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The Airborne Path During Take-off for Constant Rate-of-Pitch Manoeuvres

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SUMMARY

A method of estimating the airborne path during take-off is proposed, based on the assumption that the aircraft is rotated at a constant rate of pitch from the moment of lift-off to the point at which it attains a steady climb path. The justification for this assumption is discussed. A simplified analysis, using small perturbation equations of motion, has been developed for initial project studies. Examples of the method applied to a slender wing transport aircraft, and a lightly loaded STOL aircraft are given, and the factors affecting the value of rate of pitch used are discussed.

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CONTENTS

1 INTRODUCTION

The take-off distance of an aircraft is normally measured from the starting point of the ground run to the point at which it reaches some specified height above the ground. The length of the ground run depends mainly on the physical characteristics of the aircraft and airfield, but the airborne distance is also dependent on the way in which the pilot controls the aircraft. An assumption regarding piloting technique is therefore necessary before the airborne path of a new aircraft can be predicted.

The purpose of this Report is to suggest an alternative method to those currently used for estimating airborne paths during take-off. Details of these methods are given in the references[1-8] and it is only necessary here to outline the assumptions on which they have been based. The airborne manoeuvre has previously been taken to be one in which either the lift coefficient, or the normal acceleration, remained constant; or one in which a certain speed increment was allowed to develop. There seems to be no experimental evidence to suggest that the first two of these represent manoeuvres which are consciously aimed at by the pilot, and it is believed that they were chosen largely because of their simplicity and convenience. Consequently, the values of lift coefficient or normal acceleration to be used must be arrived at empirically. The other method, which is based on the energy equation, only defines the flight path when the variation of speed during the manoeuvre has already been prescribed.

The alternative method proposed in this Report assumes that the pilot controls the aircraft so that it has a constant rate of pitch, from the moment it leaves the ground until it reaches its steady climbing flight path. The reasons for this assumption will be given in detail in section 2, but they may be summarised here as follows:

(1) There is evidence from records of real and simulated take-offs that the technique actually used by pilots results in a manoeuvre resembling one with constant rate of pitch.

(2) Rate of pitch and pitch attitude are two flight variables which can be directly sensed and controlled by the pilot. We might therefore hope to find a more rational pattern in the rates of pitch used in practice, than for a less readily perceived variable such as the lift coefficient.

(3) Control of rate of pitch is the basis upon which a Take-Off Director instrument is currently being developed.

This method of predicting the airborne path is particularly applicable to aircraft which require appreciable changes in pitch attitude during the initial climb. It also assumes that changes in lift are produced by changes in incidence, in the conventional manner. Consequently some classes of STOL aircraft cannot be treated (e.g. tilt-wings). There may also be difficulties where a conventional aircraft is operating close to its Weight, Altitude and Temperature (WAT) limits, since the climb angles involved will then be very shallow. In the latter case a simpler analysis[5] based on the energy equation may be adequate.

During the initial project stages of an aircraft design, methods of estimation of great complexity are often not warranted because of the tentative nature of the data available. Fortunately, the solutions to the equations of motion for constant rate-of-pitch manoeuvres can be presented in a simple generalised form when the same approximations as those used in current methods of take-off estimation are assumed. This simplified analysis is presented in section 3 of the Report, and its application to the take-off of a slender wing transport, and to a lightly loaded STOL aircraft, is illustrated in section 4. These cases cover the likely range of interest, conventional aircraft falling between them.

2 REASONS FOR ASSUMING A CONSTANT RATE-OF-PITCH MANOEUVRE

Figs.1 and 2 show some of the experimental evidence on which the proposed method of estimating the airborne path was based. Time histories of take-offs made on a piloted flight simulator are shown for a conventional jet transport aircraft[9], and for a large slender wing aircraft like the Concorde[10]. Numerous individual records have been superimposed on each of these figures to reveal the underlying characteristics of the take-off. The remarkable consistency shown between the individual records in Figs.1 and 2 can be achieved on a flight simulator because identical conditions can be assured for each take-off, and because random external disturbances, such as those due to atmospheric turbulence, can be eliminated. As a help in assessing the validity of such tests, Fig.3 shows a comparison[9] between real and simulated take-offs for the conventional jet transport aircraft (a Comet 3b). Much more random scatter from run to run is now evident, but the same underlying characteristics of the take-off are still apparent.

The studies from which Figs.1 and 2 are taken[9,10] were concerned, in part, with the development of the Take-Off Director instrument mentioned earlier, and records for both undirected and directed take-offs are reproduced. The probable use of Take-Off Directors in the future is an added argument in favour of the proposed method of estimation, and this is discussed subsequently.

Although the rate of pitch records of Figs.1 and 2 show numerous short period fluctuations, the pitch attitude records may be quite closely represented by three straight line segments. These correspond to the initial rotation into the lift-off attitude, the flare up to the steady climb, and the steady climb itself. A suggested model of the take-off, based on this three segment approximation, is shown in Fig.4. The present Report is concerned mainly with the second of these three phases; the flare-up to the steady climbing angle. In the case of the two aircraft studied on the simulator the mean rate of pitch during this phase was typically about 1°/sec.

From the piloting aspect, take-off procedures are usually specified in terms of achieving certain speeds at certain points in the take-off. However, the short time scale between lifting-off and achieving the steady climbing attitude probably prevents the pilot from exercising direct 'closed-loop' control over the speed during each individual take-off. Instead, it is believed, he learns during training, and through experience, a programme of pitch attitude changes which will result in the desired speeds being achieved in an 'open-loop' manner. This closed loop control of attitude, rather than speed, arises because attitude is the variable which he can sense and control most directly. The use of the same variable in the estimation method then becomes all the more desirable, since performance and handling considerations may thereby be inter-related.

Finally, the expected introduction of Take-Off Director instruments into service provides a further argument in favour of the proposed method of estimation. The director is an instrument which senses deviations from a desired flight condition, and also those factors tending to change that deviation, in order to provide continuous instructions to the pilot. A null reading is usually given when a suitable corrective response has been generated in reply to an error signal. This corrective response could be a change in control position, or it could be a change in a suitable flight variable such as pitch attitude or rate of pitch[11]. In fact, rate of pitch is the usual form of response signal in the director systems now being studied[9,10]. One important practical feature with such a system is that the error signal to the Director usually has an upper limit, irrespective of the actual value of the error, in order to avoid demanding rates of pitch larger than those which the pilot can safely handle. With such a limit in operation the demanded manoeuvre is one having a constant (maximum allowable) rate of pitch, and so is the same as that assumed in the present analysis.

There is also some evidence to suggest that pilots who have once used a Take-Off Director subsequently model their take-off technique on its laws, even when it is not in use. If this were generally so, the adoption of such instruments should lead to even more widespread use of the type of take-off manoeuvre assumed here.

3 SIMPLIFIED ANALYSIS OF THE CONSTANT RATE-OF-PITCH MANOEUVRE

The take-off manoeuvre is difficult to treat analytically because of the non-linearities introduced by ground effect, and because the aircraft configuration may be changing due to undercarriage retraction. A detailed study of any particular problem will usually involve numerical calculation of the step-by-step type[12], allowing these effects to be taken into account. The assumption of a constant rate-of-pitch manoeuvre is still readily applicable to such calculations. However, there is also the need for simpler methods, more appropriate to the early project stage of an aircraft's development, where sufficient information to warrant the more detailed type of study is not usually available. The following simplified analysis of the constant rate-of-pitch manoeuvre uses approximations which are comparable to those adopted in most of the published methods. Because this simplified analysis is based essentially on the concept of small perturbations in the variables, some care is needed in its application. Much must be left to the judgement of the user, the range of validity depending, to some extent, on whether the particular emphasis is on quantitative results or on obtaining a more qualitative assessment of the take-off manoeuvre. It is felt that its principal application will be in estimating the flight path between the lift-off point and the screen height of 35 or 50 ft.

The aircraft is initially assumed to be at the lift-off incidence α_o, with a lift coefficient C_{L_o} and at the lift-off speed V_o. Subsequent speed and incidence changes will be denoted by u and $\Delta\alpha$. The equation of motion normal to the flight path is then:

$$\dot{\gamma} = \frac{1}{m(V_o + u)} [\tfrac{1}{2}\rho \; S \; C_{L_o} \; (1 + n_\alpha \, \Delta\alpha) \; (V_o + u)^2 - mg] \tag{1}$$

where γ is the flight path angle (assumed small, so that $\cos \gamma \approx 1$ and $\sin \gamma \approx \gamma$), and $n_\alpha = a_1/C_{L_o}$ per rad, is the increment in load factor per unit change of incidence at the lift-off speed. The real value of n_α will vary continuously throughout the manoeuvre, due to changes in ground effect, but for this simplified analysis it is necessary to assume a mean value of n_α which takes this variation into account. It is difficult to generalise on the variation of n_α with height. For instance wind tunnel tests on a typical subsonic jet transport model[13] showed a decrease in lift curve slope of 10-15% between the 'on ground' and 'free air' conditions when the flaps were retracted, but a negligible difference with the flaps at the take-off setting. In choosing a value for n_α, consideration must also be given to the size of the aircraft, since appreciable ground effect is only experienced below a height of about one semi-span. Small aircraft will therefore be affected much less than large ones when operating to the same screen height. It should also be pointed out that this analysis does not take into account the variation with height of the lift coefficient at the datum incidence.

