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Improper Integrals in Theoretical Aerodynamics

By

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Improper integrals in theoretical aerodynamics

by

K.W. Mangler

SUMMARY

Improper integrals occur in theoretical aerodynamics, if one determines the velocities, induced by a vortex sheet, which e.g. may represent the influence of a thin wing at incidence. The velocity at any point can be expressed as an integral involving an influence function, which becomes singular as this point approaches the vortex sheet. As is well known, the principal value, as defined by Cauchy, is to be taken in this case.

This paper deals with an integral, involving a "principal value of the order n". It was first introduced by Hadamard and is a generalization of Cauchy's principal value. It occurs, if one determines the derivatives of an integral, involving Cauchy's principal value. These integrals can usefully be applied in many theoretical problems, in particular in the supersonic theory of conical fields (ref.2).

In order to deal with the singularities as they occur near the leading edge of a thin wing due to the vorticity tending to infinity there, "principal values of a fractional order" are also introduced. After Hadamard all these principal values can be interpreted in such a way that only "the finite part" of the integral in question has a physical meaning. The rules for the evaluation, the differentiation and integration by parts of such an improper integral are derived and summarized (Appendix I) together with a theorem (section 5), which will be useful to workers in this field.

LIST OF CONTENTS

LIST OF ILLUSTRATIONS

1 Introduction

In many problems of applied mathematics we encounter certain singularities, which do not directly correspond to physical singularities, but are due to certain simplifications, introduced in the physical or technical problem, in order to emphasize the most important features of the problem under consideration. Sometimes only such a simplification makes the theoretical treatment of the problem at all possible.

To give an example, we consider the calculation of the pressure distribution over a wing. Here one usually separates the thickness effects from the incidence effects by introducing the conception of a very thin wing at incidence and adding the thickness effects afterwards. The thin wing is replaced by a distribution of vorticity in the plane $z = 0$ (corresponding to zero incidence of the flat wing), the strength of which must be determined in such a way, that the induced downwash, together with the undisturbed flow, produces a velocity parallel to the surface of the thin wing. As is well known, the downwash, due to a vortex element, at an arbitrary point P is proportional to a negative power of the distance from the vortex, and thus we have to deal with a mathematical singularity, if this distance tends to zero, i.e. if the point P approaches the vortex.

The three-dimensional downwash condition reduces to a two-dimensional one in the following important cases:

(1) The two-dimensional subsonic theory of a thin wing, as it was treated by Glauert (ref.1), where the wing section is the chord.

(2) The conical field theory for a wing in a supersonic flow, as treated recently in an extensive paper by Multhopp (ref.2), where the wing section must be taken spanwise.

In both cases we may write the downwash integral in the form

$$2\pi\, w(x,z) = \int_a^b f(\xi)\, \frac{x-\xi}{(x-\xi)^2+z^2}\, d\xi$$

where the x-coordinate is taken along the section, z perpendicular to it, and $f(\xi)$ denotes the strength of the vorticity for $a < \xi < b$. The function $w(x,z)$ satisfies the Laplace equation. The problem in its usual form is, that $w(x,o)$ is known along the section $a < x < b$ for $z = 0$ and $f(\xi)$ is to be determined either from $w(x,o)$ or from one of its derivatives. This leads to

$$2\pi\, w(x,o) = \lim_{z\to 0} \int_a^b \frac{f(\xi)\ (x-\xi)}{(x-\xi)^2+z^2}\, d\xi$$

In evaluating this integral we clearly have to define a principal value (i.e. finite value) for the integral

$$\int_a^b \frac{f(\xi)\ d\xi}{x-\xi}$$

in order to deal with the singularity $\xi = x$. The result, due to Cauchy, is well known. Similarly, in considering its n^{th} derivative with respect to x, we have to define the principal value of the order n for the integral

$$\int_a^b \frac{f(\xi)\,d\xi}{(x-\xi)^{n+1}} .$$

Moreover, it is well known, that in replacing the rounded nose of a profile by a sharp leading edge, we must admit an infinite velocity and so an infinite value for the vorticity at the sharp leading edge. Thus the integrand in the downwash integral may become infinite not only at $\xi = x$, but also at the ends $\xi = a$ and $\xi = b$ of the integration interval. Thus we have also to define the principal value for integrals of the form

$$\int_a^b \frac{h(\xi,x)\,d\xi}{(\xi-a)^{\alpha+m}}$$

where α is fractional (usually $\frac{1}{2}$) and m is a positive integer. We shall call them principal values of a fractional order (section 4). They occur at the ends of an integration interval and are due to a singularity in the vorticity distribution $f(\xi)$, whereas the principal values of integer order (section 3) occur in the interior of the integration interval $(\xi = x)$.

Fairly often these principal values of higher order can be avoided by an integration by parts, which reduces their order. Thus the downwash behind a wing can be determined in the lifting line theory, where the wing is replaced by a single bound vortex, from the lift distribution $\Gamma(\xi)$ by means of the influence function $(x-\xi)^{-2}$, although it is more usual to determine it from the derivative $d\Gamma/d\xi = \Gamma'(\xi)$ by means of the influence function $(x-\xi)^{-1}$. But in the theory of the supersonic conical fields the higher order principal values are very useful, and Multhopp (ref.2) makes frequent use of such integrals correctly, but without proof.

Thus the technique for working out the finite values of such integrals becomes increasingly important. In this paper the calculus of integrals of this kind, which was initiated by Hadamard (ref.3), is summarised and extended, together with a theorem (section 5), in a form which may be of practical use to the worker in this field.

It may be mentioned, that Hadamard (ref.3) called the principal value the "finite part of an infinite integral", a notation, which will become clearer in section 2. Robinson (ref.4) used the idea of the principal value, in order to generalize the concept of the subsonic source or doublet for a linearized supersonic flow and to establish some theorems for a supersonic flow, which are equivalent to Green's formula and Stokes' theorem in subsonic flow. A number of American papers (e.g. Heaslet and Lomax, ref.5) on the calculation of the pressure distribution over a wing in supersonic flow are based on this conception of the finite part of an infinite integral.

2 A contour integral

In the example of a two-dimensional flow around a profile, which was mentioned above, we can replace the distribution of sources and sinks along the profile chord, which was used by Glauert in order to determine the disturbance caused by the wing in a parallel flow, by a (different) distribution of sources and vortices along the profile contour itself. In the limiting case of thickness ratio zero both distributions will be the same, but for a profile of a small but finite thickness, we have to deal with finite velocities only along the contour and in the exterior of the profile and therefore with source and vortex distributions, which are finite along the entire contour.

Then the boundary condition along the contour can be put in the following form

$$U(P) + F(P) + F^{*}(P) = 0 \qquad (1)$$

Here $U(P)$ is the contribution of the undisturbed flow to the normal component at the point P of the contour and

$$F(P) = \oint f(s)\ g(s,P)\ ds \qquad (2)$$

and

$$F^{*}(P) = \oint f^{*}(s)\ g^{*}(s,P)\ ds \qquad (3)$$

are the normal components, induced by the vorticity distribution $f(s)$ and the source distribution $f^{*}(s)$ respectively, both given as functions of the arc length s along the contour. $g(s,P)$ and $g^{*}(s,P)$ are the normal components, induced in the point P by a vortex or a source respectively of the strength 1. The integrals have to be taken along the entire closed contour of the profile.

From now on we shall deal only with the integral in (2) since all the results can easily be extended to equation (3). The integral (2) contains a singularity at the value of s which corresponds to the pivotal point P. Here we determine the integral first for a point P just outside the contour and let the point P move on to the contour after the integration. Another possibility is, to consider (see fig.1) instead of the given contour another contour with a small dent, so that P is outside the new contour and let the dent vanish after the integration has been performed. We denote the contribution of the dent to the integral F by I_P.

In a similar way we add a bulge to the contour at the nose A and another one at the trailing edge B, so that in the limiting case of thickness ratio zero, the profile consists of a flat or cambered plate with two bulges at each end. Each of these bulges consists of a circle of a small radius ε. Their contribution to the integral may be denoted by I_A and I_B respectively. Thus we have for the thin profile:

$$F(P) = \lim_{\varepsilon \to 0} [I_A + I_B + I_P] + I \qquad (4)$$

where I is the remainder of the contour integral.

The contribution of the bulges at A, B, and P can be zero in many cases, e.g. for the flat plate at a constant incidence so that the velocity $F(P) = I$ can be obtained, without considering the bulges.

But this is not generally true. In certain cases, e.g. if $F(P)$ in (4) does not denote an induced velocity, but a derivative of such a velocity, the contributions of the bulges may be infinite, although the physical quantity $F(P)$ (the velocity derivative) may be finite. In such a case I is infinite, but the complete sum of equation (4) is finite.

This function $F(P)$ is often called the <u>"finite part" of the integral I</u>. The integral I, which is infinite, if considered by itself, represents the finite physical quantity $F(P)$, if the missing contributions of the three bulges A, B and P are added. In other words: The infinite part of the integral I is cancelled by the bulge-integrals, and only the "finite part" of I has a physical meaning. We shall prefer the notation "principal value" of I for the function F(P) in (4), since for the particular case of a flat plate and an influence function $g(s,P) = {}^1/(s-x)$, equation (4) reduces to the definition of Cauchy's principal value, as will be seen later on (section 3).

Since the vorticity distribution is of an opposite sign on both faces of a flat plate, we may write

$$I = \lim_{\varepsilon \to 0} \left(2 \int_{A+\varepsilon}^{P-\varepsilon} f \; g \; ds + 2 \int_{P+\varepsilon}^{B-\varepsilon} f \; g \; ds \right) \qquad (5)$$

In order to obtain from (5) the physically significant quantity F(P) we have to take the principal value:

$$F(P) = \fint_A^B f(s) \; g(s,P) \; ds \qquad (6)$$

which is defined by means of equation (4). (The crossed integral sign indicates that the "finite part" or the principal value must be taken.)

From our argument, it is very likely that operations such as differentiation (with respect to P) or integration by parts (with respect to s) are permissible for the integral I in (5), provided that the principal value is always taken, i.e. the contribution of the bulge integrals is allowed for. This will be proved in the following paragraphs and a number of rules for the treatment of such principal-value-integrals will be established.

Since the order of the singularities of g at the point P or of f at A or B can be altered by integration by parts, we shall establish these rules by starting from lower order singularities and proving the rules for higher order singularities by means of an integration by parts.

