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# The Derivation of Power Spectra of Density Variations in Hypersonic Wakes from Schlieren Photographs

*by*

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# THE DERIVATION OF POWER SPECTRA OF DENSITY VARIATIONS IN HYPERSONIC WAKES FROM SCHLIEREN PHOTOGRAPHS

by

J. P. Thompson

SUMMARY

An analysis is given of the relationship between power spectra of image density variations in a schlieren film of a hypersonic wake and the power spectra of gas density variations in the wake. Some results are given for the application of this theory to the schlieren film produced in an R.A.R.D.E. hypervelocity range test.

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*Replaces R.A.E. Technical Report 67226 - A.R.C. 29905

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## 1 INTRODUCTION

This paper describes some work on the deduction of density variations in the hypersonic wake from schlieren photographs taken in the R.A.R.D.E. No.2 hypervelocity range. A method is described by means of which the autocorrelation and spectrum functions of the density variations in the wake may be calculated from the autocorrelation and spectrum functions of the schlieren film density variations.

Clay, Herrmann and Slattery[1] at Lincoln Laboratory have reported work done on similar lines, in which a much larger quantity of experimental results was analysed.

## 2 THE EXPERIMENTS

The wake studied in this report was that of a 3/8 in aluminium sphere fired at 12000 ft/sec into air at 100 mmHg pressure. The Reynolds number based on body diameter was $3.4 \times 10^5$. The wake was photographed at a point 268 body diameters (8 ft 4.5 inches) behind the body; at this point it was 1.8 inches (4.7 body diameters) wide. Photographs were taken using a spark source and a single pass schlieren system giving a full-size image. The knife-edge was parallel to the wake axis. Typical photographs are shown in Fig.1.

A Joyce Loebl double-beam recording microdensitometer, model E12 Mark III, was used to measure schlieren film density. Traverses were taken parallel to the wake axis. The microdensitometer output for each traverse was a pen recording on white paper. These recordings were photographed on 35 mm film in R.A.E. Printing Department and the films were read on the digital film reader at R.R.E. Malvern. The output of this film reader was an 8-hole paper tape which could be read by the R.A.E. ICT Mercury computer.

## 3 MATHEMATICAL RELATIONSHIPS

For a single-pass schlieren system[2]

$$h(x_1, x_2) = \frac{f}{a} \int_{-\ell}^{\ell} \frac{1}{n} \frac{\partial n(x_1, x_2, x_3')}{\partial x_2} \, dx_3' \quad . \qquad (1)$$

For air,

$$n = 1 + C\rho \tag{2}$$

so

$$h(x_1,x_2) = \frac{f\,C}{a} \int_{-\ell}^{\ell} \frac{\partial\rho(x_1,x_2,x_3')}{\partial x_2}\,dx_3' \quad . \tag{3}$$

Uberoi and Kovasznay[3] give a discussion of the problem of determining the statistical properties of a quantity from the statistical properties of the output of a measuring system. They consider the case in which a quantity to be measured, $U(\underset{\sim}{s})$, is related to the output of the measuring system, $\Omega(\underset{\sim}{x})$, by the equation

$$\Omega(\underset{\sim}{x}) = \int K(\underset{\sim}{x}, \underset{\sim}{s})\, U(\underset{\sim}{s})\, d\underset{\sim}{s} \tag{4}$$

where the integration is over the whole of $\underset{\sim}{s}$ space. $K(\underset{\sim}{x}, \underset{\sim}{s})$, which expresses the operation of the measuring system, may be a generalised function[4]. For example,

$$K(\underset{\sim}{x}, \underset{\sim}{s}) = \delta(\underset{\sim}{s} - \underset{\sim}{x}) \tag{5}$$

makes $\Omega(\underset{\sim}{x})$ equal to $U(\underset{\sim}{x})$. This is the case of a perfect instrument.

Uberoi and Kovasznay consider particularly the case where $U(\underset{\sim}{x})$ is statistically homogeneous and infinite in extent and the response characteristics of the measuring system are independent of position in the field.

Then

$$\Omega(\underset{\sim}{x}) = \int K(\underset{\sim}{s} - \underset{\sim}{x})\, U(\underset{\sim}{s})\, d\underset{\sim}{s} \tag{6}$$

or, with a change of variable,

$$\Omega(\underset{\sim}{x}) = \int K(\underset{\sim}{s})\, U(\underset{\sim}{x} + \underset{\sim}{s})\, d\underset{\sim}{s} \quad . \tag{7}$$

It is possible to express the schlieren relationship (3) in the form (7).

Defining

$$S_3 = x_3' + x_3 \tag{8}$$

$$\rho^*(x_1, x_2, \{x_3' + x_3\}) = \rho(x_1, x_2, x_3') \tag{9}$$

and

$$h(x_1, x_2, x_3) = h(x_1, x_2) \tag{10}$$

then it is possible to rewrite (3) in the form

$$h(x_1, x_2, x_3) = \frac{f\ C}{a} \int_{x_3 - \ell}^{x_3 + \ell} \frac{\partial \rho^*(x_1, x_2, s_3)}{\partial x_2}\, ds_3 \quad . \tag{11}$$

It is shown in Appendix A that if a function $K(\underset{\sim}{s})$ is defined by

$$K(\underset{\sim}{s}) = \begin{cases} -\frac{f\ C}{a}\, \delta'(s_2)\, \delta(s_1) &, \quad |s_3| \le \ell \\ 0 &, \quad |s_3| > \ell \end{cases} \tag{12}$$

then

$$h(\underset{\sim}{x}) = \int K(\underset{\sim}{s})\, \rho^*(\underset{\sim}{x} + \underset{\sim}{s})\, d\underset{\sim}{s} \quad . \tag{13}$$

