A Scheme of Notation and Nomenclature for Aircraft Dynamics and Associated Aerodynamics

By H. R. Hopkin

Aero. Dept., R.A.E., Farnborough

Part 5—Appendices

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PREFACE

For many years the notation and nomenclature used in the UK for aircraft dynamics has consisted of a basic scheme introduced by Bryant and Gates (R. & M. 1801, 1937) together with various additions and amendments due to Neumark, Mitchell, and others. Modifications were not co-ordinated and resulted in a complex mixture having at least two serious drawbacks. First, further rational extensions would be extremely difficult to make and probably confusing. Secondly, some parts of the scheme appeared to possess a pattern which in fact they did not possess, and this hidden ambiguity sometimes led to mistakes.

The present Report is the fifth in a series of five separate parts of R. & M. 3562 in which a unified system of notation and nomenclature is described. The system will present few difficulties to those familiar with the scheme of Bryant and Gates and its variants, and has the great advantage that it has a built-in potential for extension. At the same time, reliable patterns are incorporated and furthermore a great deal of freedom is available to an author who wishes in special cases to simplify the notation without ambiguity—for example, by omitting suffixes.

The new system is the outcome of many years of discussion at the Royal Aircraft Establishment, in co-operation with the National Physical Laboratory. It has been adopted by the Royal Aeronautical Society for its Data Items on Dynamics. Moreover, agreements reached by the International Organisation for Standardisation on terms and symbols used in flight dynamics have so far been completely consistent with the principles of the system.

The Aeronautical Research Council hopes that publication in the R. & M. Series will encourage the acceptance of the new notation and nomenclature and its use in the general field of aircraft dynamics by workers in research establishments and universities and in industry.
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Part 5—Appendices

Reports and Memoranda No. 3562
Part 5*, June, 1966

Summary.

A scheme of notation and nomenclature applicable to the dynamics and associated aerodynamics of both aeroplanes and missiles is proposed. The proposals are intended to supersede the attempts made, notably by Bryant and Gates, to revise and extend the existing standard reference in this field, namely R. & M. 1801.

Part 1 contains an extensive introduction describing the main objectives and summarising a considerable amount of historical background. It also lists the symbols, references, and most of the tables for the whole report, and provides an index. All illustrations are appended to Part 1, and copies included in the remaining parts where required.

Parts 2, 3, and 4 deal with basic notation and nomenclature, aircraft dynamics, and associated aerodynamic data respectively, and they can be read almost independently of Part 1. A great deal of reference information is included in the main text and in the twelve appendices which comprise Part 5.

Parts 2 to 5 do not contain separate lists of symbols, references or indexes.

LIST OF CONTENTS

Appendix.

A Key for Conversion from 'Old Notation' for Aeroplanes to the Notation of this Document

B Direction Cosines between Earth Axes and Body Axes in Terms of Attitude Angles

C Notation for Motion of the Atmosphere

D Relations between Attitude and Incidence Angles in a Wind Tunnel

E Relations between Higher Derivatives of Forces and Moments and Aerodynamic-Coefficient Derivatives

F Equations of Motion for Small Disturbances of an Aircraft with Four-Fold Symmetry from Straight Steady Spinning Flight

G Equations of Motion in Terms of Expansions with Respect to 'Displacement' Variables

H Subsidiary Notation for Primary Aerofoils, Flaps, and Tabs

J Miscellaneous Aerodynamic Coefficients

K Miscellaneous Derivatives of Forces and Moments
   K.1. Oscillatory derivatives
   K.2. Mean derivatives
   K.3. Shorthand expressions

L Partial Derivatives of Functions of the Velocity and its Components

M Matrix Notation for Vector Components, Kinematics, and Equations of Motion
   M.1. General
   M.2. Kinematics
   M.3. Some useful properties and identities
   M.4. Perturbations appropriate to various axes
   M.5. Kinematics in terms of perturbations
   M.6. Equations of motion for a system of particles
   M.7. Equations of motion for an aircraft
   M.8. Linearized equations of motion for small disturbances of a rigid object
   M.9. Equations of motion for a deformable object of constant mass
APPENDIX A

Key for Conversion from 'Old Notation' for Aeroplanes to the Notation of this Document.

The purpose of this key is two-fold: first to list corresponding symbols denoting the same physical quantity, and second to give the conversion factors required when numerical equivalence is sought. The value denoted by the old symbol is equal to the value denoted by the new symbol multiplied by the factor in the third column. For example, the second block in Section A.3 implies that \( x_u \) (old) = \( \frac{1}{2} X_w \) and so on. Owing to variations in old reports the conversion factors should not be accepted without checking the definitions of old symbols.

'Old notation' means not only that of R. & M. 1801 but also a number of additional symbols used in various later papers of the R.A.E.\(^{10,11,12,48,49,50,51}\) in the Aerodynamics Data Sheets of the R.Ae.S.\(^{52}\), and also in some leaflets of Av.P. 970, Vol. 253.

The symbol \( \delta \) represents any motivator deflection such as \( \xi, \eta, \zeta \), and \( \tau_s \) (or \( \tau \)) any tab deflection. Symbols in brackets are alternatives.

A.1. Variables.

<table>
<thead>
<tr>
<th>(1) Old symbol</th>
<th>(2) New symbol</th>
<th>Multiplying factor to obtain (1) from (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x ) ( y ) ( z )</td>
<td>( x ) ( y ) ( z )</td>
<td></td>
</tr>
<tr>
<td>( U ) ( V ) ( W ) (( u ) ( v ) ( w ))</td>
<td>( u ) ( v ) ( w )</td>
<td></td>
</tr>
<tr>
<td>( V )</td>
<td>( V )</td>
<td></td>
</tr>
<tr>
<td>( u ) ( v ) ( w )</td>
<td>( u' ) ( v' ) ( w' )</td>
<td>1</td>
</tr>
<tr>
<td>( P ) ( Q ) ( R ) (( p ) ( q ) ( r ))</td>
<td>( p ) ( q ) ( r )</td>
<td></td>
</tr>
<tr>
<td>( p ) ( q ) ( r )</td>
<td>( p' ) ( q' ) ( r' )</td>
<td></td>
</tr>
<tr>
<td>( \Phi ) ( \Theta ) ( \Psi ) (( \phi ) ( \theta ) ( \psi ))</td>
<td>( \Phi ) ( \Theta ) ( \Psi )</td>
<td></td>
</tr>
<tr>
<td>( \phi ) ( \theta ) ( \psi ) not the same as ( \phi ) ( \theta ) ( \psi )</td>
<td>( \Phi' ) ( \Theta' ) ( \Psi' )</td>
<td></td>
</tr>
<tr>
<td>( \alpha ) ( \beta )</td>
<td>( \alpha_t ) ( \beta_s )</td>
<td>1</td>
</tr>
<tr>
<td>( \xi ) ( \eta ) ( \zeta ) ( \beta )</td>
<td>( \xi ) ( \eta ) ( \zeta ) ( \tau_s(\tau) )</td>
<td>1</td>
</tr>
<tr>
<td>( X ) ( Y ) ( Z ) ( L ) ( R ) ( D )</td>
<td>( X ) ( Y ) ( Z ) ( L ) ( R(\Theta) ) ( D )</td>
<td>1</td>
</tr>
<tr>
<td>( L ) ( M ) ( N )</td>
<td>( L ) ( M ) ( N ) (( L ) ( M ) ( N ))</td>
<td>1</td>
</tr>
<tr>
<td>( H ) ( H_t(H_\tau) )</td>
<td>( B^x(B) ) ( B^\tau )</td>
<td></td>
</tr>
</tbody>
</table>
### A.2. Force and Moment Coefficients and their Derivatives.

<table>
<thead>
<tr>
<th>(1) Old symbol</th>
<th>(2) New symbol</th>
<th>Multiplying factor to obtain (1) from (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_L), (C_y), (C_D)</td>
<td>(C_{Lx}), (C_{Ya}), (C_D)</td>
<td>1</td>
</tr>
<tr>
<td>(C_i), (C_n)</td>
<td>(C_i), (C_n)</td>
<td>(l_2/b)</td>
</tr>
<tr>
<td>(C_m)</td>
<td>(C_m)</td>
<td>(l_1/l_T) or (l_1/\bar{c})</td>
</tr>
<tr>
<td>(a_1), (a_2), (a_3)</td>
<td>(C_{Lx}), (C_{L\delta}), (C_{Lx})</td>
<td>1</td>
</tr>
</tbody>
</table>

\[
C_H = \frac{H}{\frac{1}{2} \rho V^2 S \bar{c}} \left( \frac{H}{\frac{1}{2} \rho V^2 S \bar{c}} \right) \\
b_1\ b_2\ b_3
\]

The raised suffix \(\delta\) may be omitted.

\[
C_H' = \frac{B^6}{\frac{1}{2} \rho V^2 S \bar{c}} \left( \frac{H}{\frac{1}{2} \rho V^2 S \bar{c}} \right) \\
c_1\ c_2\ c_3
\]

\[
C_B = \frac{B^c}{\frac{1}{2} \rho V^2 S \bar{c}} \left( \frac{H}{\frac{1}{2} \rho V^2 S \bar{c}} \right) \\
c_1\ c_2\ c_3
\]

\[
C_B' = \frac{B^c}{\frac{1}{2} \rho V^2 S \bar{c}} \left( \frac{H}{\frac{1}{2} \rho V^2 S \bar{c}} \right) \\
c_1\ c_2\ c_3
\]

- Depends on definition of \(C_H\)  
- Same as \(C_H/C_B^6\)  
- Depends on definition of \(C_H'\)  
- Same as \(C_H'/C_B^c\)

<table>
<thead>
<tr>
<th>(1) Old symbol</th>
<th>(2) New symbol</th>
<th>Multiplying factor to obtain (1) from (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_w, Y_p, N_w, \text{ etc.}$</td>
<td>$X_u, Y_p, N_w, \text{ etc.}$</td>
<td>1</td>
</tr>
<tr>
<td>$x_w, y_w, z_w$</td>
<td>$\dot{x}_u, \dot{y}_p, \dot{z}_w$</td>
<td>$rac{1}{2}$</td>
</tr>
<tr>
<td>$x_q, z_q$</td>
<td>$\dot{x}_q, \dot{z}_q$</td>
<td>$l_4/2l_7$</td>
</tr>
<tr>
<td>$x_\omega, z_\omega$</td>
<td>$\ddot{x}<em>\omega, \ddot{z}</em>\omega$</td>
<td>$l_4/2l_7$</td>
</tr>
<tr>
<td>$m_n, m_w$</td>
<td>$\ddot{M}_n, \ddot{M}_w$</td>
<td>$l_2/b$</td>
</tr>
<tr>
<td>$y_p, y_r$</td>
<td>$\dot{y}_p, \dot{y}_r$</td>
<td></td>
</tr>
<tr>
<td>$y_\omega, n_\omega$</td>
<td>$\dot{y}<em>\omega, \dot{n}</em>\omega$</td>
<td>$l_4/2l_7$</td>
</tr>
<tr>
<td>$l_\xi, n_\xi$</td>
<td>$\dot{L}<em>\xi, \dot{N}</em>\xi$</td>
<td>$2l_3/b^2$</td>
</tr>
</tbody>
</table>
A.4. Quantities Occurring in 'Non-Dimensional' Equations of Motion.

<table>
<thead>
<tr>
<th>(1) Old symbol</th>
<th>(2) New symbol</th>
<th>Multiplying factor to obtain (1) from (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-x_u$</td>
<td>$\xi_u$</td>
<td></td>
</tr>
<tr>
<td>$-x_w$</td>
<td>$\xi_w$</td>
<td></td>
</tr>
<tr>
<td>$\bar{y}_v$</td>
<td>$\bar{y}_v$</td>
<td></td>
</tr>
<tr>
<td>$-z_u$</td>
<td>$\zeta_u$</td>
<td></td>
</tr>
<tr>
<td>$-z_w$</td>
<td>$\zeta_w$</td>
<td></td>
</tr>
<tr>
<td>$-z_q$</td>
<td>$\zeta_q$</td>
<td></td>
</tr>
<tr>
<td>$-\frac{z_q}{\mu_1}$</td>
<td>$\zeta_q$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$-\frac{y_p}{\mu_2}$</td>
<td>$\zeta_q$</td>
<td>$1$</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\kappa &= \frac{-\mu_1 m_u}{i_B} \quad m_u \\
\omega &= \frac{-\mu_1 m_w}{i_B} \quad m_w \\
\delta &= \frac{-\mu_1 m_q}{i_B} \quad m_q \\
\omega_t &= L = \frac{-\mu_2 l_v}{i_A} \quad \bar{l}_v \\
\omega_n &= N = \frac{\mu_2 l_n}{i_c} \quad \bar{n}_v = -\bar{n}_v \quad \frac{1}{2} \\
\delta_{n\epsilon} &= N = \frac{\mu_2 l_n}{i_c} \quad \bar{n}_v = \frac{1}{2} \\
\delta_{l\epsilon} &= L = \frac{\mu_2 l_q}{i_A} \quad \bar{l}_v \quad \frac{1}{2} \\
\delta_{n\epsilon} &= -N = \frac{-\mu_2 l_n}{i_c} \quad \bar{n}_v \quad \frac{1}{2} \\
\delta_{l\epsilon} &= -L = \frac{-\mu_2 l_q}{i_A} \quad \bar{l}_v \quad \frac{1}{2} \\
\nu &= \frac{-m_u}{i_B} \quad \bar{m}_q \\
\chi &= \frac{-m_w}{i_B} \quad \bar{m}_w \\
v_1 &= l_1 = \frac{-l_p}{i_A} \quad \bar{l}_p \\
v_n &= n_2 = \frac{-n_r}{i_c} \quad \bar{n}_r \\
v_{lr} &= l_2 = \frac{l_p}{i_A} \quad \bar{p} = -\bar{l}_p \\
v_{np} &= n_1 = \frac{-n_p}{i_c} \quad \bar{n}_p
\end{align*}
\]
A.4. Quantities Occurring in ‘Non-Dimensional’ Equations of Motion—continued

<table>
<thead>
<tr>
<th>(1) Old symbol</th>
<th>(2) New symbol</th>
<th>Multiplying factor to obtain (1) from (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\ddot{u}$ $\ddot{\theta}$ $\ddot{\psi}$</td>
<td>$\ddot{u}$ $\ddot{\theta}$ $\ddot{\psi}$</td>
<td>1</td>
</tr>
<tr>
<td>$\ddot{\rho}$ $\ddot{\eta}$ $\ddot{\varphi}$</td>
<td>$\ddot{\rho}$ $\ddot{\eta}$ $\ddot{\varphi}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$i = m/\rho SV_e$</td>
<td>$\tau$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$\tau = t/\dot{i}$</td>
<td>$\dot{i} = t/\tau$</td>
<td>2</td>
</tr>
</tbody>
</table>

| $\mu_1 = \frac{m}{\rho S I_T}$ | $\mu_1$ | $\frac{l_1}{2l_T}$ |
| $\mu_2 = \frac{2m}{\rho S b}$ | $\mu_2$ | $\frac{l_2}{b}$ |
| $\mu_3 = \frac{m}{\rho S I_F}$ | $I_x I_y I_z$ | $I_{yx} I_{zx} I_{xy}$ | 1 |

| $A$ | $B$ | $C$ | $e_A = -e_A = -\frac{E_A}{A}$ | $e_x$ |
| $D$ | $E$ | $F$ | $e_C = -e_C = -\frac{E_C}{C}$ | $e_z$ |

| $i_A = \frac{4A}{mb^2}$ | $i_x$ | $\frac{4l_2^2}{b^2}$ |
| $i_B = \frac{B}{ml^2}$ | $i_y$ | $\frac{l_1}{l_2}$ |
| $i_C = \frac{4C}{mb^2}$ | $i_z$ | $\frac{4l_2^2}{b^2}$ |
A.5. Stability quantities.

<table>
<thead>
<tr>
<th>(1) Old symbol</th>
<th>(2) New symbol</th>
<th>Multiplying factor to obtain (1) from (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$K_n$</td>
<td>1</td>
</tr>
<tr>
<td>$B_1$</td>
<td>$J_n$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$C_1$</td>
<td>$K_{n-1}$</td>
<td>$\frac{1}{4}$</td>
</tr>
<tr>
<td>$D_1$</td>
<td>$J_{n-1}$</td>
<td>$\frac{1}{8}$</td>
</tr>
<tr>
<td>$E_1$</td>
<td>etc.</td>
<td>$\frac{1}{16}$</td>
</tr>
<tr>
<td>$\lambda = -R + iJ$</td>
<td>$\lambda = -\hat{k} + i\hat{\nu}$</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>$(-r + ij)$</td>
<td>$(-\hat{r} + i\hat{\nu})$</td>
<td></td>
</tr>
<tr>
<td>$(-R + iS)$</td>
<td>$(-\hat{r} + i\hat{\nu})$</td>
<td></td>
</tr>
<tr>
<td>$(-r + is)$</td>
<td>$(-\hat{r} + i\hat{\nu})$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$h$</th>
<th>$h_n$</th>
<th>$h_m$</th>
<th>$h_G$</th>
<th>$h_n$</th>
<th>$h_m$</th>
<th>$h_G$</th>
<th>$h_n$</th>
<th>$h_m$</th>
<th>$l_1/\tilde{c}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$\bar{H}_G$</td>
<td>$\bar{H}_n$</td>
<td>$\bar{H}_m$</td>
<td>$\bar{H}_G$</td>
<td>$\bar{H}_n$</td>
<td>$\bar{H}_m$</td>
<td></td>
</tr>
<tr>
<td>$K_n$</td>
<td>$K_n'$</td>
<td>$k_s$</td>
<td>$k_{sf}$</td>
<td>$k_s$</td>
<td>$k_{sf}$</td>
<td>$k_s$</td>
<td>$k_{sf}$</td>
<td>$l_1/\tilde{c}$</td>
<td>on certain assumptions (see Section 14)</td>
</tr>
<tr>
<td>$H$</td>
<td>$H_m'$</td>
<td>$k_m$</td>
<td>$k_{mf}$</td>
<td>$k_m$</td>
<td>$k_{mf}$</td>
<td>$k_m$</td>
<td>$k_{mf}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Multiplying factor to obtain (1) from (2)
APPENDIX B

Direction Cosines between Earth Axes and Body Axes in Terms of Attitude Angles.

(see Section 5.4, 5.6, M.2)

The direction cosines $l_i, m_i, n_i$ ($i = 1, 2, 3$) are defined in Section 5.6. The following expressions for them are in terms of attitude angles defined in Sections 5.2 and 5.3. The three Tables relate in turn to attitude angles for datum planes $O_x y_0 z_0, O_y z_0 x_0, O_z x_0 y_0.$

<table>
<thead>
<tr>
<th>$l_1$</th>
<th>$m_1$</th>
<th>$n_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos \Theta \cos \Psi$</td>
<td>$\sin \Phi \sin \Theta \cos \Psi$</td>
<td>$\cos \Phi \sin \Psi_y$</td>
</tr>
<tr>
<td>$-\cos \Phi \sin \Psi$</td>
<td>$\cos \Psi_y$</td>
<td>$-\sin \Phi \sin \Theta \sin \Psi_y$</td>
</tr>
<tr>
<td>$\cos \Phi \sin \Theta \cos \Psi$</td>
<td>$\sin \Phi \sin \Psi_y$</td>
<td>$\cos \Psi z \sin \Theta z \cos \Psi z$</td>
</tr>
<tr>
<td>$+ \sin \Phi \sin \Psi$</td>
<td>$+ \cos \Phi \sin \Theta \sin \Psi_y$</td>
<td>$-\sin \Phi \sin \Psi z$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$l_2$</th>
<th>$m_2$</th>
<th>$n_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cos \Theta \sin \Psi$</td>
<td>$\sin \Phi \sin \Theta \sin \Psi$</td>
<td>$\cos \Phi \sin \Psi_y$</td>
</tr>
<tr>
<td>$+ \cos \Phi \cos \Psi$</td>
<td>$\cos \Theta \cos \Psi_y$</td>
<td>$+ \sin \Phi \sin \Theta \sin \Psi_y$</td>
</tr>
<tr>
<td>$\cos \Phi \sin \Theta \sin \Psi$</td>
<td>$\sin \Phi \sin \Psi_y$</td>
<td>$-\sin \Phi \sin \Theta \cos \Psi_y$</td>
</tr>
<tr>
<td>$- \sin \Phi \cos \Psi$</td>
<td>$- \cos \Phi \sin \Theta \cos \Psi_y$</td>
<td>$+ \cos \Phi \sin \Psi z$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$l_3$</th>
<th>$m_3$</th>
<th>$n_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$- \sin \Theta$</td>
<td>$\sin \Phi \cos \Theta$</td>
<td>$\cos \Phi \cos \Theta z$</td>
</tr>
<tr>
<td>$\sin \Theta_y$</td>
<td>$\sin \Theta$</td>
<td>$\sin \Phi \cos \Theta z$</td>
</tr>
<tr>
<td>$\cos \Phi \cos \Theta$</td>
<td>$\sin \Theta_y$</td>
<td>$\sin \Theta z$</td>
</tr>
</tbody>
</table>
### Appendix B

<table>
<thead>
<tr>
<th>$l_1$</th>
<th>$-\sin \Pi_{xx}$</th>
<th>$\cos \Phi_{yx} \cos \Pi_{yx}$</th>
<th>$\cos \Phi_{zx} \cos \Pi_{zx}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>$\sin \Phi_{xx} \cos \Pi_{xx}$</td>
<td>$-\sin \Pi_{yx}$</td>
<td>$-\sin \Phi_{zx} \cos \Pi_{zx}$</td>
</tr>
<tr>
<td>$n_1$</td>
<td>$\cos \Phi_{xx} \cos \Pi_{xx}$</td>
<td>$\sin \Phi_{yx} \cos \Pi_{yx}$</td>
<td>$\sin \Pi_{zx}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$l_2$</th>
<th>$\cos \Pi_{xx} \sin \Theta_{xx}$</th>
<th>$\cos \Phi_{yx} \sin \Pi_{yx} \cos \Theta_{yx} + \sin \Phi_{yx} \sin \Theta_{yx}$</th>
<th>$\cos \Phi_{zx} \sin \Pi_{zx} \sin \Theta_{zx} + \sin \Phi_{zx} \cos \Theta_{zx}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_2$</td>
<td>$\sin \Phi_{xx} \sin \Pi_{xx} \sin \Theta_{xx} + \cos \Phi_{xx} \cos \Theta_{xx}$</td>
<td>$\cos \Pi_{yx} \cos \Theta_{yx}$</td>
<td>$-\sin \Phi_{zx} \sin \Pi_{zx} \sin \Theta_{zx} + \cos \Phi_{zx} \cos \Theta_{zx}$</td>
</tr>
<tr>
<td>$n_2$</td>
<td>$\cos \Phi_{xx} \sin \Pi_{xx} \sin \Theta_{xx} - \sin \Phi_{xx} \cos \Theta_{xx}$</td>
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### Appendix B

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APPENDIX C

Notation for Motion of the Atmosphere.