Substituting for the lift-off condition, $\tfrac{1}{2}\rho \; V_o^2 \; S \; C_{L_o} = mg$ in equation (1) and neglecting terms of the second order, we have:

$$\dot{\gamma} = \frac{g}{V_o} n_\alpha \Delta\alpha + \frac{2gu}{V_o^2} \quad . \tag{2}$$

The equation of motion along the flight path, with the same approximations, is

$$\dot{u} = g \left[\frac{T - D}{W} - \gamma \right] \quad . \tag{3}$$

The term $\frac{T - D}{W}$, representing the balance of thrust and drag forces, will again vary in some complex manner during the manoeuvre, but must be given an equivalent constant value in this simplified analysis.

The third relationship between the variables u, γ and $\Delta\alpha$ arises from the condition that the rate of pitch shall be constant during the manoeuvre:

$$\dot{\gamma} + \dot{\Delta\alpha} = Q \;\; \text{(constant)} \quad . \tag{4}$$

Then, equations (2), (3) and (4) may be solved (Appendix A) to yield the following expression for the variation of climb angle with time:

$$\frac{\gamma}{\left[\frac{T - D}{W} + \frac{V_o\,Q\,n_\alpha}{2g}\right]} = \frac{\lambda_2}{\lambda_1 - \lambda_2}\, e^{\lambda_1\left(\frac{gt}{V_o}\right)} - \frac{\lambda_1}{\lambda_1 - \lambda_2}\, e^{\lambda_2\left(\frac{gt}{V_o}\right)} + 1 \qquad (5)$$

where $\lambda_1,\ \lambda_2 = \dfrac{-n_\alpha \pm \sqrt{n_\alpha^2 - 8}}{2}$.

The left hand side of (5) is simply proportional to γ by a factor which depends only on the constants of a given problem. The right hand side is a function of the non-dimensional time variable $\left(\frac{gt}{V_o}\right)$ and of the problem parameter n_α. Generalized climb gradient time histories for different values of n_α may be drawn by plotting the function $F(\gamma)\left(= \dfrac{\gamma}{\frac{T - D}{W} + \frac{V_o\,Q\,n_\alpha}{2g}}\right)$ against $\left(\frac{gt}{V_o}\right)$. Values of this climb gradient function for the range of n_α and $\left(\frac{gt}{V_o}\right)$ likely to be of interest are listed in Table 1, and the generalized time histories are plotted in Fig.5.

The variation of height with time is found by integration of the equation:

$$\frac{dh}{dt} = V_o\,\gamma \quad . \qquad (6)$$

This neglects the effect of the increase in speed during the manoeuvre so that the height will tend to be underestimated. The resulting height expression (derived in Appendix A) is:

$$\frac{h}{\frac{V_o^2}{g}\left[\frac{T - D}{W} + \frac{V_o\,Q\,n_\alpha}{2g}\right]} = \frac{\lambda_2}{\lambda_1(\lambda_1 - \lambda_2)}\, e^{\lambda_1\left(\frac{gt}{V_o}\right)} - \frac{\lambda_1}{\lambda_2(\lambda_1 - \lambda_2)}\, e^{\lambda_2\left(\frac{gt}{V_o}\right)} + \frac{\lambda_1 + \lambda_2}{\lambda_1\,\lambda_2} + \frac{gt}{V_o} \quad . \qquad (7)$$

Generalized height time histories for different values of n_α may be drawn, in a similar manner to those for the climb gradient, by plotting the function $F(h) \left(= \dfrac{h}{\dfrac{V_o^2}{g}\left[\dfrac{T - D}{W} + \dfrac{V_o Q n_\alpha}{2g}\right]} \right)$ against the time variable $\left(\dfrac{gt}{V_o}\right)$. Values of this height function are listed in Table 2, and the generalized time histories are plotted in Fig.6.

The variation of speed with time is most readily found from the energy equation. The net work done up to any point in the take-off must equal the increase in potential and kinetic energy. Since the resultant longitudinal force $(T - D)$ has been assumed constant, this leads to the following expression (neglecting second order terms):

$$(T - D) V_o t = m V_o u + mgh$$

or, rearranging:

$$\frac{u}{V_o} = \frac{T - D}{W}\left(\frac{gt}{V_o}\right) - \frac{gh}{V_o^2} \quad . \qquad (8)$$

An expression for the variation of the height h with time has already been given in equation (7). Hence, the speed increment at any point can be calculated from (8).

Once the time to reach a given height has been found, the airborne distance may be calculated simply as

$$S_a = V_o t \quad , \qquad (9)$$

which again neglects the increase in speed during the manoeuvre.

A further feature of interest during the take-off is the variation in incidence and, more particularly, the maximum value of incidence used. Integration of equation (4) gives the expression for the incidence at any point:

$$\alpha = \alpha_o + Qt - \gamma \qquad (10)$$

where γ, as a function of time, is given by equation (5). It is further shown in Appendix A that the incidence has a maximum (or minimum) value when the following expression is satisfied:

$$\frac{Q\,V_o}{g\left[\frac{T-D}{W} + \frac{V_o\,Q\,n_\alpha}{2g}\right]} = \frac{\lambda_1\,\lambda_2}{(\lambda_1 - \lambda_2)}\left\{e^{\lambda_1\left(\frac{gt}{V_o}\right)} - e^{\lambda_2\left(\frac{gt}{V_o}\right)}\right\} \quad . \qquad (11)$$

To enable the point in time $t_{\hat{\alpha}}$, at which equation (11) is satisfied to be found rapidly, values of the right hand side of the expression have been tabulated in Table 3 and plotted in Fig.7 for a range of $\left(\frac{gt}{V_o}\right)$ and n_α. For any particular problem $\frac{Q\,V_o}{g\left[\frac{T-D}{W} + \frac{V_o\,Q\,n_\alpha}{2g}\right]}$ may be evaluated and compared with the plot of $F(t_{\hat{\alpha}})$ to find the time at which the maximum incidence occurs. The actual value of the incidence can then be found from equation (10).

4 APPLICATIONS OF THE ANALYSIS

The use of this simplified analysis is illustrated in this section by two examples. The first concerns the take-off of a large slender wing transport aircraft, of the type studied in the simulator tests mentioned earlier. The second is for an STOL aircraft, in which the required take-off performance is achieved by low wing loading and high thrust-weight ratio. In this example the proposed method of estimation is also compared with that which assumes constant lift coefficient during the manoeuvre.

4.1 The airborne path for a slender wing transport aircraft

For this example a wing loading of 85 lb/ft^2 was chosen together with a C_{L_o} at lift-off of 0.625, giving a lift-off speed of 200 kt (338 ft/sec). Although the lift curve slope of this low aspect ratio aircraft is small, (a_1 = 3.75 per rad) the lift-off speed is high relative to other aircraft of similar wing loading, so that the increment in load factor per unit change in incidence, (n_α = 6 per rad) is at the top of the range considered. A typical value of $\frac{T-D}{W}$ = 0.12 was used. Time histories of the airborne part of the take-off, calculated according to the analysis given in section 3, are shown in Fig.8 for two values of rate of pitch; Q = 0.5°/sec and Q = 1.0°/sec.

With most of the simplified methods of take-off estimation a difficulty occurs when the aircraft reaches the steady climbing path, since an instantaneous adjustment in incidence is theoretically required to settle it on the steady climb. This is illustrated in Fig.8 where step changes in attitude and incidence are indicated. In practice such adjustments are unlikely to be required, since the pilot will have anticipated them by fairing smoothly into the steady climb. At this stage the continued validity of several of the approximations in the simplified analysis is, in any case, becoming doubtful. For the slender wing aircraft, lift-off and initial climb are made at speeds well below minimum drag speed, so that the factor $\frac{T - D}{W}$ is actually increasing with increasing speed. This makes it desirable to adopt an accelerating climb, rather than a steady climb, for a period immediately after the flare up. Such limitations in the analysis are probably relatively unimportant in the context of initial project studies for which it is intended.

The choice of rate of pitch to be used in the flare up depends on several factors. Firstly, there are various overriding limitations on the maximum rate of pitch which can be used, irrespective of performance considerations. One of these is the fastest rate which a pilot will feel that it is safe to use from the point of view of maintaining adequate control. More experimental study is needed to find values for this limit, and the factors which affect it. In the absence of such studies, a value of $1\frac{1}{2}°$/sec is suggested as being appropriate to commercial transport operations. Another limiting factor may be the maximum incidence reached during the manoeuvre. Fig.8 shows that doubling the rate of pitch more than doubles the increase in incidence during the flare up. Furthermore, this peak in incidence is dependent on the initial longitudinal acceleration, and a combination of high rate of pitch with low thrust-weight ratio may bring the incidence to dangerous levels. A further point which must be considered is the danger of striking the tail on the ground. Pinsker[14] has shown that this may be more critical a second or so after lift-off than at lift-off itself.