It now becomes possible to apply the analysis of Uberoi and Kovasznay to the schlieren problem. They show that if $\psi(\underset{\sim}{\tau})$ and $S(\underset{\sim}{k})$ are defined by

$$\psi(\underset{\sim}{\tau}) = \int K(\underset{\sim}{s} + \underset{\sim}{\tau})\, K(\underset{\sim}{s})\, d\underset{\sim}{s} \tag{14}$$

and

$$S(\underset{\sim}{k}) = \int \psi(\underset{\sim}{\tau})\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{\tau}}\, d\underset{\sim}{\tau} \tag{15}$$

then the relationship between the spectra $G^*(\underset{\sim}{k})$ and $\Gamma(\underset{\sim}{k})$ is

$$\Gamma(\underset{\sim}{k}) = S(\underset{\sim}{k})\, G^*(\underset{\sim}{k}) \quad . \tag{16}$$

In Appendix B the functions $\psi(\underset{\sim}{\tau})$ and $S(\underset{\sim}{k})$ are found for the $K(\underset{\sim}{s})$ defined by (12). The result is

$$\Gamma(\underset{\sim}{k}) = 4\, \frac{f^2 C^2}{a^2}\, k_2^2 \left(\frac{\sin k_3\, \ell}{k_3}\right)^2 G^*(\underset{\sim}{k}) \quad . \tag{17}$$

If the gas density field is homogeneous and isotropic then

$$\Gamma(\underset{\sim}{k}) = 4\, \frac{f^2 C^2}{a^2}\, k_2^2 \left(\frac{\sin k_3\, \ell}{k_3}\right)^2 \frac{G(k)}{4\pi\, k^2} \quad . \tag{18}$$

The relationship between $\gamma_1(k_1)$ and $\Gamma(\underset{\sim}{k})$ is

$$\gamma_1(k_1) = \iint \Gamma(\underset{\sim}{k})\, dk_2\, dk_3 \quad . \tag{19}$$

It is shown in Appendix C that (18), (19) yield the relation

$$G(k) = \frac{2\, a^2}{\pi\, f^2 C^2\, \ell}\, k\, \frac{d}{dk}\left\{k^{-2} \int_k^\infty \frac{d}{dk_1}\left[k_1\, \gamma_1(k_1)\right] \frac{k_1\, dk_1}{(k_1^2 - k^2)^{\frac{1}{2}}}\right\} . \tag{20}$$

A similar analysis (Appendix D) yields

$$G(k) = -\frac{2\, a^2}{\pi\, \ell\, f^2 C^2} \int_k^\infty \frac{d}{dk_2}\left[k_2^{-1}\, \gamma_2(k_2)\right] \frac{k_2\, dk_2}{(k_2^2 - k^2)^{\frac{1}{2}}} \tag{21}$$

(20) and (21) are relations by means of which the one-dimensional spectrum function of the density variations in the flow field may be found from the spectrum functions measured on the schlieren film.

For the cases where $\gamma_1(k_1)$ or $\gamma_2(k_2)$ may be expressed as series in negative powers of $k_1$ and $k_2$, the integrals in (20) and (21) may be evaluated analytically to give expressions for $G(k)$, which are series in negative powers of $k$. (Appendix E.) If $G(k)$ has the form of an inverse $q$ power law for all values of $k \geq$ some value $k_o$, then $\gamma_1(k_1)$ follows an inverse $(q-1)$ power law if $q > 2$, and $\gamma_2(k_2)$ follows an inverse $(q-1)$ power law if $q > 1$, in both cases over the range $\geq k_o$. This relationship is the same as that found by Clay, Herrmann and Slattery[1]; their function $S_\rho$ is, apart from a constant factor, equivalent to the $G^*(\underset{\sim}{k}) = \frac{G(k)}{4\pi k^2}$ of the present report.

The relationship between the one-dimensional and three-dimensional spectra of density variations in the flow is

$$F(k_1) = \frac{1}{2} \int_{k_1}^{\infty} \frac{G(k)}{k} \, dk \tag{22}$$

for a homogeneous isotropic field. This result is proved in Appendix F.

The integral in (22) may be evaluated analytically for the case where $G(k)$ is expressible as a series in negative powers of $k$. (Appendix F.) If $G(k)$ follows an inverse $q$ power law for all $k \geq$ some value $k_o$, then $F(k_1)$ follows the same power law for $k_1 \geq k_o$.

## 4 DATA REDUCTION AND RESULTS

A programme was written for the R.A.E. ICT Mercury computer which analysed the data on the tape produced by the film reader. Autocorrelation and spectrum functions of the film density variations were calculated from the standard formulae given by Blackman and Tukey[5] with a hanning window of 15% of the length of the wake traverse. The spectrum function $G(k)$ of the density variations in the wake was then calculated by means of equation (20) on the assumption that the field was homogeneous and isotropic. The one-dimensional spectrum function $F(k_1)$ was calculated from equation (22). Power law lines were fitted to the spectrum functions by a least squares process. All these operations on each wake traverse were done by one run of the computer programme.

Values obtained for the power of wave number giving the best fit to the spectra were as follows.