3 Principal values of integer order

3.1 Cauchy's principal value

In many fields of applied mechanics we have to deal with an integral of the form $(a < x < b)$

$$F(x) = \int_a^b \frac{f(\xi) \; d\xi}{x-\xi} \qquad (7)$$

Since the integrand tends to infinity for $\xi = x$, this integral has no meaning, if considered as an integral in the ordinary sense. Therefore we have to go back to a more general integral, which contains (1) as a limiting case.

In aerodynamics the integral F(x) usually occurs, if one determines the x-component of the velocity, which is induced by a distribution $f(\xi)$ of sources and sinks along the interval $a < \xi < b$, of the x-axis in a two-dimensional flow. At any point (x,z) of the plane this velocity is given (apart from a factor ${}^1/2\pi$) by the function

$$F(x,z) = \int_a^b \frac{f(\xi)\ (x-\xi)\ d\xi}{(x-\xi)^2+z^2} \qquad (8)$$

and $F(x)$ in equation (7) is the limiting case, if we put $z = 0$ in equation (8). Now it is apparently a reasonable procedure to define the so far meaningless integral (7) as the limiting case for $z \to 0$ of the integral $F(x,z)$ in (8):

$$F(x) = \lim_{z \to 0} F(x,z) \qquad (9)$$

Physically speaking, the velocity induced by the source distribution along the axis $z = 0$ is a continuous function of z including $z = 0$.

In order to perform this limiting process $z \to 0$, we divide the integral (8) from a to b into three parts. In the first part from a to $x - \varepsilon$ and the third part from $x + \varepsilon$ to b the difference $u = \xi - x$ is never zero and we may safely go to the limit $z \to 0$. The second part from $x-\varepsilon$ to $x+\varepsilon$ contains the value $\xi = x$. It may be written as

$$I_p = \int_{x-\varepsilon}^{x+\varepsilon} \frac{f(\xi)\ (x-\xi)\ d\xi}{(x-\xi)^2+z^2} = - \int_{-\varepsilon}^{\varepsilon} \frac{f\ (x+u)\ u\ du}{u^2+z^2}$$

or, expanding $f(x+u)$ in a Taylor series for small values of $u = \xi - x$:

$$I_p = - \int_{-\varepsilon}^{\varepsilon} \left[f(x) \pm u\ f'(x) + \frac{u^2}{2} f''(x) + \dots \right] \frac{u\ du}{u^2+z^2}$$

The integrals over the first and third term vanish for symmetry reasons and we obtain from the second term

$$I_p = -f'(x)\ (2\varepsilon - 2z \tan^{-1} \frac{\varepsilon}{z}) + \dots$$

which in the limit $z \to 0$ tends to a finite value. All the omitted terms contain even higher powers of ε.

Since all these considerations hold for any value $\varepsilon > 0$ (which is not too big as to invalidate the expansion of $f(\xi)$ in a Taylor series), we may go to the limit $\varepsilon \to 0$ and obtain

$$F(x) = \lim_{z \to 0} F(x,z) = \lim_{\varepsilon \to 0} \left\{ \int_a^{x-\varepsilon} \frac{f(\xi)\ d\xi}{x-\xi} + \int_{x+\varepsilon}^{b} \frac{f(\xi)\ d\xi}{x-\xi} \right\} \qquad (10)$$

since the third contribution I_p vanishes for $\varepsilon \to 0$. This is the well-known principal value of the integral (7), as first introduced by Cauchy.

For a wing with a finite thickness $z = \pm z_o(x)$ we require the velocity field only for points $z^2 \geqq z_o^2$ outside the profile and on its surface, where $(x-\xi)^2+z^2$ is always positive, so that no singularity occurs in $F(x,z)$ and no principal value is required.

3.2 The derivatives of the function F(x)

After having repeated all the well-known results for Cauchy's principal value, we proceed now to determine the derivatives of the function $F(x)$ in (7). As before, we have to go back to the general function $F(x,z)$ in (8), have to determine first the derivatives of $F(x,z)$ with respect to x and have to go to the limit $z \to 0$ afterwards.

Since in many problems not $f(\xi)$, but either $F(x)$ in (7) or one of its derivatives $\frac{d^nF(x)}{dx^n}$ is known, and $f(\xi)$ is the unknown function (the vorticity distribution is usually determined by a condition imposed on the downwash or sidewash), we need a relation between d^nF/dx^n which holds (on the wing surface) for $z = 0$ without going back to the velocity field outside the wing. This can be achieved by means of a generalization of Cauchy's idea of the principal value of an integral, as it was suggested first by Hadamard.

A differentiation of (7) under the integral sign would lead to

$$- F'(x) = ⨍_a^b \frac{f(\xi)\, d\xi}{(x-\xi)^2} \tag{11}$$

and a repeated differentiation to

$$\frac{(-1)^n}{n!} \frac{d^nF(x)}{dx^n} = ⨍_a^b \frac{f(\xi)\, d\xi}{(x-\xi)^{n+1}}, \quad (a < x < b) \tag{12}$$

These integrals have no meaning if they are considered as ordinary integrals. But they lead to a finite result, if we take their "principal value" (as indicated by the crossed integral sign), which can be defined in the following way:

$$⨍_a^b \frac{f(\xi)\, d\xi}{(x-\xi)^{n+1}} = \lim_{\varepsilon \to 0} \left\{ \int_a^{x-\varepsilon} \frac{f(\xi)\, d\xi}{(x-\xi)^{n+1}} + \int_{x+\varepsilon}^{b} \frac{f(\xi)\, d\xi}{(x-\xi)^{n+1}} + (-1)^n K_n(x,\varepsilon) \right\}$$

$$(a < x < b) \tag{13}$$

where

$$K_n(x,\varepsilon) = \sum_{j=0}^{n-1} \frac{f^{(j)}(x)}{j!} \frac{1-(-1)^{n-j}}{(n-j) \cdot \varepsilon^{n-j}}\ ; \ K_0(x,\varepsilon) \equiv 0 \tag{14}$$

The function $K_n(x,\varepsilon)$ depends on the coefficients of the first n terms of a Taylor series for the function $f(\xi) = f(x+u)$ at the point $\xi = x$:

$$f(\xi) = f(x+u) = f(x) + u\, f'(x) + \ldots = \sum_{j=0}^{\infty} \frac{u^j}{j!} f^{(j)}(x) \tag{15}$$

$$(\xi = x+u)$$

In order to prove the equations (13), (14), we go back to the more general function $F(x,z)$ as defined in (8) and determine the derivatives $\frac{d^nF(x,z)}{dx^n}$ first for $z \neq 0$ and go afterwards to the limit $z \to 0$. We have

$$\frac{(-1)^n}{n!} \frac{d^n}{dx^n} F(x,z) = \frac{(-1)^n}{n!} \int_a^b f(\xi) \frac{d^n}{dx^n}\left(\frac{x-\xi}{(x-\xi)^2+z^2}\right) d\xi$$

$$= \frac{1}{n!} \int_a^b f(\xi) \frac{d^n}{d\xi^n}\left(\frac{x-\xi}{(x-\xi)^2+z^2}\right) d\xi \tag{16}$$

We divide the integral from a to b into three parts, integrating first from a to $x-\varepsilon$, then from $x-\varepsilon$ to $x+\varepsilon$, and finally from $x+\varepsilon$ to b. In the first and in the third integral $u = \xi-x$ is always different from zero, so that the integration over ξ, the differentiation with respect to x or ξ and the limitation process $z \to 0$ can be carried out in any arbitrary order, $(\varepsilon>0)$, and we obtain the first two terms in the bracket of equation (13). It remains to be shown that the third contribution, namely the integral

$$\frac{1}{n!} \int_{x-\varepsilon}^{x+\varepsilon} f(\xi) \frac{d^n}{d\xi^n}\left(\frac{x-\xi}{(x-\xi)^2+z^2}\right) d\xi = \frac{-1}{n!} \int_{-\varepsilon}^{\varepsilon} f(x+u) \frac{d^n}{du^n}\left(\frac{u}{u^2+z^2}\right) du$$

tends in the limit $z \to 0$ to the term $(-1)^n K_n(x,\varepsilon)$ as defined in (14). We introduce the Taylor series (15) and obtain the integral sum

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(x)}{j!} (-1)^n L_{n,j} = K(x,\varepsilon) \tag{17}$$

where

$$L_{n,j} = \lim_{z\to 0} \frac{(-1)^{n-1}}{n!} \int_{-\varepsilon}^{\varepsilon} u^j \frac{d^n}{du^n}\left(\frac{u}{u^2+z^2}\right) du \tag{18}$$

We can see, immediately, that $L_{n,j} = 0$ for symmetry reasons, if $n-j$ is an even number or zero. Furthermore we have

$$L_{1,0} = \lim_{z\to 0}\left(\frac{u}{u^2+z^2}\right)_{-\varepsilon}^{\varepsilon} = \frac{2}{\varepsilon} \tag{19}$$

Now we conclude by means of an integration by parts, that

$$L_{n,j} = \lim_{z\to 0}\left\{ \frac{(-1)^n}{n!}\left(u^j \frac{d^{n-1}}{du^{n-1}}\left(\frac{u}{u^2+z^2}\right)\right)_{-\varepsilon}^{+\varepsilon} + \frac{j}{n}\frac{(-1)^{n-1}}{(n-1)!}\int_{-\varepsilon}^{\varepsilon} u^{j-1}\frac{d^{n-1}}{dn^{n-1}}\left(\frac{u}{u^2+z^2}\right) du \right\}$$

$$= \left(\frac{u^{j-n}}{n}\right)_{-\varepsilon}^{\varepsilon} + \frac{j}{n} L_{n-1,j-1}$$

or

$$L_{n,j} - \frac{j}{n} L_{n-1,j-1} = \frac{1-(-1)^{n-j}}{n\,\varepsilon^{n-j}} \tag{20}$$

From this reduction formula, we have, remembering (19),

$$L_{n,j} = \frac{1-(-1)^{n-j}}{(n-j)\,\varepsilon^{n-j}}, \quad j \neq n \tag{21}$$

$$L_{n,n} = 0$$

Inserting this result (21) into equation (17) we obtain the function $K_n(x,\varepsilon)$, as defined in (14), since it is sufficient to extend the sum only between 0 and $n-1$, because for $n < j$ we obtain positive powers of ε, which will vanish in the limit $\varepsilon \to 0$.