If none of these factors are overriding, the correct rate of pitch will be determined by the need to keep a proper balance between the gain in height and the gain in speed. A high rate of pitch will result in a rapid gain in height, and therefore a shorter take-off distance, but it may mean that the required safety speeds are not attained. Too low a rate of pitch, on the other hand, will result in an excessive gain in speed at the expense of height and will lead to an over-long take-off distance. Just as the best rate of pitch must be found during the prototype flight trials by a process of trial and error, so the estimation should cover a range of pitch rates to yield the most acceptable value.

As an example, one take-off procedure suggested for this type of aircraft[10] required a gain in speed of 5 kt (8.5 ft/sec) at the screen height of 35 ft, and of 13 kt (22 ft/sec) by the time the aircraft had reached a height of 200 ft. Fig.8 shows that the second requirement determines the maximum rate of pitch which can be used; in this case about 0.75°/sec.

4.2 The airborne path for a lightly loaded STOL aircraft

In this example we shall consider the case of an aircraft specially designed to take-off in short distances; of the order of 1000 ft, compared with the 8-9000 ft needed for the slender wing transport aircraft. This is achieved by a low wing loading $\omega = 20\ \text{lb/ft}^2$; a high take-off lift coefficient $C_{L_0} = 2$, and a high thrust-weight ratio, resulting in a value of $\frac{T - D}{W} = 0.35$. It is assumed that the aircraft would be of large aspect ratio, giving a high lift curve slope ($a_1 = 6$ per rad), but because of the low take-off speed of 54 kt (92 ft/sec) the increment in load factor per unit change in incidence is only half that of the slender wing aircraft ($n_\alpha = 3$ per rad).

Fig.9 shows time histories of the airborne path, calculated by the methods given in section 3, for constant rates of pitch of 1°/sec, 3°/sec and 6°/sec. Once again, more experimental data is required to establish the maximum rates of pitch which pilots would be prepared to use for this type of STOL operation. Until recently a maximum of 3°/sec would have been considered likely, but a U.S. flight study with a COIN type STOL aircraft[15] has shown rates of pitch as high as 11°/sec being used.

It must be admitted at once that, with the magnitude of thrust-weight ratio chosen for this example, the concept of small perturbations used in the simplified analysis can hardly be justified, at least from the point of view of obtaining results which have good absolute accuracy. However the results are of interest in showing the relative magnitude of the variables and for giving a general impression of the manoeuvre. The main differences to be seen in the manoeuvres of Figs.8 and 9, apart from the obvious differences in time scale, concern the degree to which incidence must be reduced as the manoeuvre progresses. For the STOL aircraft, with its low thrust loading and lower value of n_α, the incidence must be reduced quite rapidly if the increase in lift caused by the build up in speed is not to produce an excessive steepening of the flight path. In fact, there seems to be little danger of stalling this type of aircraft during take-off, provided that power failure does not occur.

It should be remembered that the method of estimating height in this analysis ignores the effect of the speed increment in the kinematic equation (6) so that the height will tend to be underestimated.

The main differences between the constant rate of pitch method of estimation, and that assuming constant C_L during the manoeuvre, are shown in Fig.10. For the latter case it was assumed that the lift coefficient was held at the same value as for lift-off, $C_L = 2$ (and that the incidence remained constant). The climb angle and pitch attitude would then be equal during the manoeuvre. The constant C_L manoeuvre starts off more gently than that at constant rate of pitch, since curvature of the flight path depends entirely on the additional lift generated by the increase in speed. By the end of the manoeuvre however, it is much more violent, with rates of pitch of nearly 10°/sec. The adjustment in incidence then theoretically required to settle the aircraft in the steady climb is also, of course, much larger. For the two cases shown in Fig.10 the actual differences in flight path are not great, but it will be appreciated that one is obtained by a manoeuvre which is scarcely credible in practice, while the other is based on actual pilot behaviour. This may be of some importance when attempting to estimate the airborne path for novel situations.

5 CONCLUSIONS

A basis for estimating airborne paths during take-off is provided by the assumption that the pilot controls the aircraft so that it has a constant rate of pitch, from the moment it leaves the ground until it reaches its steady climbing flight path. This assumption is supported by evidence of pilots' actual behaviour, by consideration of the variables which can be perceived and controlled most easily, and by modes of operation which will result from the introduction of Take-Off Director instruments.

The choice of an appropriate value for rate of pitch may be determined by a number of factors. There is the need to achieve a correct balance between the gain in height and the gain in speed, so that the take-off distance is a minimum, consistent with achieving required safety speeds. There are also overriding limitations in rate of pitch due to the maximum incidence which can be used, to the need for avoiding tail strikes on the ground, and to the maximum values of rate of pitch which pilots feel that it is safe to use from the handling point of view. Further experimental data on this handling feature are needed. Until this is forthcoming, maximum rates of pitch of $1\frac{1}{2}$°/sec for commercial transport operations, and 3°/sec for STOL operations are suggested, although there is some recent U.S. evidence that much higher rates of pitch might be used in maximum effort STOL take-offs.

The assumption of a constant rate-of-pitch manoeuvre has been used, in conjunction with a small perturbation solution of the equations of motion, to give a simple method of estimating the airborne path during take-off. This method is felt to be adequate for initial project studies and is thought to be superior to some currently used methods, such as those which assume a constant lift coefficient or a constant normal acceleration, since it represents a manoeuvre which is more realistic.

Appendix A

(Reference: section 3)

A SIMPLIFIED ANALYSIS OF THE CONSTANT RATE-OF-PITCH MANOEUVRE

The three equations containing the variables u, γ and $\Delta\alpha$ are given in section 3 of the text as follows:

For motion normal to the flight path, (equation 2),

$$\dot{\gamma} = \frac{g}{V_o} n_\alpha \Delta\alpha + \frac{2gu}{V_o^2} \quad . \tag{A-1}$$

For motion along the flight path, (equation 3),

$$\dot{u} = g\left[\frac{T - D}{W} - \gamma\right] \tag{A-2}$$

and from the condition of constant rate of pitch (equation 4)

$$\dot{\gamma} + \Delta\dot{\alpha} = Q \quad . \tag{A-3}$$

Then differentiating (A-1) w.r.t. time gives

$$\ddot{\gamma} = \frac{g}{V_o} n_\alpha \Delta\dot{\alpha} + \frac{2g\dot{u}}{V_o^2} \tag{A-4}$$

and substituting for $\Delta\dot{\alpha}$ from (A-3) and for $\dot{u}$ from (A-2) leads to the following differential equation in γ.

$$\ddot{\gamma} + \frac{gn_\alpha}{V_o}\dot{\gamma} + \frac{2g^2}{V_o^2}\gamma = \frac{gn_\alpha Q}{V_o} + \frac{2g^2}{V_o^2}\left(\frac{T - D}{W}\right) \tag{A-5}$$

which has the solution; for initial conditions $\gamma = 0$ when $t = 0$:

$$\gamma = \left[\frac{T - D}{W} + \frac{V_o Q n_\alpha}{2g}\right]\left\{\frac{\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_1\left(\frac{gt}{V_o}\right)} - \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_2\left(\frac{gt}{V_o}\right)} + 1\right\} \tag{A-6}$$

where $\lambda_1, \lambda_2 = \dfrac{-n_\alpha \pm\sqrt{n_\alpha^2 - 8}}{2}$.

The height variation is found by substituting this relationship for γ into equation (6) of the text:

$$\dot{h} = V_o \left[\frac{T - D}{W} + \frac{V_o\, Q\, n_\alpha}{2g}\right] \left\{ \frac{\lambda_2}{\lambda_1 - \lambda_2} e^{\lambda_1\left(\frac{gt}{V_o}\right)} - \frac{\lambda_1}{\lambda_1 - \lambda_2} e^{\lambda_2\left(\frac{gt}{V_o}\right)} + 1 \right\} \quad . \qquad \text{(A-7)}$$

Then integrating w.r.t. t, and with initial conditions $h = 0$ when $t = 0$

$$h = \frac{V_o^2}{g} \left[\frac{T - D}{W} + \frac{V_o\, Q\, n_\alpha}{2g}\right] \left\{ \frac{\lambda_2}{\lambda_1(\lambda_1 - \lambda_2)} e^{\lambda_1\left(\frac{gt}{V_o}\right)} - \frac{\lambda_1}{\lambda_2(\lambda_1 - \lambda_2)} e^{\lambda_2\left(\frac{gt}{V_o}\right)} + \frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} + \frac{gt}{V_o} \right\} \quad . \qquad \text{(A-8)}$$

The variation of speed, distance and incidence with time during the manoeuvre are given by equations (8), (9) and (10) of the text.