Table 1

Wave number index of spectrum functions

| Traverse | Film one-dimensional | Flow three-dimensional | Flow one-dimensional |
|---|---|---|---|
| | $\gamma_1(k_1)$ | G(k) | $F(k_1)$ |
| AA1 | -1.64 | -1.15 | -2.84 |
| AA2 | -1.47 | -1.10 | -2.56 |
| BB1 | -1.70 | -0.97 | -2.86 |
| BB2 | -1.72 | -1.09 | -2.42 |
| Mean for AA | -1.56 | -1.12 | -2.70 |
| Mean for BB | -1.71 | -1.03 | -2.64 |
| Overall mean | -1.63 | -1.08 | -2.67 |

Measurements were made on two negatives, 1 and 2, derived from a schlieren photograph reproduced as the lower picture in Fig.1. Traverses AA1, AA2 were taken at a distance 0.75 of the wake radius from the wake axis; traverses BB1 and BB2 were taken at 0.55 of the wake radius from the wake axis. It is seen that the spectrum derived directly from the film density comes much closer to the -5/3 power law expected for homogeneous isotropic turbulence than does that allowing for the 3-dimensionality of the field (for which -1 is a better approximation to the data). In fact however the assumption of homogeneous isotropic turbulence is not necessarily a good one especially for an off-axial slice of an axisymmetric wake, which might be expected to include a fairly large proportion of low-wave-number eddies for which a power-law representation of the spectral function is inappropriate.

More accurate values for the spectral density especially at the low-wave-number end could only be obtained by analysis of longer strips of wake (in terms of wake diameter), but these were not obtainable with the existing schlieren windows at R.A.R.D.E.

In conclusion, the technique of analysing films by means of a densitometer to obtain turbulent spectra appears a valuable one, but the interpretation is very sensitive to the particular assumption made about the type of distribution, and an improvement on that of spatial homogeneity would be desirable.

## Appendix A

$$\int K(\underset{\sim}{s})\, \rho^*(\underset{\sim}{x} + \underset{\sim}{s})\, d\underset{\sim}{s} = -\frac{f\,C}{a} \int_{|s_3| \le \ell} \delta'(s_2)\, \delta(s_1)\, \rho^*(\underset{\sim}{x} + \underset{\sim}{s})\, d\underset{\sim}{s}$$

$$= -\frac{f\,C}{a} \int_{|s_3 - x_3| \le \ell} \delta'(s_2 - x_2)\, \delta(s_1 - x_1)\, \rho^*(\underset{\sim}{s})\, d\underset{\sim}{s}$$

by change of variable

$$= -\frac{f\,C}{a} \iint_{|s_3 - x_3| \le \ell} \delta(s_1 - x_1) \left[\int \delta'(s_2 - x_2)\, \rho^*(\underset{\sim}{s})\, ds_2\right] ds_1\, ds_3$$

$$= \frac{f\,C}{a} \iint_{|s_3 - x_3| \le \ell} \delta(s_1 - x_1) \left[\int \delta(s_2 - x_2)\, \frac{\partial \rho^*(\underset{\sim}{s})}{\partial s_2}\, ds_2\right] ds_1\, ds_3$$

by integration by parts

$$= \frac{f\,C}{a} \iint_{|s_3 - x_3| \le \ell} \delta(s_1 - x_1)\, \frac{\partial \rho^*(s_1, x_2, s_3)}{\partial x_2}\, ds_1\, ds_3$$

$$= \frac{f\,C}{a} \int_{s_3 = x_3 - \ell}^{s_3 = x_3 + \ell} \frac{\partial \rho^*(x_1, x_2, s_3)}{\partial x_2}\, ds_3 \quad .$$

## Appendix B

$$\psi(\underset{\sim}{\tau}) = \int K(\underset{\sim}{s} + \underset{\sim}{\tau})\, K(\underset{\sim}{s})\, d\underset{\sim}{s}$$

$$= \frac{f^2 c^2}{a^2} \int [\delta'(s_2 + \tau_2) \quad \delta(s_1 + \tau_1)]\, [\delta'(s_2)\, \delta(s_1)]\, d\underset{\sim}{s}$$

the range of integration being

$$|s_3| \leq \ell, \quad |s_3 + \tau_3| \leq \ell$$

and all values of $s_1$ and $s_2$

$$= \frac{f^2 c^2}{a^2} \int_{s_3 \text{range}} ds_3 \int \delta'(s_2 + \tau_2)\, \delta'(s_2)\, \delta(\tau_1)\, ds_2$$

$$= -\frac{f^2 c^2}{a^2} \int_{s_3 \text{range}} ds_3\, \delta(\tau_1) \int \delta(s_2)\, \delta''(s_2 + \tau_2)\, ds_2$$

$$= -\frac{f^2 c^2}{a^2} \int_{s_3 \text{range}} ds_3\, \delta(\tau_1)\, \delta''(\tau_2) \quad .$$

The dependence on $\tau_3$ of the range of integration with respect to $s_3$ is shown in Fig.2. This range is

$$+ (2\ell - |\tau_3|) \quad \text{if} \quad |\tau_3| \leq 2\ell$$

$$0 \quad \text{if} \quad |\tau_3| > 2\ell \quad .$$

Therefore

$$\psi(\underset{\sim}{\tau}) = -\frac{f^2c^2}{a^2}(2\ell - |\tau_3|)\,\delta(\tau_1)\,\delta''(\tau_2)$$

$$\text{if } |\tau_3| \leq 2\ell$$

$$= 0 \quad \text{if } |\tau_3| > 2\ell$$

$$S(\underset{\sim}{k}) = \int \psi(\underset{\sim}{\tau})\, e^{-i\underset{\sim}{k}\cdot\underset{\sim}{\tau}}\, d\underset{\sim}{\tau}$$