Thus we obtain a finite result for the derivatives of $F(x,z)$ when z tends to zero for any arbitrary value of $\varepsilon > 0$. It now remains to be shown, that we may go to the limit $\varepsilon \to 0$, as indicated in (13), and shall obtain a finite result even in the limit $\varepsilon = 0$. For this purpose we denote the polynomial, consisting of the first $(n+1)$ terms of the Taylor series (15) for $f(\xi)$ by $f_n(\xi)$:

$$f_n(\xi) = \sum_{j=0}^{n} \frac{(\xi-x)^j}{j!} f^{(j)}(x) \tag{22}$$

Thus the functions $f(\xi)$ and $f_n(\xi)$ have the first n derivatives at $\xi = x$ in common, so that the difference

$$f(\xi) - f_n(\xi) = \sum_{j=n+1}^{\infty} \frac{(\xi-x)^j}{j!} f^{(j)}(x)$$

can be divided by $(x-\xi)^{n+1}$ and the integral

$$\int_a^b \frac{f(\xi) - f_n(\xi)}{(x-\xi)^{n+1}} d\xi$$

is a regular integral. We introduce the indefinite integrals

$$G(x,\xi) = \int^{\xi} \frac{f(\xi)\, d\xi}{(x-\xi)^{n+1}} \tag{23}$$

$$G_n(x,\xi) = \int^{\xi} \frac{f_n(\xi)\, d\xi}{(x-\xi)^{n+1}}$$

$$= (-1)^{n-1} \sum_{j=0}^{n-1} \frac{f^{(j)}(x)}{j\,!} \frac{(\xi-x)^{j-n}}{(j-n)} + \frac{f^{(n)}(x)}{n\,!} \log|\xi-x| \tag{24}$$

(where the log of the modulus of (ξ-x) must be taken), and have

$$\int_a^b \frac{f(\xi) - f_n(\xi)}{(x-\xi)^{n+1}} d\xi = \Big(G(x,b) - G_n(x,b)\Big) - \Big(G(x,a) - G_n(x,a)\Big) \tag{25}$$

For the integral over the polynomial, we have to apply our definition (13):

$$\fint_a^b \frac{f_n(\xi)\, d\xi}{(x-\xi)^{n+1}} = \lim_{\varepsilon \to 0} \Big\{ G_n(x,x-\varepsilon) - G_n(x,a)$$

$$+ G_n(x,b) - G_n(x,x+\varepsilon) + (-1)^n K_n(x,\varepsilon) \Big\}$$

$$= \lim_{\varepsilon \to 0} \Big\{ G_n(x,b) - G_n(x,a) + (-1)^{n-1} \frac{f^{(n)}(x)}{n\,!} \log\left|\frac{\varepsilon}{\varepsilon}\right| \Big\} \tag{26}$$

since the remaining terms cancel each other according to (14) and (24). This result is independent of ε and the limiting process $\varepsilon \to 0$ is thus justified and leads to a finite answer.

It may be pointed out that $G_n(x,b) - G_n(x,a)$ in general also contains logarithmic terms, and the result in (26) is correct provided that always the log of the modulus is taken.

Equation (26) permits the result of the integration (13), (14) to be written in a shorter way, provided that the indefinite integral (23) is known. We obtain by adding (25) and (26) the result:

$$\fint_a^b \frac{f(\xi)\, d\xi}{(x-\xi)^{n+1}} = [G(x,b) - G(x,a)] \tag{27}$$

i.e. the integral may be formally treated as an ordinary integral, without any regard to the singularity at $\xi = x$, provided that the log-terms are treated in the way mentioned above.

Equation (27) explains, why Hadamard (ref.3) who first introduced this type of integral, called the principal value, the "finite part" of an infinite integral. All the terms, which near the pivotal point $\xi = x$ would give an infinite contribution to the integral, are omitted (cf. equation (13) and (14)). One takes only the "finite part" of the integral.

Thus it has been proved that we can obtain the derivatives of the function $F(x)$ in equation (7) by means of equations (13) and (14), using only values of $f(\xi)$ and its derivatives along the axis $z = 0$ without either determining the values of $F(x)$ or the values of the function $F(x,z)$ and its derivatives outside the axis $z = 0$.

3.3 The z derivative of the function H(x,z)

It may be pointed out that for the applications in aerodynamics only integrals of the form

$$F_n(x,z) = \int_a^b f(\xi)\, g_n(x,\xi,z)\, d\xi \tag{28}$$

with

$$\lim_{z\to 0} g_n(x,\xi,z) = \frac{1}{(x-\xi)^n} \tag{29}$$

will occur, where

$$g_n(x,\xi,z) = \frac{\partial^n}{\partial x^n}\left\{\frac{x-\xi}{(x-\xi)^2+z^2}\right\} \tag{30}$$

and thus $F_n(x,z)$ are solutions of the Laplace equation

$$\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial z^2} = 0 \tag{31}$$

Since

$$\frac{\partial}{\partial x}\left(\frac{x-\xi}{(x-\xi)^2+z^2}\right) = \frac{\partial^2}{\partial x^2}\left(\tfrac{1}{2}\log\left[(x-\xi)^2 + z^2\right]\right)$$

$$= -\frac{\partial^2}{\partial z^2}\left(\tfrac{1}{2}\log\left[(x-\xi)^2 + z^2\right]\right)$$

$$= -\frac{\partial}{\partial z}\left(\frac{z}{(x-\xi)^2+z^2}\right)$$

the influence function may be also of the form

$$g_n(x,\xi;z) = -\frac{\partial^{n-1}}{\partial x^{n-1}}\frac{\partial}{\partial z}\left(\frac{z}{(x-\xi)^2+z^2}\right) \tag{32}$$

This enables us to express the "normal" derivative $\frac{\partial H}{\partial z}$ of the function

$$H(x,z) = \int_a^b \frac{h(\xi)\ z\ d\xi}{(x-\xi)^2+z^2} \qquad (33)$$

along the axis $z = 0$ in the form of a principal value integral

$$\frac{\partial H}{\partial z}(x,0) = \fint_a^b \frac{h\ (\xi)d\xi}{(x-\xi)^2} \qquad (34)$$

which has to be determined according to the rules given in equations (13) and (14).

An important application of equation (34) is the case of a lifting surface with a discontinuity of the potential-function $H(x,0) = \pm h(x)$ $(a < x < b)$ along the axis $z = 0$. Equation (34) determines the downwash $\frac{\partial H}{\partial z}$ along the lifting surface, or vice versa, since usually the downwash is prescribed by the wing plan form, equation (34) can be used as an integral equation to determine the discontinuity $h(x)$ of the potential function and the load on the wing.

Finally, it may be pointed out that all these relations which hold, if the function $g_n(x,\xi;z)$ is a solution of the Laplace equation, are not generally true, for any influence function $g_n(x,\xi;z)$ which satisfies the condition (29). This can be seen from the following example. The integral

$$\int_a^b \frac{(x-\xi)^2+\lambda z^2}{[(x-\xi)^2+z^2]^2}\,d\xi = \int_{a-x}^{b-x} \frac{u^2+\lambda z^2}{(u^2+z^2)^2}\,du$$

$$= \frac{\lambda-1}{2}\int \frac{d}{du}\left(\frac{u}{u^2+z^2}\right)du + \frac{\lambda+1}{2}\int \frac{du}{u^2+z^2}$$

$$= \frac{\lambda-1}{2}\left[\frac{b-x}{(b-x)^2+z^2} + \frac{x-a}{(x-a)^2+z^2}\right] + \frac{\lambda+1}{2z}\left[\tan^{-1}\frac{b-x}{z} + \tan^{-1}\frac{x-a}{z}\right] \qquad (35)$$

tends to infinity for $z \to 0$, if $(\lambda+1) \neq 0$, and only for $\lambda = -1$ does it tend to the same result, as would have been obtained by putting $z \to 0$ first and using the rules (13) and (14) for the evaluation of a principal value integral.

3.4 Integration by parts of a principal value integral

Now we derive the rules, governing the integration by parts of an integral of the form (12). For this purpose we go back to the function $F(x,z)$ and obtain there by integrating by parts $(n \geq 1)$:

$$\frac{d^n}{dx^n} F(x,z) = \frac{d^{n-1}}{dx^{n-1}} \int_a^b f(\xi) \frac{d}{d\xi}\left(\frac{\xi - x}{(x-\xi)^2 + z^2}\right) d\xi$$

$$= \frac{d^{n-1}}{dx^{n-1}} \left\{ \left(f(\xi) \frac{(\xi - x)}{(\xi - x)^2 + z^2} \right)_a^b + \int_a^b f'(\xi) \frac{(x-\xi)}{(x-\xi)^2 + z^2} d\xi \right\}$$

When taking the limit $z \to 0$, we have

$$\frac{(-1)^n}{n!} \frac{d^n}{dx^n} F(x) = \fint_a^b \frac{f(\xi)\, d\xi}{(x-\xi)^{n+1}}$$

$$= \frac{1}{n} \left[\frac{f(b)}{(x-b)^n} - \frac{f(a)}{(x-a)^n} - \fint_a^b \frac{f'(\xi)\, d\xi}{(x-\xi)^n} \right] \qquad (36)$$

This shows that the rules apply in the same way as for an ordinary proper integral, provided that the principal values are taken as defined by equation (13). From (36) and (13) we have the following rule for the differentiation of such an integral ($n \geq 1$):

$$\frac{d}{dx} \fint_a^b \frac{f(\xi)\, d\xi}{(x-\xi)^n} = - n \fint_a^b \frac{f(\xi)\, d\xi}{(x-\xi)^{n+1}}$$

$$= \fint_a^b \frac{f'(\xi)\, d\xi}{(x-\xi)^n} + \frac{f(a)}{(x-a)^n} - \frac{f(b)}{(x-b)^n} \qquad (37)$$

4 Principal values of fractional order

In most applications the function $f(\xi)$ which occurs in the integral equation (7) or (12), is not regular at the ends a and b of the integration interval. It may be of the form

$$f(\xi) = \frac{A(\xi)}{(\xi - a)^\alpha} \qquad \text{near } \xi = a$$

$$f(\xi) = \frac{B(\xi)}{(b-\xi)^\beta} \qquad \text{near } \xi = b \qquad (38)$$

where $A(\xi)$ and $B(\xi)$ are regular functions of ξ, i.e. expandable in a Taylor series and α and β are not integers. Then the relations (13), (14), (36), (37) require certain modifications, which allow for these singularities.

