small. Thus the variation in front or aft of the wing is neglected; it is not anticipated that this assumption will lead to any serious error.

For a lifting aerofoil the approximating function is chosen to be, for $\bar{\zeta} \geqslant 0$,

$$\bar{u}(\bar{\xi},\bar{\zeta}) = \frac{\bar{u}(\bar{\xi}, +0)}{\left[1 - \dfrac{\bar{\zeta}\, \bar{Z}_+''(\bar{\xi})}{\bar{u}(\bar{\xi}, +0)}\right]} \tag{23a}$$

This function has the correct qualitative behaviour as $|\bar{\zeta}| \to \infty$ and if expanded in powers of $\bar{\zeta}$ for small $\bar{\zeta}$ is identical to Eq. (22b) to first order in $\bar{\zeta}$. For $\bar{\zeta} \leqslant 0$

$$\bar{u}(\bar{\xi},\bar{\zeta}) = \frac{\bar{u}(\bar{\xi}, -0)}{\left[1 - \dfrac{\bar{\zeta}\, \bar{Z}_-''(\bar{\xi})}{\bar{u}(\bar{\xi}, -0)}\right]} \tag{23b}$$

For a non-lifting aerofoil the approximating function is, for $\bar{\zeta} \geqslant 0$,

$$\bar{u}(\bar{\xi},\bar{\zeta}) = \frac{\bar{u}(\bar{\xi}, +0)}{\left[1 - \dfrac{\bar{\zeta}\, \bar{Z}_+''(\bar{\xi})}{2\, \bar{u}(\bar{\xi}, +0)}\right]^2} \tag{24a}$$

while for $\bar{\zeta} \leqslant 0$

$$\bar{u}(\bar{\xi},\bar{\zeta}) = \frac{\bar{u}(\bar{\xi}, -0)}{\left[1 - \dfrac{\bar{\zeta}\, \bar{Z}_-''(\bar{\xi})}{2\, \bar{u}(\bar{\xi}, -0)}\right]^2} \tag{24b}$$

## 2.2 Derivation of the Integral Equations

If the approximation for $\bar{u}(\bar{\xi},\bar{\zeta})$ given by Eq. (23) is substituted into the double integrals of Eq. (20) and then integrated with respect to $\bar{\zeta}$ using Eq. (16) then Eq. (20) can be written as

$$\frac{1}{2}\left[\bar{u}(\bar{x}, +0) + \bar{u}(\bar{x}, -0)\right] - \frac{\left[\bar{u}^2(\bar{x}, +0) + \bar{u}^2(\bar{x}, -0)\right]}{8}$$

$$= \frac{1}{2\pi}\int_0^1 \frac{\Delta\, \bar{\phi}_{\bar{\zeta}}(\bar{\xi})}{(\bar{x}-\bar{\xi})}\, d\bar{\xi} + I_\tau(\bar{x},\bar{u}(\bar{x},\pm 0)) \tag{25}$$

where

$$I_\tau(\bar{x},\bar{u}(\bar{x}, 0)) = \frac{\left[\bar{u}^2(\bar{x}, +0) + \bar{u}^2(\bar{x}, -0)\right]}{8}$$

$$+ \frac{1}{4\pi}\int_0^1 \{\bar{u}(\bar{\xi},+0)\, \bar{Z}_+''(\bar{\xi})\, E_T\left[\frac{-\bar{Z}_+''(\bar{\xi})(\bar{x}-\bar{\xi})}{\bar{u}(\bar{\xi},+0)}\right]$$

$$- \bar{u}(\bar{\xi}, -0)\, \bar{Z}_-''(\bar{\xi})\, E_T\left[\frac{-\bar{Z}_-''(\bar{\xi})(\bar{x}-\bar{\xi})}{\bar{u}(\bar{\xi},-0)}\right]\}d\bar{\xi} \tag{26}$$

and

$$E_T(x) = \frac{1}{(1+x^2)}\{2(3x^2-1)\ln|x| + (1+x^2)(x^2-3)-(x^2-3)|x|\pi\} \tag{27}$$

On applying a similar procedure to Eq.(21)

$$\frac{1}{2}\left[\bar{w}(\bar{x}, +0) + \bar{w}(\bar{x}, -0)\right] = \frac{-1}{2\pi}\int_0^1 \frac{\left[\bar{u}(\bar{\xi},+0) - \bar{u}(\bar{\xi}, -0)\right]}{(\bar{x} - \bar{\xi})}\, d\bar{\xi}$$

$$+ \frac{1}{8\pi}\int_0^1 \frac{\left[\bar{u}^2(\bar{\xi},+0) - \bar{u}^2(\bar{\xi},-0)\right]}{(x - \bar{\xi})}\, d\bar{\xi} + I_c(\bar{x},\bar{u}(\bar{x},\pm 0)) \tag{28}$$

where

$$I_c(\bar{x},\bar{u}(\bar{x},\pm 0)) = \frac{1}{8\pi}\int_0^1 \frac{|\bar{u}^2(\bar{\xi},+0) - \bar{u}^2(\bar{\xi},-0)|}{(\bar{x} - \bar{\xi})} d\bar{\xi}$$

$$+ \frac{1}{4\pi}\int \{\bar{u}(\bar{\xi},+0)\ \bar{Z}_+''(\bar{\xi})\ E_c\left[\frac{+\bar{Z}_+''(\bar{\xi})(\bar{x}-\bar{\xi})}{\bar{u}(\bar{\xi},+0)}\right]$$

$$+ \bar{u}(\bar{\xi},-0)\ \bar{Z}_-''(\bar{\xi})\ E_c\left[\frac{-\bar{Z}_-''(\bar{\xi})(\bar{x}-\bar{\xi})}{\bar{u}(\bar{\xi},-0)}\right]\} d\bar{\xi} \qquad (29)$$

and

$$E_c(x) = \frac{1}{(1+x^2)^3}\{(1-3x^2)\pi \mathrm{sqn}(\bar{x}-\bar{\xi}) + x(1+x^2)(5+x^2) + 2x(3-x^2)\ \ln|x|\ \} \qquad (30)$$

It is shown later that the effect of the integrals $I_\tau(\bar{x},\bar{u}(\bar{x},\pm 0))$ and $I_c(\bar{x},\bar{u}(\bar{x},\pm 0))$ is small compared to the effect of the remaining non-linear terms and thus the integrals can be regarded as correction terms.

For a non-lifting aerofoil the approximation function of Eq.(24) should be used, and $I_\tau(\bar{x},\bar{u}(\bar{x},\pm 0)$ is given by Eq.(26) but with $E_T(x)$ replaced by $E_{T1}(\frac{x}{2})$ where

$$E_{T1}(x) = \frac{1}{(1+x^2)^5}\{\frac{(1+x^2)(x^6+x^4+71\ x^2-25)}{6}$$

$$+ (5-10x^2+x^4)|x|\pi - 2(5x^4-10x^2+1)\ln|x|\ \} \qquad (31)$$

Also, since for a non-lifting aerofoil

$$I_c(\bar{x},\bar{u}(\bar{x},\pm 0)) = 0\ ,$$

both sides of Eq. (28) are then identically zero.

The functions $E_T(x)$ and $E_{T1}(\frac{x}{2})$ are shown in Fig. 3 and it can be seen that $E_T(x)$ does not differ much from $E_{T1}(\frac{x}{2})$, thus strengthening the assumption that only the variation of $\bar{u}(\bar{\xi},\bar{\zeta})$ for small $\bar{\zeta}$ is irportant.