$$= -\frac{f^2c^2}{a^2}\int_{|\tau_3|\leq 2\ell} e^{-i\underset{\sim}{k}\cdot\underset{\sim}{\tau}}\,(2\ell - |\tau_3|)\,\delta(\tau_1)\,\delta''(\tau_2)\, d\underset{\sim}{\tau}$$

$$= -\frac{f^2c^2}{a^2}\int_{-2\ell}^{2\ell}(2\ell - |\tau_3|)\, e^{-ik_3\tau_3}\, d\tau_3$$

$$\times \int e^{-ik_1\tau_1}\,\delta(\tau_1)\, d\tau_1$$

$$\times \int e^{-ik_2\tau_2}\,\delta''(\tau_2)\, d\tau_2$$

$$= \frac{f^2c^2}{a^2}\left\{\int_0^{2\ell}(2\ell - \tau_3)\, e^{-ik_3\tau_3}\, d\tau_3 + \int_{-2\ell}^{0}(2\ell + \tau_3)\, e^{-ik_3\tau_3}\, d\tau_3\right\}$$

$$\times \int \delta'(\tau_2)\,(-ik_2)\, e^{-ik_2\tau_2}\, d\tau_2$$

$$= -\frac{f^2c^2}{a^2}\int_0^{2\ell}(2\ell - \tau_3)\,(e^{-ik_3\tau_3} + e^{ik_3\tau_3})\, d\tau_3$$

$$\times \int \delta(\tau_2)\,(-k_2^2)\, e^{-ik_2\tau_2}\, d\tau_2$$

$$= \frac{f^2 c^2}{a^2} \int_0^{2\ell} 2\,(2\ell - \tau_3)\cos k_3 \tau_3 \, d\tau_3 \, k_2^2$$

$$= \frac{2f^2 c^2}{a^2} k_2^2 \left\{ \left[ \frac{1}{k_3} (\sin k_3 \tau_3)\,(2\ell - \tau_3) \right]_0^{2\ell} + \int_0^{2\ell} \frac{1}{k_3} \sin k_3 \tau_3 \, d\tau_3 \right\}$$

$$= \frac{2f^2 c^2}{a^2} k_2^2 \left[ - \frac{1}{k_3^2} \cos k_3 \tau_3 \right]_0^{2\ell}$$

$$= - \frac{2f^2 c^2}{a^2} \frac{k_2^2}{k_3^2} (\cos 2\ell k_3 - 1)$$

$$= \frac{4f^2 c^2}{a^2} \frac{k_2^2}{k_3^2} \sin^2 \ell \; k_3$$

$$= \frac{4f^2 c^2}{a^2} k_2^2 \left( \frac{\sin k_3 \ell}{k_3} \right)^2 .$$

Therefore

$$\Gamma(\underset{\sim}{k}) = \frac{4f^2 c^2}{a^2} k_2^2 \left( \frac{\sin k_3 \ell}{k_3} \right)^2 G^*(\underset{\sim}{k})$$

## Appendix C

$$\gamma_1(k_1) = \iint \Gamma(\underset{\sim}{k})\, dk_2\, dk_3$$

$$= \frac{f^2 C^2}{\pi a^2} \iint k_2^2 \left(\frac{\sin k_3 \ell}{k_3}\right)^2 \frac{G(k)}{k^2}\, dk_2\, dk_3 \quad .$$

If $\ell$ is much larger than the scale of the density variations, $[(\sin k_3 \ell)/k_3]^2$ has almost the effect of a Dirac delta function, so that

$$\gamma_1(k_1) \simeq \frac{f^2 C^2 \ell}{a^2} \int k_2^2 \frac{G([k_1^2 + k_2^2]^{\frac{1}{2}})}{k_1^2 + k_2^2}\, dk_2 \quad .$$

Let

$$k'^2 = k_1^2 + k_2^2 \quad .$$

Then

$$\gamma_1(k_1) = \frac{f^2 C^2 \ell}{a^2} \int_{k_1}^{\infty} (k'^2 - k_1^2) \frac{G(k')}{k'^2} \frac{k'\, dk'}{\sqrt{k'^2 - k_1^2}}$$

$$= \frac{f^2 C^2 \ell}{a^2} \int_{k_1}^{\infty} \sqrt{k^2 - k_1^2}\, k^{-1}\, G(k)\, dk$$

$$= \frac{f^2 C^2 \ell}{a^2} \left[\int_{\infty}^{k} k^{-1}\, G(k)\, dk \sqrt{k^2 - k_1^2}\,\right]_{k_1}^{\infty}$$

$$- \frac{f^2 C^2 \ell}{a^2} \int_{k_1}^{\infty} \left(\int_{\infty}^{k} k^{-1}\, G(k)\, dk\right) \tfrac{1}{2}\, (k^2 - k_1^2)^{-\frac{1}{2}}\, 2k\, dk \, .$$

The first of these two terms is zero.

This equation may now be inverted by the method given by Uberoi and Kovasznay in their appendix.

$$\int_{\infty}^{k_1} k^{-1} \, G(k) \, dk = \frac{2a^2}{\pi \, f^2 C^2 \, \ell \, k_1^2} \int_{k_1}^{\infty} \frac{d}{dk} \left[ k \, \gamma_1(k) \right] \frac{k \, dk}{(k^2 - k_1^2)^{\frac{1}{2}}} \quad .$$

Therefore

$$G(k_1) = \frac{2a^2}{\pi \, f^2 C^2 \, \ell} \, k_1 \, \frac{d}{dk_1} \left\{ \frac{1}{k_1^2} \int_{k_1}^{\infty} \frac{d}{dk} \left[ k \, \gamma_1(k) \right] \frac{k \, dk}{(k^2 - k_1^2)^{\frac{1}{2}}} \right\} \quad .$$

Equation (20) may be obtained by interchanging k and $k_1$.