One way of generalizing the results of section 3 for such integrals would be to replace the integral (7) or (12) by a contour integral, which contains the interval $a < \xi < b$ in its interior and then to go to the limiting value of the contour, which would consist of twice the interval $a < \xi < b$ and three bulges or dents respectively at a, b and P, as was explained in section 1 (see fig.1). The contributions of the bulges at a and b have to be allowed for in order to generalize the results of the last section.

We shall follow a different and perhaps simpler method by which the contributions of the bulges at a and b is determined by means of an integration by parts, which are permissible for the contour integral and thus are still valid in the limiting case.

4.1 Singularity at the lower limit of the integral

If $\alpha < 1$ and $\beta < 1$ in equation (38), the rule for the evaluation of the principal value, as given in equations (13) and (14), applies in the same way as before, provided that the integral is, as usual, defined as the limit

$$\not\int_a^b \frac{f(\xi)\ d\xi}{(x-\xi)^{n+1}} = \lim_{\delta\to 0} \not\int_{a+\delta}^{b-\delta} \frac{f(\xi)\ d\xi}{(x-\xi)^{n+1}} \tag{39}$$

The limit $\delta \to 0$ can be taken, if $\alpha < 1$, $\beta < 1$, and leads to a finite answer. In order to obtain the contribution of the "bulge"-integrals at $\xi = a$ and $\xi = b$ for any (not integer) value of α and β, we proceed as follows.

We split the integral from a to b into two integrals, so that only the first part

$$\int_a^c \frac{f(\xi)\ d\xi}{(x-\xi)^{n+1}} = \int_a^c \frac{A(\xi)\ d\xi}{(\xi-a)^\alpha\ (x-\xi)^{n+1}} \tag{40}$$

is influenced by the singularity at a. We may assume that

$$a < c < x < b$$

so that, for the time being, we may forget about the singularity at $\xi = x$. Then (40) is of the form

$$I = \lim_{\delta\to 0} \int_{a+\delta}^{c} \frac{h(\xi)\ d\xi}{(\xi-a)^\alpha} \tag{41}$$

where

$$h(\xi) = \frac{f(\xi)\ (\xi-a)^\alpha}{(x-\xi)^{n+1}} = \frac{A(\xi)}{(x-\xi)^{n+1}}$$

is a regular function of ξ near a. We introduce

$$H(\xi) = H(a) + \int_a^\xi h(\xi)\ d\xi$$

which is also a regular function near $\xi = a$ and obtain by an integration by parts from (41):

$$I = \lim_{\delta\to 0}\left\{\int_{a+\delta}^{c} \frac{H'(\xi)\,d\xi}{(\xi-a)^{\alpha}}\right\}$$

$$= \lim_{\delta\to 0}\alpha\left\{\int_{a+\delta}^{c} \frac{H(\xi)\,d\xi}{(\xi-a)^{\alpha+1}} - \frac{H(a+\delta)}{\alpha\delta^{\alpha}}\right\} + \frac{H(c)}{(c-a)^{\alpha}} \qquad (42)$$

For this we write

$$I = \alpha \fint_{a}^{b} \frac{H(\xi)\,d\xi}{(\xi-a)^{\alpha+1}} + \frac{H(c)}{(c-a)^{\alpha}} \qquad (43)$$

where the first integral does not exist in the ordinary sense, since the exponent $\alpha+1$ is greater than 1. But we obtain a finite answer for I, if we take the principal value of the integral, which is defined by the bracket on the right of equation (42).

By generalizing this, i.e. by repeating the partial integration in (42) again and again, we arrive at the following definition of a principal value (m = integer number):

$$\fint_{a}^{c} \frac{H(\xi)\,d\xi}{(\xi-a)^{\alpha+m}} = \lim_{\delta\to 0}\left\{\int_{a+\delta}^{c} \frac{H(\xi)\,d\xi}{(\xi-a)^{\alpha+m}} - \sum_{j=0}^{m-1} \frac{H^{(j)}(a)}{j!\,\delta^{\alpha+m-1-j}(\alpha+m-1-j)}\right\} \qquad (44)$$

$$(0 < \alpha < 1)$$

Here we have used the Taylor series for the function $H(\xi)$ near $\xi = a$:

$$H(\xi) = H(a) + (\xi-a)\,H'(a) + \ldots = \sum_{j=0}^{\infty} (\xi-a)^{j}\,\frac{H^{(j)}(a)}{j!} \qquad (45)$$

The limit as defined by equation (44) is finite, since these terms of the integral, which in the limit $\delta \to 0$ would tend to infinity, are cancelled by the additional terms in the bracket. This can be seen in the following way:

We denote the polynomial which consists of the first (m+1) terms of the series (45) by $H_1(\xi)$:

$$H_1(\xi) = \sum_{j=0}^{m} \frac{(\xi-a)^{j}}{j!}\,H^{(j)}(a) \qquad (46)$$

and obtain:

$$\int_{a+\delta}^{c} \frac{H(\xi)\,d\xi}{(\xi-a)^{\alpha+m}} = \int_{a+\delta}^{c} \frac{H(\xi) - H_1(\xi)}{(\xi-a)^{\alpha+m}}\,d\xi + \sum_{j=0}^{m} \frac{H^{(j)}(a)}{j\,!} \int_{a+\delta}^{c} (\xi-a)^{j-\alpha-m}\,d\xi$$

Here the first integral on the right is regular in the ordinary sense, since the factor $(\xi-a)^{m+1}$ is cancelled out. For the second term we have

$$\sum_{j=0}^{m} \frac{H^{(j)}(a)}{j\,!}\left(\frac{(\xi-a)^{1+j-\alpha-m}}{1+j-\alpha-m}\right)_{a+\delta}^{c} = -\sum_{j=0}^{m} \frac{H^{(j)}(a)}{j\,!}\left(\frac{\delta^{1+j-\alpha-m}}{1+j-\alpha-m} - \frac{(c-a)^{1+j-\alpha-m}}{1+j-\alpha-m}\right)$$

The first terms in this sum, which (for $j=0,1\ldots m-1$) would, in the limit $\delta \to 0$, tend to infinity, are cancelled by the additional terms in the bracket of (44), and therefore the result is finite.

Now we are going to show, that the definition (44) is a reasonable generalization of equation (42). For $m = 1$ equation (44) reduces to equation (42). Now we derive the rule for an integration by parts of the integral (44). This will enable us to reduce (44) by a succession of integrations by parts to the form (42) or even (41).

4.2 Integration by parts

The rule for the integration by parts of the integral (44) can be written down as follows:

$$\fint_{a}^{c} \frac{H(\xi)\,d\xi}{(\xi-a)^{\alpha+m}} = \frac{1}{\alpha+m-1}\left[\fint_{a}^{c} \frac{H'(\xi)\,d\xi}{(\xi-a)^{\alpha+m-1}} - \frac{H(c)}{(c-a)^{\alpha+m-1}}\right] \qquad (47)$$

It means that the formal procedure for the integration by parts applies also in this case, provided that the principal value of the integrals is taken as defined in (44) and all terms are omitted, which would formally give infinity.

In order to prove this equation (which for $m = 1$ becomes identical with equation (42)), we have to go back to the definition (40) of the principal value. We integrate by parts:

$$\int_{a+\delta}^{c} \frac{H(\xi)\,d\xi}{(\xi-a)^{\alpha+m}} = \left\{\frac{H(a+\delta)}{\delta^{\alpha+m-1}} + \int_{a+\delta}^{c} \frac{H'(\xi)\,d\xi}{(\xi-a)^{\alpha+m-1}}\right\}\frac{1}{(\alpha+m-1)} - \frac{H(c)}{(\alpha+m-1)(c-a)^{\alpha+m-1}}$$

In the first term on the right we replace $H(a+\delta)$ by its Taylor series (comp. equation (45)) and obtain

$$\frac{H(a+\delta)}{\delta^{\alpha+m-1}} = \sum_{j=0}^{\infty} \frac{H^{(j)}(a)}{j\,!}\,\delta^{j+1-m-\alpha}\left(\frac{-\,j}{\alpha+m-1-j} + \frac{\alpha+m-1}{\alpha+m-1-j}\right)$$

Here the second term in the bracket can be taken to the other side, which yields the definition for the principal value of the integral on the left of equation (47). The first term in the bracket gives, together with the integral, the principal value of the integral on the right of equation (47), if one replaces j, which runs now from 1 to $m-1$, by $(K+1)(K=0,\ldots m-2)$. Thus the proof of equation (47) is complete.

4.3 Differentiation with respect to the limit

Now we can show that the derivative with respect to a limit of an integral as defined in equation (44) can be obtained by differentiating the integrand only:

$$\frac{d}{da} \fint_a^c \frac{H(\xi)\, d\xi}{(\xi-a)^{\alpha+m}} = (\alpha+m) \fint_a^c \frac{H(\xi)\, d\xi}{(\xi-a)^{\alpha+m+1}} \qquad (48)$$

This will be proved by going back to the definition (44). At first we have

$$\frac{d}{da} \int_{a+\delta}^c \frac{H(\xi)\, d\xi}{(\xi-a)^{\alpha+m}} = (\alpha+m) \int_{a+\delta}^c \frac{H(\xi)\, d\xi}{(\xi-a)^{\alpha+m+1}} - \frac{H(a+\delta)}{\delta^{\alpha+m}}$$

We introduce again the expansion (45), this time in the form

$$- \frac{H(a+\delta)}{\delta^{\alpha+m}} = \sum_{K=0}^{\infty} \frac{H^{(K)}(a)}{K!} \delta^{K-\alpha-m} \left(- \frac{\alpha+m}{\alpha+m-K} + \frac{K}{\alpha+m-K} \right)$$

Differentiating the additional sum in (44) yields

$$- \sum_{j=0}^{m-1} \frac{H^{(j+1)}(a)}{j!\, \delta^{\alpha+m-1-j} (\alpha+m-1-j)} = - \sum_{K=1}^{m} \frac{H^{(K)}(a)}{(K-1)!\, \delta^{\alpha+m-K!} (\alpha+m-K)}$$

which cancels the second term of the bracket in the last equation but one. The first term of this bracket yields together with the integral above in the limit $\delta \to 0$ the principal value integral on the right of equation (48) q.e.d.