If $I_c(\bar{x},\bar{u}(\bar{x},\pm 0))$ is assumed known then Eq.(28) can be inverted by the standard formula to give

$$\left[\bar{u}(\bar{x},+0) - \bar{u}(\bar{x},-0)\right] - \frac{\left[\bar{u}^2(\bar{x},+0) - \bar{u}^2(\bar{x},-0)\right]}{4}$$

$$= \frac{1}{\pi}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} \int_0^1 \frac{\left[\bar{w}(\bar{\xi},+0) + \bar{w}(\bar{\xi},-0)\right]}{(\bar{x}-\bar{\xi})}\left(\frac{\bar{\xi}}{1-\bar{\xi}}\right)^{\frac{1}{2}} d\bar{\xi}$$

$$- \frac{2}{\pi}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} \int_0^1 \frac{I_c(\bar{\xi},\bar{u}(\bar{\xi},\pm 0))}{(\bar{x}-\bar{\xi})}\left(\frac{\bar{\xi}}{1-\bar{\xi}}\right)^{\frac{1}{2}} d\bar{\xi} \qquad (32)$$

Eqs. (25, 32) can be combined to give

$$\bar{u}(\bar{x},\pm 0) - \frac{\bar{u}^2(\bar{x},\pm 0)}{4} = \frac{1}{2\pi}\int_0^1 \frac{\Delta\phi_{\bar{\zeta}}(\bar{\xi})}{(\bar{x}-\bar{\xi})} d\bar{\xi} + I(\bar{x},\pm 0)$$

$$\pm \frac{1}{2\pi}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} \int_0^1 \frac{\left[\bar{w}(\bar{\xi},+0) + \bar{w}(\bar{\xi},-0)\right]}{(\bar{x}-\bar{\xi})}\left(\frac{\bar{\xi}}{1-\bar{\xi}}\right)^{\frac{1}{2}} d\bar{\xi} \qquad (33)$$

where

$$I(\bar{x},\pm 0) = I_\tau(\bar{x},\bar{u}(\bar{x},\pm 0)) \mp \frac{1}{\pi}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} \int_0^1 \frac{I_c(\bar{\xi},\bar{u}(\bar{\xi},\pm 0))}{(\bar{x}-\bar{\xi})}\left(\frac{\bar{\xi}}{1-\bar{\xi}}\right)^{\frac{1}{2}} d\bar{\xi} \qquad (34)$$

The "+" and "-" signs denote values on the upper and lower surfaces respectively.

The linearised solution can be obtained from Eq.(34) by neglecting all second order terms and applying the linearised boundary conditions Eq. (11, 12). If the linearised solution for $\bar{u}(\bar{x},\pm 0)$ is denoted by $\bar{u}_\ell(\bar{x},\pm 0)$ then

$$\bar{u}_\ell(\bar{x},\pm 0) = \frac{1}{\pi}\int_0^1 \frac{\bar{Z}_T(\bar{\xi})}{(\bar{x}-\bar{\xi})} d\bar{\xi} \pm \left\{\bar{A}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} + \frac{1}{\pi}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} \int_0^1 \frac{\bar{Z}_c'(\bar{\xi})}{(\bar{x}-\bar{\xi})}\left(\frac{\bar{\xi}}{1-\bar{\xi}}\right)^{\frac{1}{2}} d\bar{\xi}\right\} \qquad (35)$$

Using Eq.(35), Eq.(33) can be written in the alternative form

$$\bar{u}(\bar{x},\pm 0) - \frac{\bar{u}^2(\bar{x},\pm 0)}{4} = \bar{u}_\ell(\bar{x},\pm 0) + I(\bar{x},\pm 0) \qquad (36)$$

## 2.3 Some Second Order Corrections

One of the features of linearised theory is that the solution $\bar{u}_{\ell}(\bar{x},\pm 0)$ given by Eq.(35), is singular at the leading edge; the inversion integral used in the derivation of $I(\bar{x},\pm 0)$ in Eq.(34) is also singular and thus the solution $\bar{u}(\bar{x},\pm 0)$ obtained from Eq.(36) will be singular at the aerofoil nose. But $\bar{u}(\bar{x},\pm 0)$ will no longer contain the conventional leading edge singularity (i.e. ~ $\underline{0}(\bar{x}^{-\frac{1}{2}})$) for it will be changed to something more like $\underline{0}(\bar{x}^{-\frac{1}{4}})$

The fact that the fundamental equations for this study are non-linear implies that a more accurate solution to that given by linearised theory should be sought in the region of the leading edge, preferably without any singular behaviour.

Attention is therefore turned temporarily to the linearised form of the basic differential equation, Eq.(10), namely

$$\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{z}\bar{z}} = 0 \qquad (37)$$

The exact boundary conditions given by Eq.(4) can be expressed in the transformed variables of Eq.(9) as

$$\frac{\bar{w}(\bar{x},\bar{z}_{su}(\bar{x}))}{\frac{k}{\beta^2} + \bar{u}(\bar{x},\bar{z}_{su}(\bar{x}))} = \frac{\bar{z}'_{su}(\bar{x})}{\beta^2}$$

$$\frac{\bar{w}(\bar{x},\bar{z}_{sl}(\bar{x}))}{\frac{k}{\beta^2} + \bar{u}(\bar{x},\bar{z}_{sl}(\bar{x}))} = \frac{\bar{z}'_{sl}(\bar{x})}{\beta^2} \qquad (38)$$

where

$$\bar{z}_{su}(\bar{x}) = \beta z_{su}(x) \quad \text{and} \quad \bar{z}_{sl}(\bar{x}) = \beta z_{sl}(x)$$

On using Eq.(38) the total resultant velocity $\bar{q}(\bar{x},\bar{z}_{su}(\bar{x}))$ on the upper surface of the wing is given by

$$\bar{q}(\bar{x},\bar{z}_{su}(\bar{x})) = \left[\frac{k}{\beta^2} + \bar{u}(\bar{x},\bar{z}_{su}(\bar{x}))\right]\left[1 + \left(\frac{\bar{z}_{su}(\bar{x})}{\beta^2}\right)^2\right]^{\frac{1}{2}} \qquad (39)$$

The exact flow remains finite everywhere so that at the nose

$$\bar{u}(\bar{x},\bar{z}_{su}(\bar{x})) \to -\frac{k}{\beta^2} \quad \text{as} \quad \bar{z}'_{su}(\bar{x}) \to \infty$$

Eq.(39) can be written in the form

$$\bar{q}(\bar{x},\bar{z}_{su}(\bar{x})) = \frac{\left[\frac{k}{\beta^2} + \bar{u}(\bar{x},\bar{z}_{su}(\bar{x}))\right]\left[1 + \left(\frac{\bar{z}'_{su}(\bar{x})}{\beta^2}\right)\right]}{\left[1 + \left(\frac{\bar{z}'_{su}(\bar{x})}{\beta^2}\right)^2\right]^{\frac{1}{2}}} \tag{40}$$

where both the numerator and the denominator tend to infinity at the nose. Although $\bar{z}_{su}(\bar{x})$ is made up from contributions from thickness, camber and incidence, only thickness contributes to the infinity at the nose.

If the standard linearised solution $\bar{u}_{\ell}(\bar{x},+0)$ given by Eq.(35) is to be used as an approximation to $\bar{u}(\bar{x},\bar{z}_{su}(\bar{x}))$ and since this linearised solution already incorporates an infinity at the leading edge then an approximation to $\bar{q}(\bar{x},\bar{z}_{su}(\bar{x}))$ is taken to be

$$\bar{q}(\bar{x},\bar{z}_{su}(\bar{x})) = \frac{\left[\frac{k}{\beta^2} + \bar{u}_{\ell}(\bar{x},+0)\right]}{\left[1 + \left(\frac{\beta^2 \bar{Z}'_{\tau}(\bar{x})}{k}\right)^2\right]^{\frac{1}{2}}} \tag{41}$$

Eq.(41) ensures that $\bar{q}(\bar{x},\bar{z}_{su}(\bar{x}))$ becomes finite in the leading edge region; although it is not expected that the accuracy will be good it will at least be an improvement on the linearised infinity.

It is noted that only the thickness term is retained in the above correction term; this is done for two reasons. First it is only the thickness term which gives the necessary infinity at the leading edge; second it is necessary to preserve the same denominator for both the upper and lower surfaces to ensure that the Kutta trailing edge condition is satisfied, since $\bar{u}_{\ell}(\bar{x},\pm 0)$ already satisfies the Kutta condition.