## Appendix D

$$\gamma_2(k_2) = \iint \Gamma(\underset{\sim}{k})\ dk_1\ dk_3$$

$$= \frac{f^2 C^2}{\pi a^2} \iint k_2^2 \left(\frac{\sin k_3 \ell}{k_3}\right)^2 \frac{G(k)}{k^2}\ dk_1\ dk_3 \quad .$$

If $\ell$ is much larger than the scale of the density variations, $[(\sin k_3\ell)/k_3]^2$ may be replaced by a Dirac delta function as in Appendix C to give

$$\gamma_2(k_2) \simeq \frac{f^2 C^2 \ell}{a^2} k_2^2 \int \frac{G([k_1^2 + k_2^2]^{\frac{1}{2}})}{k_1^2 + k_2^2}\ dk_1 \quad .$$

Let

$$k'^2 = k_1^2 + k_2^2 \quad .$$

Then

$$\gamma_2(k_2) = \frac{f^2 C^2 \ell}{a^2} k_2^2 \int_{k_2}^{\infty} \frac{G(k')}{k'^2} \frac{k'}{\sqrt{k'^2 - k_2^2}}\ dk'$$

$$= \frac{f^2 C^2 \ell}{a^2} k_2^2 \int_{k_2}^{\infty} k^{-1}\ G(k)\ (k^2 - k_2^2)^{-\frac{1}{2}}\ dk \quad .$$

Inverting,

$$k_2^{-4}\ G(k_2) = -\frac{2a^2}{\pi\ f^2 C^2\ \ell\ k_2^4} \int_{k_2}^{\infty} \frac{d}{dk}\left[k^{-1}\ \gamma_2(k)\right] \frac{k\ dk}{(k^2 - k_2^2)^{\frac{1}{2}}} \quad .$$

Therefore

$$G(k_2) = -\frac{2a^2}{\pi f^2 C^2 \ell} \int_{k_2}^{\infty} \frac{d}{dk}\left[k^{-1} \gamma_2(k)\right] \frac{k\, dk}{(k^2 - k_2^2)^{\frac{1}{2}}} \quad .$$

Transposition of $k$ and $k_2$ gives equation (21).

## Appendix E

If $\gamma_1(k_1)$ is of the form $\sum_{i=1}^{n} \frac{a_{i1}}{k_1^{p_i}}$ for all $k_1$ greater than or equal to a certain value $k_{10}$, the powers $p_i$ being positive, then

$$\int_k^\infty \frac{d}{dk_1}\left[k_1\, \gamma_1(k_1)\right] \frac{k_1\, dk_1}{(k_1^2 - k^2)^{\frac{1}{2}}} = \sum_{i=1}^{n} a_{i1}\, (1 - p_i) \int_k^\infty k_1^{1-p_i} (k_1^2 - k^2)^{-\frac{1}{2}}\, dk_1$$

for $k \geq k_{10}$.

Let

$$k_1 = k \cosh \theta$$

$$dk_1 = k \sinh \theta \; d\theta$$

$$\cosh^2\theta - \sinh^2\theta = 1 \quad .$$

Then the expression equals

$$\sum_{i=1}^{n} a_{i1}\, (1 - p_i)\, k^{1-p_i} \int_0^\infty \cosh^{1-p_i} \theta \; d\theta \quad .$$

Gradshteyn and Ryzhik[6] give an expression for the integral in terms of the beta function.

The above expression equals

$$\sum_{i=1}^{n} a_{i1}\, (1 - p_i)\, k^{1-p_i}\, 4^{\frac{p_i - 3}{2}}\, B\left(\frac{p_i - 1}{2}, \frac{p_i - 1}{2}\right)$$

if $p_i > 1$ for all i.

Therefore

$$G(k) = \frac{2a^2}{\pi f^2 C^2 \ell} \sum_{i=1}^{n} 4^{\frac{p_i - 3}{2}} B\left(\frac{p_i - 1}{2}, \frac{p_i - 1}{2}\right) \frac{a_{i1} (p_i^2 - 1)}{k^{p_i + 1}}$$

for $k \geq k_{10}$

$p_i > 1$ for all $i$ .

Therefore if $G(k) = \sum_{i=1}^{n} \frac{b_{i1}}{k^{q_i}}$ for all $k \geq k_o$, where $q_i > 2$ for all $i$,

$$\gamma_1(k_1) = \frac{\pi f^2 C^2 \ell}{2a^2} \sum_{i=1}^{n} 4^{\frac{4-q_i}{2}} B^{-1}\left(\frac{q_i - 2}{2}, \frac{q_i - 2}{2}\right) \frac{b_{i1}}{q_i (q_i - 2) k^{q_i - 1}}$$

for $k_1 \geq k_o$ .

If $\gamma_2(k_2)$ is of the form $\sum_{i=1}^{n} \frac{a_{i2}}{k_2^{r_i}}$ for all $k_2 \geq$ some value $k_{20}$, the powers $r_i$ being positive,

$$G(k) = \frac{2a^2}{\pi f^2 C^2 \ell} \sum_{i=1}^{n} a_{i2} (r_i + 1) \int_k^{\infty} k_2^{-1-r_i} (k_2^2 - k^2)^{-\frac{1}{2}} dk_2$$

for $k \geq k_{20}$

$$= \frac{2a^2}{\pi f^2 C^2 \ell} \sum_{i=1}^{n} a_{i2} (r_i + 1) k^{-1-r_i} 4^{\frac{r_i - 1}{2}} B\left(\frac{r_i + 1}{2}, \frac{r_i + 1}{2}\right)$$

$$G(k) = \frac{2a^2}{\pi f^2 C^2 \ell} \sum_{i=1}^{n} 4^{\frac{r_i - 1}{2}} B\left(\frac{r_i + 1}{2}, \frac{r_i + 1}{2}\right) \frac{a_{i2} (r_i + 1)}{k^{r_i + 1}}$$

for $k \geq k_{20}$ .