(see Sections 6, 7, 10, M.5)

When the air is not at rest relative to the earth care must be taken in interpreting quantities represented by \( V, u, v, \ldots, x, \alpha, \beta, \ldots, X, x, \ldots \). In this Report the symbols \( V, u, v, w \) always denote speeds relative to the air, and it is proposed that the corresponding speeds relative to the earth by denoted by \( V^k, u^k, v^k, w^k \). Thus, if the velocity of the air relative to the earth is represented by \( \mathbf{V}^w \) (with components \( u^w, v^w, w^w \)), we have the equations

\[
\mathbf{V}^k = \mathbf{V} + \mathbf{V}^w, \quad (C.1)
\]

\[
(u^k, v^k, w^k) = (u + u^w, v + v^w, w + w^w), \quad (C.2)
\]

and likewise for components along earth axes:

\[
(u_0^k, v_0^k, w_0^k) = (u_0 + u_0^w, v_0 + v_0^w, w_0 + w_0^w). \quad (C.3)
\]

The last equation can also be expressed in terms of direction angles (see Section 6), and we obtain

\[
\begin{align*}
\frac{u^k}{V^k} &= \frac{1}{V^k} \cos \gamma \cos \chi = \frac{1}{V^k} \cos \gamma a \cos \chi a + \frac{V^w}{V^k} \cos \gamma w \cos \chi w, \\
\frac{v^k}{V^k} &= \frac{1}{V^k} \cos \gamma \sin \chi = \frac{1}{V^k} \cos \gamma a \sin \chi a + \frac{V^w}{V^k} \cos \gamma w \sin \chi w, \\
\frac{w^k}{V^k} &= \frac{-1}{V^k} \sin \gamma = -\frac{1}{V^k} \sin \gamma a - \frac{V^w}{V^k} \sin \gamma w.
\end{align*}
\]

The choice of *capital letters* \( K \) and \( W \) gives the possibility of using them as raised or lowered suffixes. It is suggested that handwritten work should employ raised suffixes, since these capital letters are often misinterpreted as lower case, and lowered suffixes \( k \) and \( w \) are used for other purposes. Occasionally a raised suffix \( k \) is required (see Section M.2). The alternative to \( V_0^k \) would be \( V_{k0} \) rather than \( V_{0k} \).
Appendix C

since components of any vector along normal earth axes and body axes are related as always through the direction cosine matrix $S$. Thus $\{u_0^b\} = S^T \{u_b^k\}$, and $\{u^k\} = S \{u_0^b\}$. When $\Phi_e, \Theta_e, \Psi_e$ are zero and only small deviations $\phi, \theta, \psi$ in attitude are considered, we have $S \approx I - A_\phi, S^T \approx I + A_\phi$, and hence

$$\{u_0^b\} = (I + A_\phi) \{u_b^k\} = \{u_0\} + \{u_0^w\}$$

$$\{u^k\} = (I - A_\phi) \{u_0^b\}.$$

The expanded forms of these relations are

$$u_0^k = u^k - v^k \psi + w^k \theta = u_0 + u_0^w$$
$$v_0^k = v^k + u^k \psi - w^k \phi = v_0 + v_0^w$$
$$w_0^k = w^k - u^k \theta + v^k \phi = w_0 + w_0^w$$

If we also assume that $\gamma, \chi, \eta, \chi_a$ are small, then to first order accuracy equations (C.4) become

$$u_0^k = V^k = V_0 + u_0^w$$
$$v_0^k = V^k \chi_a = V \chi_a + v_0^w$$
$$w_0^k = -V^k \gamma = -V \gamma_a + w_0^w.$$

This implies small $\alpha, \beta$ and from equations (7.2):

$$u \approx V,$$
$$v \approx V \beta \approx V (\chi_a - \psi)$$
$$w \approx V \alpha \approx V (\theta - \gamma_a).$$

Hence

$$V^k \chi_a = (V^k - u_0^w)(\psi + \beta) + v_0^w,$$
$$V^k \gamma = (V^k - u_0^w)(\theta - \alpha) - w_0^w.$$
i.e.

\[
\chi = \left[ 1 - \frac{u_0^W}{v^K} \right] (\psi + \beta) + \frac{v_0^W}{v^K},
\]

\[
\gamma = \left[ 1 - \frac{u_0^W}{v^K} \right] (\theta - \alpha) - \frac{w_0^W}{v^K}.
\]

When the air is at rest, these reveal the well known approximations \( \chi = \psi + \beta, \gamma = \theta - \alpha \). Using the expressions for \( \chi \) and \( \gamma \), we can rewrite the equations for \( u^K, v^K, w^K \) in the form

\[
\frac{u^K}{v^K} = 1 + \chi \psi + \gamma \theta \approx 1,
\]

\[
\frac{v^K}{v^K} = \chi - \psi - \gamma \phi \approx \chi - \psi = \beta - \frac{u_0^W}{v^K} (\psi + \beta) + \frac{v_0^W}{v^K},
\]

\[
\frac{w^K}{v^K} = -\gamma + \theta - \chi \phi \approx \theta - \gamma = \alpha + \frac{u_0^W}{v^K} (\theta - \alpha) + \frac{w_0^W}{v^K}.
\]

Turning now to expressions that bring in the increments in velocity components, we consider for example the equivalent of equations (7.5). We obtain the flight-path velocity components \( \{u_0^K\} \) in terms of the airspeed scalar increments \( \{u'\} \) merely by substituting for \( \{u_0^-\} \), given by (7.5), into the equation

\[
\{u_0^K\} = \{u_{de}\} + \{u_0^+\} + \{u_0^W\}.
\]

Variations in wind velocity may sometimes be represented as increments, and then a full incremental notation can be applied:

\[
\{u_0^{+K}\} = \{u_0^+\} + \{u_0^W\}.
\]

These symbols denote the vector increments of the components of the flight-path velocity, and of the airspeed and windspeed vectors along earth axes.

The terms ground speed and ground windspeed may be used for the horizontal components of the flight-path velocity and wind velocity respectively.

Since \( x, y, z \) are defined relative to the earth, equations such as (7.7) are generally valid provided we replace \( \{u_0^+\} \) by \( \{u_0^{+K}\} \). Similar modifications would be required in the kinematic relations given in Appendix M.

In equations of motion such as (10.1) the kinematic terms representing acceleration must be written in terms of \( u^K, v^K \) etc. instead of \( u, v \) etc. The left-hand sides of (10.1), for example, are replaced by \( m(u^K + qw^K - rv^K) \), etc. The rotary equations of motion are unchanged. The consequent modifications to the forms of the equations of motion given elsewhere in Part 3 and in Appendices F, G, M are not set out here, for the procedure is straightforward.
APPENDIX D

Relations Between Attitude and Incidence Angles in a Wind Tunnel.
(see Sections 5, 6.2, 7, 4, M)

The general relations between the attitude and incidence of the aircraft and the direction of the air path
(i.e. the reverse of the relative wind direction) are given by equations (7.2). The attitude angles \( \Phi, \Theta, \Psi \), and
the direction angles \( \Theta_w, \Psi_w \) are by definition with respect to normal earth axes, and they may not be the
most convenient for a tunnel whose airstream is not intended to be horizontal. If we define a datum
attitude \( \Phi_d, \Theta_d, \Psi_d \) in which the body x-axis lies along the tunnel axis, and the other axes are in any
convenient directions, we can introduce the attitude-deviation angles \( \phi, \theta, \psi \) (see Section 5.6) and analogous
direction-deviation angles \( \theta_a, \psi_a \) for the airstream. The latter will normally be small and are zero if
the airstream is exactly parallel to the tunnel axis.

The expressions for \( \cos \sigma, \sin \beta, \sin \alpha \) in terms of \( \phi, \theta, \psi, \theta_a, \psi_a \) have the same form as equations (7.2),
where the capital letters are merely replaced by lower case letters:

\[
\begin{align*}
\cos \sigma &= \cos \theta \cos \theta_a \cos (\psi - \psi_a) + \sin \theta \sin \theta_a, \\
\sin \lambda \sin \sigma &= \sin \beta_a = \sin \phi \left[ \sin \theta \cos \theta_a \cos (\psi - \psi_a) - \cos \theta \sin \theta_a \right] \\
&\quad - \cos \phi \cos \theta_a \sin (\psi - \psi_a), \\
\cos \lambda \sin \sigma &= \sin \alpha_a = \cos \phi \left[ \sin \theta \cos \theta_a \cos (\psi - \psi_a) - \cos \theta \sin \theta_a \right] \\
&\quad + \sin \phi \cos \theta_a \sin (\psi - \psi_a).
\end{align*}
\]

(D.1)

From the last two equations \( \tan \lambda \) is obtained by division, and the other incidence angles \( \alpha, \beta \) are defined
by

\[
\tan \alpha = \frac{\sin \alpha}{\cos \sigma}, \quad \tan \beta = \frac{\sin \beta}{\cos \sigma}.
\]

From the last two equations (D.1) we also derive

\[
\begin{align*}
\sin (\phi - \lambda) \sin \sigma &= \cos \theta \sin (\psi - \psi_a), \\
\cos (\phi - \lambda) \sin \sigma &= \sin \theta \cos \theta_a \cos (\psi - \psi_a) - \cos \theta \sin \theta_a,
\end{align*}
\]

(D.2)

and hence expressions for \( \sin^2 \sigma \) and \( \tan (\phi - \lambda) \).

Various approximations may be used in practice. When \( \theta_a, \psi_a \) are small, for example,

\[
\begin{bmatrix}
\cos \sigma \\
\sin \beta_a \\
\sin \alpha_a
\end{bmatrix} = S_\phi \begin{bmatrix}
1 \\
\psi_a \\
-\theta_a
\end{bmatrix},
\]

where \( S_\phi \) is the direction cosine matrix in terms of \( \phi, \theta, \psi \) (see Section 5.6). This matrix can be replaced
by any equivalent one such as \( S_{\phi_x} \) or \( S_{\psi_x} \), whose elements are expressed in terms of \( \phi_x, \theta_x, \psi_x \), or \( \phi_{xy}, \theta_{xy}, \psi_{xy} \) which denote attitude-deviation angles analogous to \( \Phi_x \), etc., just as \( \phi, \theta, \psi \) are analogous to
\( \Phi, \Theta, \Psi \) (see Sections 5.3 and M.1, also Appendix B).
Appendix D

In a wind tunnel a model is often put at an incidence by rotating it about two axes in succession. This leads to an attitude at which (in an idealised experiment) one of the angles \( \phi, \theta, \psi \) is zero—usually \( \phi \) or \( \psi \). In a real experiment the mounting of the model may deflect when the airstream is applied and the aerodynamic forces are acting, and we are therefore interested in approximations corresponding to a small \( \phi, \theta, \) or \( \psi \). These are given below. Each of the two attitude angles that are not necessarily small is most conveniently expressed as the sum of the normal value and a perturbation due to deflection. For example, if \( \theta \) is not small we write \( \theta = \theta_e + \theta' \), where \( \theta' \) is small and \( \theta_e \) is the angle through which the model is rotated by the mounting mechanism and equal to the value of \( \theta \) when the airstream is off.

Because the incidence magnitude \( \sigma \) is always positive, care must be taken in applying trigonometrical identities. The value of \( \lambda \) must be such that the incidence can be made zero by a rotation \( -\lambda \) about the \( x \)-axis followed by a rotation \( -\sigma \) about the (carried) \( y \)-axis, where \( \sigma \leq \pi \) and \( -\pi \leq \lambda \leq \pi \).

(a) \( \theta_w, \psi_w, \phi \) small \((\psi, \theta \text{ rig})\)

\[
\sin \beta_s \approx \phi \sin \theta \cos \psi - \sin (\psi - \psi_a)
\]

\[= -\sin (\psi_e + \psi'(1) - \psi_a - \phi \sin \theta_e)\]

i.e.

\[
\beta_s \approx -(\psi_e + \psi'(1) - \psi_a - \phi \sin \theta_e)
\]

\[
\sin \alpha_s \approx \sin \theta \cos (\psi - \psi_a) - \theta_a \cos \theta + \phi \sin \psi
\]

\[= \sin \theta_a \cos \psi_e + (\psi'(1) - \theta_a \sin \psi)(\psi_a \sin \theta_e + \phi \sin \psi_e)
\]

\[
\sin (\phi - \psi_a \sin \sigma \approx \sin (\psi_e + \psi'(1) - \psi_a)
\]

\[
\cos (\phi - \psi_a \sin \sigma \approx \sin \theta_a \cos \psi_e + (\psi'(1) - \theta_a \sin \psi)(\psi_a \cos \theta_e + \phi \cos \psi_e)
\]

It should be noted that extreme attitudes such as \( \theta_e = \frac{\pi}{2} \) or \( \psi_e = \frac{\pi}{2} \) imply difficulties in defining \( \alpha_s \) or \( \beta_s \), but for conditions other than these we may also write

\[
\cos \beta_s \sin (\alpha_s - \theta) \approx \phi \cos \theta_e \sin (\psi - \psi_a) - \theta_a,
\]

i.e.

\[
\sin (\alpha_s - \theta) \approx \frac{\phi \cos \theta_e \sin (\psi - \psi_a) - \theta_a}{\cos (\psi - \psi_a) + \phi \sin \theta_e \sin (\psi - \psi_a)}
\]

when \( \theta_a, \psi_a, \phi \) are small.

(b) \( \theta_w, \psi_w, \psi \) small \((\theta, \phi \text{ rig})\)

\[
\sin \beta_s \approx \sin \phi_e \sin \theta_e + (\theta_a - \theta') \sin \phi_e \cos \theta_e + (\psi_e - \psi) \cos \phi_e
\]

\[
\sin \alpha_s \approx \cos \phi_e \sin \theta_e + (\theta_a - \theta') \cos \phi_e \cos \theta_e + (\psi - \psi_a) \sin \phi_e
\]

\[
\sin (\phi_e + \phi' - \lambda) \sin \sigma \approx \psi - \psi_a
\]

\[
\cos (\phi_e + \phi' - \lambda) \sin \sigma \approx \sin (\theta_e + \theta' - \theta')
\]
(c) $\theta_a, \psi_a, \theta$ small ($\psi, \phi$ rig)

$$\sin \beta_s \approx -\cos \phi \sin (\psi - \psi_a) + (\theta \cos \psi_a - \theta_a) \sin \phi_e$$
$$\sin \alpha_s \approx \sin \phi \sin (\psi - \psi_a) + (\theta \cos \psi_a - \theta_a) \cos \phi_e$$
$$\sin (\phi_e + \phi' - \lambda) \sin \sigma \approx \sin (\psi_e + \psi' - \psi_a)$$
$$\cos (\phi_e + \phi' - \lambda) \sin \sigma \approx \theta \cos \psi_e - \theta_a$$

In many applications the incidence magnitude is small, and hence also $\alpha_s, \beta_s, \alpha_t, \beta_t$, and we then have for small $\theta_a, \psi_a, \theta, \phi$:

$$\beta_s \approx \beta_t \approx - (\psi - \psi_a) \cos \phi + (\theta - \theta_a) \sin \phi,$$
$$\alpha_s \approx \alpha_t \approx (\psi - \psi_a) \sin \phi + (\theta - \theta_a) \cos \phi,$$
$$\sigma^2 \approx \alpha_s^2 + \beta_s^2 \approx (\psi - \psi_a)^2 + (\theta - \theta_a)^2,$$
$$\tan (\phi - \lambda) \approx \frac{\psi - \psi_a}{\theta - \theta_a}.$$

It is interesting to contrast the approximate formulae above with the exact ones for the idealised conditions that $\theta_a$ and $\psi_a$ are zero together with one of $\phi, \theta, \psi$.

(a) $\theta_a = \psi_a = \phi = 0$

$$\cos \sigma = \cos \theta \cos \psi$$
$$\sin \beta_s = -\sin \psi \quad \text{(e.g. $\beta_s = -\psi$)}$$
$$\sin \alpha_s = \sin \theta \cos \psi$$
$$\sin \alpha_t \cos \beta_s = \pm \sin \phi \quad \text{(e.g. $\alpha_t = \theta$ if $\beta_s = -\psi$)}$$
$$\tan \lambda = -\frac{\tan \psi}{\sin \theta}.$$

(b) $\theta_a = \psi_a = \psi = 0$

$$\cos \sigma = \cos \theta \quad \text{(e.g. $\sigma = |\theta|$)}$$
$$\sin \beta_s = \sin \phi \sin \theta$$
$$\sin \alpha_s = \cos \phi \sin \theta$$
$$\tan \lambda \approx \tan \phi \quad \text{(e.g. $\lambda = \phi$ for $\theta$ positive,}$$
$$\lambda = \phi + \pi \text{ for $\theta$ negative).}$$
Appendix D

\( \theta_\alpha = \psi_\alpha = \theta = 0 \)

\[
\begin{align*}
\cos \sigma &= \cos \psi \\
\sin \beta_\alpha &= -\cos \phi \sin \psi \\
\sin \alpha_\beta &= \sin \phi \sin \psi \\
\tan \lambda &= -\cot \phi \\
\end{align*}
\]

(e.g. \( \lambda = \phi - \frac{1}{2} \pi \) for \( \psi \) positive
\( \lambda = \phi + \frac{1}{2} \pi \) for \( \psi \) negative).

The incidence angles are usually visualised in terms of sequences of rotations. If the aircraft is initially at a particular attitude and zero incidence and it is given sequences of rotations corresponding to the matrices \( P_{\alpha \beta}, Y_{-\beta}, R_\alpha P_\phi \), or \( Y_{-\beta} P_{\alpha \beta} \), the resulting attitudes though different will produce the same incidence. Initial 'redundant' rotations about the x-axis could therefore bring the aircraft to modified starting attitudes (still zero incidence) such that the final attitude as well as the incidence is the same. The incidence matrix can thus be written

\[
S_\alpha = P_{\alpha \beta} Y_{-\beta} = R_\phi P_\phi R_{\rho_1} = Y_{-\beta} P_{\alpha \beta} R_{\rho_2},
\]

where \( \rho_1, \rho_2 \) represent redundant rotations. It is found that

\[
\begin{align*}
\sin \rho_1 &= -\frac{\tan \beta_\alpha}{\tan \sigma} = -\frac{\cos \alpha_\beta \sin \beta_\alpha}{\sin \sigma}, \\
\cos \rho_1 &= \frac{\cos \lambda}{\cos \beta_\alpha} = \frac{\sin \alpha_\beta}{\sin \sigma}, \\
\tan \rho_1 &= -\tan \lambda \cos \sigma = -\frac{\sin \beta_\alpha}{\tan \alpha_\beta} = -\frac{uv}{wV}, \\
\sin \rho_2 &= \sin \alpha_\beta \sin \beta_\alpha, \\
\cos \rho_2 &= \frac{\cos \alpha_\beta}{\cos \beta_\alpha}, \\
\tan \rho_2 &= \tan \beta_\alpha \sin \alpha_\beta = \tan \alpha_\beta \sin \beta_\alpha = \frac{vw}{wV}.
\end{align*}
\]

Each of the matrices \( P_{\alpha \beta}, Y_{-\beta}, R_\phi P_\phi, Y_{-\beta} P_{\alpha \beta} \) is the transformation matrix for converting the body-axes components of any vector into the components for a system of axes \( O_{x_\alpha y_\alpha z_\alpha} \) having the \( x_\alpha \)-axis parallel to the relative airstream (air-path axes), but the orientation of the \( y_\alpha \) and \( z_\alpha \)-axes is different in each case. The first matrix is used when the \( z_\alpha \)-axis lies in the \( zx \)-plane, the second when the \( y_\alpha \)-axis lies in the \( yz \)-plane, and the third when the \( y_\alpha \)-axis lies in the \( yx \)-plane. The first is therefore the most useful, since the \( O_{x_\alpha y_\alpha z_\alpha} \) system is then aligned with the drag, cross-stream force, and lift axes (see Section 4.3(b)). To illustrate the second and third matrices we consider a horizontal wind-tunnel with the airstream exactly along the longitudinal axis (\( \theta_\alpha \) and \( \psi_\alpha \) zero). We also assume that the model could be brought to a level attitude by rotations through \( -\lambda, -\sigma \) in one case and \( \beta_\alpha, -\alpha_\beta \) in the other. It follows that the matrix \( R_\phi P_\phi \) in the first case, and \( Y_{-\beta} P_{\alpha \beta} \) in the second, will convert body-axes components into components along the tunnel longitudinal axis, and vertical axis. This is not of course generally true.

The full expression for the matrices \( R_\phi P_\phi \) and \( Y_{-\beta} P_{\alpha \beta} \) can be written down immediately by reference to the first section of Appendix M, and \( P_{\alpha \beta} Y_{-\beta} \) is displayed explicitly towards the end of the same section. The first column in each of these three matrices is equal to \( \{ \cos \sigma, \sin \beta_\alpha, \sin \alpha_\beta \} \), since the \( x_\alpha \)-axis is always along the relative airstream.
APPENDIX E

Relations between Higher Derivatives of Forces and Moments and Aerodynamic-Coefficient Derivatives.