An approximation to the resultant velocity may be written in the form

$$\bar{q}(\bar{x},\bar{z}_{su}(\bar{x})) = \frac{k}{\beta^2} + \bar{u}_L(\bar{x},+0)$$

$$\bar{q}(\bar{x},\bar{z}_{sl}(\bar{x})) = \frac{k}{\beta^2} + \bar{u}_L(\bar{x},-0) \tag{42}$$

where $\bar{u}_L(\bar{x},\pm 0)$ can be regarded as the linearised solution modified for leading edge corrections; $\bar{u}_\ell(\bar{x},\pm 0)$ is given by Eq.(35).

Combining Eqs.(41,42)

$$\bar{u}_L(\bar{x},\pm 0) = \frac{\left[\frac{k}{\beta^2} + \bar{u}_\ell(\bar{x},\pm 0)\right]}{\left[1 + \left(\frac{\beta^2 \bar{Z}'_\tau(\bar{x})}{k}\right)^2\right]^{\frac{1}{2}}} - \frac{k}{\beta^2} \tag{43}$$

If Eq.(43) is written in terms of the original variables then

$$u_L(x,\pm 0) = \frac{\left[1 + u_\ell(x,\pm 0)\right]}{\left|1 + \left(\frac{Z'_\tau(x)}{\beta}\right)^2\right|^{\frac{1}{2}}} - 1 \tag{44}$$

where

$$u_\ell(x,\pm 0) = \frac{\beta^2 \bar{u}_\ell(\bar{x},\pm 0)}{k} \tag{45}$$

and

$$u_L(x,\pm 0) = \frac{\beta^2 \bar{u}_L(\bar{x},\pm 0)}{k}$$

The first order term $u_\ell(x,\pm 0)$ given by Eqs.(35,45) is identical to the perturbation velocity found by using the Gothert rule. According to this rule the perturbation velocity $u_\ell(x,\pm 0)$ in compressible flow is related to the incompressible perturbation velocity $u_{\ell a}(x,\pm 0)$ on an analogous aerofoil, obtained from the original by shrinking all dimensions normal to the freestream by the factor $(1 - M_\infty^2)^{\frac{1}{2}}$, by

$$u_\ell(x,\pm 0) = \frac{u_{\ell a}(x,\pm 0)}{\beta^2} \tag{46}$$

Some estimate of the accuracy of the modified linearised velocity given by Eq.(43) can be made by examining the incompressible limit as $M_\infty \to 0$ since exact solutions are available for many incompressible flow problems.

The incompressible limit of Eq.(44) is

$$\lim_{M_\infty \to 0} u_L(x,\pm 0) = \frac{\left[1 + u_i(x,\pm 0)\right]}{\left[1 + (z_\tau'(x))^2\right]^{\frac{1}{2}}} \tag{47}$$

where

$$u_i(x,\pm 0) = \lim_{M_\infty \to 0} u_\ell(x,\pm 0) = \frac{1}{\pi}\int_0^1 \frac{z_\tau'(\xi)}{(x-\xi)}\,d\xi \pm \{\alpha(\frac{1-x}{x})^{\frac{1}{2}} + \frac{1}{\pi}(\frac{1-x}{x})^{\frac{1}{2}}\int_0^1 \frac{z_c'(\xi)}{(x-\xi)}(\frac{\xi}{1-\xi})^{\frac{1}{2}}d\xi\} \tag{48}$$

For aerofoils with camber and incidence the incompressible solution given by Eq.(48,48) is found to be inaccurate near the leading edge. The accuracy can be improved however by retaining some of the second order terms in the boundary conditon eq.(4). For incompressible flow the tangency boundary condition can be written to second order by using Eq.(5); thus

$$w(x,\ 0) = z_{su}'(x) + \frac{d}{dx}\left[u(x,+0)z_{su}(x)\right] + ----$$

$$w(x,-0) = z_{sl}'(x) + \frac{d}{dx}\left[u(x,-0)z_{sl}(x)\right] + ---- \tag{49}$$

using the condition that in incompressible flow

$$\frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} = 0$$

It is found[10][11] that only the symmetric components of the second order terms in Eq.(49) significantly affect the overall solution; a good approximation to the exact incompressible flow is given by

$$\lim_{M_\infty \to 0} u_L(x,\pm 0) = \frac{\{1+u_i(x,\pm 0) \pm \frac{1}{2\pi}(\frac{1-x}{x})^{\frac{1}{2}}\int_0^1 \frac{d}{d\xi}\frac{\left[u_i(\xi,+0)z_{su}(\xi)+u_i(\xi,-0)z_{sl}(\xi)\right]}{(x-\xi)}(\frac{\xi}{1-\xi})^{\frac{1}{2}}d\xi\}_{-1}}{\left[1 + (z_\tau'(x))^2\right]^{\frac{1}{2}}} \tag{50}$$

In Refs. (10, 11) it is assumed that in addition the second order terms due to camber can be neglected.

In order to improve the accuracy of the compressible solution the Gothert rule is now applied to the numerator in the expression for the incompressible $u_L(x,\pm 0)$, Eq.(50). Although strictly the Gothert rule should only be applied to the first order term it is assumed that the application to the second order term will improve the linearised subsonic solution since in the incompressible limit the revised subsonic solution at least gives an accurate representation of the flow. Thus, reverting to the transformed variables,

$$u_L(x,\pm 0) = \frac{\left\{\frac{k}{\beta^2} + \bar{u}_L(\bar{x},\pm 0) \pm \frac{1}{\pi\beta}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} \int_0^1 \frac{\Lambda(\bar{\xi})}{(\bar{x}-\bar{\xi})} \left(\frac{\bar{\xi}}{1-\bar{\xi}}\right)^{\frac{1}{2}} d\bar{\xi}\right\}}{\left[1 + \left(\frac{\beta^2 \bar{Z}_\tau'(\bar{x})}{k}\right)^2\right]^{\frac{1}{2}}} - \frac{k}{\beta^2} \qquad (51)$$

where

$$\Lambda(\bar{\xi}) = \frac{1}{2}\frac{d}{d\bar{\xi}}\left[\bar{u}\,(\bar{\xi},+0)\,\bar{z}_{su}(\bar{\xi}) + \bar{u}\,(\bar{\xi},-0)\bar{z}_{sl}(\bar{\xi})\right] \qquad (52)$$

## 2.4 Summary of Basic Equations

Returning to the basic equation, Eq. (36), incorporating now the various modifications

$$\bar{u}(\bar{x},\pm 0) - \frac{\bar{u}^2(\bar{x},\pm 0)}{4} = \bar{u}_L(\bar{x},\pm 0) + I(\bar{x},\pm 0) \qquad (53)$$

where $\bar{u}_L(\bar{x},\pm 0)$ is given by Eqs.(35,51,52) and $I(\bar{x},\pm 0)$ is given by

$$I(\bar{x},\pm 0) = I_\tau(\bar{x},\bar{u}(\bar{x},\pm 0)) \pm \frac{\left\{\frac{1}{\pi}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} \int_0^1 \frac{I_c(\bar{\xi},\bar{u}(\bar{\xi},\pm 0))}{(\bar{x}-\bar{\xi})} \left(\frac{\bar{\xi}}{1-\bar{\xi}}\right)^{\frac{1}{2}} d\bar{\xi}\right\}}{\left[1 + \left(\frac{\beta^2 \bar{Z}_\tau'(\bar{x})}{k}\right)^2\right]^{\frac{1}{2}}} \qquad (54)$$

where $I_\tau(\bar{x},\bar{u}(\bar{x},\pm 0))$ and $I_c(\bar{x},\bar{u}(\bar{x},\pm 0))$ are given by Eqs.(26,29) respectively.