Therefore if $G(k) = \sum_{i=1}^{n} \frac{b_{i2}}{k^{t_i}}$ for all $k \geq k_o$, where $t_i > 1$ for all i,

$$\gamma_2(k_2) = \frac{\pi f^2 C^2 \ell}{2a^2} \sum_{i=1}^{n} 4^{\frac{2-t_i}{2}} B^{-1}\left(\frac{t_i}{2}, \frac{t_i}{2}\right) \frac{b_{i2}}{t_i k^{t_i - 1}} .$$

## Appendix F

$$F(k_1) = \iint G^*(\underset{\sim}{k})\, dk_2\, dk_3$$

$$G(k) = \int G^*(\underset{\sim}{k})\, d\sigma\,(k)$$

where $d\sigma\,(k)$ is a surface element of a sphere of radius $k$ in $\underset{\sim}{k}$ space.

For homogeneous isotropic density variations $G^*(\underset{\sim}{k})$ is a function of $k$ only, so

$$G(k) = 4\pi k^2\, G^*(\underset{\sim}{k}) \quad .$$

Therefore

$$F(k_1) = \iint \frac{G(k)}{4\pi k^2}\, dk_2\, dk_3$$

$k_2$ and $k_3$ may be replaced by $k$ and $\varphi$, the polar angle in the $k_2$, $k_3$ plane.

$$k_2 = \sqrt{k^2 - k_1^2}\, \cos\varphi$$

$$k_3 = \sqrt{k^2 - k_1^2}\, \sin\varphi \quad .$$

The Jacobian is

$$\begin{vmatrix} \dfrac{\partial k_2}{\partial k} & \dfrac{\partial k_2}{\partial \varphi} \\ \dfrac{\partial k_3}{\partial k} & \dfrac{\partial k_3}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} \tfrac{1}{2}(k^2 - k_1^2)^{-\frac{1}{2}}\, 2k \cos\varphi & -(k^2 - k_1^2)^{\frac{1}{2}} \sin\varphi \\ \tfrac{1}{2}(k^2 - k_1^2)^{-\frac{1}{2}}\, 2k \sin\varphi & (k^2 - k_1^2)^{\frac{1}{2}} \cos\varphi \end{vmatrix}$$

$$= k \quad .$$

Therefore

$$F(k_1) = \iint\limits_{\substack{0<\varphi<2\pi \\ k>k_1}} \frac{G(k)}{4\pi k^2}\, k \, dk \, d\varphi$$

$$F(k_1) = \tfrac{1}{2}\int_{k_1}^{\infty} \frac{G(k)}{k}\, dk \quad .$$

If

$$G(k) = \sum_{i=1}^{n} \frac{b_i}{k^{q_i}}$$

for all $k \geqq k_o$, the powers $q_i$ being positive,

$$F(k_1) = \tfrac{1}{2}\int_{k_1}^{\infty} k^{-1} \sum_{i=1}^{n} \frac{b_i}{k^{q_i}}\, dk$$

$$= \sum_{i=1}^{n} \frac{b_i}{2q_i\, k_1^{q_i}}$$

for $k_1 \geqq k_o$ .

Therefore if $F(k_1) = \sum_{i=1}^{n} \frac{c_i}{k_1^{q_i}}$, ($q_i$ + ve for all i) for all $k_1 \geqq k_o$,

$$G(k) = \sum_{i=1}^{n} \frac{2q_i\, c_i}{k^{q_i}}$$

for all $k \geqq k_o$ .

## SYMBOLS

| | |
|---|---|
| $a$ | width of the slit image at the knife edge of the schlieren system |
| $a_{11}$, $a_{12}$ | constants defined in Appendix E |
| $B(a,b)$ | beta function of $a, b$ |
| $b_{11}$, $b_{12}$ | constants defined in Appendix E |
| $b_i$ | constant defined in Appendix F |
| $C$ | Gladstone-Dale constant |
| $c_i$ | constant defined in Appendix F |
| $F(k_1)$ | one-dimensional spectrum of density variations in the flow |
| $f$ | focal length of schlieren mirror |
| $G(k)$ | three-dimensional $k$ spectrum of the flow density field |
| $G^*(\underset{\sim}{k})$ | three-dimensional $\underset{\sim}{k}$ spectrum of the flow density field |
| $h(x_1, x_2)$ | fractional change in intensity at the schlieren film at a point corresponding to coordinates $x_1, x_2$ in the flow |
| $h(x_1, x_2, x_3)$ | defined by equation (10) |
| $\underset{\sim}{k}$ | vector wave number |
| $k_1, k_2, k_3$ | components of $\underset{\sim}{k}$ corresponding to $x_1, x_2, x_3$ |
| $k_{10}, k_{20}$ | defined in Appendix E |
| $k_o$ | defined in Appendix E |
| $K(\underset{\sim}{x}, \underset{\sim}{s})$ | kernel defined by (4) |
| $K(\underset{\sim}{s} - \underset{\sim}{x})$ | kernel defined by (6) |
| $K(\underset{\sim}{s})$ | kernel defined by (12) |
| $\ell$ | $\frac{1}{2}$ length of light path in density field |
| $n$ | refractive index |
| $p_i$ | defined in Appendix E |
| $q_i$ | defined in Appendices E, F |
| $r_i$ | defined in Appendix E |
| $\underset{\sim}{s}$ | a position vector in the flow with components $s_1, s_2, s_3$, where $s_1 = x_1$ and $s_2 = x_2$ |
| $S(\underset{\sim}{k})$ | a function defined by (15) |
| $t_i$ | defined in Appendix E |
| $U(\underset{\sim}{s})$ | a field to be measured |
| $x_1, x_2, x_3'$ | position coordinates in the flow; $x_1$ and $x_2$ define a corresponding point on the schlieren film |
| $x_3$ | a constant dummy variable |