4.4 Singularity at the upper limit

For an integral with a singularity at the upper limit b the definition of the principal value reads

$$\fint_c^b \frac{H(\xi)\, d\xi}{(b-\xi)^{\beta+m}} = \lim_{\delta \to 0} \left\{ \int_c^{b-\delta} \frac{H(\xi)\, d\xi}{(b-\xi)^{\beta+m}} - \sum_{K=0}^{m-1} \frac{(-1)^K H^{(K)}(b)}{K!\, (\beta+m-1-K)\, \delta^{\beta+m-1-K}} \right\} \qquad (49)$$

and all the rules for differentiation and integration by parts can be put down in a corresponding way.

We may interpret the definitions (44) and (49) in the following way. Instead of taking the integral from a to c which would not converge, we expand the integrand near $\xi = a$ and retain only those terms of the series, which after the integration yield a finite answer. In other words: We take only the "finite part" of the integral (Hadamard). It may be mentioned that the additional terms in (44) and (49), which cancel the "infinite contributions" to the integral, are, for any given function, determined in a unique way, since no term of the order δ^0, which would give a finite contribution, may occur because of our assumption that α is always between 0 and 1.

4.5 Representation of the solutions of the Laplace equation by principal value integrals

From now on the "crossed" integral sign is to denote the principal value of the integral in question, in the following sense: At the point $\xi = x$, the principal value must be taken according to equation (13) and equation (14). At the ends $\xi = a$ and $\xi = b$ of the interval, the principal value must be taken according to the rules equation (44) and equation (49).

We consider the integral

$$F_n(x) = \frac{(-1)^n}{n!} \frac{d^n F(x)}{dx^n} = \fint_a^b \frac{f(\xi)\, d\xi}{(x-\xi)^{n+1}} \qquad (12)$$

and assume that $f(\xi)$ is singular, at the ends of the integration interval, i.e. the function $f(\xi)$ is of the form (38) near $\xi = a$ and $\xi = b$, with α and β denoting any (positive or negative) not integer number. Thus $f(\xi)$ is either zero or infinity there.

Integration by parts of (12) leads to

$$F_n(x) = -\frac{1}{n} \fint_a^b \frac{df(\xi)}{d\xi} \frac{d\xi}{(x-\xi)^n}, \qquad (n \geqq 1) \qquad (50)$$

In order to prove this we divide the integral into three parts by means of the two points c_1 and c_2 which are chosen in such a way that

$$a < c_1 < x < c_2 < b .$$

For the middle interval $c_1 < \xi < c_2$ we apply equation (36), for the two intervals $a < \xi < c_1$ and $c_2 < \xi < b$, we apply equation (47), which means that the rule for integration by parts applies in the same way as for ordinary integrals, if one omits all the terms which would give the contribution infinity. Thus we are left with equation (50). We repeat this argument and obtain:

$$\fint_a^b \frac{f(\xi)\, d\xi}{(x-\xi)^{n+1}} = \frac{1}{n(n-1)} \fint_a^b \frac{d^2 f(\xi)}{d\xi^2} \frac{d\xi}{(x-\xi)^{n-1}}, \qquad (n \geqq 2) \qquad (51)$$

or in general

$$\fint_a^b \frac{f(\xi)\, d\xi}{(x-\xi)^{n+1}} = \frac{(-1)^j\,(n-j)!}{n!} \fint_a^b \frac{d^j f(\xi)}{d\xi^j} \frac{d\xi}{(x-\xi)^{n-j+1}} \qquad (52)$$

$$(j = 0,1,2,\dots n)$$

By means of equation (52) we are in a position to reduce the higher order principal values at $\xi = x$ to Cauchy's principal value ($j = n$), but then we have to deal with higher order singularities at the limits of the integral (see equations (44) and (49)). We may also read equation (52) in the opposite direction, i.e. we can use (52) to reduce the singularities at the ends a and b of the integration interval, but have to deal then with a higher order principal value at $\xi = x$.

Since $F(x,z)$ in (8) satisfies the Laplace equation (31), $F(x)$ in (7) represents the boundary values for $z = 0$ of a solution of the Laplace equation. The same is true for $F_n(x)$ in (15). Thus (52) gives $(n+1)$ possibilities to represent the same solution $F_n(x)$ by means of a singularity distribution

$$h_j(\xi) = \frac{(-1)^j\,(n-j)!}{n!}\,\frac{d^j f(\xi)}{d\,\xi^j} \qquad (j = 0,1,\ldots n)$$

in the form

$$F_n(x) = \fint_a^b \frac{h_j(\xi)\,d\xi}{(x-\xi)^{n+1-j}} \tag{53}$$

where the principal value is to be taken for $\xi = a$, $\xi = x$, and $\xi = b$.

By means of a similar argument we can prove that the normal derivative $\frac{\partial H}{\partial z}$ of the function $H(x,z)$ in equation (33) is given along the axis $z = 0$ by equation (34), even in the case that the function $h(\xi)$ becomes singular at both ends $\xi = a$ and $\xi = b$, provided that the principal value of the integral is always taken.

4.6 Solution of an integral equation

From equation (52) we can draw the following important conclusion: If $f(\xi)$ is a solution of the integral equation

$$\fint_a^b \frac{f(\zeta)\,d\xi}{x-\xi} = F(x), \qquad (a < x < b) \tag{54}$$

then $\frac{d^n f(\xi)}{d\,\xi^n}$ is a solution of the equation $(n = 1,2,\ldots)$

$$\fint_a^b \frac{d^n f(\xi)}{d\,\xi^n}\,\frac{d\xi}{x-\xi} = \frac{d^n F(x)}{dx^n}, \qquad (a < x < b) \tag{55}$$

Thus we conclude $(F \equiv 0)$, that any derivative $\frac{d^n f_o(\xi)}{d\,\xi^n}$ of $f_o(\xi)$ is a solution of the equation

$$\fint_a^b \frac{d^n f_o(\xi)}{d\,\xi^n}\,\frac{d\xi}{x-\xi} = 0, \qquad (a < x < b) \tag{56}$$

if $f_o(\xi)$ itself is a solution of (56). Any such solution may be added to the solution $f(\xi)$ of equation (54). Thus a unique solution of an integral equation of the type (54) or (56) can be obtained only by additional conditions which specify the type of singularity which is admissible at the ends $\xi = a$ and $\xi = b$ of the interval.

5 Integration of functions, which involve principal values of higher order

5.1 A theorem

In the applications (comp. ref. 2) we often encounter an integral of the form

$$I_n(X) = \int_{a_o}^{X} h(x)\ Fn(x)\ dx, \qquad (a_o < a < b) \tag{57}$$

where $h(x)$ is a regular function of x (it can be expanded in a Taylor series at any point of the integration interval) and

$$F_n(x) = \fint_{a}^{b} \frac{f(\xi)\ d\xi}{(x-\xi)^{n+1}} \tag{58}$$

is an integral involving principal values at $\xi = x$ and possibly at $\xi = a$ (or $\xi = b$). In this section we shall always assume that $f(\xi)$ is near $\xi = a$ of the form

$$f(\xi) = A(\xi)\ (\xi-a)^{-\frac{1}{2}-m}$$

(m = interger, A = regular).

Since any function $F_n(x)$, for which $(\xi-a)^{m+\frac{1}{2}}\ f(\xi)$ is a regular function, can be reduced by means of integration by parts to the form, where

$$\frac{f(\xi)}{\sqrt{\xi-a}} = A(\xi) = \sum_{K=0}^{\infty} \frac{A^{(K)}(a)}{K!}\ (\xi-a)^K \tag{59}$$

is a regular function, we may assume that $f(\xi)$ has the form (59).

Since $F_n(x)$ is finite for $a_o \leqq x < a$ and $a < x < b$ and $h(x)$ is regular, the integral $I_n(X)$ can easily be determined if $X < a$. For $X > a$, the integration in (57) cannot be performed immediately, since the integral might be meaningless due to the singularity of $F(x)$ near $X = a$.

But the integral $I_n(X)$ has a definite meaning if we define it as the limit for $z \to 0$ of the more general function $I_n(x,z)$:

$$I_n(X) = \lim_{z\to 0} I_n(x,z) \tag{60}$$

where

$$I_n(x,z) = \int_{a_o}^{X} h(x,z)\ F_n(x,z)\ dx \tag{61}$$

with $h(x,0) \equiv h(x)$ and

$$F_n(x,z) = \frac{(-1)^n}{n\,!} \frac{d^n}{dx^n} \int_a^b \frac{f(\xi)\ (x-\xi)\ d\xi}{(x-\xi)^2 + z^2} \tag{62}$$

All these integrals have a definite meaning and are regular, if the integration is performed along a line $z = \text{const} \neq 0$ and the limit $z \to 0$ is taken afterwards. But sometimes the integrand is known only along the x-axis and cannot easily be determined for $z \neq 0$. Here the following theorem is very useful:

The integral $I_n(X)$ in (57) which is defined as the limit for $z \to 0$ of the function $I_n(X,z)$ in (61), can be determined for $X \leqq a$ as an ordinary integral. For $X \geqq a$ we have to take the principal value of the integral between the limits a_o and a and have to add the integral between the limits a and X.

5.2 Proof of the theorem

To prove this theorem, we split the integral $I_n(X,z)$ between the limits a_o and $X > a$ into three parts, namely the integral from a_o to $a - \varepsilon_1$, the integral from $a - \varepsilon_1$ to $a + \varepsilon_1$ and the integral from $a + \varepsilon_1$ to X. In the first and third part we may go to the limit $z \to 0$ before the integration, since $h(x,z)$ tends to $h(x)$ and $F_n(x,z)$ tends to $F_n(x)$, which is regular for $x < a$, and for $a < x < b$ is defined by means of a principal value.