The various components of $\bar{u}_{\ell}(\bar{x},\pm 0)$ can be evaluated either analytically or by using numerical techniques such as that of Weber[10,11]. $I_{\tau}(\bar{x},\bar{u}(\bar{x},\pm 0))$ and $I_C(\bar{x},\bar{u}(\bar{x},\pm 0))$ can be evaluated using straightforward numerical methods. The integral in Eq.(54) can be evaluated using the method of Weber[11] regarding

$$\int_0^{\bar{\xi}} I_C(\bar{\xi}',\bar{u}(\bar{\xi}',\pm 0))\, d\bar{\xi}'$$

as an effective camber distribution. After performing the necessary integration it may be shown that

$$\int_0^{\bar{x}} I_C(\bar{\xi},\bar{u}(\bar{\xi},\pm 0))\, d\bar{\xi} = \frac{1}{4\pi}\int_0^1 \left\{\bar{u}^2(\bar{\xi},+0)\left[E_W\left[\frac{-(\bar{x}-\bar{\xi})\bar{Z}_+''(\bar{\xi})}{\bar{u}(\bar{\xi},+0)}\right] - E_W\left[\frac{\bar{\xi}\bar{Z}_+''(\bar{\xi})}{\bar{u}(\bar{\xi},+0)}\right]\right]\right.$$
$$\left. - \bar{u}^2(\bar{\xi},-0)\left[E_W\left[\frac{-(\bar{x}-\bar{\xi})\bar{Z}_-''(\bar{\xi})}{\bar{u}(\bar{\xi},-0)}\right] - E_W\left[\frac{\bar{\xi}\bar{Z}_-''(\bar{\xi})}{\bar{u}(\bar{\xi},-0)}\right]\right]\right\} d\bar{\xi} \tag{55}$$

where

$$E_W(x) = \frac{1}{(1+x^2)^2}\{x^2(3+x^2)\ \ln|x| - (1+x^2) + \pi\ x\ \mathrm{sgn}(\bar{x}-\bar{\xi})\}$$

## 3. APPLICATIONS

The final equation for the perturbation velocity is Eq.(53). When $\bar{u}(\bar{x},\pm 0)$ is small then, neglecting the second order terms,

$$\bar{u}(\bar{x},\pm 0) = \bar{u}_L(\bar{x},\pm 0),$$

so that linearised theory is adequate. But at higher values of $\bar{u}(\bar{x},\pm 0)$ the second order terms are important.

A formal solution to Eq.(53) is

$$\bar{u}(\bar{x},\ 0) = 2\{1 - \left[1 - \bar{u}_L(\bar{x},\pm 0) - I(\bar{x},\pm 0)\right]^{\frac{1}{2}}\} \tag{56}$$

The negative sign is taken since $\bar{u}(\bar{x},\pm 0)$ must equal $\bar{u}_L(\bar{x},\pm 0)$ when $u(x,\pm 0)$ is small.

First it is of interest to study the order of magnitude of the second order term $I(\bar{x},\pm 0)$. The values of $I(\bar{x},\pm 0)$ have been computed for two cases, the NACA 0012 aerofoil at zero incidence and $M_\infty = 0.72$ and the NACA 0012 aerofoil at $2^\circ$ incidence at $M_\infty = 0.63$. The results are shown in Table (1) and Table (2) respectively.

Considering first the zero incidence case, it can be seen from Table 1 that over the front half of the wing, apart from the leading edge region, $I(\bar{x},+0)$ is small compared to $\frac{\bar{u}^2(\bar{x},+0)}{4}$; over the rear half of the wing $I(\bar{x},+0)$ is large compared with $\frac{\bar{u}^2(\bar{x},+0)}{4}$ but $\frac{\bar{u}^2(\bar{x},+0)}{4}$ itself is small compared to $\bar{u}(\bar{x},+0)$ so all second order terms are negligible in any case.

It can be seen from Table 2 that in the case of the aerofoil at incidence similar conclusions as those reached for the non-lifting case apply to the upper surface of the wing. On the lower surface of the wing $I(\bar{x},-0)$ is not small compared to $\frac{\bar{u}^2(\bar{x},-0)}{4}$ but $\frac{\bar{u}^2(\bar{x},-0)}{4}$ is itself small compared with $\bar{u}(\bar{x},-0)$ and the second order effects are then negligible.

Thus to a first approximation a simple relation between the perturbation velocity and the linearised solution is

$$\bar{u}(\bar{x},\pm 0) - \frac{\bar{u}^2(\bar{x},\pm 0)}{4} = \bar{u}_L(\bar{x},\pm 0)$$

The solution to this equation is

$$\bar{u}(\bar{x},\pm 0) = 2\left[1 - (1 - \bar{u}_L(\bar{x},\pm 0))^{\frac{1}{2}}\right] \tag{57}$$

To proceed to a second approximation, if the solution given by Eq.(57) is denoted by $\bar{u}_1(\bar{x},\pm 0)$ then a second approximation is given by

$$\bar{u}(\bar{x},\pm 0) = 2\left[1 - (1 - \bar{u}_L(\bar{x},\pm 0) - I_1(\bar{x},\pm 0))^{\frac{1}{2}}\right] \tag{58}$$

where

$$I_1(\bar{x},\pm 0) = I_\tau(\bar{x},\bar{u}_1(\bar{x},\pm 0)) \pm \frac{\left\{\frac{1}{\pi}\left(\frac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} \int_0^1 \frac{I_c(\bar{\xi},\bar{u}_1(\bar{\xi},\pm 0))}{(\bar{x}-\bar{\xi})} \left(\frac{\bar{\xi}}{1-\bar{\xi}}\right)^{\frac{1}{2}} d\bar{\xi}\right.}{\left[1 + \left(\frac{\beta^2\ \bar{Z}'_\tau(\bar{x})}{k}\right)^2\right]^{\frac{1}{2}}} \tag{59}$$

The pressure distribution can then be calculated using Eqs.(7,9,42) and either Eq.(57) or Eq.(58).

The pressure distributions from the first and second approximations have been calculated for the NACA 0012 aerofoil at zero incidence and $M_\infty = 0.72$ and for the NACA 0012 aerofoil at $2^\circ$ incidence and $M_\infty = 0.63$; these are shown in Fig. 4 and Fig. 5 respectively.

It is seen that in both cases the first approximation leads to an improvement on modified linearised theory except in the immediate neighbourhood of the leading edge. The results from the second approximation are in good agreement with the exact results of Sells[12].

The lift coefficient, $C_L$, and the pitching moment coefficient $C_M$ about the leading edge have been calculated from both the first and the second approximations; these are compared in Table 3 to the values obtained by the method of Sells[12]. Also shown are the coefficients obtained from the linearised velocity distribution $\bar{u}_\ell(\bar{x},\pm 0)$ (Gothert) and from the modified linearised velocity distribution $\bar{u}_L(\bar{x},\pm 0)$ (modified Gothert). The aerodynamic centre has also been determined. In the calculation of the linearised (Gothert) coefficients the first approximation to the pressure coefficient given by Eq.(8) is used.

The second approximation gives values that are in good agreement with those of Sells[12]. The first approximation does not give a significant improvement over the results using the modified linear theory; this is almost completely due to the errors in the pressure distribution in the neighbourhood of the leading edge.

## 5. CONCLUDING REMARKS

An integral method has been developed which extends and uses the results of linearised wing theory for the calculation of the pressure distribution on lifting and non-lifting aerofoils at high subsonic speeds in shock free flow. The reduction of the fundamental equation to integral equation form requires some assumptions on the behaviour of the perturbation field away from the aerofoil surface but it is shown that the resulting equations are rather insensitive to the form of the assumptions. Because the problem is non-linear, two integral equations are derived in which the usual symmetric and antisymmetric perturbation velocity distributions are coupled.

Simple first approximation solutions are derived, which merely involve the linearised thickness and camber velocity distribution, and which seem to give reasonable accuracy. Improved accuracy can be obtained from a second approximation.