The point at which the incidence is a maximum or minimum may be found by setting $\Delta\dot{\alpha} = 0$ in equation (A-3), showing that the required condition is

$$\dot{\gamma} = Q$$

or, by differentiating (A-6);

$$\frac{g}{V_o} \left[\frac{T - D}{W} + \frac{V_o\, Q\, n_\alpha}{2g}\right] \left\{ \frac{\lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)} \left[e^{\lambda_1\left(\frac{gt}{V_o}\right)} - e^{\lambda_2\left(\frac{gt}{V_o}\right)} \right] \right\} = Q \quad . \qquad \text{(A-9)}$$

Table 1

VALUES OF THE FUNCTION F(γ) USED IN THE CLIMB GRADIENT EQUATION

$\frac{gt}{V_o}$	F(γ) for values of n_α (per rad)			
	3	4	5	6
0.10	0.0091	0.0088	0.0085	0.0072
0.20	0.0329	0.0310	0.0293	0.0278
0.30	0.0672	0.0618	0.0571	0.0529
0.40	0.1087	0.0979	0.0888	0.0810
0.50	0.1548	0.1369	0.1223	0.1102
0.60	0.2036	0.1773	0.1565	0.1396
0.70	0.2534	0.2179	0.1904	0.1687
0.80	0.3032	0.2581	0.2237	0.1971
0.90	0.3522	0.2971	0.2562	0.2247
1.00	0.3996	0.3348	0.2875	0.2516
1.25	0.5091	0.4225	0.3608	0.3148
1.50	0.6035	0.5000	0.4270	0.3729
1.75	0.6827	0.5675	0.4859	0.4260
2.00	0.7476	0.6262	0.5397	0.4747

Table 2

VALUES OF THE FUNCTION F(h) USED IN THE HEIGHT EQUATION

$\frac{gt}{V_o}$	F(h) for values of n_α (per rad)			
	3	4	5	6
0.10	0.0003	0.0003	0.0003	0.0003
0.20	0.0023	0.0022	0.0021	0.0020
0.30	0.0072	0.0068	0.0063	0.0060
0.40	0.0160	0.0147	0.0136	0.0127
0.50	0.0291	0.0264	0.0242	0.0223
0.60	0.0470	0.0422	0.0381	0.0347
0.70	0.0699	0.0619	0.0555	0.0502
0.80	0.0977	0.0857	0.0762	0.0685
0.90	0.1305	0.1135	0.1002	0.0896
1.00	0.1681	0.1451	0.1274	0.1134
1.25	0.2820	0.2400	0.2086	0.1843
1.50	0.4214	0.3555	0.3072	0.2703
1.75	0.5824	0.4891	0.4226	0.3703
2.00	0.7615	0.6385	0.5499	0.4830

Table 3

VALUES OF THE FUNCTION $F(t_{\hat{\alpha}})$ USED TO FIND THE POINT OF MAXIMUM INCIDENCE

$\frac{gt}{V_o}$	$F(t_{\hat{\alpha}})$ for values of n_α (per rad)			
	3	4	5	6
0.10	0.1722	0.1643	0.1569	0.1500
0.20	0.2968	0.2717	0.2495	0.2299
0.30	0.3840	0.3393	0.3018	0.2704
0.40	0.4420	0.3789	0.3288	0.2885
0.50	0.4773	0.3993	0.3400	0.2941
0.60	0.4952	0.4064	0.3415	0.2928
0.70	0.5000	0.4045	0.3370	0.2877
0.80	0.4949	0.3965	0.3290	0.2806
0.90	0.4825	0.3846	0.3189	0.2724
1.00	0.4651	0.3704	0.3078	0.2639
1.25	0.4088	0.3301	0.2788	0.2424
1.50	0.3467	0.2895	0.2508	0.2221
1.75	0.2871	0.2519	0.2253	0.2033
2.00	0.2340	0.2184	0.2018	0.1861

SYMBOLS

Symbol	Meaning	Units
a_1	lift curve slope	per rad
C_{L_o}	take-off lift coefficient	
D	aerodynamic drag	lb
$F(\gamma)$	climb gradient function defined in section 3	rad
$F(h)$	height function defined in section 3	
$F(t_{\hat{\alpha}})$	function for determining the point of maximum incidence	rad
g	acceleration due to gravity	ft/sec^2
h	height above ground	ft
m	aircraft mass	slug
n_α	increase in load factor per unit change in incidence	per rad
Q	rate of pitch used in the manoeuvre	rad/sec
S	wing area	sq ft
S_a	airborne distance during take-off	ft
T	thrust	lb
t	time from lift-off	sec
$t_{\hat{\alpha}}$	time at which maximum incidence occurs	sec
u	speed increment during take-off	ft/sec
V_o	lift-off speed	ft/sec
W	aircraft weight	lb
α	incidence	rad
α_o	lift-off incidence	rad
$\Delta\alpha$	incidence increment during airborne phase	rad
γ	flight path angle	rad
θ	aircraft pitch attitude	rad
$\lambda_1\ \lambda_2$	roots of the characteristic equation, section 3	
ρ	air density	$slug/ft^3$
ω	wing loading	lb/ft^2

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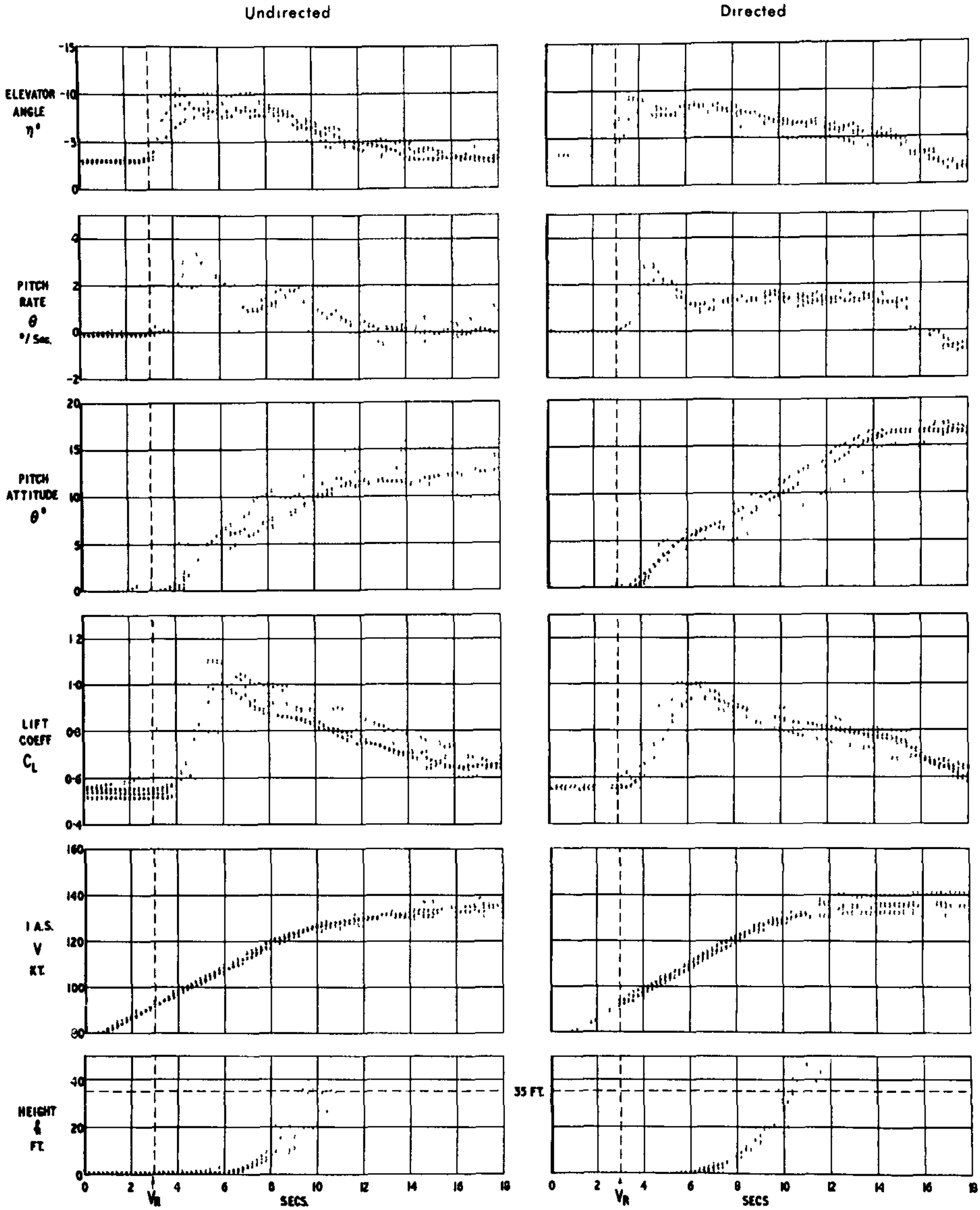

Fig.1 Time histories of simulated take-offs for a conventional jet transport aircraft (from ref.9)