## SYMBOLS (Contd)

| | |
|---|---|
| $\underset{\sim}{x}$ | $(x_1, x_2, x_3)$ |
| $\Gamma(\underset{\sim}{k})$ | three-dimensional $\underset{\sim}{k}$ spectrum of the mapped field on the schlieren film |
| $\gamma_1(k_1)$ | one-dimensional spectrum of the mapped field along a direction parallel to the schlieren knife-edge |
| $\gamma_2(k_2)$ | one-dimensional spectrum of the mapped field along a direction perpendicular to knife-edge |
| $\delta(x)$, $\delta(\underset{\sim}{x})$ | Dirac delta functions of $x$ and $\underset{\sim}{x}$ |
| $\delta'(x)$, $\delta''(x)$ | first and second derivatives of the Dirac delta function of $x$ |
| $\rho$ | gas density |
| $\rho^*$ | defined by equation (9) |
| $\underset{\sim}{\tau}$ | a position vector introduced in equation (14) |
| $\varphi$ | a polar angle in the $k_2, k_3$ plane (Appendix F) |
| $\psi(\underset{\sim}{\tau})$ | a function defined by equation (14) |
| $\Omega(\underset{\sim}{x})$ | a mapped field (equation (4)) |

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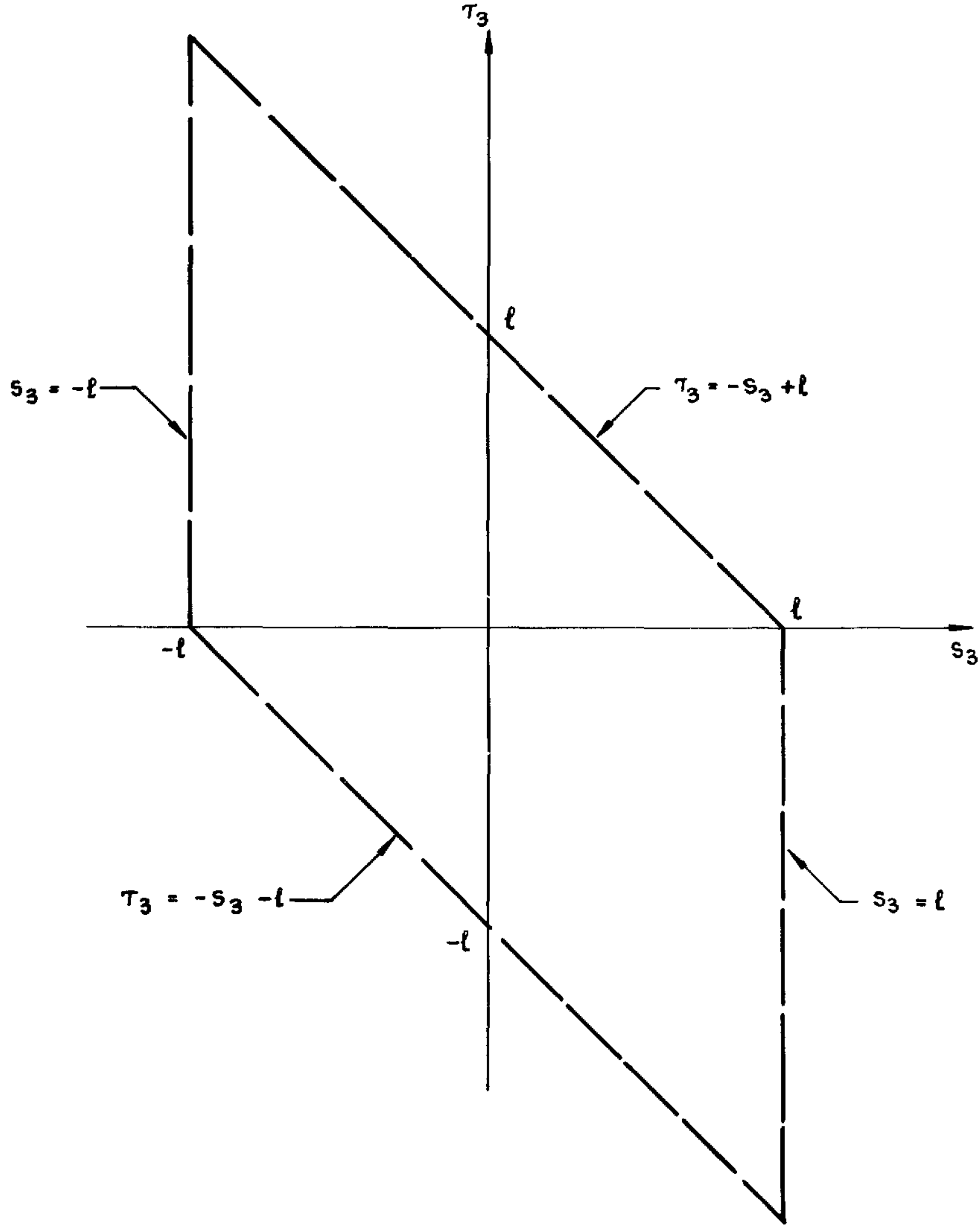