The basic idea is not new, Spreiter and Alksne(1) developed a similar approach for the non-lifting case. However, the method presented here seems to have certain advantages over this previous work in that a fairly accurate first approximation has been found. The reason arises essentially from the method of dealing with a singularity; in the method presented here a singular point is excluded from a space by a small circle of radius r, which is allowed to go to zero, whereas Spreiter et al excluded the same singularity by an infinite strip whose width is allowed to go to zero. Consequently the different expressions which subsequently appear in the analysis allow a convenient simplified formula to be developed.

The basic approach presented in this paper is capable of extension of finite wings and unsteady flow problems and the results presented give some confidence that any extension will produce equally satisfactory results.

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## APPENDIX

If $\Omega(\bar{x},\bar{z})$ and $\psi(\bar{x},\bar{z})$ are two continuous functions with continuous first and second derivatives in some domain S bounded by a curve C, then Green's Theorem states

$$\iint_S \left[\psi\nabla^2\Omega - \Omega\nabla^2\psi\right]ds = -\int_C \left[\psi\,\frac{\partial\Omega}{\partial n} - \Omega\,\frac{\partial\psi}{\partial n}\right]dC \tag{A1}$$

where $\nabla^2$ is the Laplacian operator $(\frac{\partial^2}{\partial\bar{x}^2} + \frac{\partial}{\partial\bar{z}^2})$, and where n is the inward normal to the curve C, around which the integration must be taken in an anticlockwise direction.

In Eq.(A1) $\Omega$ is to be identified with the perturbation velocity potential $\bar{\phi}$. And $\psi$ is chosen as the elementary source solution of Laplace's equation

$$\nabla^2\psi = 0$$

that is

$$\psi(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = \ln\left[(\bar{x}-\bar{\xi})^2 + (\bar{z}-\bar{\zeta})^2\right]^{\frac{1}{2}} \tag{A2}$$

where $(\bar{\xi},\bar{\zeta})$ are running co-ordinates in the $\bar{x}$ and $\bar{z}$ direction respectively.

Now $\psi(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})$ is singular at the point $(\bar{x},\bar{z})$. And, to the present order of approximation, $\bar{\phi}(\bar{\xi},\bar{\zeta})$ and its derivatives can be discontinuous across the slit $(0 \leqslant \bar{\xi} \leqslant 1,\ \bar{\zeta} = \pm 0)$ on the $\bar{\xi}$ axis where the boundary conditions are to be applied. Since $\psi(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})$ and $\bar{\phi}(\bar{\xi},\bar{\zeta})$ and their derivatives must be continuous throughout the domain S, the point $(\bar{x},\bar{z})$ and the slit $(0 \leqslant \bar{\xi} \leqslant 1,\ \bar{\zeta} = \pm 0)$ must be excluded from S. This domain S is shown in Fig.2.

On substitution of Eq.(10) and Eq.(A2), Eq.(A1) becomes

$$\iint_S \psi(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\ \bar{\phi}_{\bar{\xi}}(\bar{\xi},\bar{\zeta})\ \bar{\phi}_{\bar{\xi}\bar{\xi}}(\bar{\xi},\bar{\zeta})\ dS$$

$$= -\int_{C_1+C_W+C_\infty}\left[\psi(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\ \frac{\partial\bar{\phi}}{\partial n} - \bar{\phi}(\bar{\xi},\bar{\zeta})\ \frac{\partial\psi}{\partial n}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\right]dC \tag{A3}$$

where

$C_1$ is that part of the boundary C surrounding the point $(\bar{x},\bar{z})$ and is taken to be a small circle of radius $\varepsilon$,

$C_W$ is that part of the boundary C surrounding the $\bar{\xi}$ axis, and

$C_\infty$ completes the boundary C thus denoting the outer limiting boundary of S, taken to be a large circle, centre at the origin, of radius R.

These boundaries, $C_1$, $C_W$, $C_\infty$, together with the sense of integration, are shown in Fig. 2.

The limiting operations as $\varepsilon \to 0$ and $R \to \infty$ are determined next.

The radius of the circle surrounding the point $(\bar{x},\bar{z})$ is $\varepsilon$ and so

$$\delta C = \varepsilon\delta\theta,$$

taking $\theta$ to increase in the clockwise direction. The radius of this circle is now made infinitely small. Since $\psi = \ln(\varepsilon)$ it follows that as $\varepsilon \to 0$ all the terms in the line integral over $C_1$ vanish except those involving derivatives of $\psi$. As $\varepsilon \to 0$, $\bar{\phi}(\bar{\xi},\bar{\zeta})$ and its derivatives take their value at $(\bar{x},\bar{z})$. Thus, since the inward normal $\frac{\partial}{\partial n}$ is equivalent to $\frac{\partial}{\partial \varepsilon}$,

$$\lim_{\varepsilon\to 0} \{\int_{C_1}\left[\psi(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) \frac{\partial\bar{\phi}(\bar{\xi},\bar{\zeta})}{\partial n} - \bar{\phi}(\bar{\xi},\bar{\zeta}) \frac{\partial\psi}{\partial n}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\right]dC_1 = - 2\pi\bar{\phi}(\bar{x},\bar{z}) \tag{A4}$$

For the integral over $C_\infty$, the velocity potential $\bar{\phi}(\bar{\xi},\bar{\zeta})$ will be of the form

$$\bar{\phi}(\bar{\xi},\bar{\zeta}) = \underline{O}(\theta + \frac{1}{R})$$

for a lifting aerofoil (for a non-lifting aerofoil the $\theta$ term should be omitted) where $(R, \theta)$ are the polar co-ordinates on $C_\infty$. Thus the line integral over $C_\infty$ is a constant for a lifting aerofoil and zero for a non-lifting aerofoil; the exact value of this constant is immaterial since it will disappear in the application of a differential operator later on.

Since $C_w$ lies around the $\bar{\xi}$ axis as shown in Fig.2, Eq.(A3) then becomes, on substitution of Eq.(A4),

$$\bar{\phi}(\bar{x},\bar{z}) = \frac{1}{2\pi}\int_0^1\left[\psi(\bar{x},\bar{\xi};\bar{z},0)\ \Delta\bar{\phi}_{\bar{\zeta}}(\bar{\xi}) - \psi_{\bar{\zeta}}(\bar{x},\bar{\xi};\bar{z},0)\ \Delta\bar{\phi}(\bar{\xi})\right] d\bar{\xi} + \frac{1}{2\pi}\iint_S \psi(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\ \bar{\phi}_{\bar{\xi}}(\bar{\xi},\bar{\zeta})\ \bar{\phi}_{\bar{\xi}\bar{\xi}}(\bar{\xi},\bar{\zeta})\ dS + \text{constant}, \tag{A5}$$

where

$$\Delta\bar{\phi}(\bar{\xi}) = \bar{\phi}(\bar{\xi},+0) - \bar{\phi}(\bar{\xi},-0)$$

$$\Delta\bar{\phi}_{\bar{\zeta}}(\bar{\xi}) = \bar{\phi}_{\bar{\zeta}}(\bar{\xi},+0) - \bar{\phi}_{\bar{\zeta}}(\bar{\xi},-0)$$