(see Sections 10.2.1, 19)

As in Section 19 we define a \( \gamma \) set of variables appropriate for aerodynamic coefficients: \( M, R, \Theta, \alpha, \beta, q, \eta, V, \beta, \gamma \), etc. (where \( \alpha, \beta \) stand for \( \alpha_e, \beta_e \)); and an \( \omega \) set appropriate for the forces and moments: \( h, \Theta, u, v, q, \eta, \dot{u}, \dot{v}, \dot{q}, \dot{\eta}, \ddot{u}, \ddot{v}, \ddot{q}, \ddot{\eta} \), etc. If \( \omega_1 \) and \( \omega_2 \) are any two of the second set (\( \omega_1 = \omega_2 \) is permissible), and a first-order derivative \( \partial Z / \partial \omega_1 \) has been determined as a function of \( V \), the incidence magnitude \( \sigma \), the \( \gamma \) variables, \( C_Z \) and its derivatives, then the second-order derivative \( \partial^2 Z / \partial \omega_2 \partial \omega_1 \) can be obtained from the relation

\[
\frac{\partial^2 Z}{\partial \omega_2 \partial \omega_1} = \frac{\partial Z_{\omega_1}}{\partial \omega_2}
\]

\[
= \frac{\partial Z_{\omega_1}}{\partial V} \frac{\partial V}{\partial \omega_2} + \frac{\partial Z_{\omega_1}}{\partial \sigma} \frac{\partial \sigma}{\partial \omega_2} + \sum \frac{\partial Z_{\omega_1}}{\partial \gamma} \frac{\partial \gamma}{\partial \omega_2}.
\]

(E.1)

Useful expressions for many \( \partial \gamma / \partial \omega \) terms are given in Appendix L.

As examples consider derivatives at a datum condition in which all the \( \gamma \) variables other than \( M, R, \alpha, \beta \) are zero. We then have

\[
Z_u = \frac{1}{2} \rho VS \cos \sigma \left[ 2C_Z + M \frac{\partial C_Z}{\partial M} + R \frac{\partial C_Z}{\partial R} \right] - \frac{1}{2} \rho VS \cos \sigma \left( C_{Za} \tan \alpha + C_{Z\beta} \tan \beta \right)
\]

\[
Z_v = \frac{1}{2} \rho VS \sin \beta \left[ 2C_Z + M \frac{\partial C_Z}{\partial M} + R \frac{\partial C_Z}{\partial R} \right] + \frac{1}{2} \rho VS \left( C_{Z\beta} \cos \beta - C_{Za} \tan \alpha \sin \beta \right),
\]

\[
Z_w = \frac{1}{2} \rho VS \sin \alpha \left[ 2C_Z + M \frac{\partial C_Z}{\partial M} + R \frac{\partial C_Z}{\partial R} \right] + \frac{1}{2} \rho VS \left( C_{Za} \cos \alpha - C_{Z\beta} \tan \beta \sin \alpha \right),
\]

\[
Z_q = \frac{1}{2} \rho VS \hat{C}_4 \quad \text{etc.}
\]

It is important at this stage to omit the suffix \( e \) and to begin with expressions for \( Z_u, \) etc. rather than \( Z_{uv} \) etc.

By applying (E.1) we obtain, for example,

\[
\dot{Z}_{uv} = \frac{Z_{uv}}{\frac{1}{2} \rho e S}
\]

\[
= C_{Z\alpha a} \left( \cos \alpha \cos \beta + \sin \alpha \tan \alpha \sin \beta \tan \beta \right) +
\]

\[
+ \sec \alpha \cos 2\alpha \sin \beta \left[ C_{Za} + M \frac{\partial C_{Za}}{\partial M} + R \frac{\partial C_{Za}}{\partial R} \right] +
\]

19
Appendix E

\[ + \sec \beta \cos 2 \beta \sin \alpha \left[ C_{z\beta} + M \frac{\partial C_{z\beta}}{\partial M} + R \frac{\partial C_{z\beta}}{\partial R} \right] + \]

\[ + \sin \alpha \sin \beta \left[ 3 \left( M \frac{\partial C_z}{\partial M} + R \frac{\partial C_z}{\partial R} \right) + \left( M \frac{\partial}{\partial M} + R \frac{\partial}{\partial R} \right)^2 C_z \right] - \]

\[- \sin \alpha \sin \beta (C_{zxx} + C_{z\beta \beta}), \]

where all quantities on the right-hand side are evaluated at the datum condition. If \( \beta_e = 0 \) the expression simplifies to

\[ \ddot{Z}_{wv} = C_{z\beta} \cos \alpha + \sin \alpha \left[ C_{z\beta} + M \frac{\partial C_{z\beta}}{\partial M} + R \frac{\partial C_{z\beta}}{\partial R} \right], \]

and similarly it can be found that (for \( \beta_e = 0 \))

\[ \ddot{Z}_{ww} = 2C_z + (1 + 3 \cos^2 \alpha - \sin 2\alpha) \left[ M \frac{\partial C_z}{\partial M} + R \frac{\partial C_z}{\partial R} \right] + \cos^2 \alpha \left[ M \frac{\partial}{\partial M} + R \frac{\partial}{\partial R} \right]^2 C_z - \]

\[- 2C_z \sin \alpha \cos^3 \alpha + C_{zxx} \sin^2 \alpha, \]

\[ \ddot{Z}_{wv} = \cos \alpha \left[ C_{z\beta} + M \frac{\partial C_{z\beta}}{\partial M} + R \frac{\partial C_{z\beta}}{\partial R} \right] - C_{z\beta} \sin \alpha, \]

\[ \ddot{Z}_{vw} = \cos 2\alpha \left[ C_{z\alpha} + M \frac{\partial C_{z\alpha}}{\partial M} + R \frac{\partial C_{z\alpha}}{\partial R} \right] + \]

\[ + \sin \alpha \cos \alpha \left[ 3 \left( M \frac{\partial C_z}{\partial M} + R \frac{\partial C_z}{\partial R} \right) + \left( M \frac{\partial}{\partial M} + R \frac{\partial}{\partial R} \right)^2 C_z - C_{zxx} \right]. \]

Third-order derivatives should be determined by a similar process: for example the relation

\[ \dddot{Z}_{wv} = \frac{1}{2} \rho \ V S \ddot{Z}_{wv} \]

leads to

\[ \dddot{Z}_{wvn} = \frac{1}{2} \rho \ S \dddot{Z}_{wn} \frac{\partial V}{\partial v} + \frac{1}{2} \rho \ V S \dddot{Z}_{wn} \frac{\partial \sigma}{\partial \nu} + \frac{1}{2} \rho \ V S \sum_{y} \frac{\partial \ddot{Z}_{wn}}{\partial y} \frac{\partial y}{\partial v}, \]

and

\[ \dddot{Z}_{vwn} = \frac{Z_{wvn}}{\frac{1}{2} \rho \ S}, \]

where finally all quantities on the right-hand side are given their datum values.

Expressions for derivatives of any order can be built up in this way, but the usefulness of a Taylor expansion with derivatives above second-order is extremely dubious.
APPENDIX F

Equations of Motion for Small Disturbances of an Aircraft with Four-Fold Symmetry from Straight Steady Spinning Flight
(see Section 10.2.1)

Consider an aircraft having identical pitch and yaw properties. Charters has shown that this will be true for all aircraft having geometric and mass symmetry of the fourth order or higher, that is having four or more planes of symmetry through its longitudinal axis: it is also true for three-fold symmetry if linear aerodynamics is assumed. Because of the mass symmetry body axes having the x-axis coincident with the longitudinal axis will be principal axes, and these are naturally adopted for expressing the equations of motion. We then take $d_{x}, d_{y}, d_{z}, e_{x}, e_{y}, e_{z}, f_{x}, f_{y}, f_{z}$ to be zero, and also $b_{x} = -b_{z}$.

The opening paragraph of Section 10.2.2 is relevant to this example, and it will be convenient to take arbitrary but roughly corresponding datum values for $p_{e}$ and $\xi_{e}$. For example, we may specify that

$$L_{q} \xi_{e} + L_{p} p_{e} = 0,$$

this equation being a crude approximation to the rolling-moment equation*. We then interpret $\xi_{e}$ as the approximate aileron deflection initially imposed in order to achieve a rate of roll of about $p_{e}$, further control being applied to keep perturbations small. Apart from $u_{e}, \xi_{e}, p_{e}$, the datum values of all the other variables can be taken as zero, although a non-zero value of $w_{e}$ might be assumed in order to represent approximately the incidence at which the lift would be equal to the weight.

The forces and moments are expanded in the usual way with respect to their values when $u = u_{e}$, $\xi = \xi_{e}$, etc., and the perturbations will be assumed small and the equations linearized. The conditions of symmetry impose special relations between some pitch and yaw derivatives, and make anti-symmetric derivatives zero. We have

$$y_{v} = z_{w}, \quad y_{\phi} = z_{\psi}, \quad n_{u} = -m_{w}, \quad n_{\phi} = -m_{\psi},$$

$$y_{w} = -z_{v}, \quad y_{\psi} = -z_{\phi}, \quad n_{w} = m_{v}, \quad n_{\psi} = m_{\phi},$$

$$y_{r} = -z_{q}, \quad y_{q} = z_{r}, \quad n_{r} = m_{q}, \quad n_{q} = -m_{r},$$

$$y_{\zeta} = -z_{\eta}, \quad y_{\eta} = z_{\zeta}, \quad n_{\zeta} = m_{\eta}, \quad n_{\eta} = -m_{\zeta},$$

and the zero derivatives are

$$X_{v}, X_{\phi}, X_{w}, X_{\psi}, X_{q}, X_{\eta}, X_{\zeta},$$

$$Y_{v}, Y_{\phi}, Y_{w}, Y_{\psi}, Y_{q}, Y_{\eta}, Y_{\zeta},$$

$$L_{v}, L_{\phi}, L_{w}, L_{\psi}, L_{q}, L_{\eta}, L_{\zeta},$$

$$m_{v}, m_{\phi}, m_{w}, m_{\psi}, m_{q}, m_{\eta}, m_{\zeta},$$

With the values above, the equations of motion (10.6) take the form

*For this purpose the derivatives $L_{q}$ and $L_{p}$ are evaluated for $u = u_{e}$ and $\xi = 0 = p$, but in the equations of motion (F.1) all derivatives are evaluated for datum values $u_{e}, \xi_{e}, p_{e}$. 

21
Appendix F

\[
\begin{align*}
(D + x_u) u' + x_p p' + x_\xi \xi' + x_v v' &= g_x, \\
-(z_\alpha D + z_\alpha + p_\alpha) w' + [(1 + z_\omega) D + z_\omega] v' + (u_\omega - z_q) q' + z_q r' + z_q q' + z_\eta q' + z_\xi = g_y, \\
(z_\beta D + z_\beta + p_\beta) v' + [(1 + z_\omega) D + z_\omega] w' - (u_\omega - z_q) q' + z_q r' + z_q q' + z_\eta = g_z, \\
l_u u' + (D + l_p) p' + l_\xi \xi' + l_v v' &= 0, \\
(m_o D + m_o) v' + (m_o D + m_o) w' + (D + m_q) q' + (m_r - b_o p_o) r' + m_r q' + m_\xi q' &= 0, \\
(m_o D + m_o) w' - (m_o D + m_o) v' + (D + m_q) r' - (m_r - b_o p_o) q' + m_r q' - m_\xi q' &= 0.
\end{align*}
\]

\(F.1\)

In these equations the terms arising from the derivatives \(x_p, x_v, z_\alpha, z_\omega, z_q, z_\xi, l_u, l_v, m_o, m_r, m_\xi\) include the so-called Magnus terms due to the spin \(p_\alpha\). For example, \(z_\alpha = -Z_{\alpha \omega}/m_o\) and \(Z_{\alpha \omega}\) is the value of \(\partial Z/\partial \omega\) when \(p = p_\alpha\) and \(\xi = \xi_\alpha\), which is

\[Z_{\alpha \omega} p_\alpha + Z_{\alpha \omega} \xi_\alpha,
\]

on the assumption that \(\partial Z/\partial \omega\) varies linearly with \(p\) and \(\xi\), \(Z_{\omega \omega}\) and \(Z_{\xi \omega}\) being evaluated for \(\xi = 0 = p\).

The third and sixth of equations (F.1) may be multiplied by \(i = -1\) and added to the second and fifth equations respectively to give

\[
\begin{align*}
(1 + z_\omega D + z_\omega + i(z_\alpha + i \xi_\alpha)) u' + [v' + iw'] + [z_\alpha - i(u_\omega - z_q)] [q' + ir'] + (z_\eta + i \xi_\eta) (\eta' + i \xi') &= g_x + ig_z, \\
(m_o D + m_o) v' + (m_o D + m_o) w' + (D + m_q) q' + (m_r - b_o p_o) r' + m_r q' + m_\xi q' &= 0, \\
(m_o D + m_o) w' - (m_o D + m_o) v' + (D + m_q) r' - (m_r - b_o p_o) q' + m_r q' - m_\xi q' &= 0.
\end{align*}
\]

\(F.2\)

The first and fourth equations remain unchanged:

\[
\begin{align*}
(D + x_u) u' + x_p p' + x_\xi \xi' + x_v v' &= g_x, \\
l_u u' + (D + l_p) p' + l_\xi \xi' + l_v v' &= 0.
\end{align*}
\]

\(F.3\)

To solve equations (F.2) and (F.3) the gravitational terms must be expressed in terms of attitude angles (usually \(\Phi, \Theta, \Psi\)), and the rates of change of the latter must be expressed in terms of \(p_\alpha, q', r', r'\). Also, the motivator deflections \(\xi', \eta', \zeta', \zeta\) must be specified.

The term \((g_\alpha + ig_\xi)\) may be written as \(ig \exp(-i \Phi) \cos \Theta\). When \(\Theta\) is small this is approximately the same as \(ig \exp(-ip_\alpha)\), and \(g_\alpha\) may be neglected if in addition the aircraft is travelling sufficiently fast. The kinematic equations are obtained directly from (5.2):

\[
\begin{align*}
D \Phi &= p_\alpha + p' + (q' \sin \Phi + r' \cos \Phi) \tan \Theta, \\
D \Theta &= q' \cos \Phi - r' \sin \Phi,
\end{align*}
\]

but the \(q'\) and \(r'\) terms can be neglected in the first of these if we assume that \(\tan \Theta\) is small.

With the approximations mentioned, equations (F.2) and (F.3) can be solved independently provided that \(\xi'\) and \(v'\) depend only on \(u'\) and \(p'\), and \(\eta'\) and \(\zeta'\) depend only on \(v', w', q', r'\).
In aeroelastic applications it is often convenient to expand the forces $X$, $Y$, $Z$ and the moments $\mathcal{L}$, $\mathcal{M}$, $\mathcal{N}$ on the right-hand sides of equations (10.3) and (10.4) in terms of variables defining the position of the aircraft, of their derivatives with respect to time, and of motivator deflections. This method of expanding with respect to 'displacement' variables is similar and alternative to that given in Section 10.2.1, where 'velocity' variables are employed.

It is usual in aeroelastic applications to consider small disturbances from a datum condition of straight symmetric flight at constant velocity in a uniform atmosphere which is at rest, and to choose body axes such that the $xz$-plane is the longitudinal plane of symmetry. The datum values $v_c$, $p_c$, $q_c$, $r_c$, $\Phi_c$ are then zero. Also, ground effects are usually ignored and one-fold symmetry of the aircraft implies that the first-order derivatives of $X$, $Z$, $\mathcal{M}$ with respect to the displacements $y$, $\phi$, $\psi$ and the motivator deflections $\xi$, $\xi$, $\delta$ are zero, and similarly the derivatives of $Y$, $\mathcal{L}$, $\mathcal{N}$ with respect to the displacements $x$, $z$, $\theta$ and the motivator deflections $\eta$, $\nu$, $\kappa$ are zero.

As explained in Section 7 and M.4 it is convenient to express perturbations in aircraft position in terms of $x^+$, $y^+$, $z^+$: these are the components, along body axes, of the displacement of the origin from its position in undisturbed flight. It is shown in Section M.5 that, when the datum angular velocity is zero and the atmosphere is at rest,

\[ u' = \dot{x}^+ - w_c \theta, \]
\[ v' = \dot{y}^+ - u_c \psi + w_c \phi, \]
\[ w' = \dot{z}^+ + u_c \theta, \]

and that the acceleration of the origin has components $\ddot{x}^+$, $\ddot{y}^+$, $\ddot{z}^+$ along body axes (see equations (M.14c) and (M.15c)).

The immediate purpose is to display the notation for derivatives of the forces and moments with respect to displacement variables, rather than discuss equations of motion that include aeroelastic effects—the latter aspect is considered in Section M.9. For simplicity we therefore take the aircraft to be a rigid object with origin at the centre of gravity. Equations of motion analogous to (10.9) are then obtained, which break-up into longitudinal and lateral groups as follows.

\[
\begin{align*}
    m\ddot{x}^+ - mg'_x &= \Sigma X_\theta, \\
    m\ddot{z}^+ - mg'_z &= \Sigma Z_\theta, \\
    I_x \ddot{\phi} &= \Sigma M_\phi, \\
\end{align*}
\]  
\[
\begin{align*}
    m\ddot{y}^+ - mg'_y &= \Sigma Y_\phi, \\
    I_y \ddot{\phi} - I_{xy} \ddot{\psi} &= \Sigma L_\phi, \\
    I_z \ddot{\psi} - I_{xz} \ddot{\phi} &= \Sigma N_\phi. \\
\end{align*}
\]  

(G.1) 

(G.2)
Substituting for the gravitational terms, and retaining a representative selection of derivatives, we have as examples of (G.1) and (G.2) the following equations.

\[
\begin{align*}
    m\ddot{x} &= (X_0 - g_x) \theta + X_y \eta' + X_z \xi' + X_z \phi' + X_\theta \gamma' + X_{\phi} \theta', \\
    m\ddot{z} &= (Z_0 - g_z) \theta + Z_y \eta' + Z_z \xi' + Z_\theta \gamma' + Z_{\theta} \phi' + Z_{\phi} \theta', \\
    I_x \ddot{\phi} &= M_\theta \theta' + M_\phi \phi' + M_\xi \xi' + M_\gamma \gamma' + M_{\phi} \phi' + M_{\theta} \theta', \\
    I_z \ddot{\psi} &= I_{\phi} \phi' + I_{\psi} \psi' + I_{\xi} \xi' + I_{\gamma} \gamma' + I_{\phi} \phi' + I_{\psi} \psi'.
\end{align*}
\]

\[\text{Equations (G.3)}\]

\[
\begin{align*}
    m\ddot{y} &= (Y_\phi + g_{\phi}) \phi + (Y_\phi + g_{\phi}) \psi + Y_{\xi} \xi' + Y_{\gamma} \gamma' + Y_{\phi} \phi' + Y_{\psi} \psi', \\
    I_x \ddot{\phi} - I_z \ddot{\psi} &= L_\phi \phi' + L_\psi \psi' + L_{\xi} \xi' + L_{\gamma} \gamma' + L_{\phi} \phi' + L_{\psi} \psi' + L_{\xi} \xi' + L_{\gamma} \gamma', \\
    I_z \ddot{\psi} - I_x \ddot{\phi} &= N_\phi \phi' + N_\psi \psi' + N_{\xi} \xi' + N_{\gamma} \gamma' + N_{\phi} \phi' + N_{\psi} \psi' + N_{\xi} \xi' + N_{\gamma} \gamma'.
\end{align*}
\]

\[\text{Equations (G.4)}\]

No kinematic relationships are required to solve these equations, but the variations in motivator deflections such as \(\xi', \eta'\) must of course be known. The derivatives can be expressed in terms of those with respect to velocity variables, and the relations are established in Section 22. As mentioned there, equations of motion in terms of displacements are sometimes unnecessarily complicated. Moreover (see Appendix C) they are generally unsuitable when the air itself is in motion.

When simple harmonic oscillations are being considered it is usual to introduce oscillatory derivatives (see Appendix K). Letting \(D\) denote \(d/dt\) we then write the first of equations (G.3), for example, as

\[
(mD^2 - \ddot{x} - \ddot{x}) x' - (\ddot{x} D + \ddot{x}) x' - (\ddot{x} D + \ddot{x}) \eta' - (\ddot{x} D + \ddot{x}) g_1) \theta = 0.
\]

Sometimes the equations of motion are developed with reference to earth axes, but Woodcock has shown that if second order terms are neglected the resulting equations are identical with those with reference to body axes, provided the origin is at the centre of gravity.
APPENDIX H

Subsidiary Notation for Primary Aerofoils, Flaps, and Tabs.

(see Sections 15.1, 8, Appendix J)

The dressings $W$, $T$, $F$ introduced below for distinguishing quantities that refer to the wing, tailplane, and fin, can also be used for the forces of lift and drag, their coefficients, and their derivatives, and so on. For example the coefficient of lift on the tailplane may be symbolised as $C_L^T$, $C_{LT}$, or $C_{LT}$. When derivatives like $C_{La}$ have to be distinguished, the form $C_{La}^T$ seems good, in that it is compact and also the position of the $T$ suggests (as it should) that it qualifies both the $C_L$ and the $a$. In other words

$$C_{La}^T = \frac{\partial C_L^T}{\partial a_T}.$$  

$C_L^T$ is of course given by

$$L_T = Q_T S_T C_L^T,$$

where $L_T$ denotes the lift on the tailplane, and $Q_T$ the local kinetic pressure.

It should be remembered that the dressing $T$ is also used for distinguishing forces due to the propulsive system (see Section 3), and $W$ for quantities specifying the wind (see Section 6 and Appendix C).

The symbols given below may receive further dressings. For example, the mean chord of a fin is denoted by $\bar{c}_F$. The area of a primary aerofoil is usually taken to be the gross area, whereas the representative area of a flap or tab is the area aft of the hinge line. The chord of a flap is also taken aft of the hinge line. The arm of a tailplane or fin is the distance of its aerodynamic centre from the centre of gravity of the whole aircraft.