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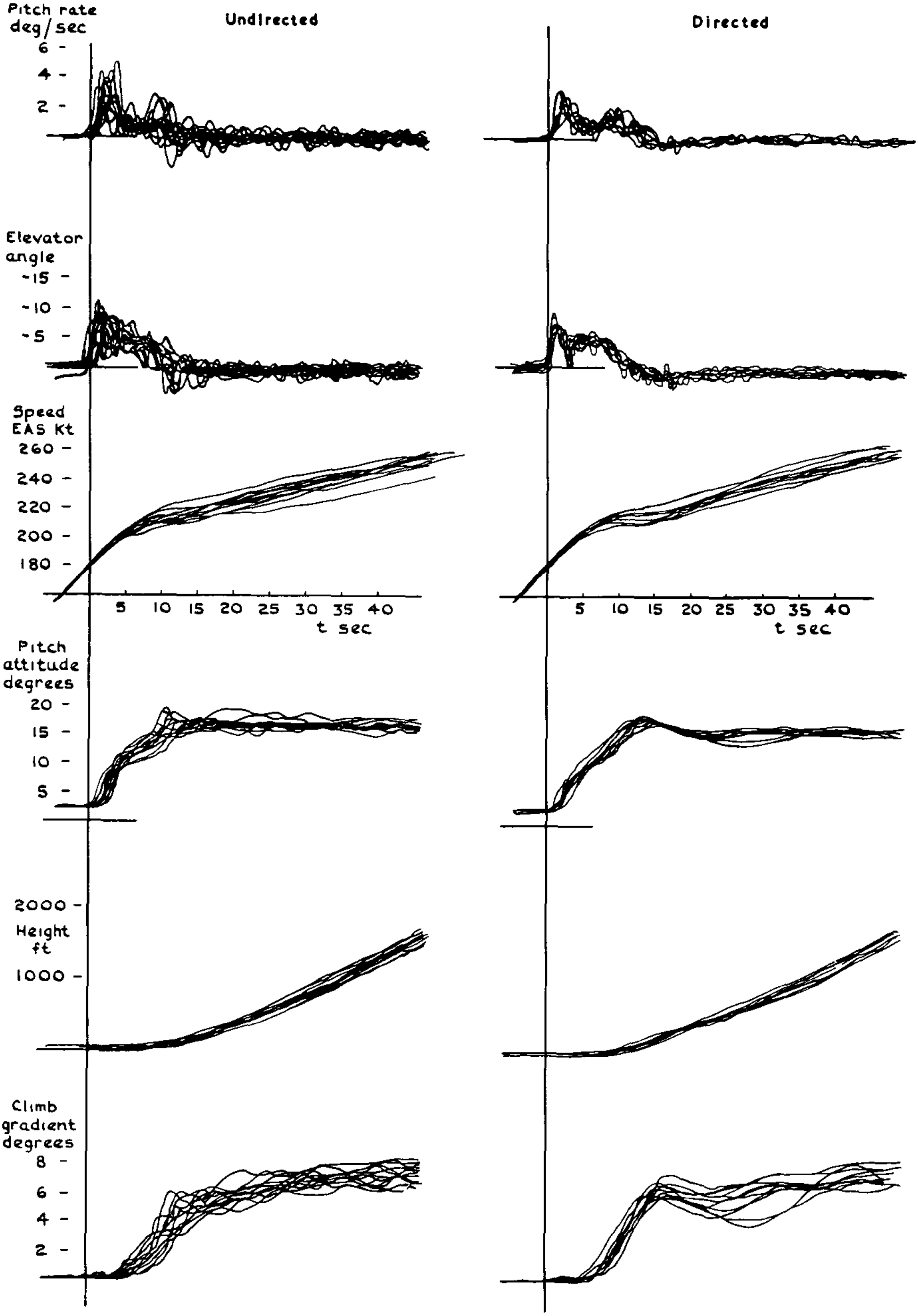

Fig. 2 Time histories of simulated take-offs for a slender wing transport aircraft (from ref. 10)

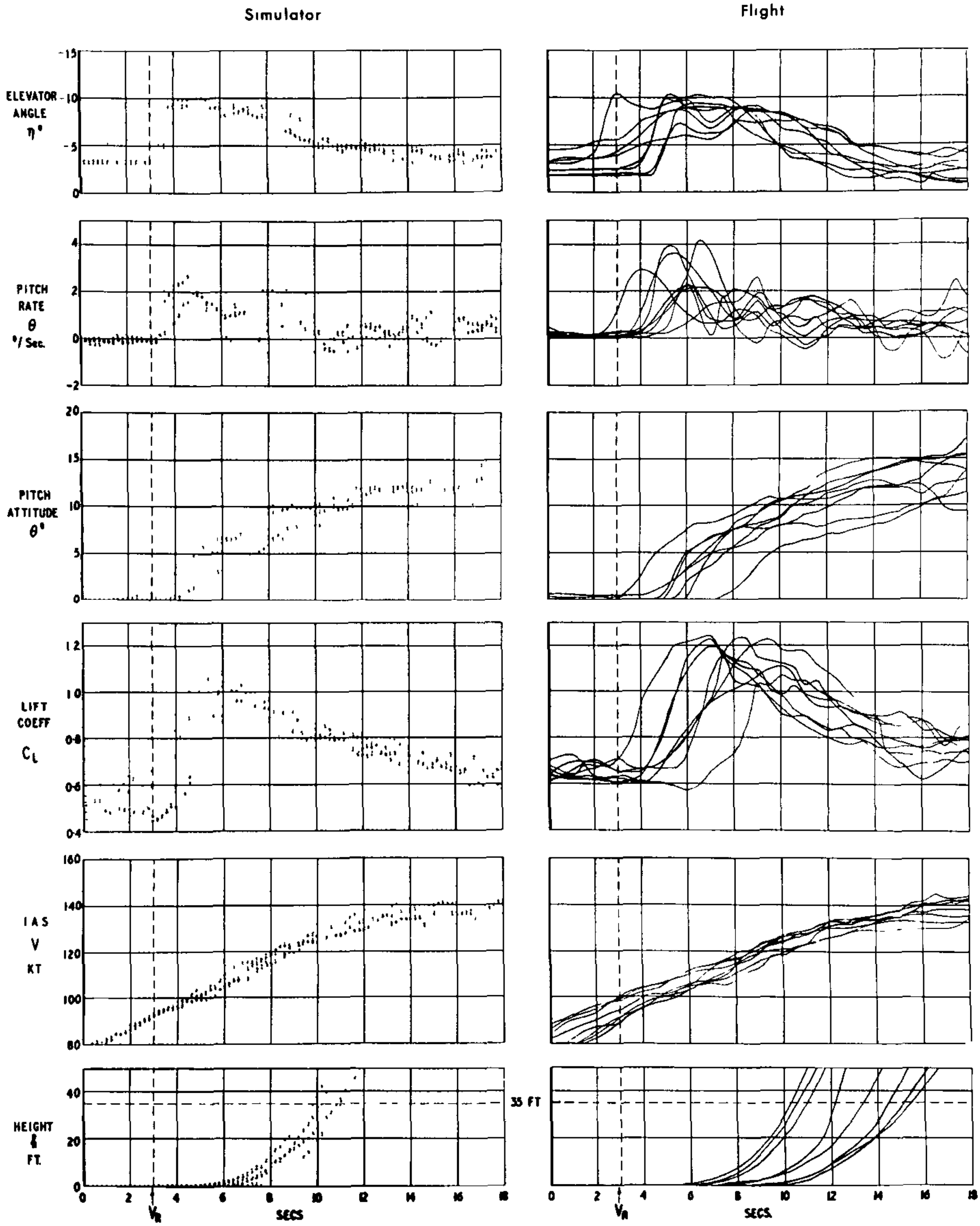

Fig.3. Comparison of real and simulated take-offs for the conventional jet transport aircraft (from ref.9)