The surface integral in Eq.(A5) is therefore defined, for $\bar{z} > 0$, as

$$\iint_S F dS = \lim_{\varepsilon\to 0}\Big\{\int_{-\infty}^{\bar{x}-\varepsilon}\Big(\int_0^\infty F d\bar{\zeta}\Big)d\bar{\xi} + \int_{\bar{x}+\varepsilon}^{\infty}\Big(\int_0^\infty F d\bar{\zeta}\Big)d\bar{\xi} + \int_{-\infty}^{\infty}\Big(\int_{-\infty}^{0} F d\bar{\zeta}\Big)d\bar{\xi} + \int_{\bar{x}-\varepsilon}^{\bar{x}+\varepsilon}\Big[\int_0^{\bar{z}-[\varepsilon^2-(\bar{x}-\bar{\xi})^2]^{\frac{1}{2}}} F d\bar{\zeta} + \int_{\bar{z}-[\varepsilon^2-(\bar{x}-\bar{\xi})^2]^{\frac{1}{2}}}^{\infty} F d\bar{\zeta}\Big] d\bar{\xi}\Big\} \tag{A6}$$

while, for $\bar{z} < 0$,

$$\iint_S F d\bar{\zeta} = \lim_{\varepsilon\to 0}\Big\{\int_{-\infty}^{\infty}\Big(\int_0^\infty F d\bar{\zeta}\Big)d\bar{\xi} + \int_{-\infty}^{\bar{x}-\varepsilon}\Big(\int_{-\infty}^{0} F d\bar{\zeta}\Big)d\bar{\xi} + \int_{\bar{x}+\varepsilon}^{\infty}\Big(\int_{-\infty}^{0} F d\bar{\zeta}\Big)d\bar{\xi} + \int_{\bar{x}-\varepsilon}^{\bar{x}+\varepsilon}\Big[\int_{-\infty}^{\bar{z}-[\varepsilon^2-(\bar{x}-\bar{\xi})^2]^{\frac{1}{2}}} F d\bar{\zeta} + \int_{\bar{z}+[\varepsilon^2-(\bar{x}-\bar{\xi})^2]^{\frac{1}{2}}}^{0} F d\bar{\zeta}\Big] d\bar{\xi}\Big\} \tag{A7}$$

The double integral in Eq.(A5) can be integrated by parts with respect to $\bar{\xi}$; by reference to Eq.(A6,A7) Eq.(A5) becomes

$$\bar{\phi}(\bar{x},\bar{z}) = \frac{1}{2\pi}\int_0^1\left[\psi(\bar{x},\bar{\xi};\bar{z},0)\ \Delta\ \bar{\phi}_{\bar{\zeta}}(\bar{\xi}) - \psi_{\bar{\zeta}}(\bar{x},\bar{\xi};\bar{z},0)\ \Delta\ \bar{\phi}(\bar{\xi})\right] d\bar{\xi} - \frac{1}{4\pi}\iint_S \psi_{\bar{\xi}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\ \bar{u}^2(\bar{\xi},\bar{\zeta})\ dS + \text{constant} \tag{A8}$$

| $\bar{x}$ | $\dfrac{I(\bar{x},+0)}{\frac{\bar{u}^2(\bar{x},+0)}{4}}$ | $\dfrac{\bar{u}^2(\bar{x},+0)}{4}$ | $\bar{u}(\bar{x},+0)$ |
|---|---|---|---|
| 0.381 | - 3.192 | 0.0341 | 0.3694 |
| 0.0844 | - 0.501 | 0.1261 | 0.7101 |
| 0.1465 | - 0.1532 | 0.1704 | 0.8256 |
| 0.2223 | - 0.1052 | 0.1525 | 0.7809 |
| 0.3087 | - 0.169 | 0.1096 | 0.6621 |
| 0.4025 | - 0.33 | 0.069 | 0.5261 |
| 0.500 | - 0.641 | 0.0393 | 0.3967 |
| 0.691 | - 3.04 | 0.0078 | 0.1762 |
| 0.853 | - 146.8 | 0.0001 | - 0.019 |
| 0.99 | 1.222 | 0.0438 | - 0.4188 |

Results from NACA 0012 at Zero Incidence, $M_\infty = 0.72$

Table 1

| $\bar{x}$ | Upper Surface | | | Lower Surface | | |
|---|---|---|---|---|---|---|
| | $\dfrac{I(\bar{x},+0)}{\frac{\bar{u}^2(\bar{x},+0)}{4}}$ | $\dfrac{\bar{u}^2(\bar{x},+0)}{4}$ | $\bar{u}(\bar{x},+0)$ | $\dfrac{I(\bar{x},-0)}{\frac{\bar{u}^2(\bar{x},-0)}{4}}$ | $\dfrac{\bar{u}^2(\bar{x},-0)}{4}$ | $\bar{u}(\bar{x},-0)$ |
| 0.038 | - 0.456 | 0.153 | 0.7834 | 11.14 | 0.0028 | -0.106 |
| 0.084 | - 0.176 | 0.154 | 0.785 | - 0.824 | 0.0020 | 0.090 |
| 0.1465 | - 0.115 | 0.121 | 0.695 | - 0.324 | 0.0080 | 0.1785 |
| 0.222 | - 0.113 | 0.087 | 0.589 | - 0.269 | 0.0106 | 0.206 |
| 0.309 | - 0.175 | 0.058 | 0.482 | - 0.262 | 0.0099 | 0.199 |
| 0.403 | - 0.283 | 0.037 | 0.384 | - 0.412 | 0.0074 | 0.172 |
| 0.3 | - 0.485 | 0.021 | 0.295 | - 0.699 | 0.0046 | 0.136 |
| 0.691 | - 1.615 | 0.0055 | 0.148 | - 4.25 | 0.0008 | 0.057 |
| 0.853 | -35.4 | 0.0001 | 0.0189 | - 9.5 | 0.00021 | -0.029 |
| 0.99 | 1.036 | 0.0148 | -0.243 | 0.942 | 0.016 | -0.253 |

Results from NACA 0012 at 2° Incidence, $M_\infty = 0.63$

Table 2

| | Sells | 1st Approx. | 2nd Approx. | Gothert | Modified Gothert |
|---|---|---|---|---|---|
| $C_L$ | 0.335 | 0.355 | 0.335 | 0.282 | 0.299 |
| $C_M$ about the leading edge | -0.0826 | -0.0890 | -0.0840 | -0.0705 | -0.0777 |
| a/c aft of the leading edge | 0.246 | 0.251 | 0.251 | 0.25 | 0.26 |