<table>
<thead>
<tr>
<th>Primary aerofoils</th>
<th>Wing*</th>
<th>area</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>chord</td>
<td>$c$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>span</td>
<td>$b$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>aspect ratio $(b^2/S)$</td>
<td>$A$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>sweepback angle</td>
<td>$\Lambda$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>dihedral angle</td>
<td>$\Gamma$</td>
<td></td>
</tr>
<tr>
<td>Tailplane</td>
<td>area</td>
<td>$S_T$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>chord</td>
<td>$c_T$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>arm</td>
<td>$l_T$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>volume ratio $(S_T l_T/S_l)$</td>
<td>$V_T$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>angle of attack (local stream)</td>
<td>$\alpha_T$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>setting (positive in the same sense as elevator deflection)</td>
<td>$\eta_T$</td>
<td></td>
</tr>
<tr>
<td>Fin</td>
<td>area</td>
<td>$S_F$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>chord</td>
<td>$c_F$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>arm</td>
<td>$l_F$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>volume ratio $(S_F l_F/S_l)$</td>
<td>$V_F$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>angle of sideslip (local stream)</td>
<td>$\beta_F$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>setting (positive in the same sense as rudder deflection)</td>
<td>$\zeta_F$</td>
<td></td>
</tr>
</tbody>
</table>

*The suffix $W$ is attached to the symbols for wing quantities only when essential.
Appendix H

Flaps and tabs

Control surface ($\delta$ stands for $\xi$, $\eta$ or $\zeta$)

- area $S_\delta$
- chord $c_\delta$
- deflection $\delta$
- moment of inertia about hinge line $I_\delta$

Tab (add suffix $\zeta$, $\eta$, or $\xi$ when necessary)

- area $S_t$
- chord $c_t$
- deflection $\tau$

Angle of downwash (positive when it decreases $\alpha_T$)

- $\varepsilon$

Angle of sidewash (positive when it decreases $\beta_T$)

- $\sigma$

NOTE

The effective local angles of attack and sideslip depend on local linear velocities, and when the angles are small they are often written as $(\alpha_T + qL/V)$, $(\beta_T - rL/V)$, where $\alpha_T = \alpha - \varepsilon$, $\beta_T = \beta - \sigma$, on the basis that the effects of $q$ and $r$ are equivalent to incidence effects, and may be used with the derivative $C_L$ to obtain associated increments in $C_L^T$.

APPENDIX J

Miscellaneous Aerodynamic Coefficients

(see Sections 15.2, 16)

The use of coefficients based on $\rho V^2$ instead of $\frac{1}{2} \rho V^2$ lingered in this country until about 1935. Such coefficients are obsolete, but if used they should be denoted by the symbol $K$: for example

$$K_Z = \frac{Z}{\rho V^2 S}, \quad K_m = \frac{M}{\rho V^2 S}.$$

Coefficients are often required to denote 'local' values, and are usually defined for a spanwise strip. Such local coefficients should be indicated by the use of the small symbols $c$ or $k$, as appropriate. Taking a strip whose length is the local chord and whose width is $\delta y$, a local force coefficient such as $C_Z$ is defined as the limit of $\frac{\delta Z}{2 \rho V^2 c} \delta y$, where $\delta Z$ is the z-force acting on the strip:

$$C_Z = \frac{\delta Z/\delta y}{2 \rho V^2 c},$$

where $y$ specifies the spanwise station of the strip. Similarly a moment coefficient such as $c_m$ is given by

$$c_m = \frac{\delta M/\delta y}{2 \rho V^2 c^2}.$$
Appendix J

If it were necessary to distinguish between spanwise and chordwise local coefficients, one way would be to add a raised suffix s or c. Thus

\[ c'_m = \frac{\partial M/\partial y}{\frac{1}{2} \rho V^2 c^2}, \quad c''_m = \frac{\partial M/\partial x}{\frac{1}{2} \rho V^2 b^2}, \]

where \( b \) here represents the length of a strip at chordwise station \( x \). The force and moment gradients such as \( \partial M/\partial y, \partial M/\partial x \) may be denoted by \( M_y, M_x \) respectively. When coefficients defined as above are used for two-dimensional aerofoil theory they are usually denoted by a capital \( C \).

In problems involving moveable aerofoils, such as flap-type motivators and tabs, coefficients of the hinge moments may be required. These are almost invariably based on the appropriate individual reference areas and lengths, which we denote here by \( S_b, l_b \) (\( \delta \) standing for any motivator deflection symbol) and by \( S_t, l_t \) (\( \tau \) standing for any tab deflection symbol). If we denote the hinge moment for a motivator by \( B \) (or \( B^t \) if necessary), and the hinge moment for a tab by \( B^t \), the corresponding coefficients are defined as

\[ C_B = \frac{B}{\frac{1}{2} \rho V^2 S_b l_b}, \quad C_B^t = \frac{B^t}{\frac{1}{2} \rho V^2 S_t l_t}. \]

Choosing the symbol for hinge moment is no easy task, particularly on the international front, where \( H \) has no particular relevance in most languages. A glance at the list of symbols shows that \( H \) is the symbol for distances of significant points, aerodynamic or dynamic, from a datum on the \( x \)-axis, and \( H \) is also the symbol for autopilot parameters referring to rudder movements; \( H \) is also the symbol for coefficients in response polynomials concerned with the variable \( h \) (height). It could be argued that most of the \( H \) symbols usually bear suffixes, and that \( H \) and \( C_H \) might often be acceptable for hinge-moment quantities. The letter \( B \) is proposed mainly because it is hardly used for any other purpose.

The choice of the representative area and representative length is based on considerations similar to those brought out in Section 16. Derivatives of hinge moments or of hinge-moment coefficients are written in general as

\[ B_u = \partial B/\partial u, \quad B'_u = \partial B'/\partial w, \]
\[ C_{B_u} = \partial C_B/\partial u, \quad C_{B'_u} = \partial C_B'/\partial w, \]

and so on, but it seems practicable, when there is no ambiguity, to retain the existing symbols \( b_1, b_2, b_3, \]
\[ c_1, c_2, c_3, \]
and hence also \( a_1, a_2, a_3 \). We thus have

\[ C_{L_x} = a_1, \quad C_{B_b} = b_1, \quad C_{B_b}^t = c_1, \]
\[ C_{L_z} = a_2, \quad C_{B_b}^t = b_2, \quad C_{B_b}^t = c_2, \]
\[ C_{L_{zt}} = a_3, \quad C_{B_t} = b_3, \quad C_{B_t}^t = c_3. \]

It is recommended that a statement of identification be made when the older symbols are introduced. When the \( b \)'s or \( c \)'s refer to a particular motivator (e.g. elevator), an additional dressing may be used (e.g. \( C_{b_1} = b_{1_b} \)).

When the \( a \)'s refer to a part of the aircraft, such as the tailplane, they may need an additional dressing, such as \( T \). It seems preferable to write \( C_{L_{zt}}^T \) rather than \( (C_{L_{zt}})^T \) or \( C_{L_{zt}T} \), but \( a_{1T} \) and \( a_{1T}^T \) seem equally acceptable, and likewise \( b_{1p}, c_{1p}, c_{1T} \). When it is necessary to emphasize that the whole aircraft is being considered, the dressing \( \Sigma \) can be used in a similar way to that described in Section 3. For example, if the contributions from the wing and tailplane are the only ones considered, then

\[ a_{1T} = a_{1w}^T + \frac{S_T}{S} a_{1T}, \]
Appendix J

if the kinetic pressure at the tailplane is the same as at the wing.

For propellers and rotors we require coefficients based not only on $V$ but also on $\Omega$, the rate of rotation of the propeller or rotor in radians per unit time. At present the forms of coefficients for propellers and rotors are different, and moreover there is a diversity of definitions in both cases. Proposals are made here for a unified notation which can cover both fields.

It is recommended that coefficients based on $V$ be denoted by $K$ because they seem more analogous with the $K$ coefficients formerly used for aeroplane wings: coefficients based on $\Omega$, however, should be allotted a new symbol, and $G$ is proposed. It is further recommended that the representative area be the disc area $\pi R^2$, and that the representative length be the disc radius $R$. Thrust, torque, and power coefficients are then defined as follows.

\[
\begin{align*}
\text{Thrust:} & \quad K_T = \frac{T}{\rho V^2 \pi R^2}, \quad G_T = \frac{T}{\rho \Omega^2 \pi R^2} \\
\text{Torque:} & \quad K_Q = \frac{Q}{\rho V^2 \pi R^2}, \quad G_Q = \frac{Q}{\rho \Omega^2 \pi R^2} \\
\text{Power:} & \quad K_P = \frac{P}{\rho V^3 \pi R^2}, \quad G_P = \frac{P}{\rho \Omega^3 \pi R^2}.
\end{align*}
\]

It will be noted that

\[
G_T = J^2 K_T, \quad G_Q = J^2 K_Q, \quad G_P = J^3 K_P,
\]

where the approximate tip-speed ratio $J$ is given by

\[
J = \frac{V}{\Omega R}.
\]

APPENDIX K

Miscellaneous Derivatives of Forces and Moments

(see Sections 15.2, 17.1, G, 10.2.1)


If all the variables are simple harmonic with a common constant frequency, then within a framework of linear aerodynamics the forces and moments will also be simple harmonic, and it is convenient to introduce pairs of constant derivatives denoted by $Z_u$, $Z_w$, etc. They refer to in-phase and quadrature components, since the forcing $Z$ can be expressed as

\[
Z = Z_u + (Z_{uu} + Z_{u\dot{u}}) + (Z_{uw} + Z_{u\dot{w}}) + \ldots
\]

(K.1)

where $Z_u$ incorporates the dependence on $D^2 u'$, $D^4 u'$, etc., and $Z_{u\dot{u}}$ the dependence on $D^3 u'$, $D^5 u'$, etc., and so on, this being feasible because $D^2 u' = -\omega^2 u'$, etc., where $\omega$ is the angular frequency of the simple harmonic motions.

Quantities like $Z_u$, $Z_w$ are functions of the datum values of $u$, $\dot{u}$, $w$, $\dot{w}$, etc., but of course they also depend
on the frequency. They are known as oscillatory derivatives. Other examples are \( \tilde{C}_{Zw} \), etc. which are relevant to the expansion of \( C_Z \) in terms of aerodynamic-coefficient derivatives. When it is clear that oscillatory derivatives are the only ones being used, the dressing \( \sim \) may be omitted. Otherwise, however, a plain derivative symbol should always represent a quasi-steady derivative; \( Z_m \) for example, being the value of \( \partial Z / \partial u \) when all other variables are constant, and steady values of \( Z \) are determined for steady values of \( u \). Changes in aircraft flight conditions are often considered to be similar enough to the idealised ones on which quasi-steady or oscillatory derivatives are based, in order to make an approximate calculation feasible in terms of these 'constant' derivatives. For example, oscillatory derivatives may be used in problems where a poorly damped oscillatory mode is most significant, the other modes having relatively small amplitudes throughout the motion. Difficulties arise when more than one oscillatory mode is important, unless all frequencies are so low that the aircraft can be considered to be passing gradually through a series of steady non-oscillatory states and these can be represented as small perturbations from a datum so that quasi-steady derivatives may be used.

K.2. Mean derivatives.

When non-linearities are present it may be convenient to define mean derivatives symbolised by \( Z_m \), etc. The definitions will depend on the problem, and in general we can say only that

\[
Z_w = \frac{1}{w} \int_{w_0}^{w} f_1(Z_w, w) \, dw,
\]

\[
Z_{wq} = \frac{1}{w'q} \int_{w_0}^{w} \int_{q_0}^{q} f_2(Z_{wq}, w, q) \, dw \, dq,
\]

where the functions \( f_1, f_2 \), etc. allow for various weighting functions.

When the weighting is uniform we would have the unweighted mean derivatives

\[
Z_w = \frac{1}{w} \int_{w_0}^{w} Z_w dw, \text{ etc.,}
\]

and hence

\[
Z_w = Z_w + \frac{1}{2} Z_{ww} w' + \frac{1}{8} Z_{www} w'^2 + \ldots,
\]

\[
Z_{wq} = Z_{wq} + \frac{1}{2} Z_{wwq} w' + \frac{1}{8} Z_{wwqq} q' + \ldots,
\]

and so on. In this case the general Taylor expansion can be replaced by a more compact form. Thus for three variables \( w, q, \eta \) we would have

\[
Z = Z_0 + Z_w w' + Z_q q' + Z_\eta \eta' + Z_{wq} w' q' + Z_{w\eta} w' \eta' + Z_{qq} q' \eta' + Z_{wq} w' q' \eta' + Z_{ww} w' \eta' + Z_{wwq} w' q' \eta' + \ldots,
\]

Mean derivatives of force and moment coefficients can be defined in a similar way.


The general Taylor expansion can be expressed in various compact forms, (K.3) above being one
Appendix K

example. Another scheme is to write

\[ Z_w' = Z_{(w)} , \]
\[ Z_{q(w)}' = Z_{(q(w))} , \text{ etc.,} \]

where \( Z_w' \ldots \) stand here for the unweighted mean derivatives. Equation (K.3) then becomes

\[ Z = Z_d + Z_{(w)} + Z_{(q)} + Z_{(wq)} + Z_{(qq)} + Z_{(qw)} + Z_{(wqq)} . \quad (K.4) \]

Complementary parts of (K.4) can be distinguished by means of raised suffixes: for example,

\[ Z = Z_{(w)} + Z^{(w)} \]
\[ = Z_{(q)} + Z^{(q)} \]
\[ = Z_{(wq)} + Z^{(wq)} . \quad (K.5) \]

Thus, \( Z_{(w)} \) represents the sum of all terms depending solely on \( w' \), and \( Z^{(w)} \) represents the sum of all other contributions to \( Z \).

Another scheme is to write

\[ Z = Z_{(w)} + Z^{(w)} , \quad (K.6) \]

where

\[ Z_{[w]} = Z_{(w)} + Z_{(wq)} + Z_{(qw)} + Z_{(wwq)} , \]
\[ Z_{[wq]} = Z_{(wq)} + Z_{(wwq)} , \quad (K.7) \]

and so on. Thus \( Z_{[w]} \) represents the sum of all terms depending wholly or partially on \( w' \). In the absence of coupling terms \( Z_{[w]} = Z_{(w)} \).

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APPENDIX L

Partial Derivatives of Functions of the Velocity and its Components

(see Sections 19, E)

For some purposes it is necessary to know the derivatives of quantities such as \( V, \dot{V} \), the incidence angles \( \sigma, \alpha, \beta \), and the rates of change \( \dot{\sigma}, \dot{\alpha}, \dot{\beta} \), with respect to \( u, v, w \). Many of these are listed below, and it should be noted that \( \alpha \) may stand for \( \alpha_x \) or \( \alpha_y \), and that \( \beta \) stands for \( \beta_x \). Some derivatives depend on the choice of \( \alpha \), and these are given separately for the two cases.

The basic equations are

\[ V^2 = u^2 + v^2 + w^2 , \]

and

\[ V\dot{V} = uu + v\dot{v} + w\dot{w} , \]
Appendix L

or

\[ \dot{V} = \dot{u} \cos \sigma + \dot{v} \sin \beta + \dot{w} \sin \alpha, \]

and likewise

\[ V' = u' \cos \sigma + v' \sin \beta + w' \sin \alpha, \]

where \( V' \) is a small increment in \( V \), and so on. By definition, \( \cos \sigma = u/V, \sin \beta = v/V, \sin \alpha = w/V, \tan \alpha \equiv w/u, \) and normalised variables are

\[ \dot{V}_r = \frac{\dot{V}}{V^2}, \quad \dot{\beta}_r = \frac{\dot{\beta}}{V}, \quad \dot{\alpha}_r = \frac{\dot{\alpha}}{V}. \]

The following expressions can be established and it should be remembered that

\[ \cos \sigma = \cos \alpha \cos \beta, \]

\[ \sin \alpha = \sin \alpha \cos \beta. \]

\[ \left( \frac{\partial V}{\partial u}, \frac{\partial V}{\partial v}, \frac{\partial V}{\partial w} \right) = \left( \frac{\partial \dot{V}}{\partial \dot{u}}, \frac{\partial \dot{V}}{\partial \dot{v}}, \frac{\partial \dot{V}}{\partial \dot{w}} \right) \]

\[ = (\cos \sigma, \sin \beta, \sin \alpha) \]

\[ \left( \frac{\partial M}{\partial u}, \frac{\partial M}{\partial v}, \frac{\partial M}{\partial w} \right) = \frac{M}{V} (\cos \sigma, \sin \beta, \sin \alpha) \]

\[ \left( \frac{\partial R}{\partial u}, \frac{\partial R}{\partial v}, \frac{\partial R}{\partial w} \right) = \frac{R}{V} (\cos \sigma, \sin \beta, \sin \alpha) \]

\[ \left( \frac{\partial \dot{V}}{\partial u}, \frac{\partial \dot{V}}{\partial v}, \frac{\partial \dot{V}}{\partial w} \right) = (\dot{\sigma} \sin \sigma, \dot{\beta} \cos \beta, \dot{\alpha} \cos \alpha) \]

\[ \left( \frac{\partial \sigma}{\partial \dot{u}}, \frac{\partial \sigma}{\partial \dot{v}}, \frac{\partial \sigma}{\partial \dot{w}} \right) = \left( \frac{\partial \dot{\sigma}}{\partial \dot{u}}, \frac{\partial \dot{\sigma}}{\partial \dot{v}}, \frac{\partial \dot{\sigma}}{\partial \dot{w}} \right). \]

\[ = \frac{1}{V} (-\sin \sigma, \cot \sigma \sin \beta, \cot \sigma \sin \alpha) \]

\[ \left( \frac{\partial \beta}{\partial \dot{u}}, \frac{\partial \beta}{\partial \dot{v}}, \frac{\partial \beta}{\partial \dot{w}} \right) \]

\[ = \frac{1}{V} (\cos \sigma \tan \beta, \cos \beta, -\tan \beta \sin \alpha) \]

\[ \frac{\partial \beta}{\partial u} = \dot{\beta} \sin \sigma \tan \beta \frac{\beta \cos \sigma \sec \beta}{V} \frac{\dot{V} \cos \sigma \tan \beta}{V^2} \]

\[ = \frac{\beta}{V} \left( \frac{\sin^2 \beta - \cos \sigma}{\cos \sigma - \cos^2 \beta} \right) + \frac{\dot{\alpha}_r \sin \alpha \cos \alpha \tan \beta}{V \cos \sigma} + \frac{\dot{\alpha} \cos \alpha \tan \beta}{V^2} \]

31
Appendix L

\[ \frac{\partial \beta}{\partial \nu} = -\frac{1}{V} \left( \beta \sin \beta + \frac{\dot{V} \cos \beta}{V} \right) \]

\[ \frac{\partial \beta}{\partial \omega} = \tan \beta \left( \frac{\dot{V} \sin \alpha}{V} - \dot{\alpha} \cos \alpha \right) - \frac{\beta \sin \alpha \sec^2 \beta}{V} \]

\[ \frac{\partial \dot{V}}{\partial u} = \frac{2V \cos \sigma - l \dot{\alpha} \sin \sigma}{V^2} \]

\[ = -\frac{2V \cos \sigma}{V} - \sec \sigma \left( \frac{l}{l_1} \dot{\alpha} \sin \alpha + \frac{l}{l_2} \dot{\beta} \sin \beta \cos \beta \right) \]

\[ \frac{\partial \dot{V}}{\partial v} = \frac{2V \sin \beta + \dot{\beta} \cos \beta}{V} \]

\[ \frac{\partial \dot{V}}{\partial \omega} = \frac{2V \sin \alpha + \dot{\alpha} \cos \alpha}{V} \]

\[ \frac{\partial \beta_\nu}{\partial \nu} = \frac{l_2 \dot{V}_\nu \tan \beta \cos \sigma - \dot{\beta} \cos \sigma (2 + \tan^2 \beta \sec^2 \sigma) + \frac{l_2 \dot{\alpha}}{l_1} \sin \alpha \cos \alpha \tan \beta}{l_1 V} \]

\[ \frac{\partial \beta_\omega}{\partial \omega} = -\frac{1}{V} \left( 2 \dot{\beta}_\omega \sin \beta + \frac{l_2}{l_1} \dot{V}_\omega \cos \beta \right) \]

\[ \frac{\partial \dot{\beta}_\nu}{\partial \nu} = -\frac{\beta_\nu \sin \alpha (2 + \tan^2 \alpha)}{V} + \frac{\tan \beta}{V} \left( \frac{l_2}{l_1} \dot{V}_\nu \sin \alpha - \frac{l_2}{l_1} \dot{\alpha} \cos \alpha \right) \]

In the last six equations \( \alpha \) stands for \( \alpha_\nu \) and \( \dot{\alpha} \), for \( l_1 \dot{x}_\nu/V \), i.e. \( l_1 \sec \alpha_\nu (\dot{x}_\nu \cos \alpha \sec \beta - \dot{\beta} \sin \alpha \sin \beta) / V \).

\[ \left( \frac{\partial \dot{V}_\nu}{\partial \nu}, \frac{\partial \dot{V}_\nu}{\partial \omega}, \frac{\partial \dot{V}_\omega}{\partial \omega} \right) = \frac{1}{V^2} (\cos \sigma, \sin \beta, \sin \alpha) \]

\[ \left( \frac{\partial \beta_\nu}{\partial \nu}, \frac{\partial \beta_\nu}{\partial \omega}, \frac{\partial \beta_\omega}{\partial \omega} \right) = \frac{l_2}{V^2} (-\tan \beta \cos \sigma, \cos \beta, -\tan \beta \sin \alpha) \].