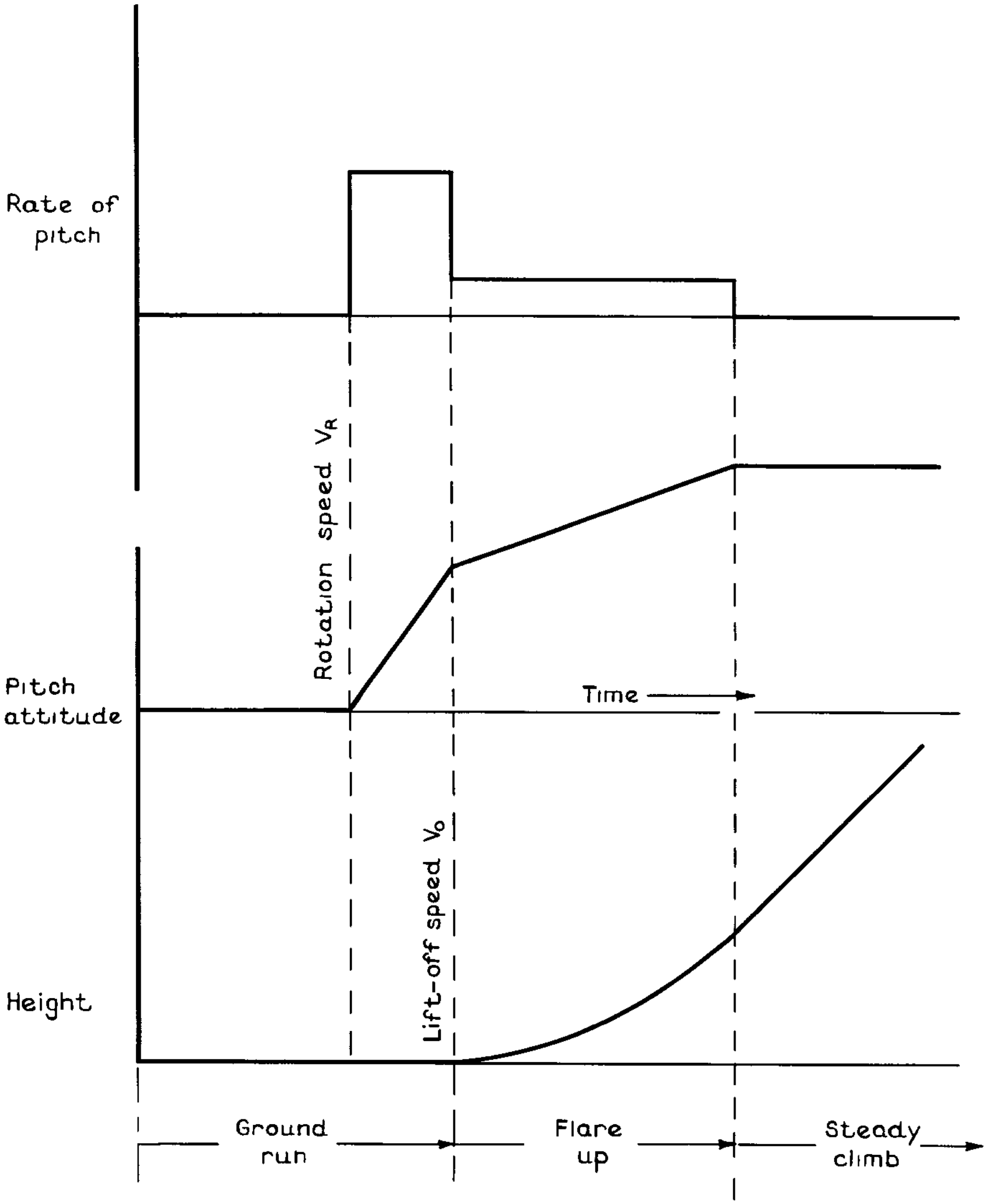

Fig. 4 Time history of the take-off manoeuvre assumed in the analysis

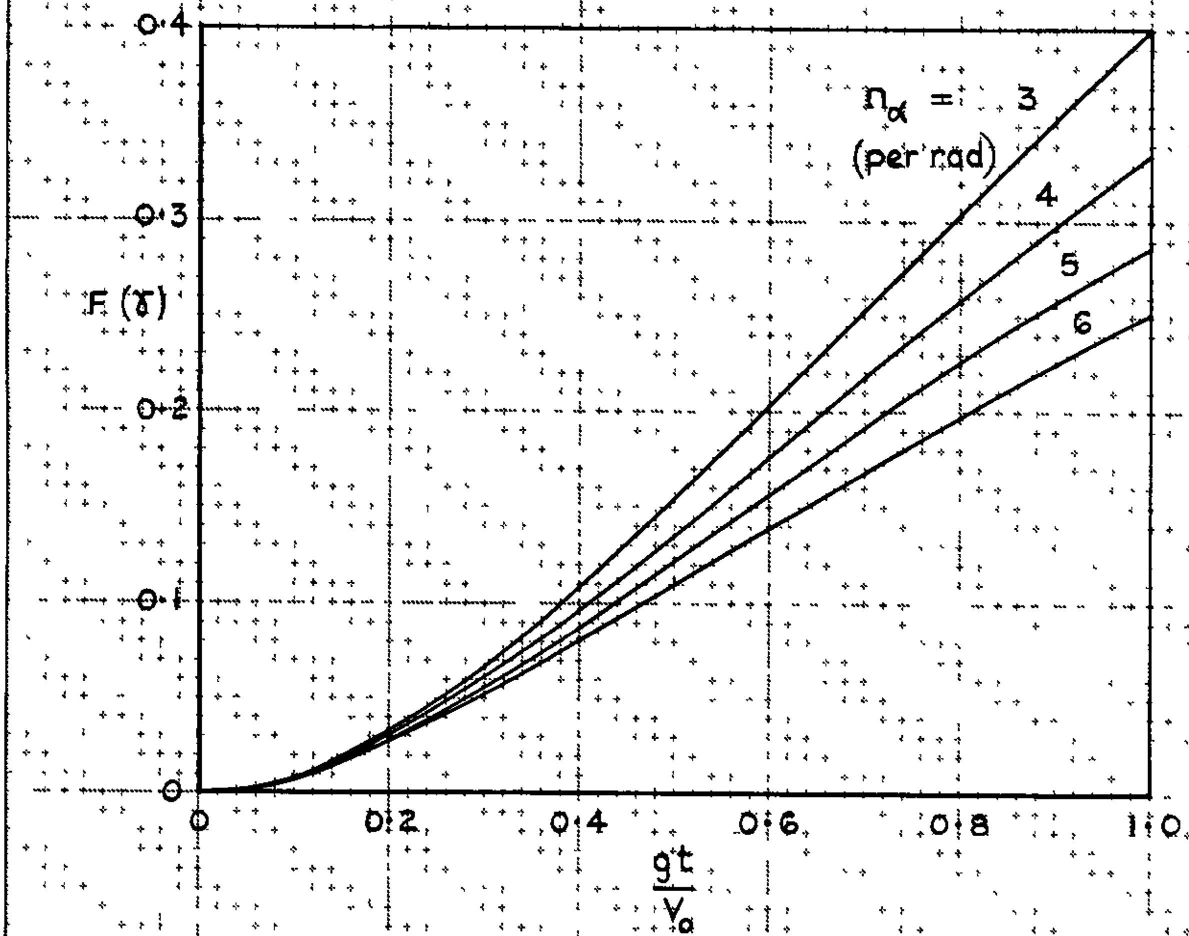

Fig. 5 The function $F(\gamma)$ used in the climb gradient equation

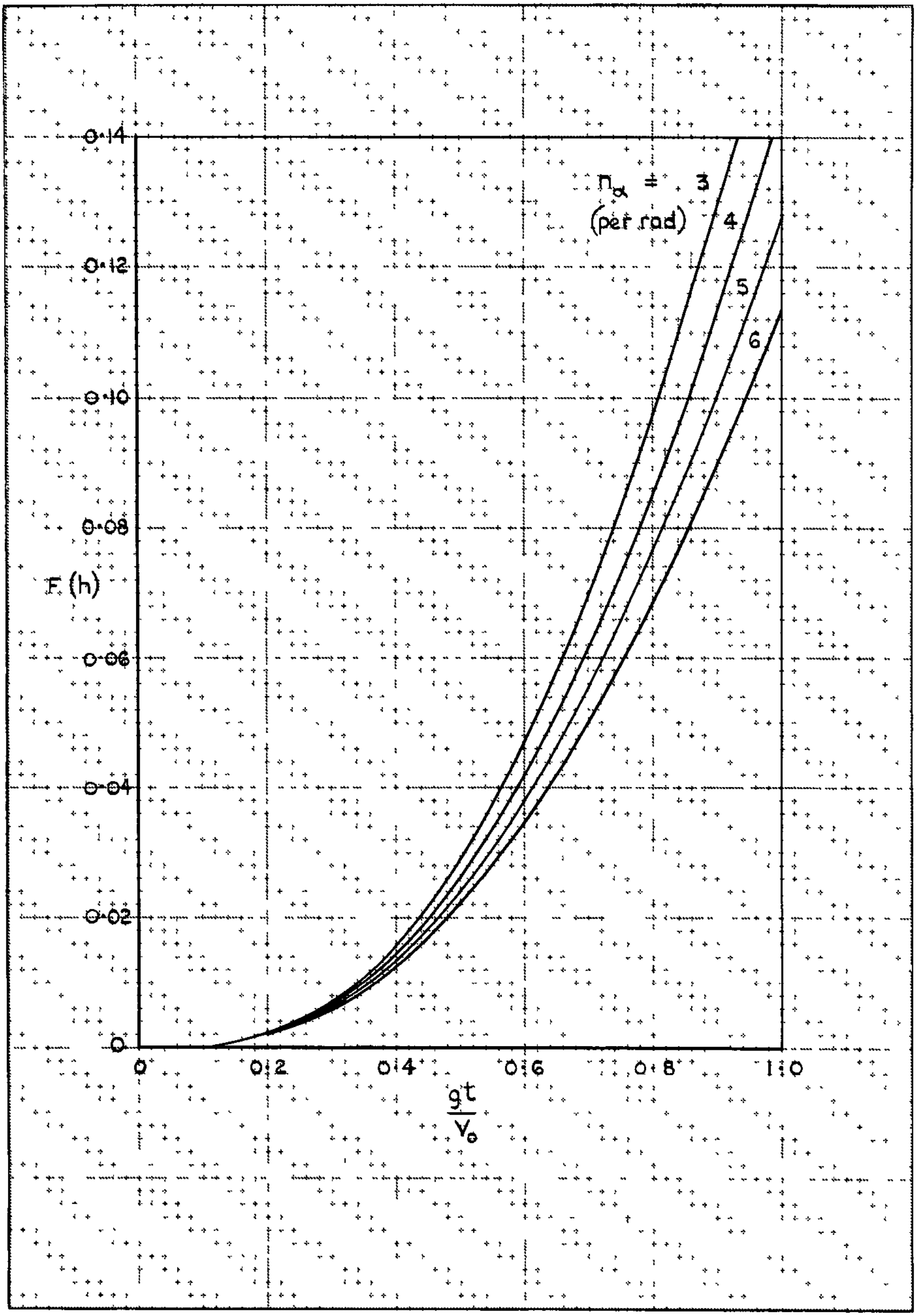

Fig.6 The function F(h) used in the height equation

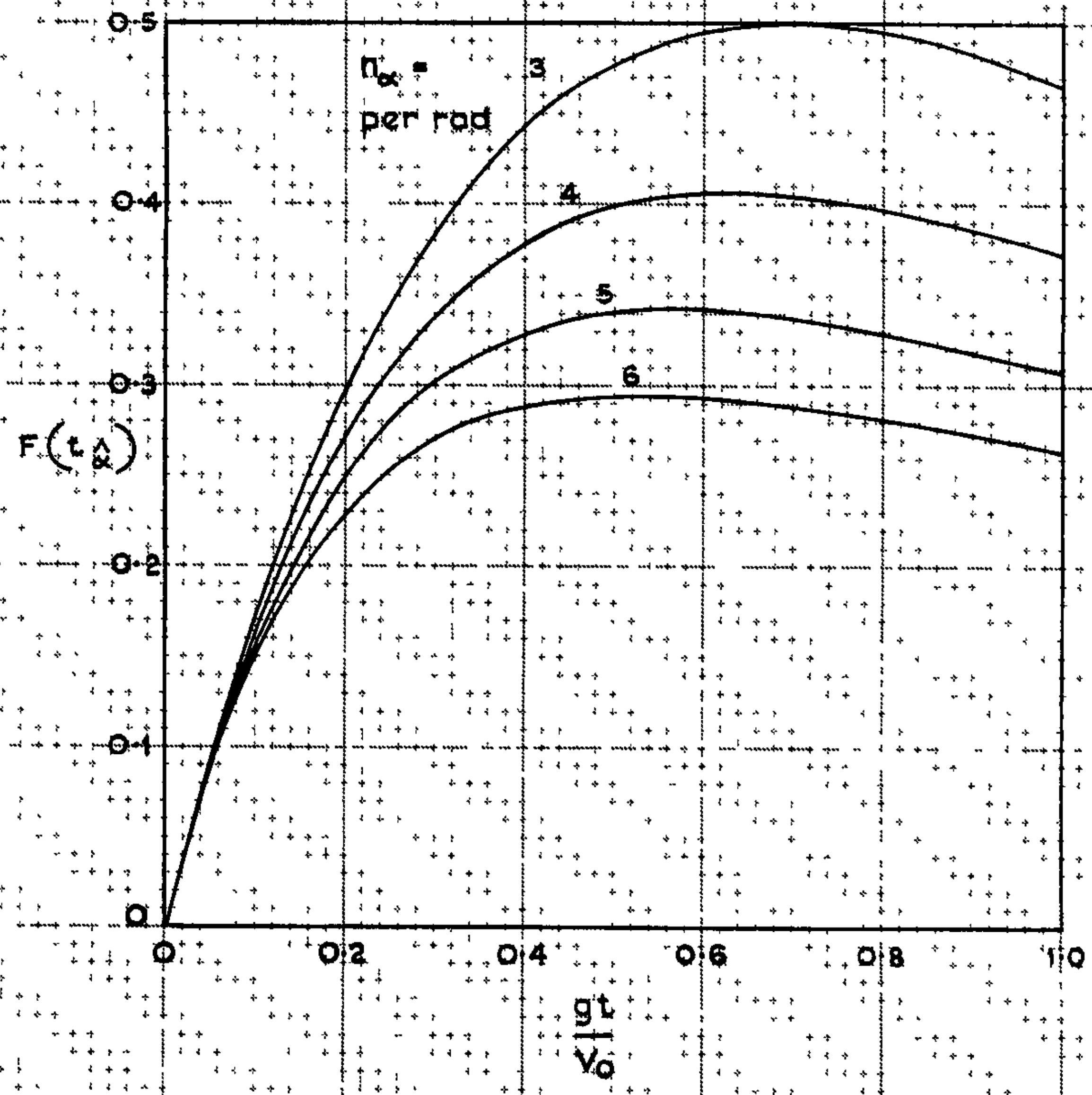