Table 3

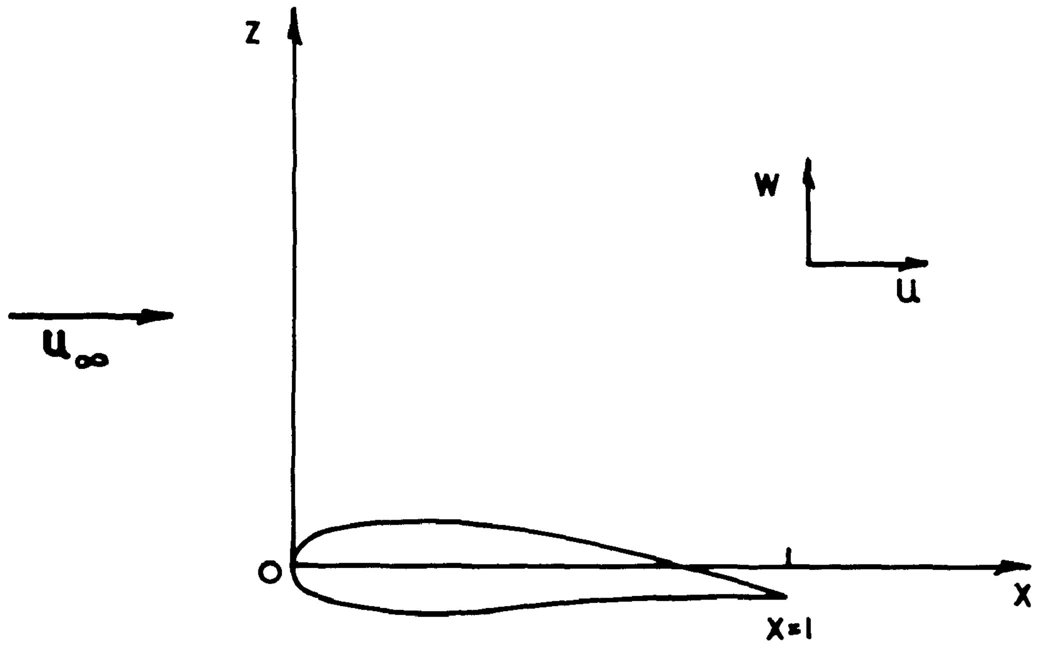



Fig. 1 CO-ORDINATE SYSTEM

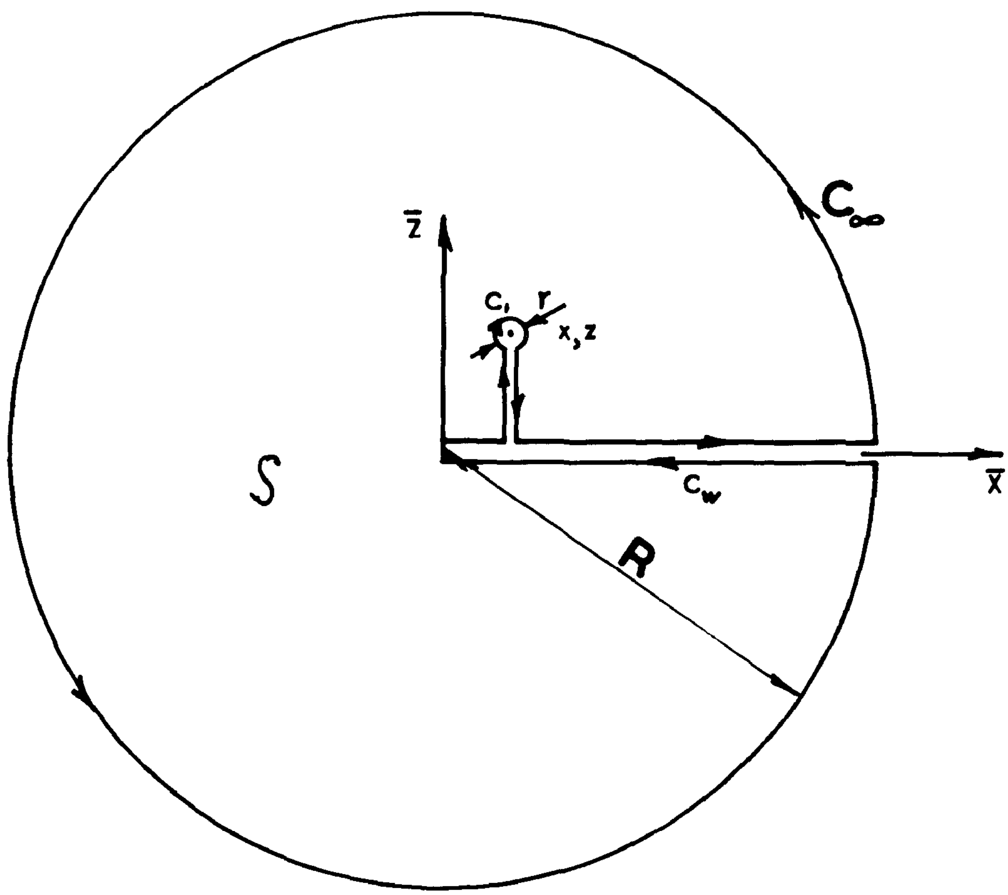
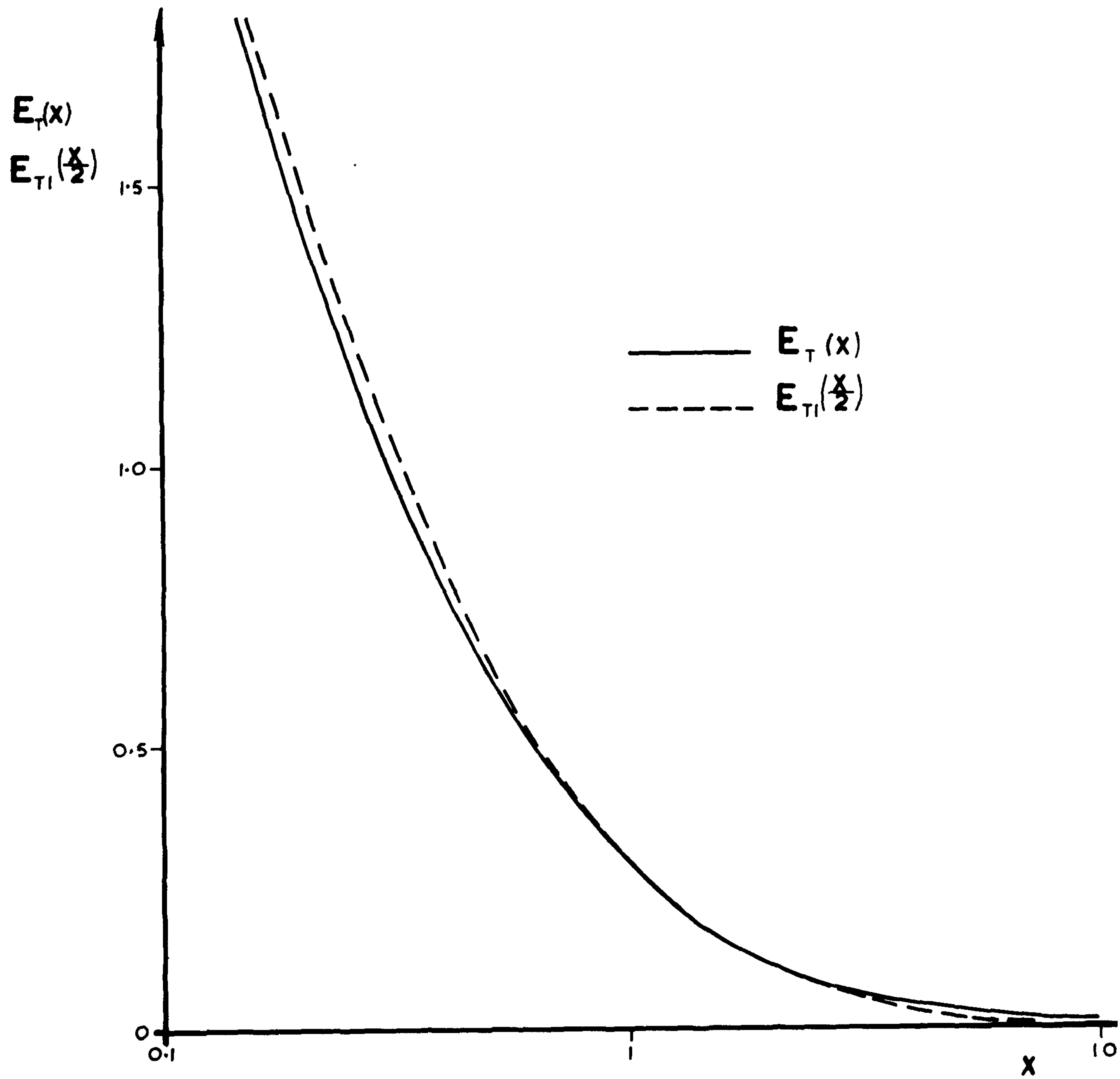
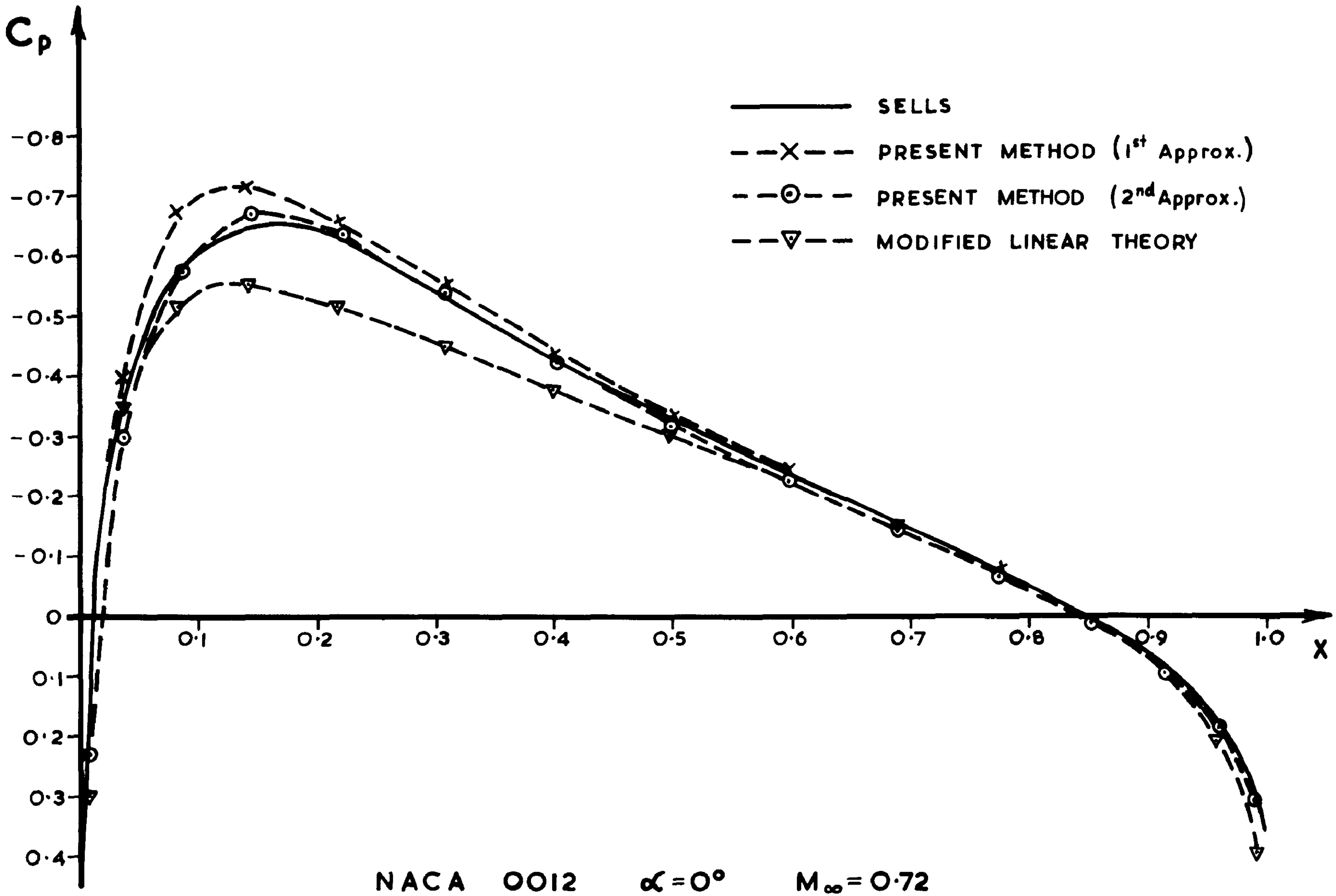