From the second part of the integral, namely

$$I_n^{*} = \int_{a-\varepsilon_1}^{a+\varepsilon_1} h(x,z)\ F_n(x,z)\ dx \tag{63}$$

we obtain for $n \geqq 2$ by successive partial integrations, using (62):

$$(-1)^n\, n!\ I_n^{*}(x,z) = \sum_{j=0}^{n-2} (-1)^j \left[h^{(j)}(x,z) \frac{d^{n-j-1}}{dx^{n-j-1}} \int_a^b \frac{f(\xi)\ (x-\xi)\ d\xi}{(x-\xi)^2 + z^2} \right]_{a-\varepsilon_1}^{a+\varepsilon_1}$$

$$+ (-1)^{n-1} \int_{a-\varepsilon_1}^{a+\varepsilon_1} h^{(n-1)}(x,z) \frac{d}{dx} \left(\int_a^b \frac{f(x)\ (x-\xi)\ d\xi}{(x-\xi)^2 + z^2} \right) dx \tag{64}$$

where $h^{(j)}$ is the j^{th} derivative of h with respect to x.

We shall prove later on (see 5.3) that $F_o(x,z)$ tends to a finite function, which is a regular function $R(x-a)$ of $(x-a)$ for $x < a$ and the sum of the same regular function $R(x-a)$ and the product of $\pi \sqrt{a-x}$ times another regular function $S(x-a)$ of $(x-a)$:

$$F_o(x,0) = F_o(x) = \begin{cases} R(x-a)\ , & x > a, \\ R(x-a) + \pi \sqrt{x-a}\ S(x-a), & x < a, \end{cases} \tag{65}$$

Assuming for the time being that (65) is correct, we can see that $I_o^*(x,z)$ tends to a regular function $I_o^*(x)$ and $I_1^*(x,z)$ to a regular function $I_1^*(x)$, which depends on $F_1(x) = -\frac{dF_o}{dx}$. Since both I_o^* and I_1^* are finite they vanish for $\varepsilon_1 \to 0$ and our theorem is proved for $n = 0$ and $n = 1$.

Since for the last integral in (64) the same remarks hold, we can go to the limit $z \to 0$ in (64) and obtain for $n \geqq 2$:

$$(-1)^n\, n!\, I_n^*(x) = -\sum_{j=0}^{n-2} (-1)^j \left[h^{(j)}(x) \frac{d^{n-j-1}}{dx^{n-j-1}} \left(\pi \sqrt{a-x}\, S(x-a) \right) \right]_{x=a-\varepsilon_1}$$

$$+ \sum_{j=0}^{n-2} (-1)^j \left[h^{(j)}(x) \frac{d^{n-j-1}}{dx^{n-j-1}} R(x-a) \right]_{x=a-\varepsilon_1}^{a+\varepsilon_1}$$

$$+ (-1)^{n-1} \int_{a-\varepsilon_1}^{a+\varepsilon_1} h^{(n-1)}(x) \frac{d}{dx} F_o(x)\, dx \qquad (66)$$

The second and the third term of this expression are finite and tend to zero for $\varepsilon_1 \to 0$, whereas the first term tends to infinity. By adding the first part of the integral (57), namely

$$\int_{a_o}^{a-\varepsilon_1} h(x)\, F_n(x)\, dx$$

which can be integrated by parts in the same way, we would, apart from finite terms, obtain the same terms at the upper limit $a-\varepsilon_1$, as are given above in equation (66), but with the opposite sign. It follows that "integrating from $a-\varepsilon_1$ to $a+\varepsilon_1$ and taking the limit $\varepsilon_1 \to 0$" means "taking away the infinite terms" or in other words "taking the principal value of the integral". Since these "infinite terms" are determined in a unique way, we may apply the same procedure for determining this principal value, as was explained in the preceding paragraph on principal values of a fractional order (section 4).

Thus we have proved that the integral

$$I_n(x) = \int_{a_o}^{a} h(x)\, F_n(x)\, dx, \qquad (a_o < a < b) \qquad (57)$$

where $F_n(x)$ is defined by

$$F_n(x) = \fint_a^b f(\xi) \frac{d\xi}{(x-\xi)^{n+1}} \qquad (58)$$

and

$$\frac{f(\xi)}{(\xi-a)^{-m+\frac{1}{2}}} = A(\xi)$$

is a regular function (m = integer number) gives the limiting value of the integral

$$\lim_{\varepsilon\to 0} \int_{a_0}^{a-\varepsilon} h(x)\ F_n(x)\ dx,$$

if taken in the ordinary sense, and gives the limit

$$\lim_{\varepsilon\to 0} \int_{a_0}^{a+\varepsilon} h(x)\ F_n(x)\ dx$$

if we take the "principal value" of it, as explained in section 4.

5.3 An auxiliary theorem

Finally, we have to prove equation (65). We choose a value c between $a+\varepsilon_1$ and b, so that for $a \leqslant \xi \leqslant c$ a finite number of terms of the series (59) represents $f(\xi)$ with sufficient accuracy. Then we write

$$F_0(x,z) = \int_a^b \frac{f(\xi)\ (x-\xi)\ d\xi}{(x-\xi)^2 + z^2} = \int_a^c + \int_c^b \tag{67}$$

This tends to $L(x) + M(x)$ for $z \to 0$, where

$$M(x) = \int_c^b \frac{f(\xi)\ d\xi}{x-\xi}$$

is a regular function of $(x-a)$, if $x < a+\varepsilon_1 < c$, and therefore contributes to $R(x-a)$ in (65).

Thus it remains to be shown that

$$L(x) = \lim_{z\to 0} \int_a^c \sqrt{\xi-a}\ A(\xi) \frac{(x-\xi)\ d\xi}{(x-\xi)^2+z^2}$$

$$= \int_a^c A(\xi) \frac{\sqrt{\xi-a}\ d\xi}{x-\xi} = \sum_{K=0}^{N} \frac{A^{(K)}(a)}{K!} \int_0^{c-a} \frac{u^{K+\frac{1}{2}}\ du}{(x-a) - u} \tag{68}$$

$(u = \xi-a)$ has the required properties, as described by (65). The sum in (68) is extended over a finite number N of terms of the series (59), which gives the function with sufficient accuracy for $a < \xi < c$.

It can be seen by differentiating

$$\frac{d}{du}\left\{\sum_{K=0}^{N-1} B_K\, u^{K+\frac{3}{2}} + D\int \frac{\sqrt{u}\,du}{(x-a)-u}\right\}$$

$$= \sum_{K=0}^{N-1}\left(K+\frac{3}{2}\right) B_K\, u^{K+\frac{1}{2}} + \frac{D\sqrt{u}}{x-a-u} \tag{69}$$

and equating this expression to

$$\sum_{K=0}^{N} \frac{A^{(K)}(a)}{K\,!}\,\frac{u^{K+\frac{1}{2}}}{(x-a)-u}$$

that the integral $L(x)$ in (68) can be written as

$$L(x) = p_1(x-a)\int_0^{c-a} \frac{\sqrt{u}\,du}{x-a-u} + p_2(x-a) \tag{70}$$

where p_1 and p_2 are polynomials in $(x-a)$. Furthermore we have

$$\int \frac{\sqrt{u}\,du}{x-a-u} = -2\sqrt{u} + \sqrt{x-a}\,\log\frac{\sqrt{u}+\sqrt{x-a}}{\sqrt{u}-\sqrt{x-a}} \qquad \text{if } x > a$$

$$= -2\sqrt{u} - 2\sqrt{a-x}\,\tan^{-1}\sqrt{\frac{a-x}{u}} \qquad \text{if } x < a$$

and

$$\int_0^{c-a}\frac{\sqrt{u}\,du}{x-a-u} = -2\sqrt{c-a} + \sqrt{x-a}\,\log\frac{1+\sqrt{\frac{x-a}{c-a}}}{1-\sqrt{\frac{x-a}{c-a}}}, \qquad x > a$$

$$= -2\sqrt{c-a} + 2\sqrt{a-x}\left(\frac{\pi}{2} - \tan^{-1}\sqrt{\frac{a-x}{c-a}}\right), \qquad x < a \tag{71}$$

where the principal value at $\xi = x$, $u = x-a$, has been allowed for in the first integral $(x > a)$.

The result for $x > a$ can be expanded in a power series which contains only (not negative) powers of $(x-a)$. The same applies for the difference

$$\int_0^{c-a}\frac{\sqrt{u}\,du}{x-a-u} - \pi\sqrt{a-x}$$

if $x < a$, and both series are identical as can be seen by comparing the coefficients or by means of the relation:

$$\sqrt{x-a}\,\log\frac{1+\sqrt{\frac{x-a}{c-a}}}{1-\sqrt{\frac{x-a}{c-a}}} = -2\sqrt{a-x}\,\tan^{-1}\sqrt{\frac{a-x}{c-a}}$$

Thus we have the result

$$\begin{aligned}\int_0^{c-a}\frac{\sqrt{u}\,du}{(x-a)-u} &= p_3(x-a), \qquad (x > a)\\ &= p_3(x-a) + \pi\sqrt{a-x}, \qquad (x < a)\end{aligned} \tag{72}$$

where p_3 is a regular function of $x-a$, and finally

$$\begin{aligned}L(x) &= p_2 + p_1\,p_3 \equiv R(x-a), \qquad x > a\\ &= p_2 + p_1\,p_3 + p_1\,\pi\sqrt{x-a} \equiv R(x-a) + S(x-a)\,\pi\sqrt{x-a}, \qquad x < a\end{aligned} \tag{73}$$

is a regular function R of $(x-a)$ for $x > a$, and the product $\sqrt{x-a}$ times a regular function S of $(x-a)$ has to be added to obtain its value for $x < a$. Thus equation (65) is correct and the proof of our theorem is complete.

6 Some examples

In order to illustrate the rules derived in the preceding sections, we consider the integral

$$F_o(x) = \fint_{-a}^{a}\frac{d\xi}{\sqrt{a^2-\xi^2}\,(x-\xi)} \tag{74}$$

The indefinite integral is

$$\frac{1}{\sqrt{a^2-x^2}}\log\left(\frac{a^2-x\xi+\sqrt{a^2-x^2}\,\sqrt{a^2-\xi^2}}{(\xi-x)\,\sqrt{a^2-x^2}}\right) \quad \text{if} \quad x^2 < a^2$$

and

$$\frac{-1}{\sqrt{x^2-a^2}}\sin^{-1}\left(\frac{a^2-x\xi}{a|\xi-x|}\right) \quad \text{if} \quad x^2 > a^2$$

and we obtain (using (13) for $x^2 < a^2$) the result

$$F_o(x) = 0 , \qquad \text{if} \quad x^2 < a^2$$

$$F_o(x) = \frac{\pi}{\sqrt{x^2-a^2}} , \qquad \text{if} \quad x > a \tag{75}$$

$$F_o(x) = - \frac{\pi}{\sqrt{x^2-a^2}} , \qquad \text{if} \quad x < -a$$

If we replace $(\xi-x)$ in the logarithmic term by its modulus $|\xi-x|$, we need not consider the principal value at $\xi = x$. According to equation (27) it is sufficient to take the indefinite integral at the limits $\xi = \pm a$.