Fig.7 The function $F(t_{\hat{\alpha}})$ used to find the point of maximum incidence

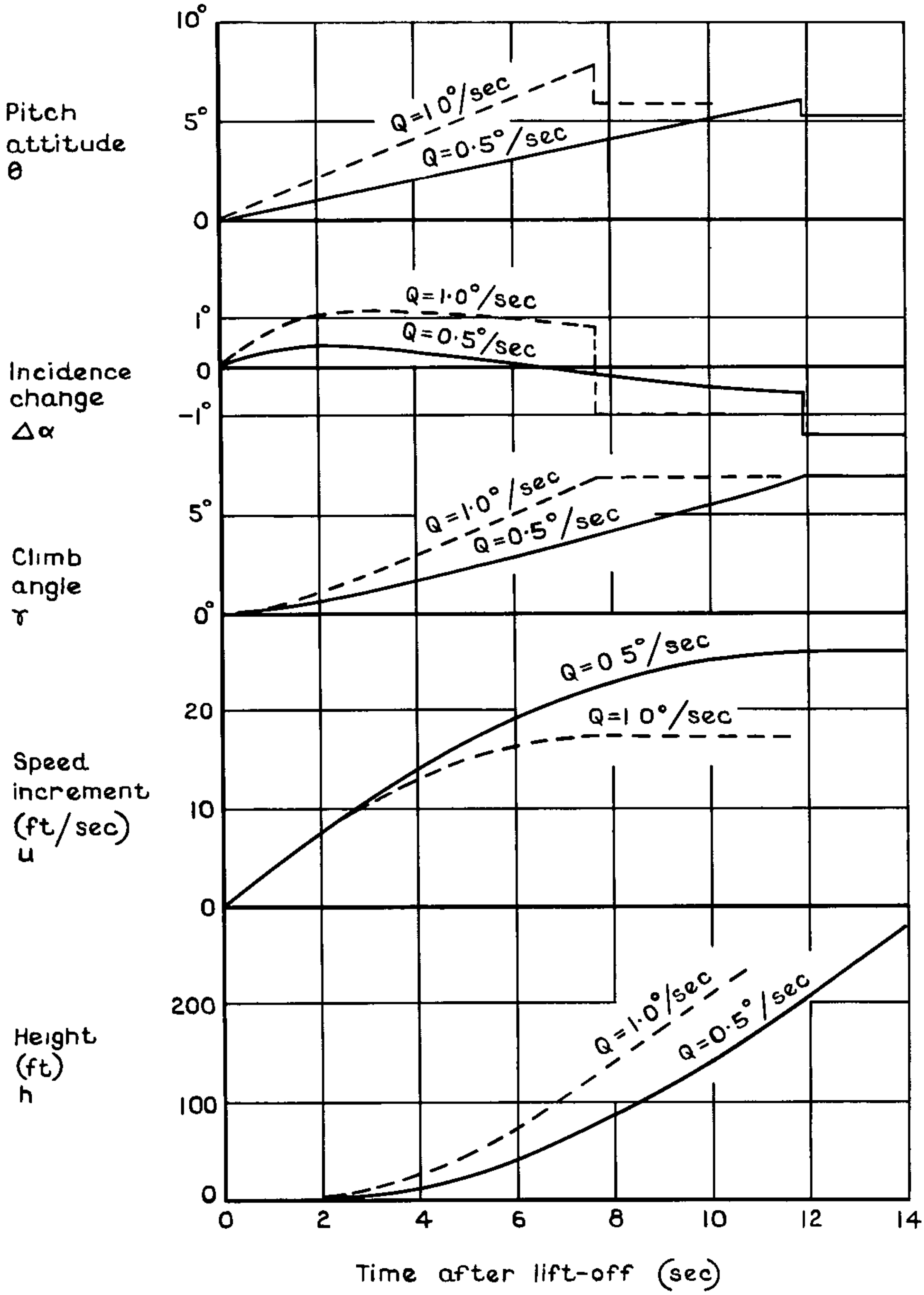

Fig 8 Comparison of take-offs for the slender wing aircraft with different pitch rates

$$\left[w = 85\,lb/ft^2,\ V = 200kt,\ n_\alpha = 6 \text{ per rad},\ \frac{T-D}{W} = 0{\cdot}12\right]$$