Fig. 2 DOMAIN OF INTEGRATION FOR GREEN'S THEOREM




VARIATION OF $E_T(x)$ AND $E_{T1}(\frac{x}{2})$ WITH X

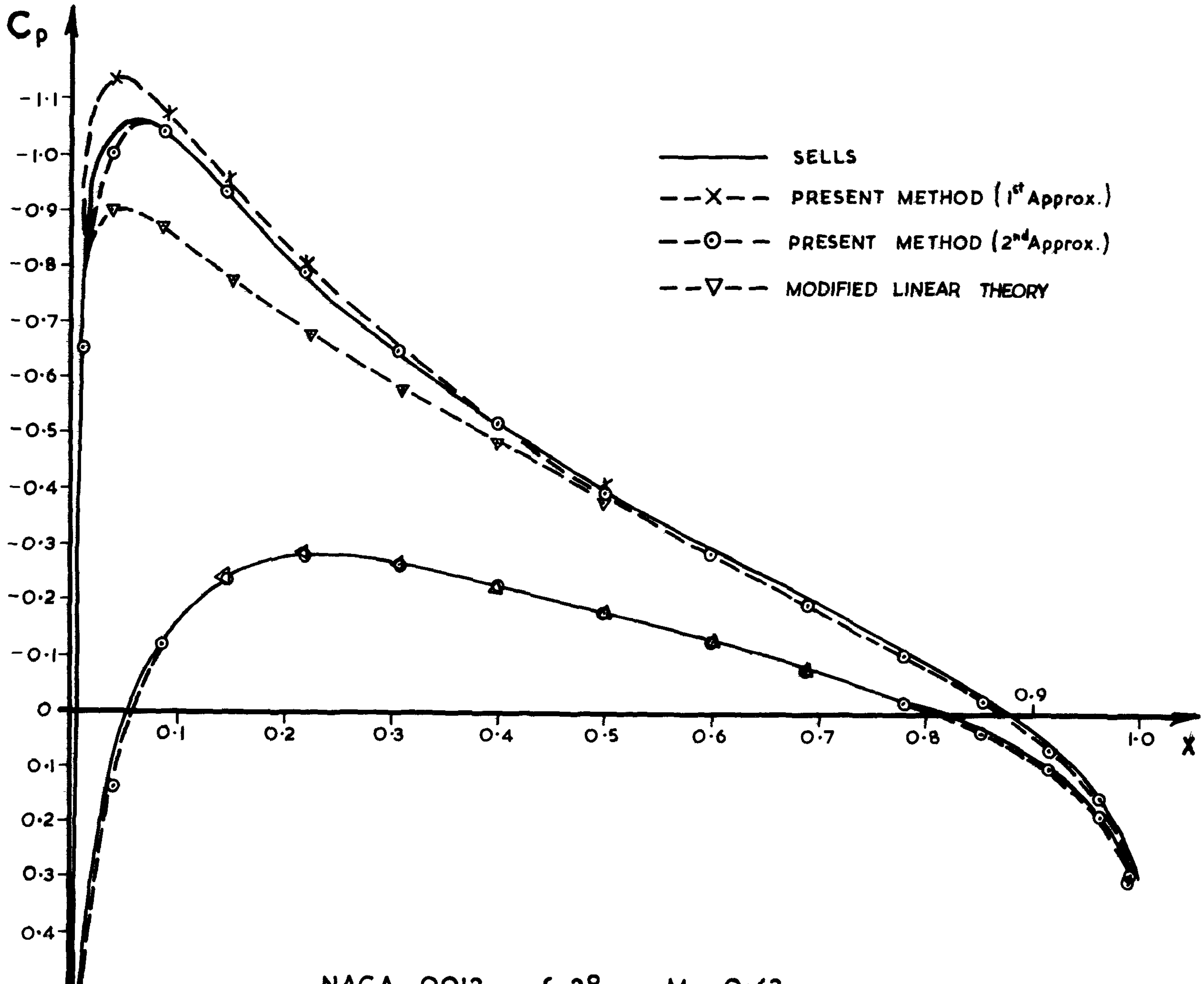
Fig. 3



Cp
SELLS
PRESENT METHOD (1st Approx.)
PRESENT METHOD (2nd Approx.)
MODIFIED LINEAR THEORY
-0·8
-0·7
-0·6
-0·5
-0·4
-0·3
-0·2
-0·1
0
0·1
0·2
0·3
0·4
0·1
0·2
0·3
0·4
0·5
0·6
0·7
0·8
0·9
1·0
X
NACA 0012 α=0° M∞=0·72


Fig. 4

R 72580/R54 400 1/74 P




NACA 0012 $\alpha = 2°$ $M_\infty = 0{\cdot}63$

Fig. 5

34260

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HIGH SUBSONIC FLOW PAST A STEADY TWO-DIMENSIONAL AEROFOIL

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DETACHABLE ABSTRACT CARDS

A simple first approximation to the flow, can be expressed in the form

$$\bar{u} - \frac{\bar{u}^2}{4} = \bar{u}_L ,$$

where $\bar{u}$ is the perturbation velocity and $\bar{u}_L$ is a modified linearised value, gives a fair degree of accuracy except in the immediate neighbourhood of the leading edge. An iterated second step approximation gives good results for a NACA 0012 aerofoil.

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