FIG. 2 DEPENDENCE ON $\tau_3$ OF THE RANGE OF INTEGRATION WITH RESPECT TO $S_3$. THE RANGE IS BOUNDED BY THE BROKEN LINES

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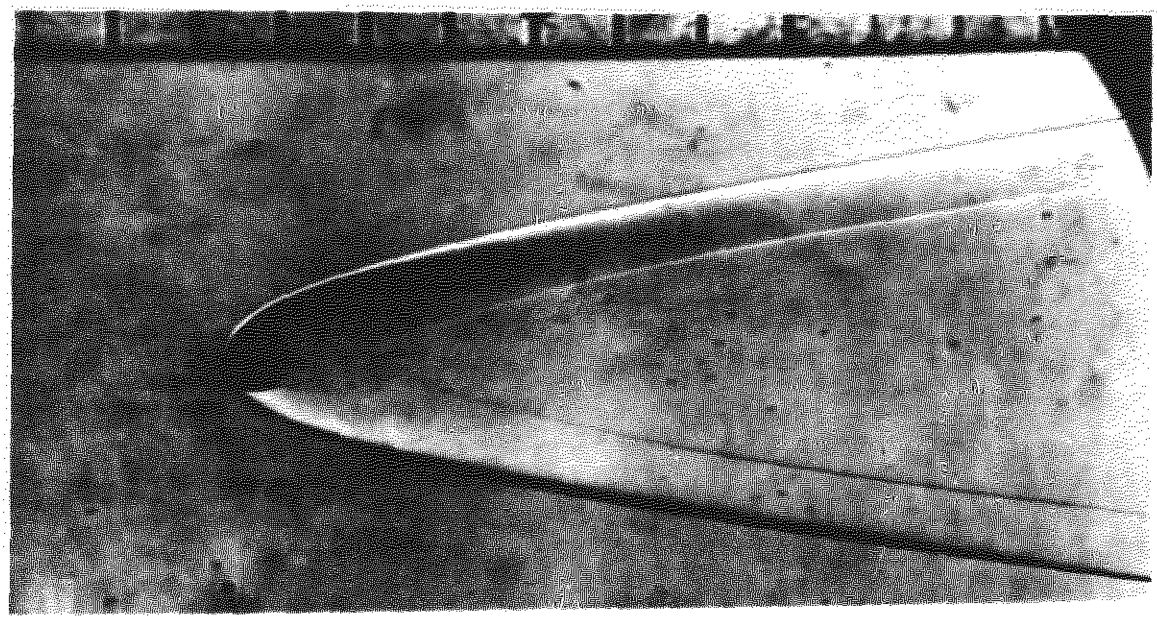
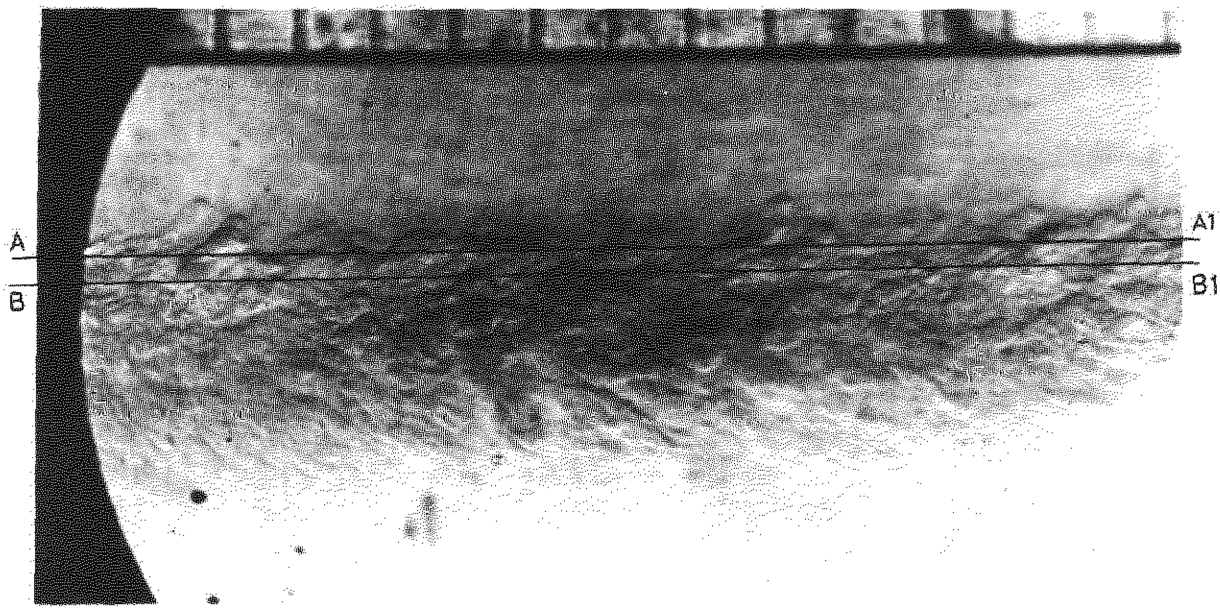


3/8-inch sphere V=14,000 ft/sec p=100mm Hg




Wake 8 feet behind 3/8-inch sphere p=100mm Hg

Fig.1. Typical schlieren photographs

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533.6.011.55 :
532.517 :
519.242 :
533.12 :
533.6.071.31

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