For \( \alpha = \alpha_\nu \):

\[ \left( \frac{\partial \alpha}{\partial \nu}, \frac{\partial \alpha}{\partial \omega}, \frac{\partial \alpha}{\partial \omega} \right) = \left( \frac{\partial \dot{\alpha}}{\partial \nu}, \frac{\partial \dot{\alpha}}{\partial \nu}, \frac{\partial \dot{\alpha}}{\partial \nu} \right) \]

\[ = \frac{1}{V} (-\cos \sigma \tan \alpha, -\sin \beta \tan \alpha, \cos \alpha) \]

\[ \frac{\partial \dot{\alpha}}{\partial \nu} = \frac{\sigma \sin \sigma \tan \alpha - \dot{x}_\nu \cos \sigma \sec^2 \alpha + \dot{V} \cos \sigma \tan \alpha}{V} \]

\[ = \frac{\dot{x}_\nu \sin^2 \alpha \cos \sigma}{V \cos \sigma + \cos \sigma \sec^2 \alpha} \]

\[ + \frac{\beta \sin \beta \cos \beta \tan \alpha}{V \cos \sigma} + \frac{V \cos \sigma \tan \alpha}{V^2} \]

32
Appendix L

\[ \frac{\partial \dot{x}}{\partial v} = \tan \alpha \left( \frac{\dot{V} \sin \beta}{V} - \beta \cos \beta \right) - \frac{\dot{x} \sin \beta \sec^2 \alpha}{V} \]

\[ \frac{\partial \dot{x}}{\partial w} = - \frac{1}{V} \left( \dot{x} \sin \alpha + \frac{\dot{V} \cos \alpha}{V} \right) \]

\[ \frac{\partial \ddot{x}}{\partial u} = \frac{l_1 \dot{V}}{l V} \tan \alpha \cos \sigma - \frac{\dot{\alpha} \cos \sigma}{l V} - \left( 2 + \tan^2 \alpha - \sin^2 \alpha \sec^2 \sigma \right) \frac{\dot{\beta}}{l_2 V \cos \sigma} \]

\[ \frac{\partial \ddot{x}}{\partial v} = \frac{\dot{\alpha} \sin \beta}{V} \left( 2 + \tan^2 \alpha \right) + \tan \alpha \left( \frac{l_1}{l} \dot{V} \sin \beta - \frac{l_1}{l_2} \beta \cos \beta \right) \]

\[ \frac{\partial \ddot{x}}{\partial w} = - \frac{1}{V} \left( 2 \dot{x} \sin \alpha + \frac{l_1}{l} \dot{V} \cos \alpha \right) \]

\[ \left( \frac{\partial \ddot{x}}{\partial u}, \frac{\partial \ddot{x}}{\partial v}, \frac{\partial \ddot{x}}{\partial w} \right) = \frac{l_1}{V^2} (\sin \alpha \cos \sigma, - \sin \alpha \sin \beta, \cos \alpha) \]

For \( \alpha = \alpha' \):

\[ \left( \frac{\partial \ddot{x}}{\partial u}, \frac{\partial \ddot{x}}{\partial v}, \frac{\partial \ddot{x}}{\partial w} \right) = \left( \frac{\partial \ddot{x}}{\partial u}, \frac{\partial \ddot{x}}{\partial v}, \frac{\partial \ddot{x}}{\partial w} \right) \]

\[ = \frac{1}{V} (-\sin \alpha \sec \beta, 0, \cos \alpha \sec \beta) \]

\[ \frac{\partial \dot{x}}{\partial u} = \frac{\dot{V}}{V^2} \sin \alpha \sec \beta - \frac{\sec \beta}{V} (\dot{\alpha} \cos \alpha + \dot{\beta} \sin \alpha \tan \beta) \]

\[ \frac{\partial \dot{x}}{\partial v} = 0 \]

\[ \frac{\partial \dot{x}}{\partial w} = - \frac{\dot{V}}{V^2} \cos \alpha \sec \beta + \frac{\sec \beta}{V} (-\dot{\alpha} \sin \alpha + \dot{\beta} \cos \alpha \tan \beta) \]

\[ \frac{\partial \ddot{x}}{\partial u} = \frac{l_1}{l} \dot{V} \sin \alpha \sec \beta - \frac{\dot{\alpha} \cos \alpha \sec \beta}{l} \left( \dot{\alpha} \cos \beta + \dot{\beta} \sin \beta \right) - \frac{l_1}{l_2} \frac{\dot{\beta}}{V} \sin \alpha \tan \beta \sec \beta \]

\[ \frac{\partial \ddot{x}}{\partial v} = - \frac{\dot{\alpha}}{V} \sin \beta \]

\[ \frac{\partial \ddot{x}}{\partial w} = - \frac{l_1}{l} \dot{V} \cos \alpha \sec \beta - \frac{\dot{\alpha} \sin \alpha \sec \beta}{l} \left( \dot{\alpha} \cos \beta + \dot{\beta} \sin \beta \right) + \frac{l_1}{l_2} \frac{\dot{\beta}}{V} \cos \alpha \tan \beta \sec \beta \]

\[ \left( \frac{\partial \ddot{x}}{\partial u}, \frac{\partial \ddot{x}}{\partial v}, \frac{\partial \ddot{x}}{\partial w} \right) = \frac{l_1}{V^2} (-\sin \alpha \sec \beta, 0, \cos \alpha \sec \beta) \]
Appendix L

If the normalised variables \( w_a, v_a, \dot{w}_a, \dot{v}_a \) are used instead of \( \alpha, \beta, \dot{\alpha}, \dot{\beta} \), the following relations will be needed:

\[
(u_{\lambda}, v_{\lambda}, w_{\lambda}) = \left( \frac{u}{V}, \frac{v}{V}, \frac{w}{V} \right)
\]

\[
(V_{\lambda}, \dot{v}_{\lambda}, \dot{w}_{\lambda}) = \frac{1}{V^2} (lV, l_2 \dot{v}, l_1 \dot{w})
\]

\[
\left( \frac{\partial V}{\partial u}, \frac{\partial V}{\partial v}, \frac{\partial V}{\partial w} \right) = \left( \frac{\partial \dot{V}}{\partial \dot{u}}, \frac{\partial \dot{V}}{\partial \dot{v}}, \frac{\partial \dot{V}}{\partial \dot{w}} \right) = (u_{\lambda}, v_{\lambda}, w_{\lambda})
\]

\[
\left( \frac{\partial u_{\lambda}}{\partial u}, \frac{\partial u_{\lambda}}{\partial v}, \frac{\partial u_{\lambda}}{\partial w} \right) = \frac{1}{V} (1 - u_{\lambda}^2, -u_{\lambda} v_{\lambda}, -u_{\lambda} w_{\lambda})
\]

\[
\left( \frac{\partial v_{\lambda}}{\partial u}, \frac{\partial v_{\lambda}}{\partial v}, \frac{\partial v_{\lambda}}{\partial w} \right) = \frac{1}{V} (-v_{\lambda} u_{\lambda}, 1 - v_{\lambda}^2, -v_{\lambda} w_{\lambda})
\]

\[
\left( \frac{\partial w_{\lambda}}{\partial u}, \frac{\partial w_{\lambda}}{\partial v}, \frac{\partial w_{\lambda}}{\partial w} \right) = \frac{1}{V} (-w_{\lambda} u_{\lambda}, -w_{\lambda} v_{\lambda}, 1 - w_{\lambda}^2)
\]

\[
\left( \frac{\partial \dot{u}_{\lambda}}{\partial u}, \frac{\partial \dot{u}_{\lambda}}{\partial v}, \frac{\partial \dot{u}_{\lambda}}{\partial w} \right) = -\frac{2 \dot{u}_{\lambda}}{V} (u_{\lambda}, v_{\lambda}, w_{\lambda})
\]

\[
\left( \frac{\partial \dot{v}_{\lambda}}{\partial u}, \frac{\partial \dot{v}_{\lambda}}{\partial v}, \frac{\partial \dot{v}_{\lambda}}{\partial w} \right) = -\frac{2 \dot{v}_{\lambda}}{V} (u_{\lambda}, v_{\lambda}, w_{\lambda})
\]

\[
\left( \frac{\partial \dot{w}_{\lambda}}{\partial u}, \frac{\partial \dot{w}_{\lambda}}{\partial v}, \frac{\partial \dot{w}_{\lambda}}{\partial w} \right) = \left( 0, \frac{l_2}{V^2}, 0 \right)
\]

\[
\left( \frac{\partial \ddot{u}_{\lambda}}{\partial u}, \frac{\partial \ddot{u}_{\lambda}}{\partial v}, \frac{\partial \ddot{u}_{\lambda}}{\partial w} \right) = \left( 0, 0, \frac{l_1}{V^2} \right)
\]

\[
\frac{\partial \dot{V}_{\lambda}}{\partial u} = \frac{1}{V} \left( \frac{l \dot{u}_a}{l_1} - 3 \dot{V}_{\lambda} u_{\lambda} \right)
\]

\[
= \frac{\dot{V}_{\lambda}}{V} \left( \frac{1}{u_{\lambda}} - 3 u_{\lambda} \right) - \frac{l}{V u_{\lambda}} \left( \frac{\dot{V}_{\lambda} v_{\lambda} + \dot{w}_a w_{\lambda}}{l_2} \right)
\]

\[
\frac{\partial \dot{V}_{\lambda}}{\partial v} = \frac{1}{V} \left( \frac{l \dot{v}_a}{l_2} - 3 \dot{V}_{\lambda} v_{\lambda} \right)
\]

\[
= \frac{\dot{V}_{\lambda}}{V} \left( \frac{1}{v_{\lambda}} - 3 v_{\lambda} \right) - \frac{l}{V u_{\lambda}} \left( \frac{\dot{V}_{\lambda} u_{\lambda} + \dot{w}_a w_{\lambda}}{l_1} \right)
\]

\[
\frac{\partial \dot{V}_{\lambda}}{\partial w} = \frac{1}{V} \left( \frac{l \dot{w}_a}{l_1} - 3 \dot{V}_{\lambda} w_{\lambda} \right)
\]

\[
\left( \frac{\partial \dot{V}_{\lambda}}{\partial u}, \frac{\partial \dot{V}_{\lambda}}{\partial v}, \frac{\partial \dot{V}_{\lambda}}{\partial w} \right) = \frac{1}{V^2} (u_{\lambda}, v_{\lambda}, w_{\lambda})
\]
APPENDIX M

Matrix Notation for Vector Components, Kinematics, and Equations of Motion.

(see Sections 3, 5.5, 5.6, 6.2, 7, 9, 10, 17.2, 20.3, 20.4, 22, D, G)

M.1. General.

In the main text we have used the symbols u, v, w for the scalar values of the component vectors of a resultant velocity vector denoted by \( \vec{V} \). It is, in general, difficult to devise a consistent symbolism for vectors, their components, and some associated matrices, and the following artifice is proposed for application only when matrices are employed.

A column matrix such as

\[
\begin{bmatrix}
    u \\
    v \\
    w
\end{bmatrix}
\]

will be denoted by the abbreviated form \{u\}. If \( u, v, w \) are the components of a vector*, the latter is represented as \((u)\). An associated anti-symmetric matrix

\[
\begin{bmatrix}
    0 & -w & v \\
    w & 0 & -u \\
    -v & u & 0
\end{bmatrix}
\]

is denoted by \( A_{uu} \), and the row matrix \([u\ v\ w]\) by \([u]\). The use of square and curly brackets follows common practice\(^{55,56}\), although it is sometimes a disadvantage that square brackets are also usual for rectangular matrices. A special bracket for use with row matrices would be very convenient, and may in fact be adopted**. Since in this report some matrices (for example \( A_{uu}, I_m, S \)) are not written with brackets except when expressed fully, it is as well to reserve round brackets for algebraic manipulations.

Occasional use is made in this report of the anti-symmetry operator \( A \), which is defined by the equation \( A\{u\} = A_{uu} \). The inverse operation corresponding to \( A^{-1} \) consists of forming a column matrix from the elements located as (row 3, column 2), (row 1, column 3), (row 2, column 1) in an anti-symmetric matrix. It is debatable whether we should express the corresponding operations referring to row matrices in terms of the operator \( A \), that is, whether we should write \( A[u] = A_{uu} \) and \( A^{-1}A_{uu} = [u] \). There seems no chance

---

*The term vector is here restricted to its old-fashioned meaning of a physical quantity, and column matrices or row matrices are not called vectors.

**Some writers, mainly in U.S.A., have adopted the brackets \([\ ]\) to denote a row matrix, as suggested by Frazer, Duncan and Collar\(^{56}\), but misprints are not uncommon. The author thinks that perhaps \([\ ]\) are more suggestive of a row, more distinctive, and they are moreover easy to type if square brackets are available. It is worth reminding the reader that there seems to be no agreement on whether \([\ ]\) should distinguish a principal diagonal matrix and \([\ ]\) the other kind, or vice versa.
of ambiguity about the direct operation, and if need be the inverse one could always be identified merely by writing \( \{ A^{-1} A_u \} \) or \( \{ A^{-1} A_v \} \).

No great advocacy for the choice of the letter \( A \) is implied: it merely corresponds to ‘anti-symmetric’, while the letter \( S \) which could have been likewise associated with ‘skew-symmetric’ is preferred for denoting a transformation matrix. It is a worthwhile advantage to be able to use the forms \( A_u \) or \( A \{ u \} \), and \( A^{-1} \). Other symbolisms that we have seen either use dressings (such as an overscript \( \sim \) ), or embody a rule that cannot be applied universally (such as corresponding capital and lower-case letters). A large \( \sim \) placed over a complicated matrix expression stretching half across the page is a printer’s nightmare, and the inverse operation presents a problem. The convention that, for example, \( \Omega \) denotes \( A \omega \), where \( \omega \) is a column matrix, will be satisfactory when suitable letters are available. The method employed in this Report has no limitations other than the sacrifice of the one letter chosen.

By analogy with the familiar \([a_{ij}]\), which stands for a matrix having \( a_{ij} \) as the element in the row \( i \)/column \( j \) position, we may use \([u_{ij}]\) to stand for the square matrix \[
\begin{bmatrix}
u_{ij} & u_{ij} & ur \\
u_{ij} & u_{ij} & vr \\
u_{ij} & u_{ij} & wr
\end{bmatrix}
\]

product \( \{u\} [p] \). A similar notation is particularly useful when the elements are derivatives, such as \( X_u \equiv \partial X / \partial u \). Thus the Jacobian matrix* \([\partial (X, Y, Z) / \partial (u, v, w)]\) is written

\[
[X_u] \equiv \begin{bmatrix}X_u & X_v & X_w \\
Y_u & Y_v & Y_w \\
Z_u & Z_v & Z_w
\end{bmatrix}
\]

When second-order derivatives are involved, such as \( X_{u'p} \equiv \partial^2 X / \partial u \partial p \), the abbreviated form should resemble the \( a_{ij} \) pattern with regard to the suffixes, and we write

\[
[X_{up}] \equiv \begin{bmatrix}X_{up} & X_{up} & X_{up} \\
X_{up} & X_{up} & X_{up} \\
X_{up} & X_{up} & X_{up}
\end{bmatrix}
\]

rather than

\[
\begin{bmatrix}X_{up} & X_{up} & X_{up} \\
Y_{up} & Y_{up} & Y_{up} \\
Z_{up} & Z_{up} & Z_{up}
\end{bmatrix}
\text{ or }
\begin{bmatrix}X_{up} & X_{up} & X_{up} \\
Y_{up} & Y_{up} & Y_{up} \\
Z_{up} & Z_{up} & Z_{up}
\end{bmatrix}
\]

Angular displacements are not vectors, but to a first order they can be handled like vectors when they are small. Therefore, although matrices such as \( \{ \Phi \} \), which represents attitude, do not belong to the vector family, we can include \( \{ \phi \} \), \( \{ \Phi \} \) and \( A_{\phi}, A_{\Phi} \) when the deviations \( \phi, \theta, \psi \) and the increments \( \Phi, \Theta, \Psi \) are small.

Other useful matrices are those associated with axes transformation\(^{57,58}\). For two systems of axes, such as a body system \( Oxyz \) and an earth system \( Ox'0 y'0 z'0 \), the components of any vector at any instant

*The reader is reminded that an associated quantity the ‘Jacobian’ is the determinant of the transposed matrix, but written as \( \partial (X, Y, Z) / \partial (u, v, w) \). The Jacobian is not used in this Report.
Appendix M.1

are related through a transformation matrix which is expressed in terms of Euler angles or equivalents. For example, as in Section 5.6, we have

\[ \{u\} = S\{u_0\}, \]

with \( S \) often expressed in terms of the attitude angles \( \Phi, \Theta, \Psi \), and in particular as the product of three simpler matrices, each representing one rotation in the Euler sequence. It is convenient to ascribe the term rotation matrix to simple transformation matrices of this kind, which can be compounded in many ways to obtain more general transformation matrices.

Rotation matrices are denoted by \( R, P, \) or \( Y \) according as the rotation is considered about an \( x, y, \) or \( z \)-axis respectively, these letters being suggested by the names given to body axes (roll, pitch, yaw), which are mainly used in aircraft dynamics. Such matrices thus have the forms

\[
R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{bmatrix}, \quad P = \begin{bmatrix} c & 0 & -s \\ 0 & 1 & 0 \\ -s & c & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} c & s & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

where \( c \) and \( s \) denote the cosine and sine of the angle of rotation. We may then write, for example,

\[
R_{\Phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \Phi & \sin \Phi \\ 0 & -\sin \Phi & \cos \Phi \end{bmatrix}, \quad P_{\theta} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix}, \quad \text{etc.,}
\]

where the suffix indicates the angle of rotation, or

\[
Y_{2} = \begin{bmatrix} \ c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{etc.,}
\]

where the suffix indicates the position of a particular rotation in a sequence of rotations, involving angles such as \( \theta_1, \theta_2, \theta_3, \ldots \), and \( c_2 = \cos \theta_2, s_2 = \sin \theta_2, \text{etc.} \)

Any transformation matrix can in fact be expressed in terms of a single rotation about a suitably chosen line, but it seems valuable to restrict the term rotation matrix as already described. The single-rotation interpretation is described briefly below.

Consider the rotation of an auxiliary system of axes through an angle \( \delta \) about a line having direction cosines \( l, m, n \) with respect to earth axes, and starting from a position of coincidence with the earth axes. If this rotation results in coincidence with the body axes, then the transformation matrix \( S \) can be written

\[
\begin{bmatrix}
  c^2 - s^2 + 2s^2 l^2 & 2s (slm + cn) & 2s (nl - cm) \\
  2s (slm - cn) & c^2 - s^2 + 2s^2 m^2 & 2s (smn + cl) \\
  2s (nl + cm) & 2s (smn - cl) & c^2 - s^2 + 2s^2 n^2
\end{bmatrix},
\]

37
where \( c, s \) denote \( \cos \frac{1}{2} \delta, \sin \frac{1}{2} \delta \) respectively. It is worth remarking that a compact expression for the matrix is given by

\[
S = I - A_1 \sin \delta + 2A_1^2 \sin^2 \frac{1}{2} \delta,
\]

where \( I \) denotes a unit matrix, and \( A_i \) is the anti-symmetric matrix based on \( \{ l \} = \{ l \ m \ n \} \). These forms of \( S \) in terms of \( \delta \) and \( l, m, n \) are connected with the representation of attitude in terms of quaternions or other sets of four parameters (see Section 5.5). If the axis of rotation is chosen to be the \( x_0 \)-axis, the simple \( R \) form of a roll-rotation matrix is obtained, since then \( l = 1 \), with \( m \) and \( n \) zero. Similarly, \( P \) or \( Y \) forms result when \( m = 1 \) with \( l \) and \( n \) zero or when \( n = 1 \) with \( l \) and \( m \) zero.

It is useful to have a list of the matrices representing sequences of two rotations or three rotations. Such a list is given below, and the suffixes 1, 2, 3 refer to the first, second, and third rotation respectively: the corresponding matrix product proceeds in the reverse order. Sequences such as \( R_3P_2R_1 \), involving any basic type of rotation matrix more than once, are not often of interest, and are not considered here. The expressions are valid only if each rotation takes place about a carried axis position that resulted from the preceding rotation.

\[
\begin{align*}
P_2R_1 &= \begin{bmatrix} c_2 & s_2s_1 & -s_2c_1 \\ 0 & c_1 & s_1 \\ s_2 & -c_2s_1 & c_2c_1 \end{bmatrix}, & Y_2R_1 &= \begin{bmatrix} c_2 & s_2c_1 & s_2s_1 \\ -s_2 & c_2c_1 & c_2s_1 \\ 0 & -s_1 & c_1 \end{bmatrix}, \\
P_2Y_1 &= \begin{bmatrix} c_1 & s_1 & 0 \\ -c_2s_1 & c_2c_1 & s_2 \\ s_2s_1 & -s_2c_1 & c_2 \end{bmatrix}, & P_2Y_1 &= \begin{bmatrix} c_2c_1 & c_2s_1 & -s_2 \\ -s_1 & c_1 & 0 \\ s_2c_1 & s_2s_1 & c_2 \end{bmatrix}, \\
R_3P_2Y_1 &= \begin{bmatrix} c_2c_1 & c_2s_1 & -s_2 \\ s_3s_2c_1 - c_3s_1 & s_3s_2s_1 + c_3c_1 & s_3c_2 \\ c_3s_2c_1 + s_3s_1 & c_3s_2s_1 - s_3c_1 & c_3c_2 \end{bmatrix},
\end{align*}
\]

38
Appendix M.1

\[
R_3 Y_2 P_1 = \begin{bmatrix}
c_2 c_1 & s_2 & -c_2 s_1 \\
-c_3 s_2 c_1 + s_3 s_1 & c_3 c_2 & c_3 s_2 s_1 + s_3 c_1 \\
s_3 s_2 c_1 + c_3 s_1 & -s_3 c_2 & -s_3 s_2 s_1 + c_3 s_1 \\
\end{bmatrix}.
\]

\[
Y_3 R_2 P_1 = \begin{bmatrix}
s_3 s_2 s_1 + c_3 c_1 & s_3 c_2 & s_3 s_2 c_1 - c_3 s_1 \\
c_3 s_2 s_1 - s_3 c_1 & c_3 c_2 & c_3 s_2 c_1 + s_3 s_1 \\
c_2 s_1 & -s_2 & c_2 c_1 \\
\end{bmatrix}.
\]

\[
Y_3 P_2 R_1 = \begin{bmatrix}
c_3 c_2 & c_3 s_2 s_1 + s_3 c_1 & -c_2 s_2 c_1 + s_3 s_1 \\
-s_3 c_2 & -s_3 s_2 s_1 + c_3 s_1 & c_2 s_2 c_1 + s_3 s_1 \\
-s_2 & -c_2 s_1 & c_2 c_1 \\
\end{bmatrix}.
\]

\[
P_3 Y_2 R_1 = \begin{bmatrix}
c_3 c_2 & c_3 s_2 c_1 + s_3 s_1 & c_3 s_2 s_1 - s_3 c_1 \\
-s_3 c_2 & s_3 s_2 c_1 - c_3 s_1 & s_3 s_2 s_1 + c_3 c_1 \\
\end{bmatrix}.
\]

It should be noted that, when one of the rotation sequences starts from the horizontal reference attitude which was specified in Section 5 for the definition of attitude angles, the resultant matrix is equal to \(S\), the matrix of direction cosines

\[
\begin{bmatrix}
l_1 & l_2 & l_3 \\
m_1 & m_2 & m_3 \\
n_1 & n_2 & n_3 \\
\end{bmatrix}.
\]

When, however, a sequence starts from the vertical reference attitude of Section 5, the resultant matrix is equal to \(S^*\), which is

\[
\begin{bmatrix}
-l_3 & l_2 & l_1 \\
-m_2 & m_2 & m_1 \\
-n_3 & n_2 & n_1 \\
\end{bmatrix}.
\]

We can thus write

\[
S = R_\phi P_\phi Y_\psi = P_{\phi_x} R_{\phi_y} Y_{\psi_y}
\]

\[
= R_{\phi_{xy}} Y_{\Pi_{xy}} P_{\phi_{xy}} = Y_{\phi_{xy}} R_{\Pi_{xy}} P_{\phi_{xy}}
\]

\[
= P_{\phi_{xy}} Y_{\Pi_{xy}} R_{\phi_{xy}} = Y_{\phi_{xy}} P_{\Pi_{xy}} R_{\phi_{xy}}.
\]

39
Appendix M.1

\[ S^* = Y_{\theta_s} P_{-\theta_s} R_{-\psi_s} = P_{(\theta_s, -\psi_s)} Y_{\theta_s} R_{-\psi_s} = R_{\theta_{n_s}} P_{n_{n_s}} Y_{\theta_{e_s}}. \]

It may be convenient sometimes to write

\[ S = S_{\theta} = R_{\theta} Y_{\psi} = S_{\psi}, \text{ etc.} \]

where the suffix \( \Phi \) attached to \( S \) implies the angles \( \Phi, \Theta, \Psi \); \( \phi \) implies \( \theta, \psi \); and similarly for other trios.