Next, we consider the integral

$$F_1(x) = - \frac{dF_o(x)}{dx} = -\!\!\!\!\!\!\int_{-a}^{a} \frac{d\xi}{\sqrt{a^2-\xi^2}\,(x-\xi)^2} \tag{76}$$

which. by an integration by parts, can be written as

$$F_1(x) = - -\!\!\!\!\!\!\int_{-a}^{a} \frac{\xi}{(a^2-\xi^2)^{\frac{3}{2}}} \frac{d\xi}{(x-\xi)} \tag{77}$$

The integral can be evaluated by differentiating equation (75)

$$F_1(x) = 0 , \qquad \text{if} \quad x^2 < a^2,$$

$$F_1(x) = \left| \frac{\pi x}{(x^2-a^2)^{\frac{3}{2}}} \right| , \qquad \text{if} \quad x^2 > a^2 \tag{78}$$

Working out equation (76) directly, we have

$$\int \frac{d\xi}{\sqrt{a^2-\xi^2}\,(x-\xi)^2} = \frac{1}{a^2-x^2} \left[\frac{\sqrt{a^2-\xi^2}}{x-\xi} - x \int \frac{d\xi}{\sqrt{a^2-\xi^2}\,(x-\xi)} \right] \tag{79}$$

which for $x^2 > a^2$ leads, by means of equation (75), directly to the answer (78). For $x^2 < a^2$ the integral in the bracket vanishes, as shown above and we are left with

$$\frac{1}{a^2-x^2} \left[\frac{\sqrt{a^2-x^2}}{\varepsilon} - \frac{\sqrt{a^2-x^2}}{(-\varepsilon)} \right] - \frac{2}{\varepsilon\sqrt{a^2-x^2}} = 0$$

(compare equation (13)). This result could have been obtained directly from equation (76) by inserting the limits $\xi = \pm a$ into (79), without considering the integral near $\xi = x$.

We may also determine $F_1(x)$ from equation (77). The indefinite integral is

$$-\int \frac{\xi \, d\xi}{(a^2-\xi^2)^{\frac{3}{2}} \, (x-\xi)} = \frac{1}{a^2-x^2} \left[\frac{x+\xi}{\sqrt{a^2-\xi^2}} - x \int \frac{d\xi}{(x-\xi)\sqrt{a^2-\xi^2}} \right]$$

Here the second term yields in the same way as before the answer (equation (78)). The first term has to be taken at the limits $\xi = a - \delta$ and $\xi = - a + \delta$ with the appropriate correction terms according to equation (44) and equation (49). We obtain $(\alpha = \frac{1}{2}, \, m = 1)$

$$\left[\frac{1}{a^2-x^2} \cdot \frac{x+a}{\sqrt{2a}\sqrt{\delta}} + \frac{a}{\frac{1}{2}\,\delta^{\frac{1}{2}} \, (x-a) \, (2a)^{\frac{3}{2}}} \right]$$

$$+ \left[- \frac{1}{a^2-x^2} \, \frac{x-a}{\sqrt{2a}\sqrt{\delta}} + \frac{(-a)}{\frac{1}{2}\,\delta^{\frac{1}{2}} \, (x+a) \, (2a)^{\frac{3}{2}}} \right] = 0$$

Thus the result (78) has been confirmed once again.

Another integral, which occurs fairly often in supersonic aerodynamics, is the following

$$\overset{*}{F}(x) = \times\!\!\!\!\!\!\int_{-a}^{a} \frac{\sqrt{a^2-\xi^2} \, d\xi}{(x-\xi)^2} \qquad (80)$$

It also represents the downwash behind a lifting line with an elliptic load distribution in subsonic flow.

The indefinite integral is

$$\int \frac{\sqrt{a^2-\xi^2} \, d\xi}{(x-\xi)^2} = - \left[\sin^{-1} \frac{\xi}{a} + \frac{(x+\xi)\sqrt{a^2-\xi^2}}{a^2-x^2} - \frac{(a^2-\xi^2)^{\frac{3}{2}}}{(a^2-x^2) \, (x-\xi)} \right]$$

$$+ \, x \int \frac{d\xi}{(x-\xi)\sqrt{a^2-\xi^2}} \qquad (81)$$

where the last integral has been treated above. For $x^2 > a^2$ we obtain by means of equations (74) and (75)

$$\int_{-a}^{a} \frac{\sqrt{a^2-\xi^2} \, d\xi}{(x-\xi)^2} = -\pi + \left| \frac{\pi x}{\sqrt{x^2-a^2}} \right|, \qquad (x^2 > a^2) \qquad (82)$$

For $x^2 < a^2$ we have

$$\times\!\!\!\!\!\!\int_{-a}^{a} \frac{\sqrt{a^2-\xi^2}\, d\xi}{(x-\xi)^2} = -\pi , \qquad (x^2 < a^2) \tag{83}$$

since the last integral in (81) vanishes according to (75) and because the second and third term in the bracket contribute nothing at the limits $\xi = \pm a$, and near the pivotal point $\xi = x$. The latter fact follows either from equation (27) or from equation (13) by means of

$$\frac{(a^2-x^2)^{\frac{1}{2}}}{\varepsilon} - \frac{(a^2-x^2)^{\frac{1}{2}}}{(-\varepsilon)} - \frac{2\sqrt{a^2-x^2}}{\varepsilon} = 0 .$$

In order to illustrate the theorem in section 5, we determine now the integral (comp. ref. 2):

$$I(X) = \int_{+1}^{X} \frac{\sqrt{1-x^2}}{x} F_1(x)\, dx, \qquad (0 < a < 1;\ 0 < x < 1) \tag{84}$$

where $F_1(x)$ is given by equation (76). Thus we have with (78)

$$I(X) = \pi \int_{1}^{X} \frac{\sqrt{1-x^2}\, dx}{(x^2-a^2)^{\frac{3}{2}}} \tag{85}$$

We may integrate by parts and obtain for $a < x < 1$:

$$I = \pi \int_{1}^{x} \frac{\sqrt{1-x^2}}{x} \frac{d}{dx}\left(\frac{-1}{\sqrt{x^2-a^2}}\right) dx ,$$

$$I(x) = -\pi \left[\frac{\sqrt{1-x^2}}{x\sqrt{x^2-a^2}} + \int_{1}^{x} \frac{dx}{x^2\sqrt{1-x^2}\,\sqrt{x^2-a^2}} \right] \tag{86}$$

The second term can be reduced to the standard elliptic integral $E(k,\phi)$ of the second kind by means of the substitution

$$\sin\phi = \frac{\sqrt{1-\frac{a^2}{x^2}}}{\sqrt{1-a^2}} , \qquad d\phi = \frac{a\, dx}{x\sqrt{x^2-a^2}\,\sqrt{1-x^2}} \tag{87}$$

$$k = \sqrt{1-a^2}$$

which yields

$$\begin{aligned} -\pi \int_{1}^{x} \frac{dx}{x^2\sqrt{1-x^2}\sqrt{x^2-a^2}} &= \frac{\pi}{a^2} \int_{\phi}^{\pi/2} \sqrt{1-(1-a^2)\sin^2\phi}\; d\phi \\ &= +\frac{\pi}{a^2}\left[E\left(k,\frac{\pi}{2}\right) - E(k,\phi)\right] \end{aligned} \tag{88}$$

Thus for $a < x < 1$ the integral $I(x)$ is given by (86) and (88). The result tends to infinity for $x \rightarrow a$.

In order to obtain the value of the integral $I(x)$ for $x^2 < a^2$, we notice that $F_1(x) \equiv 0$ for $x^2 < a^2$. Thus $I(x)$ is constant for $x^2 < a^2$ and according to section 5 is equal to the principal value of the integral for $x = a$,

$$I(x) = - \fint_a^1 \frac{\sqrt{1-x^2}}{x} F_1(x)\, dx$$

$$= \lim_{\delta \to 0} \left[- \int_{a+\delta}^{+1} \frac{\sqrt{1-x^2}}{x} F_1(x)\, dx + \frac{\pi \sqrt{1-a^2}}{\frac{1}{2}\, \delta^{\frac{1}{2}}\, (2a)^{\frac{3}{2}}} \right]$$

We introduce (86) and obtain

$$I(x) = + \pi \int_a^1 \frac{dx}{x^2 \sqrt{1-x^2}\, \sqrt{x^2-a^2}} \qquad (x^2 < a^2)$$

$$= \frac{\pi}{a^2} E(k) \tag{89}$$

where $E(k)$ denotes the complete elliptic integral of the second kind with the parameter $k = \sqrt{1-a^2}$.

7 Concluding remarks

The derivatives of the integral

$$F(x) = \int_a^b \frac{f(\xi)\, d\xi}{x-\xi} \tag{7}$$

are represented in the form of an integral, involving a principal value of higher order:

$$F_n(x) = \frac{(-1)^n}{n\,!} \frac{d^n F(x)}{dx^n} = \fint_a^b \frac{f(\xi)\, d\xi}{(x-\xi)^{n+1}} \tag{12}$$

where, following Hadamard, the "finite part" of the integral must be taken according to equation (13).

This applies only if $F(x)$ is to be understood as the limiting value for $z = 0$ of a more general function $F(x,z)$

$$F(x) = \lim_{z \to 0} F(x,z) = \lim_{z \to 0} \int_a^b f(\xi) \frac{x-\xi}{(x-\xi)^2 + z^2}\, d\xi \tag{8}$$

which is a solution of the Laplace differential equation in two variables. $F_n(x)$ yields the limiting values for $z \to 0$ of the derivatives of these functions $F(x,z)$, which are again solutions of the Laplace equation. Such a "crossed" integral can be integrated in parts according to equation (36) and can be differentiated according to equation (37).