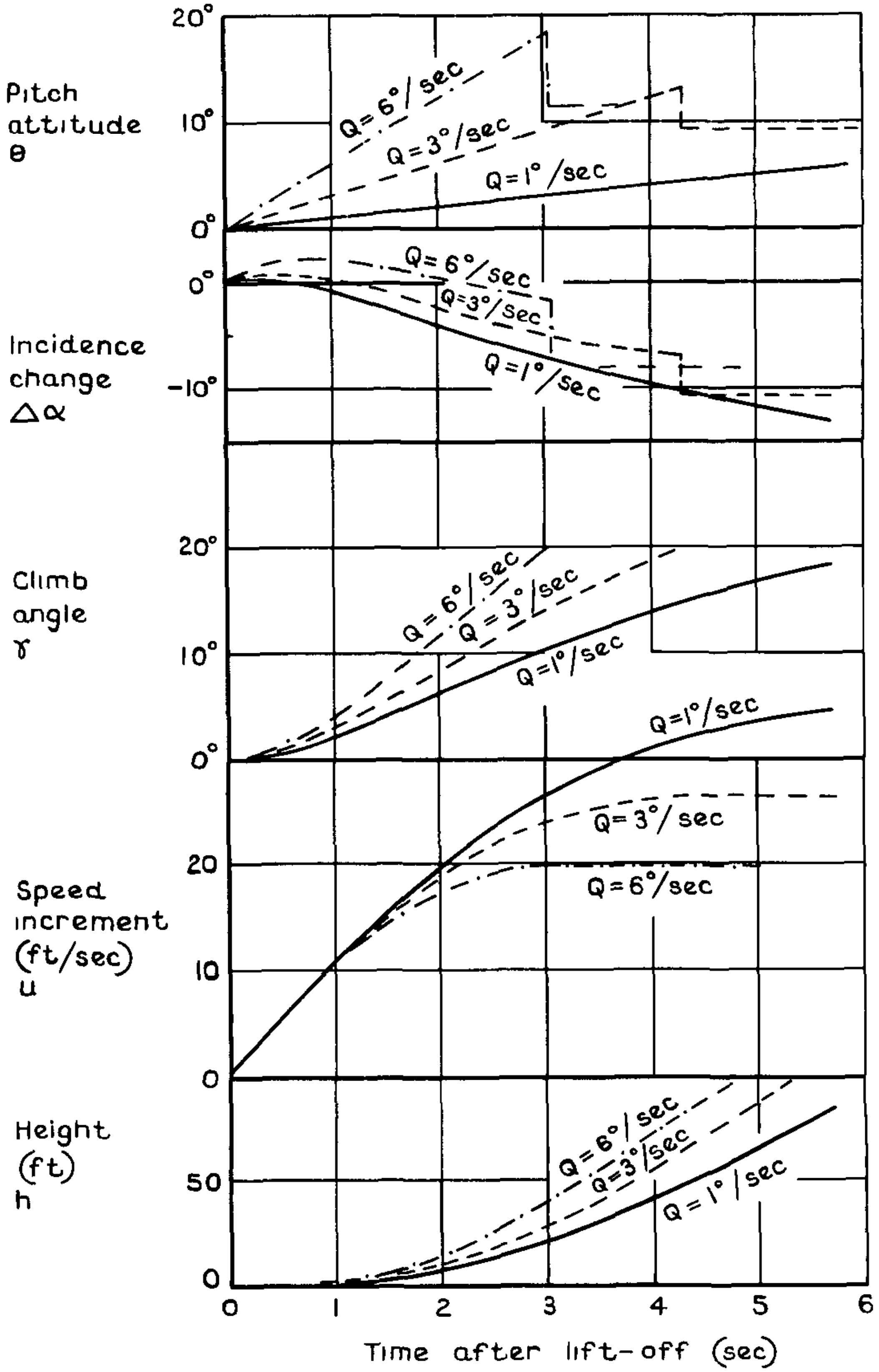

Fig.9 Comparison of take-offs for the S.T.O.L aircraft with different pitch rates

$\left[w = 20\,\text{lb/ft}^2, V = 54\,\text{kt}, n_\alpha = 3\text{ per rad}, \frac{T-D}{W} = 0{\cdot}35\right]$

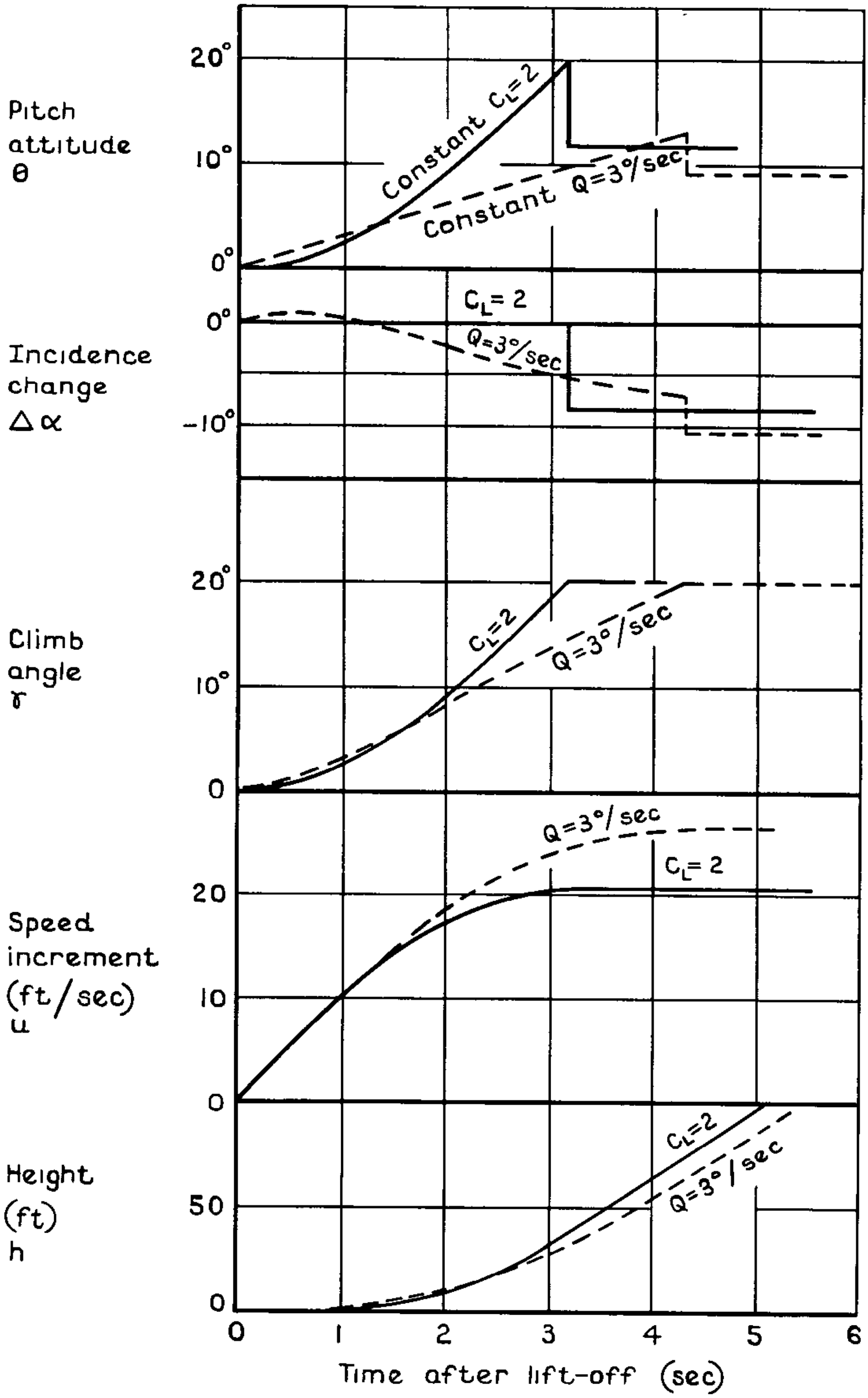

Fig.10 Comparison of manoeuvres with constant C_L and constant pitch rate for the S.T.O.L aircraft

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A.R.C. C.P. No.1042
March 1968

Perry, D.H.

THE AIRBORNE PATH DURING TAKE-OFF FOR CONSTANT RATE-OF-PITCH MANOEUVRES

533.6.013.152 :
533.6.015.1 :
629.13.077 :
532.511 :
629.137.1 :
533.693.3 :
629.136-118

A method of estimating the airborne path during take-off is proposed, based on the assumption that the aircraft is rotated at a constant rate of pitch from the moment of lift-off to the point at which it attains a steady climb path. The justification for this assumption is discussed. A simplified analysis, using small perturbation equations of motion, has been developed for initial project studies. Examples of the method applied to a slender wing transport aircraft, and a lightly loaded STOL aircraft are given, and the factors affecting the value of rate of pitch used are discussed.

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