It is suggested that \( S \) may be used in general to denote any axes transformation matrix, with a suffix or suffixes to indicate the axes involved. Thus \( S_{\theta\phi} \) would refer to a transformation from \( 'b' \) axes to \( 'a' \) axes, and

\[ \{u_a\} = S_{\theta\phi} \{u_b\}. \]

This notation has in effect been proposed by Merson in unpublished work, and he also introduces raised suffixes to clarify the meanings of the column matrices. He might write, for example, \( u^{BW} \) instead of our plain \( u \), since it represents the velocity of \( B \) (the body) relative to \( W \) (the wind). Other equivalents would be \( u^{WE} \) and \( u^W, u^BE \) and \( u^E \), where \( E \) stands for earth. We thus have the equivalent equations

\[ u^{BE} = u^{BW} + u^{WE}, \]

and

\[ u^E = u + u^W, \]

which are true for any axes system (any suffixes). We also have

\[ S_{ac} = S_{ab} S_{bc}, \]

for any three axes systems \( a, b, c \).

Although Merson's notation is general it is desirable to eliminate as many dressings as possible, and the simplified notation put forward here will be adequate in many problems whilst still conforming in spirit with the general scheme. We write

\[ \{u\} = S_g \{u_g\} = S_a \{u_a\} = S_k \{u_k\} = S_l \{u_l\}, \text{ etc.} \]

where the suffixes \( g, a, k \) imply earth, air-path, and body-path axes respectively. Other suffixes may be used instead of \( g \) when particular earth axes are invoked, such as suffix \( o \) for normal earth axes, \( A \) for datum-attitude earth axes, and \( E \) for datum-path earth axes (see Section 4.2). To eliminate as many suffixes as possible, the plain \( S \) has on the whole been used instead of \( S_o \) in this Report. We thus write

\[ \{u\} = S\{u_o\} = S_e\{u_e\}, \text{ etc.} \]
Appendix M.1

and

\[ \{u_A\} = S_{A0}\{u_0\}, \quad \{u_B\} = S_{E0}\{u_0\}. \]

By definition \(S_\phi, S_{\phi b}, S_{\phi a}, S_{\phi a b}, \text{ and the corresponding datum matrices } S_{\phi b}, \text{ etc. and also } S_\phi \text{ all have the same form. } S_\phi \text{ is a particular expression of } S, S_{\phi b}, S_{\phi a}, S_{\phi a b}, S_{\phi a b c}, S_{\phi a b c d}, S_{\phi}, \text{ and also } S_\phi \text{ of } S_{A0}, S_{E0} \text{ of } S_{E0} \text{ of } S_{A0}, S_{\phi b}, S_{\phi a}, S_{\phi a b}, S_{\phi a b c}, S_{\phi a b c d}, S_{\phi}, \text{ of } S_\phi \text{ of } S_{A0}, \text{ As the suffix } e \text{ does not imply an axes system, } S_e \text{ could denote the datum value of } S, S_{\phi b}, \text{ again being a particular expression for it.}

It is proposed that \(S_\phi \) be called the (aircraft) attitude matrix, and \(S_\phi \) the attitude-deviation matrix. Similarly \(S_a \) should be termed the incidence matrix, and this is given by

\[
S_a = P_a Y - \beta_a
\]

\[
= \begin{bmatrix}
\cos \alpha_t \cos \beta_s & -\cos \alpha_t \sin \beta_s & -\sin \alpha_t \\
\sin \beta_s & \cos \beta_s & 0 \\
\sin \alpha_t \cos \beta_s & -\sin \alpha_t \sin \beta_s & \cos \alpha_t
\end{bmatrix}
\]

\[
= \begin{bmatrix}
\cos \sigma & -\cos \sigma \tan \beta_s & -\sin \alpha_s \sec \beta_s \\
\sin \beta_s & \cos \beta_s & 0 \\
\sin \alpha_s & -\sin \alpha_s \tan \beta_s & \cos \sigma \sec \beta_s
\end{bmatrix},
\]

where \(\alpha_t, \beta_s, \sigma, \alpha_s \) are the incidence angles defined in Section 6.2. Other expressions for the incidence matrix are given in Appendix D.

Now consider any two systems of axes, \(b\) and \(c\), with attitude angles \(\{\Phi_b\}\) and \(\{\Phi_c\}\). It is convenient to introduce symbols for relative attitude angles, which define the attitude of one system relative to the other in exactly the same way as the attitude of each is defined relative to normal earth axes. For defining the attitude of \(b\) relative to \(c\) we may use \(\{\Phi_{bc}\}\), and it follows that

\[
S_{bc} = S_{\Phi_{bc}} = R_{\Phi_{bc}} P_{\Phi_{bc}} Y_{\Phi_{bc}},
\]

just as

\[
S = S_\Phi = R_\Phi P_\Phi Y_\Phi.
\]

Examples of the interpretation of relative attitude angles are given below, with \(b\) temporarily implying body axes.

\[
\Phi_{b0} = \Phi, \quad \Phi_{a0} = \Phi_a, \quad \Phi_{k0} = \Phi_k.
\]

When \(\Phi_{bc} = 0\), it can be convenient to write

\[
\{\Phi_{bc}\} = \{0 \quad \alpha_c \quad -\beta_c\},
\]

by analogy with

\[
\{\Phi_{ba}\} = \{0 \quad \alpha \quad -\beta\},
\]

41
Appendix M.1

where $\alpha$, $\beta$ are the incidence angles $\alpha_b$, $\beta_b$, i.e. angles of attack and sideslip. The condition $\Phi_{bc} = 0$ implies that the $z_c$-axis lies in the $zx$-plane of axes system $b$. The symbols $\alpha_{bc}$, $\beta_{bc}$ would be used if $b$ did not refer to body axes, and a second method of reducing the suffixes in special cases would be to write $\alpha_c$, $\beta_c$ for $\alpha_{bc}$, $\beta_{bc}$ in other words to use $\alpha_c$, $\beta_c$ for specifying the attitude of axes $c$ relative to the air-path. This does not seem so useful as the first simplification, which is designed for relating body axes to the $x$-axis of another system, and is particularly convenient when the other system is based on the flight path. The advantage of using the basic symbols $\alpha$ and $\beta$ in this way is that corresponding matrices of the same form are then represented by similar symbols. For example, we have the analogous relations

$$S_{a} = P_{a} Y_{-\beta},$$

and

$$S_{b} = P_{a} Y_{-\beta_b}.$$

It should be noted that $\{u_{a}\} = \{V \ 0 \ 0\}$ and $\{u_{b}\} = \{V^K \ 0 \ 0\}$, which imply that $\tan \alpha = w/u$, $\sin \beta = v/V$, and $\tan \beta = w^K/u^K$, $\sin \beta = v^K/V^K$. From one point of view it would therefore be sensible to use the symbols $\alpha_K$, $\beta_K$ or $\alpha^K$, $\beta^K$, instead of $\alpha_c$, $\beta_c$.

As described in Section 5.6, the deviation of a system of body axes from its datum attitude can be expressed by the equation

$$S_\phi = S_\phi S_\phi_f,$$

and this concept can be extended to any system (say system $c$) by writing

$$S_{\phi_c} = S_{\phi_c} S_{\phi_c f}.$$

The angles $\{\phi_c\}$ are the attitude-deviation angles for system $c$. We could even write

$$S_{bc} = S_{\phi_c} (S_{bc})_f,$$

where $\{\phi_{bc}\}$ are the relative attitude-deviation angles of system $b$ with respect to $c$. When the datum angles are constant the suffix $f$ is replaced by $e$.

M.2. Kinematics.

In the following analysis body axes and normal earth axes are considered, since these are commonly used and the associated notation is comparatively free from suffixes. Similar results are in fact obtained for any two sets of axes, and examples are given at the end of the section, mainly in order to emphasize the somewhat subtle distinctions in the symbols for angular velocity components of various sorts (see also Section 5.5).

For small perturbations in attitude we find that $A_\phi = I - S_\phi$, where $I$ is the appropriate unit matrix, and it can be shown that in general

$$\dot{S} = -A_\phi S,$$

(M.1)

or, in another guise in the form of equation (5.4a):

$$\{I\} = -A_\phi \{I\},$$

42
where \( i = 1, 2, \) or \( 3. \)

If \( S^T \) denotes the transpose\(^*\) of \( S \), which is orthogonal, then \( S S^T = S^T S = I \), and hence

\[
A_p = S S^T = -S S^T = \frac{1}{2}(S S^T - S S^T),
\]

\[
A_p = A_p = \frac{1}{2}(S S^T - S S^T).
\]

These expressions may be used when \( S \) is known but explicit calculation of the angular velocity is not required.

The transpose of any rotation matrix, or of any product of rotation matrices (such as \( S \)), is also the inverse of the matrix.

From these equations may be obtained an explicit expression for the angular velocity components \( \{p\} \) in terms of the time derivatives of Euler angles appearing in the expression for \( S \) (or \( S^* \)). However, it is more convenient to deduce it by appropriate resolving, and its most compact form is

\[
\{p\} = R_i \dot{\theta}_i [ \delta_{k1} \delta_{k2} \delta_{k3} ] + R_j \dot{\theta}_j [ \delta_{j1} \delta_{j2} \delta_{j3} ] + R_k \dot{\theta}_k [ \delta_{i1} \delta_{i2} \delta_{i3} ],
\]

(M.2)

where each of \( i, j, k \) is 1, 2, or 3 \((i \neq j \neq k \neq i)\), and \( \delta_{kn} \delta_{pm} \delta_{ln} \) are Kronecker delta functions \((\delta_{nn} = 1)\) when \( m = n \), but is zero when \( m \neq n \). In equation (M.2) the symbol \( R \) denotes any rotation matrix of the roll, pitch, or yaw type, these being identified by the suffixes 1, 2, 3, respectively. The resultant matrix \( S \) (or \( S^* \)) is equal to \( R_i R_j R_k \), and \( R_k \) is associated with the first angle of rotation \((\theta_1)\), \( R_j \) with the second one \((\theta_2)\), and \( R_k \) with the third one \((\theta_3)\). These angles are identified with some trio of attitude angles as defined in Section 5. For example, if \( i = 1, j = 2, k = 3 \), we have \( S = RPY \), with \( \theta_1 = \Psi, \theta_2 = \Theta, \theta_3 = \Phi \), or with \( \theta_1 = \Theta_{xx}, \theta_2 = \Pi_{xx}, \theta_3 = \Phi_{xx} \). Taking the first of these we have

\[
\{p\} = R_\phi P_\phi Y_\psi [ 0 ] + R_\phi P_\phi [ 0 ] + R_\phi [ \Phi ]
\]

\[
\begin{bmatrix}
\psi \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\delta_{11} \\
\delta_{12} \\
\delta_{13}
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\cos \Phi \\
\sin \Phi \\
0
\end{bmatrix}
\]

and this is equivalent to the first three of equations (5.3).

Equation (M.2) can always be reduced to this form, and we have:

\*The transpose of a matrix is the matrix resulting from an interchange of rows and columns, and is variously represented as \( S' \), \( S^T \), \( 'S \), \( S \). We avoid the prime here because we have adopted it for denoting increments, and we avoid the overscript \( ~ \) because this would not be convenient for complicated matrix expressions and it is also used for oscillatory derivatives. We also avoid the use of a prefix.
Appendix M.2

For $k = 1$,

$$
\{ \hat{p} \} = \hat{\theta}_1 \{ c_3 c_2 \ - (\delta_{i2}s_2 + \delta_{i3}s_3 c_2) \ (\delta_{i2}s_2 + \delta_{i2}s_3 c_2) \} + \\
+ \hat{\theta}_2 \{ (\delta_{i3} - \delta_{i2}) s_3 \ \delta_{i3} c_3 \ \delta_{i2} c_3 \} + \\
+ \hat{\theta}_3 \{ 0 \ \delta_{i2} \ \delta_{i3} \}.
$$

For $k = 2$,

$$
\{ \hat{p} \} = \hat{\theta}_1 \{ (\delta_{i1}s_2 + \delta_{i3}s_3 c_2) \ c_3 c_2 \ - (\delta_{i3}s_2 + \delta_{i1}s_3 c_2) \} + \\
+ \hat{\theta}_2 \{ \delta_{i3} c_3 \ (\delta_{i1} - \delta_{i3}) s_3 \ \delta_{i1} c_3 \} + \\
+ \hat{\theta}_3 \{ \delta_{i1} \ 0 \ \delta_{i3} \}.
$$

For $k = 3$,

$$
\{ \hat{p} \} = \hat{\theta}_1 \{ -(\delta_{i1}s_2 + \delta_{i2}s_3 c_2) \ (\delta_{i2}s_2 + \delta_{i1}s_3 c_2) \ c_3 c_2 \} + \\
+ \hat{\theta}_2 \{ \delta_{i2} c_3 \ \delta_{i1} c_3 \ (\delta_{i2} - \delta_{i1}) s_3 \} + \\
+ \hat{\theta}_3 \{ \delta_{i1} \ \delta_{i2} \ 0 \}.
$$

The column matrices giving the coefficients of $\hat{\theta}_1$ in fact represent the $k^{th}$ column of the matrix $S$ or $S^*$, whichever is appropriate. For the usual attitude angles $\Phi, \Theta, \Psi$ we take $i = 1, j = 2, k = 3$, and hence

$$
\{ \hat{p} \} = \Phi \{ m_3 \ n_3 \} + \Theta \{ 0 \cos \Phi \ - \sin \Phi \} + \Psi \{ 1 \ 0 \ 0 \}.
$$

For the alternative angles $\Theta_{xx}, \Pi_{xx}, \Phi_{xx}$ already mentioned $S^*$ is the associated matrix, and

$$
\{ \hat{p} \} = \hat{\Theta}_{xx} \{ m_1 \ n_1 \} + \Pi_{xx} \{ 0 \cos \Phi_{xx} \ - \sin \Phi_{xx} \} + \Phi_{xx} \{ 1 \ 0 \ 0 \}.
$$

When the expression for $\{ \hat{p} \}$ is known for a particular set of Euler angles such as $\Phi, \Theta, \Psi$, it may be written as

$$
\{ \hat{p} \} = Q_{\Phi} \{ \hat{\Phi} \}
$$

but it should be remembered that the form of the $Q$ matrix is changed according to the Euler set. For $S_{\Phi}$ and similar matrices such as $S_{\Phi}, S_{\Phi_{xx}},$ etc., the form of the $Q$ matrix is given by

$$
Q_{\Phi} \equiv \begin{bmatrix}
1 & 0 & -\sin \Theta \\
0 & \cos \Phi & \sin \Phi \cos \Theta \\
0 & -\sin \Phi & \cos \Phi \cos \Theta
\end{bmatrix},
$$

and the inverse matrix is

$$
Q_{\Phi}^{-1} \equiv \begin{bmatrix}
1 & \sin \Phi \tan \Theta & \cos \Phi \tan \Theta \\
0 & \cos \Phi & -\sin \Phi \\
0 & \sin \Phi \sec \Theta & \cos \Phi \sec \Theta
\end{bmatrix},
$$

so that $\{ \hat{\Phi} \} = Q_{\Phi}^{-1} \{ \hat{p} \}$. More generally the relations between $\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3$ and $\{ \hat{p} \}$ are as follows.

For $k = 1$,

$$
c_2 \hat{\theta}_1 = [c_3 \ - \delta_{i3}s_3 \ \delta_{i2}s_3] \{ \hat{p} \},
$$

$$
\hat{\theta}_2 = [(\delta_{i3} - \delta_{i2}) s_3 \ \delta_{i3} c_3 \ \delta_{i2} c_3] \{ \hat{p} \},
$$

$$
c_2 \hat{\theta}_3 = [(\delta_{i2} - \delta_{i3}) c_3 s_2 \ (\delta_{i2} c_2 + \delta_{i3}s_2 s_2) \ (\delta_{i3} c_2 + \delta_{i2}s_2 s_2)] \{ \hat{p} \}.
$$
For \( k = 2 \),
\[
c_2 \theta_1 = \begin{bmatrix} \delta_{13}s_3 & c_3 & -\delta_{11}s_3 \end{bmatrix} \{ p \},
\]
\[
\theta_2 = \begin{bmatrix} \delta_{13}c_3 & (\delta_{12} - \delta_{11})s_3 & \delta_{11}c_3 \end{bmatrix} \{ p \},
\]
\[
c_2 \theta_3 = \begin{bmatrix} (\delta_{11}c_2 + \delta_{13}s_3s_2) & (\delta_{13} - \delta_{11})c_3s_2 & (\delta_{13}c_2 + \delta_{11}s_3s_2) \end{bmatrix} \{ p \}.
\]

For \( k = 3 \),
\[
c_2 \theta_1 = \begin{bmatrix} -\delta_{12}s_3 & \delta_{11}s_3 & c_3 \end{bmatrix} \{ p \},
\]
\[
\theta_2 = \begin{bmatrix} \delta_{12}c_3 & \delta_{11}c_3 & (\delta_{12} - \delta_{11})s_3 \end{bmatrix} \{ p \},
\]
\[
c_2 \theta_3 = \begin{bmatrix} (\delta_{11}c_2 + \delta_{13}s_3s_2) & (\delta_{13} - \delta_{11})c_3s_2 & (\delta_{13}c_2 + \delta_{11}s_3s_2) \end{bmatrix} \{ p \}.
\]

Taking the same two examples as before with \( i = 1, j = 2, k = 3 \), we have
\[
\Psi \cos \Theta = q \sin \Phi + r \cos \Phi,
\]
\[
\dot{\Theta} = q \cos \Phi - r \sin \Phi,
\]
\[
\Phi \cos \Theta = p \cos \Theta + (q \sin \Phi + r \cos \Phi) \sin \Theta,
\]
which are equivalent to the first three of (5.2); and
\[
\dot{\Theta}_{xx} \cos \Pi_{xx} = q \sin \Phi_{xx} + r \cos \Phi_{xx},
\]
\[
\dot{\Pi}_{xx} = q \cos \Phi_{xx} - r \sin \Phi_{xx},
\]
\[
\dot{\Phi}_{xx} \cos \Pi_{xx} = p \cos \Pi_{xx} + (q \sin \Phi_{xx} + r \cos \Phi_{xx}) \sin \Pi_{xx}.
\]

Many expressions for the direction cosines \( l_i, m_i, n_i \) are given in Appendix B.

When considering a rigid object with angular velocity components \( \{ p \} \) we may express the linear velocity of a particle in it as the sum of the velocity of the origin and a contribution due to the rotation of the object with respect to the origin. The components of the latter contribution are represented by \( A_p \{ x \} \), where \( \{ x \} \) are the position coordinates of the particle. Similarly a vector \( \{ \vec{u} \} \) whose line of action passes through the point \( \{ x \} \) will have a moment \( A_x \{ \vec{u} \} \) about the origin.

By definition of the transformation matrix we write
\[
\{ u \} = S \{ u_0 \},
\]
where \( \{ \vec{u} \} \) here denotes any vector having components \( \{ u \} \) along body axes and components \( \{ u_0 \} \) along earth axes. Differentiating this equation and using (M.1), we establish the well-known rotating-axes theorem:
\[
\{ \dot{u} \} + A_p \{ u \} = S \{ \dot{u}_0 \}. \tag{M.4}
\]

The right-hand side represents the components along body axes of the rate of change of any vector \( \{ \vec{u} \} \). For example, we also have:

45
Appendix M.2

\[ \{\dot{x}\} + A_p \{x\} = S \{\dot{x}_0\}. \] (M.4a)

If \( \{x\} \) and \( \{u\} \) denote the components of linear displacement and velocity respectively of some particle, so that \( \{u_0\} = \{\dot{x}_0\} \), we can write the components of linear acceleration along body axes as

\[ S \{u_0\} = S \{\dot{x}_0\} = \{\dot{u}\} + A_p \{u\} = \{\ddot{x}\} + 2A_p \{x\} + A_p^2 \{x\} + \dot{A}_p \{x\}. \] (M.5)

The term \( 2A_p \{x\} \) is the Coriolis term, and \( A_p^2 \{x\} \) is the centripetal term. The matrix \( \dot{A}_p \) is the same as \( A_p \), that is \( \dot{A}\{\dot{p}\} \).