In the applications, the function $f(\xi)$, which determines $F(x)$, is usually singular at the ends a and b of the integration interval, i.e. $f(\xi)$ tends to infinity at one or both ends. In order to cope with these cases the "principal value of a fractional order" is defined in equations (44) and (49). The integration and differentiation of such an integral are governed by equations (47) and (48). The rules for an integral $F_n(x)$ with singularities at both ends of the integration interval become particularly simple (e.g. (52)).

Finally, the integral

$$I_n(x) = \int_{a_o}^{x} h(x)\ F_n(x)\ dx \qquad (57)$$

of the product of a regular function $h(x)$ and a function $F_n(x)$ is evaluated. It is proved (section 5), that this integral, taken from a point $a_o < a$ up to $x = a$ gives the value $I_n(a)$ just outside the interval $a < \xi < b$, where $f(\xi)$ is defined. But the finite part of this integral, taken in the meaning of equations (44) and (49), gives the value $I_n(a)$ just inside this interval. In other words: The integration over a small interval $a-\varepsilon$ to $a+\varepsilon$ takes out the "infinite part" of the integral and leads to a finite answer. This is identical with the answer, which would be obtained if the integration would be performed along a path $z \neq 0$ (with the appropriate values for $h(x,z)$ and $F_n(x,z)$, which tend to $h(x)$ and $F_n(x)$ for $z \to 0$), thus avoiding the singularity at $x = a$, and the limit $z \to 0$ would be taken afterwards. This result is very important since it only requires the functions h and F to be known along the axis $z = 0$.

The most important rules for dealing with these principal value integrals are collected in Appendix I.

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APPENDIX I

Summary of the important equations

1. Principal values of integer order:

$$F_n(x) = \frac{(-1)^n}{n\,!}\,\frac{d^nF(x)}{dx^n} = \fint_a^b \frac{f(\xi)\,d\xi}{(x-\xi)^{n+1}} \qquad (n = 0,1,2,\ldots) \qquad (12)$$

$$\fint_a^b \frac{f(\xi)\,d\xi}{(x-\xi)^{n+1}} = \lim_{\varepsilon\to 0}\left\{\int_a^{x-\varepsilon}\frac{f(\xi)\,d\xi}{(x-\xi)^{n+1}} + \int_{x+\varepsilon}^b \frac{f(\xi)\,d\xi}{(x-\xi)^{n+1}} + (-1)^n\sum_{j=0}^{n-1}\frac{f^{(j)}(x)}{j\,!}\,\frac{1-(-1)^{n-j}}{(n-j)\;\varepsilon^{n-j}}\right\} \qquad (13),(14)$$

$$\fint_a^b \frac{f(\xi)\,d\xi}{(x-\xi)^{n+1}} = G(x,b) - G(x,a) \qquad (27)$$

with

$$\frac{d}{d\xi}\,G(x,\xi) = \frac{f(\xi)}{(x-\xi)^{n+1}} \qquad (23)$$

(Any terms $\log(\xi-x)$, which might occur in $G(x,\xi)$ have to be understood as $\log|\xi-x|$).

Partial integration:

$$\fint_a^b \frac{f(\xi)\,d\xi}{(x-\xi)^{n+1}} = \frac{1}{n}\left[\frac{f(b)}{(x-b)^n} - \frac{f(a)}{(x-a)^n} - \fint_a^b \frac{f'(\xi)\,d\xi}{(x-\xi)^n}\right] \qquad (36)$$

$$(n = 1,2,\ldots)$$

Differentiation:

$$\frac{d}{dx}\fint_a^b \frac{f(\xi)\cdot d\xi}{(x-\xi)^n} = -\,n\fint_a^b \frac{f(\xi)\,d\xi}{(x-\xi)^{n+1}}$$

$$= \fint_a^b \frac{f'(\xi)\,d\xi}{(x-\xi)^n} + \frac{f(a)}{(x-a)^n} - \frac{f(b)}{(x-b)^n} \qquad (37)$$

2. Principal values of fractional order $(a < c < b)$:

$$\fint_{a}^{c} \frac{H(\xi)\, d\xi}{(\xi-a)^{\alpha+m}} = \lim_{\delta\to 0} \left\{ \int_{a+\delta}^{c} \frac{H(\xi)\, d\xi}{(\xi-a)^{\alpha+m}} - \sum_{j=0}^{m-1} \frac{H^{(j)}(a)}{j!\, \delta^{\alpha+m-1-j} (\alpha+m-1-j)} \right\} \qquad (44)$$

$$(0 < \alpha < 1;\ m = 1,2,\ldots)$$

$$\fint_{c}^{b} \frac{H(\xi)\, d\xi}{(b-\xi)^{\beta+m}} = \lim_{\delta\to 0} \left\{ \int_{c}^{b-\delta} \frac{H(\xi)\, d\xi}{(b-\xi)^{\beta+m}} - \sum_{j=0}^{m-1} \frac{(-1)^{j}\, H^{(j)}(b)}{j!\, \delta^{\beta+m-1-j} (\beta+m-1-j)} \right\} \qquad (49)$$

$$(0 < \beta < 1;\ m = 1,2,\ldots)$$

Partial integration:

$$\fint_{a}^{c} \frac{H(\xi)\, d\xi}{(\xi-a)^{\alpha+m}} = \frac{1}{\alpha+m-1} \left[\fint_{a}^{c} \frac{H'(\xi)\, d\xi}{(\xi-a)^{\alpha+m-1}} - \frac{H(c)}{(c-a)^{\alpha+m-1}} \right] \qquad (47)$$

Differentiation:

$$\frac{d}{da} \fint_{a}^{c} \frac{H(\xi)\, d\xi}{(\xi-a)^{\alpha+m}} = (\alpha+m) \fint_{a}^{c} \frac{H(\xi)\, d\xi}{(\xi-a)^{\alpha+m+1}} \qquad (48)$$

3. Integrals with singularities at the ends:

$$\fint_{a}^{b} \frac{f(\xi)\, d\xi}{(x-\xi)^{n+1}} = \frac{(-1)^{j}\,(n-j)!}{n!} \fint_{a}^{b} \frac{d^{j} f(\xi)}{d\,\xi^{j}} \frac{d\xi}{(x-\xi)^{n-j+1}} \qquad (52)$$

$$(j = 0,1,2,\ldots,n;\ n = 1,2,\ldots)$$

($f(\xi)$ is either zero or infinity at $\xi = a$ and at $\xi = b$).

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FIG.1.

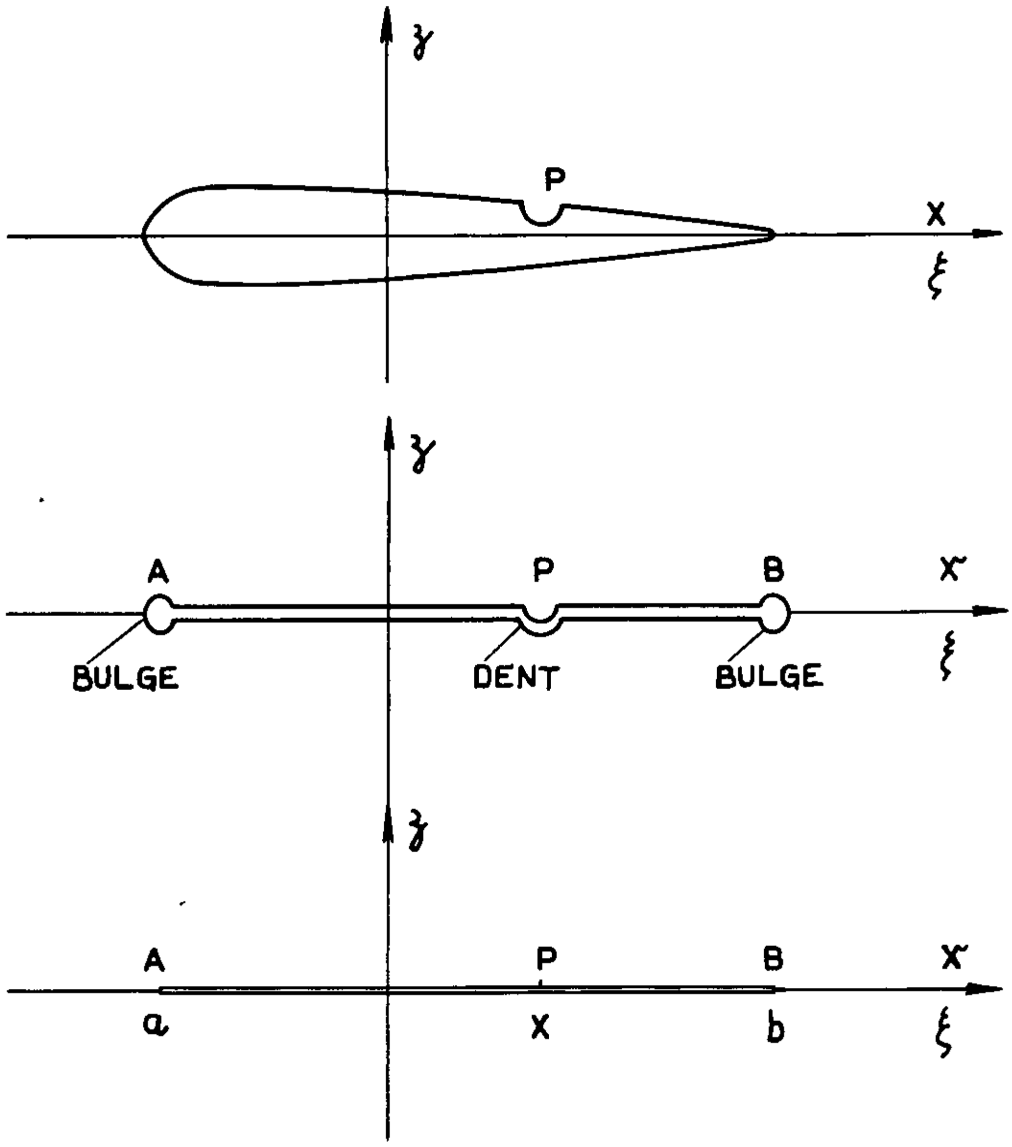

FIG.1. CONTRIBUTIONS OF THE INTEGRALS OVER THE DENT AT P AND THE BULGES AT A AND B TO THE CONTOUR INTEGRAL F (P) IN EQUATION (2)

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