It should be noted that for any vector with components \( \{u\} \) and \( \{u_0\} \) along two systems of axes and related through the transformation matrix \( S \),

\[ SA_{u_0} = A_u S, \]

i.e.

\[ SA_{u_0} S^T = A_u. \] (M.6)

Terms such as \( A_p \{u\} \) are equivalent to those written in vector notation as \( \vec{\Omega} \times \vec{V} \), or \( \vec{\Omega}_\lambda \vec{V} \), where \( p, q, r \) are the components of \( \Omega \) and \( u, v, w \), the components of \( \vec{V} \). Indeed, Nicolaides\(^{59}\) has mixed these notations and in effect written

\[ \left[ \begin{array}{c} p \\ q \\ r \end{array} \right] \times \left[ \begin{array}{c} u \\ v \\ w \end{array} \right], \]

that is \( \{p\} \times \{u\} \). Perhaps both ways of expressing the matrix product may be convenient as circumstances change, and there seems no merit in a rigid adherence to one of them. The vector \( \times \) operator is not introduced elsewhere in this Report, and furthermore it should be noted that the operator \( \times \) has already been used in matrix theory for denoting a Kronecker product\(^{60}\).

If one wished, the vector operators grad, div, and curl could be represented as \( \{D_x\} \), \( [D_x] \), and \( A_{D_x} \) (or \( A_{V} \)) respectively, where \( D_x \equiv \partial/\partial x \), since, if \( \phi \) is a scalar quantity,

\[ \{D_x\} \phi \equiv \{\partial \phi/\partial x \ \partial \phi/\partial y \ \partial \phi/\partial z\}, \]

which is the column matrix corresponding to grad \( \phi \), and so on. On the other hand, following Nicolaides, we might apply the vector operator notation to matrices, and write for example

\[ \text{curl } \{u\} = \left\{ \begin{array}{c} \partial w/\partial y - \partial v/\partial z \\ \partial u/\partial z - \partial w/\partial x \\ \partial v/\partial x - \partial u/\partial y \end{array} \right\}, \]

and \( \text{curl}_0 \{u\} \) for the same matrix when \( x, y, z \) are replaced by \( x_0, y_0, z_0 \). Care should be taken when manipulating complicated matrix expressions that involve differential operator matrices, and standard relations in vector algebra will sometimes be useful, for example

\[ \text{div } (\vec{V} \times \vec{\Omega}) = \vec{\Omega} \text{ curl } \vec{V} - \vec{V} \text{ curl } \vec{\Omega}, \]

i.e.

\[ [D_x] A_u \{p\} = [p] A_{D_x} \{u\} - [u] A_{D_x} \{p\}. \]
Appendix M.2

The possibilities of exploiting vector notation when handling matrices that represent vector components are mentioned here only in passing, and are not relevant to the main purpose of this document.

Returning now to the generalised forms of the kinematic equation (M.1) and the rotating-axes theorem as given by (M.4), we consider any two axes systems b and c. If the transformation matrix is \( S_{bc} \), and \( \{p_b^k\} \) denotes the components, along axes \( b \), of the angular velocity of system \( b \) relative to system \( c \), then it can be shown that

\[
\dot{S}_{bc} = -A \{p_b^k\} \cdot S_{bc},
\]

which implies the relationship

\[
\{p_b^k\} = Q_{\Phi_{bc}} \{\Phi_{bc}\},
\]

where \( Q_{\Phi_{bc}} \) has the same form as \( Q_\Phi \). For instance,

\[
\dot{S}_{a0} = -A \{p_a^k\} \cdot S_{a0}, \quad \dot{S}_{k0} = -A \{p_k^k\} \cdot S_{k0},
\]

and

\[
\{p_a^a\} = Q_{\Phi_a} \{\Phi_a\}, \quad \{p_k^k\} = Q_{\Phi_k} \{\Phi_k\}.
\]

It should be noted that \( \{p^c\} \) and \( \{p^b\} \) denote the components, along body axes, of the angular velocities of the air-path and flight-path systems of axes relative to the earth, whereas \( \{p_a^a\} \) and \( \{p_k^k\} \) represent the components of the first angular velocity along air-path axes and of the second along flight-path axes, respectively. The components \( \{p_a^a\} \) and \( \{p_k^k\} \), which of course are not the same, may also occur in analysis.

Another application of (M.3a) is to the relative rotation of body axes \( b \) and body axes \( b' \) in a datum motion (distinguished below as axes \( b' \)). If the attitudes of these axes are defined by \( \{\Phi\} \) and \( \{\Phi_f\} \) respectively, and the components of the angular velocity of system \( b \) along axes \( b \) are \( \{p\} \), and those of system \( b' \) along axes \( b' \) are \( \{p_f\} \), then

\[
\{p\} = Q_{\Phi} \{\Phi\} \text{ and } \{p_f\} = Q_{\Phi_f} \{\Phi_f\},
\]

and

\[
\{p_b^{bf}\} = \{p\} - S_{\Phi} \{p_f\} = Q_{\Phi} \{\phi\},
\]

where

\[
S_{\Phi} = S_{\Phi_a} S_{\Phi_f}.
\]

When dealing with relative attitude angles of the incidence type, it is reasonable to write

\[
\{p_b^{bc}\} = \{p-p^c\} \text{ if } b \text{ implies body axes}
\]

\[
= Q_{\Phi_a} \begin{bmatrix} 0 & \dot{\alpha}_c & -\beta_c \end{bmatrix},
\]

where \( Q_{\Phi_a} \) is equal to \( Q_{\Phi_{bc}} \), with \( \Phi_{bc} = 0, \Theta_{bc} = \alpha_c, \Psi_{bc} = -\beta_c \). This means that

\[
Q_{\Phi_a} = \begin{bmatrix} 1 & 0 & -\sin \alpha_c \\ 0 & 1 & 0 \\ 0 & 0 & \cos \alpha_c \end{bmatrix},
\]

47
Appendix M.2

and

$$Q_{x}^{-1} = \begin{bmatrix} 1 & 0 & \tan \alpha_c \\ 0 & 1 & 0 \\ 0 & 0 & \sec \alpha_c \end{bmatrix}.$$ 

An example of relations of this type is given by

$$0 = p - p' + (r - r') \tan \alpha_c,$$

$$\dot{\alpha} = q - q',$$

$$\dot{\beta} = (r' - r) \sec \alpha_c,$$

where \( \{p\} = S_c \{p_c\} \) and \( \{p'\} = S_c \{p'_c\} \), with \( S_c = P_{\alpha} Y_{\beta} \).

It follows from (M.1a) that differentiation of the equation

$$\{\dot{u}_b\} = S_{bc} \{u_c\}$$

leads to

$$\{\dot{u}_b\} + A \{p_b'\} \cdot \{u_b\} = S_{bc} \{u_c\},$$

and this relates the rates of change of the two sets of velocity components, in terms of the relative angular velocity components and the transformation matrix. This more general expression of the rotating-axes theorem is not useful unless the right-hand side is physically meaningful, and it should be remembered that \( \dot{\alpha} \) stands merely for \( du_c/dt \) and does not necessarily represent the acceleration along the \( x_c \)-axis.

M.3. Some useful properties and identities.

Anti-symmetric matrices like \( A_x \) and \( A_u \) have many useful properties, such as

$$A_x \{x\} = 0,$$  \hspace{1cm} (M.7)

$$A_x \{u\} + A_u \{x\} = 0,$$  \hspace{1cm} (M.8)

$$A_u A_x \{p\} + A_x A_p \{u\} + A_p A_u \{x\} = 0,$$  \hspace{1cm} (M.9)

$$[x] A_x = 0,$$

and so on. The matrix product \( A_u \{x\} \) is equal to the column matrix \( \{(vz - wy) \ (wx - uz) \ (uy - vx)\} \), and \( [u] A_x \) is equal to its transpose. Therefore, since \( [x] \{x\} = \{x\}^T \{x\} \) and is equal to the quadratic form \( (x^2 + y^2 + z^2) \), we have

$$[u] A_x A_u \{x\} = (vz - wy)^2 + (wx - uz)^2 + (uy - vx)^2$$

$$= -[x] A_x^2 \{x\}$$

$$= -[u] A_x^2 \{u\}.$$
It is also found that

\[ [x] A_u \{p\} = x(vr - wq) + y(wp - ur) + z(uq - vp) \]

\[ = [p] A_x \{u\} \]

\[ = [u] A_p \{x\}. \]

The product of two anti-symmetric matrices \( A_p, A_u \) is given by

\[
A_p A_u = \begin{bmatrix}
-(rw + qv) & qu & ru \\
pv & -(rw + pu) & rv \\
pw & qw & -(qv + pu)
\end{bmatrix},
\]

and it follows that


\[ = (up + vq + wr) I, \quad (M.10) \]

and in particular that

\[ [uu] - A_u^2 = (u^2 + v^2 + w^2) I, \quad (M.10a) \]

where \( I \) is a unit matrix.

It is also evident that \( A_u A_p \) is the transpose of \( A_p A_u \) and hence the matrix \( (A_p A_u - A_u A_p) \) is equal to minus its own transpose, and must be anti-symmetric. Thus

\[
A_p A_u - A_u A_p = \begin{bmatrix}
0 & -(pv - qu) & (ru - pw) \\
(pv - qu) & 0 & -(qw - rv) \\
-(ru - pw) & (qw - rv) & 0
\end{bmatrix},
\]

and can also be equated to \( A \{A_p \{u\}\} \), where \( A \) is the anti-symmetry operator, since \( A_p \{u\} \) is the column matrix \((qw - rv)(ru - pw)(pv - qu))\). It is suggested that \( A \{A_p \{u\}\} \) may be called a second-order anti-symmetric matrix and denoted by \( A_{pu} \). It follows that

\[ A_{pu} \{x\} = (A_p A_u - A_u A_p) \{x\} \]

\[ = - A_x A_p \{u\} \]

\[ = A_x A_u \{p\}. \quad (M.11) \]

Higher-order anti-symmetric matrices can similarly be derived, and it will be found that

\[ A_{pu \ldots x} = A \{A_{pu \ldots \{x\}}\} \]

\[ = A_{pu \ldots} A_x A_{pu \ldots}, \]
Appendix M.3

\[ A_{pv} \ldots \times \{l\} = -A_lA_{pv} \ldots \{x\} = (-)^n A_lA_x \ldots A_{v\{p\}}, \]

where \( n \) is the order of the matrix \( A_{pv} \ldots \times \). Various useful identities may be obtained quickly by application of these general equations. For example, since \( A_x \{x\} \) is zero, we also have

\[ A_{xu} \{A_x \{u\}\} = 0, \]
\[ A_{xup} \{A_{xu} \{p\}\} = 0, \]

and so on, and hence

\[ (A_x A_u - A_u A_x) A_x \{u\} = 0, \]

i.e.

\[ A_x A_u A_x \{u\} = A_u A_x^2 \{u\} = -A_x A_u^2 \{x\}, \text{ etc.} \]

It is worth noting that

\[ A_x A_u A_x = -[x] \{u\} A_x = -(xu + yv + zw) A_x, \]

and that

\[ S A_u S^T = A \{S \{u\}\}, \]

where \( S \) is any direction cosine matrix. The latter equation yields (M.6) when \( u \) is replaced with \( u_0 \), or \( S \) interchanged with \( S^T \).

M.4. Perturbations appropriate to various axes.

The definition of perturbations presents some difficulty, owing to the need for distinguishing between scalars and vectors and the components of the latter along various axes. Consider any vector such as \( \{\vec{u}\} \), which for convenience we shall also represent as \( \vec{V} \). If it has a datum value \( \vec{V}_d \), the perturbation from this is denoted by \( \vec{V}^+ \), so that \( \vec{V} = \vec{V}_d + \vec{V}^+ \). We can resolve these vectors along any axes, for example earth axes, and obtain the related vector or scalar equations of the components:

\[ \vec{u}_g = \vec{u}_{d_0} + \vec{u}_g^+, \text{ etc.}, \]

and

\[ \{u_g\} = \{u_{d_0}\} + \{u_g^+\}. \]

The vector forms are of course unnecessary since the directions of the axes are known. For body axes we would have

\[ \{u\} = \{u_d\} + \{u^+\}. \]

The perturbations \( u^+, u_g^+ \), etc. in the components are called vector increments, while the perturbation
Appendix M.4

in the resultant vector $\vec{V}$ is termed the incremental vector.

Now consider perturbations in attitude angles. We write

$$\Phi = \Phi_f + \Phi', \text{ etc.}$$

and similarly

$$S = S_f + S',$$

where the suffix $f$ denotes datum value and $\Phi', S'$ are the increments in $\Phi, S$. When a datum value is constant, $f$ is replaced by $e$—a non-essential but convenient practice. As previously mentioned, an alternative way of representing perturbations in attitude is to write

$$S = S_\phi S_f,$$

where $\{\phi\}$ are the attitude-deviation angles.

There is also an alternative way of expressing vector components in terms of perturbations. If we define $\{u_f\}$ as the components of $\vec{V}_d$ along directions defined by $S_f$, and write

$$\{u\} = \{u_f\} + \{u'\},$$

then the perturbations $\{u'\}$ are termed the scalar increments. The datum attitude is usually constant, and so is $\vec{V}_d$, and then $u_f$ is constant and written $u_e$, $V_d$ becomes $V_e$, and $\Phi_f$ becomes $\Phi_e$, etc. It is important to realise that even if $\vec{V}_d$ is constant, the components $\{u_d\}$ along body axes are not constant, owing to axes rotations. This is why the scalar increments $\{u'\}$ are usually used rather than the vector increments $\{u^+\}$. They are related as follows:

$$\{u^+\} - \{u'\} = (I - S_\phi) \{u_f\},$$

and when the deviation angles $\phi, \theta, \psi$ are small the right-hand side is approximately equal to $A_\phi \{u_f\}$.

Fig. 3 illustrates the difference between vector and scalar increments. The latter are in the nature of hybrid quantities and there is no associated equation of component vectors: the expression $(\vec{u}_f + \vec{u})$ is meaningless. As explained in Section 7, it is convenient to use vector increments for representing perturbations of velocity in relation to earth axes and for representing perturbations in displacement.

M.5. Kinematics in terms of perturbations.

In order to derive the equations of motion applicable to small perturbations in the variables, we require expressions for linear and angular velocities and accelerations in terms of the perturbations. As explained in the previous section we shall use the scalar increments of velocities, $\{u'\}$ and $\{p'\}$, and the vector increments of linear displacements, $\{x^+\}$. It is assumed that the atmosphere is at rest so that $\{u\}$ also represents the components of velocity relative to the earth. When the air is moving we must replace $\{u\}$ by $\{u^e\}$ in kinematic expressions—see Appendix C.

Perturbations in attitude angles will be represented by the deviation angles $\{\phi\}$ rather than the increments $\{\Phi\}$. We then have

$$S = S_\phi S_f,$$  \hspace{1cm} (M.12)

where $S_f$ stands for $S_{\Phi_f}, \text{ and } \Phi_f, \Theta_f, \Psi_f$ are datum values of $\Phi, \Theta, \Psi$. Differentiating (M.12) and using (M.1) to express $S$ and $S_f$, we also obtain

$$\dot{S}_\phi = S_\phi A_{\Phi_f} - A_\phi S_\phi,$$  \hspace{1cm} (M.13)

51
where \( \{p_r\} \) are the components, along the datum directions of the axes, of the datum angular velocity of the system of axes, i.e. \( \{p_r\} = Q_{0r} \{\Phi_r\} \).

The general relationship between \( \{u'\} \) and \( \{x'\} \) can be derived from that between \( \{u\} \) and \( \{x\} \), as follows. Since \( \{u_0\} = \{\tilde{x}_0\} \), and \( \{x\} \) are the components of a vector, the equivalent of equation (M.4a) is

\[
\{x'\} + A_p \{x\} = S \{\tilde{x}_0\}
\]

\[
= S \{u_0\}
\]

\[
= \{u\} .
\]

Then, since

\[
\{u\} = \{u_f\} + \{u'\}
\]

\[
= S_\Phi \{u_f\} + \{u\} ,
\]

\[
\{u'\} = \{u\} + (S_\Phi - I) \{u_f\}
\]

\[
= \{x'\} + A_p \{x\} + (S_\Phi - I) \{u_f\} .
\]

(M.14)

The acceleration components along body axes are given by

\[
S \{\tilde{u}_0\} = \{\tilde{u}_f\} + \{\tilde{u}'\} + A_p \{u_f\} + A_p \{u'\}
\]

\[
= \{\tilde{x}'\} + 2A_p \{\tilde{x}\} + A_p^2 \{x\} + A_{\Phi} \{x\} + S_\Phi \{u_f\} + S_\Phi A_p \{u_f\} .
\]

(M.15)

If the perturbations are small and their products are neglected, the last two equations become

\[
\{u'\} = \{\tilde{x}'\} + A_p \{x\} - A_{\Phi} \{u_f\}
\]

\[
= \{\tilde{x}'\} + A_p \{x\} + A_{\Phi} \{\Phi\} .
\]

(M.14a)

\[
S \{\tilde{u}_0\} = \{\tilde{x}'\} + 2A_p \{\tilde{x}\} + A_p^2 \{x\} + A_{\Phi} \{x\} +
\]

\[
+ \{u_f\} + A_p \{u_f\} + A_{\Phi} \{\Phi\} + (A_{\mu}) \{\Phi\} .
\]

(M.15a)

If in addition we have steady datum conditions for which \( \tilde{u}_f \) and \( \tilde{p}_f \) are zero, and we therefore write \( u_f = u_0, p_f = p_0 \), these equations become

\[
\{u\} = \{\tilde{x}'\} + A_p \{x\} + A_{\Phi} \{\Phi\} ,
\]

(M.14b)

\[
S \{\tilde{u}_0\} = \{\tilde{x}'\} + 2A_p \{\tilde{x}\} + A_p^2 \{x\} + A_{\Phi} \{x\} + A_{\Phi} \{u_0\} + (A_{\mu} \Phi) \{\Phi\} .
\]

(M.15b)

Finally, if in addition the angular velocity is zero in the datum condition, that is \( \{p_r\} = 0 \), the equations become

\[
\{u'\} = \{\tilde{x}'\} + A_{\Phi} \{\Phi\} ,
\]

(M.14c)

\[
S \{\tilde{u}_0\} = \{\tilde{x}'\} .
\]

(M.15c)

Now consider angular velocities and accelerations. It has already been stated in Section M.2 by way of
Appendix M.5

illustrating equation (M.3a) that \( \{p\} = Q_\phi \{\phi\}, \{p_f\} = Q_{p_f} \{\phi_f\} \) and

\[
\{p\} = Q_\phi \{\phi\} + S_\phi \{p_f\};
\]

and hence

\[
\{p'\} = Q_\phi \{\phi\} + (S_\phi - I) \{p_f\},
\]

where

\[
Q_\phi = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \cos \theta \sin \phi \\ 0 & -\sin \phi & \cos \theta \cos \phi \end{bmatrix},
\]

and \(Q_\phi, Q_{p_f}\) also have this form. The angular acceleration components along body axes are therefore given by

\[
S_f \{i_0\} = \{p\} + A_p \{p\} = \{p\}
\]

\[
\begin{align*}
&= Q_\phi \{\phi\} + \dot{Q}_\phi \{\phi\} + S_\phi \{p_f\} + \dot{S}_\phi \{p_f\} \\
&= Q_\phi \{\phi\} + \dot{Q}_\phi \{\phi\} + S_\phi \{p_f\} - A_p S_\phi \{p_f\}.
\end{align*}
\]

\[
(M.18)
\]

\( \dot{S}_\phi \) being obtained from (M.13), and \( A_p \{p\} \) and \( A_{p_f} \{p_f\} \) put equal to zero. If the perturbations \( \phi, \dot{\phi}, p' \), etc. are small and their products are neglected, equations (M.17) and (M.18) reduce to

\[
\{p'\} = \{\phi\} - A_p \{p_f\} = \{\phi\} + A_{p_f} \{\phi\},
\]

\[
\begin{align*}
S_f \{i_0\} &= \{\phi\} + A_p \{\phi\} + \dot{A}_{p_f} \{\phi\} + \{p_f\}
\end{align*}
\]

\[
(M.18a)
\]

use being made of the approximations \( S_\phi \approx I - A_{p_f} Q_\phi \approx I, \)

\[
\dot{S}_\phi = -A \{Q_\phi \{\phi\}\} \quad S_\phi \approx -A_{p_f} (I - A_{p_f}) \approx -A_{p_f}.
\]

APPENDIX M.5


Consider a system of particles of mass \( m_i \), the total mass being \( m = \Sigma m_i \), and each \( m_i \) constant. Let the components of linear displacement and velocity of any particle be denoted by \( \{x_i\} \) and \( \{u_i\} \) respectively. The components of linear momentum and angular momentum are then given by \( m_i \{u_i\} \) and \( m_i A_{xi} \{u_i\} \) respectively, where \( A_{xi} \) denotes the anti-symmetric matrix based on \( \{x_i\} \), namely \( A \{x_i\} \).

In the most general case the components of vectors are specified with respect to a rotating axes system \( Oxyz \), whose origin is moving with velocity \( \{u\} \), say. Also, if we are dealing with a localised collection of particles, the origin is likely to be at or near the centre of gravity, and it is convenient to define the co-ordinates \( \{x_i\} \) relative to this point. If then the angular velocity components of the axes system are \( \{\phi\} \), we have from equations (M.4a),

\[
\{u\} = \{\dot{x}\} + A_p \{x\},
\]

where

\[
\{u\} = \{\dot{x}\} + A_p \{x\},
\]

\[
\begin{align*}
\{x\} &= \{x\} + \{\dot{x}\} + A_{p_f} \{x\} \\
\{\phi\} &= \{\phi\} + A_{p_f} \{\phi\} + \dot{A}_{p_f} \{\phi\}
\end{align*}
\]

\[
(M.19)
\]
if \( \{ x \} \) represents the position of 0 relative to a point \( O_0 \) fixed with respect to the earth.

We can equate the rate of change of linear momentum to the applied forces, and, if the latter are denoted by \( \{ X_i \} \) for the whole system, we have

\[
\sum_{i} m_i \{ \dot{u}_i \} + A_p \sum_{i} m_i \{ u_i \} = \{ X_i \} .
\]  
(M.20)

We cannot, however, equate the rate of change of the angular momentum about 0 to the applied torque about the same axis, because the origin 0 is not at rest. To obtain an expression for \( \{ L_i \} \), which represents the components of the applied torques about 0 for the whole system, we determine the difference between the moments about \( O_0 \) and 0, that is between \( \Sigma (A_x + A_{xi}) \{ X_i \} \) and \( \Sigma A_{xi} \{ X_i \} = \{ L_i \} \). The first of these is equal to the rate of change of angular momentum about \( O_0 \), which is

\[
\frac{d}{dt} \sum m_i (A_x + A_{xi}) \{ u_i \} + A_p \sum m_i (A_x + A_{xi}) \{ u_i \} ,
\]

and of course the forces \( \{ X_i \} \) acting on the particle \( m_i \) are given by

\[
\{ X_i \} = m_i \{ \dot{u}_i \} + A_p m_i \{ u_i \} .
\]

By combining these expressions, and using the kinematic relation

\[
\frac{d}{dt} (A_x + A_{xi}) = A_{ui} (A_x + A_{xi}) A_p - A_p (A_x + A_{xi}) ,
\]

we establish that

\[
\sum m_i A_{xi} \{ \dot{u}_i \} + \sum m_i A_{xi} A_p \{ u_i \} = \{ L_i \} .
\]  
(M.21)

If we substitute for \( \{ u_i \} \), equations (M.20) and (M.21) become

\[
m \{ \dot{u} \} + mA_p \{ u \} + \sum m_i (\{ \dot{x}_i \} + 2A_p \{ \dot{x}_i \} + A_p + A_{p}^2) \{ x_i \} = \{ X_i \} ,
\]  
(M.20a)

and

\[
\sum m_i A_{xi} \{ \dot{u}_i \} + \sum m_i A_{xi} \{ \dot{u}_i \} + \sum m_i A_{xi} (A_p \{ x_i \} + 2A_p \{ \dot{x}_i \} + A_p^2 \{ x_i \})
= \sum m_i A_{xi} \{ \dot{u}_i \} + A_p \{ u \} + \{ \dot{x}_i \} - \sum m_i (A_{xi}^2 \{ \dot{p} \} + 2A_{xi} A_{xi} \{ p \} + A_p A_{xi}^2 \{ p \}) = \{ L_i \} .
\]  
(M.21a)

The moment equation can be put in terms of \( I_n \), the inertia matrix of the whole system (see Section 9), which is defined as

\[
I_n = \begin{bmatrix}
I_x & -I_{xy} & -I_{xz} \\
-I_{xy} & I_y & -I_{yz} \\
-I_{xz} & -I_{yz} & I_z
\end{bmatrix} ,
\]  
(M.22)

where

\[
I_x = \Sigma m_i (y_i^2 + z_i^2) , \quad I_{xy} = \Sigma m_i x_i y_i , \text{ etc.}
\]
Appendix M.6

As
\[
A_x^2 = \begin{bmatrix}
-(y^2 + z^2) & xy & xz \\
xy & -(z^2 + x^2) & yz \\
xz & yz & -(x^2 + y^2)
\end{bmatrix},
\] (M.23)

it follows that
\[
I_n = -\Sigma m_i A^2_{x_i},
\]
and
\[
\dot{\mathbf{i}}_n = -\Sigma m_i \left( A_{x_1} A_{x_1} + A_{x_2} A_{x_2} \right),
\]
so that equation (M.21a) becomes
\[
\Sigma m_i A_{x_i} \{ u_{x_i} \} + \dot{I}_n \{ p \} + \dot{\mathbf{i}}_n \{ p \} + \Sigma m_i \left( A_{x_1} \{ \dot{x}_i \} + A_{p} \dot{A}_{x_1} \{ x_i \} \right) = \{ L^2 \}. \tag{M.21b}
\]

Equations (M.20a) and (M.21b) are simplified when we take the origin 0 at the centre of gravity of the system. In general, \( \Sigma m_i \{ x_i \} = m \{ x_0 \} \), \( \Sigma m_i A_{x_1} = mA_{x_0} \), and so on, where \( \{ x_0 \} \) denotes the position of the centre of gravity relative to 0, so that these terms (and also their derivatives \( \Sigma m_i \{ \dot{x}_i \} \), etc.) become zero when \( \{ x_0 \} = 0 \). We then have
\[
m \{ \dot{u} \} + mA_{p} \{ u \} = \{ X^2 \}, \tag{M.20b}
\]
and
\[
I_n \{ \dot{p} \} + A_{p} I_n \{ p \} + \dot{I}_n \{ p \} + \Sigma m_i \left( A_{x_1} \{ \dot{x}_i \} + A_{p} \dot{A}_{x_1} \{ x_i \} \right) = \{ L^2 \}. \tag{M.21c}
\]

If, on the other hand, the \( \{ x_i \} \) are constant (this implies a rigid object as defined in the next Section), we have
\[
m \{ \dot{u} \} + mA_{p} \{ u \} + mA_{p} \{ x_0 \} = \{ X^2 \}, \tag{M.20c}
\]
and
\[
I_n \{ \dot{p} \} + A_{p} I_n \{ p \} = \{ L^2 \}. \tag{M.21d}
\]

The force equation (M.20c) reduces to (M.20b) if the origin is at the centre of gravity, and (M.20b) together with (M.21d) are the matrix forms of equations (10.2).

M.7. Equations of motion for an aircraft.

The general equations of motion for a system of particles are given as (M.20a) and (M.21a). When we consider an aircraft it is not profitable to try and take account of every individual mass particle, and we invent mathematical abstractions of the aircraft, which are the simplest that can be used without losing any essential features of a particular investigation. The simplest of all such abstractions is an aircraft which is a rigid object, and it is important to distinguish between this and the concept of a rigid aircraft. A rigid object* is an assembly of particles that behave as an entity; in other words there is no relative movement (\( \langle \dot{x}_i \rangle = 0 \)), and the mass distribution is constant (\( \bar{m} = \dot{I}_n = 0 \)). In a rigid aircraft there is a

*The traditional mathematical term is 'rigid body' but the word body has a special significance for aerodynamicists.
Appendix M.7

rigid structural framework (in itself a rigid object), which contains non-rigid substances such as fuel and engine gases, and moving elements such as machinery, stores, and passengers.

In general, equations (M.20a) and (M.21b) would be applied to a rigid aircraft, although the rigid parts would be considered as a whole, so that terms similar to the left-hand sides of (M.20c) and (M.21d) would arise in which the mass and inertia constants of the rigid portion alone feature.

If the ejection of engine gases is being included, the particles of gas are taken in a separate group, and if, as is usual, we postulate a constant mass flow \( k \), the contribution to summation terms such as \( \Sigma m_i \) can be expressed in terms of \( k \), of the exit velocity of the gas relative to the structure, and of the co-ordinates of the exit point. In some cases the co-ordinates of the point at which fuel enters the engine must be included, also the point at which air is drawn into the aircraft.

Moving elements are often ignored, but must sometimes play a significant part. There is no requirement for a general notation except perhaps when the moving elements are entirely rotational and possess an angular momentum relative to the remainder of the aircraft. The angular-momentum components may be denoted by \( J_x, J_y, J_z \), and if the aircraft is otherwise rigid the moment equations become

\[
I_n \{ \dot{p} \} + \{ J_x \} + A_p I_n \{ p \} + A_p \{ J_x \} = \{ L^x \},
\]

where \( I_n \) includes the effect of the mass of the rotating elements, which for this purpose only are taken as fixed to the rigid part of the aircraft. When the left-hand side of (M.21e) is written in component form, and \( \{ J_x \} \) taken to be zero, the three remaining expressions are the same as the left-hand sides of equations (10.2) apart from additional terms \( (J_z q - J_y r), (J_x r - J_y p), (J_y p - J_z q) \) in the \( L, M, N \) equations respectively. The reduced form of the moment equations—corresponding to (10.4)—may be written with terms \( (c_{xy} q - c_{yx} r), (c_{yx} r - c_{xy} p), (c_{yz} p - c_{zy} q) \) on the left-hand sides, where the angular-momentum constants are given by

\[
c_{xy} = \frac{J_x}{I_y}, \quad c_{yx} = \frac{J_y}{I_x}, \quad c_{xz} = \frac{J_z}{I_x},
\]

For dealing with a deformable aircraft we are likely to employ equations (M.20a) and (M.21b), since the centre of gravity may not be a convenient origin for defining deformation variables. In the following Section we discuss the procedure for dealing with a deformable object of constant mass. It does not seem worthwhile to distinguish between a deformable object and a deformable aircraft in a similar manner to that described earlier for rigid objects and aircraft.

M.8. Linearized equations of motion for small disturbances of a rigid object.

The general equations of motion for a rigid object referred to body axes having the origin at the centre of gravity have been given as (M.20b) and (M.21d). If we define a datum motion in which \( \{ \dot{u} \} = \{ \dot{u}_d \}, \{ u \} = \{ u_d \}, \) etc. and the motion during a disturbance is defined in terms of perturbations \( \{ u' \} = \{ u \} - \{ u_d \}, \) etc. then the linearized forms of these equations are

\[
m \{ \dot{u}_d \} + m A_{p_d} \{ u_d \} + m A_{p'} \{ u' \} + m A_{p''} \{ u_d \} = \{ X^d \},
\]

\[
I_n \{ \dot{p}_d \} + A_p I_n \{ p_d \} + I_n \{ p' \} + A_p I_n \{ p'' \} = \{ L^d \}.
\]

The linearization is incomplete until we write

\[
\{ X^d \} = \{ X_d^d \} + \{ X^e \},
\]

\[
\{ L^d \} = \{ L_d^d \} + \{ L^e \},
\]

56
Appendix M.8

where

\[ m \{ \dot{u}_f \} + m A_{p_f} \{ u_f \} = \{ X^F \}, \]
\[ I_n \{ \dot{p}_f \} + A_{p_f} I_n \{ p_f \} = \{ L^F \}. \]

The completely linearized equations thus become

\[ m \{ \dot{u}' \} + m A_{p_f} \{ u' \} - m A_{w_f} \{ p' \} = \{ X^Z \}, \quad (M.27) \]
\[ I_n \{ \dot{p}' \} + (A_{p_f} I_n - A_{H_f}) \{ p' \} = \{ L^Z \}, \quad (M.28) \]

where for convenience \( A_{H_f} \) stands for the anti-symmetric matrix based on \( \{ H_f \} = I_n \{ p_f \} \), which represents the angular momentum during the datum motion. An alternative form of the equations is obtained in terms of the perturbations \( \{ x^+ \} \) and \( \{ \phi \} \) by applying equations \( (M.14a) \) and \( (M.17a) \). We obtain

\[ m \{ \dot{x}^+ \} + 2m A_{p_f} \{ \dot{x}^+ \} + m (A^2_{p_f} + A_{p_f}) \{ x^+ \} + m A_{w_f} \{ \phi \} + m (A_{w_f})_f \{ \phi \} = \{ X^Z \}, \quad (M.29) \]
\[ I_n \{ \dot{\phi} \} + B_{1f} \{ \phi \} + B_{0f} \{ \phi \} = \{ L^Z \}, \quad (M.30) \]

where \( B_1 \) is the anti-symmetric matrix

\[ (A_{p_f} I_n + I_n A_{r} - A_{H_f}), \]

and \( B_0 \) is \( (A_{p_f} I_n - A_{H_f}) A_{p_f} \).

The more complicated matrix expressions are given below in more detail in terms of \( a, b, c; d, e, f \), which stand temporarily for \( I_x, I_y, I_z; I_{xy}, I_{xz}, I_{xy} \)

\[ A_{p_f} \{ H \} = \begin{bmatrix} (c-b) qr + (fr-eq) p + d (r^2 - q^2) \\ (a-c) rp + (dp-fr) q + e (p^2 - r^2) \\ (b-a) pq + (eq-dp) r + f (q^2 - p^2) \end{bmatrix} \]

\[ A_H = \begin{bmatrix} 0 & -(cr - ep - dq) & (bq - dr - fp) \\ (cr - ep - dq) & 0 & -(ap - fq - er) \\ -(bq - dr - fp) & (ap - fq - er) & 0 \end{bmatrix} \]

\[ A_{p_f} I_n = \begin{bmatrix} (fr - eq) & -(br + dq) & (cq + dr) \\ (ar + ep) & (dp-fr) & -(cp + er) \\ -(aq+fp) & (bp+fq) & (eq-dp) \end{bmatrix} \]

\[ = -(I_n A_{p})^T \]
Appendix M.8

\[
A_p I_n + I_n A_p = A \begin{bmatrix}
(b+c) p + fg + er \\
(c+a) q + dr + fp \\
(a+b) r + ep + dq
\end{bmatrix}
\]

\[
B_1 = A \begin{bmatrix}
(b+c-a) p + 2(fq+er) \\
(c+a-b) q + 2(dr+fp) \\
(a+b-c) r + 2(ep+dq)
\end{bmatrix}
\]

\[
B_0 = \begin{bmatrix}
(b-c)(q^2-r^2) & (c-b) pq & (b-c) pr \\
(c-a) pq & (c-a)(r^2-p^2) & (a-c) qr \\
(b-a) pr & (a-b) qr & (a-b)(p^2-q^2)
\end{bmatrix}
- \begin{bmatrix}
4dq r + p(fq+er) & f(r^2-p^2)-2dpr-eqr & e(q^2-p^2)-2dpq-fqr \\
f(r^2-q^2)-2eqr-drp & 4erp+q(dr+fp) & d(p^2-q^2)-2epq-frp \\
e(q^2-r^2)-2fqr-dpq & d(p^2-r^2)-2frp-epq & 4fpq+r(ep+dq)
\end{bmatrix}
\]

If \{\dot{u}_f\} and \{\dot{p}_f\} are zero the linearized equations simplify: the terms \(A_{p_f}\) and \(A_{u_f}\) become zero, and the suffix \(f\) is replaced by \(e\) to show that \{\(u\)\} and \{\(p\)\} are constant during the datum motion. If in addition \{\(p_e\)\} is zero, the equations become

\[
m \{\ddot{u}'\} - m A_{u_e} \{p'\} = \{X_e^2\}, \quad (M.27a)
\]

\[
I_n \{p'\} = \{L_e^2\}, \quad (M.28a)
\]

\[
m \{\ddot{x}'\} = \{X_e^2\}, \quad (M.29a)
\]

\[
I_n \{\ddot{p}'\} = \{L_e^2\}. \quad (M.30a)
\]

The earth contact and gravitational contributions to the right hand sides are usually expressed separately, and the remainders (denoted by \{\(X'\)\} and \{\(L'\)\}) are expanded in a Taylor series in terms of (for example) the \(\omega\) set of variables (see Section 17.2). The matrix form of the expansion is

\[
\{X'\} = [X_{\omega}] \{\omega'\} + \frac{1}{2} \begin{bmatrix}
[\omega'] [X_{\omega}\omega] & [\omega'] [Y_{\omega\omega}] & [\omega'] [Z_{\omega\omega}]
\end{bmatrix} \{\omega'\} + \text{higher order terms},
\]

where \(\{\omega\} \equiv \{\omega_1, \omega_2, \omega_3, \ldots\}\) corresponds to the \(\omega\) variables here denoted by \(\omega_1, \omega_2, \ldots\). The matrix derivative notation follows the pattern explained earlier:

\[
[X_{\omega}] = \begin{bmatrix}
X_{\omega_1} & X_{\omega_2} & \ldots & \ldots \\
Y_{\omega_1} & Y_{\omega_2} & \ldots & \ldots \\
Z_{\omega_1} & Z_{\omega_2} & \ldots & \ldots
\end{bmatrix}
\]

Appendix M.8

\[ [X_{o,o}] = \begin{bmatrix}
X_{o_1,o_1} & X_{o_1,o_2} & X_{o_1,o_3} \\
X_{o_2,o_1} & X_{o_2,o_2} & \cdot \\
X_{o_3,o_1} & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{bmatrix} \]

\( X_{o,o} \) is of course equal to \( X_{o_{o,o}} \). If the \( o \) variables concerned in the Taylor expansion are \( h; u, v, w; p, q, r; \ldots \), the matrix of second derivatives can be written in the partitioned form shown below, where for convenience \( \{h\} \) is defined to be \( \{h \ 0 \ 0\} \), and so on.

\[ [X_{o,o}] = \begin{bmatrix}
[X_h] & [X_u] & [X_p] \\
[X_u] & [X_v] & [X_q] \\
[X_p] & [X_w] & [X_r] \\
\cdot & \cdot & \cdot \\
\end{bmatrix} \]

M.9. Equations of motion for a deformable object of constant mass.

In a deformable object we usually consider convenient assemblies of individual particles of mass \( m_i \), and express the variations in the co-ordinates of the particles (relative to a reference point 0) in terms of as few deformation variables as possible. For example, one variable may describe the relative movement of two rigid parts such as a fin and the remainder of the aircraft. Body axes are defined as described in Section 4.1. They rotate in unison with a reference trihedron (see Section 3) whose vertex is at the origin 0. The motion of the body axes is specified as usual by velocity components \( \{u\} \) and \( \{p\} \), which may be further expressed in terms of datum values \( \{u_d\}, \{p_d\}, \) and of perturbations \( \{u'\}, \{p'\}, \) or \( \{x^+\}, \{\phi\} \). The co-ordinates \( \{x\} \) of 0 relative to an earth-fixed point \( 0_0 \) are unlikely to be needed, since in trajectory calculations the co-ordinates \( \{x_0\} \) referred to earth axes will be of more interest. No confusion need arise therefore if we use \( \{x_i\} \), as in Section M.6, to denote the co-ordinates of any particle \( m_i \) relative to 0.

The general equations of motion are as given already, namely (M.20a) and (M.21a), and, as before, it is convenient to separate the forces and moments of gravitational origin from the others by writing

\[ \begin{align*}
\{X^C\} &= \{X\} + m \{g_x\}, \\
\{L^C\} &= \{L\} + mA_{xo} \{g_x\}. 
\end{align*} \]  

(M.27)

The applied force \( \{X^C\} \) is the resultant of the external forces, such as the aerodynamic and gravitational ones, and of internal forces caused by the deformation of the aircraft. Earth contact forces and moments, \( \{X^C\} \) and \( \{L^C\} \), have not been specifically included in equations (M.27), but they could be added when necessary. Additional equations of motion must be established connecting the relative motion of parts of the aircraft with the forces and moments acting on those portions.

Before the equations can be solved all the forces and moments must be expressed as functions of the variables. The nature of these functions and a suitable general notation are not considered in this Report, but it is useful to discuss some aspects of the problem when small perturbations from a datum condition are assumed.

59
Appendix M.9

In a linearized treatment we would apply equations (M.20a) and (M.21a), and also write

\[ \{ x_1 \} = \{ x_{ei} \} + \sum_{j=1}^{n} \{ a_j q_j \}, \]

where \( a_j \) are constants, and \( q_j \) are generalised perturbation co-ordinates (including main and subsidiary control deflections). 'Rigid-body' variables of the velocity (or displacement) type may then for convenience be denoted by \( q_{a+1}, q_{a+2}, \ldots, q_{a+n} \). It should be noted that the shape of the aircraft is taken to be constant in the datum condition, so that \( x_{ei} \) is written rather than \( x_{fi} \).

It is easily seen that in the datum condition equations (M.20a) and (M.21a) reduce to

\[ m \{ \dot{u}_f \} + m A_{pf} \{ u_f \} + m (A_x + A_y^2) \{ x_0 \} = \{ X \} + m \{ g_x \} \]

\[ m (A_{xc}) \{ \dot{u}_f \} + A_{pf} \{ u_f \} + (I_x) \{ \dot{p}_f \} + A_p(I_y) \{ p_f \} = \{ L \} + m (A_{xc}) \{ g_x \} \]\n
(M.28)

(M.29)

When \( \{ u_f \} \) and \( \{ p_f \} \) are constant, the left-hand sides reduce to

\[ m A_{pe} \{ u_e \} + m A_{pe}^2 \{ x_0 \} \]

and

\[ m (A_{xc}) \{ u_e \} + A_{pe} \{ I_x \} \{ p_e \} \]

respectively,

and if in addition \( \{ p_e \} \) is zero, they both vanish.

If we are mainly interested in the 'rigid-body modes' and deem it sufficient to take account of only one or two 'aeroelastic modes', we may prefer to retain the conventional symbols for the rigid-body variables and introduce special ones for the body deformation variables. The choice of these will depend on the circumstances. For instance, perhaps \( \varepsilon \) and \( \sigma \) or \( \phi \) (Russian letter pronounced like a long 'eh' as in 'air') would in general be convenient, but in a purely longitudinal problem the symbols \( \phi \) and \( \psi \) might be used, and in a purely lateral problem, \( \theta \) and \( \eta \). Symbols would also be needed to represent the forces or moments involved in the deformation equations. Perhaps \( E \) and \( T \) could be used for this purpose, and we would then end up with derivatives of the form \( X_w, X_v, E_w, E_v, T_w, T_v \), and so on, if we were to expand the expressions in Taylor series. In addition we would have concise quantities denoted by \( x_w, x_v, e_w, e_v, t_w, t_v \), and so on, although the concise quantities \( e_w, t_w \) etc. might (in a longitudinal problem, for example,) be defined as \( -E_w/i_w - T_w/i_v \), rather than \( -E_w/i_w - T_v/i_v \), etc., where \( i_w \) and \( i_v \) represent inertia parameters which would reduce the coefficient of \( \varepsilon \) to unity in the \( E \) force equation, and the coefficient of \( \delta \) to unity in the \( T \) equation. Additional inertia parameters would appear in the \( \{ X \} \) and \( \{ L \} \) equations, but there should be no difficulty in choosing suitable suffixes and using the letters \( I \) and \( i \) in all inertia terms.

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